The plane of the orbit is inclined to the xy-plane by an angle i, called the inclination of the orbit. The two planes intersect along the node line. The angle in the equatorial plane between the x-axis and the node line is called the right ascension, Ω. In the orbit plane, r is the radius vector to the moving body and r_p is the vector to the perigee. The angle between the node line and r_p is ω, the argument of perigee. These 3 parameters, together with the parameters (a, e, M) of the orbit within its own plane complete a system of 6 parameters that suffices to determine the location in space of a body moving in any Keplerian orbit. These parameters are known as the classical orbit parameters. We summarize them:

(i) a, the semi-major axis

(ii) e, the eccentricity

(iii) i, the inclination

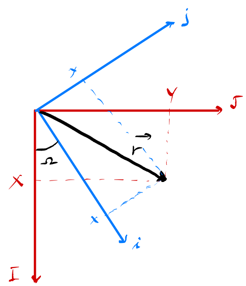
(iv) Ω, the right ascension of the ascending node

(v) ω, the argument of perigee

(vi) $M = n(t - t_0)$, the mean anomaly

Define the vector $\alpha = (a, e, i, \Omega, \omega, M)^T$. Although these parameters completely define an orbit in space, some of them (e.g. Ω) are poorly defined when i is small, as with geostationary orbits. In such cases a variation of the six ~~listed~~ parameters will be preferred. Orbits with very small inclination are called equatorial orbits.

It will be useful to know how to transform between coordinate systems which are related by rotations. Consider:



We have two sets of orthonormal vectors $\{\hat{\imath}, \hat{\jmath}\}$ and $\{\hat{I}, \hat{J}\}$ with respect to which we may decompose a vector $\vec{r}$:

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} = X\hat{I} + Y\hat{J}.$$

Notice that

$$\hat{I}\cdot\hat{i} = \cos\Omega, \qquad \hat{J}\cdot\hat{i} = \sin\Omega,$$
$$\hat{I}\cdot\hat{j} = -\sin\Omega, \qquad \hat{J}\cdot\hat{j} = \cos\Omega.$$

So,

$$X\hat{I}\cdot\hat{i} + Y\hat{J}\cdot\hat{i} = x\hat{i}\cdot\hat{i} + y\hat{j}\cdot\hat{i}$$
$$= x$$

$$\Rightarrow \boxed{x = \cos\Omega\, X + \sin\Omega\, Y}$$

and

$$X\hat{I}\cdot\hat{j} + Y\hat{J}\cdot\hat{j} = x\hat{i}\cdot\hat{j} + y\hat{j}\cdot\hat{j}$$
$$= y$$

$$\Rightarrow \boxed{y = -\sin\Omega\, X + \cos\Omega\, Y}$$

This gives the rotation transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\Omega & \sin\Omega \\ -\sin\Omega & \cos\Omega \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}.$$

Secretly we may view this as a rotation about a third unit vector $\hat{K}$ (which would coincide for the two coordinate systems. Defining z, Z as the respective coordinates we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

Write a rotation about the μ^{th} axis $R_\mu(\Omega)$. It suffices to string together rotations about the x- and z-axes to obtain an arbitrary rotation in $SO(3)$. The transformation between the inertial coordinates $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ and the orbit coordinates $\begin{pmatrix} P \\ Q \\ W \end{pmatrix}$, where

- P is directed from the center of the orbit to the perigee
- W is directed along the momentum axis of the orbit, $h = r \times V$
- Q is $\perp$ to both others

is

$$\begin{pmatrix} P \\ Q \\ W \end{pmatrix} = R_z(\omega)\, R_x(i)\, R_z(\Omega)\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

We can define a local coordinate system (polar coordinates for the orbit) as

$$\begin{pmatrix} R \\ S \\ W \end{pmatrix} = R_z(\omega+\theta)\, R_x(i)\, R_z(\Omega)\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.$$

We want to construct the transformation from the 6 classical orbit parameters to the inertial coordinate system X, Y, Z. Since a Keplerian orbit lies in a plane, we can define a coordinate system x, y in a plane with $z = 0$

$$x = a\cos\psi - c = a(\cos\psi - e)$$
$$y = b\sin\psi = a\sqrt{1-e^2}\sin\psi$$

where ψ is the angle introduced with the auxiliary circle. Then the position is $r = x\hat{i} + y\hat{j}$ for some $\{\hat{i}, \hat{j}\}$. The value of ψ is determined by $\psi - e\sin\psi = M = nt$, which can be solved for perturbatively in e. Set $\hat{i} = P$ and $\hat{j} = Q$. Then

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = R_z^{-1}(\Omega)\, R_x^{-1}(i)\, R_z^{-1}(\omega) \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

The velocity vector is

$$v = \frac{dr}{dt} = \frac{dr}{d\psi}\frac{d\psi}{dt}$$

where

$$\frac{d\psi}{dt}\left(1 - e\cos\psi\right) = \frac{dM}{dt} = n$$

$$\Rightarrow \frac{d\psi}{dt} = \frac{an}{r}.$$

So

$$\begin{aligned} \frac{dr}{dt} &= -a\sin\psi \frac{d\psi}{dt}\hat{P} + a\sqrt{1-e^2}\cos\psi\frac{d\psi}{dt}\hat{Q} \\ &= \frac{a^2 n}{r}\left(\sin\psi\,\hat{P} + \sqrt{1-e^2}\cos\psi\,\hat{Q}\right) \\ &\equiv v_P\hat{P} + v_Q\hat{Q}. \end{aligned}$$

We can apply the inverse transformation to get the velocity in the inertial system as well. If we knew r, v in the inertial coordinates, it is straightforward to find the 6 orbit parameters.

So far we have only considered ideal Keplerian orbits which derived from an assumption of only two bodies experiencing mutual gravitation. This ideal situation does not exist in practice. For example, there are 8 other planets influencing any planet's orbit in addition to the sun. For an earth-orbiting satelite it turns out that the sun and moon must both be taken into account. These are examples of conservative perturbing forces, but there are non-conservative ones as well. Satelites experience solar pressure and atmospheric drag.

Write N2L for the ideal Kepler situation as

$$\frac{d^2\vec{r}}{dt^2} = -\mu\frac{\vec{r}}{r^3} \equiv \vec{\gamma}_K$$

with initial conditions $\vec{r}(0), \vec{v}(0)$. For Keplerian orbits, the 6 orbit parameters are constant:

$$\frac{da}{dt} = \frac{de}{dt} = \frac{d\Omega}{dt} = \frac{d\omega}{dt} = \frac{di}{dt} = 0, \quad \frac{dM}{dt} = n.$$

We represent the perturbed EOM by

$$\frac{d^2\vec{r}}{dt^2} = \vec{\gamma}_K + \vec{\gamma}_P$$

$$\vec{r}(t_0) = \vec{r}_0, \quad \vec{v}(t_0) = \vec{v}_0.$$

The magnitude of $\vec{\gamma}_p$ should be appreciably smaller than $\vec{\gamma}_K$ for it to be a perturbation. The full solution to the EOM is called the true orbit. If we were to remove the perturbation at some time t_0 and find the subsequent Keplerian solution, that would be called an osculating orbit.

In general the perturbing force will depend on $\vec{r}, \vec{v}, t$ (e.g. the moon's impact on an earth-orbitting satellite depends on the satellite's position relative to the moon). The EOM are

$$\frac{d\vec{v}}{dt} = \vec{F}(\vec{r}, \vec{v}, t), \quad \frac{d\vec{r}}{dt} = \vec{v}.$$

We can think of $\vec{r}$ and $\vec{v}$ as depending on the 6 α_i as well as t. Therefore we have the 6 implicit EOM

$$\frac{\partial \vec{v}}{\partial \alpha_i} \frac{d\alpha_i}{dt} = \vec{F}(\alpha, t)$$
$$\frac{\partial \vec{r}}{\partial \alpha_i} \frac{d\alpha_i}{dt} = \vec{v}(\alpha, t)$$

which can be inverted to give EOM for each α_i

$$\frac{da}{dt} = F_a(\alpha, t)$$
$$\frac{de}{dt} = F_e(\alpha, t)$$
$$\frac{d\Omega}{dt} = F_\Omega(\alpha, t)$$
$$\frac{d\omega}{dt} = F_\omega(\alpha, t)$$
$$\frac{di}{dt} = F_i(\alpha, t)$$
$$\frac{dM}{dt} = F_M(\alpha, t)$$

Now it remains to find expressions for the RHS of these equations. Decompose the perturbing force $\vec{\gamma}_p$ along the axes of a moving Cartesian frame defined so that $\hat{R}$ points along the position vector $\vec{r}$, $\hat{S}$ is in the local plane of the osculating orbit ($\perp$ to $\hat{R}$); $\hat{W}$ is $\perp$ to both $\hat{R}$ and $\hat{S}$. Then

$$\vec{\gamma}_p = R\hat{R} + S\hat{S} + W\hat{W}.$$

Start with the equation for $\frac{da}{dt}$. Recall that

$$E = \int \vec{F} \cdot d\vec{r} = \int \vec{F} \cdot \vec{v}\, dt.$$

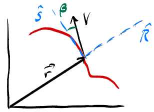
Then since $E = -\frac{\mu}{2a}$, we find that

$$\frac{dE}{dt} = \frac{\mu}{2a^2} \frac{da}{dt} = \vec{v} \cdot (\vec{\gamma}_p + \vec{\gamma}_K).$$

However in a Keplerian orbit $\vec{v} \cdot \vec{\gamma}_K = 0$, so

$$\frac{\mu}{2a^2} \frac{da}{dt} = \vec{v} \cdot \vec{\gamma}_p.$$

The velocity vector is in the plane of the osculating orbit and therefore has an expansion in terms of $\hat{R}$ and $\hat{S}$. If $\hat{v} \cdot \hat{S} = \cos\beta$, then



$$\vec{v} = v\sin\beta\,\hat{R} + v\cos\beta\,\hat{S}$$

$$\Rightarrow \quad \frac{dE}{dt} = \frac{\mu}{2a^2}\frac{da}{dt} = R v\sin\beta + S v\cos\beta$$

The radial component of the velocity is $\vec{v}\cdot\hat{R} = v\sin\beta = \frac{dr}{dt}$. But

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\frac{h}{r^2}$$

Using $h = \sqrt{p\mu}$ and $r = \frac{p}{1+e\cos\theta}$, this means

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{\sqrt{p\mu}}{r^2} = \sqrt{\frac{\mu}{p}}\; e\sin\theta$$

$$\Rightarrow \sin\beta = \frac{e\sin\theta}{v}\sqrt{\frac{\mu}{p}}$$

Using $\frac{v^2}{2} = \mu\left(\frac{1}{r} - \frac{1}{2a}\right)$ to eliminate $\frac{\sqrt{\mu}}{v}$ it follows that

$$\sin\beta = \frac{e\sin\theta}{\sqrt{1+2e\cos\theta+e^2}}, \qquad \cos\beta = \frac{1+e\cos\theta}{\sqrt{1+2e\cos\theta+e^2}}.$$

Thus

$$\begin{aligned}\frac{dE}{dt} &= R v\sin\beta + S v\cos\beta \\ &= \sqrt{\frac{\mu}{p}}\left(e\sin\theta\, R + (1+e\cos\theta)\, S\right)\end{aligned}$$

and

$$\begin{aligned}\frac{da}{dt} &= \frac{2a^2}{\mu}\frac{dE}{dt} \\ &= \frac{2a^2}{\sqrt{\mu p}}\left(e\sin\theta\, R + (1+e\cos\theta)\, S\right)\end{aligned}$$

Recall that

$$p = a(1-e^2)$$

which means

$$\begin{aligned}\dot{a} &= \frac{2a^2}{\sqrt{\mu a(1-e^2)}}\left(e\sin\theta\, R + (1+e\cos\theta) S\right) \\ &= \frac{2}{\sqrt{1-e^2}}\sqrt{\frac{a^3}{\mu}}\left(e\sin\theta\, R + (1+e\cos\theta) S\right) \\ &= \frac{2}{n\sqrt{1-e^2}}\left(e\sin\theta\, R + (1+e\cos\theta)\, S\right)\end{aligned}$$

This is the Gauss equation for the time evolution of a. There are analogous ones for the other 5 parameters:

$$\begin{aligned}\dot{e} &= \frac{\sqrt{1-e^2}}{na}\left(\sin\theta\, R + (\cos\psi + \cos\theta)\, S\right) \\ \dot{i} &= \frac{r}{na^2\sqrt{1-e^2}}\cos(\theta+\omega)\, W \\ \dot{\Omega} &= \frac{r}{na^2\sqrt{1-e^2}}\frac{\sin(\theta+\omega)}{\sin i}\, W \\ \dot{\omega} &= \frac{\sqrt{1-e^2}}{nae}\left[-R\cos\theta + \left(1+\frac{1}{1+e\cos\theta}\right)\sin\theta\, S - \dot{\Omega}\cos i\right] \\ \dot{M} &= n + \frac{1-e^2}{nae}\left[\left(\cos\theta - \frac{2e}{1+e\cos\theta}\right) R - \left(1+\frac{1}{1+e\cos\theta}\right)\sin\theta\, S\right]\end{aligned}$$

The Gauss equations allow us to update global properties of the orbit captured by the orbit parameters as the system evolves in time under the perturbation.

Ex: For low-orbit satellites, the air density is high enough to produce a perturbing force, which in turn tends to decrease orbit altitude. Suppose the orbit is circular and the air density ρ is constant for the entire orbit period. The perturbing force F_p due to drag is given by Stokes' Law

$$F_p = \frac{1}{2} \rho v^2 C_d S,$$

where v is the velocity, C_d is the drag coefficient, and S is the equivalent satellite surface in the direction of motion. The perturbing acceleration is $\vec{\gamma}_p = \vec{F}_p / m_s$, where m_s is the satellite mass. Then

$$\frac{\vec{v}}{|\vec{v}|} = - \frac{\vec{\gamma}_p}{|\vec{\gamma}_p|} \qquad \& \qquad \vec{v} \cdot \vec{\gamma}_p = - \frac{\rho v^3 C_d S}{2 m_s}.$$

The term $C_d S / 2m_s$ is known as the ballistic coefficient. The Gauss equation is

$$\dot{a} = - \rho \sqrt{a \mu} \; \frac{C_d S}{m_s}.$$

The rate of decrease of the semi-major axis is proportional to the air density and the geometric properties of the satellite, but inversely proportional to the ~~satellite~~ mass. To maintain a constant altitude, the satellite must provide a force to balance the perturbing force $\vec{F}_p$, with an inevitable fuel consumption.