Algorithms

Lecture Topic: Linear Programming (Part 2)

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Roadmap of this lecture:

1. Linear Programming (LP)

1.1 SIMPLEX Algorithm for LP when the initial basic solution is feasible.

1.2 Bland's Rule for SIMPLEX Algorithm.

SIMPLEX Algorithm

maximize $3x_1 + x_2 + 2x_3$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$

$$2x_1 + 2x_2 + 5x_3 \leq 24$$

$$4x_1 + x_2 + 2x_3 \leq 36$$

$$x_1, x_2, x_3 \geq 0$$

SIMPLEX Algorithm

1) Turn the LP into Standard Form

SIMPLEX Algorithm

maximize $3x_1 + x_2 + 2x_3$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$
$$2x_1 + 2x_2 + 5x_3 \leq 24$$
$$4x_1 + x_2 + 2x_3 \leq 36$$
$$x_1, x_2, x_3 \geq 0$$

Already in Standard Form

SIMPLEX Algorithm

1) Turn the LP into Standard Form

SIMPLEX Algorithm

maximize $3x_1 + x_2 + 2x_3$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$

$$2x_1 + 2x_2 + 5x_3 \leq 24$$

$$4x_1 + x_2 + 2x_3 \leq 36$$

$$x_1, x_2, x_3 \geq 0$$

SIMPLEX Algorithm

1) Turn the LP into Standard Form

2) Turn the LP into Slack Form

SIMPLEX Algorithm

maximize $3x_1 + x_2 + 2x_3$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$

$$2x_1 + 2x_2 + 5x_3 \leq 24$$

$$4x_1 + x_2 + 2x_3 \leq 36$$

$$x_1, x_2, x_3 \geq 0$$

$$\Downarrow$$

$$z = 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

SIMPLEX Algorithm

1) Turn the LP into Standard Form

2) Turn the LP into Slack Form

SIMPLEX Algorithm

maximize $3x_1 + x_2 + 2x_3$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$
$$2x_1 + 2x_2 + 5x_3 \leq 24$$
$$4x_1 + x_2 + 2x_3 \leq 36$$
$$x_1, x_2, x_3 \geq 0$$

$$z = 3x_1 + x_2 + 2x_3$$
$$x_4 = 30 - x_1 - x_2 - 3x_3$$
$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$
$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$

SIMPLEX Algorithm

1) Turn the LP into Standard Form
2) Turn the LP into Slack Form
3) Get a basic solution: set all non-basic variables to 0

SIMPLEX Algorithm

maximize $3x_1 + x_2 + 2x_3$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$
$$2x_1 + 2x_2 + 5x_3 \leq 24$$
$$4x_1 + x_2 + 2x_3 \leq 36$$
$$x_1, x_2, x_3 \geq 0$$

$$z = 3x_1 + x_2 + 2x_3$$
$$x_4 = 30 - x_1 - x_2 - 3x_3$$
$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$
$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Non-basic variables = 0

Basic solution:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$

SIMPLEX Algorithm

1) Turn the LP into Standard Form

2) Turn the LP into Slack Form

3) Get a basic solution: set all non-basic variables to 0

SIMPLEX Algorithm

maximize $3x_1 + x_2 + 2x_3$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$
$$2x_1 + 2x_2 + 5x_3 \leq 24$$
$$4x_1 + x_2 + 2x_3 \leq 36$$
$$x_1, x_2, x_3 \geq 0$$

$$z = 3x_1 + x_2 + 2x_3$$
$$x_4 = 30 - x_1 - x_2 - 3x_3$$
$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$
$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic variables

Basic solution:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$

SIMPLEX Algorithm

1) Turn the LP into Standard Form

2) Turn the LP into Slack Form

3) Get a basic solution: set all non-basic variables to 0

SIMPLEX Algorithm

maximize $3x_1 + x_2 + 2x_3$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$
$$2x_1 + 2x_2 + 5x_3 \leq 24$$
$$4x_1 + x_2 + 2x_3 \leq 36$$
$$x_1, x_2, x_3 \geq 0$$

$$z = 3x_1 + x_2 + 2x_3$$
$$x_4 = 30 - x_1 - x_2 - 3x_3$$
$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$
$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$

SIMPLEX Algorithm

1) Turn the LP into Standard Form

2) Turn the LP into Slack Form

3) Get a basic solution: set all non-basic variables to 0

4) If the basic solution if feasible (i.e., all variables are non-negative), continue.

SIMPLEX Algorithm

maximize $3x_1 + x_2 + 2x_3$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$
$$2x_1 + 2x_2 + 5x_3 \leq 24$$
$$4x_1 + x_2 + 2x_3 \leq 36$$
$$x_1, x_2, x_3 \geq 0$$

$$z = 3x_1 + x_2 + 2x_3$$
$$x_4 = 30 - x_1 - x_2 - 3x_3$$
$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$
$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$

SIMPLEX Algorithm

1) Turn the LP into Standard Form

2) Turn the LP into Slack Form

3) Get a basic solution: set all non-basic variables to 0

4) If the basic solution if feasible (i.e., all variables are non-negative), continue.

Lucky day: this basic solution is feasible.
(If not, we will discuss how to handle it later.)

SIMPLEX Algorithm

maximize $3x_1 + x_2 + 2x_3$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$

$$2x_1 + 2x_2 + 5x_3 \leq 24$$

$$4x_1 + x_2 + 2x_3 \leq 36$$

$$x_1, x_2, x_3 \geq 0$$

$$\Downarrow$$

Objective value = 0

$$z = 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

SIMPLEX Algorithm

maximize $3x_1 + x_2 + 2x_3$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$
$$2x_1 + 2x_2 + 5x_3 \leq 24$$
$$4x_1 + x_2 + 2x_3 \leq 36$$
$$x_1, x_2, x_3 \geq 0$$

Objective value = 0

$$\boxed{z = 3x_1 + x_2 + 2x_3}$$
$$x_4 = 30 - x_1 - x_2 - 3x_3$$
$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$
$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

Here the non-basic variables

$x_1,\ x_2,\ x_3$

all have positive coefficients in the objective function.

So we can pick any of them.

SIMPLEX Algorithm

maximize $3x_1 + x_2 + 2x_3$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$
$$2x_1 + 2x_2 + 5x_3 \leq 24$$
$$4x_1 + x_2 + 2x_3 \leq 36$$
$$x_1, x_2, x_3 \geq 0$$

Objective value = 0

$$\boxed{z = 3x_1 + x_2 + 2x_3}$$
$$x_4 = 30 - x_1 - x_2 - 3x_3$$
$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$
$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

Why a non-basic variable?

The objective value is a function of non-basic variables.

So if we want to greedily increase the objective value, we want to adjust the value of a non-basic variable.

Note: we see not only the objective value, but also the Basic Variables, as functions of the non-basic variables.

SIMPLEX Algorithm

maximize $3x_1 + x_2 + 2x_3$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$
$$2x_1 + 2x_2 + 5x_3 \leq 24$$
$$4x_1 + x_2 + 2x_3 \leq 36$$
$$x_1, x_2, x_3 \geq 0$$

Objective value = 0

$$\boxed{z = 3x_1 + x_2 + 2x_3}$$
$$x_4 = 30 - x_1 - x_2 - 3x_3$$
$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$
$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

Why does the non-basic variable need to have a positive coefficient in the objective function?

In the current solution (the basic solution), all non-basic variables have value 0. So we cannot decrease its value (to keep the solution feasible). So we can only increase its value (from 0 to something bigger). To increase the objective value at the same time, the coefficient needs to be positive.

SIMPLEX Algorithm

maximize $3x_1 + x_2 + 2x_3$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$
$$2x_1 + 2x_2 + 5x_3 \leq 24$$
$$4x_1 + x_2 + 2x_3 \leq 36$$
$$x_1, x_2, x_3 \geq 0$$

Objective value = 0

$$z = 3x_1 + x_2 + 2x_3$$
$$x_4 = 30 - x_1 - x_2 - 3x_3$$
$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$
$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

Why do we care about this?

Because we need to keep the solution feasible.

Note that when we increase the value of the above non-basic variable:

1) The other non-basic variables are still 0.
2) The objective value increases its value.
3) The basic variables will change their values.

SIMPLEX Algorithm

maximize $3x_1 + x_2 + 2x_3$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$
$$2x_1 + 2x_2 + 5x_3 \leq 24$$
$$4x_1 + x_2 + 2x_3 \leq 36$$
$$x_1, x_2, x_3 \geq 0$$

Objective value = 0

$$\boxed{z = 3x_1 + x_2 + 2x_3}$$
$$x_4 = 30 - x_1 - x_2 - 3x_3$$
$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$
$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

Here the non-basic variables

$x_1,\ x_2,\ x_3$

all have positive coefficients in the objective function.

So we can pick any of them.

Let's pick x_1

SIMPLEX Algorithm

$$z = 3\boxed{x_1}^{\,0} + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$

Objective value = 0

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

By how much can we increase x_1 ?

SIMPLEX Algorithm

$$z = 3\boxed{x_1}^{\,0} + x_2 + 2x_3$$

$$\boxed{x_4} = 30 - x_1 - x_2 - 3x_3 \longrightarrow x_1 \leq 30$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$

Objective value = 0

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

By how much can we increase x_1 ?

SIMPLEX Algorithm

$$z = 3\boxed{x_1}^{\,0} + x_2 + 2x_3$$

$$\boxed{x_4} = 30 - x_1 - x_2 - 3x_3 \longrightarrow x_1 \leq 30$$

$$\boxed{x_5} = 24 - 2x_1 - 2x_2 - 5x_3 \longrightarrow x_1 \leq 12$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$

Objective value = 0

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

By how much can we increase x_1 ?

SIMPLEX Algorithm

$$z = 3\boxed{x_1}^{0} + x_2 + 2x_3$$

$$\boxed{x_4} = 30 - x_1 - x_2 - 3x_3 \rightarrow x_1 \leq 30$$

$$\boxed{x_5} = 24 - 2x_1 - 2x_2 - 5x_3 \rightarrow x_1 \leq 12$$

$$\boxed{x_6} = 36 - 4x_1 - x_2 - 2x_3 \rightarrow x_1 \leq 9$$

Basic solution:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$

Objective value = 0

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

By how much can we increase x_1 ?

SIMPLEX Algorithm

9

0

$$z = 3\boxed{x_1} + x_2 + 2x_3$$

30

21 $\boxed{x_4} = 30 - x_1 - x_2 - 3x_3$ $x_1 \leq 30$

24

6 $\boxed{x_5} = 24 - 2x_1 - 2x_2 - 5x_3$ $x_1 \leq 12$

36

0 $\boxed{x_6} = 36 - 4x_1 - x_2 - 2x_3$ $x_1 \leq 9$

Basic solution:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$

Objective value = 0

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

By how much can we increase x_1 ?

9

SIMPLEX Algorithm

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

9

0

$$z = 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

36

0

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$

Objective value = 0

SIMPLEX Algorithm

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

9

0

$$z = 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

36

0

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Pivot

(So that the new solution will be the basic solution for the new LP.)

Basic solution:

$$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$$

Objective value = 0

SIMPLEX Algorithm

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

9

0

$$z = 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

36

0

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

$$4x_1 = 36 - x_2 - 2x_3 - x_6$$

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6$$

Basic solution:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$

Objective value = 0

SIMPLEX Algorithm

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

9

0

$$z = 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

36

0

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

$$4x_1 = 36 - x_2 - 2x_3 - x_6$$

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6$$

Plug in

Basic solution:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$

Objective value = 0

SIMPLEX Algorithm

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

9
0

$$z = 3x_1 + x_2 + 2x_3$$
$$x_4 = 30 - x_1 - x_2 - 3x_3$$
$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

36
0

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

$$z = 27 + \frac{1}{4}x_2 + \frac{1}{2}x_3 - \frac{3}{4}x_6$$
$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6$$
$$x_4 = 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6$$
$$x_5 = 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6$$

Basic solution:

$$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$$

Objective value = 0

SIMPLEX Algorithm

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

9
0

$$
\begin{aligned}
z &= 3x_1 + x_2 + 2x_3 \\
x_4 &= 30 - x_1 - x_2 - 3x_3 \\
x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
x_6 &= 36 - 4x_1 - x_2 - 2x_3
\end{aligned}
$$

36
0

$$
\begin{aligned}
z &= 27 + \frac{1}{4}x_2 + \frac{1}{2}x_3 - \frac{3}{4}x_6 \\
x_1 &= 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6 \\
x_4 &= 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6 \\
x_5 &= 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6
\end{aligned}
$$

Basic solution:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 30,\ x_5 = 24,\ x_6 = 36$

Objective value = 0

Basic solution:

$x_1 = 9,\ x_2 = 0,\ x_3 = 0,\ x_4 = 21,\ x_5 = 6,\ x_6 = 0$

Objective value = 27

SIMPLEX Algorithm

$$z = 27 + \frac{1}{4}x_2 + \frac{1}{2}x_3 - \frac{3}{4}x_6$$

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6$$

$$x_4 = 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6$$

$$x_5 = 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6$$

We can increase either x_2 or x_3.

Let's increase x_3.

Basic solution:

$$x_1 = 9,\ x_2 = 0,\ x_3 = 0,\ x_4 = 21,\ x_5 = 6,\ x_6 = 0$$

Objective value = 27

SIMPLEX Algorithm

$$z = 27 + \frac{1}{4}x_2 + \frac{1}{2}\boxed{x_3} - \frac{3}{4}x_6$$

0

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6$$

$$x_4 = 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6$$

$$x_5 = 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6$$

By how much can we increase x_3 ?

Basic solution:

$x_1 = 9,\ x_2 = 0,\ x_3 = 0,\ x_4 = 21,\ x_5 = 6,\ x_6 = 0$

Objective value = 27

SIMPLEX Algorithm

$$z = 27 + \frac{1}{4}x_2 + \frac{1}{2}\boxed{x_3} - \frac{3}{4}x_6$$

0

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6 \quad \Rightarrow \quad x_3 \leq 18$$

$$x_4 = 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6$$

$$x_5 = 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6$$

By how much can we increase x_3 ?

Basic solution:

$x_1 = 9,\ x_2 = 0,\ x_3 = 0,\ x_4 = 21,\ x_5 = 6,\ x_6 = 0$

Objective value = 27

SIMPLEX Algorithm

$$z = 27 + \frac{1}{4}x_2 + \frac{1}{2}\boxed{x_3} - \frac{3}{4}x_6$$

0

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6 \quad \Rightarrow \quad x_3 \leq 18$$

$$x_4 = 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6 \quad \Rightarrow \quad x_3 \leq \frac{42}{5}$$

$$x_5 = 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6$$

By how much can we increase x_3 ?

Basic solution:

$x_1 = 9,\ x_2 = 0,\ x_3 = 0,\ x_4 = 21,\ x_5 = 6,\ x_6 = 0$

Objective value = 27

SIMPLEX Algorithm

$$z = 27 + \frac{1}{4}x_2 + \frac{1}{2}\boxed{x_3} - \frac{3}{4}x_6$$

0

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6 \quad \Rightarrow \quad x_3 \leq 18$$

$$x_4 = 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6 \quad \Rightarrow \quad x_3 \leq \frac{42}{5}$$

$$x_5 = 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6 \quad \Rightarrow \quad x_3 \leq \frac{3}{2}$$

By how much can we increase x_3 ?

Basic solution:

$$x_1 = 9,\ x_2 = 0,\ x_3 = 0,\ x_4 = 21,\ x_5 = 6,\ x_6 = 0$$

Objective value = 27

SIMPLEX Algorithm

$$\begin{array}{ll}
 & \qquad\qquad\qquad\quad 1.5 \\
 & \qquad\qquad\qquad\quad 0 \\
 & z = 27 + \frac{1}{4}x_2 + \frac{1}{2}\boxed{x_3} - \frac{3}{4}x_6 \\
\downarrow & x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6 \quad \Rightarrow \quad x_3 \leq 18 \\
\downarrow & x_4 = 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6 \quad \Rightarrow \quad x_3 \leq \frac{42}{5} \\
\begin{array}{c}6\\0\end{array} \downarrow & \boxed{x_5} = 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6 \quad \Rightarrow \quad x_3 \leq \frac{3}{2}
\end{array}$$

By how much can we increase x_3 ?

$$\frac{3}{2} = 1.5$$

Basic solution:

$$x_1 = 9,\ x_2 = 0,\ x_3 = 0,\ x_4 = 21,\ x_5 = 6,\ x_6 = 0$$

Objective value = 27

SIMPLEX Algorithm

1.5

0

$$z = 27 + \frac{1}{4}x_2 + \frac{1}{2}x_3 - \frac{3}{4}x_6$$

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6 \quad \Rightarrow \quad x_3 \leq 18$$

$$x_4 = 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6 \quad \Rightarrow \quad x_3 \leq \frac{42}{5}$$

6

0

$$x_5 = 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6 \quad \Rightarrow \quad x_3 \leq \frac{3}{2}$$

Pivot

Basic solution:

$$x_1 = 9,\ x_2 = 0,\ x_3 = 0,\ x_4 = 21,\ x_5 = 6,\ x_6 = 0$$

Objective value = 27

SIMPLEX Algorithm

1.5

0

$$z = 27 + \frac{1}{4}x_2 + \frac{1}{2}\boxed{x_3} - \frac{3}{4}x_6$$

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6$$

$$x_4 = 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6$$

6

0

$$\boxed{x_5} = 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6$$

Basic solution:

$$x_1 = 9,\ x_2 = 0,\ x_3 = 0,\ x_4 = 21,\ x_5 = 6,\ x_6 = 0$$

Objective value = 27

$$z = \frac{111}{4} + \frac{1}{16}x_2 - \frac{1}{8}x_5 - \frac{11}{6}x_6$$

$$x_1 = \frac{33}{4} - \frac{1}{16}x_2 + \frac{1}{8}x_5 - \frac{5}{16}x_6$$

$$x_3 = \frac{3}{2} - \frac{3}{8}x_2 - \frac{1}{4}x_5 + \frac{1}{8}x_6$$

$$x_4 = \frac{69}{4} + \frac{3}{16}x_2 + \frac{5}{8}x_5 - \frac{1}{16}x_6$$

Basic solution:

$$x_1 = \frac{33}{4},\ x_2 = 0,\ x_3 = \frac{3}{2},\ x_4 = \frac{69}{4},\ x_5 = 0,\ x_6 = 0$$

Objective value = $\frac{111}{4} = 27.75$

SIMPLEX Algorithm

$$z = \frac{111}{4} + \frac{1}{16}x_2 - \frac{1}{8}x_5 - \frac{11}{6}x_6$$

$$x_1 = \frac{33}{4} - \frac{1}{16}x_2 + \frac{1}{8}x_5 - \frac{5}{16}x_6$$

$$x_3 = \frac{3}{2} - \frac{3}{8}x_2 - \frac{1}{4}x_5 + \frac{1}{8}x_6$$

$$x_4 = \frac{69}{4} + \frac{3}{16}x_2 + \frac{5}{8}x_5 - \frac{1}{16}x_6$$

We can increase x_2

By how much?

Basic solution:

$$x_1 = \frac{33}{4},\ x_2 = 0,\ x_3 = \frac{3}{2},\ x_4 = \frac{69}{4},\ x_5 = 0,\ x_6 = 0$$

Objective value = $\frac{111}{4} = 27.75$

SIMPLEX Algorithm

$$z = \frac{111}{4} + \frac{1}{16}\boxed{x_2} \overset{0}{} - \frac{1}{8}x_5 - \frac{11}{6}x_6$$

$$x_1 = \frac{33}{4} - \frac{1}{16}x_2 + \frac{1}{8}x_5 - \frac{5}{16}x_6 \quad \Longrightarrow \quad x_2 \leq 132$$

$$x_3 = \frac{3}{2} - \frac{3}{8}x_2 - \frac{1}{4}x_5 + \frac{1}{8}x_6$$

$$x_4 = \frac{69}{4} + \frac{3}{16}x_2 + \frac{5}{8}x_5 - \frac{1}{16}x_6$$

Basic solution:

$$x_1 = \frac{33}{4},\ x_2 = 0,\ x_3 = \frac{3}{2},\ x_4 = \frac{69}{4},\ x_5 = 0,\ x_6 = 0$$

Objective value = $\frac{111}{4} = 27.75$

SIMPLEX Algorithm

$$z = \frac{111}{4} + \frac{1}{16}\overset{\uparrow\ 0}{x_2} - \frac{1}{8}x_5 - \frac{11}{6}x_6$$

$$\downarrow\ x_1 = \frac{33}{4} - \frac{1}{16}x_2 + \frac{1}{8}x_5 - \frac{5}{16}x_6 \quad \Rightarrow \quad x_2 \leq 132$$

$$\downarrow\ x_3 = \frac{3}{2} - \frac{3}{8}x_2 - \frac{1}{4}x_5 + \frac{1}{8}x_6 \quad \Rightarrow \quad x_2 \leq 4$$

$$x_4 = \frac{69}{4} + \frac{3}{16}x_2 + \frac{5}{8}x_5 - \frac{1}{16}x_6$$

Basic solution:

$$x_1 = \frac{33}{4},\ x_2 = 0,\ x_3 = \frac{3}{2},\ x_4 = \frac{69}{4},\ x_5 = 0,\ x_6 = 0$$

Objective value = $\frac{111}{4} = 27.75$

SIMPLEX Algorithm

$$z = \frac{111}{4} + \frac{1}{16}\overset{0}{x_2} - \frac{1}{8}x_5 - \frac{11}{6}x_6$$

$$x_1 = \frac{33}{4} - \frac{1}{16}x_2 + \frac{1}{8}x_5 - \frac{5}{16}x_6 \qquad \Longrightarrow \qquad x_2 \leq 132$$

$$x_3 = \frac{3}{2} - \frac{3}{8}x_2 - \frac{1}{4}x_5 + \frac{1}{8}x_6 \qquad \Longrightarrow \qquad x_2 \leq 4$$

$$x_4 = \frac{69}{4} + \frac{3}{16}x_2 + \frac{5}{8}x_5 - \frac{1}{16}x_6 \qquad \Longrightarrow \qquad x_2 \leq \infty$$

Basic solution:

$$x_1 = \frac{33}{4},\ x_2 = 0,\ x_3 = \frac{3}{2},\ x_4 = \frac{69}{4},\ x_5 = 0,\ x_6 = 0$$

Objective value = $\frac{111}{4} = 27.75$

SIMPLEX Algorithm

4

0

$$z = \frac{111}{4} + \frac{1}{16}\boxed{x_2} - \frac{1}{8}x_5 - \frac{11}{6}x_6$$

$$x_1 = \frac{33}{4} - \frac{1}{16}x_2 + \frac{1}{8}x_5 - \frac{5}{16}x_6$$

1.5

0

$$\boxed{x_3} = \frac{3}{2} - \frac{3}{8}x_2 - \frac{1}{4}x_5 + \frac{1}{8}x_6$$

$$x_4 = \frac{69}{4} + \frac{3}{16}x_2 + \frac{5}{8}x_5 - \frac{1}{16}x_6$$

Pivot

Basic solution:

$$x_1 = \frac{33}{4},\ x_2 = 0,\ x_3 = \frac{3}{2},\ x_4 = \frac{69}{4},\ x_5 = 0,\ x_6 = 0$$

Objective value = $\frac{111}{4} = 27.75$

SIMPLEX Algorithm

4

0

$$z = \frac{111}{4} + \frac{1}{16}x_2 - \frac{1}{8}x_5 - \frac{11}{6}x_6$$

$$x_1 = \frac{33}{4} - \frac{1}{16}x_2 + \frac{1}{8}x_5 - \frac{5}{16}x_6$$

1.5

0

$$x_3 = \frac{3}{2} - \frac{3}{8}x_2 - \frac{1}{4}x_5 + \frac{1}{8}x_6$$

$$x_4 = \frac{69}{4} + \frac{3}{16}x_2 + \frac{5}{8}x_5 - \frac{1}{16}x_6$$

$$z = 28 - \frac{1}{6}x_3 - \frac{1}{6}x_5 - \frac{2}{3}x_6$$

$$x_1 = 8 + \frac{1}{6}x_3 + \frac{1}{6}x_5 - \frac{1}{3}x_6$$

$$x_2 = 4 - \frac{8}{3}x_3 - \frac{2}{3}x_5 + \frac{1}{3}x_6$$

$$x_4 = 18 - \frac{1}{2}x_3 + \frac{1}{2}x_5$$

Basic solution:

$$x_1 = \frac{33}{4},\ x_2 = 0,\ x_3 = \frac{3}{2},\ x_4 = \frac{69}{4},\ x_5 = 0,\ x_6 = 0$$

Objective value = $\frac{111}{4} = 27.75$

Basic solution:

$$x_1 = 8,\ x_2 = 4,\ x_3 = 0,\ x_4 = 18,\ x_5 = 0,\ x_6 = 0$$

Objective value = 28

SIMPLEX Algorithm

$$z = 28 - \frac{1}{6}x_3 - \frac{1}{6}x_5 - \frac{2}{3}x_6$$
$$x_1 = 8 + \frac{1}{6}x_3 + \frac{1}{6}x_5 - \frac{1}{3}x_6$$
$$x_2 = 4 - \frac{8}{3}x_3 - \frac{2}{3}x_5 + \frac{1}{3}x_6$$
$$x_4 = 18 - \frac{1}{2}x_3 + \frac{1}{2}x_5$$

This is the end of the SIMPLEX Algorithm!

The SIMPLEX Algorithm ends when in the objective function, the coefficients of the non-basic variables are all <= 0.

Basic solution:

$$x_1 = 8,\ x_2 = 4,\ x_3 = 0,\ x_4 = 18,\ x_5 = 0,\ x_6 = 0$$

Objective value = 28

Quiz questions:

1. What is the main idea of the SIMPLEX Algorithm?

2. Is the SIMPLEX Algorithm a greedy algorithm?

Roadmap of this lecture:

1. Linear Programming (LP)

   1.1 SIMPLEX Algorithm for LP when the initial basic solution is feasible.

   1.2 Bland's Rule for SIMPLEX Algorithm.

SIMPLEX Algorithm

The SIMPLEX Algorithm ends when
in the objective function, the coefficients
of the non-basic variables are all <= 0.

But can we guarantee SIMPLEX Algorithm
will end (instead of looping forever)?

Wait … if the objective value increments
monotonically each time, how could it
possibly loop forever?

What if the objective value increments by 0?

Really … can it happen?

SIMPLEX Algorithm

$$z = x_1 + x_2 + x_3$$
$$x_4 = 8 - x_1 - x_2$$
$$x_5 = x_2 - x_3$$

The SIMPLEX Algorithm ends when in the objective function, the coefficients of the non-basic variables are all <= 0.

But can we guarantee SIMPLEX Algorithm will end (instead of looping forever)?

Wait … if the objective value increments monotonically each time, how could it possibly loop forever?

What if the objective value increments by 0?

Really … can it happen?

SIMPLEX Algorithm

The SIMPLEX Algorithm ends when
in the objective function, the coefficients
of the non-basic variables are all <= 0.

But can we guarantee SIMPLEX Algorithm
will end (instead of looping forever)?

Wait … if the objective value increments
monotonically each time, how could it
possibly loop forever?

What if the objective value increments by 0?

Really … can it happen?

$$z = \boxed{x_1}^{0} + x_2 + x_3$$

$$x_4 = 8 - x_1 - x_2 \quad \rightarrow \quad x_1 \leq 8$$

$$x_5 = x_2 - x_3 \quad \rightarrow \quad x_1 \leq \infty$$

SIMPLEX Algorithm

The SIMPLEX Algorithm ends when in the objective function, the coefficients of the non-basic variables are all <= 0.

But can we guarantee SIMPLEX Algorithm will end (instead of looping forever)?

Wait … if the objective value increments monotonically each time, how could it possibly loop forever?

What if the objective value increments by 0?

Really … can it happen?

$$z = x_1 + x_2 + x_3$$
$$x_4 = 8 - x_1 - x_2$$
$$x_5 = x_2 - x_3$$

$x_1 \leq 8$

$$z = 8 + x_3 - x_4$$
$$x_1 = 8 - x_2 - x_4$$
$$x_5 = x_2 - x_3$$

SIMPLEX Algorithm

The SIMPLEX Algorithm ends when in the objective function, the coefficients of the non-basic variables are all <= 0.

But can we guarantee SIMPLEX Algorithm will end (instead of looping forever)?

Wait … if the objective value increments monotonically each time, how could it possibly loop forever?

What if the objective value increments by 0?

Really … can it happen?

$$z = x_1 + x_2 + x_3$$
$$x_4 = 8 - x_1 - x_2$$
$$x_5 = x_2 - x_3$$

$$z = 8 + \overset{0}{x_3} - x_4$$
$$x_1 = 8 - x_2 - x_4 \quad \rightarrow \quad x_3 \leq \infty$$
$$x_5 = x_2 - x_3 \quad \rightarrow \quad x_3 \leq 0$$

SIMPLEX Algorithm

The SIMPLEX Algorithm ends when in the objective function, the coefficients of the non-basic variables are all <= 0.

$$z = x_1 + x_2 + x_3$$
$$x_4 = 8 - x_1 - x_2$$
$$x_5 = x_2 - x_3$$

See the promise we made.

0

0

$$z = 8 + \boxed{x_3} - x_4$$
$$x_1 = 8 - x_2 - x_4$$

0

0

$$\boxed{x_5} = x_2 - x_3 \quad \rightarrow \quad x_3 \leq 0$$

Should we pivot?

Yes, even though here the objective value increments by 0 (i.e., it does not change).

SIMPLEX Algorithm

The SIMPLEX Algorithm ends when
in the objective function, the coefficients
of the non-basic variables are all <= 0.

But can we guarantee SIMPLEX Algorithm
will end (instead of looping forever)?

Wait … if the objective value increments
monotonically each time, how could it
possibly loop forever?

What if the objective value increments by 0?

Really … can it happen?

$$z = x_1 + x_2 + x_3$$
$$x_4 = 8 - x_1 - x_2$$
$$x_5 = x_2 - x_3$$

$$\Downarrow$$

$$z = 8 + \overset{0\,\uparrow\,0}{x_3} - x_4$$
$$x_1 = 8 - x_2 - x_4$$
$$\underset{0\,\downarrow\,0}{x_5} = x_2 - x_3$$

$$\Downarrow$$

$$z = 8 + x_2 - x_4 - x_5$$
$$x_1 = 8 - x_2 - x_4$$
$$x_3 = x_2 - x_5$$

SIMPLEX Algorithm

The SIMPLEX Algorithm ends when in the objective function, the coefficients of the non-basic variables are all <= 0.

But can we guarantee SIMPLEX Algorithm will end (instead of looping forever)?

Wait … if the objective value increments monotonically each time, how could it possibly loop forever?

What if the objective value increments by 0?

Really … can it happen?

$$z = x_1 + x_2 + x_3$$
$$x_4 = 8 - x_1 - x_2$$
$$x_5 = x_2 - x_3$$

$$z = 8 + x_3 - x_4$$
$$x_1 = 8 - x_2 - x_4$$
$$x_5 = x_2 - x_3$$

$$z = 8 + \underset{\uparrow 0}{x_2} - x_4 - x_5$$
$$x_1 = 8 - x_2 - x_4 \quad \rightarrow \quad x_2 \leq 8$$
$$x_3 = x_2 - x_5 \quad \rightarrow \quad x_2 \leq \infty$$

SIMPLEX Algorithm

The SIMPLEX Algorithm ends when in the objective function, the coefficients of the non-basic variables are all <= 0.

But can we guarantee SIMPLEX Algorithm will end (instead of looping forever)?

Wait … if the objective value increments monotonically each time, how could it possibly loop forever?

What if the objective value increments by 0?

Really … can it happen?

$$z = x_1 + x_2 + x_3$$
$$x_4 = 8 - x_1 - x_2$$
$$x_5 = x_2 - x_3$$

$$\Downarrow$$

$$z = 8 + x_3 - x_4$$
$$x_1 = 8 - x_2 - x_4$$
$$x_5 = x_2 - x_3$$

$$\Downarrow$$

$8 \uparrow 16$; $0 \uparrow 8$

$$z = 8 + \boxed{x_2} - x_4 - x_5$$
$$x_1 = 8 - x_2 - x_4 \quad \rightarrow \quad x_2 \leq 8$$
$$x_3 = x_2 - x_5 \quad \rightarrow \quad x_2 \leq \infty$$

So we can make the objective value larger again.

SIMPLEX Algorithm

The SIMPLEX Algorithm ends when in the objective function, the coefficients of the non-basic variables are all <= 0.

But can we guarantee SIMPLEX Algorithm will end (instead of looping forever)?

Wait … if the objective value increments monotonically each time, how could it possibly loop forever?

What if the objective value increments by 0?

Really … can it happen?

$$z = x_1 + x_2 + x_3$$
$$x_4 = 8 - x_1 - x_2$$
$$x_5 = x_2 - x_3$$

$$\Downarrow$$

$$z = 8 + x_3 - x_4$$
$$x_1 = 8 - x_2 - x_4$$
$$x_5 = x_2 - x_3$$

$$\Downarrow$$

$$\overset{16}{\underset{8}{\uparrow}}\ z = 8 + \underset{0\uparrow 8}{\boxed{x_2}} - x_4 - x_5$$
$$x_1 = 8 - x_2 - x_4 \quad \rightarrow \quad x_2 \leq 8$$
$$x_3 = x_2 - x_5 \quad \rightarrow \quad x_2 \leq \infty$$

This example shows that the objective value can indeed increment by 0.

If this happens continuously, we get into an infinite loop.

Can we avoid this?

SIMPLEX Algorithm

Bland's Rule:

1) When we choose a non-basic variable for incrementing its value, if there is a tie, choose the variable of the smallest index.

2) When we choose a basic variable for pivoting, if there is a tie, choose the variable of the smallest index.

Bland's Rule guarantees the SIMPLEX algorithm will end.

Quiz questions:

1. What is the Bland's Rule?

2. Why do we need Bland's Rule for the SIMPLEX Algorithm?