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Homework Assignment 2
CS 663: Digital Image Processing

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Exercise-2

Let Q_{ij} denote the intensity value at pixel (x_i, y_j). Our motive is to approximate the intensity at any intermediate location **X** using bi-cubic interpolation, where the image intensity is expressed in the form $v(x,y) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}x^i y^j$, with (x,y) being the spatial coordinates. We have to determine the coefficients of interpolation given by a_{ij}. For a location **X**, the positions of the 16 nearest neighbours can be visually represented as follows:

$$
\begin{array}{ccccc}
(x_0,y_0) & (x_0,y_1) & & (x_0,y_2) & (x_0,y_3) \\
(x_1,y_0) & (x_1,y_1) & & (x_1,y_2) & (x_1,y_3) \\
 & & \mathbf{X} & & \\
(x_2,y_0) & (x_2,y_1) & & (x_2,y_2) & (x_2,y_3) \\
(x_3,y_0) & (x_3,y_1) & & (x_3,y_2) & (x_3,y_3)
\end{array}
$$

In terms of the intensity values at the 16 nearest neighbours, the visual representation becomes:

$$
\begin{array}{ccccc}
Q_{00} & Q_{01} & & Q_{02} & Q_{03} \\
Q_{10} & Q_{11} & & Q_{12} & Q_{13} \\
 & & \mathbf{X} & & \\
Q_{20} & Q_{21} & & Q_{22} & Q_{23} \\
Q_{30} & Q_{31} & & Q_{32} & Q_{33}
\end{array}
$$

The $v(x,y)$ values, calculated at these 16 nearest neighbours give 16 linearly independent equations (in the variables a_{ij}), which can be expressed in the form of a matrix multiplication as follows:

$$
\begin{bmatrix}
1 & y_0 & y_0^2 & y_0^3 & x_0 & x_0y_0 & x_0y_0^2 & x_0y_0^3 & x_0^2 & x_0^2y_0 & x_0^2y_0^2 & x_0^2y_0^3 & x_0^3 & x_0^3y_0 & x_0^3y_0^2 & x_0^3y_0^3 \\
1 & y_1 & y_1^2 & y_1^3 & x_0 & x_0y_1 & x_0y_1^2 & x_0y_1^3 & x_0^2 & x_0^2y_1 & x_0^2y_1^2 & x_0^2y_1^3 & x_0^3 & x_0^3y_1 & x_0^3y_1^2 & x_0^3y_1^3 \\
1 & y_2 & y_2^2 & y_2^3 & x_0 & x_0y_2 & x_0y_2^2 & x_0y_2^3 & x_0^2 & x_0^2y_2 & x_0^2y_2^2 & x_0^2y_2^3 & x_0^3 & x_0^3y_2 & x_0^3y_2^2 & x_0^3y_2^3 \\
1 & y_3 & y_3^2 & y_3^3 & x_0 & x_0y_3 & x_0y_3^2 & x_0y_3^3 & x_0^2 & x_0^2y_3 & x_0^2y_3^2 & x_0^2y_3^3 & x_0^3 & x_0^3y_3 & x_0^3y_3^2 & x_0^3y_3^3 \\
1 & y_0 & y_0^2 & y_0^3 & x_1 & x_1y_0 & x_1y_0^2 & x_1y_0^3 & x_1^2 & x_1^2y_0 & x_1^2y_0^2 & x_1^2y_0^3 & x_1^3 & x_1^3y_0 & x_1^3y_0^2 & x_1^3y_0^3 \\
1 & y_1 & y_1^2 & y_1^3 & x_1 & x_0y_1 & x_1y_1^2 & x_1y_1^3 & x_1^2 & x_1^2y_1 & x_1^2y_1^2 & x_1^2y_1^3 & x_1^3 & x_1^3y_1 & x_1^3y_1^2 & x_1^3y_1^3 \\
1 & y_2 & y_2^2 & y_2^3 & x_1 & x_1y_2 & x_1y_2^2 & x_1y_2^3 & x_1^2 & x_1^2y_2 & x_1^2y_2^2 & x_1^2y_2^3 & x_1^3 & x_1^3y_2 & x_1^3y_2^2 & x_1^3y_2^3 \\
1 & y_3 & y_3^2 & y_3^3 & x_1 & x_1y_3 & x_1y_3^2 & x_1y_3^3 & x_1^2 & x_1^2y_3 & x_1^2y_3^2 & x_1^2y_3^3 & x_1^3 & x_1^3y_3 & x_1^3y_3^2 & x_1^3y_3^3 \\
1 & y_0 & y_0^2 & y_0^3 & x_2 & x_2y_0 & x_2y_0^2 & x_2y_0^3 & x_2^2 & x_2^2y_0 & x_2^2y_0^2 & x_2^2y_0^3 & x_2^3 & x_2^3y_0 & x_2^3y_0^2 & x_2^3y_0^3 \\
1 & y_1 & y_1^2 & y_1^3 & x_2 & x_2y_1 & x_2y_1^2 & x_2y_1^3 & x_2^2 & x_2^2y_1 & x_2^2y_1^2 & x_2^2y_1^3 & x_2^3 & x_2^3y_1 & x_2^3y_1^2 & x_2^3y_1^3 \\
1 & y_2 & y_2^2 & y_2^3 & x_2 & x_2y_2 & x_2y_2^2 & x_2y_2^3 & x_2^2 & x_2^2y_2 & x_2^2y_2^2 & x_2^2y_2^3 & x_2^3 & x_2^3y_2 & x_2^3y_2^2 & x_2^3y_2^3 \\
1 & y_3 & y_3^2 & y_3^3 & x_2 & x_2y_3 & x_2y_3^2 & x_2y_3^3 & x_2^2 & x_2^2y_3 & x_2^2y_3^2 & x_2^2y_3^3 & x_2^3 & x_2^3y_3 & x_2^3y_3^2 & x_2^3y_3^3 \\
1 & y_0 & y_0^2 & y_0^3 & x_3 & x_3y_0 & x_3y_0^2 & x_3y_0^3 & x_3^2 & x_3^2y_0 & x_3^2y_0^2 & x_3^2y_0^3 & x_3^3 & x_3^3y_0 & x_3^3y_0^2 & x_3^3y_0^3 \\
1 & y_1 & y_1^2 & y_1^3 & x_3 & x_3y_1 & x_3y_1^2 & x_3y_1^3 & x_3^2 & x_3^2y_1 & x_3^2y_1^2 & x_3^2y_1^3 & x_3^3 & x_3^3y_1 & x_3^3y_1^2 & x_3^3y_1^3 \\
1 & y_2 & y_2^2 & y_2^3 & x_3 & x_3y_2 & x_3y_2^2 & x_3y_2^3 & x_3^2 & x_3^2y_2 & x_3^2y_2^2 & x_3^2y_2^3 & x_3^3 & x_3^3y_2 & x_3^3y_2^2 & x_3^3y_2^3 \\
1 & y_3 & y_3^2 & y_3^3 & x_3 & x_3y_3 & x_3y_3^2 & x_3y_3^3 & x_3^2 & x_3^2y_3 & x_3^2y_3^2 & x_3^2y_3^3 & x_3^3 & x_3^3y_3 & x_3^3y_3^2 & x_3^3y_3^3
\end{bmatrix}
\begin{bmatrix} a_{00} \\ a_{01} \\ a_{02} \\ a_{03} \\ a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \\ a_{20} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{30} \\ a_{31} \\ a_{32} \\ a_{33} \end{bmatrix}
=
\begin{bmatrix} Q_{00} \\ Q_{01} \\ Q_{02} \\ Q_{03} \\ Q_{10} \\ Q_{11} \\ Q_{12} \\ Q_{13} \\ Q_{20} \\ Q_{21} \\ Q_{22} \\ Q_{23} \\ Q_{30} \\ Q_{31} \\ Q_{32} \\ Q_{33} \end{bmatrix}
$$

This humongous matrix multiplication can be written in the form $\mathbf{Xa} = \mathbf{Q}$, where **X** is the 16×16 matrix having known entries, **a** is the vector of unknown coefficients of interpolation and **Q** is the vector of known intensity values at these 16 nearest neighbors. The vector **a** can then easily be expressed as a matrix product in terms of the inverse of **X**, using the equation $\mathbf{a} = \mathbf{X}^{-1}\mathbf{Q}$.

There are a total of 16 unknowns (i.e. the variables a_{ij}) in the expression for the bi-cubic interpolation function $v(x,y)$. This requires us to have at least 16 linearly independent equations to arrive at unique determinate values for all these variables. In order to have linearly independent equations, we need to use the intensity values of at least 16 nearest neighbors.