Phys 304 Infinite Square Well Dynamics

Define:

$$\psi_1(x) = \sqrt{\frac{2}{a}} \sin(\frac{\pi x}{a})$$

$$\psi_2(x) = \sqrt{\frac{2}{a}} \sin(\frac{2\pi x}{a})$$

Then the question is whether

$$\psi(x) = \sqrt{\frac{1}{a}} (\sin(\frac{\pi x}{a}) + \sin(\frac{2\pi x}{a}))$$

is a valid wavefunction.
First we must check normalization:

$$\int_0^a \psi(x)\psi^*(x)dx = \frac{1}{a}\int_0^a \sin^2(\frac{\pi x}{a}) + \sin^2(\frac{2\pi x}{a}) + 2\sin(\frac{\pi x}{a})\sin(\frac{2\pi x}{a})dx$$

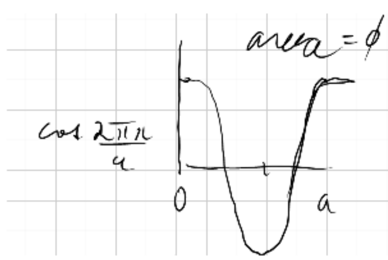
Trigonometric Identities:

$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$$

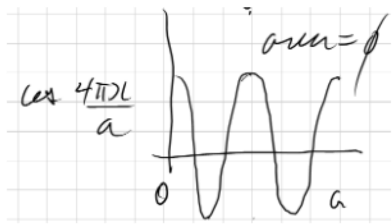
$$\sin(\theta)\sin(\phi) = \frac{\cos(\theta-\phi)-\cos(\theta+\phi)}{2}$$

Using these trig identities, we get:

$$\sin^2(\frac{\pi x}{a}) = \frac{1}{2} - \frac{\cos(\frac{2\pi x}{a})}{2}$$



$$\sin^2(\frac{2\pi x}{a}) = \frac{1}{2} - \frac{\cos(\frac{4\pi x}{a})}{2}$$



$$2\sin(\frac{\pi x}{a})\sin(\frac{2\pi x}{a}) = \cos(\frac{\pi x}{a}) - \cos(\frac{3\pi x}{a})$$

So:

$$\int_0^a \sin^2(\frac{\pi x}{a})dx = \frac{a}{2} = \int_0^a \sin^2(\frac{2\pi x}{a})dx$$

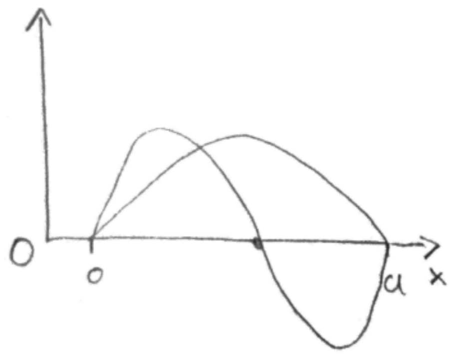
$$\int_0^a 2\sin(\frac{\pi x}{a})\sin(\frac{2\pi x}{a})dx = c_1\sin(\frac{\pi x}{a})|_0^a - c_2\sin(\frac{3\pi x}{a})|_0^a$$

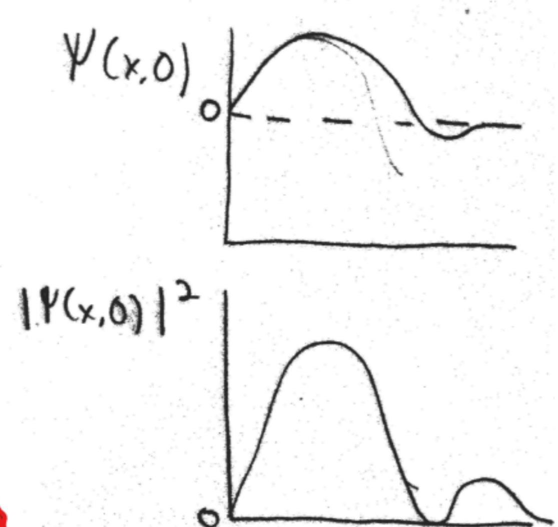
In the latter two integrals, both terms vanish, so the integral is just 0. Therefore we are left with just:

$$\int_0^a \psi(x)\psi^*(x)dx = 1$$

So yes, this wave function is normalized.

Sketch $\Psi(x, 0)$, just qualitatively:





What is $\Psi(x,t)$?

$$\Psi(x,t) = \sqrt{\frac{1}{a}}\left(\sin(\frac{\pi x}{a})e^{-i\frac{\hbar\pi^2}{2ma^2}t} + \sin(\frac{2\pi x}{a})e^{-i4\frac{\hbar\pi^2}{2ma^2}t}\right)$$

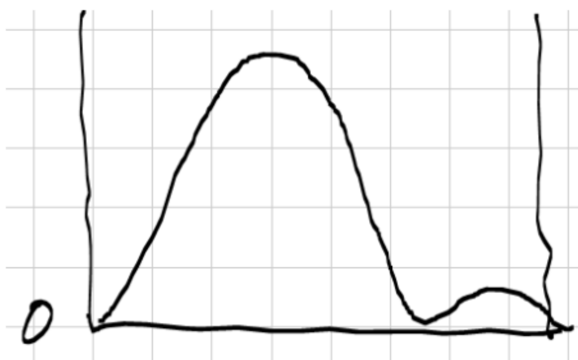
Now we can define:

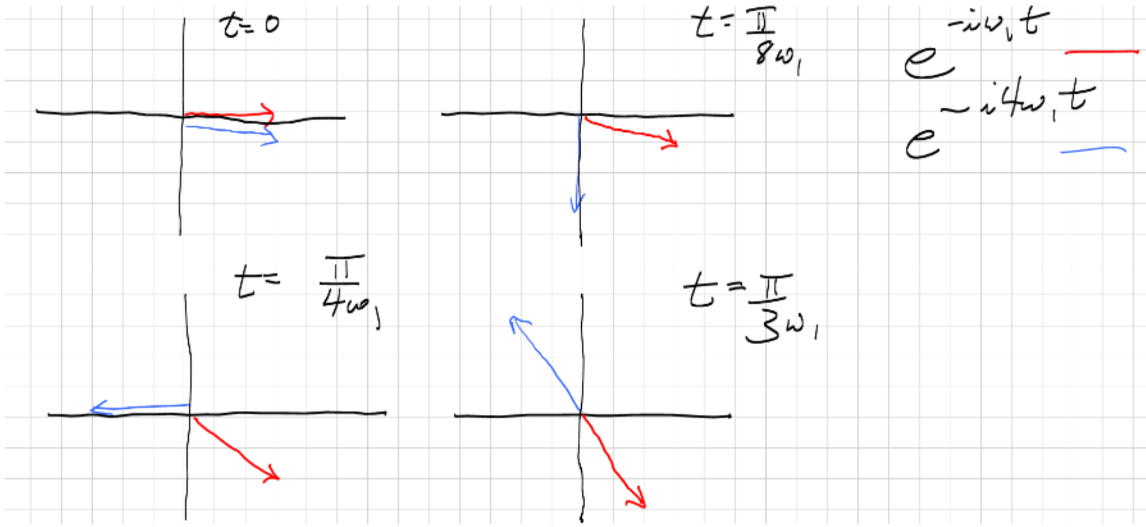
$$\omega_1 \equiv \frac{\hbar\pi^2}{2ma^2}$$
$$k \equiv \frac{\pi}{a}$$

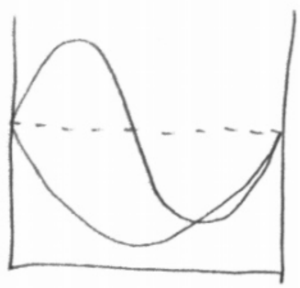
Then we get:

$$\Psi(x,t) = \sqrt{\frac{1}{a}}\left(\sin(kx)e^{-i\omega_1 t} + \sin(2kx)e^{-i4\omega_1 t}\right)$$

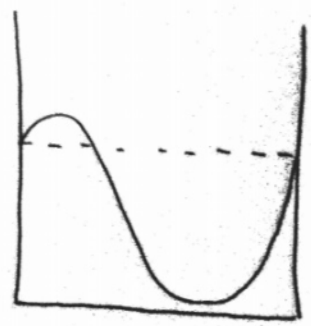
This is a periodic function in time (what is the period?), but we are interested in the dynamics of the probability density, so the question is, after t = 0, when will $|\Psi(x, t =?)|^2$ again look like:



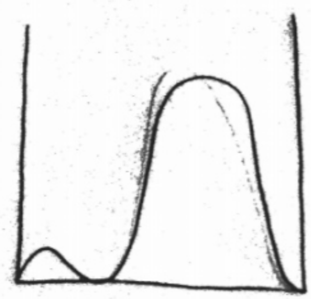




$\Rightarrow \quad \Psi\left(x, t=\frac{\pi}{3\omega_1}\right) \times e^{i\frac{4\pi}{3}}$



$\left|\Psi\left(x,t=\frac{\pi}{3\omega_1}\right)\right|^2$



So $t=?$ when $|\Psi(x,t)|^2 = |\Psi(x,0)|^2$?

$$
\begin{aligned}
\omega_1 t =& 4\omega_1 t - m2\pi \\
3\omega_1 t =& m2\pi \\
t_m =& \frac{2\pi}{3\omega_1} m
\end{aligned}
$$

- **Describe what the particle is "doing"**

- **What would you expect $\langle p\rangle(t = \frac{\pi}{6\omega_1})$ to look like?**

Recall that the operator for p is $-i\hbar\frac{d}{dx}$, so:

$$\langle p\rangle(t) = \int_0^a \Psi^*(x,t)(-i\hbar\frac{d}{dx})\Psi(x,t)dx$$

Then we have:

$$\begin{aligned}
\Psi(x,t) =& \sqrt{\frac{1}{a}}(\sin(kx)e^{-i\omega_1 t} + \sin(2kx)e^{-i4\omega_1 t}) \\
\frac{d\Psi(x,t)}{dx} =& \sqrt{\frac{1}{a}}(k\cos(kx)e^{-i\omega_1 t} + 2k\cos(2kx)e^{-i4\omega_1 t}) \\
\therefore \int_0^a \Psi^*(x,t)(-i\hbar\frac{d}{dx})\Psi(x,t)dx =& -\frac{i\hbar}{a}(\int_0^a k\sin(kx)\cos(kx)dx + \int_0^a 2k\sin(2kx)\cos(2kx)dx \\
&+ \int_0^a 2k\sin(kx)\cos(2kx)e^{-3i\omega_1 t}dx + \int_0^a k\sin(2kx)\cos(kx)e^{+3i\omega_1 t}dx
\end{aligned}$$

Now $2\sin(\alpha)\cos(\beta) = \sin(\alpha-\beta) + \sin(\alpha+\beta)$, so the first two terms vanish.
Then, since:

$$\begin{aligned}
\sin(kx)\cos(2kx) =& \frac{1}{2}(\sin(3kx) + \sin(-kx)) \\
=& \frac{1}{2}(\sin(3kx) - \sin(kx)) \\
\sin(2kx)\cos(kx) =& \frac{1}{2}(\sin(3kx) + \sin(kx))
\end{aligned}$$

The third term gives us:

$$\begin{aligned}
\int_0^a 2k\sin(kx)\cos(2kx)e^{-3i\omega_1 t}dx =& k(\frac{-\cos(\frac{3\pi x}{a})}{3k} + \frac{\cos(\frac{\pi x}{a})}{k})|_0^a e^{-3i\omega_1 t} \\
=& ((\frac{1}{3} - 1) - (-\frac{1}{3} + 1))e^{-3i\omega_1 t} \\
=& (\frac{2}{3} - 2)e^{-3i\omega_1 t} \\
=& -\frac{4}{3}e^{-3i\omega_1 t}
\end{aligned}$$

And the fourth term:

$$
\begin{aligned}
\int_0^a k\sin(2kx)\cos(kx)e^{+3i\omega_1 t}dx =&\frac{1}{2}((\frac{1}{3}+1)-(-\frac{1}{3}-1))e^{+3i\omega_1 t} \\
=&\frac{1}{2}(\frac{2}{3}+2)e^{+3i\omega_1 t} \\
=&\frac{4}{3}e^{+3i\omega_1 t}
\end{aligned}
$$

And therefore:

$$
\begin{aligned}
\langle p\rangle(t) =& -\frac{i\hbar}{a}(\frac{4}{3}e^{+3i\omega_1 t}-\frac{4}{3}e^{-3i\omega_1 t}) \\
=&\frac{8\hbar}{3a}\sin(3\omega_1 t)
\end{aligned}
$$

What effective distance does the particle travel in one half period?