

# Tensor Bible

Ashton Cole

January 26, 2026



# Contents

<b>1</b>	<b>Vectors</b>	<b>3</b>
1.1	Vector Space . . . . .	3
1.2	Dual Space . . . . .	3
1.2.1	Isometric Isomorphism . . . . .	3
1.3	Inner Product and Norm . . . . .	3
1.4	Vector Spaces in Continuum Mechanics . . . . .	3
1.5	Getting Started . . . . .	3
1.5.1	Notation . . . . .	3
1.6	Bases and Components . . . . .	4
1.6.1	Cobasis . . . . .	4
1.6.2	Independence of Basis . . . . .	5
1.6.3	Computing Components . . . . .	5
1.6.4	Curvilinear Coordinates . . . . .	5
1.7	Kronecker Delta and Levi-Civita Symbol Identities . . . . .	5
1.8	Einstein Notation . . . . .	5
<b>2</b>	<b>Tensor Algebra</b>	<b>7</b>
2.1	Dyads and Dyadics . . . . .	7
2.2	Contractions . . . . .	7
2.3	Equivalence to Linear Algebra . . . . .	7
2.4	Miscellaneous Operations . . . . .	9
2.4.1	Change of Basis . . . . .	9
2.5	Helpful Identities . . . . .	9
<b>3</b>	<b>Tensor Calculus</b>	<b>11</b>
3.1	Introducing Nabla . . . . .	11
3.2	Gradient . . . . .	11
3.2.1	Gradient Product Rules . . . . .	12
3.3	Divergence . . . . .	12
3.3.1	Divergence Product Rules . . . . .	12
3.4	Curl . . . . .	13
3.4.1	Curl Product Rules . . . . .	13
3.5	Laplacian . . . . .	13
3.5.1	Laplacian Product Rules . . . . .	13
3.6	Other Second Derivatives . . . . .	13
3.7	Function Derivatives . . . . .	13
<b>4</b>	<b>Kinematics</b>	<b>15</b>
4.1	Kinematic Variables . . . . .	15

<b>5</b>	<b>Conservation Laws</b>	<b>17</b>
5.1	Continuum Mechanics Assumptions . . . . .	17
5.2	Conservation Law Variables . . . . .	17
5.3	Reynold's Transport Theorem . . . . .	18
5.4	Generic Scalar Conservation Laws . . . . .	18
5.4.1	Material Frame . . . . .	18
5.4.2	Spatial Frame . . . . .	18
5.5	Generic Vector Conservation Laws . . . . .	18
5.5.1	Material Frame . . . . .	19
5.5.2	Spatial Frame . . . . .	19
5.6	Conservation of Mass . . . . .	19
5.6.1	Material Frame . . . . .	19
5.6.2	Spatial Frame . . . . .	19
5.7	Conservation of Momentum . . . . .	19
5.7.1	Material Frame . . . . .	19
5.7.2	Spatial Frame . . . . .	20
5.8	Conservation of Angular Momentum . . . . .	20
5.8.1	Material Frame . . . . .	20
5.8.2	Spatial Frame . . . . .	20
5.9	Conservation of Energy . . . . .	20
5.9.1	Material Frame . . . . .	20
5.9.2	Spatial Frame . . . . .	20
5.10	Entropy Inequality . . . . .	20
5.10.1	Material Frame . . . . .	20
5.10.2	Spatial Frame . . . . .	20
5.11	Advection Diffusion Reaction Equation . . . . .	21
<b>6</b>	<b>Solid Mechanics</b>	<b>23</b>
6.1	Provisional Constitutive Laws . . . . .	23
6.2	Elastic Wave Equations . . . . .	23
<b>7</b>	<b>Fluid Mechanics</b>	<b>25</b>
7.1	Shallow Water Equations . . . . .	25
7.1.1	Closures and Assumptions . . . . .	25
<b>A</b>	<b>Proofs</b>	<b>27</b>

# List of Theorems

A.1	Proof (Equation 2.13)	27
A.2	Proof (Equation 2.14)	27
A.3	Proof (Equation 2.15)	27
A.4	Proof (Equation 2.16)	28
A.5	Proof (Equation 2.17)	28
A.6	Proof (Equation 3.30)	28
A.7	Proof (Equation 3.34)	28



# Introduction

before, big picture, what is a tensor, algebraic object describing multilinear relationship between algebraic objects [1]





# Chapter 1

## Vectors

Ironically enough, this was the hardest section to write. It is easy to get lost in abstraction and pedantics.

I think that the best way to start is with vectors. Tensors are always defined relative to some kind of vector space. Scalars are so-called zeroth-order tensors, vectors are so-called first-order tensors, and any other kind of tensor can be constructed from these.

### 1.1 Vector Space

examples of vector spaces

### 1.2 Dual Space

#### 1.2.1 Isometric Isomorphism

Riesz representation and dealing with complex vector spaces

### 1.3 Inner Product and Norm

norm is also a metric

### 1.4 Vector Spaces in Continuum Mechanics

construction of euclidean geometry but don't get to lost

$\mathbb{R}^n$  is a Euclidean geometry, also cartesian coordinates

note on how velocity

also fields which are functions

### 1.5 Getting Started

For all intents and purposes of continuum mechanics, it is sufficient to start with a Euclidean space, usually two- or three-dimensional, described by a Cartesian coordinate system, because this can be used to describe the physical world at our scale quite well.

#### 1.5.1 Notation

Scalars are simply written as a letter, usually lower-case ( $a$ ). Vectors are bolded ( $\mathbf{a}$ ). When hand-written, they may have an arrow on top ( $\vec{a}$ ) or a single underline ( $\underline{a}$ ). Unit vectors, with a magnitude 1, may be

marked with a circumflex ( $\hat{\mathbf{a}}$ ). Higher-order tensors are bolded and usually upper-case ( $\mathbf{A}$ ). When written, they may have a number of underlines equal to their order.

## 1.6 Bases and Components

We start with some right-handed orthonormal basis  $\{\hat{\mathbf{e}}_i\}_{i=1}^n$  for an  $n$ -dimensional Euclidean vector space  $\mathbb{R}^n$ . If these align with a Cartesian coordinate frame, or when a Cartesian coordinate frame is defined to line up with them, they're called the canonical basis, i.e. the standard one. Using the geometric definitions for the dot and cross product, the following relations exist between the basis vectors, represented in shorthand by the Kronecker delta  $\delta$  and Levi-Civita symbol  $\epsilon$ .

These products are both bilinear. The dot product commutes, and the cross product anti-commutes.

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} := \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \quad (1.1)$$

$$(\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j) \cdot \hat{\mathbf{e}}_k = \epsilon_{ijk} := \begin{cases} 1, & (i, j, k) = (1, 2, 3), (2, 3, 1), \text{ or } (3, 1, 2), \\ -1, & (i, j, k) = (3, 2, 1), (2, 1, 3), \text{ or } (1, 3, 2), \\ 0, & \text{otherwise.} \end{cases} \quad (1.2)$$

Since any basis set is linearly independent and spans the vector space, any vector in the space may be represented in terms of such a basis. The coefficients of the linear combination are called components. It is important to note that these are not the same as coordinates, especially for non-Cartesian coordinate systems.

$$\mathbf{a} = \sum_{i=1}^n a_i \hat{\mathbf{e}}_i \quad (1.3)$$

### 1.6.1 Cobasis

If the chosen basis is not orthonormal with respect to some inner product and corresponding induced norm, then there is some more nuance. Sidestepping the excess mental infrastructure of the dual space and Riesz representation, the cobasis of some basis  $\{\mathbf{f}_i\}_{i=1}^n$ , marked with a superscript as  $\{\mathbf{f}^i\}_{i=1}^n$ , is another basis for the same space, which is bi-orthonormal to the basis.

$$\mathbf{f}^i \cdot \mathbf{f}_j = \mathbf{f}_j \cdot \mathbf{f}^i = \delta_{ij} \quad (1.4)$$

Notation for components in a non-orthonormal basis is more complicated. The contravariant components of the basis use superscripts, and the covariant components of the cobasis use subscripts. This may seem backwards, but this is the convention that was developed.

$$\mathbf{a} = \sum_{i=1}^n \tilde{a}^i \mathbf{f}_i \quad (1.5)$$

$$\mathbf{a} = \sum_{i=1}^n \tilde{a}_i \mathbf{f}^i \quad (1.6)$$

Contra in contravariant refers to how the components scale inversely with the basis vectors. For example, if a basis is changed from inches to feet, the components used to represent a vector will shrink to represent the same length. On the other hand, the covariant components would increase.

If one goes further into the weeds of tensor-world, there is a whole shorthand system of upstairs and downstairs indices used for tensors defined by a certain combination of bases and cobases. However, in continuum-world, one can fudge over all of these nuances by choosing to use an orthonormal basis, which is conveniently its own cobasis.

### 1.6.2 Independence of Basis

That freedom of choice is an important idea behind this tensor system. Up until this point, you were probably used to seeing vectors and matrices as lists and tables of numbers. Start with a unit vector  $(1, 0)$ . Rotate the coordinate frame 45 degrees clockwise and you get  $(\sqrt{2}/2, \sqrt{2}/2)$ . This would seem like a new and unequal vector, since  $(1, 0) \neq (\sqrt{2}/2, \sqrt{2}/2)$ .

Writing them in terms of components of particular bases allows for more abstraction and flexibility. The vector is no longer the tuple per se, but is represented by a tuple in the context of a particular reference frame. The vector and the physical quantity it represents stay the same conceptually, regardless of the frame of reference:  $\mathbf{a} = \sum_{i=1}^n a_i \hat{\mathbf{e}}_i = \sum_{i=1}^n \tilde{a}^i \mathbf{f}_i = \sum_{i=1}^n \tilde{a}_i \mathbf{f}^i$ . Only components vary. So we can say for the prior example, labeling the coordinate frames as A and B,  $(1, 0)_A = (\sqrt{2}/2, \sqrt{2}/2)_B$ . Operations and properties like an inner product, cross product, norm should stay the same, too. However, it may not necessarily be that  $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^n \tilde{a}^i \tilde{b}_i$ , because the dot product is defined geometrically, not algebraically.

### 1.6.3 Computing Components

differentiate coordinates and components

### 1.6.4 Curvilinear Coordinates

same vector using different coordinates

## 1.7 Kronecker Delta and Levi-Civita Symbol Identities

The Kronecker Delta and Levi-Civita Symbol have the following identities which are useful in proof.

$$a = b \quad \text{Proof ??} \quad (1.7)$$

$$a = b \quad \text{Proof ??} \quad (1.8)$$

$$\epsilon_{pqs} \epsilon_{nrs} = \delta_{pn} \delta_{qr} - \delta_{pr} \delta_{qn} \quad \text{Proof ??} \quad (1.9)$$

$$\epsilon_{pqs} \epsilon_{rqs} = 2\delta_{pr} \quad \text{Proof ??} \quad (1.10)$$

## 1.8 Einstein Notation



# Chapter 2

## Tensor Algebra

aaa

### 2.1 Dyads and Dyadics

tensor product spaces  
order and covector-vector degree

### 2.2 Contractions

### 2.3 Equivalence to Linear Algebra

isomorphic to linear algebra if appropriate combination of basis/cobasis

linearity

coordinate vs non coordinate basis

components

deltas and epsilons, symbols, only up to rank 2, etc

einstein notation

tensor as operator

basis independence

isomorphism

definitions contrived to fulfill certain properties

using dyadic symbol and not

defining property of dyadic, compare to linear algebra

simple properties of each product (distributive, associative, commutative) that may depend on the  
operands

linear algebra comparison

computing in canonical basis

introducing the dot product

introducing the cross product

introducing the dyadic product

specifically takes two vectors

non commutative

scalar rule

simple contractions

double contractions

change of basis

invariants of vectors and tensors

symmetric, skew-symmetric

complex numbers

operations between tensors (to fit with vectors)

- $\phi, \kappa \in \mathbb{R}$
- $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^n$
- $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$
- $\boldsymbol{\omega} \in \mathbb{R}^p$
- $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$
- $\mathbf{Q}, \mathbf{R} \in \mathbb{R}^{m \times m}$
- $\mathbf{S}, \mathbf{T} \in \mathbb{R}^{m \times n}$
- $\mathbf{U} \in \mathbb{R}^{n \times p}$
- $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{3 \times 3}$

## INTRODUCING VECTORS

$$\mathbf{S} = \sum_{i=1}^m \sum_{j=1}^n S_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \quad (2.1)$$

$$\mathbf{u} \times \mathbf{v} = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 u_i v_j \epsilon_{ijk} \hat{\mathbf{e}}_k \quad (2.2)$$

$$(2.3)$$

## INTRODUCING TENSORS

$$(\mathbf{a} \otimes \mathbf{x}) \mathbf{y} := \mathbf{a} (\mathbf{x} \cdot \mathbf{y}) \quad (2.4)$$

$$\mathbf{a} \cdot (\mathbf{b} \otimes \mathbf{x}) := (\mathbf{a} \cdot \mathbf{b}) \mathbf{x} \quad (2.5)$$

$$\mathbf{x} \cdot (\mathbf{y} \otimes \mathbf{a}) \mathbf{b} = \dots \quad \text{Proof ??} \quad (2.6)$$

$$(\mathbf{a} \otimes \mathbf{x})^T := \mathbf{x} \otimes \mathbf{a} \quad (2.7)$$

$$\text{tr}(\mathbf{a} \otimes \mathbf{b}) := \mathbf{a} \cdot \mathbf{b} \quad (2.8)$$

$$(\mathbf{a} \otimes \mathbf{x}) (\mathbf{y} \otimes \boldsymbol{\omega}) = (\mathbf{x} \cdot \mathbf{y}) (\mathbf{a} \otimes \boldsymbol{\omega}) \quad \text{Proof ??} \quad (2.9)$$

$$(\mathbf{a} \otimes \mathbf{x}) : (\mathbf{b} \otimes \mathbf{y}) := (\mathbf{a} \cdot \mathbf{b}) (\mathbf{x} \cdot \mathbf{y}) \quad (2.10)$$

$$\mathbf{a} \otimes \mathbf{x} = \sum_{i=1}^n \sum_{j=1}^m a_i x_j \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \quad (2.11)$$

$$(2.12)$$

## CONTRACTIONS

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^n a_i b_i \quad \text{Proof A.1 (2.13)}$$

$$\mathbf{S}\mathbf{a} = \mathbf{S} \cdot \mathbf{a} = \sum_{i=1}^m \sum_{j=1}^n S_{ij} a_j \hat{\mathbf{e}}_i \quad \text{Proof A.2 (2.14)}$$

$$\mathbf{x} \cdot \mathbf{S} = \sum_{i=1}^m \sum_{k=1}^n x_i S_{ik} \hat{\mathbf{e}}_k \quad \text{Proof A.3 (2.15)}$$

$$\mathbf{S}\mathbf{U} = \sum_{i=1}^m \sum_{j=1}^n \sum_{\ell=1}^p S_{ij} U_{j\ell} (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_\ell) \quad \text{Proof A.4 (2.16)}$$

$$\mathbf{S} : \mathbf{T} = \sum_{i=1}^m \sum_{j=1}^n S_{ij} T_{ij} \quad \text{Proof A.5 (2.17)}$$

## 2.4 Miscellaneous Operations

### 2.4.1 Change of Basis

## 2.5 Helpful Identities

$$\mathbf{S}\mathbf{a} = \mathbf{a} \cdot (\mathbf{S}^T) \quad \text{Proof ?? (2.18)}$$

$$(\mathbf{x} \cdot \mathbf{S}) \cdot \mathbf{a} = \mathbf{x} \cdot (\mathbf{S}\mathbf{a}) \quad \text{Proof ?? (2.19)}$$

$$(\mathbf{S}^T) : \mathbf{V} = \mathbf{S} : (\mathbf{V}^T) \quad \text{Proof ?? (2.20)}$$





# Chapter 3

## Tensor Calculus

differentiate nabla and grad

in different coordinate systems

expand to field...define operations...basis changes locally

algebraic and geometric definitions which conveniently coincide for a euclidean geometry described by a cartesian coordinate system

INTRODUCING NABLA OPERATOR

### 3.1 Introducing Nabla

$$\nabla := \sum_{i=1}^m \frac{\partial}{\partial x_i} \hat{\mathbf{e}}_i \quad (3.1)$$

$$\nabla[\dots] = \nabla \otimes [\dots] := \sum_{i=1}^m \frac{\partial}{\partial x_i} [\hat{\mathbf{e}}_i \otimes \dots] \quad (3.2)$$

$$\nabla \cdot [\dots] := \sum_{i=1}^m \frac{\partial}{\partial x_i} [\hat{\mathbf{e}}_i \cdot \dots] \quad (3.3)$$

$$\nabla \times [\dots] := \sum_{i=1}^m \frac{\partial}{\partial x_i} [\hat{\mathbf{e}}_i \times \dots] \quad (3.4)$$

### 3.2 Gradient

$$\nabla \phi = \dots \quad \text{Proof ?? (3.5)}$$

$$\nabla \mathbf{a} = \dots \quad \text{Proof ?? (3.6)}$$

$$\nabla \mathbf{S} = \dots \quad \text{Proof ?? (3.7)}$$

$$\text{grad } \phi = \dots \quad (3.8)$$

$$\text{grad } \mathbf{a} = \dots \quad (3.9)$$

$$\text{grad } \mathbf{S} = \dots \quad (3.10)$$

$$(3.11)$$

### 3.2.1 Gradient Product Rules

#### First Order Value, Zeroth Order Operand

$$\nabla(\phi\kappa) = \phi\nabla\kappa + \kappa\nabla\phi \quad \text{Proof ?? (3.12)}$$

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\nabla\mathbf{b})\mathbf{a} + (\nabla\mathbf{a})\mathbf{b} \quad \text{Proof ?? (3.13)}$$

$$\nabla(\mathbf{S} : \mathbf{T}) = (\nabla\mathbf{T}) : \mathbf{S} + (\nabla\mathbf{S}) : \mathbf{T} \quad \text{Proof ?? (3.14)}$$

#### Second Order Value, First Order Operand

$$\nabla(\phi\mathbf{a}) = \dots \quad \text{Proof ?? (3.15)}$$

$$\nabla(\mathbf{S}\mathbf{a}) = \dots \quad \text{Proof ?? (3.16)}$$

$$\nabla(\mathbf{y} \cdot \mathbf{S}) = \dots \quad \text{Proof ?? (3.17)}$$

$$\nabla(\mathcal{D} : \mathbf{S}) = \dots \quad \text{Proof ?? (3.18)}$$

$$\nabla(\mathbf{S} : \mathcal{D}) = \dots \quad \text{Proof ?? (3.19)}$$

$$\nabla(\mathbf{u} \times \mathbf{v}) = \dots \quad \text{Proof ?? (3.20)}$$

#### Third Order Value, Second Order Operand

$$\nabla(\phi\mathbf{S}) = \dots \quad \text{Proof ?? (3.21)}$$

$$\nabla(\mathbf{a} \otimes \mathbf{y}) = \dots \quad \text{Proof ?? (3.22)}$$

$$\nabla(\mathbf{S}\mathbf{U}) = \dots \quad \text{Proof ?? (3.23)}$$

$$\nabla(\mathcal{D} \cdot \mathbf{a}) = \dots \quad \text{Proof ?? (3.24)}$$

$$\nabla(\mathbf{y} \cdot \mathcal{D}) = \dots \quad \text{Proof ?? (3.25)}$$

## 3.3 Divergence

$$\nabla \cdot \mathbf{a} = \dots \quad \text{Proof ?? (3.26)}$$

$$\nabla \cdot \mathbf{S} = \dots \quad \text{Proof ?? (3.27)}$$

$$\text{div } \mathbf{a} = \dots \quad \text{Proof ?? (3.28)}$$

$$\text{div } \mathbf{S} = \dots \quad \text{Proof ?? (3.29)}$$

### 3.3.1 Divergence Product Rules

#### Zeroth Order Value, First Order Operand

$$\nabla \cdot (\phi\mathbf{a}) = \phi\nabla \cdot \mathbf{a} + \mathbf{a} \cdot \nabla\phi \quad \text{Proof A.6 (3.30)}$$

$$\nabla \cdot (\mathbf{S}\mathbf{a}) = \mathbf{S} : \nabla\mathbf{a} + \mathbf{a} \cdot (\nabla \cdot \mathbf{S}) \quad \text{Proof ?? (3.31)}$$

$$\nabla \cdot (\mathbf{y} \cdot \mathbf{S}) = (\mathbf{S}^T) : \nabla\mathbf{y} + \mathbf{y} \cdot (\nabla \cdot (\mathbf{S}^T)) \quad \text{Proof ?? (3.32)}$$

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \dots \quad \text{Proof ?? (3.33)}$$

#### First Order Value, Second Order Operand

$$\nabla \cdot (\phi\mathbf{S}) = \phi\nabla \cdot \mathbf{S} + \nabla\phi \cdot \mathbf{S} \quad \text{Proof A.7 (3.34)}$$

$$\nabla \cdot (\mathbf{a} \otimes \mathbf{y}) = \mathbf{a} \cdot \nabla\mathbf{y} + \mathbf{y} (\nabla \cdot \mathbf{a}) \quad \text{Proof ?? (3.35)}$$

$$\nabla \cdot (\mathbf{S}\mathbf{U}) = \mathbf{S} : (\nabla\mathbf{U}) + (\nabla \cdot \mathbf{S}) \mathbf{U} \quad \text{Proof ?? (3.36)}$$

## 3.4 Curl

$$\nabla \times \mathbf{a} = \dots \quad \text{Proof ?? (3.37)}$$

$$\nabla \times \mathbf{S} = \dots \quad \text{Proof ?? (3.38)}$$

$$\text{curl } \mathbf{a} = \dots \quad (3.39)$$

$$\text{curl } \mathbf{S} = \dots \quad (3.40)$$

### 3.4.1 Curl Product Rules

First Order Value, First Order Operand

## 3.5 Laplacian

### 3.5.1 Laplacian Product Rules

Zeroth Order Value, Zeroth Order Operand

$$\nabla^2 (\phi \kappa) = \dots \quad \text{Proof ?? (3.41)}$$

$$\nabla^2 (\mathbf{a} \cdot \mathbf{b}) = \dots \quad \text{Proof ?? (3.42)}$$

$$\nabla^2 (\mathbf{S} : \mathbf{T}) = \dots \quad \text{Proof ?? (3.43)}$$

First Order Value, First Order Operand

$$\nabla^2 (\phi \mathbf{a}) = \dots \quad \text{Proof ?? (3.44)}$$

$$\nabla^2 (\mathbf{S} \mathbf{a}) = \dots \quad \text{Proof ?? (3.45)}$$

$$\nabla^2 (\mathbf{y} \cdot \mathbf{S}) = \dots \quad \text{Proof ?? (3.46)}$$

$$\nabla^2 (\mathbf{u} \times \mathbf{v}) = \dots \quad \text{Proof ?? (3.47)}$$

Second Order Value, Second Order Operand

$$\nabla^2 (\phi \mathbf{S}) = \dots \quad \text{Proof ?? (3.48)}$$

$$\nabla^2 (\mathbf{a} \otimes \mathbf{y}) = \dots \quad \text{Proof ?? (3.49)}$$

$$\nabla^2 (\mathbf{S} \mathbf{U}) = \dots \quad \text{Proof ?? (3.50)}$$

## 3.6 Other Second Derivatives

div of curl = 0

curl of grad = 0

## 3.7 Function Derivatives

derivative of vector value/input functions

compositions thereof

derivative of determinant, etc

grad div relation



# Chapter 4

## Kinematics

Topics:

- different frames
- deformation map
- all the different kinds of tensors
- material and spatial fields
- derivatives of material and spatial fields
- integral change of variables with jacobian

### 4.1 Kinematic Variables

- $\mathbf{U}, \mathbf{u}$ : displacement in material/spatial frames
  - $\mathbf{U} := \boldsymbol{\phi}_t - \mathbf{X}$
- $\mathbf{V}, \mathbf{v}$ : velocity in material/spatial frames
  - $\mathbf{V} := \frac{\partial \boldsymbol{\phi}}{\partial t}$
- $\mathbf{L}$ : velocity gradient in spatial frame
  - $\mathbf{L} := \text{grad } \mathbf{v}$
- $\mathbf{D} = \mathbf{D}(\mathbf{X}, t)$ : symmetric component of velocity gradient in material frame???
  - $\mathbf{D} := \frac{1}{2} (\mathbf{L} + \mathbf{L}^T)$
- $\mathbf{W} = \mathbf{W}(\mathbf{X}, t)$ : skew-symmetric component of velocity gradient in material frame???
  - $\mathbf{W} := \frac{1}{2} (\mathbf{L} - \mathbf{L}^T)$



# Chapter 5

## Conservation Laws

Topics:

- continuum assumptions
- reynolds transport theorem, i.e. leibniz integral rule
- cauchy theorem
- cauchy theorem for heat flux
- general conservation laws: integral and differential, material and spatial
- mass
- momentum
- energy
- entropy inequality
- define control volume properly, relate  $v_t$  to  $v_0$ , same for  $\omega$
- explain variable capitalization is just change of domain mapping, don't confuse with piola tensors
- for applied laws, there are a million ways to express, but only including useful ones

### 5.1 Continuum Mechanics Assumptions

### 5.2 Conservation Law Variables

explain convention that  $Q(\mathbf{X}, t) = q(\boldsymbol{\varphi}(\mathbf{X}, t), t)$ , also define  $\rho$  not and show that it is constant??

- $\varrho, \rho$ : density in material/spatial frames
- $\boldsymbol{\Sigma}, \boldsymbol{\sigma}$ : Cauchy stress tensor in material/spatial frames
- $\mathbf{S}$ : second Piola-Kirchoff stress tensor in material frame
  - $\mathbf{S} := J\mathbf{F}^{-1}\boldsymbol{\Sigma}\mathbf{F}^{-T}$
- $\mathbf{P} = \mathbf{P}(\mathbf{X}, t)$ : first Piola-Kirchoff stress tensor in material frame
  - $\mathbf{P} := J\boldsymbol{\Sigma}\mathbf{F}^{-T} = \mathbf{F}\mathbf{S}$
- $\mathbf{B}, \mathbf{b}$ : body force per unit mass in material/spatial frames
- $E, e$ : internal energy per unit mass in material frame
- $I, i$ : potential? energy per unit mass in material frame
- $i = i(\mathbf{X}, t)$ : potential? energy per unit mass in spatial frame
- $\mathbf{Q}, \mathbf{q}$ : heat flux in material/spatial frames

### 5.3 Reynold's Transport Theorem

$$\frac{d}{dt} \int_{V(t)} q \, d\Omega_t = \int_{V(t)} \frac{\partial q}{\partial t} \, d\Omega_t + \oint_{\partial V(t)} q \mathbf{v} \cdot \hat{\mathbf{n}} \, dS \quad \text{Proof ?? (5.1)}$$

$$\frac{d}{dt} \int_{V(t)} \mathbf{q} \, d\Omega_t = \int_{V(t)} \frac{\partial \mathbf{q}}{\partial t} \, d\Omega_t + \oint_{\partial V(t)} \mathbf{q} (\mathbf{v} \cdot \hat{\mathbf{n}}) \, dS \quad \text{Proof ?? (5.2)}$$

### 5.4 Generic Scalar Conservation Laws

explain vars here, v sub omega, yadda yadda

#### 5.4.1 Material Frame

$$\frac{d}{dt} \int_{V_0} QJ \, d\Omega_0 = \oint_{\partial V_0} \mathbf{\Phi} \cdot \hat{\mathbf{N}} J \, dS_0 + \int_{V_0} RJ \, d\Omega_0 \quad \text{Integral form. Proof ?? (5.3)}$$

$$\frac{d(QJ)}{dt} = -\operatorname{div}_{\mathbf{X}} (\mathbf{\Phi} \cdot \mathbf{F}^{-T} J) + RJ \quad \text{Differential form. Proof ?? (5.4)}$$

If the conserved property is intensive, expressed per unit mass, the following conservation laws may be derived.

$$\int_{V_0} \rho_0 \frac{d\tilde{Q}}{dt} \, d\Omega_0 = \oint_{\partial V_0} \mathbf{\Phi} \cdot \hat{\mathbf{N}} J \, dS_0 + \int_{V_0} RJ \, d\Omega_0 \quad \text{Integral form. Proof ?? (5.5)}$$

$$\rho_0 \frac{d\tilde{Q}}{dt} = -\operatorname{div}_{\mathbf{X}} (\mathbf{\Phi} \cdot \mathbf{F}^{-T} J) + RJ \quad \text{Differential form. Proof ?? (5.6)}$$

#### 5.4.2 Spatial Frame

$$\frac{d}{dt} \int_{V(t)} q \, d\Omega_t = - \oint_{\partial V(t)} \phi \cdot \hat{\mathbf{n}} \, dS + \int_{V(t)} r \, d\Omega_t \quad \text{Integral form. Proof ?? (5.7)}$$

$$\frac{\partial q}{\partial t} + \operatorname{div} (q\mathbf{v}) = -\operatorname{div} \phi + r \quad \text{Differential form. Proof ?? (5.8)}$$

If the conserved property is intensive, expressed per unit mass, the following conservation laws may be derived.

$$\int_{V(t)} \rho \left( \frac{\partial \tilde{q}}{\partial t} + \operatorname{grad} \tilde{q} \cdot \mathbf{v} \right) \, d\Omega_t = - \oint_{\partial V(t)} \phi \cdot \hat{\mathbf{n}} \, dS + \int_{V(t)} r \, d\Omega_t \quad \text{Integral form. Proof ?? (5.9)}$$

$$\rho \left( \frac{\partial \tilde{q}}{\partial t} + \operatorname{grad} \tilde{q} \cdot \mathbf{v} \right) = -\operatorname{div} \phi + r \quad \text{Differential form. Proof ?? (5.10)}$$

### 5.5 Generic Vector Conservation Laws

explain vars here, v sub omega, yadda yadda



### 5.5.1 Material Frame

$$\frac{d}{dt} \int_{V_0} \mathbf{Q} J \, d\Omega_0 = \oint_{\partial V_0} \Phi \hat{\mathbf{N}} J \, dS_0 + \int_{V_0} \mathbf{R} J \, d\Omega_0 \quad \text{Integral form. Proof ?? (5.11)}$$

$$\frac{d(\mathbf{Q} J)}{dt} = -\operatorname{div}_{\mathbf{X}} (\Phi \mathbf{F}^{-T} J) + \mathbf{R} J \quad \text{Differential form. Proof ?? (5.12)}$$

If the conserved property is intensive, expressed per unit mass, the following conservation laws may be derived.

$$\int_{V_0} \rho_0 \frac{d\tilde{\mathbf{Q}}}{dt} \, d\Omega_0 = \oint_{\partial V_0} \Phi \hat{\mathbf{N}} J \, dS_0 + \int_{V_0} \mathbf{R} J \, d\Omega_0 \quad \text{Integral form. Proof ?? (5.13)}$$

$$\rho_0 \frac{d\tilde{\mathbf{Q}}}{dt} = -\operatorname{div}_{\mathbf{X}} (\Phi \mathbf{F}^{-T} J) + \mathbf{R} J \quad \text{Differential form. Proof ?? (5.14)}$$

### 5.5.2 Spatial Frame

$$\frac{d}{dt} \int_{V(t)} \mathbf{q} \, d\Omega_t = - \oint_{\partial V(t)} \phi \hat{\mathbf{n}} \, dS + \int_{V(t)} \mathbf{r} \, d\Omega_t \quad \text{Integral form. Proof ?? (5.15)}$$

$$\frac{\partial \mathbf{q}}{\partial t} + \operatorname{div} (\mathbf{q} \otimes \mathbf{v}) = -\operatorname{div} \phi + \mathbf{r} \quad \text{Differential form. Proof ?? (5.16)}$$

If the conserved property is intensive, expressed per unit mass, the following conservation laws may be derived.

$$\int_{V(t)} \rho \left( \frac{\partial \tilde{\mathbf{q}}}{\partial t} + (\operatorname{grad} \tilde{\mathbf{q}}) \mathbf{v} \right) \, d\Omega_t = - \oint_{\partial V(t)} \phi \hat{\mathbf{n}} \, dS + \int_{V(t)} \mathbf{r} \, d\Omega_t \quad \text{Integral form. Proof ?? (5.17)}$$

$$\rho \left( \frac{\partial \tilde{\mathbf{q}}}{\partial t} + (\operatorname{grad} \tilde{\mathbf{q}}) \mathbf{v} \right) = -\operatorname{div} \phi + \mathbf{r} \quad \text{Differential form. Proof ?? (5.18)}$$

## 5.6 Conservation of Mass

### 5.6.1 Material Frame

$$\rho_0 = \varrho J \quad \text{Proof ?? (5.19)}$$

### 5.6.2 Spatial Frame

$$\frac{d}{dt} \int_{\mathbf{x} \in V \subseteq \Omega_t} \rho(\mathbf{x}, t) \, d\Omega_t + \int_{\mathbf{x} \in \partial V} \rho(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) \cdot \hat{\mathbf{n}}(\mathbf{x}) \, d(\partial V) = 0 \quad \text{Integral form. Proof ?? (5.20)}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 \quad \text{Proof ?? (5.21)}$$

## 5.7 Conservation of Momentum

### 5.7.1 Material Frame

$$\rho_0 \frac{dV}{dt} = \operatorname{div}_{\mathbf{X}} \mathbf{P} + \rho_0 \mathbf{B} \quad \text{Proof ?? (5.22)}$$

$$(5.23)$$

### 5.7.2 Spatial Frame

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) = \operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} \quad \text{Conservative form. Proof ?? (5.24)}$$

$$\rho \left( \frac{\partial \mathbf{v}}{\partial t} + (\operatorname{grad} \mathbf{v}) \mathbf{v} \right) = \operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} \quad \text{Reduced form. Proof ?? (5.25)}$$

$$(5.26)$$

## 5.8 Conservation of Angular Momentum

### 5.8.1 Material Frame

$$\mathbf{S} = \mathbf{S}^T \quad \text{Reduced form. Proof ?? (5.27)}$$

$$(5.28)$$

### 5.8.2 Spatial Frame

$$\frac{\partial(\mathbf{x} \times \rho \mathbf{v})}{\partial t} + \operatorname{div}(\mathbf{x} \times \rho \mathbf{v} \otimes \mathbf{v}) = \operatorname{div}(\mathbf{x} \times \boldsymbol{\sigma}) + \mathbf{x} \times \rho \mathbf{b} \quad \text{Conservative form. Proof ?? (5.29)}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \quad \text{Reduced form. Proof ?? (5.30)}$$

$$(5.31)$$

## 5.9 Conservation of Energy

### 5.9.1 Material Frame

$$\rho_0 \frac{dI}{dt} + \operatorname{div}_{\mathbf{x}} \mathbf{Q} = \rho_0 \mathbf{R} + \mathbf{S} : \mathbf{D} \quad \text{Proof ?? (5.32)}$$

$$(5.33)$$

### 5.9.2 Spatial Frame

$$\frac{\partial(\rho e)}{\partial t} + \operatorname{div}(\rho e \mathbf{v}) = \operatorname{div}(\boldsymbol{\sigma} \mathbf{v}) + \rho r + \rho \mathbf{b} \cdot \mathbf{b} - \operatorname{div} \mathbf{q} \quad \text{Conservation form. Proof ?? (5.34)}$$

$$\rho \left( \frac{\partial e}{\partial t} + \operatorname{grad} e \cdot \mathbf{v} \right) = \boldsymbol{\sigma} : \mathbf{d} + \rho r + \rho \mathbf{b} \cdot \mathbf{b} - \operatorname{div} \mathbf{q} \quad \text{Reduced form. Proof ?? (5.35)}$$

$$(5.36)$$

## 5.10 Entropy Inequality

### 5.10.1 Material Frame

$$\rho_0 \frac{dN}{dt} + \operatorname{div}_{\mathbf{x}} \frac{\mathbf{Q}}{\Theta} \geq \rho_0 \frac{R}{\Theta} \quad \text{Proof ?? (5.37)}$$

### 5.10.2 Spatial Frame

$$\frac{\partial(\rho \eta)}{\partial t} + \operatorname{div}(\rho \eta \mathbf{v}) + \operatorname{div} \frac{\mathbf{q}}{\Theta} - \rho r \geq 0 \quad \text{Proof ?? (5.38)}$$

$$(5.39)$$

## 5.11 Advection Diffusion Reaction Equation

$$\frac{\partial c}{\partial t} + \nabla \cdot (\mathbf{v}c - \mathbf{D}\nabla c) = R \quad \text{Proof ?? (5.40)}$$

$$\frac{\partial \mathbf{c}}{\partial t} + \nabla \cdot (\mathbf{v} \otimes \mathbf{c} - \mathcal{D} : \nabla \mathbf{c}) = R \quad \text{Proof ?? (5.41)}$$



# Chapter 6

## Solid Mechanics

general

elastic waves

thermoelastic solid

engineering stress example

### 6.1 Provisional Constitutive Laws

- $\mathbb{C}$  = ??? : elasticity tensor in material frame

$$- \mathbb{C} := \frac{\partial \mathbf{S}}{\partial \mathbf{E}} = \frac{\partial^2 \hat{\Psi}}{\partial \mathbf{E}^2}$$

- = ??? : elasticity tensor in spatial frame

### 6.2 Elastic Wave Equations

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \operatorname{div} \boldsymbol{\sigma} = 0$$

Proof ?? (6.1)

$$\rho \frac{\partial \mathbf{u}}{\partial t} = (\lambda + 2\mu) \operatorname{grad} (\operatorname{div} \mathbf{u}) - \mu \operatorname{curl}^2 \mathbf{u} \quad \text{hm (6.2)}$$

(6.3)



# Chapter 7

## Fluid Mechanics

full navier stokes  
incompressible  
rans  
stokes flow  
euler equations  
acoustic equations  
my special acoustic equations  
shallow water equations

### 7.1 Shallow Water Equations

conservative, non-conservative,

$$\frac{\partial(\rho\eta)}{\partial t} + \operatorname{div}(\rho\eta\mathbf{u}) = 0$$

Proof ?? (7.1)

$$\frac{\partial(\rho\eta\mathbf{u})}{\partial t} + \operatorname{div}\left(\rho\eta\mathbf{u} \otimes \mathbf{u} + \frac{1}{2}\rho g\eta^2\mathbf{I}\right) = 0$$

Proof ?? (7.2)

$$\frac{\partial(\rho\eta)}{\partial t} + \nabla \cdot (\rho\eta\mathbf{u}) = 0$$

Proof ?? (7.3)

$$\frac{\partial(\rho\eta\mathbf{u})}{\partial t} + \nabla \cdot (\rho\eta\mathbf{u} \otimes \mathbf{u}) + \nabla \cdot \frac{1}{2}\rho g\eta^2 = 0$$

Proof ?? (7.4)

#### 7.1.1 Closures and Assumptions

diffusive wave equations  
bernoulli  
groundwater darcy flow





# Appendix A

## Proofs

**Proof A.1 (Equation 2.13)**

$$\mathbf{a} \cdot \mathbf{b} = \left( \sum_{i=1}^n a_i \hat{\mathbf{e}}_i \right) \cdot \left( \sum_{j=1}^n b_j \hat{\mathbf{e}}_j \right) \quad \text{Using 1.3 (A.1)}$$

$$= \sum_{i=1}^n \sum_{j=1}^n a_i b_j \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j \quad (\text{A.2})$$

$$= \sum_{i=1}^n a_i b_i \quad \text{Using 1.1 (A.3)}$$

**Proof A.2 (Equation 2.14)**

$$\mathbf{S}\mathbf{a} = \left( \sum_{i=1}^m \sum_{j=1}^n S_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \right) \left( \sum_{k=1}^n a_k \hat{\mathbf{e}}_k \right) \quad \text{Using 2.1, 1.3 (A.4)}$$

$$= \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^n S_{ij} a_k (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) \hat{\mathbf{e}}_k \quad (\text{A.5})$$

$$= \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^n S_{ij} a_k \hat{\mathbf{e}}_i (\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_k) \quad \text{Using 2.4 (A.6)}$$

$$= \sum_{i=1}^m \sum_{j=1}^n S_{ij} a_j \hat{\mathbf{e}}_i \quad \text{Using 1.1 (A.7)}$$

**Proof A.3 (Equation 2.15)**

$$\mathbf{x} \cdot \mathbf{S} = \left( \sum_{i=1}^n x_i \hat{\mathbf{e}}_i \right) \cdot \left( \sum_{j=1}^m \sum_{k=1}^n S_{jk} \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \right) \quad \text{Using 1.3, 2.1 (A.8)}$$

$$= \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^n x_i S_{jk} \hat{\mathbf{e}}_i \cdot (\hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k) \quad (\text{A.9})$$

$$= \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^n x_i S_{jk} (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j) \hat{\mathbf{e}}_k \quad \text{Using 2.5 (A.10)}$$

$$= \sum_{i=1}^m \sum_{k=1}^n x_i S_{ik} \hat{\mathbf{e}}_k \quad \text{Using 1.1 (A.11)}$$

**Proof A.4 (Equation 2.16)**

$$\mathbf{S}\mathbf{U} = \left( \sum_{i=1}^m \sum_{j=1}^n S_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \right) \left( \sum_{k=1}^n \sum_{\ell=1}^p U_{k\ell} \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell \right) \quad \text{Using 2.1 (A.12)}$$

$$= \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^n \sum_{\ell=1}^p S_{ij} U_{k\ell} (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) (\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell) \quad (\text{A.13})$$

$$= \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^n \sum_{\ell=1}^p S_{ij} U_{k\ell} (\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_k) (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_\ell) \quad \text{Using 2.9 (A.14)}$$

$$= \sum_{i=1}^m \sum_{j=1}^n \sum_{\ell=1}^p S_{ij} U_{j\ell} (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_\ell) \quad \text{Using 1.1 (A.15)}$$

**Proof A.5 (Equation 2.17)**

$$\mathbf{S} : \mathbf{T} = \left( \sum_{i=1}^m \sum_{j=1}^n S_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \right) : \left( \sum_{k=1}^m \sum_{\ell=1}^m T_{k\ell} \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell \right) \quad \text{Using 2.1 (A.16)}$$

$$= \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^m \sum_{\ell=1}^m S_{ij} T_{k\ell} (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) : (\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell) \quad (\text{A.17})$$

$$= \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^m \sum_{\ell=1}^m S_{ij} T_{k\ell} (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_k) (\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_\ell) \quad \text{Using 2.10 (A.18)}$$

$$= \sum_{i=1}^m \sum_{j=1}^n S_{ij} T_{ij} \quad \text{Using 1.1 (A.19)}$$

$$(\text{A.20})$$

**Proof A.6 (Equation 3.30)**

$$\nabla \cdot (\phi \mathbf{a}) = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left[ \hat{\mathbf{e}}_i \cdot \left( \phi \sum_{j=1}^n a_j \hat{\mathbf{e}}_j \right) \right] \quad \text{Using 3.3, 1.3 (A.21)}$$

$$= \sum_{i=1}^n \sum_{j=1}^n \frac{\partial \phi a_j}{\partial x_i} (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j) \quad (\text{A.22})$$

$$= \sum_{i=1}^n \frac{\partial \phi a_i}{\partial x_i} \quad \text{Using 1.1 (A.23)}$$

$$= \sum_{i=1}^n \left( \phi \frac{\partial a_i}{\partial x_i} + a_i \frac{\partial \phi}{\partial x_i} \right) \quad (\text{A.24})$$

$$= \phi \nabla \cdot \mathbf{a} + \mathbf{a} \cdot \nabla \phi \quad \text{Using 3.26, 2.13, 3.5 (A.25)}$$

**Proof A.7 (Equation 3.34)**

$$\nabla \cdot (\phi \mathbf{S}) = \left( \sum_{i=1}^m \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \right) \cdot \left( \phi \sum_{j=1}^m \sum_{k=1}^n S_{jk} \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \right) \quad \text{Using 3.3, 2.1 (A.26)}$$

$$= \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^n \frac{\partial (\phi S_{jk})}{\partial x_i} \hat{\mathbf{e}}_i \cdot (\hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k) \quad (\text{A.27})$$

$$= \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^n \left( \phi \frac{\partial S_{jk}}{\partial x_i} + S_{jk} \frac{\partial \phi}{\partial x_i} \right) (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j) \hat{\mathbf{e}}_k \quad \text{Using 2.5 (A.28)}$$

$$= \sum_{i=1}^m \sum_{k=1}^n \phi \frac{\partial S_{ik}}{\partial x_i} \hat{\mathbf{e}}_k + S_{ik} \frac{\partial \phi}{\partial x_i} \hat{\mathbf{e}}_k \quad \text{Using 1.1 (A.29)}$$

$$= \phi \nabla \cdot \mathbf{S} + \nabla \phi \cdot \mathbf{S} \quad \text{Using 3.27, 3.5, 2.15 (A.30)}$$



# Bibliography

- [1] Yann LeCun, Corinna Cortes, and CJ Burges. Mnist handwritten digit database. *ATT Labs [Online]*. Available: <http://yann.lecun.com/exdb/mnist>, 2, 2010.