Momentum Operator

Given $\Psi(x,t)$, it makes sense that one way to calculate the particle's velocity would be:

$$\begin{aligned}\frac{d}{dt} < x > (t) =& \frac{\partial}{\partial t}\int_{-\infty}^{\infty} x(\Psi^*(x,t)\Psi(x,t))dx \\ =& \int_{-\infty}^{\infty} x(\frac{\partial \Psi^*(x,t)}{\partial t}\Psi(x,t) + \Psi^*(x,t)\frac{\partial \Psi(x,t)}{\partial t})dx\end{aligned}$$

Now, we have from the Schrödinger equation:

$$i\hbar\frac{\partial \Psi(x,t)}{\partial t} = (\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V)\Psi(x,t)$$

We can substitute this in to obtain:

$$\int_{-\infty}^{\infty} x((\frac{-i\hbar}{2m}\frac{\partial^2 \Psi^*(x,t)}{\partial x^2} + \frac{iV}{\hbar}\Psi^*(x,t))\Psi(x,t) + \Psi^*(x,t)(\frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) - \frac{iV}{\hbar}\Psi(x,t)))dx$$

Now, note that the terms with the Potential V in them cancel out, leaving us with:

$$\int_{-\infty}^{\infty} x((\frac{-i\hbar}{2m}\frac{\partial^2 \Psi^*(x,t)}{\partial x^2})\Psi(x,t) + \Psi^*(x,t)(\frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t)))dx$$

We can now integrate by parts as follows:

$$\int x(\frac{\partial^2 \Psi^*(x,t)}{\partial x^2})\Psi dx$$

$$u = x\Psi(x,t) \qquad dv = \frac{\partial^2 \Psi^*(x,t)}{\partial x^2}dx$$

$$du = (\Psi(x,t) + x\frac{\partial \Psi(x,t)}{\partial x})dx \qquad v = \frac{\partial \Psi^*(x,t)}{\partial x}$$

$$\int_{-\infty}^{\infty} x(\frac{\partial^2 \Psi^*(x,t)}{\partial x^2})\Psi(x,t)dx = x(\frac{\partial \Psi^*(x,t)}{\partial x})\Psi(x,t)|_{-\infty}^{\infty} - \int (\Psi(x,t) + x\frac{\partial \Psi(x,t)}{\partial x})\frac{\partial \Psi^*(x,t)}{\partial x}dx$$

The first term in the integral vanishes, so we are left with:

$$\frac{d}{dt} < x > (t) = + \frac{i\hbar}{2m} \int \Psi(x,t) \frac{\partial \Psi^*(x,t)}{\partial x} + x \frac{\partial \Psi(x,t)}{\partial x} \frac{\partial \Psi^*(x,t)}{\partial x} dx$$
$$- \frac{i\hbar}{2m} \int \Psi^*(x,t) \frac{\partial \Psi(x,t)}{\partial x} + x \frac{\partial \Psi^*(x,t)}{\partial x} \frac{\partial \Psi(x,t)}{\partial x} dx$$

The $x \frac{\partial \Psi^*(x,t)}{\partial x} \frac{\partial \Psi(x,t)}{\partial x}$ terms cancel, and we are left with the expression:

$$\frac{d}{dt} < x > (t) = \frac{i\hbar}{2m} \int (\Psi(x,t) \frac{\partial \Psi^*(x,t)}{\partial x} - \Psi^*(x,t) \frac{\partial \Psi(x,t)}{\partial x}) dx \qquad *$$

Now we can integrate the first term again by parts:

$$u = \Psi(x,t) \qquad dv = \frac{\partial \Psi^*(x,t)}{\partial x} dx$$
$$du = \frac{\partial \Psi(x,t)}{\partial x} dx \qquad v = \Psi^*(x,t)$$
$$\therefore \frac{i\hbar}{2m} \int \Psi(x,t) \frac{\partial \Psi^*(x,t)}{\partial x} = |\Psi(x,t)|^2|_{-\infty}^{\infty} - \int \Psi^*(x,t) \frac{\partial \Psi(x,t)}{\partial x} dx$$

Integrating both terms now, the $|\Psi(x,t)|^2|_{-\infty}^{\infty}$ term cancels and the $-\int \Psi^*(x,t) \frac{\partial \Psi(x,t)}{\partial x} dx$ terms sum together, leaving us with:

$$\frac{d}{dt} < x > (t) = -\frac{i\hbar}{m} \int \Psi^*(x,t) \frac{\partial \Psi(x,t)}{\partial x} dx$$

BOTTOM LINE:

$$m < v > (t) = \int_{-\infty}^{\infty} \Psi^*(x,t)(-i\hbar \frac{\partial}{\partial x}) \Psi(x,t) dx$$
$$< x > (t) = \int_{-\infty}^{\infty} \Psi^*(x,t)(x)\Psi(x,t) dx$$
$$< Q(p,x) > = \int_{-\infty}^{\infty} \Psi^*(x,t) Q(-i\hbar \frac{\partial}{\partial x}, x) \Psi(x,t) dx$$

Now, define total energy $H = \frac{p^2}{2m} + V(x)$. Then we get:

$$< H > (t) = \int_{-\infty}^{\infty} \Psi^*(x,t)(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)) \Psi(x,t) dx$$