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Homework Assignment 3
CS 663: Digital Image Processing

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Exercise 3

Assuming that we have a discrete function $f(x, y)$ defined on the domain $x \in [0, 1, 2, ..., W_1],\ y \in [0, 1, 2, ..., W_2]$ and we have done zero padding on both sides of the function to make its extent infinite, the expression for its 2D Discrete Fourier transform is as follows:

$$\mathcal{F}(f(x, y))(\mu, \nu) = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x, y) exp(-j\frac{2\pi\mu}{W_1}x) exp(-j\frac{2\pi\nu}{W_2}y)$$

Consider two 2D functions with zero padding on both sides to make its extent infinite, $f(x, y)$ and $g(x, y)$. Their convolution can be written as follows:

$$(f * g)(x, y) = \sum_{t=-\infty}^{\infty} \sum_{u=-\infty}^{\infty} f(t, u) g(x - t, y - u)$$

The Fourier transform of this convolution can be written as:

$$\mathcal{F}((f * g)(x, y))(\mu, \nu) = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} \left(\sum_{t=-\infty}^{\infty} \sum_{u=-\infty}^{\infty} f(t, u) g(x - t, y - u) \right) exp(-j\frac{2\pi\mu}{W_1}x) exp(-j\frac{2\pi\nu}{W_2}y)$$

Now, we can rearrange this expression as follows:

$$\mathcal{F}((f * g)(x, y))(\mu, \nu) = \sum_{t=-\infty}^{\infty} \sum_{u=-\infty}^{\infty} f(t, u) \left(\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} g(x - t, y - u) exp(-j\frac{2\pi\mu}{W_1}x) exp(-j\frac{2\pi\nu}{W_2}y) \right)$$

From the sifting property of Fourier transforms, we know the following:

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} g(x - t, y - u) exp(-j\frac{2\pi\mu}{W_1}x) exp(-j\frac{2\pi\nu}{W_2}y) = G(\mu, \nu) exp(-j\frac{2\pi\mu}{W_1}t) exp(-j\frac{2\pi\nu}{W_2}u)$$

where $G(\mu, \nu)$ is the Discrete Fourier transform of $g(x, y)$.

Plugging this back into the original equation, we have the following:

$$\mathcal{F}((f * g)(x, y))(\mu, \nu) = \sum_{t=-\infty}^{\infty} \sum_{u=-\infty}^{\infty} f(t, u) G(\mu, \nu) exp(-j\frac{2\pi\mu}{W_1}t) exp(-j\frac{2\pi\nu}{W_2}u)$$

Pulling out $G(\mu, \nu)$ out of the summations (which are w.r.t t and u),

$$\mathcal{F}((f * g)(x, y))(\mu, \nu) = G(\mu, \nu) \sum_{t=-\infty}^{\infty} \sum_{u=-\infty}^{\infty} f(t, v) exp(-j\frac{2\pi\mu}{W_1}t) exp(-j\frac{2\pi\nu}{W_2}u)$$

Now, we know that:

$$\sum_{t=-\infty}^{\infty} \sum_{u=-\infty}^{\infty} f(t, v) exp(-j\frac{2\pi\mu}{W_1}t) exp(-j\frac{2\pi\nu}{W_2}u) = F(\mu, \nu)$$

where $F(\mu, \nu)$ is the Discrete Fourier transform of $f(x, y)$.

Plugging this backing into the original equation, we have the following:

$$\mathcal{F}((f * g)(x, y))(\mu, \nu) = G(\mu, \nu) F(\mu, \nu)$$

Hence, we have proved the convolution theorem for 2D Discrete Fourier transforms.