Kepler, etc.

Consider 2 body problem with (stationary) central mass M and (mobile) orbiting mass m. Let r be the distance between them and θ the angular position of the mass m. Then the EOM can be put in the form

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2},$$

where $\mu \equiv GM$ and h is the (constant, conserved) angular momentum of m. The coordinate $u = \frac{1}{r}$. The solution is

$$u = \frac{\mu}{h^2} + c\cos(\theta - \theta_0).$$

The integration constant c can be found from conservation of energy per unit mass

$$E = \frac{v^2}{2} - \frac{\mu}{r}.$$

We know that $\vec{v} = (\dot{x}, \dot{y})$ has squared length

$$v^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right].$$

Using $\frac{du}{d\theta} = -c\sin(\theta - \theta_0)$ we have

$$v^2 = \left[c^2\sin^2(\theta-\theta_0) + c^2\cos^2(\theta-\theta_0) + 2\frac{\mu}{h^2}c\cos(\theta-\theta_0)\right]h^2$$
$$= h^2\left[c^2 + \frac{2\mu}{h^2}c\cos(\theta-\theta_0)\right]$$

and

$$E = \frac{1}{2}v^2 - \mu u$$
$$= \frac{1}{2}h^2c^2 - \frac{1}{2}\frac{\mu^2}{h^2}$$
$$\Rightarrow c = \frac{\mu}{h^2}\sqrt{1 + 2E\frac{h^2}{\mu^2}}.$$

Define the eccentricity

$$e = \sqrt{1 + 2Eh^2/\mu^2}$$

so that

$$\boxed{E = (e^2-1)\frac{\mu^2}{2h^2}}.$$

The relationship between r and θ is found to be

$$r = \frac{h^2/\mu}{1 + e\cos(\theta-\theta_0)} = \frac{p}{1+e\cos(\theta-\theta_0)}.$$

The constant p is called the semi-latus rectum or the parameter.

The solutions to the EOM are conic sections which are distinguished based on the value of e. The possibilities are

(i) Circular orbit: $e = 0 \Rightarrow r = p$, $v^2 = \mu/r$, $E = -\mu^2/2h^2 < 0$

(ii) Elliptic orbit: $0 < e < 1 \Rightarrow E = (e^2-1)\mu^2/2h^2 < 0$. The point on the ellipse at $\theta = 0$ is called the periapsis, and the radius vector from the prime focus F of the ellipse to that point is minimal. This distance is $r_p = \frac{p}{1+e}$. For orbits around Earth, $\theta = 0$ is called the perigee; for orbits around the sun it is the perihelion. The point at $\theta = \pi$ is called the apoapsis (apogee, apohelion) which has maximal distance from F with distance $r_a = \frac{p}{1-e}$. Then

$$\frac{r_a}{r_p} = \frac{1-e}{1+e} \Rightarrow e = \frac{r_a - r_p}{r_a + r_p}.$$

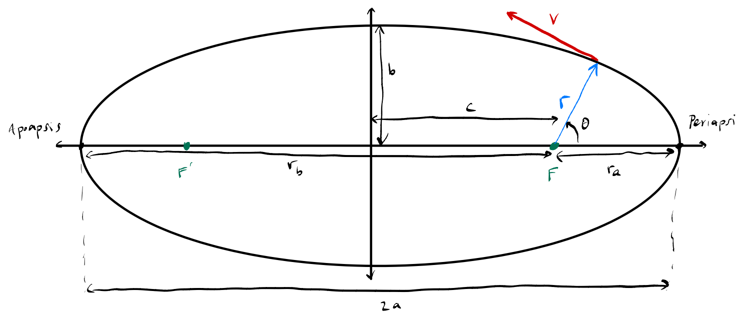
The length of the major axis of the ellipse is

$$2a = r_a + r_p = \frac{2p}{1-e^2} \Rightarrow p = a(1-e^2) = h^2/\mu.$$

The length a is called the semi-major axis. The total energy per unit mass is

$$E = -\frac{\mu}{2a}$$

which is called the energy constant.



The distance from F to the origin, c, is given by $c = ae$. The semi-minor axis b is $b = a\sqrt{1-e^2}$.

(iii) Parabolic orbits: These are not practically useful but have $E = 0$ and are on the cusp between bound and scattering states. They have $e = 1$, so

$$r = \frac{p}{1+\cos\theta}, \quad r_p = \frac{p}{2}$$

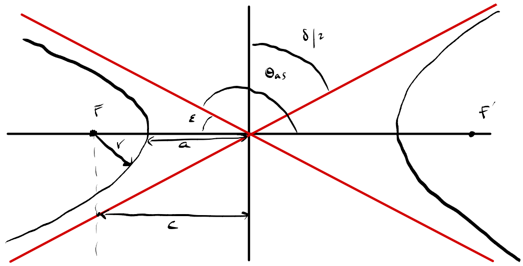
This also means $a \to \infty$ and $v^2 = 2\mu/r$.

(iv) Hyperbolic orbits: The energy is $E > 0$ so these are scattering states. They have $e > 1$ and $a = -\mu/2E < 0$. To make sense of this we can define $p = a(e^2-1)$ so that

$$r = \frac{a(e^2-1)}{1+e\cos\theta}$$

As $r \to \infty$ we want the denominator $\to 0$, giving the asymptote condition

$$\cos(\theta_\infty) = \cos(\theta_{as}) = -1/e$$



From the geometry we have

$$\varepsilon = \pi - \theta_{as} = \frac{\pi}{2} - \frac{\delta}{2}$$

$$\Rightarrow \quad \theta_{as} = \frac{\pi}{2} + \frac{\delta}{2}$$

$$\Rightarrow \quad \cos\left(\frac{\pi}{2} + \frac{\delta}{2}\right) = -\frac{1}{e}$$

$$\Rightarrow \quad \sin\left(\frac{\delta}{2}\right) = \frac{1}{e}$$

These orbits are useful for transplanetary spacecraft voyages. Consider the behavior as $r \to \infty$. The angular momentum can be written $h = V_\infty \Delta$. The total energy is

$$E = \frac{V_\infty^2}{2} - \frac{\mu}{r_\infty} = -\frac{\mu}{2a}$$

$$\Rightarrow \quad a = -\frac{\mu}{V_\infty^2} \quad \text{as } r_\infty \to \infty.$$

But $\quad p = a(e^2 - 1)$

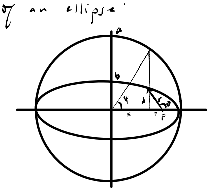
$$\Rightarrow \quad p = \frac{\mu}{V_\infty^2}(e^2 - 1) = \frac{h^2}{\mu} = \frac{V_\infty^2 \Delta^2}{\mu}$$

$$\Rightarrow \quad V_\infty \Delta = \frac{\mu}{V_\infty}\sqrt{e^2 - 1}$$

$$\& \quad e^2 = 1 + \frac{V_\infty^4 \Delta^2}{\mu^2}.$$

If Δ and V_∞ are known, we can find a, e, δ. As a spacecraft navigates toward a far planet with V_∞, we can use the direction of the motion to determine h (and therefore Δ). This allows us to determine the parameters of the orbit the craft will follow at the distant planet.

We can use Kepler's laws to locate a body in orbit either from its angular deviation from the major axis or from the time elapsed since passing the perigee. We use the following diagram to define the true and eccentric anomalies of an ellipse:



The true anomaly θ is defined as the angle between the major axis and the vector locating the body with respect to the primary focus F. To define the eccentric anomaly, draw an auxiliary circle of radius a (semi-major axis) concentric with the ellipse. The eccentric anomaly is the angle ψ obtained by extending a line perpendicular to the major axis through the location of the body and measuring the angular position of the line's intersection with the circle. Then

$$x + y = ae = c$$

$$x = a\cos\psi$$

$$y = -r\cos\theta$$

$$\Rightarrow a\cos\psi - r\cos\theta = ae$$

Using the equation of the ellipse:

$$a\cos\psi = ae + \frac{a(1-e^2)\cos\theta}{1+e\cos\theta} = \frac{ae + a\cos\theta}{1+e\cos\theta}$$

$$\Rightarrow \cos\psi = \frac{e+\cos\theta}{1+e\cos\theta}$$

$$\Rightarrow \sin\psi = \frac{\sin\theta\sqrt{1-e^2}}{1+e\cos\theta}$$

or inverting

$$\cos\theta = \frac{\cos\psi - e}{1-e\cos\psi}, \quad \sin\theta = \frac{\sin\psi\sqrt{1-e^2}}{1-e\cos\psi}.$$

Notice that we can also write

$$r = a(1-e\cos\psi).$$

Kepler's area law says

$$\frac{dA}{dt} = \frac{h}{2}$$

where A is the area swept out in time t. Therefore $A = \frac{1}{2}ht$. Because the area of an ellipse is πab, if the period of the orbit is T then

$$T = \frac{2A}{h} = 2\pi\sqrt{\frac{a^3}{\mu}} = \frac{2\pi}{n}$$

This is Kepler's 3rd law. We have defined the mean motion n. Define the mean anomaly $M = n(t-t_p)$, where t_p is a time where the body is at the perigee. We can compare this to the true anomaly θ to find t_p.

Knowing θ allows us to find ψ. Knowing ψ, the time elapsed from the perigee can be computed. Defining $t_M = t - t_p$ we find that

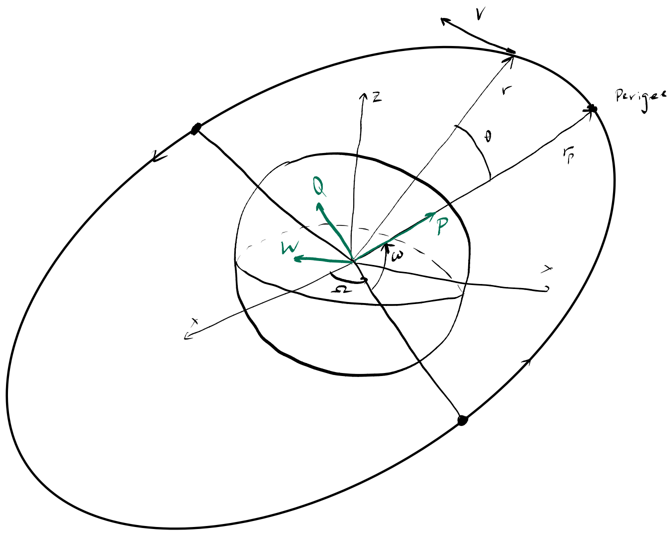
$$t_M \frac{2\pi}{T} = \psi - e\sin\psi$$

For earth-orbiting spacecraft, it is common to define an inertial coordinate system with the origin at the CoM of the earth and whose direction ~~in~~ space is fixed relative to the solar system. The earth moves in a nearly-circular orbit around the sun with a long period, so its motion is practically unaccelerated for our purposes.

The z-axis is the axis of rotation of the earth. The xy-plane is the equatorial plane. The axis of rotation is (famously) inclined with respect to the ecliptic plane (around the sun). The equatorial and ecliptic planes intersect along a line which is quasi-inertial in space, called the vernal equinox vector (or direction). We choose this to be the x-axis; the y-axis is chosen to complete the right-handed orthogonal system.

Both the equatorial and ecliptic planes move slowly with respect to the true celestial inertial coordinate system, centered in the CoM of the solar system. The planets affect the orientation of the ecliptic plane in the slow rotational motion of planetary precession. As the z-axis precesses, so does the equatorial plane. Thus the geocentric coordinate system moves slowly relative to the stars, and it is necessary to define the system with reference to a certain date.

There are now 3 more parameters necessary to describe an orbit in space. An attempt at a diagram is made:



The plane of the orbit is inclined to the xy-plane by an angle i, called the inclination of the orbit. The two planes intersect along the node line. The angle in the equatorial plane between the x-axis and the node line is called the right ascension, Ω. In the orbit plane, r is the radius vector to the moving body and r_p is the vector to the perigee. The angle between the node line and r_p is ω, the argument of perigee. These 3 parameters, together with the parameters (a, e, M) of the orbit within its own plane complete a system of 6 parameters that suffices to determine the location in space of a body moving in any Keplerian orbit. These parameters are known as the classical orbit parameters. We summarize them:

(i) a, the semi-major axis

(ii) e, the eccentricity

(iii) i, the inclination

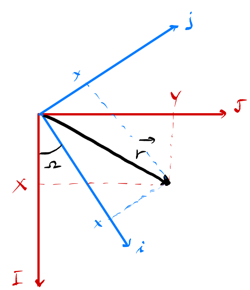
(iv) Ω, the right ascension of the ascending node

(v) ω, the argument of perigee

(vi) $M = n(t - t_0)$, the mean anomaly

Define the vector $\alpha = (a, e, i, \Omega, \omega, M)^T$. Although these parameters completely define an orbit in space, some of them (e.g. Ω) are poorly defined when i is small, as with geostationary orbits. In such cases a variation of the six listed parameters will be preferred. Orbits with very small inclination are called equatorial orbits.

It will be useful to know how to transform between coordinate systems which are related by rotations. Consider:



We have two sets of orthonormal vectors $\{\hat{i}, \hat{j}\}$ and $\{\hat{I}, \hat{J}\}$ with respect to which we may decompose a vector $\vec{r}$:

$$\vec{r} = x\hat{i} + y\hat{j} = X\hat{I} + Y\hat{J}.$$

Notice that

$$\hat{I}\cdot\hat{i} = \cos\Omega, \qquad \hat{J}\cdot\hat{i} = \sin\Omega,$$
$$\hat{I}\cdot\hat{j} = -\sin\Omega, \qquad \hat{J}\cdot\hat{j} = \cos\Omega.$$

So,

$$X\,\hat{I}\cdot\hat{i} + Y\hat{J}\cdot\hat{i} = x\,\hat{i}\cdot\hat{i} + y\,\hat{j}\cdot\hat{i}$$
$$= x$$

$$\Rightarrow \quad x = \cos\Omega\, X + \sin\Omega\, Y$$

and

$$X\,\hat{I}\cdot\hat{j} + Y\hat{J}\cdot\hat{j} = x\,\hat{i}\cdot\hat{j} + y\,\hat{j}\cdot\hat{j}$$
$$= y$$

$$\Rightarrow \quad y = -\sin\Omega\, X + \cos\Omega\, Y$$

This gives the rotation transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\Omega & \sin\Omega \\ -\sin\Omega & \cos\Omega \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.$$

Secretly we may view this as a rotation about a third unit vector $\hat{K}$ (which would coincide for the two coordinate systems. Defining z, Z as the respective coordinates we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

Write a rotation about the n^{th} axis $R_n(\Omega)$. It suffices to string together rotations about the x- and z-axes to obtain an arbitrary rotation in $SO(3)$. The transformation between the inertial coordinates $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ and the orbit coordinates $\begin{pmatrix} P \\ Q \\ W \end{pmatrix}$, where

- P is directed from the center of the orbit to the perigee
- W is directed along the momentum axis of the orbit, $h = r \times v$
- Q is $\perp$ to both others

is

$$\begin{pmatrix} P \\ Q \\ W \end{pmatrix} = R_z(\omega)\, R_x(i)\, R_z(\Omega) \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

We can define a local coordinate system (polar coordinates for the orbit) as

$$\begin{pmatrix} r \\ s \\ w \end{pmatrix} = R_z(\omega+\theta)\, R_x(i)\, R_z(\Omega) \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.$$

We want to construct the transformation from the 6 classical orbit parameters to the inertial coordinate system X, Y, Z. Since a Keplerian orbit lies in a plane, we can define a coordinate system x, y in a plane with $z = 0$

$$x = a\cos\psi - c = a(\cos\psi - e)$$
$$y = b\sin\psi = a\sqrt{1-e^2}\,\sin\psi$$

where ψ is the angle introduced with the auxiliary circle. Then the position is $r = x\hat{\imath} + y\hat{\jmath}$ for some $\{\hat{\imath}, \hat{\jmath}\}$. The value of ψ is determined by $\psi - e\sin\psi = M = nt$, which can be solved for perturbatively in e. Set $\hat{\imath} = P$ and $\hat{\jmath} = Q$. Then

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = R_z^{-1}(\Omega)\, R_x^{-1}(i)\, R_z^{-1}(\omega) \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

The velocity vector is

$$v = \frac{dr}{dt} = \frac{dr}{d\psi}\frac{d\psi}{dt}$$

where

$$\frac{d\psi}{dt}\left(1 - e\cos\psi\right) = \frac{dM}{dt} = n$$

$$\Rightarrow \frac{d\psi}{dt} = \frac{an}{r}.$$

So

$$\begin{aligned} \frac{dr}{dt} &= -a\sin\psi \frac{d\psi}{dt}\hat{P} + a\sqrt{1-e^2}\cos\psi \frac{d\psi}{dt}\hat{Q} \\ &= \frac{a^2 n}{r}\left(\sin\psi\,\hat{P} + \sqrt{1-e^2}\cos\psi\,\hat{Q}\right) \\ &\equiv v_P \hat{P} + v_Q \hat{Q}. \end{aligned}$$

We can apply the inverse transformation to get the velocity in the inertial system as well. If we knew r, v in the inertial coordinates, it is straightforward to find the 6 orbit parameters.

So far we have only considered ideal Keplerian orbits which derived from an assumption of only two bodies experiencing mutual gravitation. This ideal situation does not exist in practice. For example, there are 8 other planets influencing any planet's orbit in addition to the sun. For an earth-orbiting satellite it turns out that the sun and moon must both be taken into account. These are examples of conservative perturbing forces, but there are non-conservative ones as well. Satelites experience solar pressure and atmospheric drag.

Write N2L for the ideal Kepler situation as

$$\frac{d^2\vec{r}}{dt^2} = -\mu\frac{\vec{r}}{r^3} \equiv \vec{\gamma}_K$$

with initial conditions $\vec{r}(0), \vec{v}(0)$. For Keplerian orbits, the 6 orbit parameters are constant:

$$\frac{da}{dt} = \frac{de}{dt} = \frac{d\Omega}{dt} = \frac{d\omega}{dt} = \frac{di}{dt} = 0, \qquad \frac{dM}{dt} = n.$$

We represent the perturbed EOM by

$$\frac{d^2\vec{r}}{dt^2} = \vec{\gamma}_K + \vec{\gamma}_P$$

$$\vec{r}(t_0) = \vec{r}_0, \quad \vec{v}(t_0) = \vec{v}_0.$$

The magnitude of $\vec{\gamma}_p$ should be appreciably smaller than $\vec{\gamma}_K$ for it to be a perturbation. The full solution to the EOM is called the true orbit. If we were to remove the perturbation at some time t_0 and find the subsequent Keplerian solution, that would be called an osculating orbit.

In general the perturbing force will depend on $\vec{r}, \vec{v}, t$ (e.g. the moon's impact on an earth-orbitting satellite depends on the satellites position relative to the moon). The EOM are

$$\frac{d\vec{v}}{dt} = \vec{F}(\vec{r}, \vec{v}, t), \quad \frac{d\vec{r}}{dt} = \vec{v}.$$

We can think of $\vec{r}$ and $\vec{v}$ as depending on the 6 α_i as well as t. Therefore we have the 6 implicit EOM

$$\frac{\partial \vec{v}}{\partial \alpha_i} \frac{d\alpha_i}{dt} = \vec{F}(\alpha, t)$$
$$\frac{\partial \vec{r}}{\partial \alpha_i} \frac{d\alpha_i}{dt} = \vec{v}(\alpha, t)$$

which can be inverted to give EOM for each α_i

$$\frac{da}{dt} = F_a(\alpha, t)$$
$$\frac{de}{dt} = F_e(\alpha, t)$$
$$\frac{d\Omega}{dt} = F_\Omega(\alpha, t)$$
$$\frac{d\omega}{dt} = F_\omega(\alpha, t)$$
$$\frac{di}{dt} = F_i(\alpha, t)$$
$$\frac{dM}{dt} = F_M(\alpha, t)$$

Now it remains to find expressions for the RHS of these equations. Decompose the perturbing force $\vec{\gamma}_p$ along the axes of a moving Cartesian frame defined so that $\hat{R}$ points along the position vector $\vec{r}$; $\hat{S}$ is in the local plane of the osculating orbit ($\perp$ to $\hat{R}$); $\hat{W}$ is $\perp$ to both $\hat{R}$ and $\hat{S}$. Then

$$\vec{\gamma}_p = R\hat{R} + S\hat{S} + W\hat{W}.$$

Start with the equation for $\frac{da}{dt}$. Recall that

$$E = \int \vec{F} \cdot d\vec{r} = \int \vec{F} \cdot \vec{v}\, dt.$$

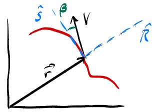
Then since $E = -\frac{\mu}{2a}$, we find that

$$\frac{dE}{dt} = \frac{\mu}{2a^2} \frac{da}{dt} = \vec{v} \cdot (\vec{\gamma}_p + \vec{\gamma}_K).$$

However in a Keplerian orbit $\vec{v} \cdot \vec{\gamma}_K = 0$, so

$$\frac{\mu}{2a^2} \frac{da}{dt} = \vec{v} \cdot \vec{\gamma}_p.$$

The velocity vector is in the plane of the osculating orbit and therefore has an expansion in terms of $\hat{R}$ and $\hat{S}$. If $\hat{v} \cdot \hat{S} = \cos\beta$, then



$$\vec{V} = v\sin\beta\,\hat{R} + v\cos\beta\,\hat{S}$$

$$\Rightarrow \quad \frac{dE}{dt} = \frac{\mu}{2a^2}\frac{da}{dt} = Rv\sin\beta + Sv\cos\beta$$

The radial component of the velocity is $\vec{V}\cdot\hat{R} = v\sin\beta = \frac{dr}{dt}$. But

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\frac{h}{r^2}$$

Using $h = \sqrt{p\mu}$ and $r = \frac{p}{1+e\cos\theta}$, this means

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{\sqrt{p\mu}}{r^2} = \sqrt{\frac{\mu}{p}}\, e\sin\theta$$

$$\Rightarrow \sin\beta = \frac{e\sin\theta}{v}\sqrt{\frac{\mu}{p}}$$

Using $\frac{v^2}{2} = \mu\left(\frac{1}{r} - \frac{1}{2a}\right)$ to eliminate $\frac{\sqrt{\mu}}{v}$ it follows that

$$\sin\beta = \frac{e\sin\theta}{\sqrt{1+2e\cos\theta+e^2}}, \qquad \cos\beta = \frac{1+e\cos\theta}{\sqrt{1+2e\cos\theta+e^2}}.$$

Thus

$$\frac{dE}{dt} = Rv\sin\beta + Sv\cos\beta$$
$$= \sqrt{\frac{\mu}{p}}\left(e\sin\theta\, R + (1+e\cos\theta)\, S\right)$$

and

$$\frac{da}{dt} = \frac{2a^2}{\mu}\frac{dE}{dt}$$
$$= \frac{2a^2}{\sqrt{\mu p}}\left(e\sin\theta\, R + (1+e\cos\theta)\, S\right)$$

Recall that

$$p = a(1-e^2)$$

which means

$$\dot{a} = \frac{2a^2}{\sqrt{\mu a(1-e^2)}}\left(e\sin\theta\, R + (1+e\cos\theta) S\right)$$
$$= \frac{2}{\sqrt{1-e^2}}\sqrt{\frac{a^3}{\mu}}\left(e\sin\theta\, R + (1+e\cos\theta) S\right)$$
$$= \frac{2}{n\sqrt{1-e^2}}\left(e\sin\theta\, R + (1+e\cos\theta)\, S\right)$$

This is the Gauss equation for the time evolution of a. There are analogous ones for the other 5 parameters:

$$\dot{e} = \frac{\sqrt{1-e^2}}{na}\left(\sin\theta\, R + (\cos\psi + \cos\theta)\, S\right)$$

$$\dot{i} = \frac{r}{na^2\sqrt{1-e^2}}\cos(\theta+\omega)\, W$$

$$\dot{\Omega} = \frac{r}{na^2\sqrt{1-e^2}}\frac{\sin(\theta+\omega)}{\sin i}\, W$$

$$\dot{\omega} = \frac{\sqrt{1-e^2}}{nae}\left[-R\cos\theta + \left(1 + \frac{1}{1+e\cos\theta}\right)\sin\theta\, S - \dot{\Omega}\cos i\right]$$

$$\dot{M} = n + \frac{1-e^2}{nae}\left[\left(\cos\theta - \frac{2e}{1+e\cos\theta}\right)R - \left(1 + \frac{1}{1+e\cos\theta}\right)\sin\theta\, S\right]$$

The Gauss equations allow us to update global properties of the orbit captured by the orbit parameters as the system evolves in time under the perturbation.

Ex: For low-orbit satellites, the air density is high enough to produce a perturbing force, which in turn tends to decrease orbit altitude. Suppose the orbit is circular and the air density ρ is constant for the entire orbit period. The perturbing force F_p due to drag is given by Stokes' Law

$$F_p = \frac{1}{2} \rho v^2 C_d S,$$

where v is the velocity, C_d is the drag coefficient, and S is the equivalent satellite surface in the direction of motion. The perturbing acceleration is $\vec{\gamma}_p = \vec{F}_p / m_s$, where m_s is the satellite mass. Then

$$\frac{\vec{v}}{|\vec{v}|} = -\frac{\vec{\gamma}_p}{|\vec{\gamma}_p|} \qquad \& \qquad \vec{v} \cdot \vec{\gamma}_p = -\frac{\rho v^3 C_d S}{2 m_s}.$$

The term $C_d S / 2m_s$ is known as the ballistic coefficient. The Gauss equation is

$$\dot{a} = -\rho \sqrt{a\mu} \, \frac{C_d S}{m_s}.$$

The rate of decrease of the semi-major axis is proportional to the air density and the geometric properties of the satellite, but inversely proportional to the ~~satellite~~ mass. To maintain a constant altitude, the satellite must provide a force to balance the perturbing force $\vec{F}_p$, with an inevitable fuel consumption.