Gravity Gradient Stabilization

We now turn to the issue of controlling the attitude of a spacecraft. The following are some examples of primary control tasks for which the attitude control system is responsible:

(1) In orbital maneuvering and adjustments, the attitude of the satellite must be pointed and held in the desired ΔV direction

(2) A spin-stabilized satellite may be designed to keep the spin axis of its body pointed at some particular direction in space

(3) A nadir-pointing three-axis-stabilized satellite must keep its three Euler angles close to null relative to the orbit reference frame (true for most communication satellites)

(4) In earth-surveying satellites, the attitude control system is designed to allow the operative payload to track defined targets on the earth's surface.

(5) A scientific satellite observing the sky must maneuver its optical instruments toward different star targets in the celestial sphere in some prescribed pattern of angular motion.

An important distinction for attitude control is between passive and active attitude control. The former is attractive because the hardware required is less complicated and relatively inexpensive. Natural physical properties of the satellite and environment are used to control the spacecraft. However, the achievable accuracies with passive control are generally much lower than can be obtained with active control, which requires more sophisticated (and expensive) control instrumentation.

Generally speaking, the satellite attitude dynamics equations are three 2^{nd}-order nonlinear equations. Automatic control theory does not provide exact analytical solutions and design procedures for such dynamic plants, so linearization of the equations is necessary. We already saw linearization for small Euler angles. The control torque equations for a PID controller are written

$$\tau_{c_i} = K_{p_i}\,(\text{error}) + K_{d_i}\,\frac{d}{dt}(\text{error}) + K_{i_i}\int(\text{error})\,dt$$

where (error) represents the error of the attitude compared to the desired configuration.

We'll start by deriving the linearized angular EOM and stability conditions for purely passive gravity gradient (GG) attitude control. Since the system is passively controlled, $\tau_c = 0$ and $h_w = 0$. Therefore the EOM is

$$\tau_{dx} = I_x \ddot{\phi} + 4\omega_o^2 (I_y - I_z)\phi - \omega_o (I_x + I_z - I_y)\dot{\psi}$$

$$\tau_{dz} = I_z \ddot{\psi} + \omega_o^2 (I_y - I_x)\psi + \omega_o (I_z + I_x - I_y)\dot{\phi}$$

$$\tau_{dy} = I_y \ddot{\theta} + 3\omega_o^2 (I_x - I_z)\theta$$

The angular EOM can only be activated by disturbing torques and initial angles of the Euler angles and their derivatives. Define

$$\sigma_x = (I_y - I_z)/I_x$$

$$\sigma_y = (I_x - I_z)/I_y$$

$$\sigma_z = (I_y - I_x)/I_z$$

By Laplace transforming the y equation, the characteristic equation for the motion about the Y_B axis is

$$s^2 + 3\omega_o^2 \sigma_y I_y = 0$$

This has one unstable root if $I_x < I_z$, so the condition for stability becomes

$$I_x > I_z.$$

There is no damping factor in this 2nd order equation, so it follows that for any initial condition or nonzero disturbance τ_{dy}, the satellite will oscillate in a stable motion about the Y_B axis with an amplitude proportional to the initial condition $\theta(0)$ and the level of the disturbance τ_{dy}.

The x & z equations are coupled (the y one is independent). We have

$$\ddot{\phi} + 4\omega_o^2 \sigma_x \phi - \omega_o (1-\sigma_x)\dot{\psi} = \frac{\tau_{dx}}{I_x}$$

$$\ddot{\psi} + \omega_o^2 \sigma_z \psi + \omega_o (1-\sigma_z)\dot{\phi} = \frac{\tau_{dz}}{I_z}$$

Keep in mind that the values of σ_x, σ_z are limited and that they are smaller than unity. To show this:

$$\sigma_x = \frac{I_x - I_z}{I_x} = \frac{\int (x^2+z^2)\,dm - \int (x^2+y^2)\,dm}{\int (y^2+z^2)\,dm} = \frac{\int (z^2-y^2)\,dm}{\int (z^2+y^2)\,dm} < 1$$

A similar argument holds for the others. Taking the Laplace transforms

$$s^2\phi + 4\omega_o^2 \sigma_x \phi - \omega_o (1-\sigma_x) s\psi = \frac{\tau_{dx}}{I_x}$$

$$s^2\psi + \omega_o^2 \sigma_z \psi + \omega_o (1-\sigma_z) s\phi = \frac{\tau_{dz}}{I_z}$$

$$\Rightarrow \quad (s^2 + 4\omega_o^2\sigma_x)\phi - \omega_o(1-\sigma_x)s\psi = 0$$

$$(s^2 + \omega_o^2\sigma_z)\psi + \omega_o(1-\sigma_z)s\phi = 0$$

$$\Rightarrow \quad \begin{pmatrix} s^2 + 4\omega_o^2\sigma_x & -\omega_o(1-\sigma_x)s \\ \omega_o(1-\sigma_z)s & s^2 + \omega_o^2\sigma_z \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0$$

Taking the determinant:

$$\det(\cdots) = (s^2 + 4\omega_0^2\sigma_x)(s^2 + \omega_0^2\sigma_z) + \omega_0^2 s^2 (1-\sigma_x)(1-\sigma_z)$$
$$= s^4 + \left[4\omega_0^2\sigma_x + \omega_0^2\sigma_z + \omega_0^2(1-\sigma_x)(1-\sigma_z)\right]s^2 + 4\omega_0^2\sigma_x\sigma_z$$
$$= s^4 + \omega_0^2\left[3\sigma_x + \sigma_x\sigma_z + 1\right]s^2 + 4\omega_0^2\sigma_x\sigma_z$$

The characteristic equation is found by setting this equal to 0. Solving for s^2:

$$\frac{s^2}{\omega_0^2} = \frac{1}{2}\left[-(3\sigma_x + \sigma_x\sigma_z + 1) \pm \sqrt{(3\sigma_x + \sigma_x\sigma_z + 1)^2 - 16\sigma_x\sigma_z}\right]$$

If s_1 is a root, then so is $-s_1$. For s_1 to be a root with no positive real part, it is necessary that s_1 be imaginary $(s^2 < 0)$ and also that the discriminant be positive. This gives us 3 conditions:

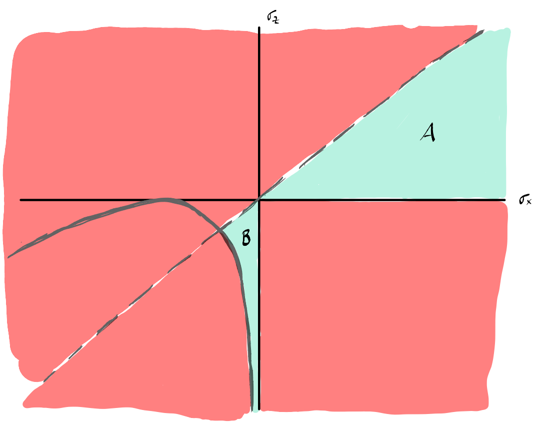
$$3\sigma_x + \sigma_x\sigma_z + 1 > 4\sqrt{\sigma_x\sigma_z}$$
$$\sigma_x\sigma_z > 0$$
$$3\sigma_x + \sigma_x\sigma_y + 1 > 0$$

Remember also that $I_y < I_x + I_z$. These inequalities can be translated to the $\sigma_x - \sigma_z$ plane.

To find the regions note that

$$(I_x - I_z)\, I_y < I_x^2 - I_z^2$$
$$\Rightarrow I_z(I_y - I_z) > I_x\,(I_y - I_x)$$
$$\Rightarrow \sigma_x > \sigma_z$$

The inequality $\sigma_x\sigma_z > 0$



The areas in red are unstable, and the ones in green are stable. The region in quadrant III is rarely used owing to structural difficulties. The region A is normally used in practical designs of GG-stabilized spacecraft. This region translates to $I_y > I_x > I_z$

To find the attitude time response of a passive GG-stabilized satellite, we use the linearized EoM and Laplace transform. Start with θ. We have

$$\theta(s) = \frac{\tau_{dy}}{I_y\, s\,(s^2 + \omega_0^2 \sigma_x)} + \frac{s\theta(0) + \dot{\theta}(0)}{s^2 + 3\omega_0^2 \sigma_y}$$

The value of σ_y depends on both I_x and I_y:

(i) For $I_x < I_z$, $\sigma_y < 0$ and one of the eigenvalues will be unstable. The pitch angle will diverge exponentially with time

(ii) For $I_x = I_z$, which is the neutral case of stability, $\sigma_y = 0$ and the time response is

$$\theta(t) = \theta(0) + \dot{\theta}(0)\,t + \tfrac{1}{2}\left(\tau_{dy}/I_y\right)t^2$$

(iii) For $I_x > I_z$, $\sigma_y > 0$ and an oscillatory motion is to be expected for an external disturbance τ_{dy}:

$$\theta(t) = \frac{\tau_{dy}}{3 I_y \omega_0^2}\left[1 - \cos\left(\sqrt{3\sigma_y}\,\omega_0 t\right)\right]$$

The time behavior is biased harmonic motion, with a constant average level amplitude of $\tau_{dy} / 3\omega_0^2 (I_x - I_z)$. The frequency depends on the relative values of the moments of inertia as well as the orbital frequency ω_0. The amplitude is proportional to the disturbance τ_{dy} and inversely proportional to the difference $I_x - I_z$. This means the only way to limit the amplitude of oscillation is by choosing appropriate values for the satellite's moments of inertia. With the set of assumptions we've made, there is no damping — in the design of a GG-stabilized satellite, it will be necessary to add some passive or active damping.

To solve the x-z EoM, we take the Laplace transform with initial conditions:

$$(s^2 + 4\omega_0^2\sigma_x)\phi - s\omega_0(1-\sigma_x)\psi = \frac{\tau_{dx}}{I_x} + s\phi_0 + \dot{\phi}_0 - \omega_0(1-\sigma_x)\psi_0$$

$$(s^2 + \omega_0^2\sigma_z)\psi + s\omega_0(1-\sigma_z)\phi = \frac{\tau_{dz}}{I_z} + \omega_0(1-\sigma_z)\phi_0 + s\psi_0 + \dot{\psi}_0$$

They have the matrix form

$$\begin{pmatrix} s^2 + 4\omega_0^2\sigma_x & -s\omega_0(1-\sigma_x) \\ s\omega_0(1-\sigma_z) & s^2 + \omega_0^2\sigma_z \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \tau_{dx}/I_x + s\phi_0 + \dot{\phi}_0 - \omega_0(1-\sigma_x)\psi_0 \\ \tau_{dz}/I_z + s\psi_0 + \dot{\psi}_0 + \omega_0(1-\sigma_z)\phi_0 \end{pmatrix}$$

which has the solution

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \frac{1}{\Delta(s)}\begin{pmatrix} s^2 + \omega_0^2\sigma_z & s\omega_0(1-\sigma_x) \\ -s\omega_0(1-\sigma_z) & s^2 + 4\omega_0^2\sigma_x \end{pmatrix}\begin{pmatrix} \tau_{dx}/I_x + s\phi_0 + \dot{\phi}_0 - \omega_0(1-\sigma_x)\psi_0 \\ \tau_{dz}/I_z + s\psi_0 + \dot{\psi}_0 + \omega_0(1-\sigma_z)\phi_0 \end{pmatrix}$$

where $\Delta(s) = s^4 + \omega_0^2(1 + \sigma_x\sigma_z + 3\sigma_x)s^2 + 4\omega_0^2\sigma_x\sigma_z$. Inverting to get the time-domain behavior is more difficult here because $\Delta(s)$ is a rank-4 polynomial, but it can be done by using partial fractions. Let's look at the behavior for some special cases:

(i) $I_x = I_y$: Here $\sigma_z = 0$ and the determinant is

$$\Delta(s)\Big|_{\sigma_z=0} = s^2\left(s^2 + 3\omega_o^2(3\sigma_x+1)\right).$$

The s^2 outside the brackets indicate two integrators, which imply neutral stability. The stability of the remaining part depend on the roots of the quadratic. They must be imaginary, which means that

$$3\sigma_x + 1 = 3(I_y - I_x)/I_x + 1 > 0.$$

Since $I_x = I_y$, this means

$$\frac{I_z}{I_x} = \frac{I_z}{I_y} < \frac{4}{3}.$$

If the inputs are disturbances of constant magnitude, then

$$\phi(s) = \frac{\tau_{dx}\left(s^2+\omega_o^2\sigma_z\right)}{I_x s^3\left(s^2+(3\sigma_x+1)\omega_o^2\right)} + \frac{\tau_{dz}\,(1-\sigma_x)\,\omega_o}{I_z s^2\left(s^2+(3\sigma_x+1)\omega_o^2\right)}$$

$$\psi(s) = -\frac{\tau_{dx}\,(1-\sigma_z)}{I_x s^2\left(s^2+(3\sigma_x+1)\omega_o^2\right)} + \frac{\tau_{dz}\,(s^2+4\omega_o^2\sigma_x)}{I_z s^3\left(s^2+(3\sigma_x+1)\omega_o^2\right)}$$

Both $\phi(t)$, $\psi(t)$ will have divergent and oscillatory terms in their time response. Hence, the symmetrical satellite with $I_x = I_y$ cannot be passively stabilized.

(ii) $I_y = I_z$: Symmetrical axis in the direction of satellite motion. In this singular case, $\sigma_x = 0$ and $\Delta(s)\big|_{\sigma_x=0} = s^2(s^2+\omega_o^2)$. As in the previous case, for constant-value disturbances about the X_B and Z_B axes, the yaw and roll Euler angles will have divergent and oscillatory motions in their time response.