Exercise-7

Consider a function defined in the original coordinate system $f \equiv f(x, y)$. Now, we are transforming the coordinate system to (u, v) where $u = x\cos\theta - y\sin\theta$ and $v = x\sin\theta + y\cos\theta$.

Using the inverse of these transformations, given by $x = u\cos\theta + v\sin\theta$ and $y = -u\sin\theta + v\cos\theta$, the function can now be defined in the new coordinate system as $f \equiv f(u, v)$.

The first derivatives of $f(u, v)$ with respect to x and y can be computed as follows (keeping in mind that $\sin\theta$ and $\cos\theta$ are just constants):

$$
\begin{aligned}
f_x &= \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = f_u\cos\theta + f_v\sin\theta \\
f_y &= \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = -f_u\sin\theta + f_v\cos\theta
\end{aligned}
$$

Now, computing the second derivative of these (again keeping in mind that $\sin\theta$ and $\cos\theta$ are constants):

$$
\begin{aligned}
f_{xx} &= \frac{\partial f_x}{\partial x} \\
&= \frac{\partial}{\partial x}(f_u\cos\theta + f_v\sin\theta) \\
&= \frac{\partial f_u}{\partial x}\cos\theta + \frac{\partial f_v}{\partial x}\sin\theta \\
&= (\frac{\partial f_u}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f_u}{\partial v}\frac{\partial v}{\partial x})\cos\theta + (\frac{\partial f_v}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f_v}{\partial v}\frac{\partial v}{\partial x})\sin\theta \\
&= (f_{uu}\cos\theta + f_{uv}\sin\theta)\cos\theta + (f_{vu}\cos\theta + f_{vv}\sin\theta)\sin\theta \\
&= f_{uu}\cos^2\theta + f_{uv}\sin\theta\cos\theta + f_{vu}\cos\theta\sin\theta + f_{vv}\sin^2\theta
\end{aligned}
$$

$$
\begin{aligned}
f_{yy} &= \frac{\partial f_y}{\partial y} \\
&= \frac{\partial}{\partial y}(-f_u\sin\theta + f_v\cos\theta) \\
&= -\frac{\partial f_u}{\partial y}\sin\theta + \frac{\partial f_v}{\partial y}\cos\theta \\
&= -(\frac{\partial f_u}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f_u}{\partial v}\frac{\partial v}{\partial y})\sin\theta + (\frac{\partial f_v}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f_v}{\partial v}\frac{\partial v}{\partial y})\cos\theta \\
&= -(-f_{uu}\sin\theta + f_{uv}\cos\theta)\sin\theta + (-f_{vu}\sin\theta + f_{vv}\cos\theta)\cos\theta \\
&= f_{uu}\sin^2\theta - f_{uv}\cos\theta\sin\theta - f_{vu}\sin\theta\cos\theta + f_{vv}\cos^2\theta
\end{aligned}
$$

Therefore, the laplacian of f with respect to the old coordinates can now be written as:

$$
\begin{aligned}
f_{xx} + f_{yy} &= (f_{uu}\cos^2\theta + f_{uv}\sin\theta\cos\theta + f_{vu}\cos\theta\sin\theta + f_{vv}\sin^2\theta) \\
&\quad + (f_{uu}\sin^2\theta - f_{uv}\cos\theta\sin\theta - f_{vu}\sin\theta\cos\theta + f_{vv}\cos^2\theta) \\
&= f_{uu}(\cos^2\theta + \sin^2\theta) + f_{vv}(\cos^2\theta + \sin^2\theta) \\
&= f_{uu} + f_{vv}
\end{aligned}
$$

Thus, we see that the Laplacian operator in the old coordinates is the same as the Laplacian operator in the new coordinates (generated through a rotation transformation). Hence, proved that the Laplacian is rotationally invariant.