Biostat 602 Winter 2017

Lecture Set 9

Point Estimation

Attainment of CRLB

Reading: CB 7.3.1–7.3.2

Attainment of CRLB

Question: How frequently can one find an unbiased estimator of $\tau(\theta)$ that attains Cramer Rao Lower bound?

Regularity condition for Cramer-Rao Theorem

$$\frac{d}{d\theta}\int_{x\in\mathcal{X}} h(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}|\theta)d\mathbf{x} = \int_{x\in\mathcal{X}} h(\mathbf{x})\frac{\partial}{\partial\theta} f_{\mathbf{X}}(\mathbf{x}|\theta)d\mathbf{x}$$

for some function $h(x)$.

- This regularity condition holds for exponential family.
- How about non-exponential family?

Example 1: Let $X_1, \cdots, X_n$ be a random sample from $Uniform(0,\theta)$. Let us check the regularity condition. We use **Leibnitz's Rule** which states

$$\frac{d}{d\theta}\int_{a(\theta)}^{b(\theta)} f(x|\theta)dx \;=\; f(b(\theta)|\theta)b'(\theta) - f(a(\theta)|\theta)a'(\theta) + \int_{a(\theta)}^{b(\theta)} \frac{\partial}{\partial\theta} f(x|\theta)dx$$

In our example of $Uniform(0,\theta)$

$$\begin{aligned}
f_X(x|\theta) &= 1/\theta \\
\frac{d}{d\theta}\int_0^\theta h(x)\left(\frac{1}{\theta}\right)dx &= \frac{h(\theta)}{\theta}\frac{d\theta}{d\theta} - h(0)f_X(0|\theta)\frac{d0}{d\theta} + \int_0^\theta \frac{\partial}{\partial\theta}h(x)\left(\frac{1}{\theta}\right)dx \\
&\neq \int_0^\theta \frac{\partial}{\partial\theta}h(x)\left(\frac{1}{\theta}\right)dx
\end{aligned}$$

Hence the interchangeability condition is not satisfied unless $h(\theta)/\theta = 0 \;\forall\theta$. This raises the following questions

1. Is there an unbiased estimator of θ?
2. Does any unbiased estimator attain CRLB?
3. Is there a best estimator in the class of unbiased estimators?

Solution:

When is the Cramer-Rao Lower Bound Attainable?

It is possible that the value of Cramer-Rao bound may be strictly smaller than the variance of any unbiased estimator

Corollary 7.3.15: Let $X_1, \cdots, X_n$ be iid with pdf/pmf $f_X(x|\theta)$, where $f_X(x|\theta)$ satisfies the assumptions of the Cramer-Rao Theorem. Let

$$L(\theta|\mathbf{x}) = \prod_{i=1}^{n} f_X(x_i|\theta)$$

denote the likelihood function. If $W(\mathbf{X})$ is unbiased for $\tau(\theta)$, then $W(\mathbf{X})$ attains the Cramer-Rao lower bound if and only if

$$\frac{\partial}{\partial\theta} \log L(\theta|\mathbf{x}) = a(\theta)[W(\mathbf{x}) - \tau(\theta)]$$

for some function $a(\theta)$.

Proof:

Revisiting the Bernoulli Example

Example 2: Let $X_1, \cdots, X_n$ be i.i.d. $Bernoulli(p)$. Is $\overline{X}$ the best unbiased estimator of p? Does it attain the Cramer-Rao lower bound?

Method Using Corollary 7.3.15:

$$\begin{aligned}
L(p|\mathbf{x}) &= \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} \\
\log L(p|\mathbf{x}) = \log \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} &= \sum_{i=1}^{n} \log[p^{x_i}(1-p)^{1-x_i}] \\
&= \sum_{i=1}^{n} [x_i \log p + (1-x_i)\log(1-p)] \\
&= \log p \sum_{i=1}^{n} x_i + \log(1-p)(n - \sum_{i=1}^{n} x_i)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial p} \log L(p|\mathbf{x}) &= \frac{\sum_{i=1}^{n} x_i}{p} - \frac{n - \sum_{i=1}^{n} x_i}{1-p} \\
&= \frac{n\overline{x}}{p} - \frac{n(1-\overline{x})}{1-p} \\
&= \frac{(1-p)n\overline{x} - np(1-\overline{x})}{p(1-p)} \\
&= \frac{n(\overline{x}-p)}{p(1-p)} = \frac{n}{p(1-p)}(\overline{x}-p) \\
&= a(p)[W(\mathbf{x}) - \tau(p)]
\end{aligned}$$

where $a(p) = \frac{n}{p(1-p)}$, $W(\mathbf{x}) = \overline{x}$, $\tau(p) = p$. Therefore, $\overline{X}$ is the best unbiased estimator for p and attains the Cramer-Rao lower bound.

Example 3: Let $X_1, \cdots, X_n$ be i.i.d. *Geometric*(p) with pmf

$$f_X(x|p) = (1-p)^{x-1}p, \quad 0 < p < 1, \ x = 1, 2, \ldots$$

Find a function $\tau(p)$ which admits an unbiased estimator that attains the CRLB.

Example 4: Let $X_1, \cdots, X_n$ be a random sample from $\mathcal{N}(\mu, \sigma^2)$. Consider estimating σ^2, assuming μ is known. Is Cramer-Rao bound attainable? What if μ is unknown?

Solution: Note that the information number equals

$$I(\sigma^2) = -\mathrm{E}\left[\frac{\partial^2}{\partial(\sigma^2)^2} \log f_X(x|\mu, \sigma)\right].$$

Now,

$$\begin{aligned}
f(x|\mu, \sigma^2) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \\
\log f(x|\mu, \sigma^2) &= -\frac{1}{2}\log(2\pi\sigma^2) - \frac{(x-\mu)^2}{2\sigma^2} \\
\frac{\partial}{\partial(\sigma^2)} \log f(x|\mu, \sigma^2) &= -\frac{1}{2}\frac{1}{\sigma^2} + \frac{(x-\mu)^2}{2(\sigma^2)^2} \\
\frac{\partial^2}{\partial(\sigma^2)^2} \log f(x|\mu, \sigma^2) &= \frac{1}{2}\frac{1}{(\sigma^2)^2} - \frac{2(x-\mu)^2}{2(\sigma^2)^3} \\
I(\sigma^2) &= -\mathrm{E}\left[\frac{1}{2\sigma^4} - \frac{2(x-\mu)^2}{2\sigma^6}\right] \\
&= -\frac{1}{2\sigma^4} + \frac{1}{\sigma^6}\mathrm{E}[(x-\mu)^2] = -\frac{1}{2\sigma^4} + \frac{1}{\sigma^6}\sigma^2 = \frac{1}{2\sigma^4}
\end{aligned}$$

Cramer-Rao lower bound is $\dfrac{1}{nI(\sigma^2)} = \dfrac{2\sigma^4}{n}$.

The unbiased estimator of $\hat{\sigma^2} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$, gives

$$\mathrm{Var}(\hat{\sigma^2}) = \frac{2\sigma^4}{n-1} > \frac{2\sigma^4}{n}$$

So, $\hat{\sigma^2}$ does not attain the Cramer-Rao lower-bound.

Is Cramer-Rao lower-bound for σ^2 attainable?

$$
\begin{aligned}
L(\sigma^2|\mathbf{x}) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right] \\
\log L(\sigma^2|\mathbf{x}) &= -\frac{n}{2}\log(2\pi\sigma^2) - \sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2\sigma^2} \\
\frac{\partial \log L(\sigma^2|\mathbf{x})}{\partial \sigma^2} &= -\frac{n}{2}\frac{2\pi}{2\pi\sigma^2} + \sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2(\sigma^2)^2} \\
&= -\frac{n}{2\sigma^2} + \sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2\sigma^4} \\
&= \frac{n}{2\sigma^4}\left(\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n} - \sigma^2\right) \\
&= a(\sigma^2)(W(\mathbf{x}) - \sigma^2)
\end{aligned}
$$

Therefore,

1. If μ is known, the best unbiased estimator for σ^2 is $\sum_{i=1}^{n}(x_i-\mu)^2/n$, and it attains the Cramer-Rao lower bound, i.e.

$$
\mathrm{Var}\left[\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{n}\right] = \frac{2\sigma^4}{n}
$$

2. If μ is not known, the Cramer-Rao lower-bound cannot be attained.

At this point, we do not know if $\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\overline{x})^2$ is the best unbiased estimator for σ^2 or not.

Result for Exponential Family

Let $X_1, \cdots, X_n$ be iid from the one parameter exponential family with pdf/pmf

$$f_X(x|\theta) = c(\theta)h(x)\exp\left[w(\theta)t(x)\right].$$

Assume that $\mathrm{E}[t(X)] = \tau(\theta)$. Then $\frac{1}{n}\sum_{i=1}^n t(x_i)$, which is an unbiased estimator of $\tau(\theta)$, attains the Cramer-Rao lower-bound. That is,

$$\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^n t(X_i)\right) = \frac{[\tau'(\theta)]^2}{I_n(\theta)}$$

Proof:

$$\mathrm{E}\left[\frac{1}{n}\sum_{i=1}^n t(X_i)\right] = \frac{1}{n}\sum_{i=1}^n \mathrm{E}\left[t(X_i)\right] = \frac{1}{n}\sum_{i=1}^n \tau(\theta) = \tau(\theta)$$

So, $\frac{1}{n}\sum_{i=1}^n t(x_i)$ is an unbiased estimator of $\tau(\theta)$.

$$\begin{aligned}
\log L(\theta|\mathbf{x}) &= \sum_{i=1}^n \log f_X(x_i|\theta) \\
&= \sum_{i=1}^n \left[\log c(\theta) + \log h(x) + w(\theta)t(x_i)\right]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \log L(\theta|\mathbf{x})}{\partial\theta} &= \sum_{i=1}^n \left[\frac{c'(\theta)}{c(\theta)} + 0 + w'(\theta)t(x_i)\right] \\
&= nw'(\theta)\left[\frac{1}{n}\sum_{i=1}^n t(x_i) - \left\{-\frac{c'(\theta)}{c(\theta)w'(\theta)}\right\}\right] \qquad (1)
\end{aligned}$$

Because $\mathrm{E}\left[\frac{\partial}{\partial\theta}\log L(\theta|\mathbf{x})\right] = 0$, from (1), we have $\tau(\theta) = -\frac{c'(\theta)}{c(\theta)w'(\theta)}$.

Hence we have,

$$\frac{\partial \log L(\theta|\mathbf{x})}{\partial\theta} = nw'(\theta)\left[\frac{1}{n}\sum_{i=1}^{n} t(x_i) - \tau(\theta)\right]$$

Thus $\frac{1}{n}\sum_{i=1}^{n} t(x_i)$ attains the CRLB and is the best unbiased estimator of $\tau(\theta)$.

Remarks

- For exponential families, CRLB approach establishes $\frac{1}{n}\sum_{i=1}^{n} t(x_i)$ to be the best estimator for its expectation $\tau(\theta)$. If the parameter of interest is non-trivially different from $\tau(\theta)$ then CRLB cannot be used to obtain the best estimator.

- For non-exponential family, it is unlikely that Cramer-Rao Theorem can help finding the best unbiased estimator, but the bound still can be calculated to determine a loose lower bound of the variance of the best unbiased estimator (provided the regularity conditions hold).