Controlling An Inverted Pendulum

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1 Inverted pendulum on a cart

1.1 Equations of motion

There are two degrees of freedom in the system:

1. the cart, whose motion is solely in the horizontal direction and whose position is x, and
2. the pendulum, whose rotation about the fulcrum where it meets the cart is described by θ; its coordinated in the x direction is denoted $\tilde{x}$.

Our goal is to keep θ within some pre-prescribed zone around the upper vertical position ($\theta = \pi$) by moving the cart to get the pendulum's center of mass over the center of mass of the cart.

We begin the analysis by defining the forces acting on the system:

- $\boldsymbol{F}$: the input/control force applied externally on the cart
- $m\boldsymbol{g}$: weight of the pendulum
- $\boldsymbol{N}$: the normal force, directed along the length of the pendulum; its component in the positive x-direction is N_x and its component in the positive y-direction is N_y
- $\boldsymbol{f}$: drag on the cart from friction

The physical constants are:

- g: acceleration due to gravity
- m: mass of the pendulum
- M: mass of the cart
- l: half the length of the pendulum; equivalently the distance from the fulcrum to the center of mass of the pendulum

We'll model the friction as $\boldsymbol{f} = -b\dot{x}\hat{x}$. The horizontal forces acting on the cart are

$$M\ddot{x} = F - N_x - b\dot{x}, \tag{1}$$

while the horizontal forces on the pendulum are

$$m\ddot{\tilde{x}} = N_x. \tag{2}$$

The horizontal position of the center of mass of the pendulum is given by $\tilde{x} = x + l\sin(\pi - \theta)$. Therefore

$$N_x = m\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta. \tag{3}$$

We can combine the two equations to eliminate N_x

$$F = (m + M)\ddot{x} + b\dot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta. \tag{4}$$

The second governing equation can be derived easily using the Lagrangian of the pendulum. As an isolated pendulum, it is

$$L_0 = \frac{1}{2}I_{\text{pivot}}\dot{\theta}^2 - mgl\cos\theta. \tag{5}$$

The moment of inertia about the fulcrum is given by $I_{\text{pivot}} = I + ml^2$ where I is the moment of inertia about the center of mass. The acceleration of the cart causes a force on the fulcrum, which exerts a torque given by $ml\ddot{x}\cos\theta$. Combining the equation of motion coming from the Lagrangian L_0 with this torque gives

$$(I + ml^2)\ddot{\theta} + mgl\sin\theta = -ml\ddot{x}\cos\theta. \tag{6}$$

We will use a small angle approximation and write $\theta \approx \pi$ to avoid changing notation. This amounts to measuring θ from the positive vertical instead of the negative one. We'll also assume the pendulum does not swing too quickly, which is to say $\dot{\theta}^2 \approx 0$. Then the two equations of motion are

$$\begin{aligned} (I + ml^2)\ddot{\theta} - mgl\theta &= ml\ddot{x}, \\ (m + M)\ddot{x} + b\dot{x} - ml\ddot{\theta} &= F. \end{aligned} \tag{7}$$

1.2 Laplace space

Define the Laplace transforms

$$\begin{aligned} \Theta(s) &= \mathcal{L}[\theta(t)] \\ X(s) &= \mathcal{L}[x(t)], \\ u(s) &= \mathcal{L}[F(t)]. \end{aligned} \tag{8}$$

The Laplace transformed equations of motion are

$$\begin{aligned} (I + ml^2)s^2\Theta - mgl\Theta &= mls^2X, \\ (m + M)s^2X + bsX - mls^2\Theta &= u. \end{aligned} \tag{9}$$

Having done this it is straightforward to calculate the transfer functions for the pendulum and cart, $G_p(s)$ and $G_c(s)$ respectively.

By rearranging the equations of motion, we find

$$\Big[\Big((m + M)\kappa(s) - ml\Big)s^2 + bs\kappa(s)\Big]\Theta = u, \tag{10}$$

where we have defined

$$\kappa(s) = \frac{I + ml^2}{ml} - \frac{g}{s^2}. \tag{11}$$

The pendulum's transfer function is

$$G_p(s) = \frac{\Theta}{u} = \frac{mls}{\mu s^3 + b(I + ml^2)s^2 - (m + M)mgls - bgl}, \tag{12}$$

where we have defined

$$\mu = (m + M)(I + ml^2) - (ml)^2. \tag{13}$$

Rearranging in a different manner, we have

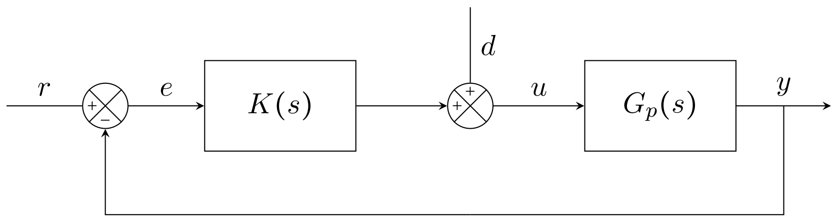
$$\left[(m + M)s^2 + bs - \left(\frac{(ml)^2 s^4}{(I + ml^2)s^2 - mgl}\right)\right] X = u, \tag{14}$$

which gives the cart's transfer function

$$G_c(s) = \frac{X}{u} = \frac{(I + ml^2)s^2 - mgl}{\mu s^4 + b(I + ml^2)s^3 - (m + M)mgls^2 - bmgls}. \tag{15}$$

2 Controller design: trial and error

Now we construct a control feedback loop for the $(\theta, \dot{\theta})$ subsystem represented by the following block diagram



The reference signal in our case is $r = 0$, since we are trying to keep $\theta = 0$ and $\dot{\theta} = 0$. The input disturbance d is used to give the system impulsive pushes (*e.g.* $d(t) = \delta(t)$). We'll choose a PID controller with transfer function

$$K(s) = K_p + \frac{K_i}{s} + K_d s. \tag{16}$$

Since $r = 0$, we find that the full transfer function for the feedback loop is

$$y = \frac{G_p}{1 + G_p K} d. \tag{17}$$

Therefore the θ part of the output is, in Laplace space,

$$y_\theta(s) = \frac{mlsd(s)}{\mu s^3 + (bml^2 + K_d ml + Ib)s^2 + (K_p - (m + M)g)mls + K_i ml - bmgl}. \tag{18}$$

The roots of the denominator are too complicated to make it worth factorizing, even though it would allow us to obtain a closed-form expression for $y(t)$. Nevertheless, let $D(s)$ be the denominator, and let z_1, z_2, and z_3 be the roots of the $D(s)/\mu$. These roots can be determined numerically in practice based on the values of the physical constants and our choices of the PID controller gains. Then

$$y_\theta(s) = \frac{ml}{\mu} \frac{sd(s)}{(s - z_1)(s - z_2)(s - z_3)}. \tag{19}$$

The general strategy for obtaining exact results for $y_\theta(t)$ will be to compute the partial fraction decomposition, yielding a sum of more manageable functions whose inverse Laplace transforms are more easily found. For simple enough functions $d(s)$, a computer algebra system (CAS) like Mathematica can compute the inverse transform on its own.

2.1 Impulse disturbance

Let us consider the real-world scenario where we manually plink the pendulum with a force of $1N$, which we model as a delta function disturbance in the time domain. The corresponding Laplace-space disturbance is $d(s) = 1$.

Assuming that $z_1 \neq z_2 \neq z_3$, taking the inverse Laplace transform gives

$$y_\theta(t) = -\frac{ml}{\mu}\left[\frac{z_1 e^{z_1 t}}{(z_2 - z_1)(z_1 - z_3)} + \frac{z_2 e^{z_2 t}}{(z_2 - z_1)(z_3 - z_2)} + \frac{z_3 e^{z_3 t}}{(z_1 - z_3)(z_3 - z_2)}\right]. \tag{20}$$

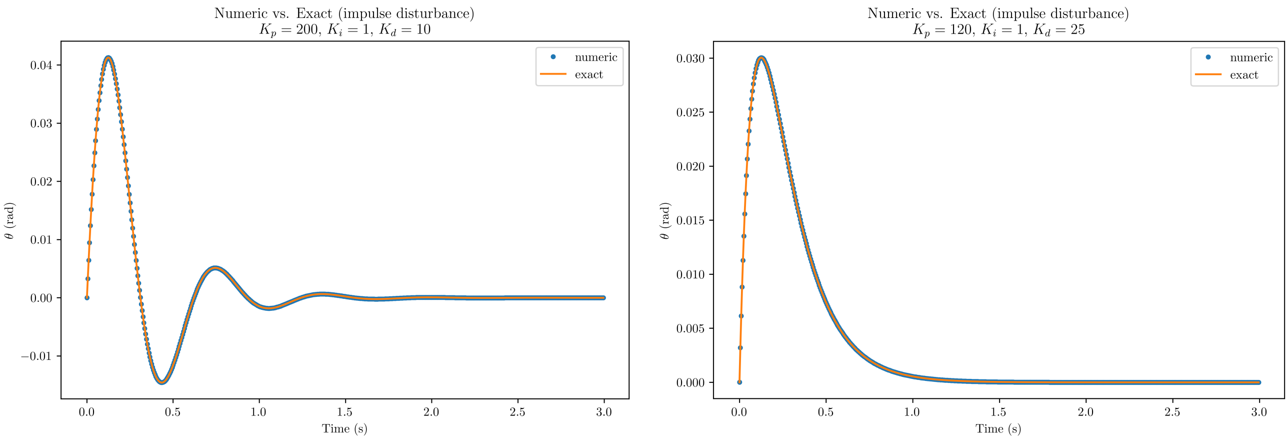
The roots z_i will in general be complex, leading to damped oscillator behavior in $\theta(t)$, as desired. There are analogous results for degenerate roots.

To visualize the response of the pendulum to the control, and to tune the controller parameters, we will move to numerics. As mentioned previously, the roots of the denominator $D(s)$ can be determined numerically once values for the physical constants and parameters are assigned specific values. Choose the following

$$m = 1, \quad M = 2, \quad l = 0.5, \quad b = 0.1, \tag{21}$$

where we've dropped standard units. The moment of inertia is $I \approx 0.08333$.

One can check easily (and quickly) that none of P, I, or D will work alone or in pairs. It turns out that setting $K_i = 1$ is a reasonable choice. Below are two attempts that use different balances between the sizes of K_p and K_d. Although we have an exact expression for the response (up to the numerically determined roots), it is worth gaining some experience with numerical inverse Laplace transforms. We compare the exact expression with a numerically determined one, obtained using the **mpmath** package in Python.



(a) Under damped response

(b) Critically damped response

Figure 1: Responses to impulse disturbance with different controller parameters

We get a reasonably good result with a large proportional gain $K_p = 200$ and a much smaller differential gain $K_d = 10$, where the pendulum swings back and forth a couple times with decreasing amplitude before settling at $\theta = 0$. We get better performance by balancing the P and D parts by decreasing K_p and increasing K_d.

2.2 Transient step disturbance

The next scenario we consider is a sudden and constant disturbance that lasts a finite length of time (perhaps an idealized gust of wind). Suppose the disturbance starts at $t = t_0$ and ends at $t = t_1$, and has unit magnitude. This is modeled in the time domain by a difference of step functions

$$d(t) = H(t - t_0) - H(t - t_1), \tag{22}$$

where H is the Heaviside function. In Laplace space this is

$$d(s) = \frac{e^{-t_0 s} - e^{-t_1 s}}{s}. \tag{23}$$

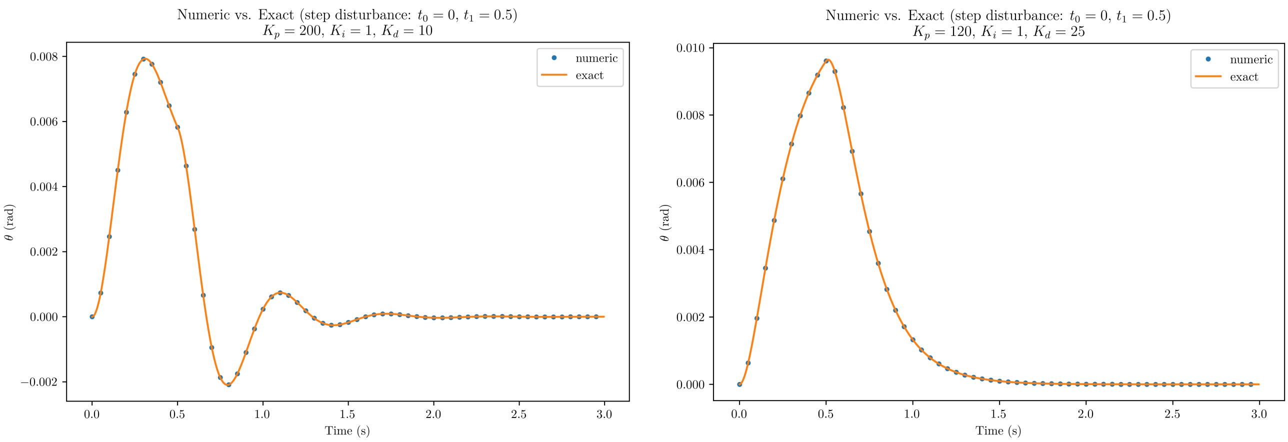
Assuming that $z_1 \neq z_2 \neq z_3$, taking the inverse Laplace transform of (19) gives

$$\begin{aligned} y_\theta(t) = \frac{ml}{\mu}\left[\frac{e^{z_1(t-t_0)}}{(z_1 - z_2)(z_1 - z_3)} + \frac{e^{z_2(t-t_0)}}{(z_2 - z_1)(z_2 - z_3)} + \frac{e^{z_3(t-t_0)}}{(z_3 - z_1)(z_3 - z_2)}\right] H(t - t_0) \\ - \frac{ml}{\mu}\left[\frac{e^{z_1(t-t_1)}}{(z_1 - z_2)(z_1 - z_3)} + \frac{e^{z_2(t-t_1)}}{(z_2 - z_1)(z_2 - z_3)} + \frac{e^{z_3(t-t_1)}}{(z_3 - z_1)(z_3 - z_2)}\right] H(t - t_1) \end{aligned} \tag{24}$$

Once again we choose the values

$$m = 1, \quad M = 2, \quad l = 0.5, \quad b = 0.1, \quad I \approx 0.08333. \tag{25}$$

Following the intuition gained from the impulse disturbance, we will presume that we need all three of P, I, and D to get good control. Two responses are shown in the fig. 2. For convenience, we use a coarser lattice for the numerical calculations.



(a) Under damped response (b) Critically damped response

Figure 2: Responses to a step disturbance (beginning at $t_0 = 0$ and ending at $t_1 = 0.5$) with different controller parameters

For comparison, we have chosen to look at the same sets of controller gains; clearly they have the same qualitative behavior as before. It is not surprising that the maximum deviation from $\theta = 0$ is larger for the impulse disturbance than for the step disturbance, as the former was modeled by an infinitely high spike.

2.3 What about the cart?

The cart's position is not being controlled, but nevertheless is impacted by applied force and is central to controlling the pendulum's position. Since the cart's position is not controlled (in fact it is not mentioned in our feedback loop diagram), its position output will differ from θ. In fact, we find

$$y_x(s) = \frac{G_c}{1 + G_p K} d, \tag{26}$$

which is nearly identical to $y_\theta(s)$. Using the controller gains $K_p = 120$, $K_i = 1$, and $K_d = 25$, we obtain the cart's position, displayed below. Evidently the cart moves with constant speed to maintain

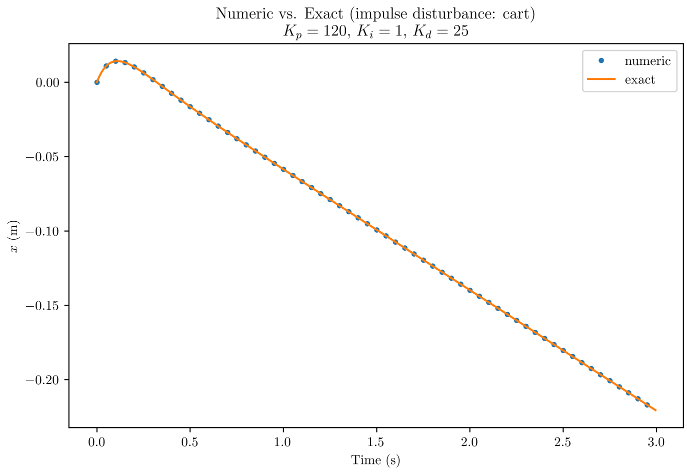


Figure 3: The cart's position while controlling the pendulum after an impulse disturbance

the pendulum's upright position. One can check that the cart behaves similarly following a step disturbance. This is not desirable behavior if we want to implement the controller in a real system; we will need to adjust the controller gains to find a balance between stable cart behavior and good control over the pendulum.

2.4 Frequency analysis

With the physical parameters set to the specific values listed above, the pendulum's transfer function $G_p(s)$ has a single zero at $s = 0$ and poles at

$$s \approx -4.4, -0.03, 4.4. \tag{27}$$

There is one pole in the right half complex plane, which indicates instability (which is, of course, not news). The Nyquist plot of the pendulum's transfer function is shown above.

In order to probe the stability of the closed loop system, we can look at the Nyquist plot of the open loop transfer function $L(s) = K(s)G_p(s)$. The goal is to change the controller gains so that the curve encloses the point $\omega = -1$ a single time going counter-clockwise. In that case, the Nyquist stability formula would read $z = n + p = -1 + 1 = 0$. Once that is accomplished, it proves advantageous to keep pushing the gains to higher values to minimize both the maximum amplitude of the pendulum

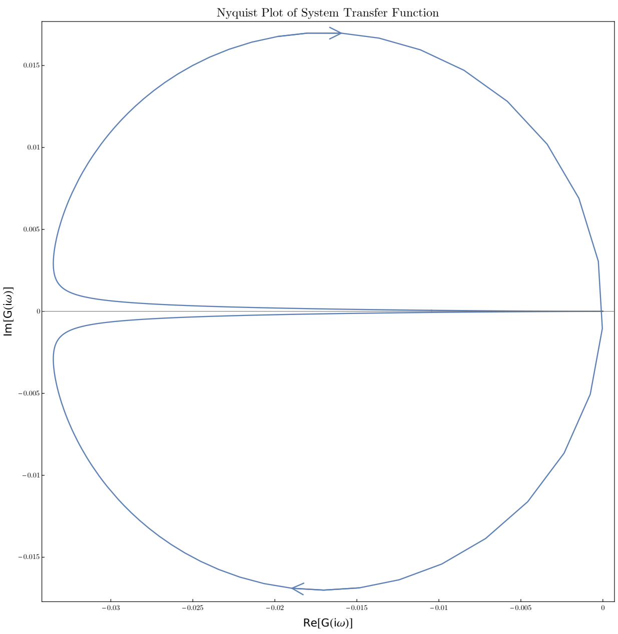


Figure 4: Nyquist plot of the pendulum's transfer function

following an impulse disturbance and the time it takes for the error to be corrected. After trial and error to adjust the Nyquist plot and the poles of the open loop transfer function, we set the gains to $K_p = 160$, $K_i = 120$, and $K_d = 40$. The Nyquist plot of the open loop transfer function is shown in fig. 5.

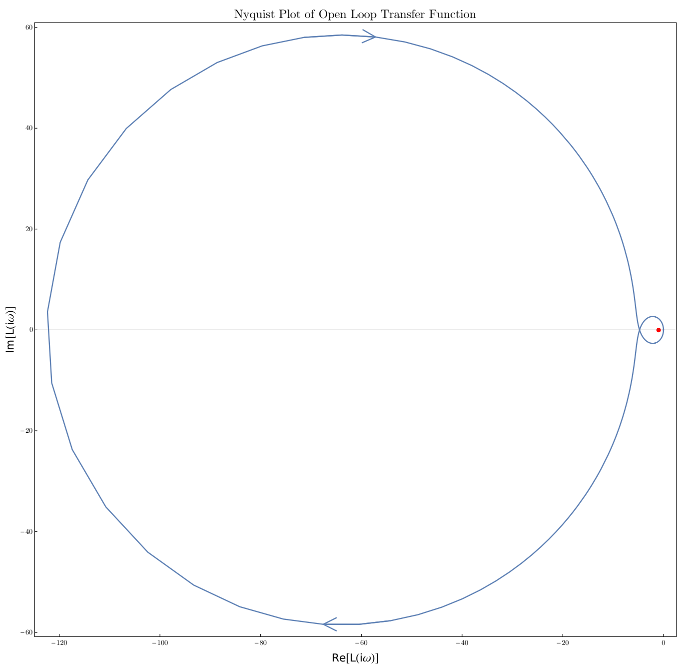
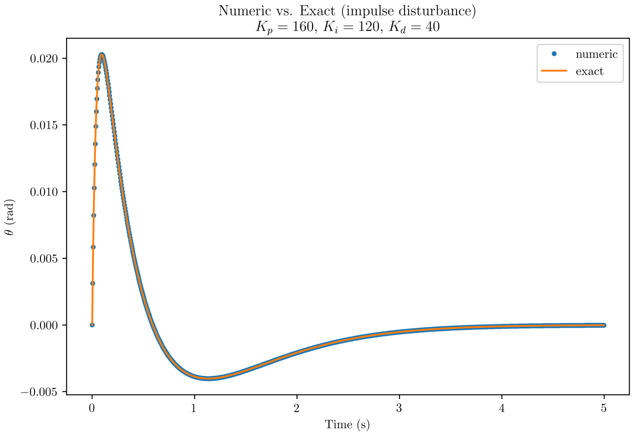


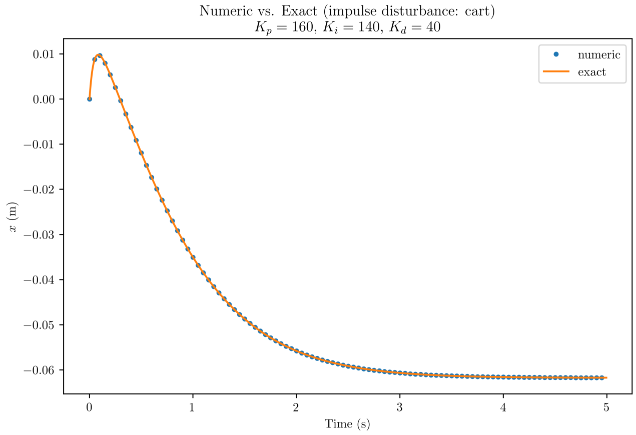
Figure 5: Nyquist plot of the open loop transfer function. The red dot indicates the point $\omega = -1$.

2.5 Revisiting the responses

With new controller gains, it is reasonable to go back and see what effect they have on the responses of the pendulum and the cart, shown in fig. 6 We can see that the pendulum now swings past $\theta = 0$



(a) Pendulum response



(b) Cart response

Figure 6: Responses to an impulse disturbance from the pendulum and the cart

from its initial disturbed position and relaxes to $\theta = 0$ in a slightly longer time than before. The cart now settles into a position around $x = -0.6$ in about the same time. We essentially get this behavior for free since we did not set out to control the carts position. One can check easily that the same behaviors persist in response to the step disturbance considered earlier.

3 Controller design: LQR and optimal control

3.1 State-space representation

The state vector of the system is

$$\xi = \begin{pmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{pmatrix}. \tag{28}$$

By isolating $\ddot{x}$ we find

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{b(I+ml^2)}{\mu} & \frac{1}{\mu}\left(\frac{ml}{I+ml^2}\right)^2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{I+ml^2}{\mu} \\ 0 \\ 0 \end{pmatrix} F. \tag{29}$$

Similarly, by isolating $\ddot{\theta}$ we get

$$\frac{d}{dt}\begin{pmatrix}0\\0\\\theta\\\dot{\theta}\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & -\frac{bml}{\mu} & \frac{(m+M)mgl}{\mu} & 0\end{pmatrix}\begin{pmatrix}x\\\dot{x}\\\theta\\\dot{\theta}\end{pmatrix} + \begin{pmatrix}0\\0\\0\\\frac{ml}{\mu}\end{pmatrix}F. \tag{30}$$

Combing the two gives the system

$$\begin{aligned}
\dot{\xi} &= A\xi + BF,\\
y &= C\xi,\\
A &= \begin{pmatrix}0 & 1 & 0 & 0\\0 & -\frac{b(I+ml^2)}{\mu} & \frac{m^2l^2g}{\mu} & 0\\0 & 0 & 0 & 1\\0 & -\frac{bml}{\mu} & \frac{(m+M)mgl}{\mu} & 0\end{pmatrix},\\
B &= \begin{pmatrix}0\\\frac{I+ml^2}{\mu}\\0\\\frac{ml}{\mu}\end{pmatrix},\\
C &= \begin{pmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\end{pmatrix}.
\end{aligned} \tag{31}$$

3.2 Linear Quadratic Regulator theory

We can take advantage of having an exact (although still an approximation in the small angle regime) model of the combined cart-pendulum system to take a more systematic approach to controller design. In particular, we define a quadratic cost functional $J[F]$ with constant symmetric cost matrices $\mathcal{Q} \in \mathbb{R}^{4\times4}$ and $\mathcal{R} \in \mathbb{R}$

$$J[F] = \frac{1}{2}\int dt\left[\xi^T\mathcal{Q}\xi + \mathcal{R}F(t)^2\right]. \tag{32}$$

To prevent runaway optimization problems (*i.e.* $J \to -\infty$) we require $\mathcal{Q} \geq 0$ and $\mathcal{R} > 0$. The equations of motion from (31) are enforced passively through the inclusion of a Lagrange multiplier $\lambda \in \mathbb{R}^4$ in an augmented cost functional

$$J'[F] = J[F] + \int dt\,\lambda^T\left(A\xi + BF - \dot{\xi}\right). \tag{33}$$

We then have an unconstrained minimization problem for J', which can be solved by using the calculus of variations. The Euler-Lagrange equations are

$$\begin{aligned}
\dot{\xi} &= A\xi + BF,\\
\dot{\lambda} &= -\mathcal{Q}\xi + A^T\lambda,\\
F &= -\mathcal{R}^{-1}B^T\lambda.
\end{aligned} \tag{34}$$

The second equation of motion implies that $\lambda \propto \xi$; define $S(t) \in \mathbb{R}^{4\times4}$ by

$$\lambda(t) = S(t)\xi(t). \tag{35}$$

The final equation of motion in (34) (which is really an algebraic constraint) then reads

$$F = -\mathcal{R}^{-1} B^T S \xi, \tag{36}$$

which takes the form of a control law $F = -K\xi$, where the gain matrix is

$$K(t) \equiv \mathcal{R}^{-1} B^T S(t). \tag{37}$$

The minimization problem for J' is solved by solving the Euler-Lagrange equations (34). However, for our purposes it suffices to solve for the matrix S. Its equation of motion is the matrix Riccati equation

$$\dot{S} = -A^T S - SA - \mathcal{Q} + SB\mathcal{R}^{-1} B^T S. \tag{38}$$

Since $\mathcal{Q}$ and $\mathcal{R}$ are each symmetric matrices, $S(t)$ is also symmetric for all t. When the problem is an infinite-horizon one (which is the case here), we can look for a steady-state solution. This amounts to solving the algebraic Riccati equation

$$-A^T S - SA - \mathcal{Q} + SB\mathcal{R}^{-1} B^T S = 0, \tag{39}$$

where we have implicitly replaced any time-dependent quantities with steady-state values. This is a system of 10 coupled quadratic equations for the independent elements of S, and while it would be prohibitively complicated to do by hand for arbitrary values of the physical and cost function parameters, it is quite fast to implement on a computer once those numbers are chosen.

Upon computing the optimal controller gain K, the new dynamics of the system is given by $A - BK$. The new (matrix) transfer function is given by

$$G_{\mathrm{LQR}}(s) = C\Big(s - (A - BK)\Big)^{-1} B, \tag{40}$$

which allows us to easily compute responses to disturbances as we did previously.

3.3 Responses

Up until this point we have resisted the urge to use a pre-designed package for computing, tuning, or plotting. However, since we have worked to a position where we could use the techniques displayed earlier, it seems fair to make the switch. Since it is open-source, we'll use the **control** package built off of **NumPy** in Python, as it is similar to the control suite in MATLAB.

Unfortunately, trial and error is still not entirely removed from the picture when employing LRQ optimal control design: we still need to manually choose the cost function parameters $\mathcal{Q}$ and $\mathcal{R}$. However, varying this parameters is more intuitive than varying, say, (K_p, K_i, K_d) in a PID controller. As we know from our previous efforts, it is actually quite easy to control the pendulum. On the other hand, we needed to change our initial controller gains because the cart wasn't being controlled at all. So, in some sense the cart is actually more difficult to control. Now, since this is a theoretical exploration of the problem, we care more about the behavior of the cart-pendulum system than we do about control effort. Therefore it makes sense to prioritize Q, and we'll set $R = 1$. Write the matrix Q as

$$Q = \begin{pmatrix} Q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Q_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{41}$$

giving us two nonnegative parameters to tune. The number Q_1 corresponds to penalizing deviations from the carts position $x = 0$, and Q_2 penalizes deviations of the pendulum's position from $\theta = 0$. Based on the preceding argument, we are going to look for values where $Q_1 > Q_2$.

In order to choose meaningful values of Q_1 and Q_2, we would need objectives based on physical constraints (like the cart only moving a certain distance, keeping the pendulum within some range of $\theta = 0$, or having both the cart and pendulum settle back to equilibrium within some amount of time). We have no such constraints, and so Q_1 and Q_2 will be chosen somewhat arbitrarily. The impulse response of the system with optimal control for the values $Q_1 = 1000$ and $Q_2 = 100$ is depicted in fig. 7

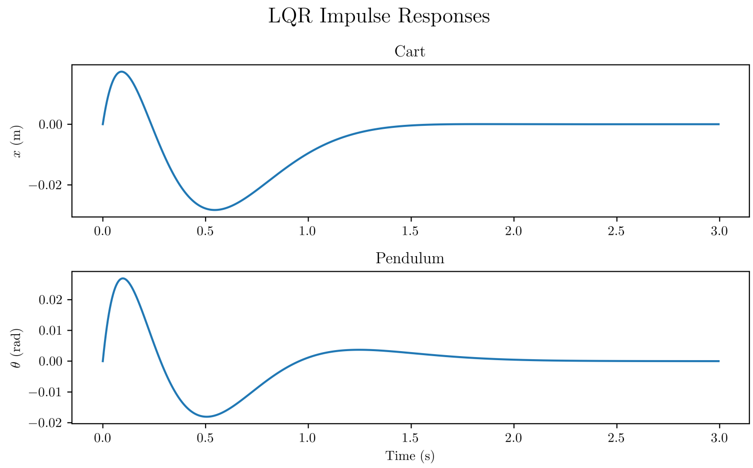


Figure 7: Impulse responses for both the cart and the pendulum. They both settle to their respective zeros in under 3 seconds.

4 Noise: Kalman filter and optimal control

Until this point we have tacitly assumed we have perfect knowledge of the state variables x and θ, and that we have perfect control over the controller. In this section we loosen those assumptions by introducing noise in the input and output of our system.

4.1 Stochastic systems

It is worth taking some time to settle notation and review the formalism of stochastic systems. A vector $x \in \mathbb{R}^n$ of real random variables is Gaussian distributed or normal if its characteristic function

ϕ has the form

$$\phi(s) = e^{i\overline{x}^T s - \frac{1}{2} s^T P s} \tag{42}$$

where $s \in \mathbb{C}^n$, $\overline{x} = \langle x \rangle$ is the mean, and $P = \left\langle (x - \overline{x})(x - \overline{x})^T \right\rangle$ is the covariance matrix. For such a random vector we write $x = \mathcal{N}(\overline{x}, P)$.

The characteristic function is the Fourier transform of the joint probability density ρ, and when P is nonsingular the inverse Fourier transform can be done in closed form to give

$$\rho(x) = \left[(2\pi)^n \det(P) \right]^{-1/2} e^{-\frac{1}{2}(x-\overline{x})^T P^{-1} (x - \overline{x})}. \tag{43}$$

When the variables in x are independent, the covariance matrix is diagonal: $P_{ij} = \sigma_i^2 \delta_{ij}$ (no sum).

A stochastic or random process is a set of random vectors $\{x(t), t \in \mathcal{I}\}$, for some indexing set $\mathcal{I}$. Such a stochastic process is Gaussian if all (countable) subsets of the vectors are jointly Gaussian (their joint probability distribution takes the form given above); the process is called white if the vectors are all independent (that is, no correlation in time).

Stochastic processes allow for the generalization of the statistical quantities defined above to have time-dependence. For example, the mean function is simply $\overline{x}(t) = \langle x(t) \rangle$. The covariance kernel is defined as $P(t, \tau) = \left\langle \Big(x(t) - \overline{x}(t) \Big) \Big(x(\tau) - \overline{x}(\tau) \Big)^T \right\rangle$. For a Gaussian white process, the covariance kernel satisfies

$$P(t, \tau) = R(t) \delta(t - \tau), \tag{44}$$

for some matrix function R.

The introduction of noise into a system described by the equations of motion (31) turns the state vector and outputs into random vectors. The new noisy equations of motion are written

$$\begin{aligned} \dot{\xi}(t) &= A(t)\xi(t) + B(t)u(t) + w(t), \\ y(t) &= C(t)\xi(t) + v(t), \end{aligned} \tag{45}$$

where $w(t)$ and $v(t)$ are Gaussian random processes. A common set of assumptions about the noisy system are:

1. The initial condition $\xi(t_0)$ is Gaussian, $\xi(t_0) = \mathcal{N}(\overline{\xi}(t_0), P(t_0))$.

2. The state disturbance $w(t)$ and the measurement noise $v(t)$ are each zero-mean Gaussian white processes that are independent of the initial condition $\xi(t_0)$. Their respective covariance kernels are determined by the matrix functions $R_w(t)$ and $R_v(t)$.

3. The processes $w(t)$ and $v(t)$ are uncorrelated, $\left\langle w(t) v^T(\tau) \right\rangle = 0$.

A powerful outcome of these assumptions is that the state vector $\xi(t)$ is a Gaussian process whose covariance matrix is determined by the Lyapunov equation

$$\dot{P}(t) = A(t)P(t) + P(t)A^T(t) + R_w(t). \tag{46}$$

4.2 State control and observers

A common strategy used to compensate for a lack of real-time knowledge of the system (*e.g.* not measuring every state variable) is to introduce a fictitious "observer" system that estimates the state vector using the past history of the outputs and inputs. Introduce the auxiliary (observer) state vector $\hat{\xi}(t)$ whose equation of motion is

$$\dot{\hat{\xi}}(t) = A(t)\hat{\xi}(t) + B(t)u(t) - L(t)\Big(C(t)\hat{\xi}(t) - y(t)\Big). \tag{47}$$

The first two terms on the right hand side reproduce the dynamics of the actual system, while the feedback term corrects differences between the actual and observer system arising from, for example, different initial conditions of the two.

When coupling an auxiliary system to a noisy system, the observer is often referred to as a filter. The equation of motion (47) has an undetermined gain $L(t)$, which can be fixed using the techniques of optimal control from the preceding section. A filter determined this way through optimality is called a Kalman filter, and the optimal gain is given by

$$L(t) = P(t)C^T(t)R_v^{-1}(t), \tag{48}$$

where P is the covariance matrix of the actual system. To take into account measurement noise, the Lyapunov equation for $P(t)$ is augmented to a differential matrix Riccati equation

$$\dot{P}(t) = A(t)P(t) + P(t)A^T(t) + R_w(t) - P(t)C^T(t)R_v^{-1}(t)C(t)P(t). \tag{49}$$

Notice that the optimal filter gain L is structurally very similar to the optimal controller gain from LQR: whereas before the gain was determined by properties of the input (the matrix B and the proportionality between the state vector and the Lagrange multiplier), here the gain is determined by properties of the output (the matrix C and the covariance of the output noise).

4.3 Noise and LQR

The task of constructing a Kalman filter and an optimal controller via LQR may be referred to as a Linear Quadratic Gaussian (LQG) problem. Since now the state vector and input are affected by a Gaussian random process (and are thus themselves random processes), the LQR cost function must be amended slightly to

$$J = \int dt\, \left\langle \xi^T \mathcal{Q} \xi + \mathcal{R} F^2 \right\rangle. \tag{50}$$

In this case, the separation principle allows us to determine the LQR gain K and the Kalman filter gain L separately. Choosing to find steady-state solutions, we have

$$\begin{aligned} K &= \mathcal{R} B^T S, \\ L &= P C^T R_v^{-1}, \\ 0 &= -A^T S - SA - \mathcal{Q} + SB\mathcal{R}^{-1}B^T S, \\ 0 &= AP + PA^T + R_w - PC^T R_v^{-1} CP. \end{aligned} \tag{51}$$

The optimal controller and filter gains are then passed to the feedback equations of motion

$$\begin{aligned}\dot{\xi} &= (A - BK)\xi + Bd,\\ y &= C\xi,\\ \dot{\hat{\xi}} &= (A - BK - LC)\hat{\xi} + Ly,\\ \hat{y} &= C\hat{\xi}.\end{aligned} \tag{52}$$

Analyzing this as a closed loop feedback system in Laplace space, we find the matrix transfer function

$$\mathcal{G}(s) = -K\Big(s - A + BK + LC\Big)^{-1} L. \tag{53}$$

A peculiar feature of this model is that this transfer function actually maps the output y to the control effort, which depends on the estimation of the state vector $\hat{\xi}$

$$u(s) = \mathcal{G}(s)y(s). \tag{54}$$

However, any confusion can be cleared up by recalling the physical definitions of these quantities. The output y represents the measurements done by sensors built into the system. State estimation is done by using the history of the input and the output to approximate the "true" state of the system.

Notice that if $y \in \mathbb{R}^p$ and $u \in \mathbb{R}^m$, then $\mathcal{G} \in \mathbb{R}^{m\times p}$.

4.4 Responses

We wish to simulate the response of our noisy system to (at least) an impulse disturbance as we have done in the preceding sections. An important thing to note when including the disturbance in the model is that the disturbance affects the real system, described by the state vector ξ. Therefore the disturbance does not directly enter the equation of motion for $\hat{\xi}$. The closed loop transfer function from the disturbance $d(s)$ to the output y is

$$y(s) = \frac{G_p(s)}{1 + G_p(s)\mathcal{G}(s)} d(s). \tag{55}$$

The transfer function

$$\mathcal{T}(s) = \frac{G_p(s)}{1 + G_p(s)\mathcal{G}(s)} \tag{56}$$

enjoys a pleasant simplification afforded to us by the dimensions of the problem. Notice that $G_p \in \mathbb{R}^{2\times1}$ and $\mathcal{G} \in \mathbb{R}^{1\times2}$. The matrix $1 + G_p\mathcal{G} \in \mathbb{R}^{2\times2}$ is simple to invert using

$$\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad\Longrightarrow\quad \mathcal{M}^{-1} = \frac{1}{\det(\mathcal{M})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{57}$$

Writing

$$G_p = \begin{pmatrix} G_{11} \\ G_{21} \end{pmatrix}, \quad \mathcal{G} = \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{12} \end{pmatrix}, \tag{58}$$

we find that

$$\mathcal{T} = \frac{G}{1 + G_{11}\mathcal{G}_{11} + G_{21}\mathcal{G}_{12}}. \tag{59}$$

This will allow us to avoid several annoyances:

- lengthy/costly symbolic computations of the matrix inverse,
- rounding/numerical errors from numerical computations of the matrix inverse,
- the **control** package's built in LQG makes computing impulse response inconvenient.

In order to actually simulate a response, we need to choose values for the covariances. Conceptually, it is easier to consider the square root of the variances, which quantify the uncertainty in the respective variables. The input represents the force applied to the cart, which when combined with the pendulum has a weight of about 30 N. The exact method by which the force is applied has not been considered[1] An uncertainty of 1N is a conservative choice, and we'll set $R_w = 1$. The uncertainties in the values of v represent errors in the measurements, and ought to be considered more carefully. We are measuring the cart's position in meters and the pendulums tilt in radians. In the previous section we saw cart displacement on the order of 1 cm and pendulum displacements on the order of 0.01 rad. Allowing even a 10% in either measurement is probably conservative. Therefore we'll choose $R_v = (0.01)^2\mathbb{I}_2$.

We will also be more deliberate about the cost function parameters. To choose the value of $\mathcal{Q}$, which penalizes deviations of the state vector from the reference point $\xi = 0$, we will separately try to enforce the behaviors

$$\begin{aligned} \dot{x} + 2\beta x &= 0, \\ \dot{\theta} + 2\alpha\theta &= 0, \end{aligned} \tag{60}$$

and $|\xi^i| \le \xi^i_{\text{max}}$ for some maximum values ξ_{max}. The parameters α and β influence the time constants of x and θ. The state penalty matrix is

$$\mathcal{Q} = \begin{pmatrix} \frac{\beta^2}{x_{\text{max}}^2} & \frac{\beta}{x_{\text{max}}\dot{x}_{\text{max}}} & 0 & 0 \\ \frac{\beta}{x_{\text{max}}\dot{x}_{\text{max}}} & \frac{1}{\dot{x}_{\text{max}}^2} & 0 & 0 \\ 0 & 0 & \frac{\alpha^2}{\theta_{\text{max}}^2} & \frac{\alpha}{\theta_{\text{max}}\dot{\theta}_{\text{max}}} \\ 0 & 0 & \frac{\alpha}{\theta_{\text{max}}\dot{\theta}_{\text{max}}} & \frac{1}{\dot{\theta}_{\text{max}}^2} \end{pmatrix}. \tag{61}$$

We can choose the maximum desired values of x and θ from the LQR results in the previous section. As for the velocities, some experimenting shows that $\dot{x}_{\text{max}}$ does not have a particularly big effect on the performance. On the other hand, $\dot{\theta}_{\text{max}}$ does, and we should choose it based on physical considerations. Recall that the model of the cart-pendulum system is predicated on the small angle and small angular velocity assumptions. We therefore make the choice

$$\begin{pmatrix} x_{\text{max}} \\ \dot{x}_{\text{max}} \\ \theta_{\text{max}} \\ \dot{\theta}_{\text{max}} \end{pmatrix} = \begin{pmatrix} 0.05 \\ 0.5 \\ 0.02 \\ 0.01 \end{pmatrix}. \tag{62}$$

Some experimenting indicates that $\alpha = 10$ and $\beta = 100$ give good results.

[1] What I have in mind is a motorized cart that moves under the pendulum, which would mean the applied force comes from acceleration caused by applying torque to the wheels with motors. Taking this point of view would take us into navigation territory, which could be interesting.

The cost parameter penalizing control effort can be determined physically by asking that the applied force not exceed a certain value. However, the relative magnitudes of $\mathcal{Q}$ and $\mathcal{R}$ are what matter. The specific value of $\mathcal{R}$ does not have a big impact in this scenario, and decreasing $\mathcal{R}$ drives the gains up while not changing the impulse response. Therefore we will still choose $\mathcal{R} = 1$.

With the physical parameters determined as before, the LQR gain and Kalman filter gain are

$$
\begin{aligned}
K &\approx \begin{pmatrix} -100 & -105 & 569 & 177 \end{pmatrix}, \\
L &\approx \begin{pmatrix} 16 & 16 \\ 257 & 364 \\ 16 & 30 \\ 374 & 570 \end{pmatrix}.
\end{aligned}
\tag{63}
$$

More precise values can be found in the accompanying Jupyter notebook. The impulse response is shown in fig. 8.

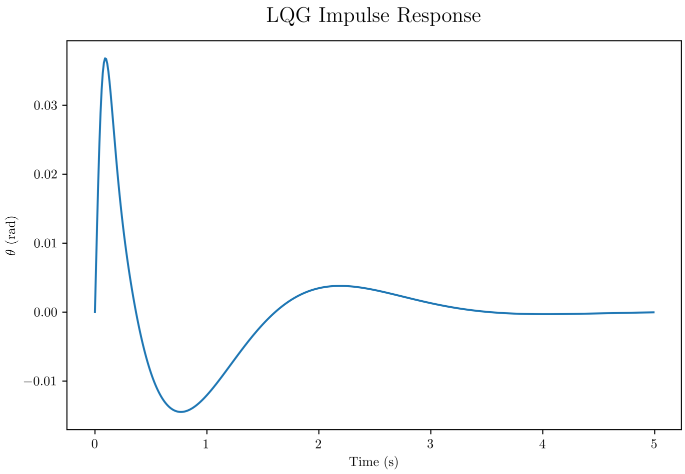


Figure 8: Impulse response for the pendulum. It settles to $\theta = 0$ in under 5 seconds.

5 Conclusion

In this project we went through the process of determining the equations of motion of an interesting physical system, designing feedback loops to control the system, and tuning the controllers through simulation and frequency analysis.

The first controller we explored was PID. We looked at different values of the gains (K_p, K_i, K_d) determined both through trial and error and heuristically from Nyquist plots. It was possible to control the pendulum through the trial-and-error efforts, but it required the cart to keep moving with a constant speed, thus making it physically impractical. It was a pleasant surprise to find that the cart's position could be controlled by choosing different values of the gains with the graphical method.

The second controller we explored was found through optimal control. We chose a quadratic cost function and assumed steady-state values for the gain. The parameters of the cost function we picked somewhat arbitrarily through trial and error, but nevertheless produced good results for the control of the pendulum and cart's positions.

Finally, we gave up the implicit assumption of noiselessness and implemented a Kalman filter. The parameters determining the noise and the cost function were chosen more carefully based on physical considerations[2] Ultimately, we obtained a controller-filter combination that yielded good results.

6 References

Apart from the documentation for the Python packages used here (**sympy**, **mpmath**, and **control**), four sources provided guidance throughout this project. Instead of including a formal bibliography here, I am simply going to list them now:

1. The textbook "Control Theory for Physicists" by Bechhoefer

2. The textbook "Fundamentals of Aerospace Navigation and Guidance" by Kabamba and Girard

3. The webpage "Control Tutorials for Matlab and Simulink" hosted by University of Michigan, found here

4. These lecture notes from LTH

[2]The impetus to switch to this was the unfortunately reality that the noisy system was much harder to control with (somewhat) arbitrarily chosen parameters.