Biostat 602 **Assignment 1** **Due: Thursday, January 12**

NOTE: The assignment is due in class on Thursday, January 12. Please see the information sheet for policies regarding assignment submission. Only worked out solutions detailing all steps will be acceptable as valid. Simply the final answer (even if correct) without any indication of the solution path will not fetch any credit.

Homework Problems

1. A coin is twice as likely to turn up tails as heads. If the coin is tossed independently, what is the probability that the third head occurs on the 5th trial?

2. Suppose X and Y are two independent variables with unit variance. Let $Z = aX + Y$, where $a > 0$. If $Cor(X, Z) = 1/3$, then obtain the value of a.

3. Let $g(x), x \geq 0$, be a valid pdf for a nonnegative random variable and define

$$f(x,y) = \frac{g(\sqrt{x^2+y^2})}{2\pi\sqrt{x^2+y^2}}.$$

for $-\infty < x, y < \infty$.

   (a) Show that $f(x, y)$ is a valid pdf.

   (b) Suppose that the pair (X, Y) has the pdf $f(x, y)$. What is $P(XY > 0)$?

4. Given independent and identically distributed random samples $X_1, X_2, \cdots, X_n$, each with finite mean μ and finite variance σ^2, define

$$\begin{aligned}
\overline{X} &= \frac{1}{n}\sum_{i=1}^{n} X_i \\
S^2 &= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 \\
W^2 &= \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2
\end{aligned}$$

   (a) Show that $S^2 \xrightarrow{P} \sigma^2$.

   (b) Derive the asymptotic distribution of $\dfrac{\sqrt{n}(\overline{X}-\mu)}{\sqrt{S^2}}$.

   (c) Use Delta method to derive the asymptotic distribution of $\left(\overline{X}\right)^2$ after you normalize it appropriately.

5. For two sets of random variables $\{X_i\}, i = 1, \dots n$ and $\{Y_j\}, j = 1, \dots m$, show that

$$Cov\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j Cov(X_i, Y_j)$$

where a_i, b_j are arbitrary constants.

6. Suppose $N \sim Poisson(\lambda)$. Given $N = n > 0$, $X_1, \dots, X_N$ are iid and follow uniform[0,1]. We define $X_0 = 0$ when $N = 0$.

   (a) Given $N = n$, find the probability $X_0, X_1, \dots, X_N$ are all less than t, where $0 < t < 1$.

   (b) find the (unconditional) probability $X_0, X_1, \dots, X_N$ are all less than t, where $0 < t < 1$.

   (c) Let $S_N = X_0 + X_1 + ... + X_N$. Compute $E(S_N)$.

7. Let X_1, X_2, X_3 be a random sample of size 3 from a $N(0, 1)$ population. In each of the following five cases, Z denotes a specific function derived from this random sample. In each case identify the distribution of the resulting random variable Z along with the associated parameters.

$$(i) \quad X_1 + X_2 + 2X_3.$$

$$(ii) \quad X_1^2 + X_2^2 + X_3^2.$$

$$(iii) \quad (X_1 - X_2)^2/2.$$

$$(iv) \quad Z = \frac{2X_1^2}{X_2^2 + X_3^2}$$

$$(v) \quad Z = \frac{(X_1 - X_2)^2}{(X_1 + X_2)^2}$$