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Homework Assignment 4
CS 663: Digital Image Processing

2021-10-26

Exercise 1

There are three possible errors that the student might be unaware of:

- The eigenvectors of $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{A}^T$ (contained in the columns of the matrix $\boldsymbol{U}$) and the eigenvectors of $\boldsymbol{L} = \boldsymbol{A}^T\boldsymbol{A}$ (contained in the columns of the matrix $\boldsymbol{V}$) are related. Let λ be an eigenvalue of $\boldsymbol{L}$ with eigenvector $\boldsymbol{w}$, then, $\boldsymbol{L}\boldsymbol{w} = \boldsymbol{A}^T\boldsymbol{A}\boldsymbol{w} = \lambda\boldsymbol{w}$. Premultiplying both sides by $\boldsymbol{A}$, we have $\boldsymbol{A}\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{w} = \lambda\boldsymbol{A}\boldsymbol{w}$. Now, $\boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{C}$, therefore, we have $\boldsymbol{C}(\boldsymbol{A}\boldsymbol{w}) = \lambda(\boldsymbol{A}\boldsymbol{w})$. This implies that all the non-zero eigenvalues of $\boldsymbol{C}$ and $\boldsymbol{L}$ are equal and their eigenvectors (contained in the columns of $\boldsymbol{U}$ and $\boldsymbol{V}$) are related by the equation $\boldsymbol{U} = \boldsymbol{A}\boldsymbol{V}$. Note that $\boldsymbol{U}$ and $\boldsymbol{V}$ are just the left and right singular matrices of $\boldsymbol{A}$ respectively.

  The problem that the student might be encountering is that all the non-zero eigenvalues of $\boldsymbol{U}$ and $\boldsymbol{V}$ (and hence the non-zero singular values of $\boldsymbol{A}$) might not be distinct! Let's say we have an eigenvalue which is repeated two times (similar arguments will hold for more than two as well). In this case, the eigenvectors corresponding to these equal eigenvalues can lie anywhere on a plane. Out of these infinite possibilities, a unique choice of eigenvectors is made by picking any one vector lying in that plane and the second eigenvector also lying in the same plane, but being orthogonal to the first.

  Now, since the columns of $\boldsymbol{U}$ and $\boldsymbol{V}$ corresponding to the repeated eigenvalue, are evaluated separately (using the eig routine separately on $\boldsymbol{A}\boldsymbol{A}^T$ and $\boldsymbol{A}^T\boldsymbol{A}$), it is highly possible that the required relation $\boldsymbol{U} = \boldsymbol{A}\boldsymbol{V}$ no longer holds, specifically for these columns. This can, in turn, lead to the problem that the matrix obtained by multiplying the SVD factorization matrices does not equal the original matrix, i.e. $\boldsymbol{A} \neq \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^T$.

- If the dimension of $\boldsymbol{A}$ is $m \times n$, the dimensions of the matrices obtained from SVD should be as follows: $dim(\boldsymbol{U}) = m \times m$, $dim(\boldsymbol{V}) = n \times n$ and $dim(\boldsymbol{S}) = m \times n$. Now, since the student computes $\boldsymbol{S}$ using the non-negative square roots of the eigenvalues computed using the eig routine on either $\boldsymbol{U}$ or $\boldsymbol{V}$, the resultant matrix will, by default, have dimension either $m \times m$ or $n \times n$, respectively. The student must take care to change the dimension to $m \times n$ by either manually deleting the appropriate number of zero columns or appending the appropriate number of zero rows (assuming $m > n$). If this is not done, properly, it can create problems in the back computation of $\boldsymbol{A}$.

- The student might not be getting the exact relation $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^T$, but the relation might still be holding approximately, i.e. $\boldsymbol{A} \sim \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^T$. In such a case there is no problem with the way the student is computing it. The problem may lie in the finite nature of the floating point computations in MATLAB which do not reveal the exact values, hence leading to slight inequality.