Kepler, etc.

Consider 2 body problem with (stationary) central mass M and (mobile) orbiting mass m. Let r be the distance between them and θ the angular position of the mass m. Then the EOM can be put in the form

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2},$$

where $\mu \equiv GM$ and h is the (constant, conserved) angular momentum of m. The coordinate $u = \frac{1}{r}$. The solution is

$$u = \frac{\mu}{h^2} + c\cos(\theta - \theta_0).$$

The integration constant c can be found from conservation of energy per unit mass

$$E = \frac{v^2}{2} - \frac{\mu}{r}.$$

We know that $\vec{v} = (\dot{x}, \dot{y})$ has squared length

$$v^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right].$$

Using $\frac{du}{d\theta} = -c\sin(\theta - \theta_0)$ we have

$$v^2 = \left[c^2\sin^2(\theta - \theta_0) + c^2\cos^2(\theta - \theta_0) + 2\frac{\mu}{h^2}c\cos(\theta - \theta_0)\right]h^2$$
$$= h^2\left[c^2 + \frac{2\mu}{h^2}c\cos(\theta - \theta_0)\right]$$

and

$$E = \frac{1}{2}v^2 - \mu u$$
$$= \frac{1}{2}h^2c^2 - \frac{1}{2}\frac{\mu^2}{h^2}$$
$$\Rightarrow c = \frac{\mu}{h^2}\sqrt{1 + 2E\frac{h^2}{\mu^2}}.$$

Define the eccentricity

$$e = \sqrt{1 + 2Eh^2/\mu^2}$$

so that

$$E = (e^2 - 1)\frac{\mu^2}{2h^2}.$$

The relationship between r and θ is found to be

$$r = \frac{h^2/\mu}{1 + e\cos(\theta - \theta_0)} = \frac{p}{1 + e\cos(\theta - \theta_0)}.$$

The constant p is called the semi-latus rectum or the parameter.

The solutions to the EOM are conic sections which are distinguished based on the value of e. The possibilities are

(i) <u>Circular orbit</u>: $e = 0 \Rightarrow r = p$, $v^2 = \mu/r$, $E = -\mu^2/2h^2 < 0$

(ii) <u>Elliptic orbit</u>: $0 < e < 1 \Rightarrow E = (e^2 - 1)\mu^2/2h^2 < 0$. The point on the ellipse at $\theta = 0$ is called the periapsis, and the radius vector from the prime focus F of the ellipse to that point is minimal. This distance is $r_p = \frac{p}{1+e}$. For orbits around Earth, $\theta = 0$ is called the perigee; for orbits around the sun it is the perihelion. The point at $\theta = \pi$ is called the apoapsis (apogee, apohelion) which has maximal distance from F with distance $r_a = \frac{p}{1-e}$. Then

$$\frac{r_a}{r_p} = \frac{1-e}{1+e} \Rightarrow e = \frac{r_a - r_p}{r_a + r_p}.$$

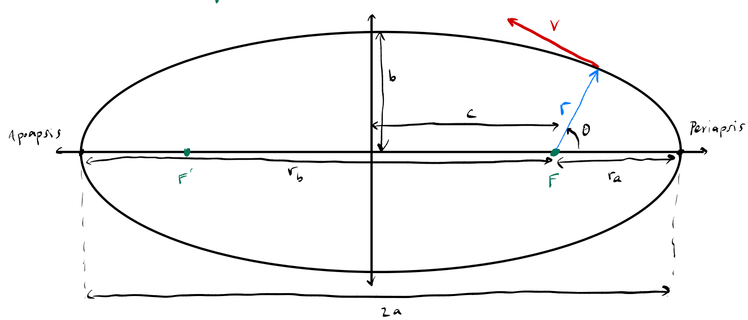
The length of the major axis of the ellipse is

$$2a = r_a + r_p = \frac{2p}{1-e^2} \Rightarrow p = a(1-e^2) = h^2/\mu.$$

The length a is called the semi-major axis. The total energy per unit mass is

$$E = -\frac{\mu}{2a}$$

which is called the energy constant.



The distance from F to the origin, c, is given by $c = ae$. The semi-minor axis b is $b = a\sqrt{1-e^2}$.

(iii) <u>Parabolic orbits</u>: These are not practically useful but have $E = 0$ and are on the cusp between bound and scattering states. They have $e = 1$, so

$$r = \frac{p}{1+\cos\theta}, \quad r_p = \frac{p}{2}$$

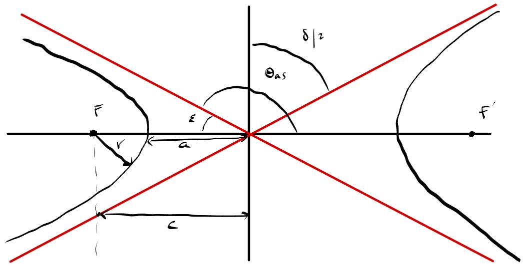
This also means $a \to \infty$ and $v^2 = 2\mu/r$.

(iv) <u>Hyperbolic orbits</u>: The energy is $E > 0$ so these are scattering states. They have $e > 1$ and $a = -\mu/2E < 0$. To make sense of this we can define $p = a(e^2 - 1)$ so that

$$r = \frac{a(e^2-1)}{1+e\cos\theta}$$

As $r \to \infty$ we want the denominator $\to 0$, giving the asymptote condition

$$\cos(\theta_\infty) = \cos(\theta_{as}) = -1/e$$



From the geometry we have

$$\varepsilon = \pi - \theta_{as} = \frac{\pi}{2} - \frac{\delta}{2}$$

$$\Rightarrow \quad \theta_{as} = \frac{\pi}{2} + \frac{\delta}{2}$$

$$\Rightarrow \quad \cos\left(\frac{\pi}{2} + \frac{\delta}{2}\right) = -\frac{1}{e}$$

$$\Rightarrow \quad \sin\left(\frac{\delta}{2}\right) = \frac{1}{e}$$

These orbits are useful for transplanetary spacecraft voyages. Consider the behavior as $r \to \infty$. The angular momentum can be written $h = V_\infty \Delta$. The total energy is

$$E = \frac{V_\infty^2}{2} - \frac{\mu}{r_\infty} = -\frac{\mu}{2a}$$

$$\Rightarrow \quad a = -\frac{\mu}{V_\infty^2} \quad \text{as } r_\infty \to \infty.$$

But $\quad p = a(e^2 - 1)$

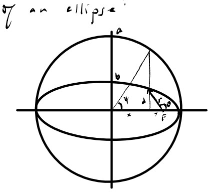
$$\Rightarrow \quad p = \frac{\mu}{V_\infty^2}(e^2 - 1) = \frac{h^2}{\mu} = \frac{V_\infty^2 \Delta^2}{\mu}$$

$$\Rightarrow \quad V_\infty \Delta = \frac{\mu}{V_\infty}\sqrt{e^2 - 1}$$

$$\& \quad e^2 = 1 + \frac{V_\infty^4 \Delta^2}{\mu^2}.$$

If Δ and V_∞ are known, we can find a, e, δ. As a spacecraft navigates toward a far planet with V_∞, we can use the direction of the motion to determine h (and therefore Δ). This allows us to determine the parameters of the orbit the craft will follow at the distant planet.

We can use Kepler's laws to locate a body in orbit either from its angular deviation from the major axis or from the time elapsed since passing the perigee. We use the following diagram to define the true and eccentric anomalies of an ellipse:



The true anomaly θ is defined as the angle between the major axis and the vector locating the body with respect to the primary focus F. To define the eccentric anomoly, draw an auxiliary circle of radius a (semi-major axis) concentric with the ellipse. The eccentric anomoly is the angle ψ obtained by extending a line perpendicular to the major axis through the location of the body and measuring the angular position of the line's intersection with the circle. Then

$$x+y = ae = c$$

$$x = a\cos\psi$$

$$y = -r\cos\theta$$

$$\Rightarrow a\cos\psi - r\cos\theta = ae$$

Using the equation of the ellipse:

$$a\cos\psi = ae + \frac{a(1-e^2)\cos\theta}{1+e\cos\theta} = \frac{ae + a\cos\theta}{1+e\cos\theta}$$

$$\Rightarrow \cos\psi = \frac{e+\cos\theta}{1+e\cos\theta}$$

$$\Rightarrow \sin\psi = \frac{\sin\theta\sqrt{1-e^2}}{1+e\cos\theta}$$

or inverting

$$\cos\theta = \frac{\cos\psi - e}{1-e\cos\psi}, \quad \sin\theta = \frac{\sin\psi\sqrt{1-e^2}}{1-e\cos\psi}.$$

Notice that we can also write

$$r = a\left(1-e\cos\psi\right).$$

Kepler's area law says

$$\frac{dA}{dt} = \frac{h}{2}$$

where A is the area swept out in time t. Therefore $A = \frac{1}{2}ht$. Because the area of an ellipse is πab, if the period of the orbit is T then

$$T = \frac{2A}{h} = 2\pi\sqrt{\frac{a^3}{\mu}} = \frac{2\pi}{n}$$

This is Kepler's 3rd law. We have defined the mean motion n. Define the mean anomoly $M = n(t-t_p)$, where t_p is a time where the body is at the perigee. We can compare this to the true anomaly θ to find t_p.

Knowing θ allows us to find ψ. Knowing ψ, the time elapsed from the perogee can be computed. Defining $t_m = t - t_p$ we find that

$$t_M \frac{2\pi}{T} = \psi - e\sin\psi$$

For earth-orbiting spacecraft, it is common to define an inertial coordinate system with the origin at the CoM of the earth and whose direction in space is fixed relative to the solar system. The earth moves in a nearly-circular orbit around the sun with a long period, so its motion is practically unaccelerated for our purposes.

The z-axis is the axis of rotation of the earth. The xy-plane is the equatorial plane. The axis of rotation is (famously) inclined with respect to the ecliptic plane (around the sun). The equatorial and ecliptic planes intersect along a line which is quasi-inertial in space, called the vernal equinox vector (or direction). We choose this to be the x-axis; the y-axis is chosen to complete the right-handed orthogonal system.

Both the equatorial and ecliptic planes move slowly with respect to the true celestial inertial coordinate system, centered in the CoM of the solar system. The planets affect the orientation of the ecliptic plane in the slow rotational motion of planetary precession. As the z-axis precesses, so does the equatorial plane. Thus the geocentric coordinate system moves slowly relative to the stars, and it is necessary to define the system with reference to a certain date.

There are now 3 more parameters necessary to describe an orbit in space. An attempt at a diagram is made: