a special order of rotations might be preferred. In some gyroscopic inertial systems this singularity might cause the phenomenon of gimbal lock.

Measurements of the body axes angular rates p, q, r relative to the reference frame, together with knowledge of the initial conditions of the Euler angles, allow us to integrate the equations for $\dot{\psi}, \dot{\theta}, \dot{\phi}$. There are also analogous equations corresponding to other orders of rotation.

Knowledge of the direction cosine matrix elements is equivalent to knowing the attitude of the spacecraft relative to the reference frame in which the transformation matrix A is defined. In general, for a rotating body, the elements of this matrix change with time. The time derivative is

$$\frac{d}{dt} A = \Omega A$$

$$\text{w/} \quad \Omega = \begin{pmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{pmatrix}$$

Here $\omega_x, \omega_y, \omega_z$ are angular velocities about the body coordinate axes. These equations are used when the angular velocity of the body can be measured (by inertial measurement instrumentation) to find the evolving direction cosine matrix. Numerical integration of $\dot{A}$ requires knowledge of $A(0)$. However, integration of this equation is time-consuming and hence seldomly carried out. Instead, the quaternion representation is more efficient.

The quaternion's basic definition is a consequence of the direction cosine matrix A. A proper real orthogonal 3×3 matrix has at least one eigenvector with eigenvalue of unity (that is, the axis of rotation is fixed). Therefore such an eigenvector, call it e_1 satisfying $Ae_1 = e_1$, has the same components along the body axes and along the reference frame axes. Any attitude transformation in space by consecutive rotations about the 3 orthogonal unit vectors of the coordinate system can be achieved by a single rotation about the eigenvector with eigenvalue $+1$. The quaternion is defined as a vector in the following way

$$q = q_4 + q_1 i + q_2 j + q_3 k \equiv q_4 + \mathbf{q}$$

where

$$i^2 = j^2 = k^2 = -1$$

$$ij = -ji = k$$

$$jk = -kj = i$$

$$ki = -ik = j$$

Note that i, j, k have a representation in terms of the Pauli matrices. The quaternion conjugate is

$$q^* = q_4 - q_1 i - q_2 j - q_3 k.$$

The vector part of q is $q = q_1 i + q_2 j + q_3 k$.

Any rotation matrix about one of the basis vectors by an angle α satisfies

$$\operatorname{tr} A_\alpha = 1 + 2\cos\alpha.$$

This turns out to be true for a rotation about any axis by an angle α. Call e the eigenvector of rotation with components e_1, e_2, e_3. In this case, the direction cosine matrix can be written

$$A_\alpha = \cos\alpha \, \mathbb{1}_3 + (1-\cos\alpha) e e^T - \sin\alpha \, E$$

Where

$$E = \begin{pmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{pmatrix}.$$

In full,

$$A = \begin{pmatrix} \cos\alpha + e_1^2(1-\cos\alpha) & e_1 e_2 (1-\cos\alpha) + e_3 \sin\alpha & e_1 e_3 (1-\cos\alpha) - e_2 \sin\alpha \\ e_1 e_2 (1-\cos\alpha) - e_3 \sin\alpha & \cos\alpha + e_2^2 (1-\cos\alpha) & e_2 e_3 (1-\cos\alpha) + e_1 \sin\alpha \\ e_1 e_3 (1-\cos\alpha) + e_2 \sin\alpha & e_2 e_3 (1-\cos\alpha) - e_1 \sin\alpha & \cos\alpha + e_3^2 (1-\cos\alpha) \end{pmatrix}$$

Let a_{ij} be the elements of A. Then the eigenvector of rotation has elements

$$e_1 = \frac{a_{23} - a_{32}}{2\sin\alpha}$$

$$e_2 = \frac{a_{31} - a_{13}}{2\sin\alpha}$$

$$e_3 = \frac{a_{12} - a_{21}}{2\sin\alpha}$$

Define the quaternion components, sometimes called the Euler symmetric parameters, are ~~expressed~~ in terms of e as

$$q_1 = e_1 \sin(\alpha/2)$$

$$q_2 = e_2 \sin(\alpha/2)$$

$$q_3 = e_3 \sin(\alpha/2)$$

$$q_4 = \cos(\alpha/2).$$

Clearly

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1 \Rightarrow |q|^2 = 1.$$

The direction cosine matrix is then

$$A(q) = (q_4^2 - q^2)\, I + 2 q q^T - 2 q_4 Q$$

w/ $$Q = \begin{pmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{pmatrix}$$

which gives

$$A = \begin{pmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1 q_2 + q_3 q_4) & 2(q_1 q_3 - q_2 q_4) \\ 2(q_1 q_2 - q_3 q_4) & -q_1^2 + q_2^2 + q_3^2 - q_4^2 & 2(q_2 q_3 + q_1 q_4) \\ 2(q_1 q_3 + q_2 q_4) & 2(q_2 q_3 - q_1 q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{pmatrix}$$

As with the direction cosine matrix, the quaternion has a simple time derivative

$$\frac{d}{dt} q = \frac{1}{2} \Omega' q$$

where

$$\Omega = \begin{pmatrix} 0 & \omega_z & -\omega_y & \omega_x \\ -\omega_z & 0 & \omega_x & \omega_y \\ \omega_y & -\omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{pmatrix}$$

Integrating this requires knowing $q(0)$.

In deriving the attitude dynamics of a spacecraft, it is important to know the inertial velocity vector ω_{BI} of the satellite, which is evaluated by inertial ~~measurement~~ instrumentation such as precision rate integrating gyros. In order to find the evolution of the Euler angles of the satellite in the orbit reference frame, we must know ω_{BR} from $\omega_{BR} = \omega_{BI} - \omega_{RIB}$. Define the basis i, j, k of the orbit reference frame in terms of the position vector r and the velocity v of the orbit as

$$k = -r/|r|$$

$$j = v \times r / |v \times r|$$

$$i = j \times k.$$

For a positive clockwise rotation δ about j we have

$$\frac{di}{dt} = \frac{di}{d\delta}\frac{d\delta}{dt} = -k\,\omega_j$$

since $\frac{di}{d\delta} = -k$. Therefore

$$\omega_j = -\frac{di}{dt} \cdot k.$$

Repeating this for the other 2,

$$\omega_i = \frac{dj}{dt} \cdot k = -\frac{dk}{dt} \cdot j$$

$$\omega_j = -\frac{di}{dt} \cdot k = \frac{dk}{dt} \cdot i$$

$$\omega_k = \frac{di}{dt} \cdot j = -\frac{dj}{dt} \cdot i.$$

Written in terms of r, v we have

$$\omega_i = -\frac{dk}{dt}\cdot j = \frac{1}{|r|}\frac{dr}{dt}\cdot\frac{v\times r}{|v\times r|}$$
$$= \frac{1}{|r||r\times v|}\, v\cdot(v\times r)$$
$$= \frac{1}{|r||r\times v|}\,(v\times v)\cdot r$$
$$= 0$$

Similarly,

$$\omega_j = \frac{dk}{dt}\cdot i = -\frac{1}{r^2|v\times r|}\, v\cdot\left[(r\cdot v)r - r^2 v\right]$$

For a circular orbit where $v\cdot r = 0 \Rightarrow v\times r = vr$,

$$\omega_j = \frac{v}{r} = \omega_0$$

where ω_0 is the angular orbital velocity of the spacecraft. For ω_k we have

$$\omega_k = -\frac{dj}{dt}\cdot i$$
$$= \frac{1}{|v\times r|^2|r|}\left(\ddot{r}\times r\right)\cdot\left[(v\times r)\times r\right]$$

For Keplerian orbits, there are no out-of-plane accelerations and so $\omega_k = 0$. Finally, for a circular orbit

$$\omega_{RI} = (0, -j\omega_0, 0)^T$$

The attitude dynamics equations will be obtained from Euler's moment equations. Previously, the body was assumed to be rigid with no moving elements inside it. However, here we will allow for the existence of rotating elements inside the satellite — known as momentum exchange devices — and for other kinds of gyroscopic devices. The most common momentum exchange devices are the reaction wheel, momentum wheel, and the control moment gyro. Let τ be the total external torque acting on the body, which is equal to inertial momentum change of the system. The moment equation is

$$\tau = \dot{h}_I = \dot{h} + \omega\times h\,.$$

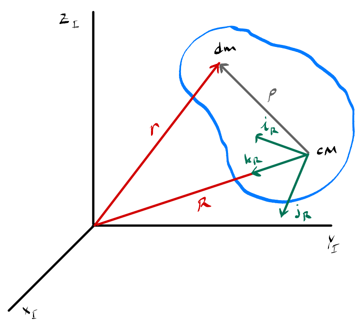
Break down the external torque τ into two principal parts: τ_c the control moments to be used for controlling the attitude of the satellite; and τ_d, the moments due to different disturbing environmental phenomena. The angular momentum of the entire system can be divided between the momentum of the rigid body $h_B = (h_x, h_y, h_z)^T$ and the momentum of the moment exchange devices $h_w = (h_{wx}, h_{wy}, h_{wz})^T$. Then $h = h_B + h_w$. The EOM are

$$\tau_x = \dot{h}_x + \dot{h}_{wx} + (\omega_y h_z - \omega_z h_y) + (\omega_y h_{wz} - \omega_z h_{wy})$$
$$\tau_y = \dot{h}_y + \dot{h}_{wy} + (\omega_z h_x - \omega_x h_z) + (\omega_z h_{wx} - \omega_x h_{wz})$$
$$\tau_z = \dot{h}_z + \dot{h}_{wz} + (\omega_x h_y - \omega_y h_x) + (\omega_x h_{wy} - \omega_y h_{wx})$$

with respect to the unit vectors of the body axis frame. These equations summarize the full attitude dynamics that must be implemented in the complete 6-DOF simulation necessary for analyzing the attitude control system.

In the first phase of the design stage, it is important to transform the moment equations into a more easily treatable form. If the design problem at hand allows working with principal axes, then the off-diagonal components of I may be eliminated, simplifying the EOM considerably. Moreover, the angular motion can be approximated by infinitesimal angular motion, which means small Euler angles and derivatives. With these assumptions, the dynamics equations can be Laplace transformed, thus gaining the important advantage of using linear control theory.

Before we can write the linearized attitude dynamics EOM, we must state and analyze one important external moment, the gravitational moment. This moment is inherent in low-orbit satellites, and cannot be neglected when dealing with passively attitude-controlled satellites. An asymmetric body subject to a gravitational field will experience a torque tending to align the axis of least inertia with the field direction. Suppose that the moving satellite is at a distance R_0 from the CM of the earth. Recall the orbit reference axis frame defined earlier:



The origin of the reference frame is at the CM of the body. The attracting gravity force is along the $\hat{k}_R$ axis; ρ is the distance between the CM and any mass element dm in the body.

We can find the components of $R = -R_0 \hat{k}_R$ in the body axis by using any one of the Euler angle transformations, for instance the transformation $A_{\psi\theta\phi}$. The components of the vector R in the body axes will be labeled R_x, R_y, R_z and are given by

$$\begin{pmatrix} R_x \\ R_y \\ R_z \end{pmatrix} = A_{\psi\theta\phi} \begin{pmatrix} 0 \\ 0 \\ -R_0 \end{pmatrix}$$

It follows that

$$R_x = R_0 \sin\theta = a_{13}(-R_0),$$

$$R_y = -R_0 \sin\phi \cos\theta = a_{23}(-R_0),$$

$$R_z = -R_0 \cos\phi \cos\theta = a_{33}(-R_0).$$

Define the gravity gradient vector $G = (G_x, G_y, G_z)^T$. The force exerted on a mass element due to gravity is

$$dF = -\frac{\mu\, dm}{|r|^3} r$$

where $r = R + \rho$. Since $\rho \ll R_0$ for realistic scenarios, the moment about the CM of the body becomes

$$dG = \rho \times dF = -\frac{\mu\, dm}{|r|^3} \rho \times r.$$

Note that

$$\frac{1}{|r|^3} \approx \frac{1}{R_0^3}\left[1 - \frac{3\, R\cdot\rho}{R_0^2}\right].$$

Integrating over the entire mass gives

$$G = \frac{3\mu}{R_0^3}\int_M dm\ (R\cdot\rho)(\rho\times R)$$

which can be written

$$G_x = \frac{3\mu}{2R_0^3}(I_z - I_y)\sin 2\phi \cos^2\theta = \frac{3\mu}{R_0^3}(I_z - I_y)\, a_{23} a_{33}$$

$$G_y = \frac{3\mu}{2R_0^3}(I_z - I_x)\sin 2\theta \cos\phi = -\frac{3\mu}{R_0^3}(I_z - I_x)\, a_{13} a_{33}$$

$$G_z = \frac{3\mu}{2R_0^3}(I_x - I_y)\sin 2\theta \sin\phi = -\frac{3\mu}{R_0^3}(I_x - I_x)\, a_{13} a_{23}.$$

These expressions can be simplified by linearization for a body in a circular orbit using small angle approximations for ϕ and θ. The lateral velocity of a body in a circular orbit of radius R_0 is $v = \sqrt{\mu/R_0}$; thus the angular orbital velocity of the body is $\omega_0 = v/R_0 = \sqrt{\mu/R_0^3}$. Hence the components of G are

$$G_x \approx 3\omega_0^2 (I_z - I_y)\phi,$$

$$G_y \approx 3\omega_0^2 (I_z - I_x)\theta,$$

$$G_z \approx 0.$$

Set $\omega = \omega_{BI}$ for simplicity. For small Euler angles

$$\begin{pmatrix} \omega_{REB\,x} \\ \omega_{REB\,y} \\ \omega_{REB\,z} \end{pmatrix} = \begin{pmatrix} 1 & \psi & -\theta \\ -\psi & 1 & \phi \\ \theta & -\phi & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -\omega_0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\psi\omega_0 \\ -\omega_0 \\ \phi\omega_0 \end{pmatrix}.$$

Then

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix} + \begin{pmatrix} -\psi\omega_0 \\ -\omega_0 \\ \phi\omega_0 \end{pmatrix}$$

Based on the expressions for p, q, r in terms of the Euler angles, we have $p \approx \dot\phi$, $q \approx \dot\theta$, $r \approx \dot\psi$. Then

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \dot\phi - \psi\omega_0 \\ \dot\theta - \omega_0 \\ \dot\psi + \phi\omega_0 \end{pmatrix}$$

In future topics we will also be concerned with momentum-biased satellites. In this type of satellite, a constant angular momentum bias h_{wy0} is applied along the Y_B axis to give inertial angular stability about the Y_B axis of the spacecraft. With this assumption, together with the gravitational moment and small angle approximation,

$$\tau_x = I_x\ddot\phi + 4\omega_0^2(I_y - I_z)\phi + \omega_0(I_y - I_z - I_x)\dot\psi + \dot h_{wx} - \omega_0 h_{wz} - \dot\psi h_{wy0} - \phi\omega_0 h_{wy0} - I_{xy}\ddot\theta - I_{xz}\ddot\psi - I_{xz}\omega_0^2\psi + 2I_{yz}\omega_0\dot\theta$$

$$\tau_y = I_y\ddot\theta + 3\omega_0^2(I_x - I_z)\theta + \dot h_{wy} - I_{xy}(\ddot\phi - 2\omega_0\psi - \omega_0^2\theta) + I_{yz}(-\ddot\psi - 2\omega_0\dot\phi + \omega_0^2\psi)$$

$$\tau_z = I_z\ddot\psi + \omega_0(I_z + I_x - I_y)\dot\phi + \omega_0^2(I_y - I_x)\psi + \dot h_{wz} + \omega_0 h_{wx} + \dot\phi h_{wy0} - \psi\omega_0 h_{wy0} - I_{yz}\ddot\theta - I_{xz}\ddot\phi - 2\omega_0 I_{xy}\dot\theta - \omega_0^2 I_{xz}\phi$$