Assignment 1 Solution

1. A coin is twice as likely to turn up tails as heads. If the coin is tossed independently, what is the probability that the third head occurs on the 5th trial?

   Solution: $P(\text{Head}) = 1/3$, while $P(\text{Tail}) = 2/3$. Therefore each toss is independent Bernoulli random variable. Therefore,

$$P(\text{3rd head occurs on the 5th trial}) = \binom{4}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^3 = \frac{8}{81}.$$

2. Suppose X and Y are two independent variables with unit variance. Let $Z = aX + Y$, where $a > 0$. If $Cor(X, Z) = 1/3$, then obtain the value of a.

   Solution: $VarZ = Var(aX + Y) = a^2 VarX + VarY = a^2 + 1$

$$\begin{aligned} Cor(X, Z) &= \frac{Cov(X, Z)}{\sqrt{VarZ}\sqrt{VarX}} = \frac{Cov(X, aX + Y)}{\sqrt{a^2+1}} \\ &= \frac{aVarX}{\sqrt{a^2+1}} = \frac{a}{\sqrt{a^2+1}} = \frac{1}{3} \end{aligned}$$

   Therefore $a = \frac{\sqrt{2}}{4}$.

3. Let $g(x)$, $x \geq 0$, be a valid pdf for a nonnegative random variable and define

$$f(x, y) = \frac{g(\sqrt{x^2+y^2})}{2\pi\sqrt{x^2+y^2}}.$$

   for $-\infty < x,\ y < \infty$.

   (a) Show that $f(x, y)$ is a valid pdf.

   (b) Suppose that the pair (X, Y) has the pdf $f(x, y)$. What is $P(XY > 0)$?

   Solution: (a) Let $x = r\cos\theta$, $y = r\sin\theta$, where $0 < r < \infty$, $0 < \theta \leq 2\pi$. Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = \int_0^{\infty}\int_0^{2\pi} \frac{g(r)}{2\pi r} r d\theta dr = \int_0^{\infty} g(r)dr = 1$$

   where $dxdy \to rd\theta dr$, because

$$J = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r.$$

Therefore $f(x,y)$ is a valid pdf.

(b) One way to do this problem is:

$$\begin{aligned}
P(XY>0) &= P(X>0,Y>0)+P(X<0,Y<0)\\
&= \int_0^\infty\int_0^\infty f(x,y)dxdy+\int_{-\infty}^0\int_{-\infty}^0 f(x,y)dxdy\\
&= \int_0^\infty\int_0^{\pi/2}\frac{g(r)}{2\pi r}rd\theta dr+\int_0^\infty\int_\pi^{3\pi/2}\frac{g(r)}{2\pi r}rd\theta dr\\
&= \frac{1}{4}\int_0^\infty g(r)dr+\frac{1}{4}\int_0^\infty g(r)dr\\
&= \frac{1}{2}.
\end{aligned}$$

4. Given independent and identically distributed random samples X_1, X_2, .., X_n, each with finite mean μ and finite variance σ^2, define

$$\begin{aligned}
\overline{X} &= \frac{1}{n}\sum_{i=1}^n X_i\\
S^2 &= \frac{1}{n-1}\sum_{i=1}^n\left(X_i-\overline{X}\right)^2\\
W^2 &= \frac{1}{n}\sum_{i=1}^n\left(X_i-\overline{X}\right)^2
\end{aligned}$$

(a) Show that $S^2 \xrightarrow{P} \sigma^2$.

(b) Derive the asymptotic distribution of $\frac{\sqrt{n}(\overline{X}-\mu)}{\sqrt{S^2}}$.

(c) Use Delta method to derive the asymptotic distribution of $\left(\overline{X}\right)^2$ after you normalize it appropriately.

Solution: (a) Let $R_n=\frac{1}{n}\sum_{i=1}^n\left(X_i-\overline{X}\right)^2$, then $S^2=\frac{n}{n-1}R_n$.

$$R_n=\frac{1}{n}\sum_{i=1}^n\left(X_i-\overline{X}\right)^2=\frac{1}{n}\sum_{i=1}^n X_i^2-(\frac{1}{n}\sum_{i=1}^n X_i)^2.$$

By weak law of large numbers, $\frac{1}{n}\sum_{i=1}^n X_i^2 \xrightarrow{P} EX_i^2=\mu^2+\sigma^2$.

Similarly, $\frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{P} \mu$. Then by continuous mapping theorem, $(\frac{1}{n}\sum_{i=1}^n X_i)^2 \xrightarrow{P} \mu^2$. Therefore,

$$R_n \xrightarrow{P} \mu^2+\sigma^2-\mu^2=\sigma^2.$$

And because $\frac{n}{n-1} \longrightarrow 1$, by Slutsky's Theorem,

$$S^2 = \frac{n}{n-1} R_n \xrightarrow{P} \sigma^2.$$

(b) By continuous mapping theorem,

$$\sqrt{\frac{\sigma^2}{S^2}} \xrightarrow{P} 1.$$

And by CLT,

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0, 1).$$

Therefore by Slutsky's Theorem,

$$\frac{\sqrt{n}(\overline{X} - \mu)}{\sqrt{S^2}} = \frac{(\overline{X} - \mu)}{\sqrt{\sigma^2}/\sqrt{n}} \sqrt{\frac{\sigma^2}{S^2}} \xrightarrow{d} N(0, 1).$$

(c) From (b) we have,

$$\sqrt{n}(\overline{X} - \mu) \xrightarrow{d} N(0, \sigma^2).$$

Then let $g(x) = x^2$ and using Delta Method, we have

$$\sqrt{n}[(\overline{X})^2 - \mu^2] \xrightarrow{d} N(0, 4\mu^2\sigma^2).$$

5. For two sets of random variables $\{X_i\}$, $i = 1, ..., n$ and $\{Y_j\}$, $i = 1, ..., m$, show that

$$Cov\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j Cov(X_i, Y_j)$$

where a_i, b_j are arbitrary constants.

Solution:

$$\begin{aligned} Cov\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j\right) &= \sum_{i=1}^{n} Cov\left(a_i X_i, \sum_{j=1}^{m} b_j Y_j\right) \\ &= \sum_{i=1}^{n} a_i Cov\left(X_i, \sum_{j=1}^{m} b_j Y_j\right) \\ &= \sum_{i=1}^{n} a_i \sum_{j=1}^{m} Cov\left(X_i, b_j Y_j\right) \\ &= \sum_{i=1}^{n} a_i \sum_{j=1}^{m} b_j Cov\left(X_i, Y_j\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j Cov(X_i, Y_j). \end{aligned}$$

6. Suppose $N \sim Poisson(\lambda)$. Given $N = n > 0$, X_1,...,X_N are iid and follow uniform [0,1]. We define $X_0 = 0$ when $N = 0$.

   (a) Given $N = n$, find the probability X_0, X_1,...,X_N are all less than t, where $0 < t < 1$.

   (b) Find the (unconditional) probability X_0, X_1,...,X_N are all less than t, where $0 < t < 1$.

   (c) Let $S_N = X_0 + X_1 + .. + X_N$. Compute $E(S_N)$.

   Solution: (a) Given $N = n$, we have

$$P(X_0 < t, X_1 < t, \ldots, X_N < t|N = n) = P(X_1 < t)P(X_2 < t)...P(X_n < t) = t^n.$$

(b)

$$\begin{aligned} P(X_0 < t, X_1 < t, \ldots, X_N < t) &= \sum_{i=0}^{\infty} P(X_0 < t, X_1 < t, \ldots, X_N < t|N = i)P(N = i) \\ &= \sum_{i=0}^{\infty} t^i \frac{\lambda^i e^{-\lambda}}{i!} \\ &= \sum_{i=0}^{\infty} \frac{(t\lambda)^i e^{-\lambda}}{i!} \\ &= e^{-\lambda(1-t)}. \end{aligned}$$

(c)

$$E(S_N) = E[E(S_N|N)] = E[N/2] = \lambda/2.$$

7. Let X_1, X_2, X_3 be a random sample of size 3 from a $N(0, 1)$ population. In each of the following five cases, Z denotes a specific function derived from this random sample. In each case identify the distribution of the resulting random variable Z along with the associated parameters.

$$\begin{aligned} (i) &\quad X_1 + X_2 + 2X_3. \\ (ii) &\quad X_1^2 + X_2^2 + X_3^2. \\ (iii) &\quad (X_1 - X_2)^2/2. \\ (iv) &\quad Z = \tfrac{2X_1^2}{X_2^2+X_3^2}. \\ (v) &\quad Z = \tfrac{(X_1-X_2)^2}{(X_1+X_2)^2}. \end{aligned}$$

Solution: (i) The sum of normal random variables is still normally distributed. With mean $0 + 0 + 2 \times 0 = 0$ and variance $1 + 1 + 2^2 = 6$, thus

$$X_1 + X_2 + 2X_3 \sim N(0, 6).$$

(ii) We know that X_1^2, X_2^2, $X_3^2 \sim \chi_1^2$, therefore

$$X_1^2 + X_2^2 + X_3^2 \sim \chi_3^2.$$

(iii) Since X_1, X_2 are independent $N(0, 1)$ variables, $X_1 - X_2 \sim N(0, 2)$. Hence

$$(X_1 - X_2)^2/2 \sim \chi_1^2.$$

(iv) $X_1^2 \sim \chi_1^2$, $X_2^2 + X_3^2 \sim \chi_2^2$. From the definition of F distribution, we have

$$Z = \frac{2X_1^2}{X_2^2 + X_3^2} = \frac{X_1^2}{(X_2^2 + X_3^2)/2} \sim F_{1,2}.$$

(v) $X_1 - X_2 \sim N(0, 2)$, $X_1 + X_2 \sim N(0, 2)$, and they are independent. Thus

$$(X_1 - X_2)^2/2 \sim \chi_1^2, \quad (X_1 + X_2)^2/2 \sim \chi_1^2$$

independently of each other. Therefore

$$Z = \frac{(X_1 - X_2)^2}{(X_1 + X_2)^2} \sim F_{1,1}.$$