Tensor Bible

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List of Theorems

Introduction

before, big picture, what is a tensor, algebraic object describing mulitilinear relationship between algebraic objects [1]

Chapter 1

Vectors

Ironically enough, this was the hardest section to write. It is easy to get lost in abstraction and pedantics.

I think that the best way to start is with vectors. Tesors are always defined relative to some kind of vector space. Scalars are so-called zeroth-order tensors, vectors are so-called first-order tensors, and any other kind of tensor can be constructed from these.

1.1 Vector Space Analysis

1.1.1 Vector Space

scalar, tuples, functions

Definition 1.1 (Abelian Group) *An Abelian group $\{A, \cdot\}$ is a set A and a binary operation $\cdot : A \times A \to A$ which satisfy the following four properties. Essentially, it generalizes the properties of addition and multiplication.*

1. Commutativity: reversing the arguments does not change the result.

$$\forall a, b \in A : a \cdot b = b \cdot a \tag{1.1}$$

2. Associativity: the order in which the operation is applied does not change the result.

$$\forall a, b, c \in A : a \cdot (b \cdot c) = (a \cdot b) \cdot c \tag{1.2}$$

3. Identity element: any element operated on with the identity results in itself.

$$\exists i \in A : \forall a \in A : i \cdot a = a \tag{1.3}$$

4. Inverse element: every element has a corresponding inverse which results in the identity.

$$\forall a \in A : \exists b \in A : a \cdot b = i \tag{1.4}$$

Definition 1.2 (Field) *A field $\{\mathbb{F}, +, \cdot\}$ is a set $\mathbb{F}$ with an addition-like operation $+ : A \times A : A$ and a multiplication-like operation $+ : A \times A : A$ which satisfy the following properties.*

1. $\{\mathbb{F}, +\}$ form an Abelian group, with an identity 0 and additive inverse marked as $-a$.

2. $\{\mathbb{F} \setminus 0, \cdot\}$ form an Abelian group, with an identity 1 and multiplicative inverse marked as a^{-1}.

3. Distributivity of multiplication

$$\forall a, b, c \in A : a \cdot (b + c) = a \cdot b + a \cdot c \tag{1.5}$$

For a field, the subtraction and division operations can also be defined.

$$a - b := a + (-b) \tag{1.6}$$
$$a/b := a \cdot b^{-1} \tag{1.7}$$

Definition 1.3 (Vector Space) *A vector space X over a field $\mathbb{F}$ is a system of sets and operations $\{X, \mathbb{F}, +, +, \cdot, *\}$ which satisfy the following axioms.*

1. *$\{X, +\}$ is an Abelian group with identity $\mathbf{0}$.*

2. *$\{\mathbb{F}, +, \cdot\}$ is a field.*

3. *Scalar multiplication $* : \mathbb{F} \times X \to X$ is a binary operation between the field and vector space which produces a vector, with the following properties.*

   (a) *Distributivity over vector addition*

   $$\forall \alpha \in \mathbb{F}, x, y \in X : \alpha * (\mathbf{x} + \mathbf{y}) = \alpha * \mathbf{x} + \alpha * \mathbf{y} \tag{1.8}$$

   (b) *Distributivity over field addition into vector addition*

   $$\forall \alpha, \beta \in \mathbb{F}, \mathbf{x} \in A : (\alpha + \beta) * \mathbf{x} = \alpha * \mathbf{x} + \beta * \mathbf{x} \tag{1.9}$$

   (c) *Associativity with field multiplication into scalar multiplication*

   $$\forall \alpha, \beta \in \mathbb{F}, \mathbf{x} \in X : (\alpha \cdot \beta) * \mathbf{x} = \alpha * (\beta * \mathbf{x}) \tag{1.10}$$

   (d) *Field identity is scalar multiplication identity*

   $$\forall \mathbf{x} \in X : 1 * \mathbf{x} = \mathbf{x} \tag{1.11}$$

can become looser with notation??
additional properties to get identity and inverse vectors

- $\forall \mathbf{x} \in X : 0 * \mathbf{x} = \mathbf{0}$

- $\forall \mathbf{x} \in X : (-1) * \mathbf{x} = -\mathbf{x}$

1.1.2   Basis

different kinds of bases, linear independence

1.1.3   Metric

1.1.4   Norm

metric norm back and forth

1.1.5   Inner Product

norm inner product back and forth

1.1.6   Dual Space

1.1.7   Riesz Representation

Riesz representation and dealing with complex vector spaces

1.2 Inner Product and Norm

norm is also a metric

1.3 Vector Spaces in Continuum Mechanics

do a proper construction here
   construction of euclidean geometry but don't get to lost
   Rn is a Euclidean geometry, also cartesian coordinates
   note on how velocity
   also fields which are functions

1.4 Getting Started

For all intents and purposes of continuum mechanics, it is sufficient to start with a Euclidean space, usually two- or three-dimensional, described by a Cartesian coordinate system, because this can be used to describe the physical world at our scale quite well.

1.4.1 Notation

Scalars are simply written as a letter, usually lower-case (a). Vectors are bolded ($\mathbf{a}$). When hand-written, they may have an arrow on top ($\vec{a}$) or a single underline ($\underline{a}$). Unit vectors, with a magnitude 1, may be marked with a circumflex ($\hat{\mathbf{a}}$). Higher-order tensors are bolded and usually upper-case ($\mathbf{A}$). When written, they may have a number of underlines equal to their order.

1.5 Bases and Components

We start with some right-handed orthonormal basis $\{\hat{\mathbf{e}}_i\}_{i=1}^n$ for an n-dimensional Euclidean vector space $\mathbb{R}^n$. If these align with a Cartesian coordinate frame, or when a Cartesian coordinate frame is defined to line up with them, they're called the canonical basis, i.e. the standard one. Using the geometric definitions for the dot and cross product, the following relations exist between the basis vectors, represented in shorthand by the Kronecker delta δ and Levi-Civita symbol ϵ.

   These products are both bilinear. The dot product commutes, and the cross product anti-commutes.

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} := \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{1.12}$$

$$(\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j) \cdot \hat{\mathbf{e}}_k = \epsilon_{ijk} := \begin{cases} 1, & (i,j,k) = (1,2,3), (2,3,1), \text{or} (3,1,2), \\ -1, & (i,j,k) = (3,2,1), (2,1,3), \text{or} (1,3,2), \\ 0, & \text{otherwise.} \end{cases} \tag{1.13}$$

   Since any basis set is linearly independent and spans the vector space, any vector in the space may be represented in terms of such a basis. The coefficients of the linear combination are called components. It is important to note that these are not the same as coordinates, especially for non-Cartesian coordinate systems.

$$\mathbf{a} = \sum_{i=1}^n a_i \hat{\mathbf{e}}_i \tag{1.14}$$

1.5.1   Cobasis

If the chosen basis is not orthonormal with respect to some inner product and corresponding induced norm, then there is some more nuance. Sidestepping the excess mental infrastructure of the dual space and Riesz representation, the cobasis of some basis $\{\mathbf{f}_i\}_{i=1}^n$, marked with a superscript as $\{\mathbf{f}^i\}_{i=1}^n$, is another basis for the same space, which is bi-orthonormal to the basis.

$$\mathbf{f}^i \cdot \mathbf{f}_j = \mathbf{f}_j \cdot \mathbf{f}^i = \delta_{ij} \tag{1.15}$$

Notation for components in a non-orthonormal basis is more complicated. The contravariant components of the basis use superscripts, and the covariant components of the cobasis use subscripts. This may seem backwards, but this is the convention that was developed.

$$\mathbf{a} = \sum_{i=1}^n \tilde{a}^i \mathbf{f}_i \tag{1.16}$$

$$\mathbf{a} = \sum_{i=1}^n \tilde{a}_i \mathbf{f}^i \tag{1.17}$$

Contra in contravariant refers to how the components scale inversely with the basis vectors. For example, if a basis is changed from inches to feet, the components used to represent a vector will shrink to represent the same length. On the other hand, the covariant components would increase.

If one goes further into the weeds of tensor-world, there is a whole shorthand system of upstairs and downstairs indices used for tensors defined by a certain combination of bases and cobases. However, in continuum-world, one can fudge over all of these nuances by choosing to use an orthonormal basis, which is conveniently its own cobasis.

1.5.2   Independence of Basis

That freedom of choice is an important idea behind this tensor system. Up until this point, you were probably used to seeing vectors and matrices as lists and tables of numbers. Start with a unit vector $(1, 0)$. Rotate the coordinate frame 45 degrees clockwise and you get $(\sqrt{2}/2, \sqrt{2}/2)$. This would seem like a new and unequal vector, since $(1, 0) \neq (\sqrt{2}/2, \sqrt{2}/2)$.

Writing them in terms of components of particular bases allows for more abstraction and flexibility. The vector is no longer the tuple per se, but is represented by a tuple in the context of a particular reference frame. The vector and the physical quantity it represents stay the same conceptually, regardless of the frame of reference: $\mathbf{a} = \sum_{i=1}^n a_i \hat{\mathbf{e}}_i = \sum_{i=1}^n \tilde{a}^i \mathbf{f}_i = \sum_{i=1}^n \tilde{a}_i \mathbf{f}^i$. Only components vary. So we can say for the prior example, labeling the coordinate frames as A and B, $(1, 0)_A = (\sqrt{2}/2, \sqrt{2}/2)_B$. Operations and properties like an inner product, cross product, norm should stay the same, too. However, it may not necessarily be that $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^n \tilde{a}^i \tilde{b}^i$, because the dot product is defined geometrically, not algebraically.

1.5.3   Computing Components

differentiate coordinates and components

1.5.4   Curvilinear Coordinates

same vector using different coordinates

1.6   Kronecker Delta and Levi-Civita Symbol Identities

The Kronecker Delta and Levi-Civita Symbol have the following identities which are useful in proof.

$$a = b \qquad \text{Proof ?? (1.18)}$$

$$a = b \qquad \text{Proof ?? (1.19)}$$

$$\epsilon_{pqs}\epsilon_{nrs} = \delta_{pn}\delta_{qr} - \delta_{pr}\delta_{qn} \qquad \text{Proof ?? (1.20)}$$

$$\epsilon_{pqs}\epsilon_{rqs} = 2\delta_{pr} \qquad \text{Proof ?? (1.21)}$$

1.7 Einstein Notation

Chapter 2

Tensor Algebra

2.1 Abstract Definitions

start with linear operators, show component representation, introduce tensor product spaces with contraction operation and co/contra basis, explain how other operations follow, introduce isomorphisms between everything

2.1.1 Linear Operators

2.1.2 Tensor Product Space

aaa

2.2 Dyads and Dyadics

tensor product spaces
   order and covector-vector degree

2.3 Contractions

2.4 Equivalence to Linear Algebra

isomorphic to linear algebra if appropriate combination of basis/cobasis
   linearity
   coordinate vs non coordinate basis
   components
   deltas and epsilons, symbols, only up to rank 2, etc
   einstein notation
   tensor as operator
   basis independence
   isomorphism
   definitions contrived to fulfill certain properties
   using dyadic symbol and not
   defining property of dyadic, compare to linear algebra
   simple properties of each product (distributive, associative, commutative) that may depend on the operands
   linear algebra comparison
   computing in canonical basis
   introducing the dot product

introducing the cross product
introducing the dyadic product
specifically takes two vectors
non commutative
scalar rule
simple contractions
double contractions
change of basis
invariants of vectors and tensors
symmetric, skew-symmetric
complex numbers
operations between tensors (to fit with vectors)

- $\phi, \kappa \in \mathbb{R}$

- $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^n$

- $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$

- $\boldsymbol{\omega} \in \mathbb{R}^p$

- $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$

- $\mathbf{Q}, \mathbf{R} \in \mathbb{R}^{m \times m}$

- $\mathbf{S}, \mathbf{T} \in \mathbb{R}^{m \times n}$

- $\mathbf{U} \in \mathbb{R}^{n \times p}$

- $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{3 \times 3}$

INTRODUCING VECTORS

$$\mathbf{S} = \sum_{i=1}^{m} \sum_{j=1}^{n} S_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \tag{2.1}$$

$$\mathbf{u} \times \mathbf{v} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} u_i v_j \epsilon_{ijk} \hat{\mathbf{e}}_k \tag{2.2}$$

$$\tag{2.3}$$

INTRODUCING TENSORS

$$(\mathbf{a} \otimes \mathbf{x}) \mathbf{y} := \mathbf{a} (\mathbf{x} \cdot \mathbf{y}) \tag{2.4}$$
$$\mathbf{a} \cdot (\mathbf{b} \otimes \mathbf{x}) := (\mathbf{a} \cdot \mathbf{b}) \mathbf{x} \tag{2.5}$$
$$\mathbf{x} \cdot (\mathbf{y} \otimes \mathbf{a}) \mathbf{b} = \ldots \qquad \text{Proof ??} \tag{2.6}$$
$$(\mathbf{a} \otimes \mathbf{x})^T := \mathbf{x} \otimes \mathbf{a} \tag{2.7}$$
$$\mathrm{tr} (\mathbf{a} \otimes \mathbf{b}) := \mathbf{a} \cdot \mathbf{b} \tag{2.8}$$
$$(\mathbf{a} \otimes \mathbf{x}) (\mathbf{y} \otimes \boldsymbol{\omega}) = (\mathbf{x} \cdot \mathbf{y}) (\mathbf{a} \otimes \boldsymbol{\omega}) \qquad \text{Proof ??} \tag{2.9}$$
$$(\mathbf{a} \otimes \mathbf{x}) : (\mathbf{b} \otimes \mathbf{y}) := (\mathbf{a} \cdot \mathbf{b}) (\mathbf{x} \cdot \mathbf{y}) \tag{2.10}$$

$$\mathbf{a} \otimes \mathbf{x} = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i x_j \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \tag{2.11}$$

$$\tag{2.12}$$

CONTRACTIONS

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i \qquad \text{Proof A.1 (2.13)}$$

$$\mathbf{Sa} = \mathbf{S} \cdot \mathbf{a} = \sum_{i=1}^{m} \sum_{j=1}^{n} S_{ij} a_j \hat{\mathbf{e}}_i \qquad \text{Proof A.2 (2.14)}$$

$$\mathbf{x} \cdot \mathbf{S} = \sum_{i=1}^{m} \sum_{k=1}^{n} x_i S_{ik} \hat{\mathbf{e}}_k \qquad \text{Proof A.3 (2.15)}$$

$$\mathbf{SU} = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{\ell=1}^{p} S_{ij} U_{j\ell} \left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_\ell \right) \qquad \text{Proof A.4 (2.16)}$$

$$\mathbf{S} : \mathbf{T} = \sum_{i=1}^{m} \sum_{j=1}^{n} S_{ij} T_{ij} \qquad \text{Proof A.5 (2.17)}$$

2.5 Miscellaneous Operations

2.5.1 Change of Basis

2.6 Helpful Identities

$$\mathbf{Sa} = \mathbf{a} \cdot \left(\mathbf{S}^T \right) \qquad \text{Proof ?? (2.18)}$$
$$(\mathbf{x} \cdot \mathbf{S}) \cdot \mathbf{a} = \mathbf{x} \cdot (\mathbf{Sa}) \qquad \text{Proof ?? (2.19)}$$
$$\left(\mathbf{S}^T \right) : \mathbf{V} = \mathbf{S} : \left(\mathbf{V}^T \right) \qquad \text{Proof ?? (2.20)}$$

Chapter 3

Tensor Calculus

differentiate nabla and grad

in different coordinate systems

expand to field...define operations...basis changes locally

algebraic and geometric definitions which conveniently coincide for a euclidean geometry described by a cartesian coordinate system

INTRODUCING NABLA OPERATOR

3.1 Introducing Nabla

$$\boldsymbol{\nabla} := \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \hat{\mathbf{e}}_i \tag{3.1}$$

$$\boldsymbol{\nabla}[\ldots] = \boldsymbol{\nabla} \otimes [\ldots] := \sum_{i=1}^{m} \frac{\partial}{\partial x_i} [\hat{\mathbf{e}}_i \otimes \ldots] \tag{3.2}$$

$$\boldsymbol{\nabla} \cdot [\ldots] := \sum_{i=1}^{m} \frac{\partial}{\partial x_i} [\hat{\mathbf{e}}_i \cdot \ldots] \tag{3.3}$$

$$\boldsymbol{\nabla} \times [\ldots] := \sum_{i=1}^{m} \frac{\partial}{\partial x_i} [\hat{\mathbf{e}}_i \times \ldots] \tag{3.4}$$

3.2 Gradient

$$\boldsymbol{\nabla}\phi = \ldots \qquad \text{Proof ?? (3.5)}$$
$$\boldsymbol{\nabla}\mathbf{a} = \ldots \qquad \text{Proof ?? (3.6)}$$
$$\boldsymbol{\nabla}\mathbf{S} = \ldots \qquad \text{Proof ?? (3.7)}$$
$$\operatorname{grad}\phi = \ldots \qquad (3.8)$$
$$\operatorname{grad}\mathbf{a} = \ldots \qquad (3.9)$$
$$\operatorname{grad}\mathbf{S} = \ldots \qquad (3.10)$$
$$(3.11)$$

3.2.1   Gradient Product Rules

First Order Value, Zeroth Order Operand

$$\boldsymbol{\nabla}\left(\phi\kappa\right) = \phi\boldsymbol{\nabla}\kappa + \kappa\boldsymbol{\nabla}\phi \qquad\qquad \text{Proof ?? (3.12)}$$

$$\boldsymbol{\nabla}\left(\mathbf{a}\cdot\mathbf{b}\right) = \left(\boldsymbol{\nabla}\mathbf{b}\right)\mathbf{a} + \left(\boldsymbol{\nabla}\mathbf{a}\right)\mathbf{b} * ** \qquad\qquad \text{Proof ?? (3.13)}$$

$$\boldsymbol{\nabla}\left(\mathbf{S}:\mathbf{T}\right) = \left(\boldsymbol{\nabla}\mathbf{T}\right):\mathbf{S} + \left(\boldsymbol{\nabla}\mathbf{S}\right):\mathbf{T} \qquad\qquad \text{Proof ?? (3.14)}$$

Second Order Value, First Order Operand

$$\boldsymbol{\nabla}\left(\phi\mathbf{a}\right) = \ldots \qquad\qquad \text{Proof ?? (3.15)}$$

$$\boldsymbol{\nabla}\left(\mathbf{Sa}\right) = \ldots \qquad\qquad \text{Proof ?? (3.16)}$$

$$\boldsymbol{\nabla}\left(\mathbf{y}\cdot\mathbf{S}\right) = \ldots \qquad\qquad \text{Proof ?? (3.17)}$$

$$\boldsymbol{\nabla}\left(\mathcal{D}:\mathbf{S}\right) = \ldots \qquad\qquad \text{Proof ?? (3.18)}$$

$$\boldsymbol{\nabla}\left(\mathbf{S}:\mathcal{D}\right) = \ldots \qquad\qquad \text{Proof ?? (3.19)}$$

$$\boldsymbol{\nabla}\left(\mathbf{u}\times\mathbf{v}\right) = \ldots \qquad\qquad \text{Proof ?? (3.20)}$$

Third Order Value, Second Order Operand

$$\boldsymbol{\nabla}\left(\phi\mathbf{S}\right) = \ldots \qquad\qquad \text{Proof ?? (3.21)}$$

$$\boldsymbol{\nabla}\left(\mathbf{a}\otimes\mathbf{y}\right) = \ldots \qquad\qquad \text{Proof ?? (3.22)}$$

$$\boldsymbol{\nabla}\left(\mathbf{SU}\right) = \ldots \qquad\qquad \text{Proof ?? (3.23)}$$

$$\boldsymbol{\nabla}\left(\mathcal{D}\cdot\mathbf{a}\right) = \ldots \qquad\qquad \text{Proof ?? (3.24)}$$

$$\boldsymbol{\nabla}\left(\mathbf{y}\cdot\mathcal{D}\right) = \ldots \qquad\qquad \text{Proof ?? (3.25)}$$

3.3   Divergence

$$\boldsymbol{\nabla}\cdot\mathbf{a} = \ldots \qquad\qquad \text{Proof ?? (3.26)}$$

$$\boldsymbol{\nabla}\cdot\mathbf{S} = \ldots \qquad\qquad \text{Proof ?? (3.27)}$$

$$\operatorname{div}\mathbf{a} = \ldots \qquad\qquad \text{Proof ?? (3.28)}$$

$$\operatorname{div}\mathbf{S} = \ldots \qquad\qquad \text{Proof ?? (3.29)}$$

3.3.1   Divergence Product Rules

Zeroth Order Value, First Order Operand

$$\boldsymbol{\nabla}\cdot\left(\phi\mathbf{a}\right) = \phi\boldsymbol{\nabla}\cdot\mathbf{a} + \mathbf{a}\cdot\boldsymbol{\nabla}\phi \qquad\qquad \text{Proof A.6 (3.30)}$$

$$\boldsymbol{\nabla}\cdot\left(\mathbf{Sa}\right) = \mathbf{S}:\boldsymbol{\nabla}\mathbf{a} + \mathbf{a}\cdot\left(\boldsymbol{\nabla}\cdot\mathbf{S}\right) \qquad\qquad \text{Proof ?? (3.31)}$$

$$\boldsymbol{\nabla}\cdot\left(\mathbf{y}\cdot\mathbf{S}\right) = \left(\mathbf{S}^T\right):\boldsymbol{\nabla}\mathbf{y} + \mathbf{y}\cdot\left(\boldsymbol{\nabla}\cdot\left(\mathbf{S}^T\right)\right) \qquad\qquad \text{Proof ?? (3.32)}$$

$$\boldsymbol{\nabla}\cdot\left(\mathbf{u}\times\mathbf{v}\right) = \ldots \qquad\qquad \text{Proof ?? (3.33)}$$

First Order Value, Second Order Operand

$$\boldsymbol{\nabla}\cdot\left(\phi\mathbf{S}\right) = \phi\boldsymbol{\nabla}\cdot\mathbf{S} + \boldsymbol{\nabla}\phi\cdot\mathbf{S} \qquad\qquad \text{Proof A.7 (3.34)}$$

$$\boldsymbol{\nabla}\cdot\left(\mathbf{a}\otimes\mathbf{y}\right) = \mathbf{a}\cdot\boldsymbol{\nabla}\mathbf{y} + \mathbf{y}\left(\boldsymbol{\nabla}\cdot\mathbf{a}\right) \qquad\qquad \text{Proof ?? (3.35)}$$

$$\boldsymbol{\nabla}\cdot\left(\mathbf{SU}\right) = \mathbf{S}:\left(\boldsymbol{\nabla}\mathbf{U}\right) + \left(\boldsymbol{\nabla}\cdot\mathbf{S}\right)\mathbf{U} \qquad\qquad \text{Proof ?? (3.36)}$$

3.4 Curl

$$\boldsymbol{\nabla} \times \mathbf{a} = \ldots \qquad \text{Proof ?? (3.37)}$$
$$\boldsymbol{\nabla} \times \mathbf{S} = \ldots \qquad \text{Proof ?? (3.38)}$$
$$\operatorname{curl} \mathbf{a} = \ldots \qquad (3.39)$$
$$\operatorname{curl} \mathbf{S} = \ldots \qquad (3.40)$$

3.4.1 Curl Product Rules

First Order Value, First Order Operand

3.5 Laplacian

3.5.1 Laplacian Product Rules

Zeroth Order Value, Zeroth Order Operand

$$\nabla^2 (\phi\kappa) = \ldots \qquad \text{Proof ?? (3.41)}$$
$$\nabla^2 (\mathbf{a} \cdot \mathbf{b}) = \ldots \qquad \text{Proof ?? (3.42)}$$
$$\nabla^2 (\mathbf{S} : \mathbf{T}) = \ldots \qquad \text{Proof ?? (3.43)}$$

First Order Value, First Order Operand

$$\nabla^2 (\phi\mathbf{a}) = \ldots \qquad \text{Proof ?? (3.44)}$$
$$\nabla^2 (\mathbf{S}\mathbf{a}) = \ldots \qquad \text{Proof ?? (3.45)}$$
$$\nabla^2 (\mathbf{y} \cdot \mathbf{S}) = \ldots \qquad \text{Proof ?? (3.46)}$$
$$\nabla^2 (\mathbf{u} \times \mathbf{v}) = \ldots \qquad \text{Proof ?? (3.47)}$$

Second Order Value, Second Order Operand

$$\nabla^2 (\phi\mathbf{S}) = \ldots \qquad \text{Proof ?? (3.48)}$$
$$\nabla^2 (\mathbf{a} \otimes \mathbf{y}) = \ldots \qquad \text{Proof ?? (3.49)}$$
$$\nabla^2 (\mathbf{S}\mathbf{U}) = \ldots \qquad \text{Proof ?? (3.50)}$$

3.6 Other Second Derivatives

div of curl = 0
    curl of grad = 0

3.7 Function Derivatives

derivative of vector value/input functions
    compositions thereof
    derivative of determinant, etc
    grad div relation

Chapter 4

Kinematics

Topics:
    different frames
    deformation map
    all the different kinds of tensors
    material and spatial fields
    derivatives of material and spatial fields
    integral change of variables with jacobian

4.1 Kinematic Variables

- $\mathbf{U}, \mathbf{u}$: displacement in material/spatial frames

    - $\mathbf{U} := \phi_t - \mathbf{X}$

- $\mathbf{V}, \mathbf{v}$: velocity in material/spatial frames

    - $\mathbf{V} := \frac{\partial \phi}{\partial t}$

- $\mathbf{L}$: velocity gradient in spatial frame

    - $\mathbf{L} := \operatorname{grad} \mathbf{v}$

- $\mathbf{D} = \mathbf{D}(\mathbf{X}, t)$: symmetric component of velocity gradient in material frame???

    - $\mathbf{D} := \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right)$

- $\mathbf{W} = \mathbf{W}(\mathbf{X}, t)$: skew-symmetric component of velocity gradient in material frame???

    - $\mathbf{W} := \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^T\right)$

Chapter 5

Conservation Laws

5.1 Continuum Mechanics Assumptions

define control volume properly, relate vt to v0, same for omega

5.2 Conservation Law Variables

explain convention that $Q(\mathbf{X}, t) = q(\boldsymbol{\varphi}(\mathbf{X}, t), t)$, also define rho not and show that it is constant??

- ϱ, ρ: density in material/spatial frames

- $\boldsymbol{\Sigma}, \boldsymbol{\sigma}$: Cauchy stress tensor in material/spatial frames

- $\mathbf{S}$: second Piola-Kirchoff stress tensor in material frame

    - $\mathbf{S} := J\mathbf{F}^{-1}\boldsymbol{\Sigma}\mathbf{F}^{-T}$

- $\mathbf{P} = \mathbf{P}(\mathbf{X}, t)$: first Piola-Kirchoff stress tensor in material frame

    - $\mathbf{P} := J\boldsymbol{\Sigma}\mathbf{F}^{-T} = \mathbf{F}\mathbf{S}$

- $\mathbf{B}, \mathbf{b}$: body force per unit mass in material/spatial frames

- E, e: energy per unit mass in material frame

- I, i: internal energy per unit mass in material frame

- $i = i(\mathbf{X}, t)$: potential? energy per unit mass in spatial frame

- $\mathbf{Q}, \mathbf{q}$: heat flux in material/spatial frames

5.3 Reynold's Transport Theorem

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} q \,\mathrm{d}\Omega_t = \int_{V(t)} \frac{\partial q}{\partial t} \,\mathrm{d}\Omega_t + \oint_{\partial V(t)} q\mathbf{v} \cdot \hat{\mathbf{n}} \,\mathrm{d}S \qquad \text{Proof ?? (5.1)}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} \mathbf{q} \,\mathrm{d}\Omega_t = \int_{V(t)} \frac{\partial \mathbf{q}}{\partial t} \,\mathrm{d}\Omega_t + \oint_{\partial V(t)} \mathbf{q} \,(\mathbf{v} \cdot \hat{\mathbf{n}}) \,\mathrm{d}S \qquad \text{Proof ?? (5.2)}$$

5.4   Closures

5.4.1   Cauchy Stress Hypothesis

something something importance for divergence theorem and converting to differential form

$$\mathbf{t} = \boldsymbol{\sigma}\hat{\mathbf{n}} \tag{5.3}$$

5.4.2   Energy of Mass

$$E = I + \frac{1}{2}\mathbf{V} \cdot \mathbf{V} \tag{5.4}$$

$$e = i + \frac{1}{2}\mathbf{v} \cdot \mathbf{v} \tag{5.5}$$

5.5   Generic Scalar Conservation Laws

explain vars here, v sub omega, yadda yadda

5.5.1   Material Frame

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V_0} QJ \, \mathrm{d}\Omega_0 = \oint_{\partial V_0} \boldsymbol{\Phi} \cdot \hat{\mathbf{N}} J \, \mathrm{d}S_0 + \int_{V_0} RJ \, \mathrm{d}\Omega_0 \qquad \text{Integral form. Proof \textbf{??}} \tag{5.6}$$

$$\frac{\mathrm{d}(QJ)}{\mathrm{d}t} = -\operatorname{div}_{\mathbf{X}}\left(\boldsymbol{\Phi} \cdot \mathbf{F}^{-T} J\right) + RJ \qquad \text{Differential form. Proof \textbf{??}} \tag{5.7}$$

If the conserved property is intensive, expressed per unit mass, the following conservation laws may be derived.

$$\int_{V_0} \rho_0 \frac{\mathrm{d}\tilde{Q}}{\mathrm{d}t} \, \mathrm{d}\Omega_0 = \oint_{\partial V_0} \boldsymbol{\Phi} \cdot \hat{\mathbf{N}} J \, \mathrm{d}S_0 + \int_{V_0} RJ \, \mathrm{d}\Omega_0 \qquad \text{Integral form. Proof \textbf{??}} \tag{5.8}$$

$$\rho_0 \frac{\mathrm{d}\tilde{Q}}{\mathrm{d}t} = -\operatorname{div}_{\mathbf{X}}\left(\boldsymbol{\Phi} \cdot \mathbf{F}^{-T} J\right) + RJ \qquad \text{Differential form. Proof \textbf{??}} \tag{5.9}$$

5.5.2   Spatial Frame

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} q \, \mathrm{d}\Omega_t = -\oint_{\partial V(t)} \boldsymbol{\phi} \cdot \hat{\mathbf{n}} \, \mathrm{d}S + \int_{V(t)} r \, \mathrm{d}\Omega_t \qquad \text{Integral form. Proof \textbf{??}} \tag{5.10}$$

$$\frac{\partial q}{\partial t} + \operatorname{div}\left(q\mathbf{v}\right) = -\operatorname{div}\boldsymbol{\phi} + r \qquad \text{Differential form. Proof \textbf{??}} \tag{5.11}$$

If the conserved property is intensive, expressed per unit mass, the following conservation laws may be derived.

$$\int_{V(t)} \rho \left(\frac{\partial \tilde{q}}{\partial t} + \operatorname{grad}\tilde{q} \cdot \mathbf{v}\right) \mathrm{d}\Omega_t = -\oint_{\partial V(t)} \boldsymbol{\phi} \cdot \hat{\mathbf{n}} \, \mathrm{d}S + \int_{V(t)} r \, \mathrm{d}\Omega_t \qquad \text{Integral form. Proof \textbf{??}} \tag{5.12}$$

$$\rho \left(\frac{\partial \tilde{q}}{\partial t} + \operatorname{grad}\tilde{q} \cdot \mathbf{v}\right) = -\operatorname{div}\boldsymbol{\phi} + r \qquad \text{Differential form. Proof \textbf{??}} \tag{5.13}$$

5.6   Generic Vector Conservation Laws

explain vars here, v sub omega, yadda yadda

5.6.1 Material Frame

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V_0} \mathbf{Q} J \, \mathrm{d}\Omega_0 = \oint_{\partial V_0} \mathbf{\Phi}\hat{\mathbf{N}} J \, \mathrm{d}S_0 + \int_{V_0} \mathbf{R} J \, \mathrm{d}\Omega_0 \qquad \text{Integral form. Proof ?? (5.14)}$$

$$\frac{\mathrm{d}(\mathbf{Q}J)}{\mathrm{d}t} = -\operatorname{div}_{\mathbf{X}}\left(\mathbf{\Phi}\mathbf{F}^{-T} J\right) + \mathbf{R}J \qquad \text{Differential form. Proof ?? (5.15)}$$

If the conserved property is intensive, expressed per unit mass, the following conservation laws may be derived.

$$\int_{V_0} \rho_0 \frac{\mathrm{d}\tilde{\mathbf{Q}}}{\mathrm{d}t} \, \mathrm{d}\Omega_0 = \oint_{\partial V_0} \mathbf{\Phi}\hat{\mathbf{N}} J \, \mathrm{d}S_0 + \int_{V_0} \mathbf{R} J \, \mathrm{d}\Omega_0 \qquad \text{Integral form. Proof ?? (5.16)}$$

$$\rho_0 \frac{\mathrm{d}\tilde{\mathbf{Q}}}{\mathrm{d}t} = -\operatorname{div}_{\mathbf{X}}\left(\mathbf{\Phi}\mathbf{F}^{-t} J\right) + \mathbf{R}J \qquad \text{Differential form. Proof ?? (5.17)}$$

5.6.2 Spatial Frame

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} \mathbf{q} \, \mathrm{d}\Omega_t = -\oint_{\partial V(t)} \phi\hat{\mathbf{n}} \, \mathrm{d}S + \int_{V(t)} \mathbf{r} \, \mathrm{d}\Omega_t \qquad \text{Integral form. Proof ?? (5.18)}$$

$$\frac{\partial \mathbf{q}}{\partial t} + \operatorname{div}\left(\mathbf{q}\otimes\mathbf{v}\right) = -\operatorname{div}\phi + \mathbf{r} \qquad \text{Differential form. Proof ?? (5.19)}$$

If the conserved property is intensive, expressed per unit mass, the following conservation laws may be derived.

$$\int_{V(t)} \rho \left(\frac{\partial \tilde{\mathbf{q}}}{\partial t} + (\operatorname{grad}\tilde{\mathbf{q}})\,\mathbf{v}\right) \mathrm{d}\Omega_t = -\oint_{\partial V(t)} \phi\hat{\mathbf{n}} \, \mathrm{d}S + \int_{V(t)} \mathbf{r} \, \mathrm{d}\Omega_t \qquad \text{Integral form. Proof ?? (5.20)}$$

$$\rho\left(\frac{\partial \tilde{\mathbf{q}}}{\partial t} + (\operatorname{grad}\tilde{\mathbf{q}})\,\mathbf{v}\right) = -\operatorname{div}\phi + \mathbf{r} \qquad \text{Differential form. Proof ?? (5.21)}$$

5.7 Conservation of Mass

for applied laws, there are a million ways to express, but only including useful ones

- Quantity: mass, $Q = \varrho, q = \rho$

- Surface fluxes: none, $\mathbf{\Phi} = \mathbf{0}, \phi = \mathbf{0}$

- Internal sources: none, $R = 0, r = 0$

5.7.1 Material Frame

$$\rho_0 = \varrho J \qquad \text{Proof ?? (5.22)}$$

5.7.2 Spatial Frame

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\left(\rho\mathbf{v}\right) = 0 \qquad \text{Conservative form. Proof ?? (5.23)}$$

5.8   Conservation of Momentum

- Quantity: momentum, $\mathbf{Q} = \varrho\mathbf{V}, \mathbf{q} = \rho\mathbf{v}$

- Surface fluxes: Cauchy stress, $\boldsymbol{\Phi} = -\boldsymbol{\Sigma}, \boldsymbol{\phi} = -\boldsymbol{\sigma}$

- Internal sources: body forces, $\mathbf{R} = \varrho\mathbf{B}, \mathbf{r} = \rho\mathbf{b}$

5.8.1   Material Frame

$$\rho_0 \frac{\mathrm{d}V}{\mathrm{d}t} = \mathrm{div}_{\mathbf{X}}\, \mathbf{P} + \rho_0\mathbf{B} \qquad\qquad \text{Proof } \mathbf{??} \;\; (5.24)$$

5.8.2   Spatial Frame

$$\frac{\partial\,(\rho\mathbf{v})}{\partial t} + \mathrm{div}\,(\rho\mathbf{v}\otimes\mathbf{v}) = \mathrm{div}\,\boldsymbol{\sigma} + \rho\mathbf{b} \qquad\qquad \text{Conservative form. Proof } \mathbf{??} \;\; (5.25)$$

$$\rho\left(\frac{\partial\mathbf{v}}{\partial t} + (\mathrm{grad}\,\mathbf{v})\,\mathbf{v}\right) = \mathrm{div}\,\boldsymbol{\sigma} + \rho\mathbf{b} \qquad\qquad \text{Reduced form. Proof } \mathbf{??} \;\; (5.26)$$

5.9   Conservation of Angular Momentum

- Quantity: angular momentum, $\mathbf{Q} = \boldsymbol{\varphi} \times \varrho\mathbf{V}, \mathbf{q} = \mathbf{x} \times \rho\mathbf{v}$

- Surface fluxes: Cauchy torque, $\boldsymbol{\Phi} = -\boldsymbol{\varphi} \times \boldsymbol{\Sigma}, \boldsymbol{\phi} = -\mathbf{x} \times \boldsymbol{\sigma}$

- Internal sources: body torques, $\mathbf{R} = \boldsymbol{\varphi} \times \varrho\mathbf{B}, \mathbf{r} = \mathbf{x} \times \rho\mathbf{b}$

5.9.1   Material Frame

$$\mathbf{S} = \mathbf{S}^T \qquad\qquad \text{Reduced form. Proof } \mathbf{??} \;\; (5.27)$$

5.9.2   Spatial Frame

$$\frac{\partial\,(\mathbf{x} \times \rho\mathbf{v})}{\partial t} + \mathrm{div}\,(\mathbf{x} \times \rho\mathbf{v}\otimes\mathbf{v}) = \mathrm{div}\,(\mathbf{x} \times \boldsymbol{\sigma}) + \mathbf{x} \times \rho\mathbf{b} \qquad\qquad \text{Conservative form. Proof } \mathbf{??} \;\; (5.28)$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \qquad\qquad \text{Reduced form. Proof } \mathbf{??} \;\; (5.29)$$

5.10   Conservation of Energy

- Quantity: energy, $Q = E, q = e$

- Surface fluxes: Cauchy power and heat flux, $\boldsymbol{\Phi} = -\mathbf{V} \cdot \boldsymbol{\Sigma} + \mathbf{Q}, \boldsymbol{\phi} = -\mathbf{v} \cdot \boldsymbol{\sigma} + \mathbf{q}$

- Internal sources: body force power and generic energy sources, $R = \varrho\mathbf{B} \cdot \mathbf{B} + \varrho R, r = \rho\mathbf{b} \cdot \mathbf{v} + \rho r$

5.10.1   Material Frame

$$\rho_0 \frac{\mathrm{d}I}{\mathrm{d}t} = \mathbf{S} : \mathbf{D} - \mathrm{div}_{\mathbf{X}}\, \mathbf{Q} + \rho_0\mathbf{R} \qquad\qquad \text{Proof } \mathbf{??} \;\; (5.30)$$

5.10.2   Spatial Frame

$$\frac{\partial (\rho e)}{\partial t} + \operatorname{div}(\rho e \mathbf{v}) = \operatorname{div}(\mathbf{v} \cdot \boldsymbol{\sigma} - \mathbf{q}) + \rho \mathbf{b} \cdot \mathbf{v} + \rho r \qquad \text{Conservative form. Proof \textbf{??} (5.31)}$$

$$\rho \left(\frac{\partial e}{\partial t} + \operatorname{grad} e \cdot \mathbf{v} \right) = ??? - \operatorname{div} \mathbf{q} + \rho \mathbf{b} \cdot \mathbf{v} + \rho r \qquad \text{Reduced form. Proof \textbf{??} (5.32)}$$

$$\rho \left(\frac{\partial i}{\partial t} + \operatorname{grad} i \cdot \mathbf{v} \right) = \boldsymbol{\sigma} : \mathbf{d} - \operatorname{div} \mathbf{q} + \rho r \qquad \text{Reduced form. Proof \textbf{??} (5.33)}$$

5.11   Entropy Inequality

- Quantity: entropy, $Q = N, q = \eta$

- Surface fluxes: entropy from heat flux, $\boldsymbol{\Phi} = \frac{\mathbf{Q}}{\Theta}, \phi = \frac{\mathbf{q}}{\theta}$

- Internal sources: entropy generation from energy sources, $R = \varrho \frac{R}{\Theta}, r = \rho \frac{r}{\theta}$

5.11.1   Material Frame

$$\rho_0 \frac{\mathrm{d}N}{\mathrm{d}t} + \operatorname{div}_{\mathbf{x}} \frac{\mathbf{Q}}{\Theta} - \rho_0 \frac{R}{\Theta} \geq 0 \qquad \text{Proof \textbf{??} (5.34)}$$

5.11.2   Spatial Frame

$$\frac{\partial (\rho \eta)}{\partial t} + \operatorname{div}(\rho \eta \mathbf{v}) + \operatorname{div} \frac{\mathbf{q}}{\Theta} - \rho \frac{r}{\theta} \geq 0 \qquad \text{Conservative form. Proof \textbf{??} (5.35)}$$

$$\rho \left(\frac{\partial \eta}{\partial t} + \operatorname{grad} \eta \cdot \mathbf{v} \right) + \operatorname{div} \frac{\mathbf{q}}{\Theta} - \rho \frac{r}{\theta} \geq 0 \qquad \text{Reduced form. Proof \textbf{??} (5.36)}$$

5.12   Advection Diffusion Reaction Equation

This is essentially a variant of the conservation of momentum. It can be applied to particles advected by a fluid, for example.

- Quantity: mass, $Q = C, q = c$

- Surface fluxes: diffusion, $\boldsymbol{\Phi} = ???, \phi = -\mathbf{D} \operatorname{grad} c$

- Internal sources: reaction, $R = R, r = r$

$$\frac{\partial c}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{v}c - \mathbf{D}\boldsymbol{\nabla}c) = R \qquad \text{Proof \textbf{??} (5.37)}$$

$$\frac{\partial \mathbf{c}}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{v} \otimes \mathbf{c} - \mathcal{D} : \boldsymbol{\nabla}\mathbf{c}) = R \qquad \text{Proof \textbf{??} (5.38)}$$

5.13   Advanced Formulations

e.g. curvature-dependent, hyperstress

Chapter 6

Solid Mechanics

general
    elastic waves
    thermoelastic solid
    engineering stress example
    eventually work to 1d stress-strain

6.1   Provisional Constitutive Laws

- $\mathbb{C} =$???: elasticity tensor in material frame

  - $\mathbb{C} := \frac{\partial \mathbf{S}}{\partial E} = \frac{\partial^2 \hat{\Psi}}{\partial \mathbf{E}^2}$

- $=$???: elasticity tensor in spatial frame

6.2   Elastic Wave Equations

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \operatorname{div} \boldsymbol{\sigma} = 0 \hspace{4cm} \text{Proof ??} \quad (6.1)$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} = (\lambda + 2\mu) \operatorname{grad}(\operatorname{div} \mathbf{u}) - \mu \operatorname{curl}^2 \mathbf{u} \hspace{2cm} \text{hm} \quad (6.2)$$

$$(6.3)$$

Chapter 7

Fluid Mechanics

full navier stokes
    incompressible
    rans
    stokes flow
    euler equations
    acoustic equations
    my special acoustic equations
    shallow water equations

7.1 Shallow Water Equations

conservative, non-conservative,

$$\frac{\partial (\rho\eta)}{\partial t} + \mathrm{div}\,(\rho\eta\mathbf{u}) = 0 \qquad \text{Proof ??} \quad (7.1)$$

$$\frac{\partial (\rho\eta\mathbf{u})}{\partial t} + \mathrm{div}\left(\rho\eta\mathbf{u}\otimes\mathbf{u} + \frac{1}{2}\rho g\eta^2\mathbf{I}\right) = 0 \qquad \text{Proof ??} \quad (7.2)$$

$$\frac{\partial (\rho\eta)}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\eta\mathbf{u}) = 0 \qquad \text{Proof ??} \quad (7.3)$$

$$\frac{\partial (\rho\eta\mathbf{u})}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\eta\mathbf{u}\otimes\mathbf{u}) + \boldsymbol{\nabla}\frac{1}{2}\rho g\eta^2 = 0 \qquad \text{Proof ??} \quad (7.4)$$

7.1.1 Closures and Assumptions

diffusive wave equations
    bernoulli
    groundwater darcy flow

Appendix A

Proofs

Proof A.1 (Equation 2.13)

$$\mathbf{a} \cdot \mathbf{b} = \left(\sum_{i=1}^{n} a_i \hat{\mathbf{e}}_i \right) \cdot \left(\sum_{j=1}^{n} b_j \hat{\mathbf{e}}_j \right) \qquad \qquad \text{\textit{Using 1.14}} \quad (A.1)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j \qquad \qquad (A.2)$$

$$= \sum_{i=1}^{n} a_i b_i \qquad \qquad \text{\textit{Using 1.12}} \quad (A.3)$$

Proof A.2 (Equation 2.14)

$$\mathbf{S}\mathbf{a} = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} S_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \right) \left(\sum_{k=1}^{n} a_k \hat{\mathbf{e}}_k \right) \qquad \text{\textit{Using 2.1, 1.14}} \quad (A.4)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{n} S_{ij} a_k \left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \right) \hat{\mathbf{e}}_k \qquad \qquad (A.5)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{n} S_{ij} a_k \hat{\mathbf{e}}_i \left(\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_k \right) \qquad \qquad \text{\textit{Using 2.4}} \quad (A.6)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} S_{ij} a_j \hat{\mathbf{e}}_i \qquad \qquad \text{\textit{Using 1.12}} \quad (A.7)$$

Proof A.3 (Equation 2.15)

$$\mathbf{x} \cdot \mathbf{S} = \left(\sum_{i=1}^{n} x_i \hat{\mathbf{e}}_i \right) \cdot \left(\sum_{j=1}^{m} \sum_{k=1}^{n} S_{jk} \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \right) \qquad \text{\textit{Using 1.14, 2.1}} \quad (A.8)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{n} x_i S_{jk} \hat{\mathbf{e}}_i \cdot \left(\hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \right) \qquad \qquad (A.9)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{n} x_i S_{jk} \left(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j \right) \hat{\mathbf{e}}_k \qquad \qquad \text{\textit{Using 2.5}} \quad (A.10)$$

$$= \sum_{i=1}^{m} \sum_{k=1}^{n} x_i S_{ik} \hat{\mathbf{e}}_k \qquad \qquad \text{\textit{Using 1.12}} \quad (A.11)$$

Proof A.4 (Equation 2.16)

$$\mathbf{SU} = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} S_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\right)\left(\sum_{k=1}^{n}\sum_{\ell=1}^{p} U_{k\ell}\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell\right) \qquad \textit{Using 2.1} \quad \text{(A.12)}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{p} S_{ij}U_{k\ell}\left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\right)\left(\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell\right) \qquad \text{(A.13)}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{p} S_{ij}U_{k\ell}\left(\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_k\right)\left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_\ell\right) \qquad \textit{Using 2.9} \quad \text{(A.14)}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{\ell=1}^{p} S_{ij}U_{j\ell}\left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_\ell\right) \qquad \textit{Using 1.12} \quad \text{(A.15)}$$

Proof A.5 (Equation 2.17)

$$\mathbf{S} : \mathbf{T} = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} S_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\right) : \left(\sum_{k=1}^{m}\sum_{\ell=1}^{m} T_{k\ell}\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell\right) \qquad \textit{Using 2.1} \quad \text{(A.16)}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{\ell=1}^{n} S_{ij}T_{k\ell}\left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\right) : \left(\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell\right) \qquad \text{(A.17)}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{\ell=1}^{n} S_{ij}T_{k\ell}\left(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_k\right)\left(\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_\ell\right) \qquad \textit{Using 2.10} \quad \text{(A.18)}$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} S_{ij}T_{ij} \qquad \textit{Using 1.12} \quad \text{(A.19)}$$

$$\text{(A.20)}$$

Proof A.6 (Equation 3.30)

$$\boldsymbol{\nabla} \cdot (\phi\mathbf{a}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[\hat{\mathbf{e}}_{\mathbf{i}} \cdot \left(\phi\sum_{j=1}^{n} a_j\hat{\mathbf{e}}_j\right)\right] \qquad \textit{Using 3.3, 1.14} \quad \text{(A.21)}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial \phi a_j}{\partial x_i}\left(\hat{\mathbf{e}}_{\mathbf{i}} \cdot \hat{\mathbf{e}}_j\right) \qquad \text{(A.22)}$$

$$= \sum_{i=1}^{n} \frac{\partial \phi a_i}{\partial x_i} \qquad \textit{Using 1.12} \quad \text{(A.23)}$$

$$= \sum_{i=1}^{n} \left(\phi\frac{\partial a_i}{\partial x_i} + a_i\frac{\partial \phi}{\partial x_i}\right) \qquad \text{(A.24)}$$

$$= \phi\boldsymbol{\nabla} \cdot \mathbf{a} + \mathbf{a} \cdot \boldsymbol{\nabla}\phi \qquad \textit{Using 3.26, 2.13, 3.5} \quad \text{(A.25)}$$

Proof A.7 (Equation 3.34)

$$\boldsymbol{\nabla} \cdot (\phi \mathbf{S}) = \left(\sum_{i=1}^{m} \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \right) \cdot \left(\phi \sum_{j=1}^{m} \sum_{k=1}^{n} S_{jk} \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \right) \qquad \textit{Using 3.3, 2.1 \; (A.26)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{\partial (\phi S_{jk})}{\partial x_i} \hat{\mathbf{e}}_i \cdot (\hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k) \qquad \text{(A.27)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{n} \left(\phi \frac{\partial S_{jk}}{\partial x_i} + S_{jk} \frac{\partial \phi}{\partial x_i} \right) (\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j) \hat{\mathbf{e}}_k \qquad \textit{Using 2.5 \; (A.28)}$$

$$= \sum_{i=1}^{m} \sum_{k=1}^{n} \phi \frac{\partial S_{ik}}{\partial x_i} \hat{\mathbf{e}}_k + S_{ik} \frac{\partial \phi}{\partial x_i} \hat{\mathbf{e}}_k \qquad \textit{Using 1.12 \; (A.29)}$$

$$= \phi \boldsymbol{\nabla} \cdot \mathbf{S} + \boldsymbol{\nabla} \phi \cdot \mathbf{S} \qquad \textit{Using 3.27, 3.5, 2.15 \; (A.30)}$$

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