BIOSTAT 602 Biostatistical Inference
Homework 04

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1. For each of the following statements, state whether it is true or false. If it is true, provide brief justification. If it is false, provide a counterexample.

   (a) Minimal sufficient statistics are always a function of the order statistics.

   (b) Any function of a minimal sufficient statistic is sufficient.

   (c) A complete statistic is always a one-to-one function of a minimal sufficient statistic.

   (d) A one-to-one function of an ancillary statistic is also ancillary

   (e) If $X_1, \dots, X_n$ is a random sample from a pdf conforming to a location family, then $\overline{X}$ also has a pdf that belongs to a location family.

2. Let X_1, X_2 be two i.i.d. random variables from the probability density function of the following form:

$$f_X(x|\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad x \in \mathbb{R},\ \sigma^2 > 0$$

   (a) Find a complete sufficient statistic for σ^2. Justify your answer.

   (b) Show that $\frac{X_1}{\sqrt{X_1^2+X_2^2}}$ is an ancillary statistic.

   (c) Show that

$$\mathbb{E}\left[\frac{X_1}{\sqrt{X_1^2+X_2^2}}\right] = 0$$

3. Let $X_1, \dots, X_n$ be i.i.d. observations from a gamma distribution $\text{Gamma}(\alpha, \beta)$ with the pdf

$$f_X(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \qquad 0 \le x \le \infty,\ \alpha > 0,\ \beta > 0$$

   (a) Show that the gamma distribution belongs to a full exponential family. Identify $h(\cdot)$, $c(\cdot)$, $w_j(\cdot)$, $t_j(\cdot)$ terms in your proof.

   (b) Find a complete sufficient statistic for ` $= (\alpha, \beta)$ and justify whether it is also a minimal sufficient statistic or not.

4. Let $X_1, \dots, X_n$ be i.i.d. random variables from the two-parameter exponential distribution with probability density function of the following form:

$$f_X(x|\mu, \sigma) = \frac{1}{\sigma} e^{-(x-\mu)/\sigma}, \quad \mu < x < \infty,\ 0 < \sigma < \infty.$$

   Consider the statistics $T_1 = X_{(1)}$ and $T_2 = X_{(1)} - \overline{X}$. Show that T_1 and T_2 are independently distributed.

5. Consider the previous problem again.

   (a) Assuming σ is known, find a method of moments estimator for μ by using the lowest-order moment of X as possible.

   (b) Assuming μ is known, find a method of moments estimator for σ by using the lowest-order moment of X as possible.

   (c) Assuming both parameters are unknown, find a method of moments estimator of σ by using the lowest-order moments of X as possible.

   (d) Assuming both parameters are unknown, find a method of moments estimator of the survival function $S(t) = P(X > t)$ for a given $t > \mu$, assuming that X follows the pdf above. You may define a new random variable and use its moments.