Assignment 5 Solution

1. Let X be a discrete random variable with pmf $f_X(x|\theta)$, where $\theta \in \{1, 2, 3\}$ and $x \in \{1, 2, 3, 4, 5, 6\}$.

$$f_X(x|\theta) = \begin{cases} x/21, & \theta = 1 \\ 1/6, & \theta = 2 \\ I(x = 3), & \theta = 3 \end{cases}$$

Find a maximum-likelihood estimator of θ. (Note that MLE is a function of x, but may not be represented as a nice-looking formula.)

Solution: $f_X(x|\theta)$ and MLE can be represented as the following table

x	$f_X(x\|1)$	$f_X(x\|2)$	$f_X(x\|3)$	$\hat{\theta}_{MLE}(x)$
1	1/21	1/6	0	2
2	2/21	1/6	0	2
3	3/21	1/6	1	3
4	4/21	1/6	0	1
5	5/21	1/6	0	1
6	6/21	1/6	0	1

$\theta_{MLE}(x)$ is obtained simply by selecting θ that maximizes $f(x|\theta)$ for each row. Therefore, the MLE $\hat{\theta}_{MLE}$ is

$$\hat{\theta}_{MLE}(x) = \begin{cases} 1, & x \in \{4, 5, 6\} \\ 2, & x \in \{1, 2\} \\ 3, & x = 3 \end{cases}$$

2. Let $X_1, \cdots, X_n$ be an *i.i.d.* random sample with pdf

$$f_X(x|\theta) = \frac{1}{\theta}, \qquad 0 \le x \le \theta$$

(a) Find a method of moments estimator of θ using the lowest-order moments as possible .

Solution: Because $\mathrm{E}X = \frac{\theta}{2}$, the method of moments estimator can be obtained by solving the following equation

$$\frac{\hat{\theta}_{MoM}}{2} = \overline{X}$$

which results in $\hat{\theta}_{MoM} = 2\overline{X}$.

(b) (10pts) Calculate the mean and variance of the method of moments estimator.

Solution: The mean and variance of $\hat{\theta}_{MoM}$ is

$$\begin{aligned}
\mathrm{E}(\hat{\theta}_{MoM}) &= \mathrm{E}(2\overline{X}) = \theta \\
\mathrm{Var}(\hat{\theta}_{MoM}) &= \mathrm{Var}(2\overline{X}) \\
&= \frac{4}{n}\mathrm{Var}(X) \\
&= \frac{4}{n}\left[\mathrm{E}X^2 - (\mathrm{E}X)^2\right] \\
&= \frac{4}{n}\left[\frac{1}{3}\theta^2 - \frac{1}{4}\theta^2\right] \\
&= \frac{1}{3n}\theta^2
\end{aligned}$$

(c) Compare the MLE $\hat{\theta}_{MLE} = X_{(n)}$ with the estimator from (a). In terms of bias and variance, which estimator is better? Justify your answer.

Solution: Because $\hat{\theta}_{MLE} = X_{(n)}$. Since $X_{(n)}/\theta \sim \mathrm{Beta}(n, 1)$, its expectation and variance is

$$\begin{aligned}
\mathrm{E}(\hat{\theta}_{MLE}) &= \frac{n}{n+1}\theta \\
\mathrm{Var}(\hat{\theta}_{MLE}) &= \frac{n}{(n+1)(n+2)}\theta^2
\end{aligned}$$

In terms of bias, because method of moments estimator is unbiased while MLE is biased, method of moments estimator is better.

$$\begin{aligned}
\mathrm{Var}(\hat{\theta}_{MoM}) &= \frac{1}{3n} = \frac{n}{3n^2} \\
&> \frac{n}{3(n+1)^2} \\
&\geq \frac{n}{(n+1)^2(n+2)} = \mathrm{Var}(\hat{\theta}_{MLE})
\end{aligned}$$

In terms of variance, MLE is better because it has smaller variance.

3. Let $X_1, \cdots, X_n$ be an *i.i.d.* random sample from a $DoubleExponential(\mu, \sigma)$ distribution with pdf

$$f_X(x|\mu, \sigma) = \frac{1}{2\sigma}\exp\left[-\frac{|x-\mu|}{\sigma}\right], \qquad x \in R, \quad \sigma > 0$$

Find MLEs of μ and σ. Show all steps.

(Hint: You may use the fact that for a set of real numbers $x_1, \cdots, x_n$ the quantity $\frac{1}{n}\sum_{i=1}^{n}|x_i - a|$ is minimized when $a = median\{x_1, \cdots, x_n\}$.)

Solution: The log likelihood function is,

$$l(\mu, \sigma|\mathbf{x}) = -n\log 2\sigma + \frac{1}{\sigma}\sum_{i=1}^{n}|x_i - \mu|$$

For fixed $\sigma > 0$, the log likelihood is maximized with respect to μ when $\hat{\mu} = med(x_i)$ from the hint, which indicates that the second term is maximized in terms of μ at the median of samples. Since it is always in real numbers, $\hat{\mu}_{MLE} = med(x_i)$.

In order to find MLE of σ, taking the derivative of log likelihood with respect to σ to find its root,

$$\begin{aligned}\frac{\partial l(\mu, \sigma|\mathbf{x})}{\partial \sigma} &= -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n}|x_i - \mu| = 0 \\ \hat{\sigma} &= \frac{1}{n}\sum_{i=1}^{n}|x_i - \hat{\mu}| > 0\end{aligned}$$

To check whether it is local maximum or not,

$$\begin{aligned}\frac{\partial^2 l(\sigma|\mathbf{x})}{\partial \sigma^2} &= -\frac{n}{\sigma^2} - \frac{2}{\sigma^3}\sum_{i=1}^{n}|x_i - \mu| \\ \left.\frac{\partial^2 l(\sigma|\mathbf{x})}{\partial \sigma^2}\right|_{\sigma=\hat{\sigma}} &= -\frac{n}{\sigma^2} - \frac{2}{\sigma^3}n\hat{\sigma} \\ &= -\frac{n}{\hat{\sigma}^2} < 0\end{aligned}$$

Since $\hat{\sigma}$ is unique and also a local maximum, it is the global maximum within the parameter space. Therefore, $\hat{\mu}_{MLE} = med(x_i)$ and $\hat{\sigma}_{MLE} = \frac{1}{n}\sum_{i=1}^{n}|x_i - \hat{\mu}|$.

4. Let $X_1, \cdots, X_n$ be an *i.i.d.* random sample from the following pdf

$$f_X(x|\theta) = \frac{x}{\theta}\exp\left(-\frac{x^2}{2\theta}\right), \qquad x > 0, \quad \theta > 0$$

(a) Find a complete sufficient statistic for θ.

Solution: The distribution of X belongs to an exponential family because

$$f_X(x|\theta) = h(x)c(\theta)\exp[w(\theta)t(x)]$$

if $h(x) = xI(x > 0)$, $c(\theta) = \frac{1}{\theta}I(\theta > 0)$, $t(x) = x^2$ and $w(\theta) = -\frac{1}{2\theta}$.
Because $\Theta = \{w(\theta) : \theta > 0\} = (-\infty, 0)$ contains an open set in $\mathbb{R}$, by Theorem 6.2.10 and 6.2.25, $\sum_{i=1}^n T(X_i) = \sum_{i=1}^n X_i^2$ is a complete sufficient statistic.

(b) Find the Cramer-Rao lower bound for the variance of any unbiased estimator of θ.

Solution: Since f_X belongs to an exponential family, by Lemma 7.3.9 and Corollary 7.3.10, Cramer-Rao lower bound is $\frac{\{\tau'(\theta)\}^2}{n\mathrm{E}\left[-\frac{\partial^2 l(\theta|x)}{\partial\theta^2}\right]}$. In this case, $\tau(\theta) = \theta$ and $l(\theta|x) = \log x - \log\theta - \frac{x^2}{2\theta}$. Then,

$$\begin{aligned}
\frac{\{\tau'(\theta)\}^2}{n\mathrm{E}\left[-\frac{\partial^2 l(\theta|x)}{\partial\theta^2}\right]} &= \frac{1}{n\mathrm{E}\left[-\frac{\partial}{\partial\theta}\{-\frac{1}{\theta} + \frac{x^2}{2\theta^2}\}\right]} \\
&= \frac{1}{n\mathrm{E}\left[-(\frac{1}{\theta^2} - \frac{x^2}{\theta^3})\right]}
\end{aligned}$$

By transformation of $y = g(x) = \frac{x^2}{2}$, $f_Y(y) = \frac{1}{\theta}e^{-\frac{y}{\theta}}$ for $y > 0$. That is, $\frac{X^2}{2} \sim \exp(\theta)$ and $\mathrm{E}\left[X^2\right] = 2\theta$. Therefore, CRLB for the variance of any unbiased estimator of θ is $\frac{\theta^2}{2n}$.

(c) Can you find a simple function (constant multiple) of the complete sufficient statistic in part (a) which is unbiased?

Solution: Since $\frac{X^2}{2} \sim \exp(\theta)$ from (b), $\sum_{i=1}^n \frac{X_i^2}{2} \sim Gamma\,(n, \theta)$. Then, $\mathrm{E}\left[\sum_{i=1}^n \frac{X_i^2}{2}\right] = n\theta$, $i.e.\mathrm{E}\left[\frac{1}{2n}\sum_{i=1}^n X_i^2\right] = \theta$. One-to-one function of a complete sufficient statistic is still a complete sufficient statistics, therefore, $\frac{1}{2n}\sum_{i=1}^n X_i^2$ is the unbiased as well as the complete sufficient statistics.

(d) Does the estimator in part(c) attain the CRLB obtained in part (b)?

Solution: Since $\frac{1}{2n}\sum_{i=1}^n X_i^2$ is the unbiased estimator of θ by (c) and $l(\theta|\mathbf{x}) = \sum_{i=1}^n \log x_i - n\log\theta - \frac{1}{2\theta}\sum_{i=1}^n x_i^2$, then

$$\begin{aligned}
\frac{\partial l(\theta|\mathbf{x})}{\partial\theta} &= -\frac{n}{\theta} + \frac{1}{2\theta^2}\sum_{i=1}^n x_i^2 \\
&= \frac{1}{\theta^2}\left(\frac{1}{2n}\sum_{i=1}^n X_i^2 - \theta\right)
\end{aligned}$$

By Corollary 7.3.15, $\sum_{i=1}^n X_i^2$ can attain the Cramer-Rao lower bound.

5. Let $X_1, \cdots, X_n$ be an *i.i.d.* random sample from pdf

$$f_X(x|\theta) = \theta x^{\theta-1} I(0 < x < 1)$$

(a) When $\theta \geq 1$, find the maximum likelihood estimator for θ.

Solution: When $0 < x_{(1)} \leq x_{(n)} < 1$, the likelihood and log likelihood functions are

$$\begin{aligned} L(\theta|\mathbf{x}) &= \prod_{i=1}^{n} f_X(x_i|\theta) = \theta^n \left(\prod_{i=1}^{n} x_i \right)^{\theta-1} \\ l(\theta|\mathbf{x}) &= n \log \theta + (\theta - 1) \sum_{i=1}^{n} \log x_i \end{aligned}$$

Taking the derivative of log-likelihood to find its root,

$$\begin{aligned} \frac{\partial l(\theta|\mathbf{x})}{\partial \theta} &= \frac{n}{\theta} + \sum_{i=1}^{n} \log x_i = 0 \\ \hat{\theta} &= -\frac{n}{\sum_{i=1}^{n} \log x_i} \end{aligned}$$

To check whether it is local maximum or not,

$$\frac{\partial^2 l(\theta|\mathbf{x})}{\partial \theta^2} = -\frac{n}{\theta^2} < 0$$

Because the likelihood function is concave, the unique interior extreme value $\hat{\theta} = -\frac{n}{\sum_{i=1}^{n} \log x_i}$ is the global maximum across $\theta > 0$.
If $-\frac{n}{\sum_{i=1}^{n} \log x_i} < 1$, then

$$\frac{\partial l(\theta|\mathbf{x})}{\partial \theta} = \frac{n}{\theta} + \sum \log x_i \leq n + \sum \log x_i < 0$$

when $\theta \geq 1$. Thus, the likelihood is monotonically decreasing, and the MLE is 1.
If $-\frac{n}{\sum_{i=1}^{n} \log x_i} \geq 1$, then, it attains global maximum likelihood, and becomes the MLE.
Therefore, the MLE of θ when $\theta \geq 1$ is

$$\hat{\theta}_{MLE} = \max \left(-\frac{n}{\sum_{i=1}^{n} \log x_i}, 1 \right)$$

(b) When $\theta > 0$, find the maximum likelihood estimator for $\tau(\theta) = 1/\theta$.

Solution: When $\theta > 0$, the MLE of θ is $\hat{\theta} = -\frac{n}{\sum_{i=1}^{n} \log x_i}$. By invariance property of MLE, the MLE of $\tau(\theta)$ is

$$\hat{\tau}(\theta) = 1/\hat{\theta} = -\frac{1}{n} \sum_{i=1}^{n} \log x_i.$$

(c) When $\theta > 0$, find the Cramer-Rao lower bound of the variance of unbiased estimators for $\tau(\theta) = 1/\theta$. Does the MLE in (b) attain the bound?

Solution: By Lemma 7.3.11, the Fisher Information Number for exponential family is

$$I_n(\theta) \quad = \quad \mathrm{E}\left[-\frac{\partial^2 l(\theta|\mathbf{x})}{\partial \theta^2}\right] = \frac{n}{\theta^2}$$

The Cramer-Rao lower bound of the variance of UMVUE for $\tau(\theta) = 1/\theta$ is

$$\mathrm{Var}\left[W(\mathbf{X})\right] \geq \frac{[\tau'(\theta)]^2}{I_n(\theta)} = \frac{\theta^2}{n}\left[\tau'(\theta)\right]^2 = \frac{\theta^2}{n} \times \frac{1}{\theta^4} = 1/(n\theta^2)$$

Define $Y = -\log X$. Then Y has an exponential distribution with mean $1/\theta$ with pdf

$$f_Y(y|\theta) = \theta \;\; \exp(\theta y), \quad y > 0, \theta > 0.$$

Hence

$$Var(1/\hat{\theta}) = Var\left(-\frac{1}{n}\sum_{i=1}^{n} \log x_i\right) = Var\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) = \frac{Var(Y)}{n} = 1/(n\theta^2).$$

Hence variance of $\tau(\hat{\theta})$ matches the CRLB.

Practice Problems

(a) C&B Exercise 7.6

(b) C&B Exercise 7.8

(c) C&B Exercise 7.10

(d) C&B Exercise 7.11

(e) C&B Exercise 7.12

(f) C&B Exercise 7.37

(g) C&B Exercise 7.38

(h) C&B Exercise 7.40

(i) C&B Exercise 7.58

(j) C&B Exercise 7.66