BIOSTAT 602 Biostatistical Inference
Homework 01

Ashton Baker

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1. A coin is twice as likely to turn up tails as heads. If the coin is tossed independently, what is the probability that the third head occurs on the 5th trial?

   Solution. This implies that $P(H) = 1/3$ and $P(T) = 2/3$. The probability of 2 heads occuring in the first 4 trials is described by a binomial distribution:

   $$\binom{4}{2}(1/3)^2(2/3)^2 = 8/27$$

   Then, the probability of heads occuring on the 5th trial is $1/3$, so the probability is

   $$(8/27)(1/3) = 8/81 \approx 0.0988$$

2. Suppose X and Y are two independent variables with unit variance. Let $Z = aX + Y$, where $a > 0$. If $\text{Cor}(X, Z) = 1/3$, then obtain the value of a.

   Solution. By the definition of correlation,

   $$\text{Cor}(X, Z) = \frac{\text{Cov}(X, Z)}{\sigma_X \sigma_Z} = 1/3$$

   We can begin by finding $\text{Cov}(X, Z)$. Note that because X and Y are independent, $\text{Cov}(X, Y) = 0$.

   $$\begin{aligned}
   \text{Cov}(X, Z) &= \mathbb{E}[XZ] - \mathbb{E}[X]\mathbb{E}[Z] \\
   &= \mathbb{E}\left[aX^2 + XY\right] - \mathbb{E}[X]\mathbb{E}[aX + Y] \\
   &= a\mathbb{E}\left[X^2\right] + \mathbb{E}[XY] - a\mathbb{E}[X]\mathbb{E}[X] + \mathbb{E}[X]\mathbb{E}[Y] \\
   &= a\text{Var}(X) + \text{Cov}(X, Y) \\
   &= a
   \end{aligned}$$

   Because X and Y have unit variance,

   $$\begin{aligned}
   \sigma_Z &= \sqrt{\text{Var}(Z)} \\
   &= \sqrt{\text{Var}(aX + Y)} \\
   &= \sqrt{a^2\text{Var}(X) + \text{Var}(Y) + 2a\text{Cov}(X, Y)} \\
   &= \sqrt{a^2 + 1}
   \end{aligned}$$

   and $\sigma_X = 1$. So $a = \sqrt{a^2 + 1}/3$.

3. Let $g(x)$, $x \geq 0$ be a valid pdf for a nonnegative random variable and define

   $$f(x, y) = \frac{g(\sqrt{x^2 + y^2})}{2\pi\sqrt{x^2 + y^2}}$$

   for $-\infty < x, y < \infty$.

   (a) Show that $f(x, y)$ is a valid pdf.

   Solution. The function f depends on x and y only in terms of $r = \sqrt{x^2 + y^2}$. So $f(x, y) = f(r)$, and

   $$\begin{aligned}
   \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\ dx\ dy &= \int_0^{\infty} 2\pi r f(r)\ dr \\
   &= \int_0^{\infty} 2\pi r \frac{g(r)}{2\pi r}\ dr \\
   &= \int_0^{\infty} g(r)\ dr \\
   &= 1
   \end{aligned}$$

   So f is a valid pdf.

(b) Suppose that the pair (X, Y) has the pdf $f(x, y)$. What is $P(XY > 0)$?
Solution. Due to the radial symmetry of f, $P(XY > 0) = 1/2$.

4. Given independent and identically distributed random samples $X_1, X_2, \dots, X_n$, each with finite mean μ and finite variance σ^2, define

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i S^2 \quad = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 W^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$$

(a) Show that $S^2 \xrightarrow{P} \sigma^2$

(b) Derive the asymptotic distribution of $\frac{\sqrt{n}(\overline{X}-\mu)}{\sqrt{S^2}}$

(c) Use the Delta method to derive the asymptotic distribution of $\overline{X}^2$ after you normalize it appropriately.

5. For two sets of random varibales $\{X_i\}$, $i = 1, \dots, n$, and $\{Y_i\}$, $j = 1, \dots, m$, show that

$$\text{Cov}\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \text{Cov}\left(X_i, Y - j\right)$$

where a_i and b_j are arbitrary constants.

6. Suppose $N \sim \text{Poisson}(\lambda)$. Given $N = n > 0$, $X_1, \dots, X_N$ are iid and follow $U[0, 1]$. We define $X_0 = 0$ when $N = 0$.

(a) Given $N = n$, find the probability that $X_0, X_1, \dots, X_N$ are all less that t, where $0 < t < 1$.

(b) Find the (unconditional) probability that $X_0, X_1, \dots, X_N$ are all less than t, where $0 < t < 1$.

(c) Let $S_N = X_0 + X_1 + \cdots + X_N$. Compute $\mathbb{E}[S_N]$.

7. Let X_1, X_2, X_3 be a random sample of size 3 from a $N(0, 1)$ population. In each of the following cases, Z denotes a specific function derived from this random sample. In each case identify the distribution of the resulting random variable Z along with the associated parameters.

(a) $X_1 + X_2 + 2X_3$.

(b) $X_1^2 + X_2^2 + X_3^3$.

(c) $(X_1 - X_2)^2/2$.

(d) $Z = \frac{2X_1^2}{X_2^2+X_3^2}$

(e) $Z = \frac{(X_1-X_2)^2}{(X_1+X_2)^2}$