BIOSTAT 602 Biostatistical Inference
Homework 05

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1. Let X be a discrete random variable with pmf $f_X(x|\theta)$ where $\theta \in \{1,2,3\}$ and $x \in \{1,2,3,4,5,6\}$.

$$f_X(x|\theta) = \begin{cases} x/21, & \theta = 1 \\ 1/6, & \theta = 2 \\ I(x=3), & \theta = 3 \end{cases}$$

Find a maximum-likelihood estimator of θ.

Solution. Given an i.i.d. sample $X_1, \dots, X_n$ from this distribution, the likelihood $\mathcal{L}$ of the sample is

$$\begin{aligned} \mathcal{L}(\theta|\mathbf{x}) &= \prod_{i=1}^{n} f_X(x_i|\theta) \\ &= \begin{cases} \left(\frac{1}{21}\right)^n \prod_{i=1}^{n} x_i, & \theta = 1 \\ (1/6)^n, & \theta = 2 \\ I(x_1, \dots, x_n = 3), & \theta = 3 \end{cases} \end{aligned}$$

So, the MLE is more of a prodedure or algorithm: Determine

$$\max \left\{ 21^{-n} \prod_{i=1}^{n} x_i, (1/6)^n, I(x_1, \dots, x_n = 3) \right\}$$

and choose $\hat{\theta}$ to be 1, 2, or 3, correspondingly.

2. Let $X_1, \dots, X_n$ be i.i.d. random variables from $U(0, \theta)$ with pdf

$$f_X(x|\theta) = \frac{1}{\theta}, \quad 0 \le x \le \theta$$

(a) Find a method of moments estimator of θ using the lowest-order moments as possible.
Solution. The first moment μ_1 can be calculated

$$\mu_1 = \mathbb{E}[X] = \int_0^\theta x \frac{1}{\theta}\, dx = \frac{\theta}{2}$$

So a method of moments estimator of θ can be determined as the solution to $\hat{\mu}_1 = \frac{\hat{\theta}}{2}$, which is of course

$$\hat{\theta} = 2\hat{\mu}_1 = \frac{2}{n} \sum_{i=1}^{n} X_i$$

(b) Calculate the mean and variance of the method of moments estimator.
Solution. The mean of $\hat{\theta}$ is

$$\begin{aligned} \mathbb{E}\left[\hat{\theta}\right] &= \mathbb{E}[2\hat{\mu}_1] \\ &= \frac{2}{n} \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] \\ &= \frac{2}{n} \sum_{i=1}^{n} \mathbb{E}[X_i] \\ &= \frac{2}{n}\left(n \frac{\theta}{2}\right) \\ &= \theta \end{aligned}$$

The variance of the given uniform distribution is $\sigma = \frac{1}{12}\theta^2$, and the variance of the sample mean is given by $\mathrm{Var}\left(\overline{X}\right) = \sigma^2/n$, so

$$\mathrm{Var}\left(\hat{\theta}\right) = \mathrm{Var}(2\hat{\mu}_1) = 4\mathrm{Var}(\hat{\mu}_1) = \frac{4\theta^2}{12n}$$

(c) Compare the MLE $\hat{\theta}_{MLE} = X_{(n)}$ with the estimator from (a) in terms of bias and variance. Which estimator is better? Justify your answer.

Solution. The CDF of the sample maximum can be computed as

$$F_{X_{(n)}}(x) = P\left(X_{(n)} \leq x\right) = \left(\frac{x}{\theta}\right)^n$$

so

$$\begin{aligned} f_{X_{(n)}}(x) &= \frac{n}{\theta}\left(\frac{x}{\theta}\right)^{n-1} \\ &= \frac{n}{\theta^n}x^{n-1} \end{aligned}$$

The mean of $X_{(n)}$ is then

$$\begin{aligned} \int_0^\theta x\frac{n}{\theta^n}x^{n-1}\ dx &= \frac{n}{\theta^n}\int_0^\theta x^n\ dx \\ &= \frac{n}{\theta^n(n+1)}\ \left[x^{n+1}\right]\Big|_0^\theta \\ &= \frac{n\theta^{n+1}}{\theta^n(n+1)} \\ &= \frac{n\theta}{n+1} \end{aligned}$$

The variance of $X_{(n)}$ is $\text{Var}\left(X_{(n)}\right) = \mathbb{E}\left[X_{(n)}^2\right] - \mathbb{E}\left[X_{(n)}\right]^2$, so we calculate

$$\begin{aligned} \mathbb{E}\left[X_{(n)}^2\right] &= \int_0^\theta x^2\frac{n}{\theta^n}x^{n-1}\ dx \\ &= \frac{n}{\theta^n}\int_0^\theta x^{n+1}\ dx \\ &= \frac{n\theta^2}{n+2} \end{aligned}$$

So

$$\text{Var}\left(X_{(n)}\right) = \mathbb{E}\left[X_{(n)}^2\right] - \mathbb{E}\left[X_{(n)}\right]^2 = \frac{n\theta^2}{n+2} - \left(\frac{n\theta}{n+1}\right)^2$$

The variance of both statistics approaches zero as the sample size increases, but $\hat{\theta}_{MLE} = X_{(n)}$ is biased, so I suppose the method of moments estimator is better.

3. Let $X_1, \ldots, X_n$ be a random sample from a double exponential distribution with pdf

$$f_X(x|\mu, \sigma^2) = \frac{1}{2\sigma}\exp[-|x-\mu|/\sigma], \qquad x \in \mathbb{R}, \quad \mu \in \mathbb{R}, \quad \sigma > 0$$

Find MLEs of μ and σ. Show all steps. You may use the fact that for a set of real numbers $x_1, x_2, \ldots, x_n$ the quantity $\frac{1}{n}\sum_{i=1}^n |x_i - a|$ is minimized when $a = \text{median}(x_1, x_2, \ldots, x_n)$

Solution. For the random sample $X_1, \ldots, X_n$, the likelihood $\mathcal{L}$ of the sample is

$$\begin{aligned} \mathcal{L}(\mu, \sigma^2|\mathbf{x}) &= \prod_{i=1}^n f(x_i|\mu, \sigma^2) \\ &= \prod_{i=1}^n \frac{1}{2\sigma}\exp[-|x_i-\mu|/\sigma] \\ &= (2\sigma)^{-n}\exp\left[-\frac{1}{\sigma}\sum_{i=1}^n |x_i - \mu|\right] \end{aligned}$$

and the log-likelihood ℓ is

$$\begin{aligned}\ell(\mu, \sigma^2|\mathbf{x}) &= \log \mathcal{L}(\mu, \sigma^2|\mathbf{x}) \\ &= -n\log(2\sigma) - \frac{1}{\sigma}\sum_{i=1}^{n}|x_i - \mu|\end{aligned}$$

If σ is constant, then it's pretty clear, given the hint, that ℓ is maximized when $\mu = \text{median}(x_1, x_2, \ldots, x_n)$. So that's the MLE for μ. For σ, we need to take the derivative:

$$\begin{aligned}\frac{\partial \ell}{\partial \sigma} &= -2n(2\sigma)^{-1} + \sigma^{-2}\sum_{i=1}^{n}|x_i - \mu| \\ &= -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n}|x_i - \mu|\end{aligned}$$

and set it equal to zero:

$$\begin{aligned}0 &= -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n}|x_i - \mu| \\ 0 &= -n\sigma + \sum_{i=1}^{n}|x_i - \mu| \\ \sigma &= \frac{1}{n}\sum_{i=1}^{n}|x_i - \mu|\end{aligned}$$

So the MLE for σ is $\hat{\sigma} = -\frac{1}{n}\sum_{i=1}^{n}|x_i - \mu|$.

4. Let $X_1, \ldots, X_n$ be an i.i.d. random sample from a distribution with the pdf

$$f_X(x|\theta) = \frac{x}{\theta}\exp\left(-\frac{x^2}{2\theta}\right), \qquad x > 0 \quad \theta > 0$$

   (a) Find a complete sufficient statistic for θ.
   (b) Find the Cramer-Rao lower bound for the variance of any ubiased estimator of θ.
   (c) Can you find a simple function (constand multiple) of the complete sufficient statistic in part (a) which is unbiased?
   (d) Does the estimator in part (c) attain the Cramer-Rao lower bound obtained in part (b)?

5. Let $X_1, \ldots, X_n$ be an i.i.d. random sample from pdf

$$f_X(x|\theta) = \theta x^{\theta-1} I(0 < x < 1)$$

   (a) When $\theta \geq 1$, find the maximum likelihood estimator for θ.
   Solution. The likelihood is

$$\begin{aligned}\mathcal{L}(\theta|\mathbf{x}) &= \prod_{i=1}^{n}\theta x_i^{\theta-1} I(0 < x < 1) \\ &= \theta^n\left(\prod_{i=1}^{n}x_i\right)^{\theta-1} I(0 < x_1, \ldots, x_n < 1)\end{aligned}$$

   and the log likelihood is

$$\ell(\theta|\mathbf{x}) = n\log(\theta) + (\theta - 1)\sum_{i=1}^{n}\log(x_i) + \log(I(0 < x_1, \ldots, x_n < 1))$$

The whole expression is meaningless if $x_1, \ldots, x_n$ are not between 0 and 1, so we will assume this is the case, and simply consider the final term to be zero. Then the MLE for θ can be found by solving $\partial\ell/\partial\theta = 0$. Note that

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \log(x_i)$$

Solving for zero gives

$$\begin{aligned} 0 &= \frac{n}{\theta} + \sum \log(x_i) \\ -\frac{n}{\theta} &= \sum_{i=1}^{n} \log(x_i) \\ \theta &= \frac{-n}{\sum_{i=1}^{n} \log(x_i)}, \end{aligned}$$

so $\hat{\theta} = -n/\sum_{i=1}^{n} \log(x_i)$ is the MLE for θ.

(b) When $\theta > 1$, find the maximum likelihood estimator for $\tau(\theta) = 1/\theta$.

Solution. First, we reformulate $\ell(\theta|\mathbf{x})$ in terms of τ. This gives

$$\ell(\tau|\mathbf{x}) = -n\log(\tau) + (\tau^{-1} - 1)\sum_{i=1}^{n} \log(x_i)$$

So the derivative is

$$\frac{\partial \ell}{\partial \tau} = \frac{-n}{\tau} - \frac{1}{\tau^2} \sum_{i=1}^{n} \log(x_i)$$

and solving for $\partial\ell/\partial\tau$ gives

$$\begin{aligned} 0 &= -\frac{n}{\tau} - \frac{1}{\tau^2}\sum_{i=1}^{n} \log(x_i) \\ 0 &= -n\tau - \sum_{i=1}^{n} \log(x_i) \\ \tau &= -\frac{1}{n}\sum_{i=1}^{n} \log(x_i) \end{aligned}$$

So $\hat{\tau} = -\frac{1}{n}\sum_{i=1}^{n} \log(x_i)$ is the MLE for τ.

(c) When $\theta > 0$, find the Cramer-Rao lower bound of the variance of unbiased estimators for $\tau(\theta) = 1/\theta$. Does the MLE in (b) attain the bound?