

BIOSTAT 602 Biostatistical Inference Homework 01

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1. A coin is twice as likely to turn up tails as heads. If the coin is tossed independently, what is the probability that the third head occurs on the 5th trial?

Solution. This implies that $P(\text{Heads}) = 1/3$ and $P(\text{Tails}) = 2/3$. The probability of 2 heads occurring in the first 4 trials is described by a binomial distribution:

$$\binom{4}{2} (1/3)^2 (2/3)^2 = 8/27$$

Then, the probability of heads occurring on the 5th trial is $1/3$, so the probability is

$$(8/27)(1/3) = 8/81 \approx 0.0988$$

2. Suppose X and Y are two independent variables with unit variance. Let $Z = aX + Y$, where $a > 0$. If $\text{Cor}(X, Z) = 1/3$, then obtain the value of a .

Solution. By the definition of correlation,

$$\text{Cor}(X, Z) = \frac{\text{Cov}(X, Z)}{\sigma_X \sigma_Z} = 1/3$$

We can begin by finding $\text{Cov}(X, Z)$. Note that because X and Y are independent, $\text{Cov}(X, Y) = 0$.

$$\begin{aligned} \text{Cov}(X, Z) &= \mathbb{E}[XZ] - \mathbb{E}[X] \mathbb{E}[Z] \\ &= \mathbb{E}[aX^2 + XY] - \mathbb{E}[X] \mathbb{E}[aX + Y] \\ &= a\mathbb{E}[X^2] + \mathbb{E}[XY] - a\mathbb{E}[X] \mathbb{E}[X] + \mathbb{E}[X] \mathbb{E}[Y] \\ &= a\text{Var}(X) + \text{Cov}(X, Y) \\ &= a \end{aligned}$$

Because X and Y have unit variance,

$$\begin{aligned} \sigma_Z &= \sqrt{\text{Var}(Z)} \\ &= \sqrt{\text{Var}(aX + Y)} \\ &= \sqrt{a^2 \text{Var}(X) + \text{Var}(Y) + 2a\text{Cov}(X, Y)} \\ &= \sqrt{a^2 + 1} \end{aligned}$$

and $\sigma_X = 1$. So $a = \sqrt{a^2 + 1}/3$.

3. Let $g(x)$, $x \geq 0$ be a valid pdf for a nonnegative random variable and define

$$f(x, y) = \frac{g(\sqrt{x^2 + y^2})}{2\pi\sqrt{x^2 + y^2}}$$

for $x, y \in \mathbb{R}$

(a) Show that $f(x, y)$ is a valid pdf.

Solution. The function f depends on x and y only in terms of $r = \sqrt{x^2 + y^2}$. So $f(x, y) = f(r)$, and

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy &= \int_0^{\infty} 2\pi r f(r) \, dr \\ &= \int_0^{\infty} 2\pi r \frac{g(r)}{2\pi r} \, dr \\ &= \int_0^{\infty} g(r) \, dr \\ &= 1 \end{aligned}$$

So f is a valid pdf.

(b) Suppose that the pair (X, Y) has the pdf $f(x, y)$. What is $P(XY > 0)$?

Solution. Due to the radial symmetry of f , $P(XY > 0) = 1/2$.

4. Given independent and identically distributed random samples X_1, X_2, \dots, X_n , each with finite mean μ and finite variance σ^2 , define

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

$$W^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2$$

(a) Show that $S^2 \xrightarrow{P} \sigma^2$

(b) Derive the asymptotic distribution of $\frac{\sqrt{n}(\bar{X} - \mu)}{\sqrt{S^2}}$

(c) Use the Delta method to derive the asymptotic distribution of \bar{X}^2 after you normalize it appropriately.

5. For two sets of random variables $\{X_i\}$, $i = 1, \dots, n$, and $\{Y_j\}$, $j = 1, \dots, m$, show that

$$\text{Cov} \left(\sum_{i=1}^n a_i X_i, \sum_{j=1}^m b_j Y_j \right) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \text{Cov}(X_i, Y_j)$$

where a_i and b_j are arbitrary constants.

Solution.

$$\begin{aligned} \text{Cov} \left(\sum_{i=1}^n a_i X_i, \sum_{j=1}^m b_j Y_j \right) &= \mathbb{E} \left[\left(\sum_{i=1}^n a_i X_i - \mathbb{E} \left[\sum_{i=1}^n a_i X_i \right] \right) \left(\sum_{j=1}^m b_j Y_j - \mathbb{E} \left[\sum_{j=1}^m b_j Y_j \right] \right) \right] \\ &= \mathbb{E} \left[\sum_{i=1}^n a_i (X_i - \mathbb{E}[X_i]) \sum_{j=1}^m b_j (Y_j - \mathbb{E}[Y_j]) \right] \\ &= \mathbb{E} \left[\sum_{i=1}^n \sum_{j=1}^m a_i b_j (X_i - \mathbb{E}[X_i]) (Y_j - \mathbb{E}[Y_j]) \right] \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i b_j \mathbb{E} [(X_i - \mathbb{E}[X_i]) (Y_j - \mathbb{E}[Y_j])] \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i b_j \text{Cov}(X_i, Y_j) \end{aligned}$$

6. Suppose $N \sim \text{Poisson}(\lambda)$. Given $N = n > 0$, X_1, \dots, X_N are iid and follow $U[0, 1]$. We define $X_0 = 0$ when $N = 0$.

(a) Given $N = n$, find the probability that X_0, X_1, \dots, X_N are all less than t , where $0 < t < 1$.

Solution. Since each X_i is uniformly distributed, $P(X_i < t) = t$. If $N = n$, then because X_1, \dots, X_n are independent,

$$P(X_1, \dots, X_n < t) = P(X_1 < t) P(X_2 < t) \cdots P(X_n < t) = t^n$$

(b) Find the (unconditional) probability that X_0, X_1, \dots, X_N are all less than t , where $0 < t < 1$.

Solution. In this case, we need to sum over all the possible values that N could take, weighted by the probability of observing each particular value n . So

$$\begin{aligned} P(X_1, \dots, X_N < t) &= \sum_{n=0}^{\infty} P(N = n) P(X_1, \dots, X_n < t) \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} t^n \\ &= e^{t\lambda/e} \end{aligned}$$

(c) Let $S_N = X_0 + X_1 + \dots + X_N$. Compute $\mathbb{E}[S_N]$.

Solution.

$$\begin{aligned} \mathbb{E}[S_N] &= \sum_{n=0}^{\infty} P(N = n) \mathbb{E}[X_1 + \dots + X_n] \\ &= \sum_{n=0}^{\infty} \left(\frac{\lambda^n e^{-\lambda}}{n!} \right) \left(\frac{n}{2} \right) \\ &= \frac{\lambda}{2} e^{\lambda/e - 1} \end{aligned}$$

7. Let X_1, X_2, X_3 be a random sample of size 3 from a $N(0, 1)$ population. In each of the following cases, Z denotes a specific function derived from this random sample. In each case identify the distribution of the resulting random variable Z along with the associated parameters.

(a) $X_1 + X_2 + 2X_3$.

(b) $X_1^2 + X_2^2 + X_3^2$.

(c) $(X_1 - X_2)^2 / 2$.

(d) $Z = \frac{2X_1^2}{X_2^2 + X_3^2}$

(e) $Z = \frac{(X_1 - X_2)^2}{(X_1 + X_2)^2}$