Exercise 3

We are given first picture g_1, which is taken by adjusting camera lens so that the scene outside (f_1) is in focus, and the reflection off the window surface (f_2) will now be defocussed or blurred. This can be written as $g_1 = f_1 + h_2 * f_2$ where h_2 stands for the blur kernel that acted on f_2. Taking Fourier Transform on both sides, we have:

$$\mathcal{F}(g_1) = \mathcal{F}(f_1 + h_2 * f_2)$$
$$G_1 = F_1 + H_2 F_2$$

where we have used the convolution theorem for Fourier transforms.

The second picture g_2 is taken by focusing the camera onto the surface of the window, with the scene outside being defocussed. This can be written as $g_2 = h_1 * f_1 + f_2$ where h_1 is the blur kernel acting on f_1. Taking Fourier Transform on both sides, we have:

$$\mathcal{F}(g_2) = \mathcal{F}(h_1 * f_1 + f_2)$$
$$G_2 = H_1 F_1 + F_2$$

where we have again used the convolution theorem for Fourier transforms.

Thus, we have a set of linear equations, which can be represented in matrix form as follows:

$$\begin{bmatrix} 1 & H_2 \\ H_1 & 1 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$$

Solving this set of equations, we can compute F_1 and F_2 as follows. Multiplying the second equation by H_2 and leaving the first equation as it is, we have:

$$G_1 = F_1 + H_2 F_2$$
$$H_2 G_2 = H_1 H_2 F_1 + H_2 F_2$$

Subtracting the second equation from the first, we have:

$$G_1 - H_2 G_2 = (1 - H_1 H_2) F_1$$

$$F_1 = \frac{G_1 - H_2 G_2}{1 - H_1 H_2}$$

Similarly, multiplying the first equation by H_1 and leaving the second equation as it is, followed by subtracting the first equation from the second, we get:

$$G_2 - H_1 G_1 = (1 - H_1 H_2) F_2$$

$$F_2 = \frac{G_2 - H_1 G_1}{1 - H_1 H_2}$$

f_1 and f_2 can now be obtained as: $f_1 = \mathcal{F}^{-1}(F_1)$ and $f_2 = \mathcal{F}^{-1}(F_2)$.

There are two inherent problems with the usage of this formula:

- The denominator of the expression for the Fourier-domain images contains $1 - H_1 H_2$. It is possible that at some frequencies, the denominator equals zero. In this case, rather than writing the explicit solution by dividing this quantity, we have to observe that the denominator is nothing but the determinant of the 2×2 coefficient matrix for the linear pair of equations. By the theory of linear algebra, we know that this either means that the equations are inconsistent with no solution (when any one of the numerators is non-zero) or the equations are consistent with infinite solutions (when both the numerators are also zero). In either case, we won't be able to get well-defined Fourier-domain images, if there exist frequencies at which the denominator goes to zero.
- In this entire computation, we have nowhere taken into account the presence of noise. The presence of noise will adversely affect the final expressions.