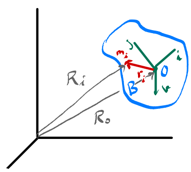
Attitude Dynamics & Kinematics

Suppose a rigid body is moving in an inertial frame. The motion can be described by the translation of its CM, together with rotation about some axis through the CM. The rate of change of a vector as observed in the inertial frame is related to the rate of change as observed in the rotating frame (the body frame B) by

$$\left.\frac{dA}{dt}\right|_I = \left.\frac{dA}{dt}\right|_B + \omega \times A$$

where ω is the angular velocity.



Suppose the body frame has its origin at the CM of the object. For any particle m_i in the body, $R_i = R_o + r_i$, and therefore (since R_o & R_i are measured in the inertial frame)

$$(\dot{R}_i - \dot{R}_o) = \dot{r}_i + \omega \times r_i$$

$$\Rightarrow \quad \dot{R}_i = \dot{R}_o + \dot{r}_i + \omega \times r_i .$$

The angular momentum of the mass m_i is

$$h_i = r_i \times (m_i \dot{R}_i)$$
$$= r_i \times m(\dot{R}_o + \dot{r}_i + \omega \times r_i).$$

But, $\dot{r}_i = 0$ for a rigid body, so

$$h_i = r_i \times m_i (\dot{R}_o + \omega \times r_i)$$
$$= -v_o \times m_i r_i + r_i \times m_i(\omega \times r_i)$$

where $\dot{R}_o \equiv v_o$. To get the angular momentum of the entire body we "sum over all masses m_i"

$$h = \sum_{m_i} (-v_o \times m_i r_i) + \sum_{m_i} r_i \times m_i(\omega \times r_i)$$
$$= -v_o \times \sum_i m_i r_i + \sum_{m_i} r_i \times m_i(\omega \times r_i)$$

The CM is located at the origin of the body frame, so $\sum_{m_i} m_i r_i = 0$. Thus

$$h = \sum_{m_i} r_i \times m_i(\omega \times r_i)$$
$$= \sum_{m_i} m_i \left(r_i^2 \omega - \omega \cdot r_i \, r_i \right)$$
$$= \left\{ \omega_x \sum_{m_i} (y_i^2 + z_i^2) m_i - \omega_y \sum_{m_i} x_i y_i m_i - \omega_z \sum_{m_i} x_i z_i m_i \right\} \hat{i}$$
$$+ \left\{ \omega_y \sum_{m_i} (x_i^2 + z_i^2) m_i - \omega_x \sum_{m_i} y_i x_i m_i - \omega_z \sum_{m_i} y_i z_i m_i \right\} \hat{j}$$
$$+ \left\{ \omega_z \sum_{m_i} (x_i^2 + y_i^2) m_i - \omega_x \sum_{m_i} z_i x_i m_i - \omega_z \sum_{m_i} z_i y_i m_i \right\} \hat{k}$$

We lump things together as follows:

$$h = h_x \hat{i} + h_y \hat{j} + h_z \hat{k}$$

$$h_x = \omega_x I_{xx} - \omega_y I_{xy} - \omega_z I_{xz}$$

$$h_y = \omega_y I_{yy} - \omega_x I_{yx} - \omega_z I_{yz}$$

$$h_z = \omega_z I_{zz} - \omega_x I_{zx} - \omega_y I_{zy}$$

$$I_{ab} = \sum_{m_i} \left(r_i^2 \delta_{ab} - r_i^a r_i^b \right) m_i$$

We construct the inertia tensor

$$I = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix}$$

so that

$$h = I\omega.$$

Now consider the more general case of this rigid body moving through space with a set of body axes where origin coincides with the CM. Let T be kinetic energy in an inertial frame. Then the velocity of a particle m_i is

$$v_i = v_0 + \omega \times r_i$$

$$\Rightarrow \quad v_i^2 = v_0^2 + 2 v_0 \cdot (\omega \times r_i) + (\omega \times r_i) \cdot (\omega \times r_i)$$

Summing over all the mass (in continuum representation)

$$T = \frac{1}{2} \int_M v^2 dm = \frac{1}{2} \int_M v_0^2 dm + \int v_0 \cdot (\omega \times r)\, dm + \frac{1}{2} \int (\omega \times r_i) \cdot (\omega \times r_i)\, dm$$

$$= \frac{1}{2} M v_0^2 + v_0 \cdot \omega \times \int_M r\, dm + \frac{1}{2} \int \|\omega \times r\|^2 dm$$

$$= T_{trans} + \frac{1}{2} \int \|\omega \times r\|^2 dm$$

where $\int r\, dm = 0$ because the CM is at the origin. Note

$$\|\omega \times r\| = \varepsilon_{abc} \varepsilon_{aij}\, \omega^b r^c \omega^i r^j$$

$$= \left(\delta_{bi} \delta_{cj} - \delta_{bj} \delta_{ci} \right) \omega^b r^c \omega^i r^j$$

$$= \omega^a \left(r^2 \delta_{ab} \right) \omega^b - \omega^a \left(r_a r_b \right) \omega^b$$

$$= \omega^a \left(r^2 \delta_{ab} - r_a r_b \right) \omega^b$$

and therefore

$$\frac{1}{2} \int \|\omega \times r\|^2 dm = \frac{1}{2} \omega^T I \omega$$

Thus the full kinetic energy is

$$T = \frac{1}{2} M v_0^2 + \frac{1}{2} \omega^T I \omega.$$

The moment of inertia of an object about an axis $\hat{\mathfrak{z}}$ going through the CM of the body is given by

$$I_{\mathfrak{z}} = a_x^2 I_{xx} + a_y^2 I_{yy} + a_z^2 I_{zz} - 2 a_x a_y I_{xy} - 2 a_x a_z I_{xz} - 2 a_y a_z I_{yz}$$

where

$$a_x = \hat{\xi}\cdot\hat{\imath}, \quad a_y = \hat{\xi}\cdot\hat{\jmath}, \quad a_z = \hat{\xi}\cdot\hat{k}.$$

The complexity of this formula owes largely to the off-diagonal elements of the inertia tensor. Therefore it's desirable to choose the body axes so that I is diagonal: these are called the principal axes. The eigenvalues of the inertia matrix are called the principal moments of inertia. The axes corresponding to the largest/smallest eigenvalues are called the major/minor axes.

With the body axes chosen to be the principal ones, the moment of inertia about another axis $\hat{\xi}$ through the CM is

$$I_\xi = a_x^2 I_x + a_y^2 I_y + a_z^2 I_z$$

where I_x, I_y, I_z are the principal moments of inertia. We also have Euler's moment equation for the angular momentum

$$M = \dot{h}_I = \dot{h}_B + \omega \times h$$

where $\dot{h}_I$ is the angular momentum measured in the inertial frame and $\dot{h}_B$ is in the body frame, which we take to be the principal axes. Then using $h = I\omega$,

$$M_x = I_x\dot{\omega}_x + \omega_y\omega_z\,(I_z - I_y)$$

$$M_y = I_y\dot{\omega}_y + \omega_x\omega_z\,(I_x - I_z)$$

$$M_z = I_z\dot{\omega}_z + \omega_x\omega_y\,(I_y - I_x).$$

Suppose the body is axisymmetric $(I_x = I_y)$ and that the rotation is about the z-axis with constant $\omega_z = n$. Note that this does not preclude $\dot{\omega}_x, \dot{\omega}_y \neq 0$. With no external torque

$$I_x\dot{\omega}_x + \omega_y n\,(I_z - I_y) = 0$$

$$I_x\dot{\omega}_y + \omega_x n\,(I_x - I_z) = 0$$

$$\dot{\omega}_z I_z = 0$$

If we define $\lambda = n\,(I_z - I_x)/I_x$ then the first two equations are

$$\dot{\omega}_x + \lambda\omega_y = 0$$

$$\dot{\omega}_y - \lambda\omega_x = 0.$$

Multiplying the first equation by ω_x and the second by ω_y, and adding the two, we have

$$\omega_x\dot{\omega}_x + \omega_y\dot{\omega}_y = 0 \leftrightarrow \frac{d}{dt}\left(\omega_x^2 + \omega_y^2\right) = 0$$

$$\rightarrow \quad \omega_x^2 + \omega_y^2 = \text{const.}$$

Because $\omega_z = n = \text{const.}$, we also have

$$\|\omega\|^2 = \omega_x^2 + \omega_y^2 + \omega_z^2 = \text{const.}$$

We can now solve for $\omega_x(t)$, $\omega_y(t)$ including the possibility of driving/disturbance. We have

$$\ddot{\omega}_x + \lambda\dot{\omega}_y = \ddot{\omega}_x + \lambda^2\omega_x = 0$$

$$\& \quad \ddot{\omega}_x - \lambda\dot{\omega}_x = \ddot{\omega}_y + \lambda\omega_y = 0$$

Then

$$\omega_x(s) = \frac{\dot{\omega}_x(0) + s\,\omega_x(0)}{s^2 + \lambda^2}$$

$$\omega_y(s) = \frac{\dot{\omega}_y(0) + s\,\omega_x(0)}{s^2 + \lambda^2}$$

which includes the initial conditions. We of course then have

$$\omega_x(t) = \omega_x(0)\cos(\lambda t) + \frac{\dot{\omega}_x(0)}{\lambda}\sin(\lambda t)$$

$$\omega_y(t) = \omega_x(0)\sin(\lambda t) - \frac{\dot{\omega}_x(0)}{\lambda}\cos(\lambda t)$$

Next suppose that the body is not necessarily axisymmetric and that $\omega_z = n + \varepsilon(t)$ for small $\varepsilon(t)$. Then

$$I_x\dot{\omega}_x + \omega_y n\,(I_z - I_y) = 0$$

$$I_y\dot{\omega}_y + \omega_x n\,(I_x - I_z) = 0$$

$$I_z\dot{\varepsilon} + \omega_x\omega_y\,(I_x - I_y) = 0$$

Then

$$\ddot{\omega}_x + n^2\left(\frac{I_z - I_y}{I_y}\right)\left(\frac{I_z - I_x}{I_x}\right)\omega_x = 0.$$

The solution is oscillatory (stable) for

$$(I_z - I_y)(I_z - I_x) > 0$$

and divergent otherwise. Therefore stability requires

$$I_z > I_x, I_y \quad \text{OR} \quad I_z < I_x, I_x.$$

If a body spins about its major or minor axis it is stable.

Since there are no external torques applied to the body, $M = \dot{h}_I = 0 \rightarrow$ the angular momentum vector is constant as measured in the inertial frame. Resolve the angular momentum and velocity into two components: one in the $x_B - y_B$ plane and another along the z_B-axis (assume once more that $I_x = I_y$).

Then

$$\omega = \omega_{xy} + \omega_z$$

$$h = I_x\omega_{xy} + I_z\omega_z.$$

Since ω and h have components in the same directions (ω_{xy} and ω_z), h, ω, ω_z are all coplanar. But, since h is constant, its direction is fixed in space. As we saw, the vector ω_{xy} rotates in the $x_B - y_B$ body plane, so the angular velocity ω must also rotate about h. Define the following angles

$$\tan\theta = \frac{h_{xy}}{h_z} = \frac{I_x \omega_{xy}}{I_z \omega_z}$$

$$\tan\gamma = \frac{\omega_{xy}}{\omega_z}.$$

θ is called the nutation angle. It follows that

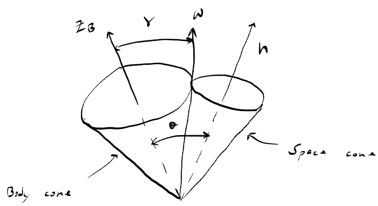
$$\tan\theta = \frac{I_x}{I_z}\tan\gamma.$$

From this we conclude that

$$\theta > \gamma \quad \text{if} \quad I_x > I_z$$

$$\theta < \gamma \quad \text{if} \quad I_x < I_z$$

We have



$I_x > I_z$

The body cone "rolls" on the space cone, which is fixed in space. The $z_B - \omega$ plane rotates about the h vector, which is also fixed in space. Whenever the body z_B axis deviates from the momentum vector h, the body is said to nutate. This nutation forces the spin axis to deviate from the nominal desired direction. Keeping the nutation angle small is one of the important tasks of attitude control systems.

The angular kinetic energy about any momentary axis $\hat{\xi}$ can be written

$$T_{rot} = \frac{1}{2} I_\xi \omega^2 = \frac{1}{2}\frac{h^2}{I_\xi}.$$

With no applied torque on the spinning body, the momentum h will remain constant. The value of I_ξ depends on the direction of the axis of rotation in the body axes frame. Since h is constant:

$$T_{max} = \frac{1}{2}\frac{h^2}{I_{min}} \quad \text{(minor axis)}$$

$$T_{min} = \frac{1}{2}\frac{h^2}{I_{max}} \quad \text{(major axis)}$$

If the body spins about the minor axis and there is some internal energy dissipation that tends to decrease the rotational kinetic energy to its minimum, then the body will transfer the spin of rotation to the major axis in order to satisfy the above max/min relations.

In the symmetric case $I_x = I_y$ with the body axes being the principal ones,

$$h^2 = (\omega_x^2 + \omega_y^2) I_x^2 + \omega_z^2 I_z^2$$

$$2T_{rot} = (\omega_x^2 + \omega_y^2) I_x + \omega_z^2 I_z$$

$$\Rightarrow \quad h^2 - 2T_{rot} I_x = \omega_z^2 I_z (I_z - I_x).$$

By definition $\cos\theta = h_z/h = I_z \omega_z / h$, and so

$$2T I_x = h^2 - h^2 \cos^2\theta \left(\frac{I_z - I_x}{I_z} \right).$$

Taking the time derivative

$$2\dot{T} I_x = 2h^2 \left(1 - \frac{I_x}{I_z}\right) \cos\theta \sin\theta \ \dot{\theta}$$

$$\Rightarrow \quad \dot{T} = \frac{h^2}{I_z} \left(\frac{I_z}{I_x} - 1 \right) \cos\theta \sin\theta \ \dot{\theta}.$$

This has interpretations:

(i) If $\dot{T} < 0$ and $I_z > I_x$ then $\dot{\theta} < 0$, which implies nutation stability

(ii) If $\dot{T} < 0$ and $I_z < I_x$ then $\dot{\theta} > 0$, which implies nutation instability

In the first scenario, an initial nutation angle will decrease to 0 under the influence of energy dissipation. In the second one, an initial nutation angle will increase until the spin is transferred to the major axis. Therefore, in the presence of energy dissipation, a spinning body is in stable angular motion only if the spin is about the major axis.

We now take up the task of attitude determination in space. For planet-orbiting spacecraft, it is most convenient to define the orbit reference frame as follows. The origin of this reference frame moves with the CM of the satellite. The z_R axis points toward the CM of the planet (R stands for reference). The x_R axis is in the plane of the orbit, perpendicular to z_R, in the direction of the velocity of the spacecraft. The y_R axis is normal to the local plane of the orbit and is orthogonal to the other 2. The angular velocity of this frame is ω_{RI} relative to the inertial axis frame of the planet. The satellite's body frame is still x_B, y_B, z_B. The satellite's attitude with respect to any reference frame is defined by a direction cosine matrix A, by its quaternion vector q, or by the Euler angles. We define the Euler angles as: ϕ, about the x_B axis; θ, about the y_B axis; ψ, about the z_B axis. The matrix A can be expressed in terms of (ϕ, θ, ψ).

Two important factors in satellite kinematics are (1) the angular velocities of the body axis frame wrt the reference axis frame and (2) the velocity of the body axis frame wrt the inertial axis frame. The angular velocity of the body frame relative to the reference frame is ω_{BR}, the angular velocity of the reference frame relative to the inertial frame is ω_{RI}. When ω_{RI} is expressed in the body frame, we denote it ω_{RIB}. The angular velocity of the body frame relative to the inertial frame is

$$\omega_{BI} = \omega_{BR} + \omega_{RIB}.$$

The vector ω_{BR} is important because it allows us to calculate the Euler angles of the moving body with respect to any defined reference frame in space. To begin with, suppose that the initial body axes were aligned with the reference axes X_R, Y_R, Z_R. When we choose the order of axes transformation as $\psi \to \theta \to \phi$, A_ψ is the first rotation about the Z_B body axis. The next rotation will be about the new $Y_{B'}$ axis by an angle θ, and so on. Finally

$$A_{\psi\theta\phi} = A_\phi A_\theta A_\psi$$
$$= \begin{pmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\theta\sin\phi\cos\psi - \cos\phi\sin\psi & \sin\theta\sin\phi\sin\psi - \cos\phi\cos\psi & \cos\theta\sin\phi \\ \sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi & \sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi & \cos\theta\cos\phi \end{pmatrix}$$

In the process of angular rotation, e.g. rotation about the Z_B axis, a derivative of the angle ψ, $d\psi/dt$, is sensed about the same axis. This derivative is subject to 3 successive angular transformations: ψ, θ, and finally one about $X_{B''}$ by the angle ϕ. The 2^{nd} transformation, about $Y_{B'}$ by the angle θ, produces the derivative $d\theta/dt$ which is subject to two angular transformations: first about $Y_{B''}$ by θ and then about $X_{B'''}$ by ϕ. The last rotation, about $X_{B'''}$, produces the derivative $d\phi/dt$. This derivative is subject only to one attitude transformation before final angular position of the body coordinate system is reached relative to the reference coordinate system. Writing $\omega_{BR} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$ we have

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \omega_{BR} = A_\phi A_\theta A_\psi \begin{pmatrix} 0 \\ 0 \\ \dot\psi \end{pmatrix} + A_\phi A_\theta \begin{pmatrix} 0 \\ \dot\theta \\ 0 \end{pmatrix} + A_\phi \begin{pmatrix} \dot\phi \\ 0 \\ 0 \end{pmatrix}$$

We get

$$p = \dot\phi - \dot\psi\sin\theta$$
$$q = \dot\theta\cos\phi + \dot\psi\cos\theta\sin\phi$$
$$r = \dot\psi\cos\theta\cos\phi - \dot\theta\sin\phi.$$

This can be inverted to give expressions for the derivatives in terms of the components of ω_{BR}:

$$\dot\phi = p + \left[q\sin\phi + r\cos\phi\right]\tan\theta$$
$$\dot\theta = q\cos\phi - r\sin\phi$$
$$\dot\psi = \left[q\sin\phi + r\cos\phi\right]\sec\theta.$$

Notice that these equations have a singularity at $\theta = \pi/2$. This is why, in certain engineering situations,