Tensor Bible

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Contents

List of Theorems

Introduction

before, big picture, what is a tensor, algebraic object describing mulitilinear relationship between algebraic objects [1]

Chapter 1

Vectors

Ironically enough, this was the hardest section to write. It is easy to get lost in abstraction and pedantics, but it is important to break away from the notion of vectors and tensors as tables of numbers. They may be represented by tables of numbers, but since vectors in continuum mechanics are geometric quantities, those components are always relative to a particular frame of reference. Careful abstraction means that operations on vectors and tensors can be defined or proven generally, and then translated to any particularly useful coordinate system or basis once it becomes necessary.

I think that the best place to start is with vectors. Tensors are always defined relative to some kind of vector space. Even sticking with vectors and tensors as tables, a matrix is only useful insofar as it operates on vectors. Otherwise, a 3×3 matrix might be better suited as a 9-dimensional vector. The two spaces are isomorphic, so the only real difference is how they are used.

1.1 Vector Analysis

bla bla bla operations past numbers which are addition-like or multiplication like

1.1.1 Group-Like Structures

Groups generalize binary operations with addition- or multiplication-like properties.

Definition 1.1 (Monoid) *A monoid $\{M, \cdot\}$ is a set M with a binary operation $\cdot : G \times G \to G$ which satisfies the following properties.*

1. *Associativity*
$$\forall m, n, p \in M : m \cdot (n \cdot p) = (m \cdot n) \cdot p \tag{1.1}$$

2. *Identity element*
$$\exists i \in M : \forall m \in M : i \cdot m = m \cdot i = m \tag{1.2}$$

Definition 1.2 (Group) *A group $\{G, \cdot\}$ is a monoid which also has inverse elements.*

$$\forall a \in G : \exists b \in A : a \cdot b = i \tag{1.3}$$

Definition 1.3 (Abelian Group) *An Abelian group $\{A, \cdot\}$ is a group which also has commutativity*

$$\forall a, b \in A : a \cdot b = b \cdot a \tag{1.4}$$

In the same spirit of generalizing addition and multiplication, the following two notation conventions are often used for such operations.

- Additive: An addition-like operation, almost always an Abelian group, with $+$ for the operation, 0 for the identity, and $-a$ for any inverses. For a group, subtraction can be defined.

$$a - b := a + (-b) \tag{1.5}$$

- Multiplicative: A multiplication-like operation, less often an Abelian group, with $\times$, $\cdot$, or $*$ for the operation, 1 for the identity, and a^{-1} for any inverses. For a group, division can be defined.

$$a/b := a \cdot b^{-1} \tag{1.6}$$

1.1.2 Group Actions

Actions generalize operations between arbitrary sets and monoids. For example, displacements are an action on positions. Compositions of displacements form an additive Abelian group using "tip to tail" addition. If one composition of displacements equals another composition, then they should get you to the same final destination.

Definition 1.4 (Left Action) *A left action $\circ : M \times X \to X$ is a binary operation between a monoid $\{M, \cdot\}$ with identity i and a set X which satisfies the following properties.*

1. *Left identity*
$$\forall x \in X : i \circ x = x \tag{1.7}$$

2. *Left compatibility*
$$\forall m, n \in M, x \in X : m \circ (n \circ x) = (m \cdot n) \circ x \tag{1.8}$$

Definition 1.5 (Right Action) *A right action $\circ : X \times M \to X$ is a binary operation between a set X and a monoid $\{M, \cdot\}$ with identity i which satisfies the following properties.*

1. *Right identity*
$$\forall x \in X : x \circ i = x \tag{1.9}$$

2. *Right compatibility*
$$\forall x \in X, m, n \in G : (x \circ m) \circ n = x \circ (m \cdot n) \tag{1.10}$$

1.1.3 Ring-Like Structures

Rings generalize cases where there is both an addition- and multiplication-like operation. Multiplication may or may not be invertible, but if it is, cannot inlcude the additive identity 0. See Proof **??**.

Definition 1.6 (Ring) *A ring $\{R, +, \cdot\}$ is a set R with an additive Abelian group $\{R, +\}$ and a multiplicative monoid $\{R, \cdot\}$ which satisfy the following properties.*

1. *Left distributivity*
$$\forall r, s, t \in R : r \cdot (s + t) = r \cdot s + r \cdot t \tag{1.11}$$

2. *Right distributivity*
$$\forall r, s, t \in R : (r + s) \cdot t = r \cdot t + s \cdot t \tag{1.12}$$

Definition 1.7 (Commutative Ring) *A commutative ring $\{R, +, \cdot\}$ is a ring where $\{R, \cdot\}$ is commutative.*

Definition 1.8 (Field) *A field $\{\mathbb{F}, +, \cdot\}$ is a ring where $\{\mathbb{F} \setminus 0, \cdot\}$ is a multiplicative group.*

1.1.4 Module-Like Structures

Modules generalize vector spaces.

Definition 1.9 (Left Module) *A left module A over a ring R is a system of sets and operations $\{A, R, \oplus, +, \cdot, *\}$ which satisfy the following properties.*

1. *$\{A, \oplus\}$ is an additive Abelian group.*

2. *$\{R, +, \cdot\}$ is a ring.*

3. *Scalar multiplication* $*: R \times A \to A$ *is a left action of the multiplicative monoid* $\{R, \cdot\}$ *of the ring on* A *which satisfies the following properties.*

   (a) *Distributivity over* A*'s addition*

$$\forall r \in R, a, b \in A : r * (a \oplus b) = r * a \oplus r * b \tag{1.13}$$

   (b) *Distributivity over* R*'s addition into* A*'s addition*

$$\forall r, s \in R, a \in A : (r + s) * a = r * a \oplus s * a \tag{1.14}$$

Definition 1.10 (Vector Space) *A vector space* V *over a field* $\mathbb{F}$ *is a left module with sets and operations* $\{V, \mathbb{F}, \oplus, +, \cdot, *\}$ *where the ring is restricted to a field. Scalar multiplication is then a left action of the multiplicative group* $\{\mathbb{F} \setminus 0, \cdot\}$.

Further, scalar multiplication can be used to relate the inverse and identity elements of the field and vector space.

$$\forall \mathbf{v} \in V : 0 * \mathbf{v} = \mathbf{0} \qquad \text{Proof } \mathbf{??} \tag{1.15}$$

$$\forall \mathbf{v} \in V : (-1) * \mathbf{v} = -\mathbf{v} \qquad \text{Proof } \mathbf{??} \tag{1.16}$$

In practice, the notation used is looser. Vector and scalar addition share the same symbol. Field and scalar multiplication symbols are usually dropped. The operations can be inferred from context.

Definition 1.11 (Affine Space) *An affine space* $\{X, V, +, \oplus\}$ *is a set* X*, vector space* V *over a field* $\mathbb{F}$*, and a right action* $\oplus : X \times V \to X$ *of the additive group* $\{V, +\}$ *which satisfies the following properties.*

1. *The action is*

   (a) *Free*

$$\forall \mathbf{v} \in V : x \oplus \mathbf{v} = x \implies \mathbf{v} = \mathbf{0} \tag{1.17}$$

   (b) *Transitive*

$$\forall x, y \in X : \exists \mathbf{v} \in V : x \oplus \mathbf{v} = y \tag{1.18}$$

This generalizes spaces with no inherent reference point, such as a Euclidean space or time. Geometric locations and dates can be considered in a relative sense. Assigning an origin or calendar is arbitrary, after all. Elements in X are called points and vectors in V are called translations or free vectors.

The definition above also implies the following properties.

$$\forall x, y \in X : \exists! \mathbf{v} \in V : x \oplus \mathbf{v} = y \qquad \text{Bijection for fixed } x\text{. Proof } \mathbf{??} \tag{1.19}$$

$$\forall \mathbf{v} \in V, y \in X : \exists! x \in X : x \oplus \mathbf{v} = y \qquad \text{Bijection for fixed } \mathbf{v}\text{. Proof } \mathbf{??} \tag{1.20}$$

Since these operations are invertible, both point-vector $\ominus : X \times V \to X$ and point-point $-- : X \times X \to V$ subtraction operations are well-defined (Proof **??**).

$$a \oplus (b - -a) = b \tag{1.21}$$

This operation also satisfies the following properties, called Weyl's axioms.

$$\forall x \in X, \mathbf{v} \in V : \exists! y \in x : y - -x = \mathbf{v} \qquad \text{Proof } \mathbf{??} \tag{1.22}$$

$$\forall x, y, z \in X : (c - -b) + (b - -a) = c - -a \qquad \text{Proof } \mathbf{??} \tag{1.23}$$

Furthermore, the parallelogram property is satisfied.

$$\forall w, x, y, x \in X : b - -a = d - -c \equiv c - -a = d - -b \tag{1.24}$$

Finally, any vector space is an affine space over itself (Proof **??**). This means that any element of V can be considered as a point or a vector. Importantly, this formalizes that notion of a reference point being arbitrary. Given a reference point $o \in A$, an isomorphism $\iota_o : X \leftrightarrow V$ can be constructed between A and V.

$$\iota_o(x) = x \ominus o \tag{1.25}$$

Definition 1.12 (Affine Subspace) *An affine subspace $W \subseteq X$ of an affine space $\{X, V, +, \oplus\}$ is a set W for which there exists some $w \in W$ such that generated vectors $\overrightarrow{W} = \{x \ominus w : x \in W\}$ is a linear subspace of V.*

It folllows that the choice of w is arbitrary, and all subspaces can be generated from linear subspaces $U \subseteq V$ as follows.

$$x + U := \{x + \mathbf{u} : \mathbf{u} \in U\} \tag{1.26}$$

The linear subspace U can be considered the affine subspace's direction, and two subspaces that share the same or subsidiary directions $T \subseteq U$ are considered parallel. The above properties imply Playfair's axiom: for any point $x \in X$ and direction $U \subseteq V$ there is a unique affine subspace of direction U (Proof **??**).

As with vector spaces, distinguishing these operations with special symbols is done for clarity. In practice, simple addition and subtraction are used, and such notation will be used moving forwards.

1.1.5 Basis

different kinds of bases, linear independence, dimension, isomorphism to Fn

1.1.6 Dual Space

algebraic, topological

1.1.7 Topological Functions

Definition 1.13 (Metric) *A metric $d : X \times X \to [0, \infty)$ is a function that generalizes distance between elements of the set X by satisfying the following properties.*

1. *Zero property*
$$\forall x, y \in X : d(x, y) = 0 \iff x = y \tag{1.27}$$
2. *Symmetry*
$$\forall x, y \in X : d(x, y) = d(y, x) \tag{1.28}$$
3. *Triangle inequality*
$$\forall x, y, z \in X : d(x, y) \leq d(x, z) + d(y, z) \tag{1.29}$$

A set with an associated metric is called a metric space.

Definition 1.14 (Norm) *A norm $\|\cdot\| : V \to [0, \infty)$ is a function that generalizes magnitude of vectors in a vector space X over a real or complex field $\mathbb{F} \subseteq \mathbb{C}$ by satisfying the following properties.*

1. *Positive-definiteness*
$$\forall \mathbf{v} \in V : \|\mathbf{v}\| = 0 \iff \mathbf{v} = \mathbf{0} \tag{1.30}$$
2. *Absolute homogeneity, i.e. scaling property*
$$\forall \alpha \in \mathbb{F}, \mathbf{v} \in V : \|\alpha \mathbf{v}\| = |\alpha| \|\mathbf{v}\| \tag{1.31}$$

3. *Triangle inequality*

$$\forall \mathbf{u}, \mathbf{v} \in V : \|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\| \tag{1.32}$$

A vector space with a norm is called a normed linear space. It is also a metric space using the induced metric.

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| \qquad \text{Proof } \mathbf{??} \tag{1.33}$$

All norms are equivalent in finite-dimensional spaces. See Proof **??**.

Definition 1.15 *An inner product* $(\cdot, \cdot) : V \times V \to \mathbb{F}$ *is a function that generalizes the similarity of vectors in a vector space* X *over a real or complex field* $\mathbb{F} \subseteq \mathbb{C}$ *by satisfying the following properties.*

1. *Positive-definiteness*

$$\forall \mathbf{x} \in X : (\mathbf{x}, \mathbf{x}) = 0 \equiv \mathbf{X} = \mathbf{0} \tag{1.34}$$

2. *Linearity in the first argument*

$$\forall \alpha_1, \alpha_2 \in \mathbb{F}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{y} \in X : (\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2, \mathbf{y}) = \alpha_1(\mathbf{x}_1, \mathbf{y}) + \alpha_2(\mathbf{x}_2, \mathbf{y}) \tag{1.35}$$

3. *Conjugate symmetry*

$$\forall \mathbf{x}, \mathbf{y} \in X : (\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})} \tag{1.36}$$

These conditions imply anti-linearity in the second argument, or linearity for functions over a real field.

$$\forall \beta_1, \beta_2 \in \mathbb{F}, \mathbf{x}, \mathbf{y}_1, \mathbf{y}_2 \in X : (\mathbf{x}, \beta_1 \mathbf{y}_1 + \beta_2 \mathbf{y}_2) = \beta_1(\mathbf{x}, \mathbf{y}_1) + \beta_2(\mathbf{x}, \mathbf{y}_2) \tag{1.37}$$

For any inner product, the Cauchy-Schwarz inequality applies.

$$(\mathbf{x}, \mathbf{y}) \leq \sqrt{(\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y})} \qquad \text{Proof } \mathbf{??} \tag{1.38}$$

A vector space with an inner product is called an inner product space. It is also a normed linear space and a metric space using the induced norm.

$$\|\mathbf{x}\| = \sqrt{(\mathbf{x}, \mathbf{x})} \qquad \text{Proof } \mathbf{??} \tag{1.39}$$

An inner product also allows for an abstract notion of angle, inspired by geometry.

$$\cos(\theta) = \frac{(\mathbf{x}, \mathbf{y})}{\|\mathbf{x}\|\|\mathbf{y}\|} \tag{1.40}$$

1.1.8 Riesz Representation

Riesz representation and dealing with complex vector spaces, dual basis cobasis link, also compare components

1.2 Geometric Spaces

Definition 1.16 (Euclidean Point Space) *The Euclidean point space*

Definition 1.17 (Euclidean Vector Space) *content...*

1.3 Vector Spaces in Continuum Mechanics

do a proper construction here

construction of euclidean geometry but don't get to lost

Rn is a Euclidean geometry, also cartesian coordinates

note on how velocity

also fields which are functions, algebra of fields is pointwise based, calculus is trickier

coordinate systems: cartesian, polar, spherical, navd88, wgs84

1.4 Getting Started

For all intents and purposes of continuum mechanics, it is sufficient to start with a Euclidean space, usually two- or three-dimensional, described by a Cartesian coordinate system, because this can be used to describe the physical world at our scale quite well.

1.4.1 Notation

Scalars are simply written as a letter, usually lower-case (a). Vectors are bolded ($\mathbf{a}$). When hand-written, they may have an arrow on top ($\vec{a}$) or a single underline ($\underline{a}$). Unit vectors, with a magnitude 1, may be marked with a circumflex ($\hat{\mathbf{a}}$). Higher-order tensors are bolded and usually upper-case ($\mathbf{A}$). When written, they may have a number of underlines equal to their order.

1.5 Bases and Components

We start with some right-handed orthonormal basis $\{\hat{\mathbf{e}}_i\}_{i=1}^n$ for an n-dimensional Euclidean vector space $\mathbb{R}^n$. If these align with a Cartesian coordinate frame, or when a Cartesian coordinate frame is defined to line up with them, they're called the canonical basis, i.e. the standard one. Using the geometric definitions for the dot and cross product, the following relations exist between the basis vectors, represented in shorthand by the Kronecker delta δ and Levi-Civita symbol ϵ.

These products are both bilinear. The dot product commutes, and the cross product anti-commutes.

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} := \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{1.41}$$

$$(\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j) \cdot \hat{\mathbf{e}}_k = \epsilon_{ijk} := \begin{cases} 1, & (i,j,k) = (1,2,3), (2,3,1), \text{or } (3,1,2), \\ -1, & (i,j,k) = (3,2,1), (2,1,3), \text{or } (1,3,2), \\ 0, & \text{otherwise.} \end{cases} \tag{1.42}$$

Since any basis set is linearly independent and spans the vector space, any vector in the space may be represented in terms of such a basis. The coefficients of the linear combination are called components. It is important to note that these are not the same as coordinates, especially for non-Cartesian coordinate systems.

$$\mathbf{a} = \sum_{i=1}^{n} a_i \hat{\mathbf{e}}_i \tag{1.43}$$

1.5.1 Cobasis

If the chosen basis is not orthonormal with respect to some inner product and corresponding induced norm, then there is some more nuance. Sidestepping the excess mental infrastructure of the dual space and Riesz representation, the cobasis of some basis $\{\mathbf{f}_i\}_{i=1}^n$, marked with a superscript as $\{\mathbf{f}^i\}_{i=1}^n$, is another basis for the same space, which is bi-orthonormal to the basis.

$$\mathbf{f}^i \cdot \mathbf{f}_j = \mathbf{f}_j \cdot \mathbf{f}^i = \delta_{ij} \tag{1.44}$$

Notation for components in a non-orthonormal basis is more complicated. The contravariant components of the basis use superscripts, and the covariant components of the cobasis use subscripts. This may seem backwards, but this is the convention that was developed.

$$\mathbf{a} = \sum_{i=1}^{n} \tilde{a}^i \mathbf{f}_i \tag{1.45}$$

$$\mathbf{a} = \sum_{i=1}^{n} \tilde{a}_i \mathbf{f}^i \tag{1.46}$$

Contra in contravariant refers to how the components scale inversely with the basis vectors. For example, if a basis is changed from inches to feet, the components used to represent a vector will shrink to represent the same length. On the other hand, the covariant components would increase.

If one goes further into the weeds of tensor-world, there is a whole shorthand system of upstairs and downstairs indices used for tensors defined by a certain combination of bases and cobases. However, in continuum-world, one can fudge over all of these nuances by choosing to use an orthonormal basis, which is conveniently its own cobasis.

1.5.2 Independence of Basis

That freedom of choice is an important idea behind this tensor system. Up until this point, you were probably used to seeing vectors and matrices as lists and tables of numbers. Start with a unit vector $(1, 0)$. Rotate the coordinate frame 45 degrees clockwise and you get $(\sqrt{2}/2, \sqrt{2}/2)$. This would seem like a new and unequal vector, since $(1, 0) \neq (\sqrt{2}/2, \sqrt{2}/2)$.

Writing them in terms of components of particular bases allows for more abstraction and flexibility. The vector is no longer the tuple per se, but is represented by a tuple in the context of a particular reference frame. The vector and the physical quantity it represents stay the same conceptually, regardless of the frame of reference: $\mathbf{a} = \sum_{i=1}^{n} a_i \hat{\mathbf{e}}_i = \sum_{i=1}^{n} \tilde{a}^i \mathbf{f}_i = \sum_{i=1}^{n} \tilde{a}_i \mathbf{f}^i$. Only components vary. So we can say for the prior example, labeling the coordinate frames as A and B, $(1, 0)_A = (\sqrt{2}/2, \sqrt{2}/2)_B$. Operations and properties like an inner product, cross product, norm should stay the same, too. However, it may not necessarily be that $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} \tilde{a}^i \tilde{b}^i$, because the dot product is defined geometrically, not algebraically.

1.5.3 Computing Components

differentiate coordinates and components

1.5.4 Curvilinear Coordinates

same vector using different coordinates

1.6 Kronecker Delta and Levi-Civita Symbol Identities

The Kronecker Delta and Levi-Civita Symbol have the following identities which are useful in proof.

$$a = b \qquad \text{Proof } \textbf{??} \tag{1.47}$$

$$a = b \qquad \text{Proof } \textbf{??} \tag{1.48}$$

$$\epsilon_{pqs}\epsilon_{nrs} = \delta_{pn}\delta_{qr} - \delta_{pr}\delta_{qn} \qquad \text{Proof } \textbf{??} \tag{1.49}$$

$$\epsilon_{pqs}\epsilon_{rqs} = 2\delta_{pr} \qquad \text{Proof } \textbf{??} \tag{1.50}$$

1.7 Einstein Notation

Chapter 2

Tensor Algebra

Scalars are so-called zeroth-order tensors, vectors are so-called first-order tensors, and any other kind of tensor can be constructed from these.

2.1 Abstract Definitions

start with linear operators, show component representation, introduce tensor product spaces with contraction operation and co/contra basis , explain how other operations follow, introduce isomorphisms between everything

2.1.1 Linear Operators

Definition 2.1 (Linear Transformation) *Given two vector spaces V, W over the same field $\mathbb{F}$, a map $T : V \to W$ is a linear transformation if it has the following properties.*

1. *Additive*

$$\forall \mathbf{x}, \mathbf{y} \in V : T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}) \tag{2.1}$$

2. *Homogeneous*

$$\forall \alpha \in \mathbb{F}, \mathbf{x} \in V : T(\alpha \mathbf{x}) = \alpha T(\mathbf{x}) \tag{2.2}$$

By convention, a linear operator's argument parentheses may be dropped: $T\mathbf{x} = T(\mathbf{x})$.

Given a finite-dimensional domain and range, a linear transformation can be represented with components with respect to a basis, similar to a vector or covector. This means that once bases have been chosen, linear transformations can be computed purely with components.

- V: vector space over $\mathbb{F}$, $\dim G = n$, with some basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$
- W: vector space over $\mathbb{F}$, $\dim G = m$, with some basis $\{\mathbf{f}_1, \ldots, \mathbf{f}_m\}$
- $T : V \to W$: linear transformation

Pick any vector $\mathbf{v} \in V$, and let $\mathbf{w} = T(\mathbf{v})$.

$$
\begin{aligned}
w_i &= \mathbf{f}_i^* \left(T(\mathbf{v})\right) \\
&= \mathbf{f}_i^* \left(T\left(\sum_{j=1}^{n} v^j \mathbf{e}_j\right)\right) \\
&= \sum_{j=1}^{n} \mathbf{f}_i^* \left(T(\mathbf{e}_j)\right) v^j \\
&= \sum_{j=1}^{n} T_{ij} v^j \\
T_{ij} &= \mathbf{f}_i^* \left(T(\mathbf{e}_j)\right)
\end{aligned}
$$

show linear operators are a vector space L(V, W)

check for upstairsing

matrix, linear algebra

Now, it's tempting to conclude $T = \sum_{i=1}^{m} \sum_{j=1}^{n} T_{ij} \mathbf{f}_i \mathbf{e}_j^*$, and that linear operators from V to W have some basis $\mathbf{f}_i \mathbf{e}_j^*$, but technically???

2.1.2 Dual/Adjoint Operator

2.1.3 Transpose Operator

maybe do some upstairs indices here

hilbert adjoint

2.1.4 Bilinear and Multilinear Operators

bilinear form gets you a tensor. Illustrate this, perhaps with a metric.

2.1.5 Tensor Product Space

aaa

2.2 Dyads and Dyadics

tensor product spaces

order and type covector-vector degree, dual = covector

build isomorphisms between tensors of same order

2.3 Contractions

2.4 Equivalence to Linear Algebra

isomorphic to linear algebra if appropriate combination of basis/cobasis

linearity

coordinate vs non coordinate basis

components

deltas and epsilons, symbols, only up to rank 2, etc

einstein notation

tensor as operator

basis independence

isomorphism
definitions contrived to fulfill certain properties
using dyadic symbol and not
defining property of dyadic, compare to linear algebra
simple properties of each product (distributive, associative, commutative) that may depend on the operands
linear algebra comparison
computing in canonical basis
introducing the dot product
introducing the cross product
introducing the dyadic product
specifically takes two vectors
non commutative
scalar rule
simple contractions
double contractions
change of basis
invariants of vectors and tensors
symmetric, skew-symmetric
complex numbers
operations between tensors (to fit with vectors)
metric tensor is in fact identity tensor in euclidean space
change of basis tensor is identity tensor
cross product is identity tensor

- $\phi, \kappa \in \mathbb{R}$
- $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^n$
- $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$
- $\boldsymbol{\omega} \in \mathbb{R}^p$
- $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$
- $\mathbf{Q}, \mathbf{R} \in \mathbb{R}^{m \times m}$
- $\mathbf{S}, \mathbf{T} \in \mathbb{R}^{m \times n}$
- $\mathbf{U} \in \mathbb{R}^{n \times p}$
- $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{3 \times 3}$

INTRODUCING VECTORS

$$\mathbf{S} = \sum_{i=1}^{m} \sum_{j=1}^{n} S_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \tag{2.3}$$

$$\mathbf{u} \times \mathbf{v} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} u_i v_j \epsilon_{ijk} \hat{\mathbf{e}}_k \tag{2.4}$$

$$ \tag{2.5}$$

INTRODUCING TENSORS

$$(\mathbf{a} \otimes \mathbf{x})\,\mathbf{y} := \mathbf{a}\,(\mathbf{x} \cdot \mathbf{y}) \tag{2.6}$$

$$\mathbf{a} \cdot (\mathbf{b} \otimes \mathbf{x}) := (\mathbf{a} \cdot \mathbf{b})\,\mathbf{x} \tag{2.7}$$

$$\mathbf{x} \cdot (\mathbf{y} \otimes \mathbf{a})\,\mathbf{b} = \ldots \qquad \text{Proof ??} \tag{2.8}$$

$$(\mathbf{a} \otimes \mathbf{x})^T := \mathbf{x} \otimes \mathbf{a} \tag{2.9}$$

$$\operatorname{tr}(\mathbf{a} \otimes \mathbf{b}) := \mathbf{a} \cdot \mathbf{b} \tag{2.10}$$

$$(\mathbf{a} \otimes \mathbf{x})\,(\mathbf{y} \otimes \boldsymbol{\omega}) = (\mathbf{x} \cdot \mathbf{y})\,(\mathbf{a} \otimes \boldsymbol{\omega}) \qquad \text{Proof ??} \tag{2.11}$$

$$(\mathbf{a} \otimes \mathbf{x}) : (\mathbf{b} \otimes \mathbf{y}) := (\mathbf{a} \cdot \mathbf{b})\,(\mathbf{x} \cdot \mathbf{y}) \tag{2.12}$$

$$\mathbf{a} \otimes \mathbf{x} = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i x_j \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \tag{2.13}$$

$$\tag{2.14}$$

CONTRACTIONS

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i \qquad \text{Proof A.1} \tag{2.15}$$

$$\mathbf{S}\mathbf{a} = \mathbf{S} \cdot \mathbf{a} = \sum_{i=1}^{m} \sum_{j=1}^{n} S_{ij} a_j \hat{\mathbf{e}}_i \qquad \text{Proof A.2} \tag{2.16}$$

$$\mathbf{x} \cdot \mathbf{S} = \sum_{i=1}^{m} \sum_{k=1}^{n} x_i S_{ik} \hat{\mathbf{e}}_k \qquad \text{Proof A.3} \tag{2.17}$$

$$\mathbf{S}\mathbf{U} = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{\ell=1}^{p} S_{ij} U_{j\ell}\,(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_\ell) \qquad \text{Proof A.4} \tag{2.18}$$

$$\mathbf{S} : \mathbf{T} = \sum_{i=1}^{m} \sum_{j=1}^{n} S_{ij} T_{ij} \qquad \text{Proof A.5} \tag{2.19}$$

2.5 Miscellaneous Operations

2.5.1 Change of Basis

The beauty of an abstract approach to tensors is recognizing that any change-of-basis matrix corresponds to the identity tensor expressed with components in a particular basis. After all, no matter what basis you use, the geometric quantity is the same, its the components and bases that change.

2.6 Helpful Identities

$$\mathbf{S}\mathbf{a} = \mathbf{a} \cdot \left(\mathbf{S}^T\right) \qquad \text{Proof ??} \tag{2.20}$$

$$(\mathbf{x} \cdot \mathbf{S}) \cdot \mathbf{a} = \mathbf{x} \cdot (\mathbf{S}\mathbf{a}) \qquad \text{Proof ??} \tag{2.21}$$

$$\left(\mathbf{S}^T\right) : \mathbf{V} = \mathbf{S} : \left(\mathbf{V}^T\right) \qquad \text{Proof ??} \tag{2.22}$$

Chapter 3

Tensor Calculus

differentiate nabla and grad

in different coordinate systems

expand to field...define operations...basis changes locally

algebraic and geometric definitions which conveniently coincide for a euclidean geometry described by a cartesian coordinate system

INTRODUCING NABLA OPERATOR

3.1 Introducing Nabla

$$\boldsymbol{\nabla} := \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \hat{\mathbf{e}}_i \tag{3.1}$$

$$\boldsymbol{\nabla}[\ldots] = \boldsymbol{\nabla} \otimes [\ldots] := \sum_{i=1}^{m} \frac{\partial}{\partial x_i} [\hat{\mathbf{e}}_i \otimes \ldots] \tag{3.2}$$

$$\boldsymbol{\nabla} \cdot [\ldots] := \sum_{i=1}^{m} \frac{\partial}{\partial x_i} [\hat{\mathbf{e}}_i \cdot \ldots] \tag{3.3}$$

$$\boldsymbol{\nabla} \times [\ldots] := \sum_{i=1}^{m} \frac{\partial}{\partial x_i} [\hat{\mathbf{e}}_i \times \ldots] \tag{3.4}$$

3.2 Gradient

$$\boldsymbol{\nabla}\phi = \ldots \qquad \text{Proof } \textbf{??} \tag{3.5}$$

$$\boldsymbol{\nabla}\mathbf{a} = \ldots \qquad \text{Proof } \textbf{??} \tag{3.6}$$

$$\boldsymbol{\nabla}\mathbf{S} = \ldots \qquad \text{Proof } \textbf{??} \tag{3.7}$$

$$\operatorname{grad}\phi = \ldots \tag{3.8}$$

$$\operatorname{grad}\mathbf{a} = \ldots \tag{3.9}$$

$$\operatorname{grad}\mathbf{S} = \ldots \tag{3.10}$$

$$\tag{3.11}$$

3.2.1 Gradient Product Rules

First Order Value, Zeroth Order Operand

$$\boldsymbol{\nabla}(\phi\kappa) = \phi\boldsymbol{\nabla}\kappa + \kappa\boldsymbol{\nabla}\phi \quad \text{Proof ??} \tag{3.12}$$
$$\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{b}) = (\boldsymbol{\nabla}\mathbf{b})\,\mathbf{a} + (\boldsymbol{\nabla}\mathbf{a})\,\mathbf{b} ∗ ∗∗ \quad \text{Proof ??} \tag{3.13}$$
$$\boldsymbol{\nabla}(\mathbf{S}:\mathbf{T}) = (\boldsymbol{\nabla}\mathbf{T}):\mathbf{S} + (\boldsymbol{\nabla}\mathbf{S}):\mathbf{T} \quad \text{Proof ??} \tag{3.14}$$

Second Order Value, First Order Operand

$$\boldsymbol{\nabla}(\phi\mathbf{a}) = \dots \quad \text{Proof ??} \tag{3.15}$$
$$\boldsymbol{\nabla}(\mathbf{S}\mathbf{a}) = \dots \quad \text{Proof ??} \tag{3.16}$$
$$\boldsymbol{\nabla}(\mathbf{y}\cdot\mathbf{S}) = \dots \quad \text{Proof ??} \tag{3.17}$$
$$\boldsymbol{\nabla}(\mathcal{D}:\mathbf{S}) = \dots \quad \text{Proof ??} \tag{3.18}$$
$$\boldsymbol{\nabla}(\mathbf{S}:\mathcal{D}) = \dots \quad \text{Proof ??} \tag{3.19}$$
$$\boldsymbol{\nabla}(\mathbf{u}\times\mathbf{v}) = \dots \quad \text{Proof ??} \tag{3.20}$$

Third Order Value, Second Order Operand

$$\boldsymbol{\nabla}(\phi\mathbf{S}) = \dots \quad \text{Proof ??} \tag{3.21}$$
$$\boldsymbol{\nabla}(\mathbf{a}\otimes\mathbf{y}) = \dots \quad \text{Proof ??} \tag{3.22}$$
$$\boldsymbol{\nabla}(\mathbf{S}\mathbf{U}) = \dots \quad \text{Proof ??} \tag{3.23}$$
$$\boldsymbol{\nabla}(\mathcal{D}\cdot\mathbf{a}) = \dots \quad \text{Proof ??} \tag{3.24}$$
$$\boldsymbol{\nabla}(\mathbf{y}\cdot\mathcal{D}) = \dots \quad \text{Proof ??} \tag{3.25}$$

3.3 Divergence

$$\boldsymbol{\nabla}\cdot\mathbf{a} = \dots \quad \text{Proof ??} \tag{3.26}$$
$$\boldsymbol{\nabla}\cdot\mathbf{S} = \dots \quad \text{Proof ??} \tag{3.27}$$
$$\operatorname{div}\mathbf{a} = \dots \quad \text{Proof ??} \tag{3.28}$$
$$\operatorname{div}\mathbf{S} = \dots \quad \text{Proof ??} \tag{3.29}$$

3.3.1 Divergence Product Rules

Zeroth Order Value, First Order Operand

$$\boldsymbol{\nabla}\cdot(\phi\mathbf{a}) = \phi\boldsymbol{\nabla}\cdot\mathbf{a} + \mathbf{a}\cdot\boldsymbol{\nabla}\phi \quad \text{Proof A.6} \tag{3.30}$$
$$\boldsymbol{\nabla}\cdot(\mathbf{S}\mathbf{a}) = \mathbf{S}:\boldsymbol{\nabla}\mathbf{a} + \mathbf{a}\cdot(\boldsymbol{\nabla}\cdot\mathbf{S}) \quad \text{Proof ??} \tag{3.31}$$
$$\boldsymbol{\nabla}\cdot(\mathbf{y}\cdot\mathbf{S}) = \left(\mathbf{S}^T\right):\boldsymbol{\nabla}\mathbf{y} + \mathbf{y}\cdot\left(\boldsymbol{\nabla}\cdot\left(\mathbf{S}^T\right)\right) \quad \text{Proof ??} \tag{3.32}$$
$$\boldsymbol{\nabla}\cdot(\mathbf{u}\times\mathbf{v}) = \dots \quad \text{Proof ??} \tag{3.33}$$

First Order Value, Second Order Operand

$$\boldsymbol{\nabla}\cdot(\phi\mathbf{S}) = \phi\boldsymbol{\nabla}\cdot\mathbf{S} + \boldsymbol{\nabla}\phi\cdot\mathbf{S} \quad \text{Proof A.7} \tag{3.34}$$
$$\boldsymbol{\nabla}\cdot(\mathbf{a}\otimes\mathbf{y}) = \mathbf{a}\cdot\boldsymbol{\nabla}\mathbf{y} + \mathbf{y}\,(\boldsymbol{\nabla}\cdot\mathbf{a}) \quad \text{Proof ??} \tag{3.35}$$
$$\boldsymbol{\nabla}\cdot(\mathbf{S}\mathbf{U}) = \mathbf{S}:(\boldsymbol{\nabla}\mathbf{U}) + (\boldsymbol{\nabla}\cdot\mathbf{S})\,\mathbf{U} \quad \text{Proof ??} \tag{3.36}$$

3.4 Curl

$$\boldsymbol{\nabla} \times \mathbf{a} = \dots \quad \text{Proof ??} \tag{3.37}$$
$$\boldsymbol{\nabla} \times \mathbf{S} = \dots \quad \text{Proof ??} \tag{3.38}$$
$$\operatorname{curl} \mathbf{a} = \dots \tag{3.39}$$
$$\operatorname{curl} \mathbf{S} = \dots \tag{3.40}$$

3.4.1 Curl Product Rules

First Order Value, First Order Operand

3.5 Laplacian

3.5.1 Laplacian Product Rules

Zeroth Order Value, Zeroth Order Operand

$$\nabla^2 (\phi\kappa) = \dots \quad \text{Proof ??} \tag{3.41}$$
$$\nabla^2 (\mathbf{a} \cdot \mathbf{b}) = \dots \quad \text{Proof ??} \tag{3.42}$$
$$\nabla^2 (\mathbf{S} : \mathbf{T}) = \dots \quad \text{Proof ??} \tag{3.43}$$

First Order Value, First Order Operand

$$\nabla^2 (\phi\mathbf{a}) = \dots \quad \text{Proof ??} \tag{3.44}$$
$$\nabla^2 (\mathbf{S}\mathbf{a}) = \dots \quad \text{Proof ??} \tag{3.45}$$
$$\nabla^2 (\mathbf{y} \cdot \mathbf{S}) = \dots \quad \text{Proof ??} \tag{3.46}$$
$$\nabla^2 (\mathbf{u} \times \mathbf{v}) = \dots \quad \text{Proof ??} \tag{3.47}$$

Second Order Value, Second Order Operand

$$\nabla^2 (\phi\mathbf{S}) = \dots \quad \text{Proof ??} \tag{3.48}$$
$$\nabla^2 (\mathbf{a} \otimes \mathbf{y}) = \dots \quad \text{Proof ??} \tag{3.49}$$
$$\nabla^2 (\mathbf{S}\mathbf{U}) = \dots \quad \text{Proof ??} \tag{3.50}$$

3.6 Other Second Derivatives

div of curl = 0
curl of grad = 0

3.7 Function Derivatives

derivative of vector value/input functions
compositions thereof
derivative of determinant, etc
grad div relation

Chapter 4

Kinematics

Topics:
different frames
deformation map
all the different kinds of tensors
material and spatial fields
derivatives of material and spatial fields
integral change of variables with jacobian

4.1 Kinematic Variables

- $\mathbf{U}, \mathbf{u}$: displacement in material/spatial frames
  - $\mathbf{U} := \boldsymbol{\phi}_t - \mathbf{X}$
- $\mathbf{V}, \mathbf{v}$: velocity in material/spatial frames
  - $\mathbf{V} := \frac{\partial \boldsymbol{\phi}}{\partial t}$
- $\mathbf{L}$: velocity gradient in spatial frame
  - $\mathbf{L} := \operatorname{grad} \mathbf{v}$
- $\mathbf{D} = \mathbf{D}(\mathbf{X}, t)$: symmetric component of velocity gradient in material frame???
  - $\mathbf{D} := \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right)$
- $\mathbf{W} = \mathbf{W}(\mathbf{X}, t)$: skew-symmetric component of velocity gradient in material frame???
  - $\mathbf{W} := \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^T\right)$

Chapter 5

Conservation Laws

5.1 Continuum Mechanics Assumptions

define control volume properly, relate vt to v0, same for omega

5.2 Conservation Law Variables

explain convention that $Q(\mathbf{X}, t) = q(\boldsymbol{\varphi}(\mathbf{X}, t), t)$, also define rho not and show that it is constant??

- ϱ, ρ: density in material/spatial frames
- $\boldsymbol{\Sigma}, \boldsymbol{\sigma}$: Cauchy stress tensor in material/spatial frames
- $\mathbf{S}$: second Piola-Kirchoff stress tensor in material frame
  - $\mathbf{S} := J\mathbf{F}^{-1}\boldsymbol{\Sigma}\mathbf{F}^{-T}$
- $\mathbf{P} = \mathbf{P}(\mathbf{X}, t)$: first Piola-Kirchoff stress tensor in material frame
  - $\mathbf{P} := J\boldsymbol{\Sigma}\mathbf{F}^{-T} = \mathbf{F}\mathbf{S}$
- $\mathbf{B}, \mathbf{b}$: body force per unit mass in material/spatial frames
- E, e: energy per unit mass in material frame
- I, i: internal energy per unit mass in material frame
- $i = i(\mathbf{X}, t)$: potential? energy per unit mass in spatial frame
- $\mathbf{Q}, \mathbf{q}$: heat flux in material/spatial frames

5.3 Reynold's Transport Theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} q \,\mathrm{d}\Omega_t = \int_{V(t)} \frac{\partial q}{\partial t}\,\mathrm{d}\Omega_t + \oint_{\partial V(t)} q\mathbf{v}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \qquad \text{Proof \textbf{??}} \quad (5.1)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \mathbf{q} \,\mathrm{d}\Omega_t = \int_{V(t)} \frac{\partial \mathbf{q}}{\partial t}\,\mathrm{d}\Omega_t + \oint_{\partial V(t)} \mathbf{q}\,(\mathbf{v}\cdot\hat{\mathbf{n}})\,\mathrm{d}S \qquad \text{Proof \textbf{??}} \quad (5.2)$$

5.4 Closures

5.4.1 Cauchy Stress Hypothesis

something something importance for divergence theorem and converting to differential form

$$\mathbf{t} = \boldsymbol{\sigma}\hat{\mathbf{n}} \tag{5.3}$$

5.4.2 Energy of Mass

$$E = I + \frac{1}{2}\mathbf{V} \cdot \mathbf{V} \tag{5.4}$$

$$e = i + \frac{1}{2}\mathbf{v} \cdot \mathbf{v} \tag{5.5}$$

5.5 Generic Scalar Conservation Laws

explain vars here, v sub omega, yadda yadda

5.5.1 Material Frame

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_0} QJ\,\mathrm{d}\Omega_0 = \oint_{\partial V_0} \boldsymbol{\Phi} \cdot \hat{\mathbf{N}} J\,\mathrm{d}S_0 + \int_{V_0} RJ\,\mathrm{d}\Omega_0 \qquad \text{Integral form. Proof ??} \tag{5.6}$$

$$\frac{\mathrm{d}(QJ)}{\mathrm{d}t} = -\operatorname{div}_{\mathbf{X}}\left(\boldsymbol{\Phi} \cdot \mathbf{F}^{-T} J\right) + RJ \qquad \text{Differential form. Proof ??} \tag{5.7}$$

If the conserved property is intensive, expressed per unit mass, the following conservation laws may be derived.

$$\int_{V_0} \rho_0 \frac{\mathrm{d}\tilde{Q}}{\mathrm{d}t}\,\mathrm{d}\Omega_0 = \oint_{\partial V_0} \boldsymbol{\Phi} \cdot \hat{\mathbf{N}} J\,\mathrm{d}S_0 + \int_{V_0} RJ\,\mathrm{d}\Omega_0 \qquad \text{Integral form. Proof ??} \tag{5.8}$$

$$\rho_0 \frac{\mathrm{d}\tilde{Q}}{\mathrm{d}t} = -\operatorname{div}_{\mathbf{X}}\left(\boldsymbol{\Phi} \cdot \mathbf{F}^{-T} J\right) + RJ \qquad \text{Differential form. Proof ??} \tag{5.9}$$

5.5.2 Spatial Frame

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} q\,\mathrm{d}\Omega_t = -\oint_{\partial V(t)} \boldsymbol{\phi} \cdot \hat{\mathbf{n}}\,\mathrm{d}S + \int_{V(t)} r\,\mathrm{d}\Omega_t \qquad \text{Integral form. Proof ??} \tag{5.10}$$

$$\frac{\partial q}{\partial t} + \operatorname{div}(q\mathbf{v}) = -\operatorname{div}\boldsymbol{\phi} + r \qquad \text{Differential form. Proof ??} \tag{5.11}$$

If the conserved property is intensive, expressed per unit mass, the following conservation laws may be derived.

$$\int_{V(t)} \rho\left(\frac{\partial \tilde{q}}{\partial t} + \operatorname{grad}\tilde{q} \cdot \mathbf{v}\right)\mathrm{d}\Omega_t = -\oint_{\partial V(t)} \boldsymbol{\phi} \cdot \hat{\mathbf{n}}\,\mathrm{d}S + \int_{V(t)} r\,\mathrm{d}\Omega_t \qquad \text{Integral form. Proof ??} \tag{5.12}$$

$$\rho\left(\frac{\partial \tilde{q}}{\partial t} + \operatorname{grad}\tilde{q} \cdot \mathbf{v}\right) = -\operatorname{div}\boldsymbol{\phi} + r \qquad \text{Differential form. Proof ??} \tag{5.13}$$

5.6 Generic Vector Conservation Laws

explain vars here, v sub omega, yadda yadda

5.6.1 Material Frame

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_0} \mathbf{Q}J \,\mathrm{d}\Omega_0 = \oint_{\partial V_0} \mathbf{\Phi}\hat{\mathbf{N}}J \,\mathrm{d}S_0 + \int_{V_0} \mathbf{R}J \,\mathrm{d}\Omega_0 \qquad \text{Integral form. Proof ?? (5.14)}$$

$$\frac{\mathrm{d}(\mathbf{Q}J)}{\mathrm{d}t} = -\operatorname{div}_{\mathbf{X}}\left(\mathbf{\Phi}\mathbf{F}^{-T}J\right) + \mathbf{R}J \qquad \text{Differential form. Proof ?? (5.15)}$$

If the conserved property is intensive, expressed per unit mass, the following conservation laws may be derived.

$$\int_{V_0} \rho_0 \frac{\mathrm{d}\tilde{\mathbf{Q}}}{\mathrm{d}t} \,\mathrm{d}\Omega_0 = \oint_{\partial V_0} \mathbf{\Phi}\hat{\mathbf{N}}J \,\mathrm{d}S_0 + \int_{V_0} \mathbf{R}J \,\mathrm{d}\Omega_0 \qquad \text{Integral form. Proof ?? (5.16)}$$

$$\rho_0 \frac{\mathrm{d}\tilde{\mathbf{Q}}}{\mathrm{d}t} = -\operatorname{div}_{\mathbf{X}}\left(\mathbf{\Phi}\mathbf{F}^{-t}J\right) + \mathbf{R}J \qquad \text{Differential form. Proof ?? (5.17)}$$

5.6.2 Spatial Frame

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \mathbf{q} \,\mathrm{d}\Omega_t = -\oint_{\partial V(t)} \boldsymbol{\phi}\hat{\mathbf{n}} \,\mathrm{d}S + \int_{V(t)} \mathbf{r} \,\mathrm{d}\Omega_t \qquad \text{Integral form. Proof ?? (5.18)}$$

$$\frac{\partial \mathbf{q}}{\partial t} + \operatorname{div}\left(\mathbf{q}\otimes\mathbf{v}\right) = -\operatorname{div}\boldsymbol{\phi} + \mathbf{r} \qquad \text{Differential form. Proof ?? (5.19)}$$

If the conserved property is intensive, expressed per unit mass, the following conservation laws may be derived.

$$\int_{V(t)} \rho\left(\frac{\partial \tilde{\mathbf{q}}}{\partial t} + (\operatorname{grad}\tilde{\mathbf{q}})\,\mathbf{v}\right) \mathrm{d}\Omega_t = -\oint_{\partial V(t)} \boldsymbol{\phi}\hat{\mathbf{n}} \,\mathrm{d}S + \int_{V(t)} \mathbf{r} \,\mathrm{d}\Omega_t \qquad \text{Integral form. Proof ?? (5.20)}$$

$$\rho\left(\frac{\partial \tilde{\mathbf{q}}}{\partial t} + (\operatorname{grad}\tilde{\mathbf{q}})\,\mathbf{v}\right) = -\operatorname{div}\boldsymbol{\phi} + \mathbf{r} \qquad \text{Differential form. Proof ?? (5.21)}$$

5.7 Conservation of Mass

for applied laws, there are a million ways to express, but only including useful ones

- Quantity: mass, $Q = \varrho, q = \rho$
- Surface fluxes: none, $\mathbf{\Phi} = \mathbf{0}, \boldsymbol{\phi} = \mathbf{0}$
- Internal sources: none, $R = 0, r = 0$

5.7.1 Material Frame

$$\rho_0 = \varrho J \qquad \text{Proof ?? (5.22)}$$

5.7.2 Spatial Frame

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\left(\rho\mathbf{v}\right) = 0 \qquad \text{Conservative form. Proof ?? (5.23)}$$

5.8 Conservation of Momentum

- Quantity: momentum, $\mathbf{Q} = \varrho \mathbf{V}, \mathbf{q} = \rho \mathbf{v}$
- Surface fluxes: Cauchy stress, $\mathbf{\Phi} = -\mathbf{\Sigma}, \boldsymbol{\phi} = -\boldsymbol{\sigma}$
- Internal sources: body forces, $\mathbf{R} = \varrho \mathbf{B}, \mathbf{r} = \rho \mathbf{b}$

5.8.1 Material Frame

$$\rho_0 \frac{\mathrm{d}V}{\mathrm{d}t} = \mathrm{div}_{\mathbf{X}}\, \mathbf{P} + \rho_0 \mathbf{B} \qquad \text{Proof ?? (5.24)}$$

5.8.2 Spatial Frame

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \mathrm{div}\,(\rho \mathbf{v} \otimes \mathbf{v}) = \mathrm{div}\, \boldsymbol{\sigma} + \rho \mathbf{b} \qquad \text{Conservative form. Proof ?? (5.25)}$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathrm{grad}\, \mathbf{v})\, \mathbf{v} \right) = \mathrm{div}\, \boldsymbol{\sigma} + \rho \mathbf{b} \qquad \text{Reduced form. Proof ?? (5.26)}$$

5.9 Conservation of Angular Momentum

- Quantity: angular momentum, $\mathbf{Q} = \boldsymbol{\varphi} \times \varrho \mathbf{V}, \mathbf{q} = \mathbf{x} \times \rho \mathbf{v}$
- Surface fluxes: Cauchy torque, $\mathbf{\Phi} = -\boldsymbol{\varphi} \times \mathbf{\Sigma}, \boldsymbol{\phi} = -\mathbf{x} \times \boldsymbol{\sigma}$
- Internal sources: body torques, $\mathbf{R} = \boldsymbol{\varphi} \times \varrho \mathbf{B}, \mathbf{r} = \mathbf{x} \times \rho \mathbf{b}$

5.9.1 Material Frame

$$\mathbf{S} = \mathbf{S}^T \qquad \text{Reduced form. Proof ?? (5.27)}$$

5.9.2 Spatial Frame

$$\frac{\partial (\mathbf{x} \times \rho \mathbf{v})}{\partial t} + \mathrm{div}\,(\mathbf{x} \times \rho \mathbf{v} \otimes \mathbf{v}) = \mathrm{div}\,(\mathbf{x} \times \boldsymbol{\sigma}) + \mathbf{x} \times \rho \mathbf{b} \qquad \text{Conservative form. Proof ?? (5.28)}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \qquad \text{Reduced form. Proof ?? (5.29)}$$

5.10 Conservation of Energy

- Quantity: energy, $Q = E, q = e$
- Surface fluxes: Cauchy power and heat flux, $\mathbf{\Phi} = -\mathbf{V} \cdot \mathbf{\Sigma} + \mathbf{Q}, \boldsymbol{\phi} = -\mathbf{v} \cdot \boldsymbol{\sigma} + \mathbf{q}$
- Internal sources: body force power and generic energy sources, $R = \varrho \mathbf{B} \cdot \mathbf{B} + \varrho R, r = \rho \mathbf{b} \cdot \mathbf{v} + \rho r$

5.10.1 Material Frame

$$\rho_0 \frac{\mathrm{d}I}{\mathrm{d}t} = \mathbf{S} : \mathbf{D} - \mathrm{div}_{\mathbf{X}}\, \mathbf{Q} + \rho_0 \mathbf{R} \qquad \text{Proof ?? (5.30)}$$

5.10.2 Spatial Frame

$$\frac{\partial(\rho e)}{\partial t} + \operatorname{div}(\rho e \mathbf{v}) = \operatorname{div}(\mathbf{v} \cdot \boldsymbol{\sigma} - \mathbf{q}) + \rho \mathbf{b} \cdot \mathbf{v} + \rho r \qquad \text{Conservative form. Proof } \textbf{??} \quad (5.31)$$

$$\rho\left(\frac{\partial e}{\partial t} + \operatorname{grad} e \cdot \mathbf{v}\right) = ??? - \operatorname{div} \mathbf{q} + \rho \mathbf{b} \cdot \mathbf{v} + \rho r \qquad \text{Reduced form. Proof } \textbf{??} \quad (5.32)$$

$$\rho\left(\frac{\partial i}{\partial t} + \operatorname{grad} i \cdot \mathbf{v}\right) = \boldsymbol{\sigma} : \mathbf{d} - \operatorname{div} \mathbf{q} + \rho r \qquad \text{Reduced form. Proof } \textbf{??} \quad (5.33)$$

5.11 Entropy Inequality

- Quantity: entropy, $Q = N, q = \eta$
- Surface fluxes: entropy from heat flux, $\boldsymbol{\Phi} = \frac{\mathbf{Q}}{\Theta}, \boldsymbol{\phi} = \frac{\mathbf{q}}{\theta}$
- Internal sources: entropy generation from energy sources, $R = \varrho \frac{R}{\Theta}, r = \rho \frac{r}{\theta}$

5.11.1 Material Frame

$$\rho_0 \frac{\mathrm{d}N}{\mathrm{d}t} + \operatorname{div}_{\mathbf{X}} \frac{\mathbf{Q}}{\Theta} - \rho_0 \frac{R}{\Theta} \geq 0 \qquad \text{Proof } \textbf{??} \quad (5.34)$$

5.11.2 Spatial Frame

$$\frac{\partial(\rho \eta)}{\partial t} + \operatorname{div}(\rho \eta \mathbf{v}) + \operatorname{div} \frac{\mathbf{q}}{\Theta} - \rho \frac{r}{\theta} \geq 0 \qquad \text{Conservative form. Proof } \textbf{??} \quad (5.35)$$

$$\rho\left(\frac{\partial \eta}{\partial t} + \operatorname{grad} \eta \cdot \mathbf{v}\right) + \operatorname{div} \frac{\mathbf{q}}{\Theta} - \rho \frac{r}{\theta} \geq 0 \qquad \text{Reduced form. Proof } \textbf{??} \quad (5.36)$$

5.12 Advection Diffusion Reaction Equation

This is essentially a variant of the conservation of momentum. It can be applied to particles advected by a fluid, for example.

- Quantity: mass, $Q = C, q = c$
- Surface fluxes: diffusion, $\boldsymbol{\Phi} = ???, \boldsymbol{\phi} = -\mathbf{D} \operatorname{grad} c$
- Internal sources: reaction, $R = R, r = r$

$$\frac{\partial c}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{v} c - \mathbf{D} \boldsymbol{\nabla} c) = R \qquad \text{Proof } \textbf{??} \quad (5.37)$$

$$\frac{\partial \mathbf{c}}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{v} \otimes \mathbf{c} - \mathcal{D} : \boldsymbol{\nabla} \mathbf{c}) = R \qquad \text{Proof } \textbf{??} \quad (5.38)$$

5.13 Advanced Formulations

e.g. curvature-dependent, hyperstress

Chapter 6

Solid Mechanics

general
elastic waves
thermoelastic solid
engineering stress example
eventually work to 1d stress-strain

6.1 Provisional Constitutive Laws

- $\mathbb{C}$ =???: elasticity tensor in material frame
  - $\mathbb{C} := \frac{\partial \mathbf{S}}{\partial E} = \frac{\partial^2 \hat{\Psi}}{\partial \mathbf{E}^2}$
- =???: elasticity tensor in spatial frame

6.2 Elastic Wave Equations

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \operatorname{div} \boldsymbol{\sigma} = 0 \qquad \text{Proof } \textbf{??} \quad (6.1)$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} = (\lambda + 2\mu) \operatorname{grad} (\operatorname{div} \mathbf{u}) - \mu \operatorname{curl}^2 \mathbf{u} \qquad \text{hm} \quad (6.2)$$

$$(6.3)$$

Chapter 7

Fluid Mechanics

full navier stokes
    incompressible
    rans
    stokes flow
    euler equations
    acoustic equations
    my special acoustic equations
    shallow water equations

7.1 Shallow Water Equations

conservative, non-conservative, cartesian, navd88, wgs84

$$\frac{\partial\left(\rho\eta\right)}{\partial t} + \operatorname{div}\left(\rho\eta\mathbf{u}\right) = 0 \qquad \text{Proof } \textbf{??} \quad (7.1)$$

$$\frac{\partial\left(\rho\eta\mathbf{u}\right)}{\partial t} + \operatorname{div}\left(\rho\eta\mathbf{u}\otimes\mathbf{u} + \frac{1}{2}\rho g\eta^2\mathbf{I}\right) = 0 \qquad \text{Proof } \textbf{??} \quad (7.2)$$

$$\frac{\partial\left(\rho\eta\right)}{\partial t} + \boldsymbol{\nabla}\cdot\left(\rho\eta\mathbf{u}\right) = 0 \qquad \text{Proof } \textbf{??} \quad (7.3)$$

$$\frac{\partial\left(\rho\eta\mathbf{u}\right)}{\partial t} + \boldsymbol{\nabla}\cdot\left(\rho\eta\mathbf{u}\otimes\mathbf{u}\right) + \boldsymbol{\nabla}\frac{1}{2}\rho g\eta^2 = 0 \qquad \text{Proof } \textbf{??} \quad (7.4)$$

7.1.1 Closures and Assumptions

diffusive wave equations
    bernoulli
    groundwater darcy flow
    telegrapher's equations

Appendix A

Proofs

Proof A.1 (Equation 2.15)

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= \left(\sum_{i=1}^{n} a_i\hat{\mathbf{e}}_i\right)\cdot\left(\sum_{j=1}^{n} b_j\hat{\mathbf{e}}_j\right) && \textit{Using 1.43} \quad \text{(A.1)}\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j \hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j && \text{(A.2)}\\
&= \sum_{i=1}^{n} a_i b_i && \textit{Using 1.41} \quad \text{(A.3)}
\end{aligned}$$

Proof A.2 (Equation 2.16)

$$\begin{aligned}
\mathbf{S}\mathbf{a} &= \left(\sum_{i=1}^{m}\sum_{j=1}^{n} S_{ij}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\right)\left(\sum_{k=1}^{n} a_k\hat{\mathbf{e}}_k\right) && \textit{Using 2.3, 1.43} \quad \text{(A.4)}\\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n} S_{ij}a_k\left(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\right)\hat{\mathbf{e}}_k && \text{(A.5)}\\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n} S_{ij}a_k\hat{\mathbf{e}}_i\left(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_k\right) && \textit{Using 2.6} \quad \text{(A.6)}\\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} S_{ij}a_j\hat{\mathbf{e}}_i && \textit{Using 1.41} \quad \text{(A.7)}
\end{aligned}$$

Proof A.3 (Equation 2.17)

$$\begin{aligned}
\mathbf{x}\cdot\mathbf{S} &= \left(\sum_{i=1}^{n} x_i\hat{\mathbf{e}}_i\right)\cdot\left(\sum_{j=1}^{m}\sum_{k=1}^{n} S_{jk}\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k\right) && \textit{Using 1.43, 2.3} \quad \text{(A.8)}\\
&= \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{n} x_i S_{jk}\hat{\mathbf{e}}_i\cdot\left(\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k\right) && \text{(A.9)}\\
&= \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{n} x_i S_{jk}\left(\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j\right)\hat{\mathbf{e}}_k && \textit{Using 2.7} \quad \text{(A.10)}\\
&= \sum_{i=1}^{m}\sum_{k=1}^{n} x_i S_{ik}\hat{\mathbf{e}}_k && \textit{Using 1.41} \quad \text{(A.11)}
\end{aligned}$$

Proof A.4 (Equation 2.18)

$$\mathbf{SU} = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} S_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\right)\left(\sum_{k=1}^{n}\sum_{\ell=1}^{p} U_{k\ell}\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell\right) \quad \textit{Using 2.3} \quad (A.12)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{p} S_{ij}U_{k\ell}\left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\right)\left(\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell\right) \quad (A.13)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{p} S_{ij}U_{k\ell}\left(\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_k\right)\left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_\ell\right) \quad \textit{Using 2.11} \quad (A.14)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{\ell=1}^{p} S_{ij}U_{j\ell}\left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_\ell\right) \quad \textit{Using 1.41} \quad (A.15)$$

Proof A.5 (Equation 2.19)

$$\mathbf{S} : \mathbf{T} = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} S_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\right) : \left(\sum_{k=1}^{m}\sum_{\ell=1}^{m} T_{k\ell}\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell\right) \quad \textit{Using 2.3} \quad (A.16)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{\ell=1}^{n} S_{ij}T_{k\ell}\left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\right) : \left(\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell\right) \quad (A.17)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{\ell=1}^{n} S_{ij}T_{k\ell}\left(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_k\right)\left(\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_\ell\right) \quad \textit{Using 2.12} \quad (A.18)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} S_{ij}T_{ij} \quad \textit{Using 1.41} \quad (A.19)$$

$$(A.20)$$

Proof A.6 (Equation 3.30)

$$\boldsymbol{\nabla} \cdot (\phi\mathbf{a}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[\hat{\mathbf{e}}_\mathbf{i} \cdot \left(\phi\sum_{j=1}^{n} a_j\hat{\mathbf{e}}_j\right)\right] \quad \textit{Using 3.3, 1.43} \quad (A.21)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial\phi a_j}{\partial x_i}\left(\hat{\mathbf{e}}_\mathbf{i} \cdot \hat{\mathbf{e}}_j\right) \quad (A.22)$$

$$= \sum_{i=1}^{n} \frac{\partial\phi a_i}{\partial x_i} \quad \textit{Using 1.41} \quad (A.23)$$

$$= \sum_{i=1}^{n}\left(\phi\frac{\partial a_i}{\partial x_i} + a_i\frac{\partial\phi}{\partial x_i}\right) \quad (A.24)$$

$$= \phi\boldsymbol{\nabla} \cdot \mathbf{a} + \mathbf{a} \cdot \boldsymbol{\nabla}\phi \quad \textit{Using 3.26, 2.15, 3.5} \quad (A.25)$$

Proof A.7 (Equation 3.34)

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot (\phi \mathbf{S}) &= \left(\sum_{i=1}^{m} \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \right) \cdot \left(\phi \sum_{j=1}^{m} \sum_{k=1}^{n} S_{jk} \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \right) && \textit{Using 3.3, 2.3} \quad \text{(A.26)} \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{\partial \left(\phi S_{jk}\right)}{\partial x_i} \hat{\mathbf{e}}_i \cdot \left(\hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k\right) && \text{(A.27)} \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{n} \left(\phi \frac{\partial S_{jk}}{\partial x_i} + S_{jk} \frac{\partial \phi}{\partial x_i} \right) \left(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j\right) \hat{\mathbf{e}}_k && \textit{Using 2.7} \quad \text{(A.28)} \\
&= \sum_{i=1}^{m} \sum_{k=1}^{n} \phi \frac{\partial S_{ik}}{\partial x_i} \hat{\mathbf{e}}_k + S_{ik} \frac{\partial \phi}{\partial x_i} \hat{\mathbf{e}}_k && \textit{Using 1.41} \quad \text{(A.29)} \\
&= \phi \boldsymbol{\nabla} \cdot \mathbf{S} + \boldsymbol{\nabla} \phi \cdot \mathbf{S} && \textit{Using 3.27, 3.5, 2.17} \quad \text{(A.30)}
\end{aligned}
$$

Bibliography

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