BIOSTAT 602 Biostatistical Inference
Homework 02

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1. Let $X_1, \dots, X_n$ be i.i.d random variables from the probability density function of the following form:

$$f_X(x|\theta) = \begin{cases} \frac{2x}{\theta^2}, & 0 < x < \theta \\ 0 & \text{otherwise} \end{cases}$$

where $\theta > 0$. Find a minimal sufficient statistic for θ.

Solution. The joint pdf for $X_1, \dots, X_n$ is

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{x}|\theta) &= \prod_{i=1}^{n} \frac{2x_i}{\theta^2} 1_{\{x_i \in (0,\theta)\}} \\ &= \left(\frac{2}{\theta^2}\right)^n 1_{\{x_{(1)} \in (0,\theta)\}} 1_{\{x_{(n)} \in (0,\theta)\}} \prod_{i=1}^{n} x_i. \end{aligned}$$

So for a second sample $\mathbf{Y}$, we get

$$\frac{f_{\mathbf{X}}(\mathbf{x}|\theta)}{f_{\mathbf{Y}}(\mathbf{y}|\theta)} = \frac{1_{\{x_{(1)} \in (0,\theta)\}} 1_{\{x_{(n)} \in (0,\theta)\}} \prod_{i=1}^{n} x_i}{1_{\{y_{(1)} \in (0,\theta)\}} 1_{\{y_{(n)} \in (0,\theta)\}} \prod_{i=1}^{n} y_i}$$

which is constant in θ when $(\min(\mathbf{X}), \max(\mathbf{X})) = (\min(\mathbf{Y}), \max(\mathbf{Y}))$, so the statistic $T(\mathbf{X}) = (\min(\mathbf{X}), \max(\mathbf{X}))$ is minimally sufficient.

2. Suppose that $X_1, \dots, X_n$ are i.i.d random variables from pdf

$$f_X(x|\theta) = \theta x^{\theta-1} \exp\left(-x^\theta\right)$$

where $\theta > 0$, $x > 0$. Show that $(\log X_{(n)})/(\log X_{(1)})$ is an ancillary statistic.

3. Let $X_1, \dots, X_n$ be i.i.d. random variables from a uniform distribution $U{-\theta}, \theta$ with the pdf given by

$$f_X(x|\theta) = \frac{1}{2\theta} 1_{\{x \in (-\theta,\theta)\}}, \quad \theta > 0$$

   (a) Is the two dimensional statistic $T_1(\mathbf{X}) = \left(X_{(1)}, X_{(n)}\right)$ a complete sufficient statistic? Justify your answer.

   (b) Is the one-dimensional statistic $T_2(\mathbf{X}) = \max_i \{|X_i|\}$ a complete sufficient statistic? Justify your answer.

4. Let $X_1, \dots, X_n$ be i.i.d. random variables from $N(\mu, \sigma^2)$ population with μ known. Find a one-dimensional minimal sufficient statistic for σ^2.

Solution. The joint distribution for $X_1, \dots, X_n$ is

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{x}|\sigma^2) &= \prod_{i=1}^{n} (2\sigma^2\pi)^{-1/2} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right) \\ &= \left(2\sigma^2\pi\right)^{-n/2} \exp\left(\sum_{i=1}^{n} -\frac{(x_i - \mu)}{2\sigma^2}\right) \end{aligned}$$

so if we have another sample $\mathbf{Y} = \{Y_1, Y_2, \dots, Y_n\}$, then the ratio

$$\begin{aligned} \frac{f_{\mathbf{X}}(\mathbf{x}|\sigma^2)}{f_{\mathbf{Y}}(\mathbf{y}|\sigma^2)} &= \frac{\exp\left(\sum_{i=1}^{n} -(x_i - \mu)^2/\sigma^2\right)}{\exp\left(\sum_{i=1}^{n} -(y_i - \mu)^2/\sigma^2\right)} \\ &= \exp\left(\frac{1}{\sigma^2}\left[\sum_{i=1}^{n} (y_i - \mu)^2 - \sum_{i=1}^{n} (x_i - \mu)^2\right]\right) \end{aligned}$$

is constant in σ^2 if and only if $\sum_{i=1}^{n} (y_i - \mu)^2 = \sum_{i=1}^{n} (x_i - \mu)^2$. So the statistic $T(\mathbf{X}) = \sum_{i=1}^{n} (x_i - \mu)^2$ is minimally sufficient for σ^2.

5. Let $X_1, \ldots, X_n$ be i.i.d. observations uniformly drawn from $\{1, 2, \ldots, \theta\}$, where θ is a positive integer. This corresponds to a discrete uniform distribution with pmf

$$f_X(x|\theta) = \begin{cases} 1/\theta & x = 1, 2, \ldots, \theta \\ 0 & \text{otherwise} \end{cases}$$

Show that $T(\mathbf{X}) = \max_i X_i$ is a complete, minimal, sufficient statistic.

Solution. The pmf of $T(\mathbf{X})$ is

$$P\left(X_{(n)} = k\right) = \binom{n}{1} \left(\frac{k}{\theta}\right)^{n-1} \left(\frac{1}{\theta}\right)$$

So if

$$\begin{aligned} \mathbb{E}\left[g(T)\right] &= \sum_{k=1}^{\theta} g(k) P\left(X_{(n)} = k\right) \\ &= \sum_{k=1}^{\theta} g(k) \binom{n}{1} \left(\frac{k}{\theta}\right)^{n-1} \left(\frac{1}{\theta}\right) \\ &= 0, \end{aligned}$$

for $\theta = 1, 2, \ldots$, then clearly $g(k) = 0$ for all k.