MATH 571 Numerical Linear Algebra
Exam 02

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1. A coin is twice as likely to turn up tails as heads. If the coin is tossed independently, what is the probability that the third head occurs on the 5th trial?

   Solution. This implies that $P(H) = 1/3$ and $P(T) = 2/3$. So

2. Suppose X and Y are two independent variables with unit variance. Let $Z = aX + Y$, where $a > 0$. If $\mathrm{Cor}(X, Z) = 1/3$,then obtain the value of a.

3. Let $g(x)$, $x \geq 0$ be a valid pdf for a nonnegative random variable and define

$$f(x, y) = \frac{g(\sqrt{x^2 + y * 2})}{2\pi\sqrt{x^2 + y^2}}$$

   for $-\infty < x, y < \infty$.

   (a) Show that $f(x, y)$ is a valid pdf.

   (b) Suppose that the pair (X, Y) has the pdf $f(x, y)$. What is $P(XY > 0)$?

4. Given independent and identically distributed random samples $X_1, X_2, \ldots, X_n$, each with finite mean μ and finite variance σ^2, define

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i S^2 \quad = \frac{1}{n-1}\sum_{i=1}^{n} (X_i - \overline{X})^2 W^2 = \frac{1}{n}\sum_{i=1}^{n} (X_i - \overline{X})^2$$

   (a) Show that $S^2 \xrightarrow{P} \sigma^2$

   (b) Derive the asymptotic distribution of $\frac{\sqrt{n}(\overline{X}-\mu)}{\sqrt{S^2}}$

   (c) Use the Delta method to derive the asymptotic distribution of $\overline{X}^2$ after you normalize it appropriately.

5. For two sets of random varibales $\{X_i\}$, $i = 1, \ldots, n$, and $\{Y_i\}$, $j = 1, \ldots, m$, show that

$$\mathrm{Cov}\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \mathrm{Cov}(X_i, Y - j)\right)$$

   where a_i and b_j are arbitrary constants.

6. Suppose $N \sim \text{Poisson}(\lambda)$. Given $N = n > 0$, $X_1, \ldots, X_N$ are iid and follow $U[0, 1]$. We define $X_0 = 0$ when $N = 0$.

   (a) Given $N = n$, find the probability that $X_0, X_1, \ldots, X_N$ are all less that t, where $0 < t < 1$.

   (b) Find the (unconditional) probability that $X_0, X_1, \ldots, X_N$ are all less than t, where $0 < t < 1$.

   (c) Let $S_N = X_0 + X_1 + \cdots + X_N$. Compute $\mathbb{E}(S_N)$.

7. Let X_1, X_2, X_3 be a random sample of size 3 from a $N(0, 1)$ population. In each of the following cases, Z denotes a specific function derived from this random sample. In each case identify the distribution of the resulting random variable Z along with the associated parameters.

   (a) $X_1 + X_2 + 2X_3$.

   (b) $X_1^2 + X_2^2 + X_3^3$.

   (c) $(X_1 - X_2)^2/2$.

   (d) $Z = \frac{2X_1^2}{X_2^2+X_3^2}$

   (e) $Z = \frac{(X_1-X_2)^2}{(X_1+X_2)^2}$