Exercise-6

Given a 1D ramp image $I(x) = cx + d$, where c and d are scalar coefficients. Assuming that the image has an infinite extent, the output image $J(x)$, obtained by filtering $I(x)$ using a zero-mean Gaussian with standard deviation σ, can be computed as:

$$J(x) = \int_{k=-\infty}^{\infty} G_\sigma(|(x+k)-x|)I(x+k)dk = \int_{k=-\infty}^{\infty} G_\sigma(|k|)I(x+k)dk$$

$$J(x) = \int_{k=-\infty}^{\infty} \frac{\exp(-\frac{k^2}{2\sigma^2})I(x+k)}{\sigma\sqrt{2\pi}}dk$$

Now, following the definition of 1D ramp, as $I(x+k) = c(x+k)+d = (cx+d)+ck = I(x)+ck$, we have:

$$J(x) = \int_{k=-\infty}^{\infty} \frac{\exp(-\frac{k^2}{2\sigma^2})(I(x)+ck)}{\sigma\sqrt{2\pi}}dk$$

Next, we separate the terms as follows:

$$J(x) = \int_{k=-\infty}^{\infty} \frac{\exp(-\frac{k^2}{2\sigma^2})I(x)}{\sigma\sqrt{2\pi}}dk + \int_{k=-\infty}^{\infty} \frac{\exp(-\frac{k^2}{2\sigma^2})ck}{\sigma\sqrt{2\pi}}dk$$

We can take $I(x)$ out from the integral, since it is independent of k.

$$J(x) = I(x)\int_{k=-\infty}^{\infty} \frac{\exp(-\frac{k^2}{2\sigma^2})}{\sigma\sqrt{2\pi}}dk + \frac{c}{\sigma\sqrt{2\pi}}\int_{k=-\infty}^{\infty} k\exp\left(-\frac{k^2}{2\sigma^2}\right)dk$$

The integral in the first term is equal to 1 since its the integral of a Gaussian PDF from $-\infty$ to ∞. In the second term, we have taken out all the constants from the integral. The integrand in the second term is an odd function, hence its integral from $-\infty$ to ∞ will be 0.

$$\therefore J(x) = I(x)$$

Hence, we get the same original image back.

Again assuming that the image has infinite extent, the output image $K(x)$, obtained by applying a bilateral filter on $I(x)$ with parameters σ_s and σ_r, can be computed as follows:

$$K(x) = \frac{1}{W}\int_{k=-\infty}^{\infty} G_{\sigma_s}(|(x+k)-x|)G_{\sigma_r}(|I(x+k)-I(x)|)I(x+k)dk = \frac{1}{W}\int_{k=-\infty}^{\infty} G_{\sigma_s}(|k|)G_{\sigma_r}(|ck|)I(x+k)dk$$

$$[\because I(x+k)-I(x) = (c(x+k)+d)-(cx+d) = (cx+d+ck)-(cx+d) = ck]$$

where,

$$\begin{aligned}
W &= \int_{k=-\infty}^{\infty} G_{\sigma_s}(|(x+k)-x|)G_{\sigma_r}(|I(x+k)-I(x)|)dk \\
&= \int_{k=-\infty}^{\infty} G_{\sigma_s}(k)G_{\sigma_r}(ck)dk \\
&= \int_{k=-\infty}^{\infty} \frac{\exp(-\frac{k^2}{2\sigma_s^2})}{\sigma_s\sqrt{2\pi}}\frac{\exp(-\frac{c^2k^2}{2\sigma_r^2})}{\sigma_r\sqrt{2\pi}}dk \\
&= \int_{k=-\infty}^{\infty} \frac{\exp\left(-k^2(\frac{1}{2\sigma_s^2}+\frac{c^2}{2\sigma_r^2})\right)}{2\pi\sigma_s\sigma_r}dk
\end{aligned}$$

Now,

$$K(x) = \frac{1}{W}\int_{k=-\infty}^{\infty} \frac{\exp(-\frac{k^2}{2\sigma_s^2})}{\sigma_s\sqrt{2\pi}} \frac{\exp(-\frac{c^2k^2}{2\sigma_r^2})}{\sigma_r\sqrt{2\pi}} I(x+k)dk$$

Combining the exponential terms and following the definition of 1D ramp, as $I(x+k) = c(x+k)+d = (cx+d)+ck = I(x)+ck$, we have:

$$K(x) = \frac{1}{W}\int_{k=-\infty}^{\infty} \frac{\exp\left(-k^2(\frac{1}{2\sigma_s^2}+\frac{c^2}{2\sigma_r^2})\right)(I(x)+ck)}{2\pi\sigma_s\sigma_r} dk$$

Next, we separate the terms as follows:

$$K(x) = \frac{1}{W}\int_{k=-\infty}^{\infty} \frac{\exp\left(-k^2(\frac{1}{2\sigma_s^2}+\frac{c^2}{2\sigma_r^2})\right)I(x)}{2\pi\sigma_s\sigma_r} dk + \frac{1}{W}\int_{k=-\infty}^{\infty} \frac{\exp(-k^2(\frac{1}{2\sigma_s^2}+\frac{c^2}{2\sigma_r^2}))ck}{2\pi\sigma_s\sigma_r} dk$$

We can take $I(x)$ out from the integral, since it is independent of k.

$$K(x) = \frac{I(x)}{W}\int_{k=-\infty}^{\infty} \frac{\exp\left(-k^2(\frac{1}{2\sigma_s^2}+\frac{c^2}{2\sigma_r^2})\right)}{2\pi\sigma_s\sigma_r} dk + \frac{c}{2\pi W\sigma_s\sigma_r}\int_{k=-\infty}^{\infty} k\exp\left(-k^2(\frac{1}{2\sigma_s^2}+\frac{c^2}{2\sigma_r^2})\right) dk$$

Observe that the integration part in the first term is just equal to W, and hence it cancels out with the W in the denominator. The integrand in the second term is odd, same as before, and hence its integral from $-\infty$ to ∞ will be 0.

$$\therefore K(x) = I(x)$$

Hence, we again get the same original image back.