Ballistic Missile

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1 Homing Guidance

We will primarily be concerned with terminal guidance, the purpose of which is to hit or come close to a preselected target. Of course, we must allow for the possibility that the target will take unpredictable evasive maneuvers, and be able to correct for navigation uncertainties. We begin with a particular geometric rule that defines parallel navigation, which will lead us to proportional navigation.

According to the parallel navigation rule, the direction of the line of sight (LOS) is to be kept constant relative to inertial space throughout the engagement. In other words, the LOS is to remain parallel to the initial LOS. Let M denote the missile/pursuer and T denote the target. Define their positions in some inertial reference frame as $\boldsymbol{r}_M$ and $\boldsymbol{r}_T$. Their relative position is $\boldsymbol{r} = \boldsymbol{r}_T - \boldsymbol{r}_M$. Let $\boldsymbol{\omega}$ be the angular velocity of $\boldsymbol{r}$ in the inertial frame. Then the parallel guidance rule can be stated as

$$\boldsymbol{\omega} = 0. \tag{1}$$

An equivalent statement is that $\boldsymbol{r} \times \dot{\boldsymbol{r}} = 0$, or that $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ are colinear throughout the engagement. An additional constraint is that $\boldsymbol{r} \cdot \dot{\boldsymbol{r}} < 0$ to ensure that M gets closer to T and not farther away.

Let $\boldsymbol{v}_M$ and $\boldsymbol{v}_T$ be the velocities of M and T respectively, measured in the inertial frame. In certain simple scenarios, these velocities remain in the same plane as $\boldsymbol{r}$: such an engagement is called planar. In the planar case, we define the angles

$$\begin{aligned} \lambda &= \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{r}}, \\ \delta &= \hat{\boldsymbol{v}}_M \cdot \hat{\boldsymbol{r}}, \\ \theta &= \hat{\boldsymbol{v}}_T \cdot \hat{\boldsymbol{r}}, \end{aligned} \tag{2}$$

where hats denote unit vectors and $\hat{\boldsymbol{x}}$ is the unit vector in the inertial frame's x-direction. The magnitude r and the angle λ satisfy the differential relationships

$$\begin{aligned} \dot{r} &= v_T \cos\theta - v_M \cos\delta, \\ \dot{\lambda} &= \frac{v_T \sin\theta - v_M \sin\delta}{r}. \end{aligned} \tag{3}$$

Then the parallel navigation rule implies

$$\begin{aligned} v_M \sin\delta &= v_T \sin\theta, \\ v_M \cos\delta &> v_T \cos\theta, \end{aligned} \tag{4}$$

which hold irrespective of whether T maneuvers.

Now consider nonplanar engagements. The geometric rule can also be expressed as

$$\hat{\boldsymbol{r}} \times (\hat{\boldsymbol{r}} \times \boldsymbol{v}_M) = \hat{\boldsymbol{r}} \times (\hat{\boldsymbol{r}} \times \boldsymbol{v}_T). \tag{5}$$

Geometrically, $\hat{\boldsymbol{r}} \times (\hat{\boldsymbol{r}} \times \boldsymbol{v}_M)$ is the component of $\boldsymbol{v}_M$ orthogonal to the LOS (and similarly for $\boldsymbol{v}_T$). Therefore the rule implies

$$\boldsymbol{v}_{M\perp} = \boldsymbol{v}_{T\perp}. \tag{6}$$

The other condition concerns the component of the velocities parallel to the LOS, *e.g.* $\boldsymbol{v}_{M||} = \boldsymbol{v}_M \cdot \hat{\boldsymbol{r}}$,

$$\boldsymbol{v}_{M||} > \boldsymbol{v}_{T||}. \tag{7}$$

Let $\boldsymbol{a}_M$ be the acceleration of M perpendicular to M's velocity. A practical way of control M is by applying various forces to it which cause it to accelerate, thus making an acceleration signal $\boldsymbol{a}_{M_c}$ crucial to the process of controlling M during pursuit. In a general, nonplanar case there are two viable control laws:

$$\begin{aligned} \boldsymbol{a}_{M_c} &= N\boldsymbol{\omega} \times \boldsymbol{v}_M, \\ \boldsymbol{a}_{M_c} &= k\boldsymbol{\omega} \times \hat{\boldsymbol{r}}, \end{aligned} \tag{8}$$

where N is called the navigation constant and k is called the gain. These laws are referred to as Pure Proportional Navigation (PPN) and True Proportional Navigation (TPN), respectively.

1.1 3D Constant Velocity Target

Let's consider a concrete example. Establish two reference frames with coincident origins, called the inertial frame and the LOS frame. The inertial frame has Cartesian unit vectors $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$; we will choose to describe vectors in this frame in terms of standard spherical coordinates (ρ, θ, γ), where θ is the azimuthal angle measured from the z-axis[1] and γ is the polar angle measured from the x-axis. The LOS frame is constructed based on the relative position of M and T; it has spherical unit vectors $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\alpha}}\}$, where $\hat{\boldsymbol{r}}$ is the LOS unit vector introduced earlier. The spherical coordinates in this frame are (r, β, α), where β is the azimuthal angle and α is the polar angle.

Let T move with a constant velocity $\boldsymbol{v}_T = v_{T,x}\hat{\boldsymbol{x}} + v_{T,y}\hat{\boldsymbol{y}} + v_{T,z}\hat{\boldsymbol{z}}$ as measured in the inertial frame. Assume M moves with constant speed v_M and with velocity

$$\boldsymbol{v}_M = v_M\Big[\sin\theta\cos\gamma\hat{\boldsymbol{x}} + \sin\theta\sin\gamma\hat{\boldsymbol{y}} + \cos\theta\hat{\boldsymbol{z}}\Big], \tag{9}$$

as measured in the inertial frame. The relative velocity is $\boldsymbol{v} \equiv \boldsymbol{v}_T - \boldsymbol{v}_M$. The LOS frame is, in general, rotating with respect to the inertial frame with angular velocity

$$\boldsymbol{\omega} = -\sin\beta\dot{\alpha}\hat{\boldsymbol{\beta}} + \dot{\beta}\hat{\boldsymbol{\alpha}}. \tag{10}$$

On the other hand, the angular velocity is also given by

$$r\boldsymbol{\omega} = \hat{\boldsymbol{r}} \times \boldsymbol{v}, \tag{11}$$

[1] This is the standard mathematical convention, which is related to the standard choice in traditional aerospace navigation by $\pi/2 - \theta$

where the right hand side is naturally computed in the inertial frame. Eqn. (11) defines the angular kinematics of the pursuit problem. In particular, we find the two relations

$$\begin{aligned} r\dot\beta &= v_{T,x}\cos\beta\cos\alpha + v_{T,y}\cos\beta\sin\alpha - v_{T,z}\sin\beta + v_M\Big[\sin\beta\cos\theta - \cos\beta\sin\theta\cos(\alpha-\gamma)\Big], \\ r\sin\beta\dot\alpha &= -v_{T,x}\sin\alpha + v_{T,y}\cos\alpha + v_M\sin\theta\sin(\alpha-\gamma). \end{aligned} \tag{12}$$

The radial kinematics come from the component of $\boldsymbol{v}$ parallel to $\hat{\boldsymbol{r}}$, namely $\dot r = \boldsymbol{v}\cdot\hat{\boldsymbol{r}}$, giving

$$\dot r = \boldsymbol{v}_T\cdot\hat{\boldsymbol{r}} - v_M\Big[\cos\beta\cos\theta + \sin\beta\sin\theta\cos(\alpha-\gamma)\Big]. \tag{13}$$

The PPN control law involves the acceleration of M perpendicular to $\boldsymbol{v}_M$, which is computed as

$$\boldsymbol{a}_M = \hat{\boldsymbol{v}}_M\times(\dot{\boldsymbol{v}}_M\times\hat{\boldsymbol{v}}_M). \tag{14}$$

It is convenient to rotate the result into the LOS frame

$$\begin{aligned} \frac{\boldsymbol{a}_M}{v_M} &= \Big[\sin\beta\sin\theta\sin(\alpha-\gamma)\dot\gamma + \sin\beta\cos\theta\cos(\alpha-\gamma)\dot\theta\Big]\hat{\boldsymbol{r}} \\ &+ \Big[\cos\beta\sin\theta\sin(\alpha-\gamma)\dot\gamma + \cos\beta\cos\theta\cos(\alpha-\gamma)\dot\theta\Big]\hat{\boldsymbol{\beta}} \\ &+ \Big[\sin\theta\cos(\alpha-\gamma)\dot\gamma - \cos\theta\sin(\alpha-\gamma)\dot\theta\Big]\hat{\boldsymbol{\alpha}}. \end{aligned} \tag{15}$$

The right hand side of the control law is given by

$$\begin{aligned} \frac{\boldsymbol{\omega}\times\boldsymbol{v}_M}{v_M} &= \Big[\sin\beta\sin\theta\sin(\alpha-\gamma)\dot\alpha + \big(\sin\beta\cos\theta - \cos\beta\sin\theta\cos(\alpha-\gamma)\big)\dot\beta\Big]\hat{\boldsymbol{r}} \\ &+ \Big[\cos\beta\cos\theta + \sin\beta\sin\theta\cos(\alpha-\gamma)\Big]\dot\beta\hat{\boldsymbol{\beta}} \\ &+ \Big[\cos\beta\cos\theta + \sin\beta\sin\theta\cos(\alpha-\gamma)\Big]\sin\beta\dot\alpha\hat{\boldsymbol{\alpha}}. \end{aligned} \tag{16}$$

The control law can be solved for expressions[2] $\dot\theta[\dot\alpha,\dot\beta]$ and $\dot\gamma[\dot\alpha,\dot\beta]$. We can then insert the kinematic expressions for $\dot\alpha$ and $\dot\beta$ to get

$$\begin{aligned} r\dot\theta &= -Nv_M\Big[\cos\beta\sin\theta - \sin\beta\cos\theta\cos(\alpha-\gamma)\Big], \\ r\sin\theta\dot\gamma &= Nv_M\sin\beta\sin(\alpha-\gamma). \end{aligned} \tag{17}$$

The equations (12), (13), and (17) define the kinematics of the controlled pursuit problem. There are 5 coupled first order differential equations for the 5 quantities $\{r,\beta,\alpha,\theta,\gamma\}$. Convenient initial data for the first 3 variables consists of specifying T's initial position (x_0,y_0,z_0), in terms of which

$$\begin{aligned} r_0 &= \sqrt{x_0^2+y_0^2+z_0^2}, \\ \beta_0 &= \cos^{-1}\left(\frac{z_0}{r_0}\right), \\ \alpha_0 &= \cos^{-1}\left(\frac{x_0}{\sqrt{x_0^2+y_0^2}}\right). \end{aligned} \tag{18}$$

The final two quantities' initial values are determined by M's initial velocity; it is convenient to specify the speed v_M and pick the direction as

$$\begin{aligned} \theta_0 &= \beta_0 + \delta\beta, \\ \gamma_0 &= \alpha_0 + \delta\alpha. \end{aligned} \tag{19}$$

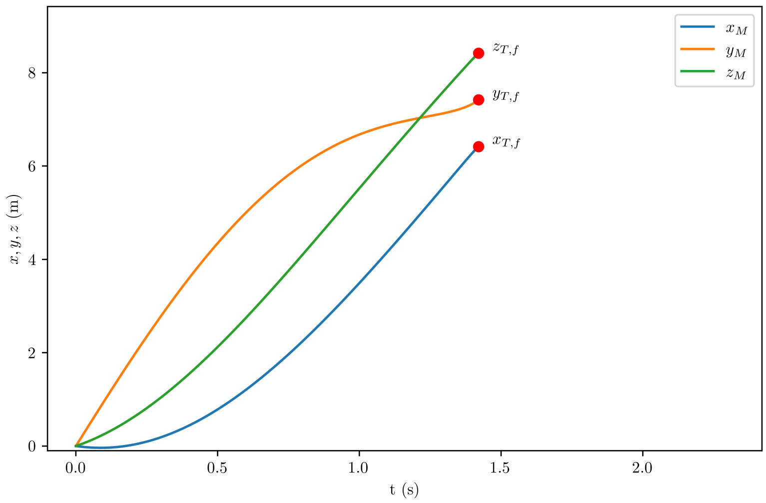

Figure 1: The components of $\boldsymbol{r}_M$

To visualize the results, suppose T's position is

$$\boldsymbol{r}_T(t) = \boldsymbol{v}_T t + \boldsymbol{r}_{T,0}, \tag{20}$$

where $\boldsymbol{v}_T = (1, 1, 1)$ and $\boldsymbol{r}_{T,0} = (5, 6, 7)$. Further, choose $v_M = 5$, $N = 2$, $\delta\beta = \pi/6$, and $\delta\alpha = \pi/4$. The components of $\boldsymbol{r}_M$ are shown in fig. (1). The full three-dimensional trajectory is shown in fig. (2). The nominal path of T is shown, with the dot indicating the point of interception.

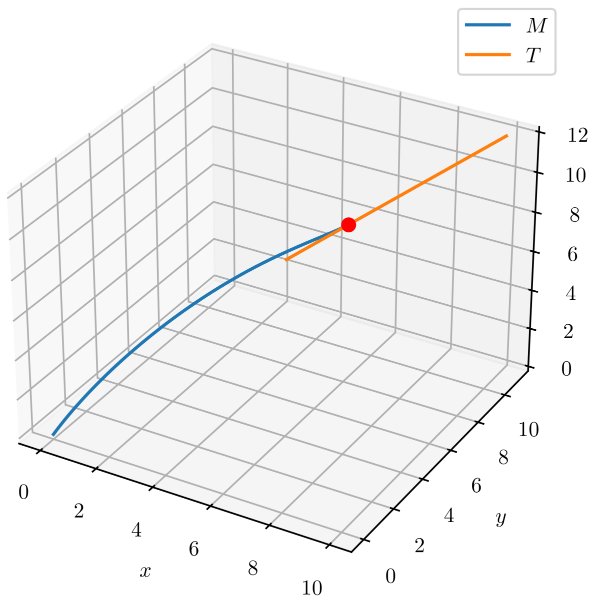

Figure 2: The 3D trajectory of M

[2]There are two unknowns to be solved for and seemingly three equations, but the $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\alpha}}$ equations are not independent.

2 Flight Dynamics

2.1 Reference Frames

We turn to a description of flight dynamics, in which we will use standard coordinate conventions for aerospace. In particular, we define a cartesian inertial reference frame $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$ where the z-axis points in the negative downward direction. We refer to the x-axis as the northerly direction, and the y-axis as the easterly direction.

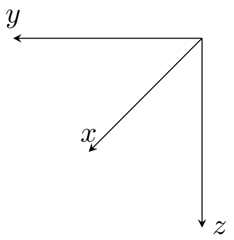

Figure 3: Inertial reference frame I

Denote the velocity of the center of mass of an aircraft as measured in the inertial frame by $\boldsymbol{v}_I$. Introduce the heading angle ξ and the flight path angle γ which determine the velocity of the aircraft in polar coordinates as

$$\boldsymbol{v}_I = V\Big[\cos\gamma\cos\xi\hat{\boldsymbol{x}} + \cos\gamma\sin\xi\hat{\boldsymbol{y}} - \sin\gamma\hat{\boldsymbol{z}}\Big]. \tag{21}$$

The reference frame with unit vectors corresponding to these polar angles is called the velocity frame.

Let $\{\hat{\boldsymbol{x}}_B, \hat{\boldsymbol{y}}_B, \hat{\boldsymbol{z}}_B\}$ be the reference frame, called the body frame, where $\{\hat{\boldsymbol{x}}_B$ aligns with the nose of the aircraft, $\hat{\boldsymbol{y}}_B$ points toward the right wing tip[3], and $\hat{\boldsymbol{z}}_B$ is downward perpendicular to the other two. The orientation of the body axis frame to the inertial frame is described by the Euler angles: ψ, the yaw angle, which represents rotation about $\hat{\boldsymbol{z}}_B$; θ, the pitch angle, which represents rotation about $\hat{\boldsymbol{y}}_B$; and ϕ, the roll angle, which represents rotation about $\hat{\boldsymbol{x}}_B$. The components of a vector in the body frame is related to the components in the inertial frame via the orthogonal matrix

$$R_I{}^B(\phi,\theta,\psi) = \begin{pmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi & \cos\phi\cos\psi + \sin\phi\cos\theta\sin\psi & \sin\phi\cos\theta \\ \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi & \cos\phi\cos\theta \end{pmatrix}. \tag{22}$$

The time derivatives of the Euler angles are related to the components of the angular velocity in the body frame by

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix}_B = \begin{pmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}_E, \tag{23}$$

or $\boldsymbol{\omega}_B = L_E{}^B\Theta_E$. The subscripts denote the reference frame in which the components are measured. We have introduced a new reference frame E (for Euler), which essentially exists to make this transformation. Note that this transformation is not orthogonal.

[3] For a perfectly cylindrically symmetric aircraft, like a missile, this choice is arbitrary.

Let the velocity in the body frame be $\boldsymbol{v}_B = u\hat{\boldsymbol{x}}_B + v\hat{\boldsymbol{y}}_B + w\hat{\boldsymbol{z}}_B$, which can be expressed in terms of spherical coordinates as

$$\boldsymbol{v}_B = V\Big[\cos\alpha\cos\beta\hat{\boldsymbol{x}}_B + \sin\beta\hat{\boldsymbol{y}}_B + \sin\alpha\cos\beta\hat{\boldsymbol{z}}_B\Big]. \tag{24}$$

These angles are α, the angle of attack, and β, the sideslip angle. They are commonly referred to as airflow angles and play an important role in determining the aerodynamic forces acting on the aircraft. In the absence of external wind, an aircraft traveling through the air can be thought of as experiencing a wind opposite the velocity. For this reason the frame defined using α and β may be called the wind frame. The transformation from the body frame to the wind frame is

$$R_W{}^B(\alpha,\beta) = \begin{pmatrix} \cos\alpha\cos\beta & -\cos\alpha\sin\beta & -\sin\alpha \\ \sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & -\sin\alpha\sin\beta & \cos\alpha \end{pmatrix}. \tag{25}$$

A transformation from the wind frame to the inertial frame is then possible through a composition of transformations

$$R_I{}^W = R_W{}^B(\alpha,\beta)^{-1}R_I{}^B(\phi,\theta,\psi). \tag{26}$$

Yet another way to express $R_I{}^W$ is by using the velocity frame. We have the transformation

$$R_I{}^V(\gamma,\xi) = \begin{pmatrix} \cos\gamma\cos\xi & \cos\gamma\sin\xi & -\sin\gamma \\ -\sin\xi & \cos\xi & 0 \\ \sin\gamma\cos\xi & \sin\gamma\sin\xi & \cos\gamma \end{pmatrix}. \tag{27}$$

Introduce the bank angle μ, which corresponds to rotation about the velocity vector. Then

$$\begin{aligned} R_I{}^W &= R_V{}^W(\mu)R_I{}^V(\gamma,\xi), \\ R_V{}^W(\mu) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\mu & \sin\mu \\ 0 & -\sin\mu & \cos\mu \end{pmatrix}. \end{aligned} \tag{28}$$

The two different expressions for $R_I{}^W$ determine the bank angle via

$$\sin\mu = \frac{\cos\theta\sin\phi\cos\beta + \sin\beta(\cos\alpha\sin\theta - \sin\alpha\cos\theta\cos\phi)}{\cos\gamma}. \tag{29}$$

By using the equality $R_W^B = R_I{}^B R_V{}^I R_W^V$ we also get the expressions

$$\begin{aligned} \sin\beta &= \cos\gamma\sin\phi\sin\theta\cos(\xi-\psi) + \cos\gamma\cos\phi\sin(\xi-\psi) - \sin\gamma\sin\phi\cos\theta, \\ \sin\alpha &= \frac{\cos\gamma\cos\phi\sin\theta\cos(\xi-\psi) - \cos\gamma\sin\phi\sin(\xi-\psi) - \sin\gamma\cos\phi\cos\theta}{\cos\beta}. \end{aligned} \tag{30}$$

2.2 Forces and Torques

Forces and torques expressed in noninertial frames can be recast to an inertial one using rotations like the ones presented in the preceding subsection. Some frames are more natural than others for certain forces and torques. For example, drag is a force that is parallel to and opposite the velocity vector, and lift is a force perpendicular to the veloctiy; these are naturally described in the wind frame. On the other hand, yawing, pitching, and rolling torques make the most sense in the body frame.

Time derivatives of vector quantities in inertial frames are related to time derivatives in rotating frames by the inclusion of the angular veloctiy. For instance, consider the angular momentum $\boldsymbol{h}$ as measured in both the inertial and body frames. Its time derivative is

$$\frac{d\boldsymbol{h}_I}{dt} = \left[\frac{d\boldsymbol{h}_B}{dt}\right]_I + \boldsymbol{\omega}_I \times \boldsymbol{h}_I = {R_B}^I \frac{d\boldsymbol{h}_B}{dt} + \boldsymbol{\omega}_I \times \boldsymbol{h}_I. \tag{31}$$

The involvement of the angular velocity represents the presence of rotation in the coordinates, a fact which almost is not worth mentioning, except that the vector is simply the Hodge dual of the corresponding rotation matrix

$$\tilde{\omega}_I = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}_I. \tag{32}$$

The term involving $\boldsymbol{\omega}$ in the time derivative is rewritten $\boldsymbol{\omega}_I \times \boldsymbol{h}_I = \tilde{\omega}_I \boldsymbol{h}_I$. The transformation properties of $\boldsymbol{\omega}$ are thus more appropriately expressed in terms of the matrix multiplication: the index contraction between $\tilde{\omega}$ and $\boldsymbol{h}_I$ is invariant, leaving only one of the indices of $\tilde{\omega}$ to transform. Put another way, $\tilde{\omega}_I$ transforms in the adjoint and $\boldsymbol{h}_I$ in the fundamental, giving

$$\tilde{\omega}_I \boldsymbol{h}_I = {R_B}^I \tilde{\omega}_B {R_I}^B {R_B}^I \boldsymbol{h}_B = {R_B}^I \tilde{\omega}_B \boldsymbol{h}_B. \tag{33}$$

Therefore

$$\frac{d\boldsymbol{h}_B}{dt} = {R_I}^B \frac{d\boldsymbol{h}_I}{dt} - \tilde{\omega}_B \boldsymbol{h}_B. \tag{34}$$

The Euler moment equations can be used to replace the time derivative of the inertial frame angular momentum with external torques $\boldsymbol{\tau}_I$. Using the definition $\boldsymbol{h}_B = I_B \boldsymbol{\omega}_B$, where I_B is the inertia tensor in the body frame, we arrive at

$$I_B \frac{d\boldsymbol{\omega}_B}{dt} = \boldsymbol{\tau}_B - \tilde{\omega}_B I_B \boldsymbol{\omega}_B. \tag{35}$$

Assuming that gravitational torque is negligible, the external torques on the aircraft come from aerodynamics and thrust effects; the total external torque can be split $\boldsymbol{\tau}_B = \boldsymbol{\tau}_{B,\text{aero}} + \boldsymbol{\tau}_{B,\text{thrust}}$.

A similar analysis can be done to handle the time derivatives of the linear momentum $\boldsymbol{p}$ in the inertial and body frames

$$\frac{d\boldsymbol{p}_B}{dt} = {R_I}^B \frac{d\boldsymbol{p}_I}{dt} - \tilde{\omega} \boldsymbol{p}_B. \tag{36}$$

Here we use Newton's Second Law to replace the time derivative of the inertial frame momentum with the external force $\boldsymbol{f}_I$ applied to the aircraft. The force due to gravity must not be ignored in this case, so the external force is

$$\boldsymbol{f} = \boldsymbol{f}_{\text{aero}} + \boldsymbol{f}_{\text{thrust}} + m\boldsymbol{g}. \tag{37}$$

The four differential equations determining the linear and rotational dynamics are

$$
\begin{aligned}
\frac{d\boldsymbol{r}_I}{dt} &= R_B{}^I \boldsymbol{v}_B \\
\frac{d\Theta}{dt} &= L_B{}^E \boldsymbol{\omega}_B \\
\frac{d\boldsymbol{v}_B}{dt} &= \frac{1}{m}\Big(\boldsymbol{f}_{B,\text{aero}} + \boldsymbol{f}_{B,\text{thrust}}\Big) + R_I{}^B \boldsymbol{g}_I - \tilde{\omega}_B \boldsymbol{v}_B \\
\frac{d\boldsymbol{\omega}_B}{dt} &= I_B^{-1}\Big(\boldsymbol{\tau}_{B,\text{aero}} + \boldsymbol{\tau}_{B,\text{thrust}} - \tilde{\omega}_B I_B \boldsymbol{\omega}_B\Big),
\end{aligned}
\tag{38}
$$

where we assume that the inertia tensor is invertible and

$$
L_B{}^E = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix}. \tag{39}
$$

2.3 Symmetric Aircraft

2.3.1 Body Frame Equations

Consider an aircraft that is reflection invariant across the $x_B z_B$-plane (*e.g.* an airplane or a missile with rear fins placed every 120 degrees). The products of inertia $I_{xy}, I_{yz} = 0$, with the other elements of I_B nonzero in general. The inverse of the inertia tensor is

$$
\begin{aligned}
I_B^{-1} &= \begin{pmatrix} I_{zz}/\Xi & 0 & I_{xz}/\Xi \\ 0 & 1/I_{yy} & 0 \\ I_{xz}/\Xi & 0 & I_{xx}/\Xi \end{pmatrix}_B, \\
\Xi &= I_{xx}I_{zz} - I_{xz}^2.
\end{aligned}
\tag{40}
$$

The hodge dual of the angular velocity in the body frame is

$$
\tilde{\omega}_B = \begin{pmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{pmatrix}_B. \tag{41}
$$

Let

$$
\begin{aligned}
\boldsymbol{f}_{B,\text{aero}} + \boldsymbol{f}_{B,\text{thrust}} &= \begin{pmatrix} X_\text{aero} + X_\text{thrust} \\ Y_\text{aero} + Y_\text{thrust} \\ Z_\text{aero} + Z_\text{thrust} \end{pmatrix}_B, \\
\boldsymbol{\tau}_{B,\text{aero}} + \boldsymbol{\tau}_{B,\text{thrust}} &= \begin{pmatrix} L_\text{aero} + L_\text{thrust} \\ M_\text{aero} + M_\text{thrust} \\ N_\text{aero} + N_\text{thrust} \end{pmatrix}_B.
\end{aligned}
\tag{42}
$$

Then the three translational equations of motion are (assuming a flat earth)

$$
\begin{aligned}
\dot{u} &= X/m - g\sin\theta + rv - qw, \\
\dot{v} &= Y/m + g\sin\phi\cos\theta - ru + pw, \\
\dot{w} &= Z/m + g\cos\phi\cos\theta + qu - pv.
\end{aligned}
\tag{43}
$$

and the rotational equations of motion are

$$\begin{aligned}
\dot{p} &= \frac{1}{\Xi}\Big[I_{zz}L + I_{xz}N - \Big(I_{xz}(I_{yy} - I_{xx} - I_{zz})p + (I_{xx}^2 + I_{zz}^2 - I_{zz}I_{yy})r\Big)q\Big], \\
\dot{q} &= \frac{1}{I_{yy}}\Big[M - (I_{xx} - I_{zz})pr - I_{xz}(p^2 - r^2)\Big], \\
\dot{r} &= \frac{1}{\Xi}\Big[I_{xz}L + I_{xx}N - \Big(I_{xz}(I_{yy} - I_{xx} - I_{zz})r + (I_{xz}^2 + I_{xx}^2 - I_{xx}I_{zz})r\Big)q\Big]
\end{aligned} \tag{44}$$

2.3.2 Velocity Frame Equations

While the previous equations of motion were expressed in the body frame (where it is most natural to compute the inertia tensor), it is also conceptually appealing to look at the translational equations in the velocity and wind axes. In both of these frames the x-axis is aligned with the inertial velocity. The velocity frame is useful for treating the aircraft as a point mass for the purpose of doing trajectory calculations (which will be important when applying the previous navigation/pursuit analysis), and the wind frame allows to find the aerodynamic angles (α, β, μ).

The angular velocity is expressed in the velocity frame in terms of the velocity frame polar angles (γ, ξ) by a transformation similar to $L_E^{\ B}$

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}_V = \begin{pmatrix} 0 & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ 0 & 0 & \cos\gamma \end{pmatrix} \begin{pmatrix} 0 \\ \dot{\gamma} \\ \dot{\xi} \end{pmatrix}. \tag{45}$$

Since the velocity is in the x_V-direction,

$$\frac{d\boldsymbol{v}_V}{dt} + \tilde{\omega}_V \boldsymbol{v}_V = \begin{pmatrix} \dot{V} \\ V\dot{\xi}\cos\gamma \\ -V\dot{\gamma} \end{pmatrix}_V , \tag{46}$$

where V is the speed. The aerodynamic forces are defined most simply in the wind frame,

$$\boldsymbol{f}_{W,\text{aero}} = \begin{pmatrix} -D \\ \mathcal{S} \\ -\ell \end{pmatrix}_W , \tag{47}$$

where D is the drag, $\mathcal{S}$ is the side force, and ℓ is the lift. This force is transformed to the velocity frame using the rotation $R_W^{\ V}$, giving

$$\boldsymbol{f}_{V,\text{aero}} = \begin{pmatrix} -D \\ \mathcal{S}\cos\mu + \ell\sin\mu \\ \mathcal{S}\sin\mu - \ell\cos\mu \end{pmatrix}_V . \tag{48}$$

Assume that the thrust is purely in the x_B-direction with magnitude T. Then the force $\boldsymbol{f}_{B,\text{thrust}}$ is transformed to the velocity frame with the composite rotation $R_B^{\ V} = R_W^{\ V} R_B^{\ W}$

$$\boldsymbol{f}_{V,\text{thrust}} = T \begin{pmatrix} \cos\alpha\cos\beta \\ \sin\alpha\sin\mu - \cos\alpha\sin\beta\cos\mu \\ -\sin\alpha\cos\mu - \cos\alpha\sin\beta\sin\mu \end{pmatrix}_V . \tag{49}$$

Finally, the gravitational force points in the z_I-direction. The vector $\boldsymbol{g}_I$ is rotated into the velocity frame with the rotation ${R_I}^V$

$$\boldsymbol{g}_V = \begin{pmatrix} -g\sin\gamma \\ 0 \\ g\cos\gamma \end{pmatrix}_V . \tag{50}$$

Therefore the velocity frame translational equations of motion are

$$\begin{aligned} m\dot{V} &= T\cos\alpha\cos\beta - D - mg\sin\gamma \\ mV\dot{\gamma} &= T\Big(\sin\alpha\cos\mu + \cos\alpha\sin\beta\sin\mu\Big) + \ell\cos\mu - \mathcal{S}\sin\mu - mg\cos\gamma \\ mV\cos\gamma\dot{\xi} &= T\Big(\sin\alpha\sin\mu - \cos\alpha\sin\beta\cos\mu\Big) + \ell\sin\mu + \mathcal{S}\cos\mu \end{aligned} \tag{51}$$

2.4 Linearized Equations

In the velocity frame, a state vector for the system consists of

$$\boldsymbol{x} = (V, \alpha, \beta, \mu, \gamma, \xi, \phi, \theta, \psi)^T. \tag{52}$$

By inspection, the force components have the dependencies

$$T = T(V,\alpha), \quad D = D(V,\alpha,\beta), \quad \mathcal{S} = \mathcal{S}(\beta,\nu), \quad \ell = \ell(V,\alpha,\mu,\vartheta), \tag{53}$$

where we have introduced the yaw rudder angle ν and the pitch rudder angle ϑ. We will linearize around the configuration

$$\boldsymbol{x}_0 = (V, \alpha, 0, 0, \gamma, \xi, \phi, \theta, \psi)^T, \quad \boldsymbol{u}_0 = (0,0)^T, \tag{54}$$

where α is small. Denote a deviation from the nominal configuration by attaching δ to the variable. Then the linearized translation equations are

$$\begin{aligned} m\delta\dot{V} &= \Big(\partial_V T - \partial_V D\Big)\delta V + \Big(\partial_\alpha T - \partial_\alpha D\Big)\delta\alpha - mg\cos\gamma\delta\gamma, \\ mV\delta\dot{\gamma} &= \Big(\alpha\partial_V T + \partial_V \ell\Big)\delta V + \Big(T + \partial_\alpha \ell\Big)\delta\alpha + mg\sin\gamma\delta\gamma + \partial_\vartheta \ell\delta\vartheta, \\ mV\cos\gamma\delta\dot{\xi} &= \Big(\partial_\beta S - T\Big)\delta\beta + \Big(\alpha T + \ell\Big)\delta\mu + \partial_\nu S\delta\nu, \end{aligned} \tag{55}$$

where it is understood that all non-delta quantities are evaluated at the nominal values.

For simplicity, assume that the product of inertia $I_{xz} = 0$. The rotational equations of motion are

$$\begin{aligned} I_x\dot{p} &= L + (I_z - I_y)qr, \\ I_y\dot{q} &= M + (I_x - I_z)pr, \\ I_z\dot{r} &= N + (I_y - I_x)pq. \end{aligned} \tag{56}$$

The dependencies of the torques on the state variables can be done by inspection

$$L = L(\beta, p, r, \nu, \varphi), \quad M = M(V, \alpha, \dot{\alpha}, q, \vartheta), \quad N = N(\beta, \dot{\beta}, p, r, \nu), \tag{57}$$

where φ is the deflection angle for the roll control surface. The time derivatives of the Euler angles are related to the body frame angular velocity by

$$\begin{aligned}\dot{\phi} &= p + \tan\theta\Big(p\sin\phi + r\cos\phi\Big),\\ \dot{\theta} &= q\cos\phi - r\sin\phi,\\ \dot{\psi} &= \sec\theta\Big(q\sin\phi + r\cos\phi\Big).\end{aligned} \tag{58}$$

We linearize these around $p, q, r, \phi = 0$ and obtain

$$\begin{aligned}I_x\delta\dot{p} &= \partial_\beta L\delta\beta + \partial_p L\delta p + \partial_r L\delta r + \partial_\nu L\delta\nu + \partial_\varphi L\delta\varphi,\\ I_y\delta\dot{q} &= \partial_V M\delta V + \partial_\alpha M\delta\alpha + \partial_{\dot{\alpha}} M\delta\dot{\alpha} + \partial_q M\delta q + \partial_\vartheta M\delta\vartheta,\\ I_z\delta\dot{r} &= \partial_\beta N\delta\beta + \partial_{\dot{\beta}} N\delta\dot{\beta} + \partial_p N\delta p + \partial_r N\delta r + \partial_\nu N\delta\nu,\\ \delta\dot{\phi} &= \delta p + \delta r\tan\theta,\\ \delta\dot{\theta} &= \delta q,\\ \delta\dot{\psi} &= \delta r\sec\theta.\end{aligned} \tag{59}$$

By varying the geometric relationships above for $\sin\mu, \sin\beta, \sin\alpha$ we also get

$$\delta\mu = \cos\theta\sec\gamma\delta\phi + \Big(\sin\theta - \alpha\cos\theta\Big)\delta\beta \tag{60}$$

The equations of motion (55) and (59) are determined in terms of the forces and torques T, D, ℓ, S, L, M, and N. For now we will ignore these dependencies and reformulate the equations of motion. Further, we will mostly drop the δs from the deviations for ease of notation: the only one we won't change is δV.

First, consider the pitching plane. We have

$$\ddot{\theta} = \left(\frac{\partial_V M}{I_y}\right)\delta V + \left(\frac{\partial_\alpha M}{I_y}\right)\alpha + \left(\frac{\partial_{\dot{\alpha}} M}{I_y}\right)\dot{\alpha} + \left(\frac{\partial_q M}{I_y}\right)\dot{\theta} + \left(\frac{\partial_\vartheta M}{I_y}\right)\vartheta, \tag{61}$$

where we've compared the equations for θ and q. By simplifying the coefficients we have

$$\ddot{\theta} = C_\theta^V\delta V + C_\theta^q\dot{\theta} + C_\theta^\alpha\alpha + C_\theta^{\dot{\alpha}}\dot{\alpha} + C_\theta^\vartheta. \tag{62}$$

For small bank angle $\mu \ll 1$, we have the geometric relationship $\theta \approx \gamma + \alpha$. For small α, this is further refined to $\theta \approx \gamma$. Using the equation of motion for γ we get

$$\begin{aligned}\dot{\theta} &= \left(\frac{\alpha\partial_V T + \partial_V\ell}{mV_0}\right)\delta V + \left(\frac{T + \partial_\alpha\ell}{mV_0}\right)\alpha + \left(\frac{\partial_\vartheta\ell}{mV_0}\right)\vartheta + \frac{g}{V_0}\sin\gamma_0\theta\\ &\equiv C_\gamma^V\delta V + C_\gamma^\alpha\alpha + C_\gamma^\vartheta\vartheta + C_\gamma^\theta\theta.\end{aligned} \tag{63}$$