Tensor Bible

Ashton Cole

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List of Theorems

Introduction

before, big picture, what is a tensor, algebraic object describing mulitilinear relationship between algebraic objects [1]

Chapter 1

Vectors

Ironically enough, this was the hardest section to write. It is easy to get lost in abstraction and pedantics.

I think that the best way to start is with vectors. Tesors are always defined relative to some kind of vector space. Scalars are so-called zeroth-order tensors, vectors are so-called first-order tensors, and any other kind of tensor can be constructed from these.

1.1 Vector Space

examples of vector spaces

1.2 Dual Space

1.2.1 Isometric Isomorphism

Riesz representation and dealing with complex vector spaces

1.3 Inner Product and Norm

norm is also a metric

1.4 Vector Spaces in Continuum Mechanics

construction of euclidean geometry but don't get to lost
Rn is a Euclidean geometry, also cartesian coordinates
note on how velocity
also fields which are functions

1.5 Getting Started

For all intents and purposes of continuum mechanics, it is sufficient to start with a Euclidean space, usually two- or three-dimensional, described by a Cartesian coordinate system, because this can be used to describe the physical world at our scale quite well.

1.5.1 Notation

Scalars are simply written as a letter, usually lower-case (a). Vectors are bolded (**a**). When hand-written, they may have an arrow on top ($\vec{a}$) or a single underline ($\underline{a}$). Unit vectors, with a magnitude 1, may be

marked with a circumflex ($\hat{\mathbf{a}}$). Higher-order tensors are bolded and usually upper-case ($\mathbf{A}$). When written, they may have a number of underlines equal to their order.

1.6   Bases and Components

We start with some right-handed orthonormal basis $\{\hat{\mathbf{e}}_i\}_{i=1}^n$ for an n-dimensional Euclidean vector space $\mathbb{R}^n$. If these align with a Cartesian coordinate frame, or when a Cartesian coordinate frame is defined to line up with them, they're called the canonical basis, i.e. the standard one. Using the geometric definitions for the dot and cross product, the following relations exist between the basis vectors, represented in shorthand by the Kronecker delta δ and Levi-Civita symbol ϵ.

These products are both bilinear. The dot product commutes, and the cross product anti-commutes.

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} := \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{1.1}$$

$$(\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j) \cdot \hat{\mathbf{e}}_k = \epsilon_{ijk} := \begin{cases} 1, & (i,j,k) = (1,2,3), (2,3,1), \text{or } (3,1,2), \\ -1, & (i,j,k) = (3,2,1), (2,1,3), \text{or } (1,3,2), \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

Since any basis set is linearly independent and spans the vector space, any vector in the space may be represented in terms of such a basis. The coefficients of the linear combination are called components. It is important to note that these are not the same as coordinates, especially for non-Cartesian coordinate systems.

$$\mathbf{a} = \sum_{i=1}^n a_i \hat{\mathbf{e}}_i \tag{1.3}$$

1.6.1   Cobasis

If the chosen basis is not orthonormal with respect to some inner product and corresponding induced norm, then there is some more nuance. Sidestepping the excess mental infrastructure of the dual space and Riesz representation, the cobasis of some basis $\{\mathbf{f}_i\}_{i=1}^n$, marked with a superscript as $\{\mathbf{f}^i\}_{i=1}^n$, is another basis for the same space, which is bi-orthonormal to the basis.

$$\mathbf{f}^i \cdot \mathbf{f}_j = \mathbf{f}_j \cdot \mathbf{f}^i = \delta_{ij} \tag{1.4}$$

Notation for components in a non-orthonormal basis is more complicated. The contravariant components of the basis use superscripts, and the covariant components of the cobasis use subscripts. This may seem backwards, but this is the convention that was developed.

$$\mathbf{a} = \sum_{i=1}^n \tilde{a}^i \mathbf{f}_i \tag{1.5}$$

$$\mathbf{a} = \sum_{i=1}^n \tilde{a}_i \mathbf{f}^i \tag{1.6}$$

Contra in contravariant refers to how the components scale inversely with the basis vectors. For example, if a basis is changed from inches to feet, the components used to represent a vector will shrink to represent the same length. On the other hand, the covariant components would increase.

If one goes further into the weeds of tensor-world, there is a whole shorthand system of upstairs and downstairs indices used for tensors defined by a certain combination of bases and cobases. However, in continuum-world, one can fudge over all of these nuances by choosing to use an orthonormal basis, which is conveniently its own cobasis.

1.6.2   Independence of Basis

That freedom of choice is an important idea behind this tensor system. Up until this point, you were probably used to seeing vectors and matrices as lists and tables of numbers. Start with a unit vector $(1,0)$. Rotate the coordinate frame 45 degrees clockwise and you get $(\sqrt{2}/2, \sqrt{2}/2)$. This would seem like a new and unequal vector, since $(1,0) \neq (\sqrt{2}/2, \sqrt{2}/2)$.

Writing them in terms of components of particular bases allows for more abstraction and flexibility. The vector is no longer the tuple per se, but is represented by a tuple in the context of a particular reference frame. The vector and the physical quantity it represents stay the same conceptually, regardless of the frame of reference: $\mathbf{a} = \sum_{i=1}^{n} a_i \hat{\mathbf{e}}_i = \sum_{i=1}^{n} \tilde{a}^i \mathbf{f}_i = \sum_{i=1}^{n} \tilde{a}_i \mathbf{f}^i$. Only components vary. So we can say for the prior example, labeling the coordinate frames as A and B, $(1,0)_A = (\sqrt{2}/2, \sqrt{2}/2)_B$. Operations and properties like an inner product, cross product, norm should stay the same, too. However, it may not necessarily be that $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} \tilde{a}^i \tilde{b}^i$, because the dot product is defined geometrically, not algebraically.

1.6.3   Computing Components

differentiate coordinates and components

1.6.4   Curvilinear Coordinates

same vector using different coordinates

1.7   Kronecker Delta and Levi-Civita Symbol Identities

The Kronecker Delta and Levi-Civita Symbol have the following identities which are useful in proof.

$$a = b \qquad\qquad \text{Proof ??} \quad (1.7)$$
$$a = b \qquad\qquad \text{Proof ??} \quad (1.8)$$
$$\epsilon_{pqs}\epsilon_{nrs} = \delta_{pn}\delta_{qr} - \delta_{pr}\delta_{qn} \qquad\qquad \text{Proof ??} \quad (1.9)$$
$$\epsilon_{pqs}\epsilon_{rqs} = 2\delta_{pr} \qquad\qquad \text{Proof ??} \quad (1.10)$$

1.8   Einstein Notation

Chapter 2

Tensor Algebra

aaa

2.1 Dyads and Dyadics

tensor product spaces
    order and covector-vector degree

2.2 Contractions

2.3 Equivalence to Linear Algebra

isomorphic to linear algebra if appropriate combination of basis/cobasis
    linearity
    coordinate vs non coordinate basis
    components
    deltas and epsilons, symbols, only up to rank 2, etc
    einstein notation
    tensor as operator
    basis independence
    isomorphism
    definitions contrived to fulfill certain properties
    using dyadic symbol and not
    defining property of dyadic, compare to linear algebra
    simple properties of each product (distributive, associative, commutative) that may depend on the
operands
    linear algebra comparison
    computing in canonical basis
    introducing the dot product
    introducing the cross product
    introducing the dyadic product
    specifically takes two vectors
    non commutative
    scalar rule
    simple contractions
    double contractions
    change of basis
    invariants of vectors and tensors
    symmetric, skew-symmetric

complex numbers
operations between tensors (to fit with vectors)

- $\phi, \kappa \in \mathbb{R}$

- $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^n$

- $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$

- $\boldsymbol{\omega} \in \mathbb{R}^p$

- $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$

- $\mathbf{Q}, \mathbf{R} \in \mathbb{R}^{m \times m}$

- $\mathbf{S}, \mathbf{T} \in \mathbb{R}^{m \times n}$

- $\mathbf{U} \in \mathbb{R}^{n \times p}$

- $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{3 \times 3}$

INTRODUCING VECTORS

$$\mathbf{S} = \sum_{i=1}^{m} \sum_{j=1}^{n} S_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \tag{2.1}$$

$$\mathbf{u} \times \mathbf{v} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} u_i v_j \epsilon_{ijk} \hat{\mathbf{e}}_k \tag{2.2}$$

$$\tag{2.3}$$

INTRODUCING TENSORS

$$(\mathbf{a} \otimes \mathbf{x}) \, \mathbf{y} := \mathbf{a} \, (\mathbf{x} \cdot \mathbf{y}) \tag{2.4}$$

$$\mathbf{a} \cdot (\mathbf{b} \otimes \mathbf{x}) := (\mathbf{a} \cdot \mathbf{b}) \, \mathbf{x} \tag{2.5}$$

$$\mathbf{x} \cdot (\mathbf{y} \otimes \mathbf{a}) \, \mathbf{b} = \ldots \qquad\qquad \text{Proof } ?? \tag{2.6}$$

$$(\mathbf{a} \otimes \mathbf{x})^T := \mathbf{x} \otimes \mathbf{a} \tag{2.7}$$

$$\operatorname{tr}(\mathbf{a} \otimes \mathbf{b}) := \mathbf{a} \cdot \mathbf{b} \tag{2.8}$$

$$(\mathbf{a} \otimes \mathbf{x}) \, (\mathbf{y} \otimes \boldsymbol{\omega}) = (\mathbf{x} \cdot \mathbf{y}) \, (\mathbf{a} \otimes \boldsymbol{\omega}) \qquad\qquad \text{Proof } ?? \tag{2.9}$$

$$(\mathbf{a} \otimes \mathbf{x}) : (\mathbf{b} \otimes \mathbf{y}) := (\mathbf{a} \cdot \mathbf{b}) \, (\mathbf{x} \cdot \mathbf{y}) \tag{2.10}$$

$$\mathbf{a} \otimes \mathbf{x} = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i x_j \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \tag{2.11}$$

$$\tag{2.12}$$

CONTRACTIONS

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i \qquad \text{Proof A.1 (2.13)}$$

$$\mathbf{Sa} = \mathbf{S} \cdot \mathbf{a} = \sum_{i=1}^{m} \sum_{j=1}^{n} S_{ij} a_j \hat{\mathbf{e}}_i \qquad \text{Proof A.2 (2.14)}$$

$$\mathbf{x} \cdot \mathbf{S} = \sum_{i=1}^{m} \sum_{k=1}^{n} x_i S_{ik} \hat{\mathbf{e}}_k \qquad \text{Proof A.3 (2.15)}$$

$$\mathbf{SU} = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{\ell=1}^{p} S_{ij} U_{j\ell} \left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_\ell \right) \qquad \text{Proof A.4 (2.16)}$$

$$\mathbf{S} : \mathbf{T} = \sum_{i=1}^{m} \sum_{j=1}^{n} S_{ij} T_{ij} \qquad \text{Proof A.5 (2.17)}$$

2.4 Miscellaneous Operations

2.4.1 Change of Basis

2.5 Helpful Identities

$$\mathbf{Sa} = \mathbf{a} \cdot \left(\mathbf{S}^T \right) \qquad \text{Proof ?? (2.18)}$$
$$(\mathbf{x} \cdot \mathbf{S}) \cdot \mathbf{a} = \mathbf{x} \cdot (\mathbf{Sa}) \qquad \text{Proof ?? (2.19)}$$
$$\left(\mathbf{S}^T \right) : \mathbf{V} = \mathbf{S} : \left(\mathbf{V}^T \right) \qquad \text{Proof ?? (2.20)}$$

Chapter 3

Tensor Calculus

differentiate nabla and grad

in different coordinate systems

expand to field...define operations...basis changes locally

algebraic and geometric definitions which conveniently coincide for a euclidean geometry described by a cartesian coordinate system

INTRODUCING NABLA OPERATOR

3.1   Introducing Nabla

$$\boldsymbol{\nabla} := \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \hat{\mathbf{e}}_i \tag{3.1}$$

$$\boldsymbol{\nabla}[\ldots] = \boldsymbol{\nabla} \otimes [\ldots] := \sum_{i=1}^{m} \frac{\partial}{\partial x_i} [\hat{\mathbf{e}}_i \otimes \ldots] \tag{3.2}$$

$$\boldsymbol{\nabla} \cdot [\ldots] := \sum_{i=1}^{m} \frac{\partial}{\partial x_i} [\hat{\mathbf{e}}_i \cdot \ldots] \tag{3.3}$$

$$\boldsymbol{\nabla} \times [\ldots] := \sum_{i=1}^{m} \frac{\partial}{\partial x_i} [\hat{\mathbf{e}}_i \times \ldots] \tag{3.4}$$

3.2   Gradient

$$\boldsymbol{\nabla}\phi = \ldots \qquad \qquad \text{Proof ?? (3.5)}$$
$$\boldsymbol{\nabla}\mathbf{a} = \ldots \qquad \qquad \text{Proof ?? (3.6)}$$
$$\boldsymbol{\nabla}\mathbf{S} = \ldots \qquad \qquad \text{Proof ?? (3.7)}$$
$$\operatorname{grad}\phi = \ldots \tag{3.8}$$
$$\operatorname{grad}\mathbf{a} = \ldots \tag{3.9}$$
$$\operatorname{grad}\mathbf{S} = \ldots \tag{3.10}$$
$$\tag{3.11}$$

3.2.1   Gradient Product Rules

First Order Value, Zeroth Order Operand

$$\boldsymbol{\nabla}\left(\phi\kappa\right) = \phi\boldsymbol{\nabla}\kappa + \kappa\boldsymbol{\nabla}\phi \qquad\qquad \text{Proof ??} \quad (3.12)$$

$$\boldsymbol{\nabla}\left(\mathbf{a}\cdot\mathbf{b}\right) = \left(\boldsymbol{\nabla}\mathbf{b}\right)\mathbf{a} + \left(\boldsymbol{\nabla}\mathbf{a}\right)\mathbf{b} * ** \qquad\qquad \text{Proof ??} \quad (3.13)$$

$$\boldsymbol{\nabla}\left(\mathbf{S}:\mathbf{T}\right) = \left(\boldsymbol{\nabla}\mathbf{T}\right):\mathbf{S} + \left(\boldsymbol{\nabla}\mathbf{S}\right):\mathbf{T} \qquad\qquad \text{Proof ??} \quad (3.14)$$

Second Order Value, First Order Operand

$$\boldsymbol{\nabla}\left(\phi\mathbf{a}\right) = \ldots \qquad\qquad \text{Proof ??} \quad (3.15)$$

$$\boldsymbol{\nabla}\left(\mathbf{S}\mathbf{a}\right) = \ldots \qquad\qquad \text{Proof ??} \quad (3.16)$$

$$\boldsymbol{\nabla}\left(\mathbf{y}\cdot\mathbf{S}\right) = \ldots \qquad\qquad \text{Proof ??} \quad (3.17)$$

$$\boldsymbol{\nabla}\left(\mathcal{D}:\mathbf{S}\right) = \ldots \qquad\qquad \text{Proof ??} \quad (3.18)$$

$$\boldsymbol{\nabla}\left(\mathbf{S}:\mathcal{D}\right) = \ldots \qquad\qquad \text{Proof ??} \quad (3.19)$$

$$\boldsymbol{\nabla}\left(\mathbf{u}\times\mathbf{v}\right) = \ldots \qquad\qquad \text{Proof ??} \quad (3.20)$$

Third Order Value, Second Order Operand

$$\boldsymbol{\nabla}\left(\phi\mathbf{S}\right) = \ldots \qquad\qquad \text{Proof ??} \quad (3.21)$$

$$\boldsymbol{\nabla}\left(\mathbf{a}\otimes\mathbf{y}\right) = \ldots \qquad\qquad \text{Proof ??} \quad (3.22)$$

$$\boldsymbol{\nabla}\left(\mathbf{S}\mathbf{U}\right) = \ldots \qquad\qquad \text{Proof ??} \quad (3.23)$$

$$\boldsymbol{\nabla}\left(\mathcal{D}\cdot\mathbf{a}\right) = \ldots \qquad\qquad \text{Proof ??} \quad (3.24)$$

$$\boldsymbol{\nabla}\left(\mathbf{y}\cdot\mathcal{D}\right) = \ldots \qquad\qquad \text{Proof ??} \quad (3.25)$$

3.3   Divergence

$$\boldsymbol{\nabla}\cdot\mathbf{a} = \ldots \qquad\qquad \text{Proof ??} \quad (3.26)$$

$$\boldsymbol{\nabla}\cdot\mathbf{S} = \ldots \qquad\qquad \text{Proof ??} \quad (3.27)$$

$$\text{div}\,\mathbf{a} = \ldots \qquad\qquad \text{Proof ??} \quad (3.28)$$

$$\text{div}\,\mathbf{S} = \ldots \qquad\qquad \text{Proof ??} \quad (3.29)$$

3.3.1   Divergence Product Rules

Zeroth Order Value, First Order Operand

$$\boldsymbol{\nabla}\cdot\left(\phi\mathbf{a}\right) = \phi\boldsymbol{\nabla}\cdot\mathbf{a} + \mathbf{a}\cdot\boldsymbol{\nabla}\phi \qquad\qquad \text{Proof A.6} \quad (3.30)$$

$$\boldsymbol{\nabla}\cdot\left(\mathbf{S}\mathbf{a}\right) = \mathbf{S}:\boldsymbol{\nabla}\mathbf{a} + \mathbf{a}\cdot\left(\boldsymbol{\nabla}\cdot\mathbf{S}\right) \qquad\qquad \text{Proof ??} \quad (3.31)$$

$$\boldsymbol{\nabla}\cdot\left(\mathbf{y}\cdot\mathbf{S}\right) = \left(\mathbf{S}^{T}\right):\boldsymbol{\nabla}\mathbf{y} + \mathbf{y}\cdot\left(\boldsymbol{\nabla}\cdot\left(\mathbf{S}^{T}\right)\right) \qquad\qquad \text{Proof ??} \quad (3.32)$$

$$\boldsymbol{\nabla}\cdot\left(\mathbf{u}\times\mathbf{v}\right) = \ldots \qquad\qquad \text{Proof ??} \quad (3.33)$$

First Order Value, Second Order Operand

$$\boldsymbol{\nabla}\cdot\left(\phi\mathbf{S}\right) = \phi\boldsymbol{\nabla}\cdot\mathbf{S} + \boldsymbol{\nabla}\phi\cdot\mathbf{S} \qquad\qquad \text{Proof A.7} \quad (3.34)$$

$$\boldsymbol{\nabla}\cdot\left(\mathbf{a}\otimes\mathbf{y}\right) = \mathbf{a}\cdot\boldsymbol{\nabla}\mathbf{y} + \mathbf{y}\left(\boldsymbol{\nabla}\cdot\mathbf{a}\right) \qquad\qquad \text{Proof ??} \quad (3.35)$$

$$\boldsymbol{\nabla}\cdot\left(\mathbf{S}\mathbf{U}\right) = \mathbf{S}:\left(\boldsymbol{\nabla}\mathbf{U}\right) + \left(\boldsymbol{\nabla}\cdot\mathbf{S}\right)\mathbf{U} \qquad\qquad \text{Proof ??} \quad (3.36)$$

3.4 Curl

$$\boldsymbol{\nabla} \times \mathbf{a} = \dots \qquad \text{Proof ?? (3.37)}$$
$$\boldsymbol{\nabla} \times \mathbf{S} = \dots \qquad \text{Proof ?? (3.38)}$$
$$\operatorname{curl} \mathbf{a} = \dots \qquad \text{(3.39)}$$
$$\operatorname{curl} \mathbf{S} = \dots \qquad \text{(3.40)}$$

3.4.1 Curl Product Rules

First Order Value, First Order Operand

3.5 Laplacian

3.5.1 Laplacian Product Rules

Zeroth Order Value, Zeroth Order Operand

$$\nabla^2 (\phi \kappa) = \dots \qquad \text{Proof ?? (3.41)}$$
$$\nabla^2 (\mathbf{a} \cdot \mathbf{b}) = \dots \qquad \text{Proof ?? (3.42)}$$
$$\nabla^2 (\mathbf{S} : \mathbf{T}) = \dots \qquad \text{Proof ?? (3.43)}$$

First Order Value, First Order Operand

$$\nabla^2 (\phi \mathbf{a}) = \dots \qquad \text{Proof ?? (3.44)}$$
$$\nabla^2 (\mathbf{S} \mathbf{a}) = \dots \qquad \text{Proof ?? (3.45)}$$
$$\nabla^2 (\mathbf{y} \cdot \mathbf{S}) = \dots \qquad \text{Proof ?? (3.46)}$$
$$\nabla^2 (\mathbf{u} \times \mathbf{v}) = \dots \qquad \text{Proof ?? (3.47)}$$

Second Order Value, Second Order Operand

$$\nabla^2 (\phi \mathbf{S}) = \dots \qquad \text{Proof ?? (3.48)}$$
$$\nabla^2 (\mathbf{a} \otimes \mathbf{y}) = \dots \qquad \text{Proof ?? (3.49)}$$
$$\nabla^2 (\mathbf{S} \mathbf{U}) = \dots \qquad \text{Proof ?? (3.50)}$$

3.6 Other Second Derivatives

div of curl = 0
    curl of grad = 0

3.7 Function Derivatives

derivative of vector value/input functions
    compositions thereof
    derivative of determinant, etc
    grad div relation

Chapter 4

Kinematics

Topics:
  different frames
  deformation map
  all the different kinds of tensors
  material and spatial fields
  derivatives of material and spatial fields

4.1   Kinematic Variables

- $\mathbf{U} = \mathbf{U}(\mathbf{X}, t)$: displacement in material frame

    - $\mathbf{U} := \boldsymbol{\phi}_t - \mathbf{X}$

- $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$: displacement in spatial frame

    - $\mathbf{u} := \mathbf{U} \circ \boldsymbol{\phi}_t^{-1} = \mathbf{x} - \boldsymbol{\phi}_t^{-1}$

- $\mathbf{V} = \mathbf{V}(\mathbf{X}, t)$: velocity in material frame

    - $\mathbf{V} := \frac{\partial \phi}{\partial t}$

- $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$: velocity in spatial frame

    - $\mathbf{v} := \mathbf{V} \circ \boldsymbol{\phi}_t^{-1} = \frac{\partial \phi}{\partial t} \circ \boldsymbol{\phi}_t^{-1}$

- $\mathbf{L} = \mathbf{L}(\mathbf{X}, t)$: velocity gradient in material frame???

    - $\mathbf{L} := \operatorname{grad}_{\mathbf{X}} \mathbf{V}$

- $\mathbf{D} = \mathbf{D}(\mathbf{X}, t)$: symmetric component of velocity gradient in material frame???

    - $\mathbf{D} := \frac{1}{2} \left(\mathbf{L} + \mathbf{L}^T \right)$

- $\mathbf{W} = \mathbf{W}(\mathbf{X}, t)$: skew-symmetric component of velocity gradient in material frame???

    - $\mathbf{W} := \frac{1}{2} \left(\mathbf{L} - \mathbf{L}^T \right)$

Chapter 5

Conservation Laws

Topics:

    continuum assumptions

    reynolds transport theorem, i.e. leibniz integral rule

    cauchy theorem

    cauchy theorem for heat flux

    general conservation laws: integral and differential, material and spatial

    mass

    momentum

    energy

    entropy inequality

    define control volume properly, relate vt to v0, same for omega

5.1 Constitutive Variables

- $\rho_0 = \rho_0(\mathbf{X})$: density in material frame

- $\rho = \rho(\mathbf{x}, t)$: density in spatial frame

- $\mathbf{S} = \mathbf{S}(\mathbf{X}, t)$: second Piola-Kirchoff stress tensor in material frame

    - $\mathbf{S} := J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}$

- $\mathbf{P} = \mathbf{P}(\mathbf{X}, t)$: first Piola-Kirchoff stress tensor in material frame

    - $\mathbf{P} := J\boldsymbol{\sigma}\mathbf{F}^{-T} = \mathbf{FS}$

- $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x}, t)$: Cauchy stress tensor in spatial frame

- $\mathbf{B} = \mathbf{B}(\mathbf{X}, t)$: body force per unit mass in material frame

- $\mathbf{b} = \mathbf{b}(\mathbf{x}, t)$: body force per unit mass in spatial frame

- $E = E(\mathbf{X}, t)$: internal energy per unit mass in material frame

- $e = e(\mathbf{x}, t)$: internal energy per unit mass in spatial frame

- $I = I(\mathbf{X}, t)$: potential? energy per unit mass in material frame

- $i = i(\mathbf{x}, t)$: potential? energy per unit mass in spatial frame

- $\mathbf{Q} = \mathbf{Q}(\mathbf{X}, t)$: heat flux in material frame

- $\mathbf{q} = \mathbf{q}(\mathbf{x}, t)$: heat flux in spatial frame

- $\Theta = \Theta(\mathbf{X}, t)$: temperature in material frame

- $\theta = \theta(\mathbf{x}, t)$: temperature energy per unit mass in spatial frame

5.2   Reynold's Transport Theorem

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbf{x} \in V(t)} \mathbf{q} \, \mathrm{d}\Omega_t = \int_{\mathbf{x} \in V(t)} \frac{\partial \mathbf{q}}{\partial t} \, \mathrm{d}\Omega_t + \int_{\mathbf{x} \in \partial(V(t))} \mathbf{q} \left(\mathbf{v} \cdot \hat{\mathbf{n}} \right) \mathrm{d}\partial\Omega_t \qquad \text{Proof ??} \quad (5.1)$$

5.3   Conservation of Mass

5.3.1   Material Frame

$$\rho_0 = \rho \circ \phi_t J \qquad \qquad \text{Proof ??} \quad (5.2)$$

5.3.2   Spatial Frame

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbf{x} \in V \subseteq \Omega_t} \rho(\mathbf{x}, t) \, \mathrm{d}V + \int_{\mathbf{x} \in \partial V} \rho(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) \cdot \hat{\mathbf{n}}(\mathbf{x}) \, \mathrm{d}(\partial V) = 0 \qquad \text{Integral form. Proof ??} \quad (5.3)$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \mathbf{v} = 0 \qquad \qquad \text{Proof ??} \quad (5.4)$$

5.4   Conservation of Momentum

5.4.1   Material Frame

$$\rho_0 \frac{\mathrm{d}V}{\mathrm{d}t} = \mathrm{div}_{\mathbf{X}} \mathbf{P} + \rho_0 \mathbf{B} \qquad \qquad \text{Proof ??} \quad (5.5)$$

$$(5.6)$$

5.4.2   Spatial Frame

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \mathrm{div} \left(\rho \mathbf{v} \otimes \mathbf{v} \right) = \mathrm{div} \, \boldsymbol{\sigma} + \rho \mathbf{b} \qquad \qquad \text{Conservative form. Proof ??} \quad (5.7)$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathrm{grad} \, \mathbf{v}) \, \mathbf{v} \right) = \mathrm{div} \, \boldsymbol{\sigma} + \rho \mathbf{b} \qquad \qquad \text{Reduced form. Proof ??} \quad (5.8)$$

$$(5.9)$$

5.5   Conservation of Angular Momentum

5.5.1   Material Frame

$$\mathbf{S} = \mathbf{S}^T \qquad \qquad \text{Reduced form. Proof ??} \quad (5.10)$$

$$(5.11)$$

5.5.2   Spatial Frame

$$\frac{\partial (\mathbf{x} \times \rho \mathbf{v})}{\partial t} + \mathrm{div} \left(\mathbf{x} \times \rho \mathbf{v} \otimes \mathbf{v} \right) = \mathrm{div} \left(\mathbf{x} \times \boldsymbol{\sigma} \right) + \mathbf{x} \times \rho \mathbf{b} \qquad \text{Conservative form. Proof ??} \quad (5.12)$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \qquad \qquad \text{Reduced form. Proof ??} \quad (5.13)$$

$$(5.14)$$

5.6 Conservation of Energy

5.6.1 Material Frame

$$\rho_0 \frac{\mathrm{d}I}{\mathrm{d}t} + \mathrm{div}_{\mathbf{X}} \, \mathbf{Q} = \rho_0 \mathbf{R} + \mathbf{S} : \mathbf{D} \qquad\qquad \text{Proof ?? (5.15)}$$

$$\text{(5.16)}$$

5.6.2 Spatial Frame

$$\frac{\partial (\rho e)}{\partial t} + \mathrm{div} \, (\rho e \mathbf{v}) = \mathrm{div} \, (\boldsymbol{\sigma} \mathbf{v}) + \rho r + \rho \mathbf{b} \cdot \mathbf{b} - \mathrm{div} \, \mathbf{q} \qquad \text{Conservation form. Proof ?? (5.17)}$$

$$\rho \left(\frac{\partial e}{\partial t} + \mathrm{grad} \, e \cdot \mathbf{v} \right) = \boldsymbol{\sigma} : \mathbf{d} + \rho r + \rho \mathbf{b} \cdot \mathbf{b} - \mathrm{div} \, \mathbf{q} \qquad \text{Reduced form. Proof ?? (5.18)}$$

$$\text{(5.19)}$$

5.7 Entropy Inequality

5.7.1 Material Frame

$$\rho_0 \frac{\mathrm{d}N}{\mathrm{d}t} + \mathrm{div}_{\mathbf{X}} \, \frac{\mathbf{Q}}{\Theta} \geq \rho_0 \frac{R}{\Theta} \qquad\qquad \text{Proof ?? (5.20)}$$

5.7.2 Spatial Frame

$$\frac{\partial (\rho \eta)}{\partial t} + \mathrm{div} \, (\rho \eta \mathbf{v}) + \mathrm{div} \, \frac{\mathbf{q}}{\Theta} - \rho r \geq 0 \qquad\qquad \text{Proof ?? (5.21)}$$

$$\text{(5.22)}$$

5.8 Advection Diffusion Reaction Equation

$$\frac{\partial c}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{v} c - \mathbf{D} \boldsymbol{\nabla} c) = R \qquad\qquad \text{Proof ?? (5.23)}$$

$$\frac{\partial \mathbf{c}}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{v} \otimes \mathbf{c} - \mathcal{D} : \boldsymbol{\nabla} \mathbf{c}) = R \qquad\qquad \text{Proof ?? (5.24)}$$

Chapter 6

Solid Mechanics

general
    elastic waves
    thermoelastic solid
    engineering stress example

6.1   Provisional Constitutive Laws

- $\mathbb{C}$ =???: elasticity tensor in material frame

  - $\mathbb{C} := \frac{\partial \mathbf{S}}{\partial E} = \frac{\partial^2 \hat{\Psi}}{\partial \mathbf{E}^2}$

- =???: elasticity tensor in spatial frame...

6.2   Elastic Wave Equations

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \operatorname{div} \boldsymbol{\sigma} = 0 \qquad\qquad \text{Proof \textbf{??}} \quad (6.1)$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} = (\lambda + 2\mu) \operatorname{grad}(\operatorname{div} \mathbf{u}) - \mu \operatorname{curl}^2 \mathbf{u} \qquad\qquad \text{hm} \quad (6.2)$$

$$(6.3)$$

Chapter 7

Fluid Mechanics

full navier stokes
   incompressible
   rans
   stokes flow
   euler equations
   acoustic equations
   my special acoustic equations
   shallow water equations

7.1 Shallow Water Equations

conservative, non-conservative,

$$\frac{\partial (\rho \eta)}{\partial t} + \text{div} (\rho \eta \mathbf{u}) = 0 \qquad \qquad \text{Proof ?? (7.1)}$$

$$\frac{\partial (\rho \eta \mathbf{u})}{\partial t} + \text{div} \left(\rho \eta \mathbf{u} \otimes \mathbf{u} + \frac{1}{2} \rho g \eta^2 \mathbf{I} \right) = 0 \qquad \qquad \text{Proof ?? (7.2)}$$

$$\frac{\partial (\rho \eta)}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \eta \mathbf{u}) = 0 \qquad \qquad \text{Proof ?? (7.3)}$$

$$\frac{\partial (\rho \eta \mathbf{u})}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \eta \mathbf{u} \otimes \mathbf{u}) + \boldsymbol{\nabla} \frac{1}{2} \rho g \eta^2 = 0 \qquad \qquad \text{Proof ?? (7.4)}$$

7.1.1 Closures and Assumptions

diffusive wave equations
   bernoulli
   groundwater darcy flow

Appendix A

Proofs

Proof A.1 (Equation 2.13)

$$\mathbf{a} \cdot \mathbf{b} = \left(\sum_{i=1}^{n} a_i \hat{\mathbf{e}}_i \right) \cdot \left(\sum_{j=1}^{n} b_j \hat{\mathbf{e}}_j \right) \qquad \qquad \text{Using 1.3} \quad \text{(A.1)}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j \qquad \qquad \text{(A.2)}$$

$$= \sum_{i=1}^{n} a_i b_i \qquad \qquad \text{Using 1.1} \quad \text{(A.3)}$$

Proof A.2 (Equation 2.14)

$$\mathbf{Sa} = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} S_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \right) \left(\sum_{k=1}^{n} a_k \hat{\mathbf{e}}_k \right) \qquad \qquad \text{Using 2.1, 1.3} \quad \text{(A.4)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{n} S_{ij} a_k \left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \right) \hat{\mathbf{e}}_k \qquad \qquad \text{(A.5)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{n} S_{ij} a_k \hat{\mathbf{e}}_i \left(\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_k \right) \qquad \qquad \text{Using 2.4} \quad \text{(A.6)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} S_{ij} a_j \hat{\mathbf{e}}_i \qquad \qquad \text{Using 1.1} \quad \text{(A.7)}$$

Proof A.3 (Equation 2.15)

$$\mathbf{x} \cdot \mathbf{S} = \left(\sum_{i=1}^{n} x_i \hat{\mathbf{e}}_i \right) \cdot \left(\sum_{j=1}^{m} \sum_{k=1}^{n} S_{jk} \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \right) \qquad \qquad \text{Using 1.3, 2.1} \quad \text{(A.8)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{n} x_i S_{jk} \hat{\mathbf{e}}_i \cdot \left(\hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \right) \qquad \qquad \text{(A.9)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{n} x_i S_{jk} \left(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j \right) \hat{\mathbf{e}}_k \qquad \qquad \text{Using 2.5} \quad \text{(A.10)}$$

$$= \sum_{i=1}^{m} \sum_{k=1}^{n} x_i S_{ik} \hat{\mathbf{e}}_k \qquad \qquad \text{Using 1.1} \quad \text{(A.11)}$$

Proof A.4 (Equation 2.16)

$$\mathbf{SU} = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} S_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \right) \left(\sum_{k=1}^{n} \sum_{\ell=1}^{p} U_{k\ell} \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell \right) \qquad \textit{Using 2.1} \quad \text{(A.12)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{p} S_{ij} U_{k\ell} \left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \right) \left(\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell \right) \qquad \text{(A.13)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{\ell=1}^{p} S_{ij} U_{k\ell} \left(\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_k \right) \left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_\ell \right) \qquad \textit{Using 2.9} \quad \text{(A.14)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{\ell=1}^{p} S_{ij} U_{j\ell} \left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_\ell \right) \qquad \textit{Using 1.1} \quad \text{(A.15)}$$

Proof A.5 (Equation 2.17)

$$\mathbf{S} : \mathbf{T} = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} S_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \right) : \left(\sum_{k=1}^{m} \sum_{\ell=1}^{m} T_{k\ell} \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell \right) \qquad \textit{Using 2.1} \quad \text{(A.16)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{\ell=1}^{n} S_{ij} T_{k\ell} \left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \right) : \left(\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell \right) \qquad \text{(A.17)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{\ell=1}^{n} S_{ij} T_{k\ell} \left(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_k \right) \left(\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_\ell \right) \qquad \textit{Using 2.10} \quad \text{(A.18)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} S_{ij} T_{ij} \qquad \textit{Using 1.1} \quad \text{(A.19)}$$

$$\text{(A.20)}$$

Proof A.6 (Equation 3.30)

$$\boldsymbol{\nabla} \cdot (\phi \mathbf{a}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[\hat{\mathbf{e}}_{\mathbf{i}} \cdot \left(\phi \sum_{j=1}^{n} a_j \hat{\mathbf{e}}_j \right) \right] \qquad \textit{Using 3.3, 1.3} \quad \text{(A.21)}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \phi a_j}{\partial x_i} \left(\hat{\mathbf{e}}_{\mathbf{i}} \cdot \hat{\mathbf{e}}_j \right) \qquad \text{(A.22)}$$

$$= \sum_{i=1}^{n} \frac{\partial \phi a_i}{\partial x_i} \qquad \textit{Using 1.1} \quad \text{(A.23)}$$

$$= \sum_{i=1}^{n} \left(\phi \frac{\partial a_i}{\partial x_i} + a_i \frac{\partial \phi}{\partial x_i} \right) \qquad \text{(A.24)}$$

$$= \phi \boldsymbol{\nabla} \cdot \mathbf{a} + \mathbf{a} \cdot \boldsymbol{\nabla} \phi \qquad \textit{Using 3.26, 2.13, 3.5} \quad \text{(A.25)}$$

Proof A.7 (Equation 3.34)

$$\boldsymbol{\nabla} \cdot (\phi \mathbf{S}) = \left(\sum_{i=1}^{m} \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \right) \cdot \left(\phi \sum_{j=1}^{m} \sum_{k=1}^{n} S_{jk} \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \right) \qquad \text{Using 3.3, 2.1 (A.26)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{\partial (\phi S_{jk})}{\partial x_i} \hat{\mathbf{e}}_i \cdot (\hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k) \qquad \text{(A.27)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{n} \left(\phi \frac{\partial S_{jk}}{\partial x_i} + S_{jk} \frac{\partial \phi}{\partial x_i} \right) (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j) \, \hat{\mathbf{e}}_k \qquad \text{Using 2.5 (A.28)}$$

$$= \sum_{i=1}^{m} \sum_{k=1}^{n} \phi \frac{\partial S_{ik}}{\partial x_i} \hat{\mathbf{e}}_k + S_{ik} \frac{\partial \phi}{\partial x_i} \hat{\mathbf{e}}_k \qquad \text{Using 1.1 (A.29)}$$

$$= \phi \boldsymbol{\nabla} \cdot \mathbf{S} + \boldsymbol{\nabla} \phi \cdot \mathbf{S} \qquad \text{Using 3.27, 3.5, 2.15 (A.30)}$$

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