HW-6 — Analysis of Algorithm

Submitted By:- ASHUTOSH CHAUHAN
UIN : 232009024

1:> Subgraph-isomorphism problem.

Two graphs A and B are isomorphic to each other if they have the same number of vertices and edges and edge connectivity is retained.

To show that the subgraph-isomorphism problem is NP-complete we need to show that :-

(i) It is in the NP complexity class
(ii) It is NP-hard.

- Proof of NP-Completeness :-

A problem P is in class NP-complete if it belongs to class NP and every problem in class NP is polynomial time reducible to problem P.

$$\underset{\text{Easier}}{\boxed{A}} \le_P \underset{\text{Harder}}{B}$$

If problem A is known to be "hard" then we can use the above relation to show that Problem B is also "hard".

We will use Clique Problem and show the polynomial time reduction from an instance of Clique problem (NP complete problem) to an instance of Subgraph-isomorphism problem.

(i) Subgraph-isomorphism problem is NP:

If given a certificate we can verify in polynomial time if it is a solution to the problem then we can say subgraph-isomorphism problem is NP.

Proof: Certificate: Let H be a subgraph of B
We also know mapping b/w vertices of A and H.

Verification: ~~to verify~~ To check if A is isomorphic to H or not we will have to verify if mapping is a bijection and if for every edge (u, v) in A there is an edge (f(u), f(v)) present in H. This will only take polynomial time.

Since It has polynomial time verifiability we can say that it belongs to the NP class.

(ii) Subgraph-isomorphism is NP-hard:

As discussed before, we will prove this using Clique Problem. We will reduce Clique problem (NP complete) to the subgraph-isomorphism problem. If this reduction is possible in polynomial time, every NP problem can be reduced to subgraph iso. problem in polynomial time which will prove it is NP-Hard.

Proof:

Let the input to clique problem (H, K). The O/P is true if graph H contains a clique of size K.

Let A be a complete graph of K vertices and B be H, where A & B are two graph of Subgraph Isomorphism Problem.

- If n is no. of vertices in H. We know $K \leq n$
- If $K > n$, then clique of size K cannot be a subgraph of H.

Since $K \leq n$, time for creating $A = O(K^2) = O(n^2)$
(Because no. of edges in complete graph of size K is $K(K-1)/2$)

H has a clique of size K, if A is subgraph of B. Hence we can say that if Clique problem is true then the subgraph isomorphism problem is also true.
Therefore, Clique problem can be reduced to the subgraph isom problem in polynomial time. So it is an NP-Hard problem.

⇒ Hence we can say that Subgraph Isomorphism Problem is NP and NP-Hard i.e <u>NP-Complete</u>.

2.) Independent Set.

Independent Set of a graph $G=(V,E)$ of E edges and V vertices is a subset $V' \subseteq V$ of vertices such that each edge in E is incident on at most one vertex in V'.

- Decision Version of problem:
Given a graph $G(V,E)$ and an positive integer

K, the problem is to determine if the graph contains independent set of size greater than K.

A problem P is in class NP complete if it belongs to class NP and every problem in class NP is polynomial time reducible to the problem P.

$$\underset{\text{easier}}{A} \leq_P \underset{\text{harder}}{B}$$

If problem A is known to be 'hard' then we can use the above relation to show that Problem B is also hard.

We will show the Clique problem is reducible to Independent Set Problem.

(i) Proof of Independent Set is NP class:

If a problem is in NP then we will be able to verify the solution to it in polynomial time. Let check this for this problem:
Let Independant Set be S.
Initial a flag as 1. For each pair (U, V) in S with V' vertices check if both are connected or not. If atleast a pair is connected then we will change flag = 0 else continue traversal.
If flag = 0 at end then our solution is not correct otherwise it is correct.
Time it took to verify solution is just traversal time of vertices & edges of graph i.e O(V+E).
∴ we can say Indepedent Set in NP problem.

(ii) Proof of Independent Set is NP-Hard:

As discussed before, we will prove this by reducing instances of Clique problem to an instance of Independent Set problem in polynomial time. We can convert every instance of Clique problem of graph G(V, E) with size K into graph G'(V', E') of size K' of Independent set, using following method:-

- V' = V ⇒ G' contain all vertices of G
- E' = complement of edge of G

Let G be a graph containing a clique of size K, meaning K vertices in G are all connected to each other. Since G' contains the complementary edges of G, these K vertices are not adjacent to each other in G'. Therefore there exists an independent set of size K.

Conversely, if the complementary graph G' contains an independent set of size K', where the edges of this set are not connected to each other, then in the original graph G (which is complement of G'), the vertices in this set are connected to each other, forming a clique of size K'. Therefore, G contains a clique of size K'.

So we can say there is an independant set of size K in graph G(V, E) if there is a clique of size K in G' (complement graph). Hence problem is NP Hard

Since Independent Set prob is both NP and NP-Hard, it is a NP-complete problem.