Assignment 2 Solution

1. Let $X_1, \cdots, X_n$ be *i.i.d.* random variables from the probability density function of the following form:

$$f_X(x|\mu,\sigma) = \frac{1}{\sigma} e^{-(x-\mu)/\sigma}, \mu < x < \infty, 0 < \sigma < \infty.$$

   (a) Assuming that μ is known, find a one-dimensional sufficient statistic for σ.
   Solution:

$$\begin{aligned}
f_X(x|\mu,\sigma) &= \frac{1}{\sigma} \exp\left(-\frac{x-\mu}{\sigma}\right) I(x_i > \mu) \\
f_{\mathbf{X}}(\mathbf{x}|\mu,\sigma) &= \prod_{i=1}^{n} \frac{1}{\sigma} \exp\left(-\frac{x-\mu}{\sigma}\right) I(x_i > \mu) \\
&= \sigma^{-n} \exp\left(-\frac{\sum_{i=1}^n x_i}{\sigma}\right) \exp\left(\frac{n\mu}{\sigma}\right) I(\min_i x_i > \mu)
\end{aligned}$$

   Because μ is a known constant, we can factorize the joint pdf as

$$\begin{aligned}
f_{\mathbf{X}}(\mathbf{x}|\mu,\sigma) &= \sigma^{-n} \exp\left(-\frac{\sum_{i=1}^n x_i}{\sigma}\right) \exp\left(\frac{n\mu}{\sigma}\right) I(\min_i x_i > \mu) \\
&= g(T(\mathbf{x})|\sigma) h(\mathbf{x}) \\
h(\mathbf{x}) &= I(\min_i x_i > \mu) \\
T(\mathbf{x}) &= \sum_{i=1}^{n} x_i \\
g(t|\sigma) &= \sigma^{-n} \exp\left(-\frac{t}{\sigma}\right) \exp\left(\frac{n\mu}{\sigma}\right)
\end{aligned}$$

   Hence, by Factorization Theorem, $T(\mathbf{x}) = \sum_{i=1}^n x_i$ is a sufficient statistic for σ.

   (b) Assuming that σ is known, find a one-dimensional sufficient statistic for μ.
   Solution: Because σ is a known constant, we can factorize the joint pdf as

$$
\begin{aligned}
f_{\mathbf{X}}(\mathbf{x}|\mu,\sigma) &= \sigma^{-n}\exp\left(-\frac{\sum_{i=1}^{n} x_i}{\sigma}\right)\exp\left(\frac{n\mu}{\sigma}\right) I(\min_i x_i > \mu) \\
&= g(T(\mathbf{x})|\mu)h(\mathbf{x}) \\
h(\mathbf{x}) &= \sigma^{-n}\exp\left(-\frac{\sum_{i=1}^{n} x_i}{\sigma}\right) \\
T(\mathbf{x}) &= \min_i x_i \\
g(t|\mu) &= \exp\left(\frac{n\mu}{\sigma}\right) I(\min_i x_i > \mu)
\end{aligned}
$$

Hence, by Factorization Theorem, $T(\mathbf{x}) = \min_i x_i$ is a sufficient statistic for σ.

(c) Assuming that that both parameters are unknown, find a two-dimensional sufficient statistic for (μ, σ).

Solution: The joint pdf can be factorized as

$$
\begin{aligned}
f_{\mathbf{X}}(\mathbf{x}|\mu,\sigma) &= \sigma^{-n}\exp\left(-\frac{\sum_{i=1}^{n} x_i}{\sigma}\right)\exp\left(\frac{n\mu}{\sigma}\right) I(\min_i x_i > \mu) \\
&= g(T_1(\mathbf{x}), T_2(\mathbf{x})|\mu)h(\mathbf{x}) \\
h(\mathbf{x}) &= 1 \\
T_1(\mathbf{x}) &= \sum_{i=1}^{n} x_i \\
T_2(\mathbf{x}) &= \min_i(x_i) \\
g(t_1, t_2|\mu,\sigma) &= \sigma^{-n}\exp\left(-\frac{t_1}{\sigma}\right)\exp\left(\frac{n\mu}{\sigma}\right) I(t_2 > \mu)
\end{aligned}
$$

Then $f_{\mathbf{X}}(\mathbf{x}|\mu,\sigma) = g(T_1(\mathbf{x}), T_2(\mathbf{x})|\mu,\sigma)h(\mathbf{x})$ holds, and $\mathbf{T}(\mathbf{X}) = (T_1(\mathbf{X}), T_2(\mathbf{X})) = (\sum_{i=1}^{n} X_i, \min_i X_i)$ is a sufficient statistic.

2. Let $X_1, \cdots, X_n$ be *i.i.d.* random variables from $N(0, \sigma^2)$ with the pdf

$$
f_X(x|\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{x^2}{2\sigma^2}\right\},\ x \in R, \sigma^2 > 0
$$

(a) Apply the Factorization Theorem to show that $\sum_{i=1}^{n} X_i^2$ is a sufficient statistic for the parameter σ^2.

Solution: The joint pdf of the sample sample is

$$
\begin{aligned}
f_{\mathbf{X}}(\mathbf{x}|\sigma^2) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{x_i^2}{2\sigma^2}\right\} \\
&= \left(\frac{1}{2\pi}\right)^{n/2} \left(\frac{1}{\sigma^2}\right)^{n/2} \exp\left\{-\frac{\sum_{i=1}^{n} x_i^2}{2\sigma^2}\right\} \\
&= h(\mathbf{x})g(T(\mathbf{x})|\sigma^2), \text{ for all } \sigma^2
\end{aligned}
$$

where $h(\mathbf{x}) = (2\pi)^{-n/2}$, $g(t|\sigma^2) = (\sigma^2)^{-n/2}\exp(-t/2\sigma^2)$ and $T(\mathbf{x}) = \sum_{i=1}^{n} x_i^2$. Thus, according to the Factorization Theorem, $T(\mathbf{x}) = \sum_{i=1}^{n} X_i^2$ is a sufficient statistic for parameter σ^2.

(b) Is $\sum_{i=1}^{n} X_i^2$ also a minimal sufficient statistic for σ^2? Justify your answer.

Solution: Let $T(\mathbf{X}) = \sum_{i=1}^{n} X_i^2$. The ratio of joint pdf

$$
\begin{aligned}
\frac{f_{\mathbf{X}}(\mathbf{x}|\sigma^2)}{f_{\mathbf{X}}(\mathbf{y}|\sigma^2)} &= \frac{\left(\frac{1}{2\pi}\right)^{n/2} \left(\frac{1}{\sigma^2}\right)^{n/2} \exp\left\{-\frac{\sum_{i=1}^{n} x_i^2}{2\sigma^2}\right\}}{\left(\frac{1}{2\pi}\right)^{n/2} \left(\frac{1}{\sigma^2}\right)^{n/2} \exp\left\{-\frac{\sum_{i=1}^{n} y_i^2}{2\sigma^2}\right\}} \\
&= \exp\left\{-\frac{\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i^2}{2\sigma^2}\right\} \\
&= \exp\left\{-\frac{T(\mathbf{x}) - T(\mathbf{y})}{2\sigma^2}\right\}
\end{aligned}
$$

Because a/x is constant as a function to x if and only if $a = 0$, the equation above is constant as a function to σ^2 if and only if $T(\mathbf{x}) = T(\mathbf{y})$. Therefore, by Theorem 6.2.13, $T(\mathbf{X}) = \sum_{i=1}^{n} X_i^2$ is a minimal sufficient statistic.

3. Let $X_1, \cdots, X_n$ be *i.i.d.* random variables from a Poisson distribution whose probability mass function is given by

$$
f_X(x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, \; x = 0, 1, 2, \ldots; \lambda > 0.
$$

(a) Find a one-dimensional sufficient statistic for parameter λ.

Solution:

$$
\begin{aligned}
f_{\mathbf{X}}(\mathbf{x}|\theta) &= \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} \\
&= \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!} \\
&= g(T(\mathbf{x})|\lambda)h(\mathbf{x})
\end{aligned}
$$

where

$$\begin{aligned} h(\mathbf{x}) &= \frac{1}{\prod_{i=1}^n x_i!} \\ T(\mathbf{x}) &= \sum_{i=1}^n x_i \\ g(t|\lambda) &= e^{-n\lambda}\lambda^t \end{aligned}$$

Hence, by Factorization theorem $\sum_{i=1}^n X_i$ is a one-dimensional sufficient statistic.

(b) Show that your answer in (a) is also a minimally sufficient statistic.

Solution: The ratio between joint pmf is

$$\begin{aligned} \frac{f_{\mathbf{X}}(\mathbf{x}|\theta)}{f_{\mathbf{X}}(\mathbf{y}|\theta)} &= \frac{\frac{e^{-n\lambda}\lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}}{\frac{e^{-n\lambda}\lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n y_i!}} \\ &= \frac{\prod_{i=1}^n y_i!}{\prod_{i=1}^n x_i!}\lambda^{\sum_{i=1}^n x_i - \sum_{i=1}^n y_i} \end{aligned}$$

Because x^a is constant as a function to x if and only if $a = 0$, the equation above is constant as a function to λ if and only if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. Therefore, by Theorem 6.2.13, $\sum_{i=1}^n X_i$ is a minimal sufficient statistic.

4. Let $X_1, \cdots, X_n$ be a random sample from $Beta(\alpha, \beta)$. Find joint sufficient statistics for (α, β).

Solution: The joint pdf of a random sample from $Beta(\alpha, \beta)$ is given by:

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{x}|\alpha,\beta) &= \left(\frac{1}{B(\alpha,\beta)}\right)^n \left(\prod_{i=1}^n x_i\right)^{\alpha-1} \left(\prod_{i=1}^n (1-x_i)\right)^{\beta-1} \prod_{i=1}^n I(0 < x_i < 1) \\ &= g(T_1(\mathbf{x}), T_2(\mathbf{x})|\alpha,\beta)h(\mathbf{x}) \end{aligned}$$

where

$$
\begin{aligned}
h(\mathbf{x}) &= \prod_{i=1}^{n} I(0 < x_i < 1) \\
T_1(\mathbf{x}) &= \prod_{i=1}^{n} x_i \\
T_2(\mathbf{x}) &= \prod_{i=1}^{n} (1 - x_i) \\
g(t_1, t_2|\alpha, \beta) &= \left(\frac{1}{B(\alpha, \beta)}\right)^n t_1^{\alpha-1} t_2^{\beta-1}
\end{aligned}
$$

Hence, by Factorization theorem (T_1, T_2) is a two-dimensional sufficient statistic.

5. Let $X_1, \cdots, X_n$ be a random sample from $Cauchy(\theta, 1)$. Find a minimal sufficient statistic for θ.

   Solution: The pdf for $Cauchy(\theta, 1)$ is

$$
f(x|\theta, 1) = \frac{1}{\pi} \frac{1}{1 + (x - \theta)^2}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.
$$

   Choose $\mathbf{x}, \mathbf{y}$ in the sample space $\mathcal{X}$. Then

$$
\begin{aligned}
\frac{f_{\mathbf{X}}(\mathbf{x}|\theta, 1)}{f_{\mathbf{X}}(\mathbf{y}|\theta, 1)} &= \prod_{i=1}^{n} \frac{1 + (y_i - \theta)^2}{1 + (x_i - \theta)^2} \\
&= \prod_{i=1}^{n} \frac{1 + (y_{(i)} - \theta)^2}{1 + (x_{(i)} - \theta)^2} \qquad (1)
\end{aligned}
$$

   where $x_{(1)} < \cdots < x_{(n)}$, $y_{(1)} < \cdots < y_{(n)}$ are the order statistics corresponding to the two sample points, respectively. The ratio in (1) is free of θ if and only if the two sequences of order statistics are the same. Hence $\mathbf{T}(\mathbf{x}) = (x_{(1)}, \cdots, x_{(n)})$ is minimal sufficient.

6. Let $X_1, \cdots, X_n$ be a random sample from $Uniform(-\theta, \theta)$. Find a minimal sufficient statistic for θ.

Solution:

$$
\begin{aligned}
f_{\mathbf{X}}(\mathbf{x}|\theta) &= \prod_{i=1}^{n} \frac{1}{\theta} I(-\theta < x_i < \theta) \\
&= (\frac{1}{2\theta})^n I(-x_{(1)} < \theta) I(x_{(n)} < \theta) \\
&= (\frac{1}{2\theta})^n I(max(-x_{(1)}, x_{(n)}) < \theta) \\
&= g(T(\mathbf{x})|\theta) h(\mathbf{x})
\end{aligned}
$$

where

$$
\begin{aligned}
h(\mathbf{x}) &= 1 \\
T(\mathbf{x}) &= max(-X_{(1)}, X_{(n)}) \\
g(t|\lambda) &= (\frac{1}{2\theta})^n I(t < \theta)
\end{aligned}
$$

Hence, by Factorization theorem $max(-X_{(1)}, X_{(n)})$ is a one-dimensional sufficient statistic. In order to show it is a minimal sufficient statistic, choose $\mathbf{x}, \mathbf{y}$ in the sample space $\mathcal{X}$. Then

$$
\frac{f_{\mathbf{X}}(\mathbf{x}|\theta)}{f_{\mathbf{X}}(\mathbf{y}|\theta)} = \frac{I(max(-x_{(1)}, x_{(n)}) < \theta)}{I(max(-y_{(1)}, y_{(n)}) < \theta)}
$$

The ratio is free of θ if and only if $max(-X_{(1)}, X_{(n)}) = max(-Y_{(1)}, Y_{(n)})$ are the same. Hence $\mathbf{T}(\mathbf{x}) = max(-X_{(1)}, X_{(n)})$ is minimal sufficient by Theorem 6.2.13.