Recap:

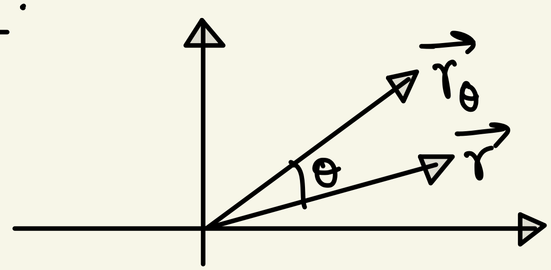
- In 2D Real space, $|\vec{r}\rangle \cong \begin{bmatrix} x \\ y \end{bmatrix}$; $\langle \vec{r}| \cong [x \;\; y] = \begin{bmatrix} x \\ y \end{bmatrix}^T$
- $\langle r||r'\rangle \equiv \langle r|r'\rangle = xx' + yy' \in \mathbb{R}$
- $\{|\alpha_1\rangle, |\alpha_2\rangle\} \Rightarrow$ Orthonormal basis.

  (a) Orthonormality $\Rightarrow \langle \alpha_i | \alpha_j \rangle = \delta_{ij}$

  (b) Completeness $\Rightarrow \sum_{i=1}^{2} \underbrace{|\alpha_i\rangle\langle\alpha_i|}_{P_i} = \mathbb{I}$ ← Resolution of the identity operator.

  P_i ← Projectors

- <u>The rotation operator</u>:



$$R_\theta |\vec{r}\rangle = |\vec{r}_\theta\rangle$$

$$= \sum_{j=1}^{2} R_\theta |\alpha_j\rangle\langle\alpha_j|\vec{r}\rangle$$

$$\Rightarrow \langle \alpha_i | \vec{r}_\theta \rangle = \sum_{j=1}^{2} \underbrace{\langle \alpha_i | R_\theta | \alpha_j \rangle}_{} \langle \alpha_j | \vec{r} \rangle$$

matrix elements of R_θ in α_i, α_j basis!

- $(\hat{R}_\theta)_{ij} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Complex Vector Spaces:

Although the space $\mathbb{R}^2$ is most intuitive to us, QM has complex numbers in built. (See TDSE!!)

- A complex inner pdt has properties: (i) $\langle u|v\rangle = \langle v|u\rangle^*$.

  (ii) $\langle u|u\rangle \geq 0$ equality satistifed iff $u=0$

  (iii) Sesquilinear: (a) $\langle u|c_1v_1+c_2v_2\rangle = c_1\langle u|v_1\rangle + c_2\langle u|v_2\rangle$

  (b) $\langle c_1u_1+c_2u_2|v\rangle = c_1^*\langle u_1|v\rangle + c_2^*\langle u_2|v\rangle$

- Concrete examples:
  - $a=\begin{pmatrix}a_1\\a_2\end{pmatrix}; b=\begin{pmatrix}b_1\\b_2\end{pmatrix}$ where $x_i, y_i \in \mathbb{C}$

    $$\langle a|b\rangle \equiv a_1^*b_1 + a_2^*b_2 \in \mathbb{C}$$

  - Consider two complex valued func^n s $f(x), g(x) \in \mathbb{C}$ in $L^2(\mathbb{R})$

    $$\langle f|g\rangle = \int_{-\infty}^{\infty} dx\, f^*(x)g(x)$$

Note that if $\underset{\text{"ket"}}{|a\rangle} = \begin{pmatrix}a_1\\a_2\\ \vdots\\ a_N\end{pmatrix} \longleftrightarrow \underset{\text{"bra"}}{\langle a|} = (a_1^*\ a_2^* \ldots\ldots a_N^*)$

- Question: if $|v\rangle = \alpha_1|v_1\rangle + \alpha_2|v_2\rangle$ then what is $\langle v|$? $= \alpha_1^*\langle v_1| + \alpha_2^*\langle v_2|$

$\Rightarrow$ Abstractly: $\langle v|u\rangle \underset{\uparrow}{=} \langle u|v\rangle^* = \langle u|\alpha_1 v_1\rangle^* \underset{\uparrow}{+} \langle u|\alpha_2 v_2\rangle^* = \alpha_1^*\langle v_1|u\rangle + \alpha_2^*\langle v_2|u\rangle$

$= (\alpha_1^*\langle v_1| + \alpha_2^*\langle v_2|)\,|u\rangle$

property of inner pdt

again inner pdt property

Here, if you have an ONB $\{|i\rangle\}$, any vector v can be expanded as:

$$|v\rangle = \sum_i |i\rangle\langle i|v\rangle = \sum_i c_i |i\rangle \text{ where } c_i \equiv \langle i|v\rangle$$

Operators and Matrix elements:

- Operators are objects that eat a vector and spit out some different vector.

$$O : V \longrightarrow V$$
$$|v\rangle \longmapsto O|v\rangle$$

- O is naturally written as $O = |u\rangle\langle v|$

- $$O = \sum_{i,j} |i\rangle \underbrace{\langle i|O|j\rangle}_{O_{ij} \to \text{ matrix elements.}} \langle j|$$

- Note this notation is self-correcting: • Matrix multiplication: $(AB)_{ij} = \langle i|AB|j\rangle$

$$= \sum_k \langle i|A|k\rangle\langle k|B|j\rangle$$
$$= \sum_k A_{ik} B_{kj} \checkmark$$

Adjoint of a linear operator

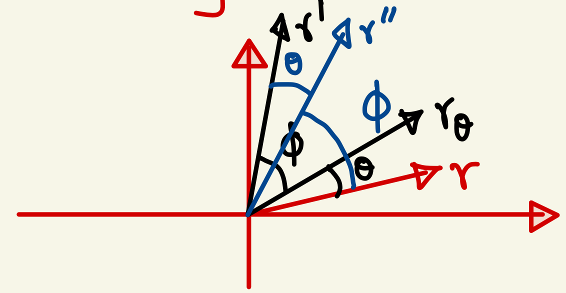
$$\langle u | O v\rangle =: \langle O^\dagger u | v\rangle \quad \forall\, u, v \in V$$

$\langle u | O | v\rangle$

Conjugate: $\langle O v | u\rangle = \langle v | O^\dagger | u\rangle$

$$\Rightarrow \quad \boxed{\langle v | O^\dagger = \langle O v|}$$

A motivating example: (2D Real Space)



$r'' = R_\theta | r'\rangle$

$$\langle r' | R_\theta r\rangle = \langle R_\theta^\dagger r' | r\rangle = \cos\phi$$

See figure: $R_\theta^\dagger = R_{-\theta}$!

$$\boxed{\text{The bra associated with } O|v\rangle \text{ is } \langle v | O^\dagger.}$$

Now note $\boxed{\langle u | O | v\rangle^* = \langle v | O^\dagger | u\rangle}$; if we choose u, v as ONB vectors:

$$\langle i | O^\dagger | j\rangle = \langle j | O | i\rangle^*$$

$$\Rightarrow (O^\dagger)_{ij} = (O_{ji})^*$$

Explicit matrix elements!

Exercise: If $O = |a\rangle\langle b|$, write $O^\dagger = ? \rightarrow |b\rangle\langle a|$

$$O|v\rangle = |a\rangle\langle b|v\rangle = \langle b|v\rangle|a\rangle \longrightarrow \langle v| O^\dagger = \langle v|b\rangle\langle a|$$

Adjoint of the derivative operator:

- The defining eq^n: $\langle O^\dagger u | v\rangle = \langle u | O v\rangle$

  Now Consider two func^{n}s $f(x), g(x)$!

$$\int_{-\infty}^{\infty} dx\, (O^\dagger f(x))^* g(x) = \int_{-\infty}^{\infty} dx\, f^*(x) \frac{d}{dx} g(x)$$

Let $dg = dv \Rightarrow v = g$

$f^* = u \Rightarrow df^* = du$

$$= f^* g \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\, \frac{df^*}{dx} g(x)$$

$$= \int_{-\infty}^{\infty} dx \left(-\frac{d}{dx} f\right)^* g(x)$$

$$\Longrightarrow \boxed{\left(\frac{d}{dx}\right)^\dagger = -\frac{d}{dx}}$$

- Problem: $O = x\frac{d}{dx}$ find the adjoint: $O^\dagger$?

- $p \cong -i\hbar \frac{d}{dx}$; what is $p^\dagger$?

Hermitian Operators:

Note expection values are given by $\langle \hat{O} \rangle_v = \langle v|\hat{O}|v\rangle \equiv \langle v|\hat{O}v\rangle$

Let us require such quantities to be real, for all v.

I.P.P. ← Inner pdt property!

$$\langle v|\hat{O}|v\rangle \underset{\text{to be real}}{=} \langle v|\hat{O}|v\rangle^{*} \underset{\text{def. of adjoint}}{=} \langle \hat{O}^{\dagger}v|v\rangle^{*} \overset{\text{I.P.P.}}{=} \langle v|\hat{O}^{\dagger}|v\rangle$$

$\Rightarrow$ $\boxed{O = O^{\dagger}}$ → Such operators are called Hermitian!

$\boxed{\text{ALL observables in QM are necessarily Hermitian.}}$

4:10

Question: If the variance of an observable is identically 0 in a particle state for an operator $\hat{Q}$, prove that the state under consideration is an eigenstate of $\hat{Q}$ with eigenvalue $\langle\hat{Q}\rangle$!

$$\langle\psi|\hat{Q}^2 - \langle\hat{Q}\rangle^2|\psi\rangle = 0 \Rightarrow \langle\psi|(\hat{Q}-\langle\hat{Q}\rangle)(\hat{Q}-\langle\hat{Q}\rangle)|\psi\rangle = 0$$

$$\Rightarrow \langle(\hat{Q}-\langle\hat{Q}\rangle)^{\dagger}\psi|(\hat{Q}-\langle\hat{Q}\rangle)|\psi\rangle = 0$$

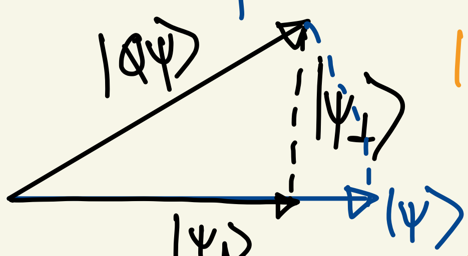
Hermitian $\hat{Q}^{\dagger} = \hat{Q}$.

$$\Rightarrow \langle(\hat{Q}-\langle\hat{Q}\rangle)\psi|(\hat{Q}-\langle\hat{Q}\rangle)|\psi\rangle = 0$$

$$\Rightarrow (\hat{Q}-\langle\hat{Q}\rangle)|\psi\rangle = 0$$

$$\Rightarrow \hat{Q}|\psi\rangle = \langle\hat{Q}\rangle|\psi\rangle$$

Geometric Interpretation.



$\|Q\psi\| = \langle\hat{Q}\rangle\|\psi\|$

Pythagoras

$\|Q\psi\|^2 = \langle\hat{Q}\rangle^2\|\psi\|^2 + \|\psi_\perp\|^2 = 0$ here!

$$|Q\psi\rangle = |\psi_\parallel\rangle + |\psi_\perp\rangle$$

where $|\psi_\parallel\rangle = P_{|\psi\rangle}|Q\psi\rangle = |\psi\rangle\langle\psi|Q|\psi\rangle = \langle Q\rangle|\psi\rangle$

Question: Prove that the eigenvectors of any Hermitian operator (ignore degeneracy for now) will form an ONB for the space of vectors the operator acts on!

Consider eigenstates of $\hat{O}$ labelled by $\{|a_i\rangle\}_{i=1}^{N}$

$$\langle a_i|\hat{O}|a_j\rangle = \langle a_i|\hat{O}^\dagger|a_j\rangle \Rightarrow a_j\langle a_i|a_j\rangle = a_i^*\langle a_i|a_j\rangle$$

Case I: $i=j \Rightarrow a_i = a_i^* \ \forall i \Rightarrow a_i \in \mathbb{R}$ for all i

$$a_j\langle a_i|a_j\rangle = a_i\langle a_i|a_j\rangle$$

Case 2: $i\neq j \Rightarrow \langle a_i|a_j\rangle = 0$

Thus they form an ONB!

one can fix the normalization for indivisual eigenvectors.

This is a fundamental fact for QM!

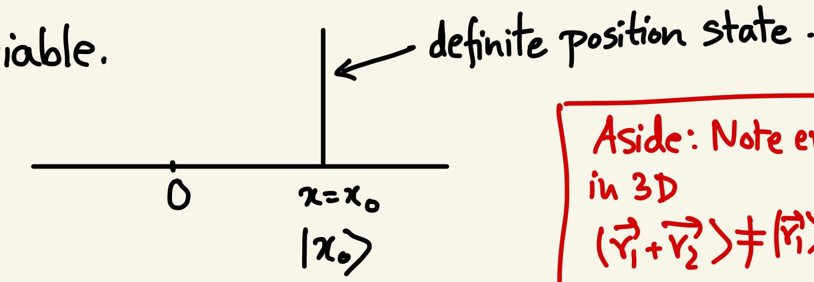
$$\boxed{\text{Any vector can be uniquely expanded in the eigenbasis of ANY observable.}}$$

later "quantum states"

Hermitian operator.

Uncountable/Continuous basis :

- Bra and Ket vectors for definite position and momentum states of a 1D particle/system.
- Start with position: $x \leftarrow$ a Continuous variable.



Basis states : $|x\rangle$, $\forall x \in \mathbb{R}$

Aside: Note even in 3D $|\vec{r}_1+\vec{r}_2\rangle \neq |\vec{r}_1\rangle + |\vec{r}_2\rangle$

- The label inside the Ket is NOT a vector.
- Inner pdt: $\langle x|y\rangle = \delta(x-y)$

$\langle x|x\rangle = \delta(0) \to \infty$

$\hookrightarrow$ non-normalisable states.

BUT, as in the free particle, they can be used to construct normalisable physical states via (continuous) superposition!

- Completeness: $\mathbb{1} = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x|$
- Sanity check: $|y\rangle = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x|y\rangle = \int_{-\infty}^{\infty} dx\, \delta(x-y)\, |x\rangle = |y\rangle$
- The position operator: $\underset{\substack{\uparrow \\ \text{operator}}}{\hat{x}} |x\rangle = \underset{\substack{\uparrow \\ \text{number}}}{x} |x\rangle$, $\leftarrow$ eigenstates.

Check: $x^\dagger = x$: $\langle x_1|\hat{x}^\dagger|x_2\rangle = \langle x_2|x|x_1\rangle^* = x_1\,\delta(x_2-x_1) = x_2\,\delta(x_1-x_2) = \langle x_1|\hat{x}|x_2\rangle$

Also note: $\langle x|\hat{x} = x\langle x|$

- Similar things for the momentum basis states: $|p\rangle$; $\forall p \in \mathbb{R}$ and operator $\hat{p}$.

$$\langle p'|p\rangle = \delta(p-p'); \quad \mathbb{1} = \int dp\, |p\rangle\langle p| ; \quad \hat{p}|p\rangle = p|p\rangle; \quad \langle p|\hat{p} = p\langle p| ; \quad \hat{p}^\dagger = \hat{p}$$

But, all this math for what? Where is the quantum state/wavefunc^n?

- The fundamental postulate of QM:

  The state of a quantum system is described by the state vector $|\psi(t)\rangle$ which lives in a Complex vector space (Hilbert space to be rigorous.)

So,

$$|\psi(t)\rangle \in \mathcal{H}.$$

Some people call this $|S(t)\rangle$

- What you have known so far as the (position-state) wavefunc^n is a mere projection of this state vector in the position basis.

$$\boxed{\psi(x,t) \underset{\text{def.}}{\equiv} \langle x|\psi(t)\rangle \in \mathbb{C}}$$

Note that:

$$\psi(x) \equiv \langle x|\psi\rangle = \int dx' \langle x|x'\rangle\langle x'|\psi\rangle = \int dx' \, \delta(x-x')\, \psi(x')$$

eigenvector x' in the x basis.

projection of $|\psi\rangle$ onto $|x'\rangle$!

May recognise this as the fourier transform!

Similarly:

$$\boxed{\tilde{\psi}(p,t) \underset{\text{def.}}{\equiv} \langle p|\psi(t)\rangle}$$

$$\psi(x) \equiv \langle x|\psi\rangle = \int dp \, \langle x|p\rangle\langle p|\psi\rangle = \int dp \, \langle x|p\rangle \, \tilde{\psi}(p)$$

This is the Fourier transform in disguise!

will turn out be $\Longrightarrow \dfrac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}$