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Homework Assignment 1
CS 663: Digital Image Processing

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Exercise-1

A function $f(z)$ is said to be linear if

$$f(kz_1 + z_2) = kf(z_1) + f(z_2) \tag{1}$$

where z_1, z_2 are two scalar values in the domain of f and k is a scalar constant. The formula for bilinear interpolation which expresses image intensities as a bilinear function of x, y (spatial coordinates) in the form

$$v(x, y) = ax + by + cxy + d \tag{2}$$

where a, b, c, d are scalar constants independent of x, y. For constant y, say $y = y_0$, the bilinear function takes the form

$$v(x) = ax + by_0 + cxy_0 + d = Ax + B \tag{3}$$

where $A = a + cy_0$ and $B = by_0 + d$. Consider the following three set of equations:

$$v(x_1) = Ax_1 + B \tag{4}$$

$$v(x_2) = Ax_2 + B \tag{5}$$

$$v(kx_1 + x_2) = A(kx_1 + x_2) + B \tag{6}$$

where eqns 4, 5 and 6 are the function values computed at x_1, x_2 and $kx_1 + x_2$. Now to check linearity of 3,

$$kv(x_1) + v(x_2) = k(Ax_1 + B) + Ax_2 + B = A(kx_1 + x_2) + B(k + 1) \neq v(kx_1 + x_2) \tag{7}$$

Hence, the function defined in eqn 3 is not linear.
Similarly, we can prove that linearity does not hold in y when x is held constant.
The definition of linearity in eqn 1 extends to the case where $\boldsymbol{z}$ is a vector, i.e. $f(\boldsymbol{z})$ is linear if

$$f(k\boldsymbol{z_1} + \boldsymbol{z_2}) = kf(\boldsymbol{z_1}) + f(\boldsymbol{z_2}). \tag{8}$$

Therefore for $\boldsymbol{z} \triangleq (x, y)$, consider the following three equations:

$$v(\boldsymbol{z_1}) = ax_1 + by_1 + cx_1y_1 + d \tag{9}$$

$$v(\boldsymbol{z_2}) = ax_2 + by_2 + cx_2y_2 + d \tag{10}$$

$$v(k\boldsymbol{z_1} + \boldsymbol{z_2}) = a(kx_1 + x_2) + b(ky_1 + y_2) + c(kx_1 + x_2)(ky_1 + y_2) + d \tag{11}$$

where eqns 9, 10 and 11 are the function values computed at $\boldsymbol{z_1} \triangleq (x_1, y_1)$, $\boldsymbol{z_2} \triangleq (x_2, y_2)$ and $k\boldsymbol{z_1} + \boldsymbol{z_2} \triangleq (kx_1 + x_2, ky_1 + y_2)$. Now to check linearity of 8,

$$\begin{aligned} kv(\boldsymbol{z_1}) + v(\boldsymbol{z_2}) &= k(ax_1 + by_1 + cx_1y_1 + d) + (ax_2 + by_2 + cx_2y_2 + d) \\ &= a(kx_1 + x_2) + b(ky_1 + y_2) + c(kx_1y_1 + x_2y_2) + (k + 1)d \\ &\neq v(k\boldsymbol{z_1} + \boldsymbol{z_2}) \end{aligned} \tag{12}$$

Hence, the function defined in eqn 2 is not linear.