

# PHYS 304 - Introduction to Quantum Mechanics

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Primary Reference:

Introduction to Quantum Mechanics, Third Edition, David J. Griffiths and Darrell F. Schroeter, Cambridge University Press 2018

The following Contents also serve as a syllabus.

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# Chapter 1

## Introduction

Everything in this course is dealt with in the non-relativistic limit, which is implied throughout.

### 1.1 Why do we need quantum mechanics?

The vast majority of our direct interactions with the physical world, through the senses of touching, hearing, and seeing, can be *adequately* and quantitatively modelled using a combination of classical mechanics and classical electrodynamics. The underlying kinetic equation of classical mechanics (pertaining to the things we touch and our ears hear) is Newton's equation, and the kinetic equations defining the classical properties of the electric and magnetic fields associated with light (we see) are the Maxwell Equations. These two sets of equations are coupled via the Lorentz equation that describes the force exerted on particles in the presence of electric and magnetic fields.

The adjective *adequately* is used because the majority of everyday observed phenomena, and most engineering design, can be explained or accomplished using only these classical equations of motion, along with the underlying concepts and principles of classical mechanics and classical optics.

Although it is not usually emphasized, or the subtleties dwelled upon at the undergraduate level, each set of equations come in "microscopic" and "macroscopic" forms. Think about how one would have to go about rigorously solving the problem of how the trajectories of two billiard balls are altered after a collision, using the classical equations of motion.....

Even if one was was ignorant of the atomic nature of matter, it is strictly necessary to break each billiard ball into a collection of differential mass elements,  $\Delta m_i$ , and solve

$\Delta m_i \frac{d^2 \Delta x_i}{dt^2} = F_i$ , given a set of initial conditions  $x_i(t=0)$ ,  $\frac{dx_i}{dt}(t=0)$ , where  $x_i(t)$  describes the position of the centre of mass of the differential mass element  $\Delta m_i$ , and  $F_i$  is the net force exerted on the  $i^{th}$  particle. This of course leads to a massive number of coupled differential equations for all of the mass elements because  $F_i$  is determined by the forces exerted on the  $i^{th}$  particle by all other mass elements in contact with it. This practically impossible mathematical problem is rendered "trivial" to solve if one assumes that the macroscopic billiard ball is infinitely rigid and undergoes no deformation as it rolls and collides, which can only be possible if all of the *internal* forces between mass elements exactly cancel one another at all times. Then one gets the familiar rigid body, or macroscopic form of Newton's equation, that relates the acceleration of the ball's *centre of mass position* to its total mass  $M$ , and the *external* force exerted on the ball as a whole. Similarly, to solve for the reflectivity of a metal mirror, strictly speaking one would have to use the Lorentz equation to solve for the effect of the electric field associated with the incident light wave on the motion of each electron in the metal surface of a mirror, and then solve the microscopic Maxwell Equations to figure out how each of those accelerating electrons radiates. In practice, by averaging all of those coupled microscopic equations of motion over macroscopic lengthscales, one obtains the macroscopic Maxwell Equations that describes the optical response of the metal mirror in terms of its complex dielectric response function,  $\epsilon$ , and macroscopic current and charge *density distributions* that average over all the discrete electrons.

What messages should we take away from this reflection? That for the majority of everyday experiences, although real material objects are composed of a huge number of elementary particles, the details of how they are interconnected via highly complex *internal* forces is largely irrelevant to how they respond, as a well-defined collection, to macroscopic external forces, and this is because when averaged over many particles, these *internal* forces largely cancel each other. In practice, Newton's equation of motion very accurately describes the dynamics of the centre of mass of bodies composed of many elementary particles.

However, this does not mean that quantum mechanics is not playing a role in the problem: one needs to use quantum mechanics to understand how the material object forms in the first place!

## 1.2 Deconstructing the fundamentals of classical mechanics

From now on in this course, to introduce the concepts and tools of quantum mechanics, we will consider the dynamics of a single non-relativistic "particle of mass  $m$ " moving in some time-independent external potential  $V$ . Until the final section of the course,  $V$  is assumed to vary in just one dimension, labeled by position  $x$ . We will not consider the quantum mechanical treatment of things like light, that are classically treated as waves.

In this context, Newton's equation of motion is a second order differential equation for the temporal variation of the particle's position, which is equal to the local negative gradient of the external potential, divided by the particle mass. The classical solution to this problem is

to define  $F(x) = -\frac{dV(x)}{dx}$ , and solve  $m\frac{d^2x}{dt^2} = F(x)$ , which requires specification of the particle position and velocity at two specific times, usually taken to be the same time.

Thus, knowing what the position and momentum of the particle is at one instant in time, and knowing the potential landscape it experiences, Newton's equation provides a recipe for unambiguously predicting its future trajectory ( $x(t)$ ) (and even recounting what its past trajectory must have been!).

Where, if anywhere, would uncertainty come into this problem? ....

From your laboratory courses, you should appreciate that any measurement has some associated uncertainty, be it due to the resolution of the instrument(s) used in the measurement, "noise", or practical limitations involved in using the apparatus, etc. Thus, the initial value of the position and velocity will never be precise, and one can truly only specify some range of possible trajectories consistent with the *measurement uncertainties* in the initial conditions. The "uncertainty" in the predicted trajectory has nothing to do with the equations of motion or their interpretation, but only to do with practical limitations imposed on specifying the initial conditions.

### 1.3 The recipe for solving the 1D particle motion problem using quantum mechanics

The solution of the 1D particle motion problem in quantum mechanics involves a different differential equation, where the function that is being solved for is **not** the particle's trajectory, but rather a *wavefunction* that *contains all the information it is possible to know about the particle's dynamics*. For instance, the particle's trajectory must somehow be evaluated from knowledge of its wavefunction. Now there are two ingredients to the recipe for solving the problem quantum mechanically; i) the differential equation that must be solved to determine the particle's wavefunction, and ii) a method of extracting information about its "trajectory" (more generally its dynamical evolution), once the wavefunction is known.

Obviously the quantum mechanical approach to solving the problem is both quantitatively and qualitatively completely different than the classical recipe: this course is devoted to familiarizing you with the underlying concepts and mathematical techniques involved in quantum mechanical approaches to solving particle kinetics problems.

The wavefunction that carries all of the information possible about the particle's dynamics is  $\Psi(x, t)$ , and the differential equation it satisfies is Schödinger's equation:

$$\frac{\partial \Psi(x, t)}{\partial t} = -\frac{i}{\hbar} \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right) \Psi(x, t), \quad (1.1)$$

a partial differential equation in both position and time. The exact recipe for extracting information about measurable quantities from knowledge of the wavefunction at all  $x, t$  will be introduced in future lectures.

## 1.4 The wavefunction

The first, and arguably the most important conceptual step towards understanding the subtleties of this process is to appreciate how the wavefunction most directly connects to our classical notion of a particle. Noting that the wavefunction is in general a complex function (i.e. it in general has both real and imaginary parts at any  $x, t$ ), the closest connection between the wavefunction and the classical notion of the corresponding particle is in the fact that the quantity,

$$\int_a^b |\Psi(x, t)|^2 dx \quad (1.2)$$

is equal to the probability of finding the particle at some position between  $x = a$  and  $x = b$ , at time  $t$ . Note that this statement is correct even if one assumes there is no measurement uncertainty in specifying the particle's position. Quantum mechanics has a built-in, intrinsic indeterminacy that only lets one specify the probability of measuring any particular value of some measurable quantity associated with the particle described by the wavefunction. Under some circumstances the wavefunction of a particle may be such that this intrinsic indeterminacy is much less than the measurement uncertainty for at least a range of times, in which case the particle in that state would be behaving much like the solution of Newton's equation would predict, for that period of time, but this is **not** generally true.

One very important implication of this interpretation of the wavefunction has to do with the effect of the measurement process itself on the wavefunction. Without knowing any of the details of how the measurement of the particle's position is made (i.e. what time-dependent additional terms are added to the potential it feels), one can deduce that if a measurement of the particle's position specifies the location of the particle at time  $t$  to be  $x = c \pm \Delta x$ , then immediately following the measurement, at time  $t + \Delta t$ , the squared magnitude of the wavefunction must be peaked at  $x = c$  and have a limited range of  $\Delta x$  around  $x = c$ . Thus the process of a position measurement is said to "collapse the wavefunction" at the location the particle is measured. Note that since the wavefunction is in general complex, this does not fully specify the wavefunction after the measurement, but it tells one a lot! (One needs to know more details about how the measurement is carried out to extract any phase information).

Pause and summarize what has been covered in this first introduction to the amazing world of quantum mechanics;

1. although quantum mechanics, like classical mechanics, involves solving a differential equation to obtain the "answer", the entity that is being solved for, the particle's



wavefunction, is vastly different than the trajectory of the particle that is solved for in classical mechanics

2. the most direct way to connect a particle's wavefunction to its more familiar classical properties is by recognizing that in quantum mechanics,

$$\int_a^b |\Psi(x, t)|^2 dx \quad (1.3)$$

is equal to the probability of finding the particle at some position between  $x = a$  and  $x = b$ , at time  $t$ .

3. since the complex wavefunction can in general spread out over a considerable distance at any given time, Eqn. 1.3 implies that there is an intrinsic indeterminacy associated with quantum mechanics: in general it is not possible to precisely predict measurements of the particle's trajectory, even if one knows the exact wavefunction
4. another implication of Eqn. 1.3 is that if one does measure its position at some time to be at a specific location, within some measurement uncertainty, immediately following the measurement the squared magnitude of the wavefunction must be localized around that position with a spread related to the measurement uncertainty. This is independent of the form of the wavefunction just before the measurement is performed.

## 1.5 exercises

## 1.6 Further insight into the interpretation of a particle dynamics through knowledge of its wavefunction

Another implication of Eqn. 1.3 is that

$$\int_{-\infty}^{+\infty} |\Psi(x, t)|^2 dx = 1 \quad (1.4)$$

which follows directly from the fact that it must be possible to find the particle somewhere.

Now let's reflect on what distinguishes a classical wave from a classical particle. The notion of a classical particle is based on it being at some well-defined (precisely *localized*) position at any given time. What is the purest form of a wave? Think of a pebble dropped in a calm pond, or the wake of a boat travelling through the water. In both cases, if one took a snapshot of the amplitude of the surface wave at some instant in time, it would look like a sinusoidal "wave" sketched in textbooks with an abscissa in units of position. Or, if one monitored the amplitude of the wave at a fixed location in space and plotted it as a function of time, the graph would again look like a "textbook" sinusoid, but now with the abscissa in units of time, rather than position. The former graph would have an