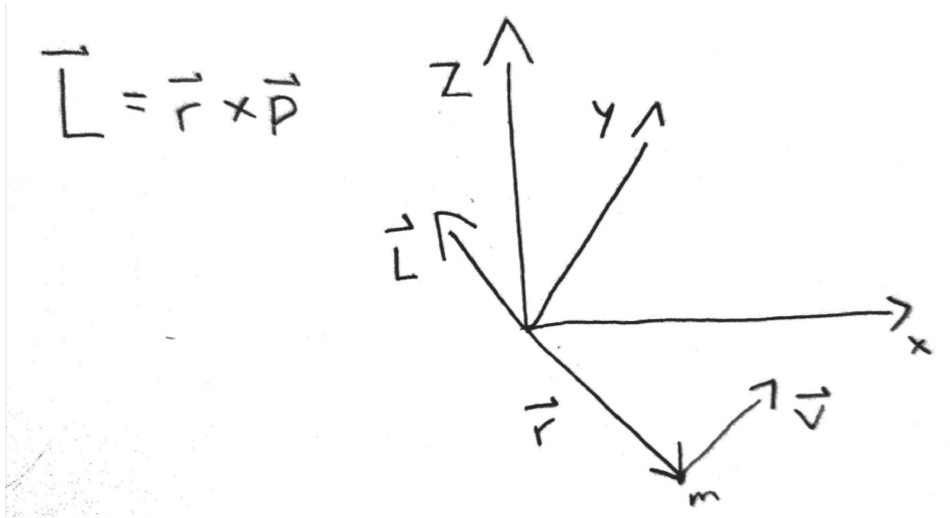
4.3.2 Orbital Angular Momentum



- Cartesian coordinates of $L_x\hat{x} + L_y\hat{y} + L_z\hat{z} = \vec{L}$ are?

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$\begin{aligned}
\implies L_x =& yp_z - zp_y \\
L_y =& zp_x - xp_z \\
L_z =& xp_y - yp_x \\
\implies \hat{\vec{L}} =& (yp_z - zp_y)\hat{x} + (zp_x - xp_z)\hat{y} + (xp_y - yp_x)\hat{z} \\
=& -i\hbar\{(y\frac{d}{dz} - z\frac{d}{dy})\hat{x} - (x\frac{d}{dz} - z\frac{d}{dx})\hat{y} + (x\frac{d}{dy} - y\frac{d}{dx})\hat{z}\} \\
[\hat{L}_x, \hat{L}_y] =& \hbar^2[(y\frac{d}{dz} - z\frac{d}{dy}), (x\frac{d}{dz} - z\frac{d}{dx})] \\
=& \hbar^2\{[y\frac{d}{dz}, x\frac{d}{dz}] - [y\frac{d}{dz}, z\frac{d}{dx}] - [z\frac{d}{dy}, x\frac{d}{dz}] + [z\frac{d}{dy}, z\frac{d}{dx}]\} \\
=& \hbar^2(-y\frac{d}{dx} + x\frac{d}{dy}) \\
=& i\hbar\hat{L}_z
\end{aligned}$$

Note that all the second order derivatives cancel each other out, leaving us with just the final expression. Now, these are cyclic permutations too; that is:

$$
\begin{aligned}
[\hat{L}_x, \hat{L}_y] =& i\hbar \hat{L}_z \\
[\hat{L}_y, \hat{L}_z] =& i\hbar \hat{L}_x \\
[\hat{L}_z, \hat{L}_x] =& i\hbar \hat{L}_y \\
|\hat{\vec{L}}|^2 =& \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2
\end{aligned}
$$

So, we can find:

$$
\begin{aligned}
[|\hat{\vec{L}}|^2, \hat{L}_x] =& [\hat{L}_x^2, \hat{L}_x] + [\hat{L}_y^2, \hat{L}_x] + [\hat{L}_z^2, \hat{L}_x] \\
=& [\hat{L}_y^2, \hat{L}_x] + [\hat{L}_z^2, \hat{L}_x]
\end{aligned}
$$

Now, note that:

$$
\begin{aligned}
[AB, C] =& ABC - CAB \\
=& A[B, C] + [A, C]B \\
=& ABC - ACB + ACB - CAB \qquad \text{Check!} \\
=& ABC - CAB \qquad \text{As desired}
\end{aligned}
$$

Now, using this, we can write:

$$
\begin{aligned}
[|\hat{\vec{L}}|^2, \hat{L}_x] =& \hat{L}_y[\hat{L}_y, \hat{L}_x] + [\hat{L}_y, \hat{L}_x]\hat{L}_y + \hat{L}_z[\hat{L}_z, \hat{L}_x] + [\hat{L}_z, \hat{L}_x]\hat{L}_z \\
=& i\hbar\{-\hat{L}_y\hat{L}_z - \hat{L}_z\hat{L}_y + \hat{L}_z\hat{L}_y + \hat{L}_y\hat{L}_z\} \\
=& 0
\end{aligned}
$$

What does this mean physically and quantum mechanically?

$$
[|\hat{\vec{L}}|^2, \hat{L}_x] = [|\hat{\vec{L}}|^2, \hat{L}_y] = [|\hat{\vec{L}}|^2, \hat{L}_z] = 0
$$

This means that $|\hat{\vec{L}}|^2$ and any of $\hat{L}_x$, $\hat{L}_y$, **or** $\hat{L}_z$ can share a complete set of eigenfunctions, but these will necessarily not be eigenfunctions of the other two orthogonal components of $\hat{\vec{L}}$ because $[\hat{L}_x, \hat{L}_y], [\hat{L}_y, \hat{L}_z], [\hat{L}_z, \hat{L}_x] \neq 0$. This implies that if I choose any one of $\hat{L}_x$, $\hat{L}_y$, or $\hat{L}_z$ to generate a shared set of eigenfunctions, it will not commute with the other $\hat{L}_i$'s. Therefore it is impossible to precisely know $\hat{\vec{L}}$ - at best, we can only know one vector component of it.

Recall:

$$\begin{aligned}
[\hat{L}_x, \hat{L}_y] =& i\hbar\hat{L}_z \\
[\hat{L}_y, \hat{L}_z] =& i\hbar\hat{L}_x \\
[\hat{L}_z, \hat{L}_x] =& i\hbar\hat{L}_y
\end{aligned}$$

What are the commutation relationships between the raising and lowering ladder operators, $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$, with $\hat{L}_z$ and $\hat{L}^2$? Note that since $[|\hat{\vec{L}}|^2, \hat{L}_x] = [|\hat{\vec{L}}|^2, \hat{L}_y] = 0$ and $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$, $[\hat{L}^2, \hat{L}_\pm] = 0$, and:

$$\begin{aligned}
[\hat{L}_z, \hat{L}_\pm] =& [\hat{L}_z, \hat{L}_x] \pm i[\hat{L}_z, \hat{L}_y] \\
=& i\hbar\hat{L}_y \pm i(-i\hbar\hat{L}_x) \\
=& i\hbar\hat{L}_y \pm \hbar\hat{L}_x \\
=& \pm\hbar(\hat{L}_x \pm i\hat{L}_y) \\
=& \pm\hbar\hat{L}_\pm
\end{aligned}$$

Now, using the same general strategy used to derive the eigen spectrum of the 1D harmonic oscillator (which used similarly defined "ladder operators"), derive the spectrum of the total angular momentum and the $\hat{L}_z$ component of the angular momentum, in a common basis.

Assume we have the eigen function of $|\hat{\vec{L}}|^2 \equiv \hat{L}^2$. Then $\hat{L}^2|\Psi_\lambda\rangle = \lambda|\Psi_\lambda\rangle$, or in position basis, $\hat{L^2}_{\vec{r}} f(\vec{r}) = \lambda f(\vec{r})$. Now, what can we say about $\hat{L}_\pm|\Psi_\lambda\rangle$?

$$\begin{aligned}
\hat{L}^2(\hat{L}_\pm|\Psi_\lambda\rangle) =& \hat{L}_\pm\hat{L}^2|\Psi_\lambda\rangle \\
=& \hat{L}_\pm\lambda|\Psi_\lambda\rangle \\
=& \lambda(\hat{L}_\pm|\Psi_\lambda\rangle
\end{aligned}$$

This means that $\hat{L}_\pm$ is also an eigenstate of $\hat{L}^2$ with same eigenvalue of $\hat{L}^2$ of λ.

Now $\hat{L}_z(\hat{L}_\pm|\Psi_\lambda\rangle) = \hat{L}_z((\hat{L}_x \pm i\hat{L}_y)|\Psi_\lambda\rangle)$, and if we assume $|\Psi_\lambda\rangle$ is an eigenfunction of $\hat{L}_z$ with eigenvalue μ, then: $\hat{L}_z|\Psi_{\lambda,\mu}\rangle = \mu|\Psi_{\lambda,\mu}\rangle$ Then using the commutation relationships

$$\begin{aligned}
[\hat{L}_z, \hat{L}_y] =& -i\hbar\hat{L}_x \\
[\hat{L}_z, \hat{L}_x] =& i\hbar\hat{L}_y
\end{aligned}$$

We see that:

$$
\begin{aligned}
\hat{L}_z\hat{L}_x =& i\hbar\hat{L}_y + \hat{L}_x\hat{L}_z \\
\hat{L}_z\hat{L}_y =& -i\hbar\hat{L}_x + \hat{L}_y\hat{L}_z \\
\implies \hat{L}_z(\hat{L}_\pm|\Psi_{\lambda,\mu}\rangle =& ((i\hbar\hat{L}_y + \hat{L}_x\hat{L}_z) \pm i(\hat{L}_y\hat{L}_z - i\hbar\hat{L}_x))|\Psi_{\lambda,\mu}\rangle \\
=& (i\hbar\hat{L}_y \pm \hbar\hat{L}_x)|\Psi_{\lambda,\mu}\rangle + \mu(\hat{L}_\pm|\Psi_{\lambda,\mu}\rangle) \\
=& (\pm\hbar + \mu)(\hat{L}_\pm|\Psi_{\lambda,\mu}\rangle)
\end{aligned}
$$

This means that $(\hat{L}_\pm|\Psi_{\lambda,\mu}\rangle)$ is also an eigenfunction of $\hat{L}_z$ with eigenvalue $\mu \pm \hbar$.

So, $\hat{L}_+$ operating on a common eigenstate of $\hat{L}^2$ and $\hat{L}_z$ generates another common eigenstate with the same eigenvalue of $\hat{L}^2$ (λ) and a higher value of the eigenvalue of $\hat{L}_z$ by $\hbar$ $(\mu + \hbar)$. However, the $\langle\hat{L}_z\rangle$ component in that state cannot physically exceed $\langle\hat{L}^2\rangle$, so there must be a "top" eigen function $|\Psi^t_{\lambda,\mu}\rangle$ such that:

$$\hat{L}_+|\Psi^t_{\lambda,\mu}\rangle = 0$$

Let $\hat{L}_z|\Psi^t_{\lambda,\mu_t}\rangle = \hbar l|\Psi^t_{\lambda,\mu_t}\rangle$, i.e. $\mu_t = \hbar l$, and we know $\hat{L}^2|\Psi^t_{\lambda,\mu}\rangle = \lambda|\Psi^t_{\lambda,\mu}\rangle$. We can then use the expansion of $\hat{L}^2$ in $\hat{L}_\pm$ and $\hat{L}_z$ as:

$$
\begin{aligned}
\hat{L}^2 =& \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \\
=& \hat{L}_-\hat{L}_+ + \hat{L}_z^2 + \hat{O} \\
\hat{L}_-\hat{L}_+ =& (\hat{L}_x - i\hat{L}_y)(\hat{L}_x + i\hat{L}_y) \\
=& \hat{L}_x^2 + \hat{L}_y^2 + i[\hat{L}_x, \hat{L}_y] \\
=& \hat{L}_x^2 + \hat{L}_y^2 - \hbar\hat{L}_z \\
\implies \hat{O} =& \hbar\hat{L}_z \\
\implies \hat{L}^2 =& \hat{L}_-\hat{L}_+ + \hat{L}_z^2 + \hbar\hat{L}_z
\end{aligned}
$$

$$
\begin{aligned}
\implies \hat{L}^2|\Psi^t_{\lambda,\mu}\rangle = (0 + (\hbar^2 l^2 + \hbar^2 l))|\Psi^t_{\lambda,\mu}\rangle& \\
=& \hbar^2 l(l+1)|\Psi^t_{\lambda,\mu}\rangle \\
\implies \lambda = \hbar^2 l(l+1)&
\end{aligned}
$$

We can perform the same analysis on $\hat{L}_z|\Psi^b_{\lambda,\mu_b}\rangle = \hbar\tilde{l}|\Psi^b_{\lambda,\mu_b}\rangle$ for state $|\Psi^b_{\lambda,\mu_b}\rangle$, defined by $\hat{L}_-|\Psi^b_{\lambda,\mu_b}\rangle = 0$, and we will see that $\lambda = \hbar^2\tilde{l}(\tilde{l} - 1)$ in this case. But this implies that $\tilde{l}(\tilde{l}-1) = l(l+1)$, which means $\tilde{l} = -l$, since the other possible solution, $\tilde{l} = l+1$, is nonsense because then $\langle \hat{L}_z \rangle_b > \langle \hat{L}_z \rangle_t$.

So, from commutator algebra and simple physical arguments (i.e. $\langle \hat{L}_z \rangle^2 \leq \langle \hat{L}^2 \rangle_t)$, we can conclude that the eigenvalues of the shared eigenstates of $\hat{L}^2$ and $\hat{L}_z$ are $\lambda = \hbar^2 l(l+1)$, for $\hat{L}^2$ operating on each of a set of distinct eigenfunctions of $\hat{L}_z$ with a set of eigenvalues μ with $-\hbar l < \mu < \hbar l$. But $\hat{L}_+$ and $\hat{L}_-$ raise or lower μ by $\hbar$, so this means $l = -l + N$, where N is an integer.

$\implies l = \frac{N}{2}$
$\implies$ l must be an integer or a half integer!
And the set of μ values for a given l value are $\mu = \hbar(-l, -l+1, ..., l-1, l)$. Thus there are $2l+1$ possible integers or half integers. Call these μ values $\hbar m$.
How would you go about deriving the common set of eigenfunctions of the operators associated with $\hat{L}_z$ and $\hat{L}^2$?

$$\hat{L}^2 |\Psi_{L^2}\rangle = \lambda |\Psi_{L^2}\rangle$$

$$\hat{L}_z |\Psi_{L_z}\rangle = \mu |\Psi_{L_z}\rangle$$

We can use the position basis representations of these and solve the Differential Equations:

$$\begin{aligned} \hat{L}^2_{\theta,\phi} \psi_{L^2}(\theta,\phi) &= \lambda \psi_{L^2}(\theta,\phi) \\ \hat{L}_{z\theta,\phi} \psi_{L_z}(\theta,\phi) &= \mu \psi_{L_z}(\theta,\phi) \end{aligned}$$

Now,

$$\begin{aligned} \hat{\vec{L}} &= \hat{\vec{r}} \times \hat{\vec{p}} \\ &= r\hat{r} \times (-i\hbar \nabla_{\vec{r}}) \\ &= -i\hbar r(\hat{r} \times \nabla_{\vec{r}}) \end{aligned}$$

Where:

$$\nabla_{\vec{r}} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin(\theta)}\frac{\partial}{\partial \phi}$$

$$\mathbf{L} = -i\hbar\left[r\left(\hat{r}\times\hat{r}\right)\frac{\partial}{\partial r} + \left(\hat{r}\times\hat{\theta}\right)\frac{\partial}{\partial\theta} + \left(\hat{r}\times\hat{\phi}\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right].$$

But $\left(\hat{r}\times\hat{r}\right) = 0$, $\left(\hat{r}\times\hat{\theta}\right) = \hat{\phi}$, and $\left(\hat{r}\times\hat{\phi}\right) = -\hat{\theta}$ (see Figure 4.1), and hence

$$\mathbf{L} = -i\hbar\left(\hat{\phi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right). \tag{4.124}$$

The unit vectors $\hat{\theta}$ and $\hat{\phi}$ can be resolved into their cartesian components:

$$\hat{\theta} = (\cos\theta\cos\phi)\,\hat{\imath} + (\cos\theta\sin\phi)\,\hat{\jmath} - (\sin\theta)\,\hat{k}; \tag{4.125}$$

$$\hat{\phi} = -(\sin\phi)\,\hat{\imath} + (\cos\phi)\,\hat{\jmath}. \tag{4.126}$$

Thus

$$\mathbf{L} = -i\hbar\left[\left(-\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}\right)\frac{\partial}{\partial\theta} - \left(\cos\theta\cos\phi\hat{\imath} + \cos\theta\sin\phi\hat{\jmath} - \sin\theta\hat{k}\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right].$$

So

$$L_x = -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right), \tag{4.127}$$

$$L_y = -i\hbar\left(+\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right), \tag{4.128}$$

and

$$\boxed{L_z = -i\hbar\frac{\partial}{\partial\phi}.} \tag{4.129}$$

We shall also need the raising and lowering operators:

$$L_\pm = L_x \pm iL_y = -i\hbar\left[(-\sin\phi \pm i\cos\phi)\frac{\partial}{\partial\theta} - (\cos\phi \pm i\sin\phi)\cot\theta\frac{\partial}{\partial\phi}\right].$$

But $\cos\phi \pm i\sin\phi = e^{\pm i\phi}$, so

$$L_\pm = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right). \tag{4.130}$$

In particular (Problem 4.24(a)):

$$L_+L_- = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right), \tag{4.131}$$

and hence (Problem 4.24(b)):

$$\boxed{L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right].} \tag{4.132}$$

We can then solve for the common eigenfunctions:

$$
\begin{aligned}
\hat{L}_{z_{\theta,\phi}}\psi_{L_z}(\theta,\phi) =& \mu\psi_{L_z}(\theta,\phi) \\
-i\hbar\frac{\partial}{\partial\phi}\psi_{L_z}(\theta,\phi) =& \mu\psi_{L_z}(\theta,\phi) \\
\implies \psi_{L_z}(\theta,\phi) =& f_\mu(\theta)e^{i\frac{\mu}{\hbar}\phi}
\end{aligned}
$$

And:

$$
-\hbar^2[\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}(\sin(\theta)\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}]\psi_{L^2}(\theta,\phi) = \lambda\psi_{L^2}(\theta,\phi)
$$

Since $\left[|\vec{\hat{L}}|^2, \hat{L}_z\right] = 0$, $\psi_{L_z}(\theta,\phi) = f_\mu(\theta)e^{i\frac{\mu}{\hbar}\phi} = \psi_{L^2}(\theta,\phi)$. This implies that:

$$
-\hbar^2[\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}(\sin(\theta)\frac{\partial}{\partial\theta}) - \frac{1}{\sin^2(\theta)}\frac{\mu^2}{\hbar^2}]f_\mu(\theta) = \lambda f_\mu(\theta)
$$

Or, multiplying through by $\frac{\sin^2(\theta)}{\hbar^2}$ and moving the right hand side to the left:

$$
\sin(\theta)\frac{\partial}{\partial\theta}(\sin(\theta)\frac{\partial f_\mu(\theta)}{\partial\theta}) + (\frac{\lambda}{\hbar^2}\sin^2(\theta) - \frac{\mu^2}{\hbar^2})f_\mu(\theta) = 0
$$

From the textbook, we have:

$$
\sin(\theta)\frac{\partial}{\partial\theta}(\sin(\theta)\frac{\partial f_\mu(\theta)}{\partial\theta}) + (l(l+1)\sin^2(\theta) - m^2)f_\mu(\theta) = 0
$$

$\implies \frac{\lambda}{\hbar^2} \equiv l(l+1)$ and $\frac{\mu}{\hbar} \equiv m$, and this is exactly what we have already solved in the general solution of spherically symmetric potentials in the Schrödinger Equation. This implies that the common set of eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$ are the spherical harmonics $Y_l^m(\theta,\phi)$! For example,

$$
\begin{aligned}
|\vec{\hat{L}}|^2 Y_l^m(\theta,\phi) =& \hbar^2 l(l+1)Y_l^m(\theta,\phi) \\
\hat{L}_z Y_l^m(\theta,\phi) =& \hbar m Y_l^m(\theta,\phi)
\end{aligned}
$$