Algorithms

Lecture Topic: Approximation Algorithms (Part 2)

Anxiao (Andrew) Jiang

Roadmap of this lecture:

1. Understand approximation algorithms by solving TSP.
   1.1 General TSP has no constant-ratio approximation unless P=NP.
2. The "Linear Programming (LP) Technique" for approximation algorithms.
   2.1 Approximation algorithm for "Weighted Vertex Cover Problem" using the "LP technique".
   2.2 Analyze the approximation ratio of the algorithm.
3. Randomized Algorithm.
   3.1 Define "Randomized Algorithm".
   3.2 Understand "Randomized Algorithm" by solving the "Max 3-CNF SAT Problem"

Approximation Algorithms

Traveling Salesman Problem (TSP)

Input: An undirected complete graph G=(V,E), where every edge $(u,v) \in E$ has a non-negative integer weight $w(u,v)$.

Output: A Hamiltonian cycle of minimum weight.

TSP with Triangle Inequality

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The edge weights satisfy the triangle inequality.

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Output: A Hamiltonian cycle of minimum weight.

➡ Does this general TSP have any approximation algorithm?

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The edge weights satisfy the triangle inequality.

Output: A Hamiltonian cycle of minimum weight.

➡ The TSP with triangle inequality has a polynomial-time 2-approximation algorithm.

Theorem: If $P \neq NP$, then for any constant $\rho \geq 1$, there is NO polynomial-time ρ-approximation algorithm for the general TSP.

Even if $\rho = 999^{999^{999^{999}}}$ (or any other huge number),

there still cannot exist a polynomial-time ρ-approximation algorithm for TSP,

unless $P = NP$ (which most people consider to be unlikely).

Theorem: If $P \neq NP,$ then for any constant $\rho \geq 1,$ there is NO polynomial-time ρ-approximation algorithm for the general TSP.

How to prove it? A hint: if we let $\rho = 1,$

the theorem becomes:

If $P \neq NP,$ then TSP has no polynomial-time 1-approximation algorithm.

$\Updownarrow$

If $P \neq NP,$ then TSP is not polynomial-time solvable.

$\Uparrow$

$TSP \in NPC$.

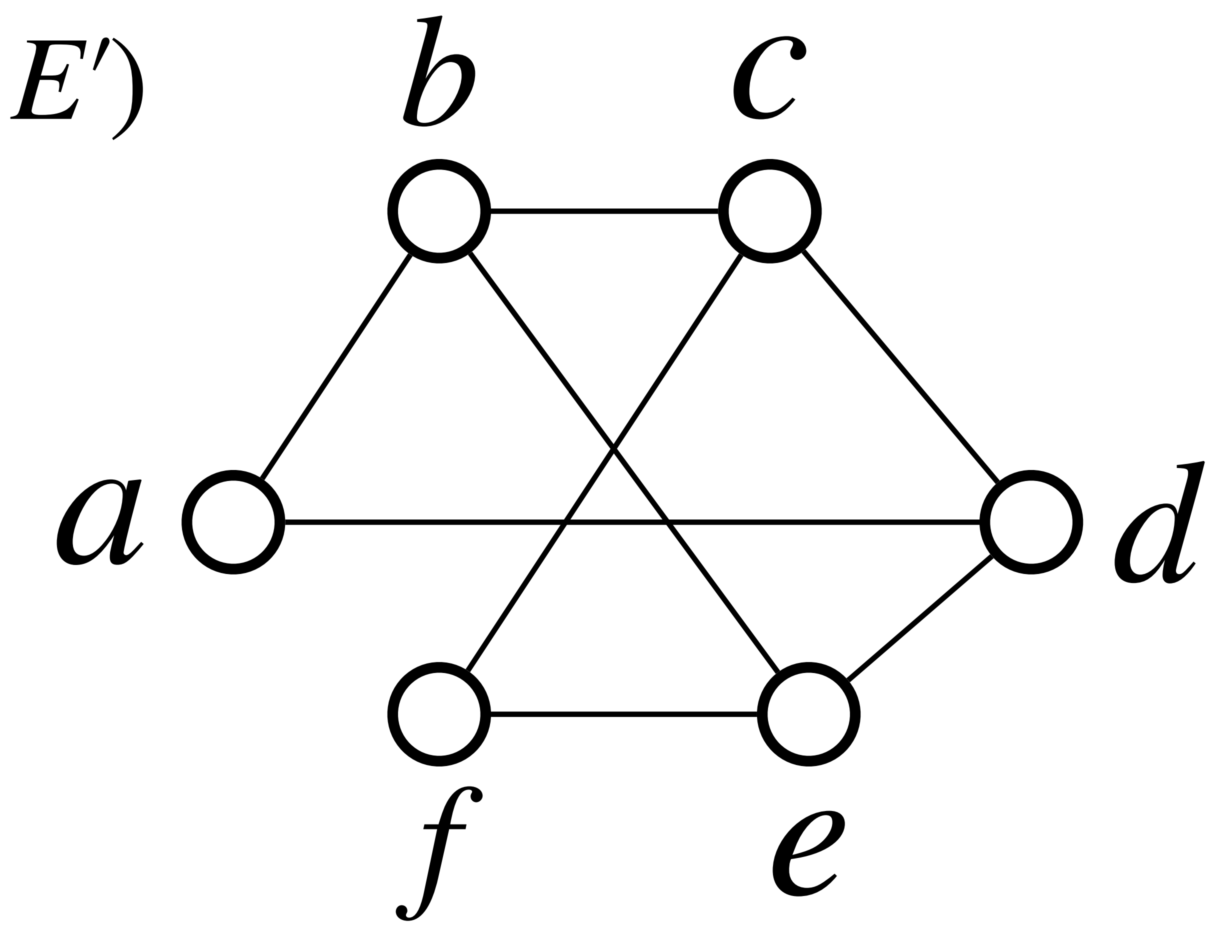
So the above theorem can actually be a generalization of proving TSP to be NP-Complete.

Theorem: If $P \neq NP$, then for any constant $\rho \geq 1$, there is NO polynomial-time ρ-approximation algorithm for the general TSP.

Proof:

Hamiltonian Cycle Problem:

$G' = (V, E')$



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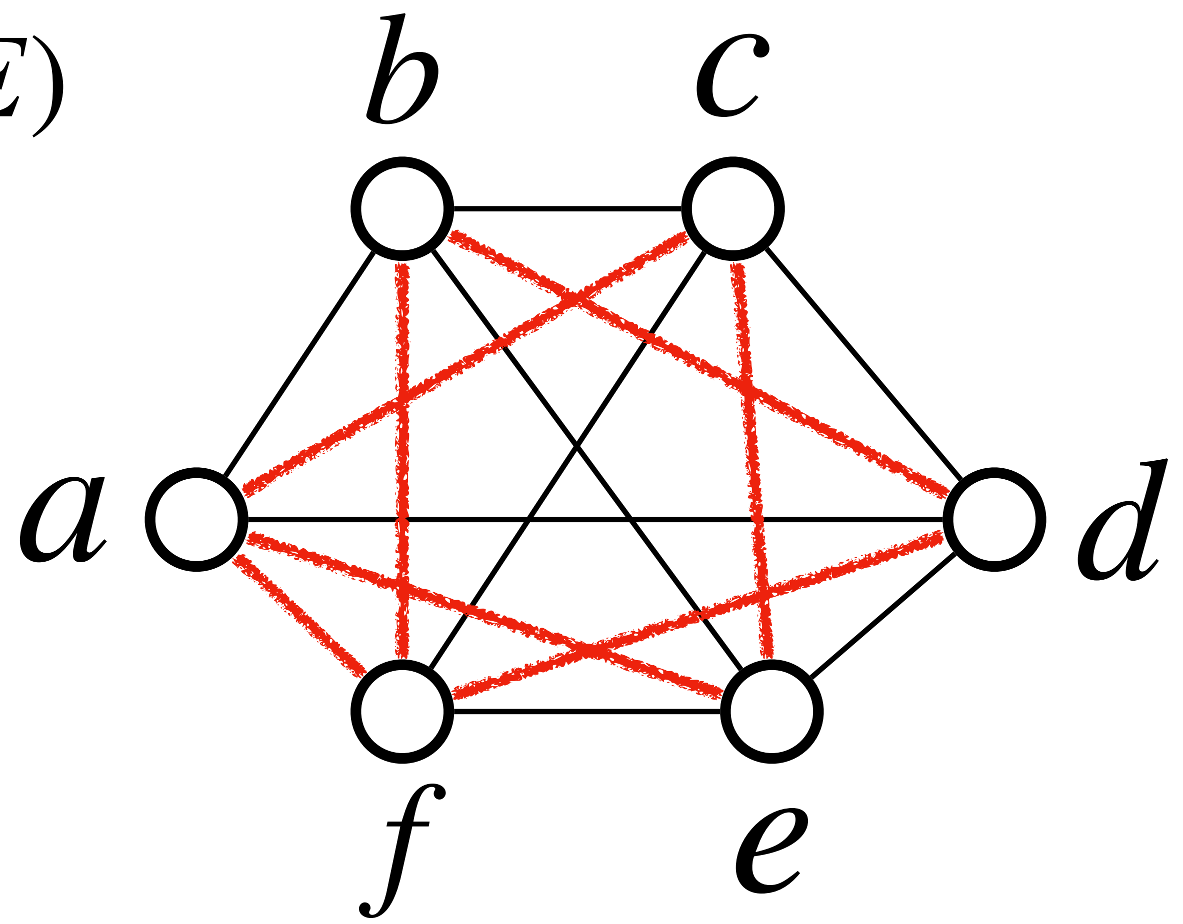

Black edges: weight 1

Red edges: weight $\rho|V| + 1$

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Proof: By contradiction: assume the ρ-approximation algorithm exists.

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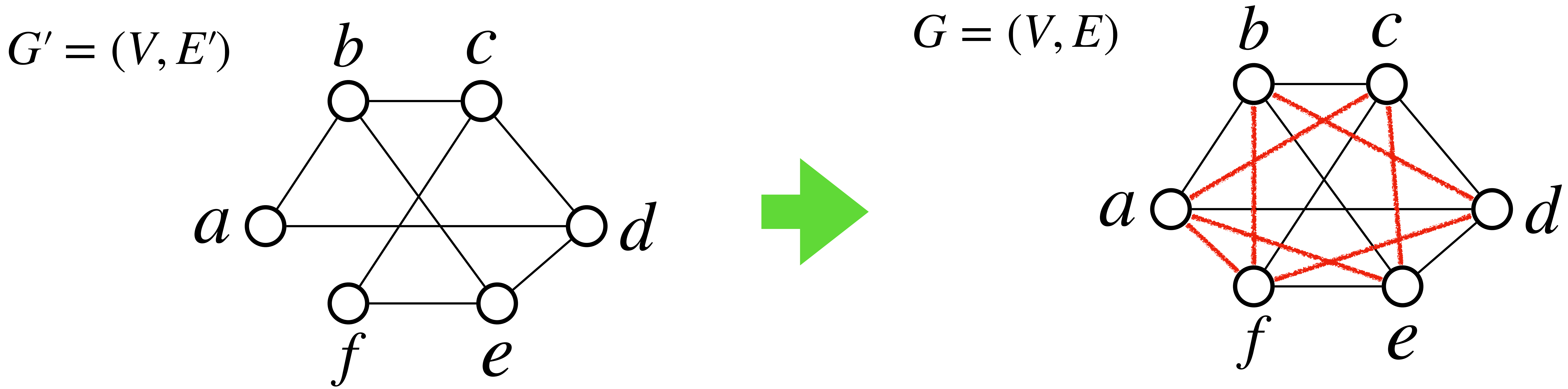

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G' has a Hamiltonian cycle ➡ G has a Hamiltonian cycle of weight |V| ➡ Using the ρ-approximation algorithm, we can find a Hamiltonian cycle of weight $\leq \rho |V|$ in polynomial time.

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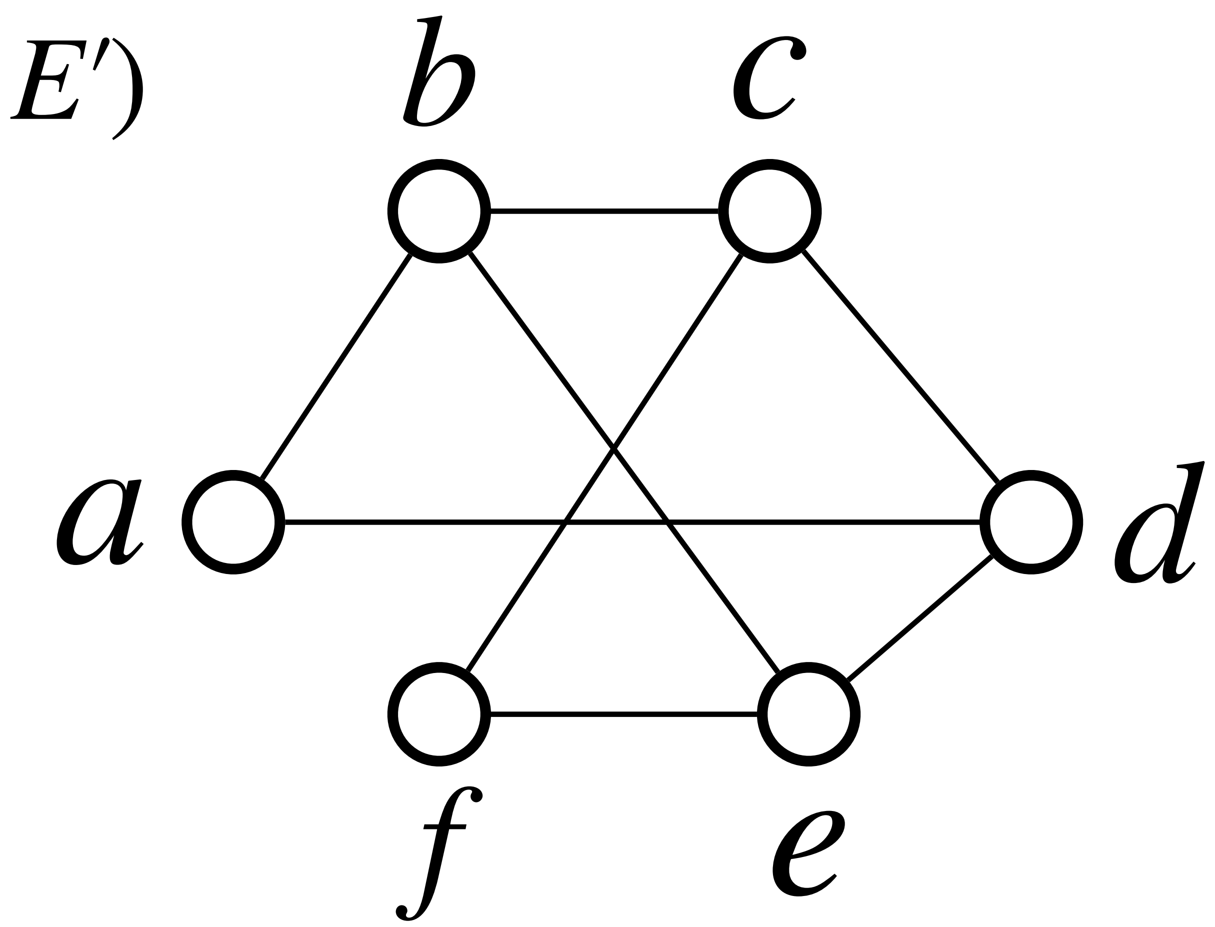

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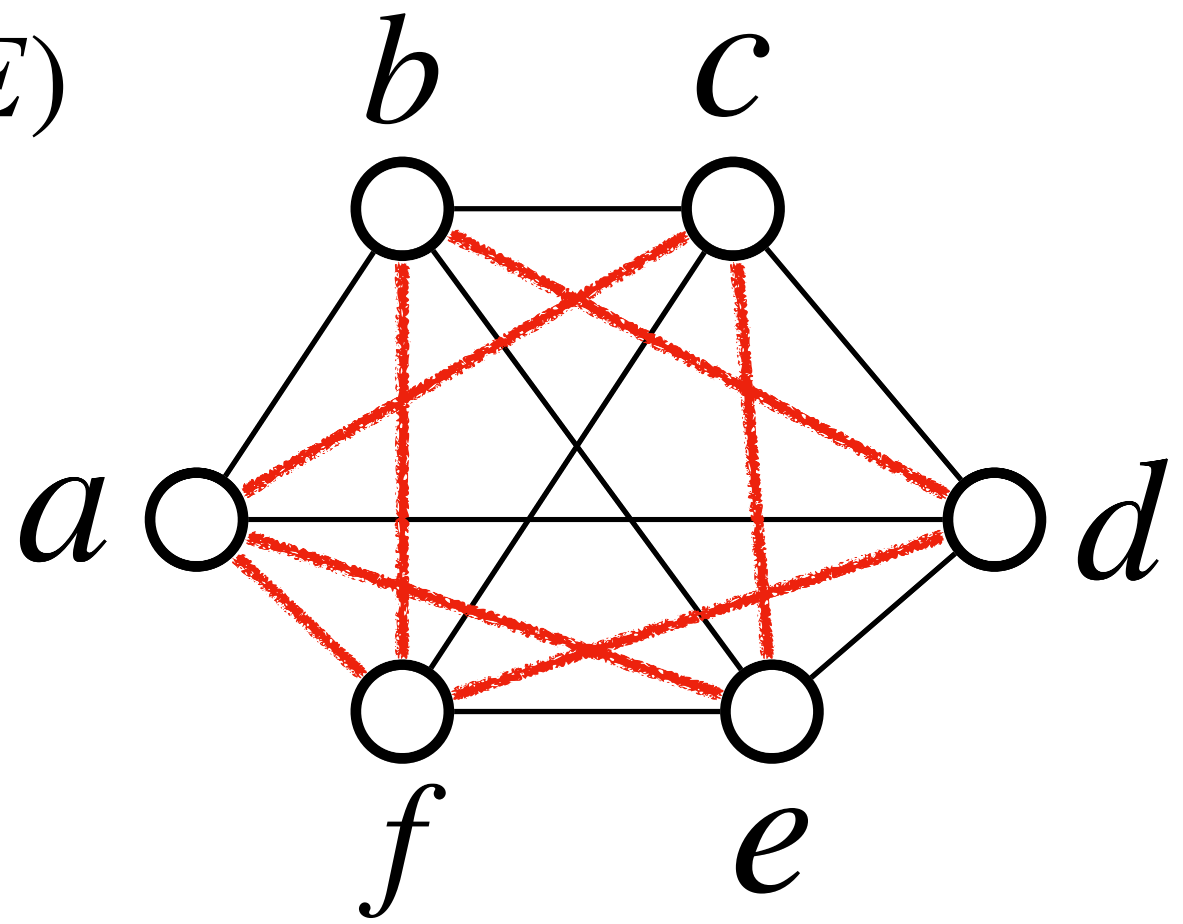

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G' has no Hamiltonian cycle ➡ Any Hamiltonian cycle in G needs to use at least one red edge ➡ Any Hamiltonian cycle in G has weight $\geq \rho|V| + 1 + |V| - 1 = (\rho + 1)|V|$ ➡ Using the ρ-approximation algorithm, we find a Hamiltonian cycle of weight $\geq (\rho + 1)|V|$ in G in polynomial time.

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We can solve the Hamiltonian Cycle Problem "exactly" in polynomial time by solving TSP approximately using the polynomial-time ρ-approximation algorithm:

1) If we find a Hamiltonian cycle of weight $\leq \rho |V|$ in G → G' has a Hamiltonian cycle

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Proof: By contradiction: assume the ρ-approximation algorithm exists.

We can solve the Hamiltonian Cycle Problem “exactly” in polynomial time.

But we know the Hamiltonian Cycle Problem in NP-complete.

So it has to be

$$P = NP$$

which is a contradiction.

Q.E.D.

Quiz questions:

1. What method did we use to prove that general TSP has no constant-approximation unless P = NP?

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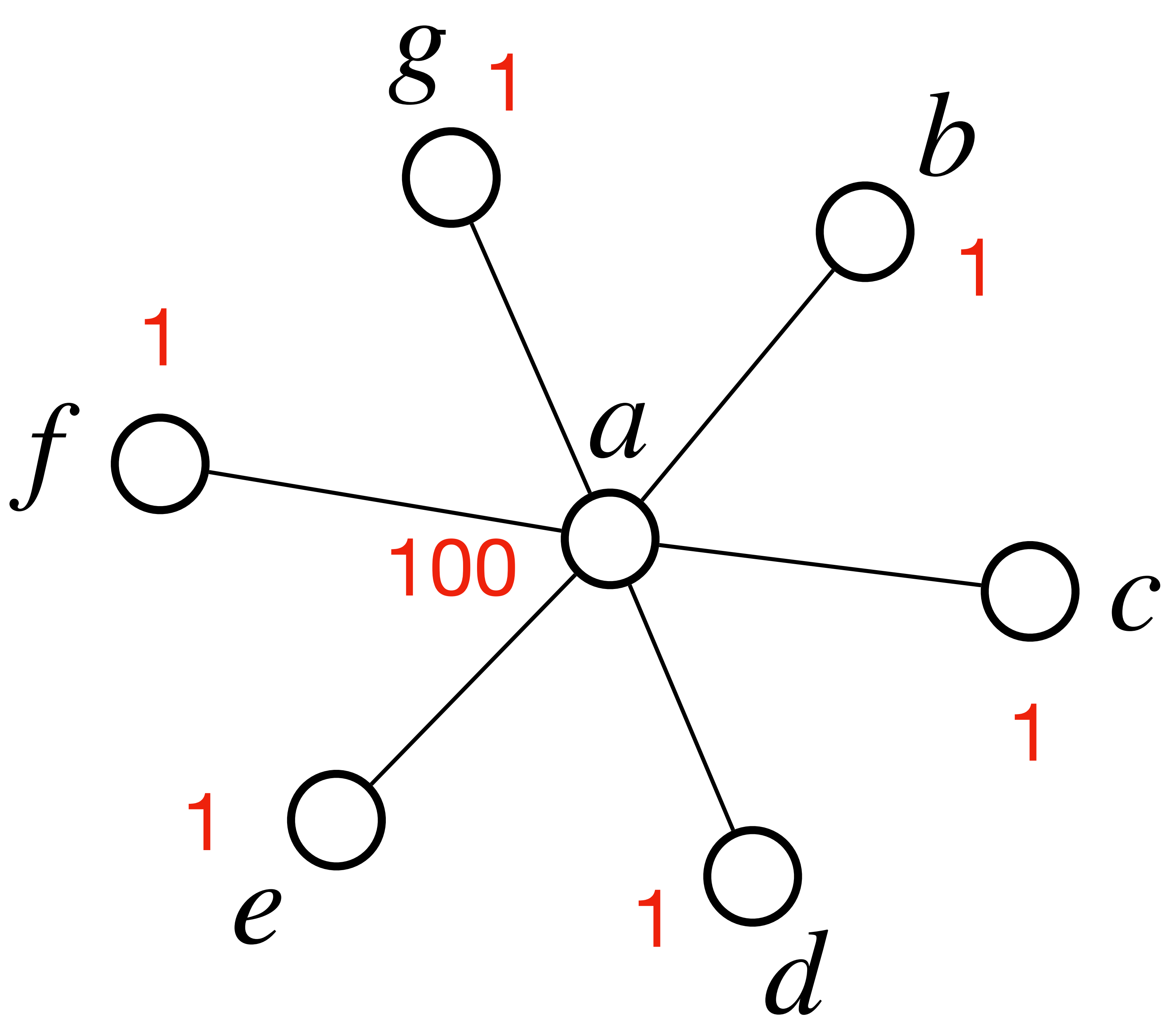
Technique: Linear Programming and Rounding for approximation

Weighted Vertex Cover Problem:

Input: An undirected graph G=(V,E), where every vertex $v \in V$ has a weight $w(v) > 0.$

Output: A vertex cover of minimum total weight.



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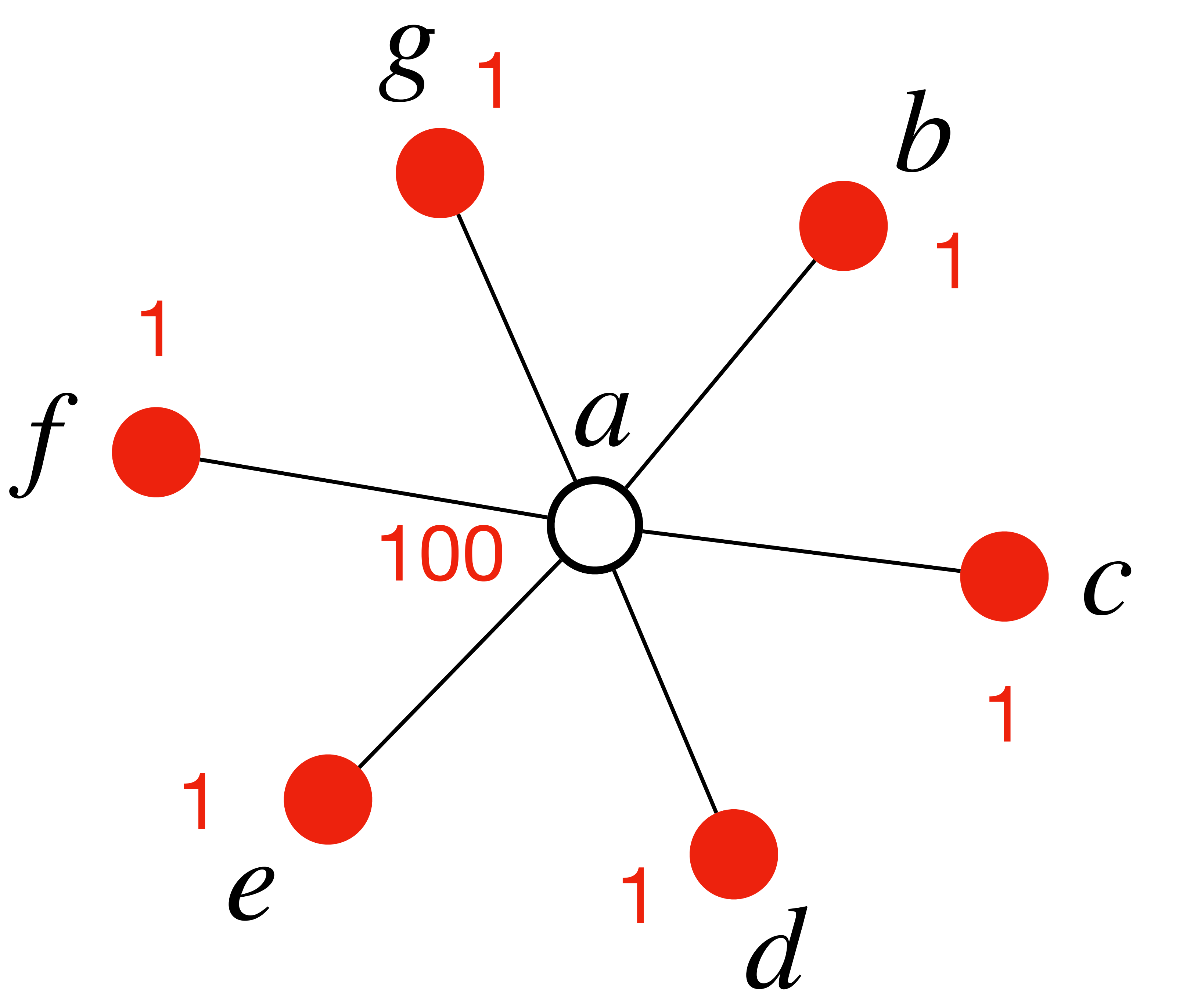
Weight of vertex cover: 6



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Our technique:

1. Formulate the problem as an integer programming problem.

Define variables:

For every node $v \in V$, define a variable

$$x(v) = \begin{cases} 1 & \text{if } v \text{ is in vertex cover} \\ 0 & \text{otherwise} \end{cases}$$

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Integer Programming Problem:

minimize $\sum_{v \in V} w(v)x(v)$

s.t. for every edge $(u, v) \in E$, $x(u) + x(v) \geq 1$

for every node $v \in V$, $x(v) \in \{0,1\}$

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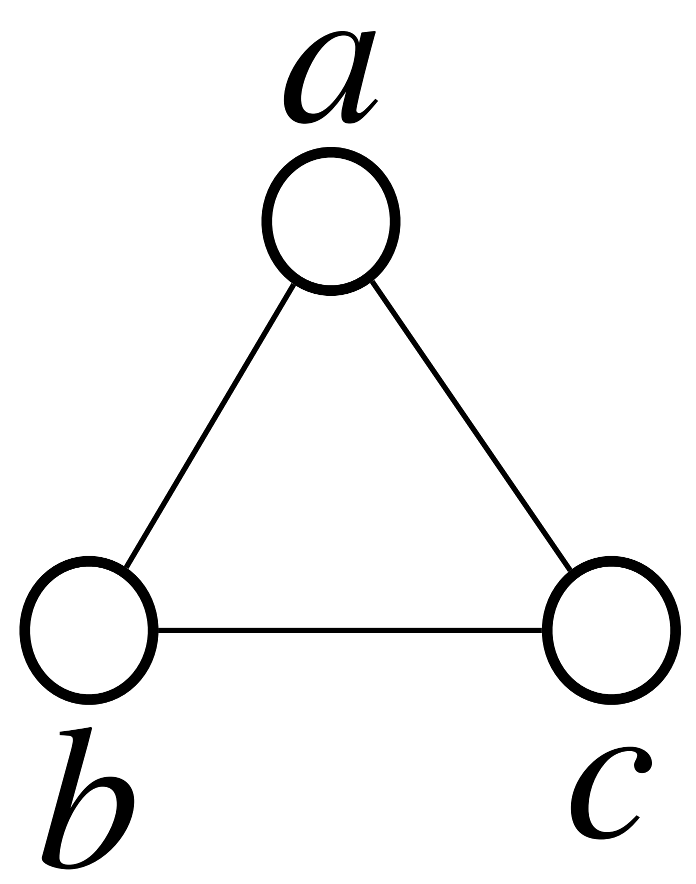

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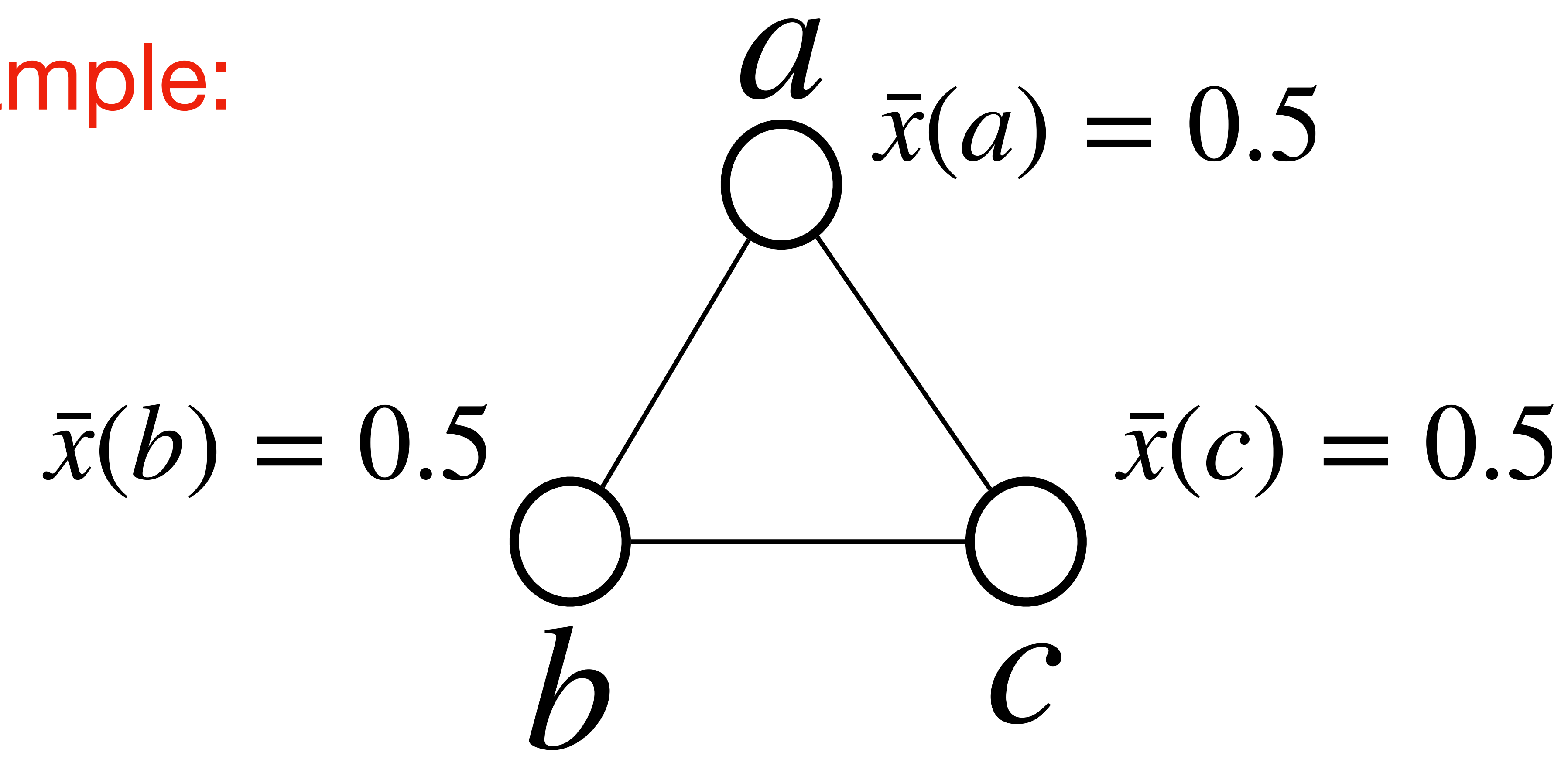

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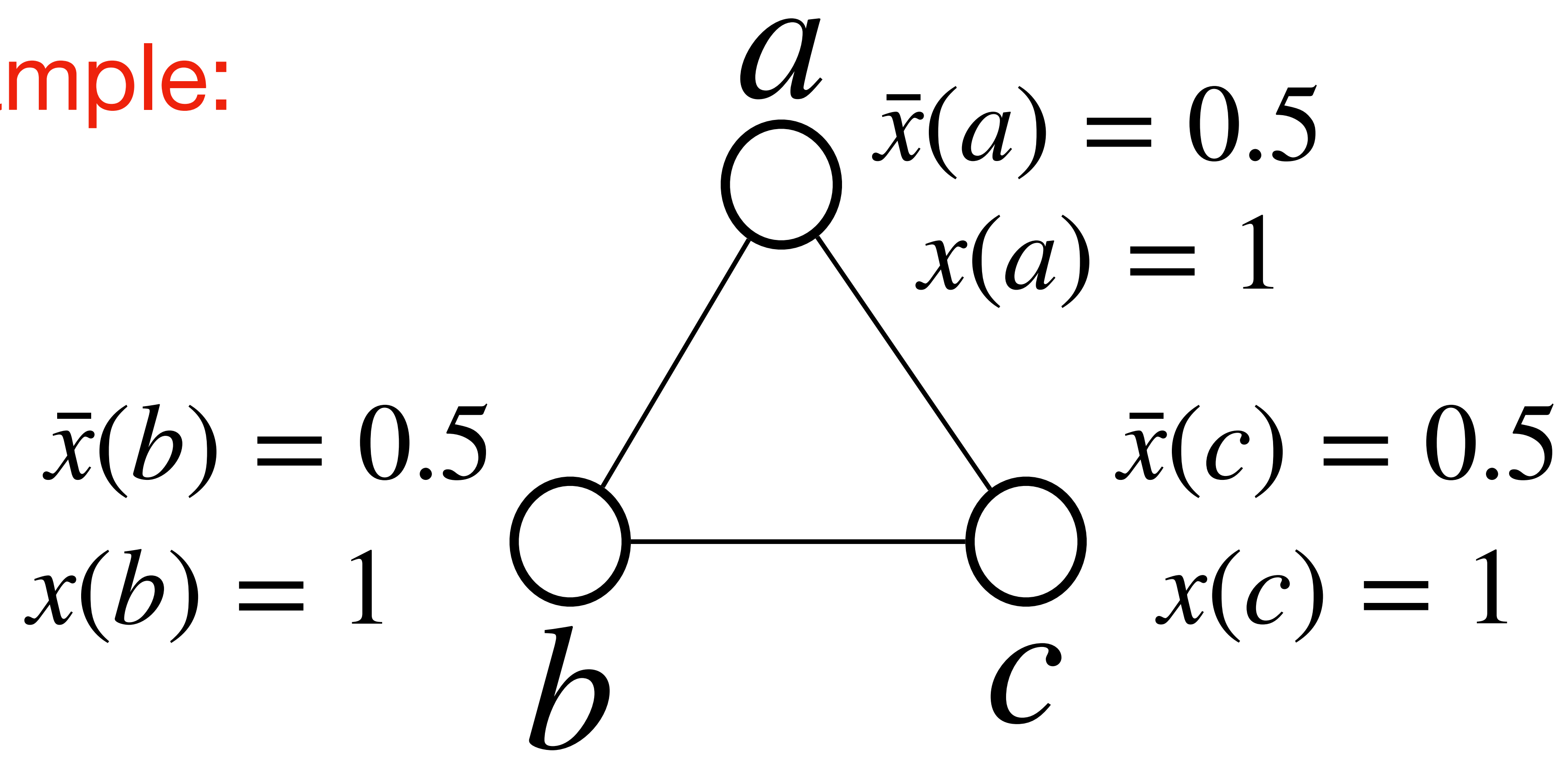

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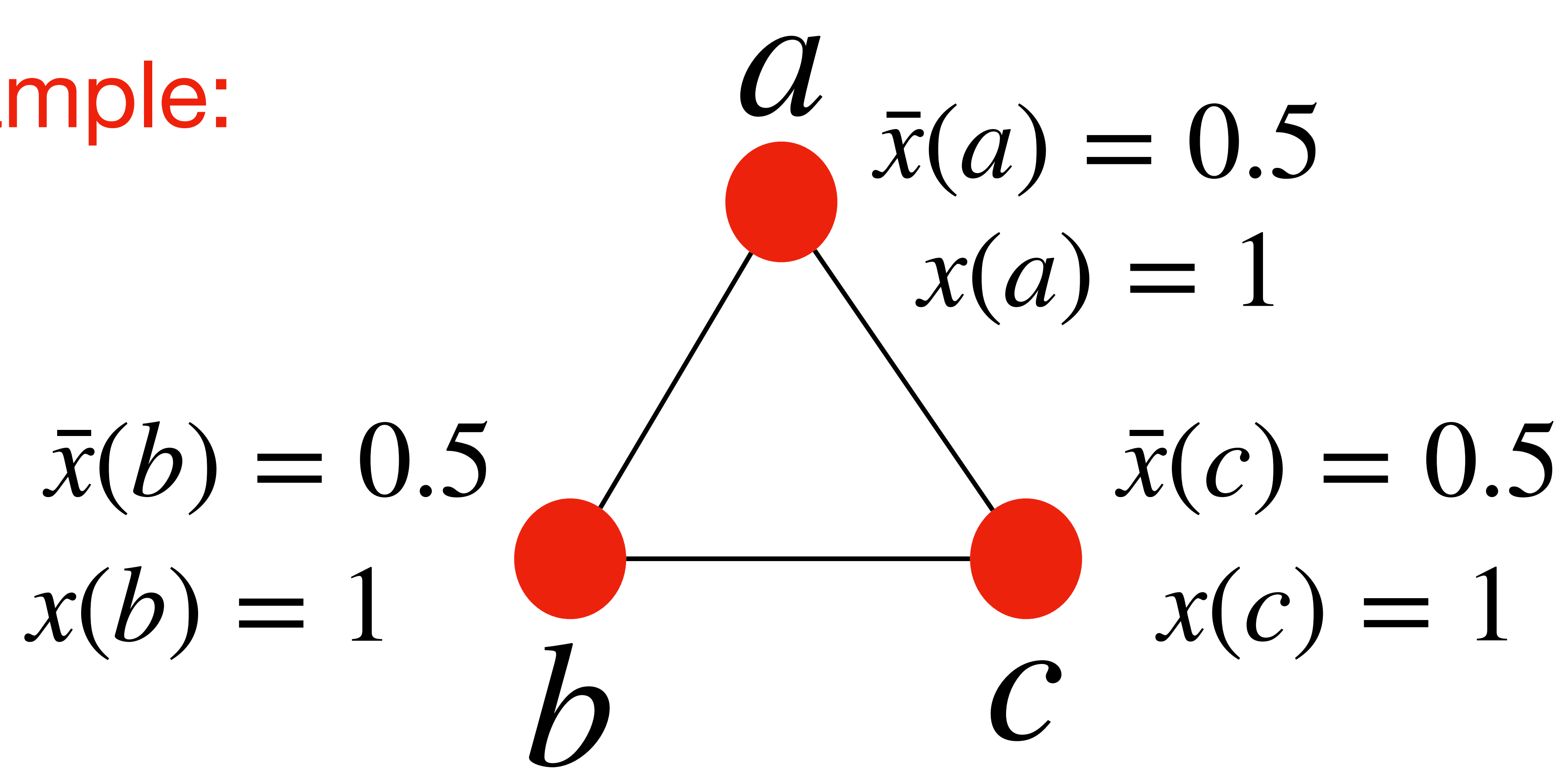

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Why is this indeed a vertex cover?

Quiz questions:

1. What is the main idea of the above LP-based approximation algorithm for "Weighted Vertex Cover"?

2. Can you think of an instance for which the above approximation algorithm outputs an optimal solution, and an instance for which it does not?

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Proof: Weight of LP solution

$$\bar{C} = \sum_{v \in V} w(v)\bar{x}(v)$$

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Optimal solution to LP: $\bar{x}(v) \quad \forall v \in V$

Get a solution to the Integer Programming Problem:

$$\forall \ v \in V, \quad x(v) = \begin{cases} 1 & \text{if } \bar{x}(v) \geq 0.5 \\ 0 & \text{if } \bar{x}(v) < 0.5 \end{cases}$$

Theorem: The above algorithm is a polynomial-time 2-approximation algorithm.

Proof: Weight of LP solution

$$\bar{C} = \sum_{v \in V} w(v)\bar{x}(v)$$

Weight of optimal vertex cover: C^*

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Vertex Cover Problem: 2-approximation.

TSP with triangle inequality: 2-approximation.

General TSP: no constant ratio approximation unless P=NP.

Set Covering Problem: $\rho(n)$ -approximation

$\rho(n) \to \infty$ as $n \to \infty$

Subset Sum Problem: $(1+\epsilon)$ -approximation

Time complexity poly$(n, \frac{1}{\epsilon})$

Quiz question:

1. How did we find the approximation ratio for the above algorithm for "Weighted Vertex Cover" without knowing the optimal cost?

Roadmap of this lecture:

1. Understand approximation algorithms by solving TSP.

    1.1 General TSP has no constant-ratio approximation unless P=NP.

2. The "Linear Programming (LP) Technique" for approximation algorithms.

    2.1 Approximation algorithm for "Weighted Vertex Cover Problem" using the "LP technique".

    2.2 Analyze the approximation ratio of the algorithm.

3. Randomized Algorithm.

    3.1 Define "Randomized Algorithm".

    3.2 Understand "Randomized Algorithm" by solving the "Max 3-CNF SAT Problem"

Randomized Algorithm

Consider a maximization problem.

Let $C^* > 0$ be the cost of an optimal solution.

Let $C > 0$ be the expected cost of the solution of a randomized algorithm.

If for all instances, we have $\dfrac{C^*}{C} \leq \rho,$

then the randomized algorithm is called a

ρ-approximation randomized algorithm.

Randomized Algorithm

Consider a minimization problem.

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If for all instances, we have $\dfrac{C}{C^*} \leq \rho,$

then the randomized algorithm is called a ρ-approximation randomized algorithm.

Quiz questions:

1. What is a "Randomized Algorithm"?
2. What is the difference between the approximation ratio of a randomized algorithm and that of a deterministic algorithm?

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1. Understand approximation algorithms by solving TSP.

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   3.1 Define "Randomized Algorithm".

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Max 3-CNF SAT Problem

Input: A 3-CNF Boolean formula of n variables and k clauses,
Where every clause is the OR of 3 literals.

The 3 variables involved in each clause are distinct.

Output: A solution to the variables that maximizes the number of satisfied clauses.

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Randomized Algorithm:

For each variable, let it be 0 or 1 with probability 0.5 and 0.5.

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Example of a clause

$$x_1 \vee \bar{x}_2 \vee \bar{x}_3$$

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Expected number of satisfied clauses $C = \frac{7}{8} \cdot k$

Number of satisfied clauses for an optimal solution $C^* \leq k$

$$\frac{C^*}{C} \leq \frac{k}{\frac{7}{8} \cdot k} = \frac{8}{7} \approx 1.14$$

Quiz questions:

1. What is the main idea of the above randomized algorithm?

2. If the number of literals in each clause increase, will the approximation ratio of the randomized algorithm increase or decrease?