

Algorithms

Lecture Topic: NP Completeness (Part 4)

Anxiao (Andrew) Jiang

Roadmap of this lecture:

1. NP Completeness

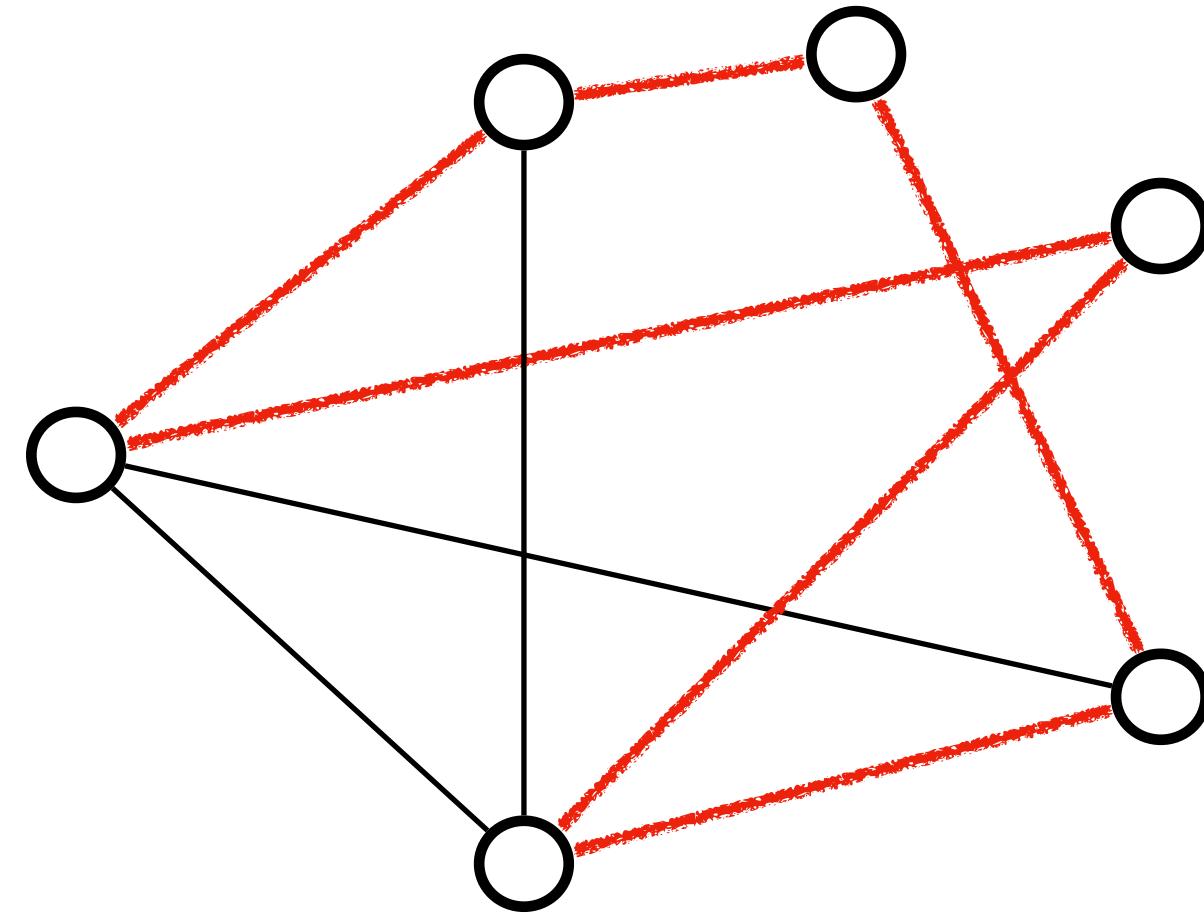
1.1 Prove the "Hamiltonian Cycle Problem" is NPC.

NP Completeness

Hamiltonian cycle: Give a graph $G=(V,E)$, a Hamiltonian cycle is a cycle in G that passes every node exactly once.

Known NPC Problems:

- 1) 3-CNF SAT Problem
- 2) Clique Problem
- 3) Vertex Cover Problem



Hamiltonian cycle Problem:

Input: A graph $G=(V,E)$.

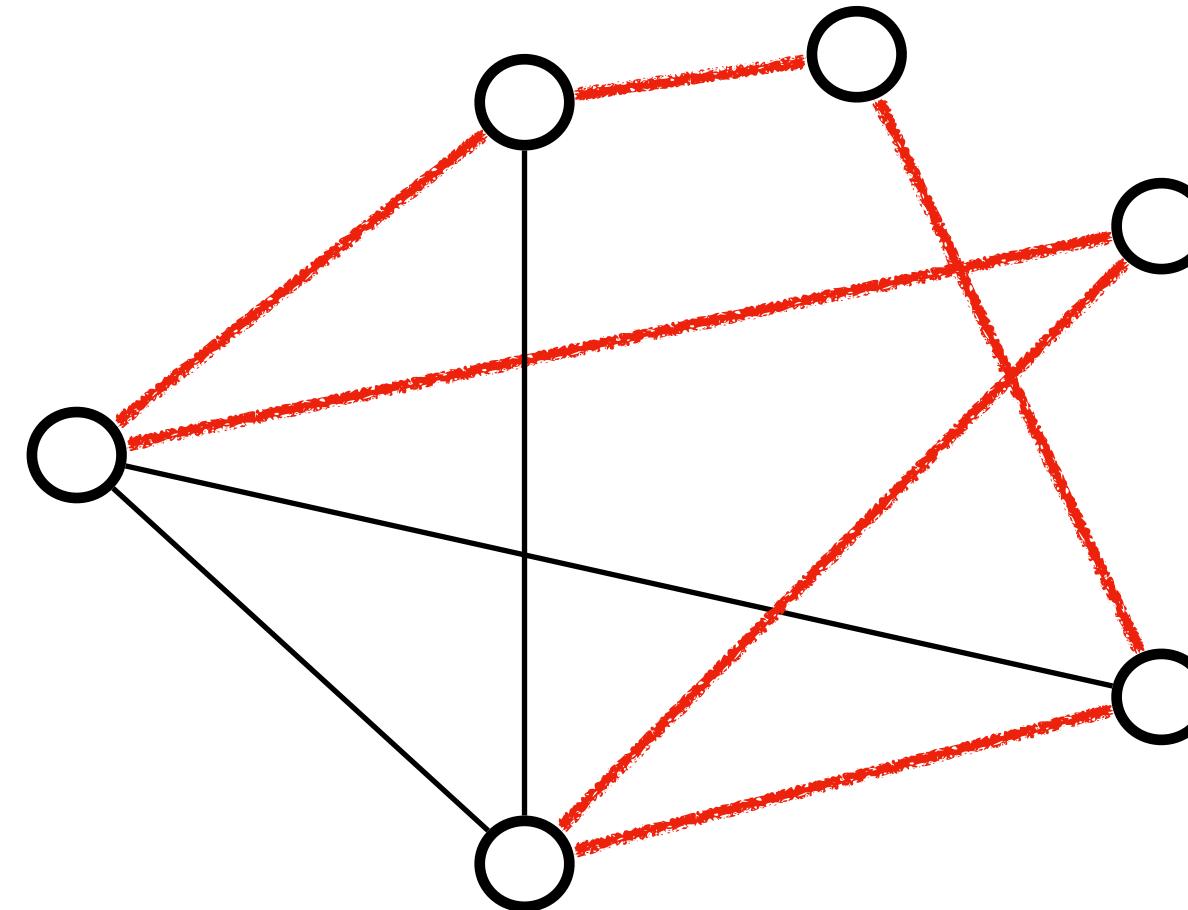
Question: Does G have a Hamiltonian cycle?

Theorem: The Hamiltonian cycle Problem is NP-Complete.

Proof: 1) Hamiltonian Cycle Problem $\in NP$.

Certificate: a Hamiltonian cycle in G .

Known NPC Problems:
1) 3-CNF SAT Problem
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Polynomial-time verification.

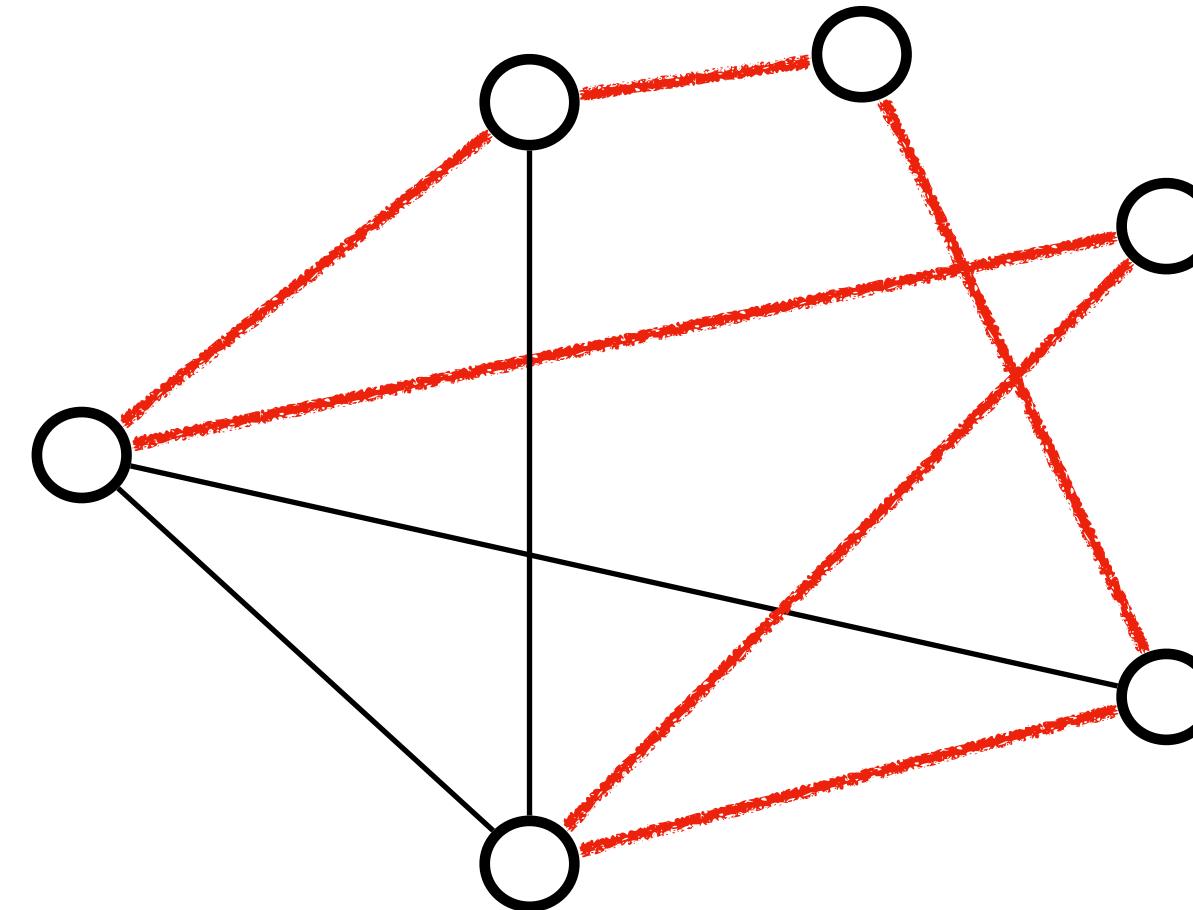
2) Which NPC problem can be reduced to the Hamiltonian Cycle Problem in polynomial time?

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Polynomial-time verification.

2) Which NPC problem can be reduced to the Hamiltonian Cycle Problem in polynomial time?

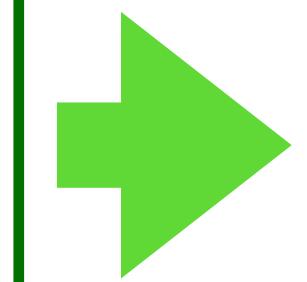
All NPC problems can. We choose the Vertex Cover Problem.

Vertex Cover Problem

Input: An undirected graph $G = (V, E)$.

An integer k .

Question: Does G have a vertex cover of size k ?



Hamiltonian cycle Problem:

Input: A graph $G' = (V', E')$.

Question: Does G' have a Hamiltonian cycle?

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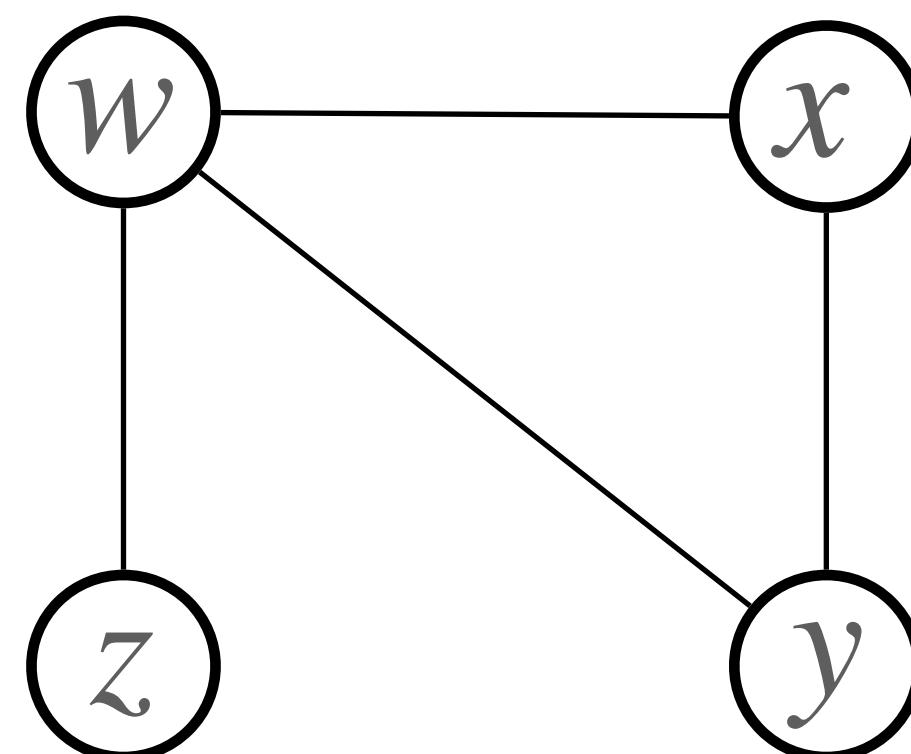
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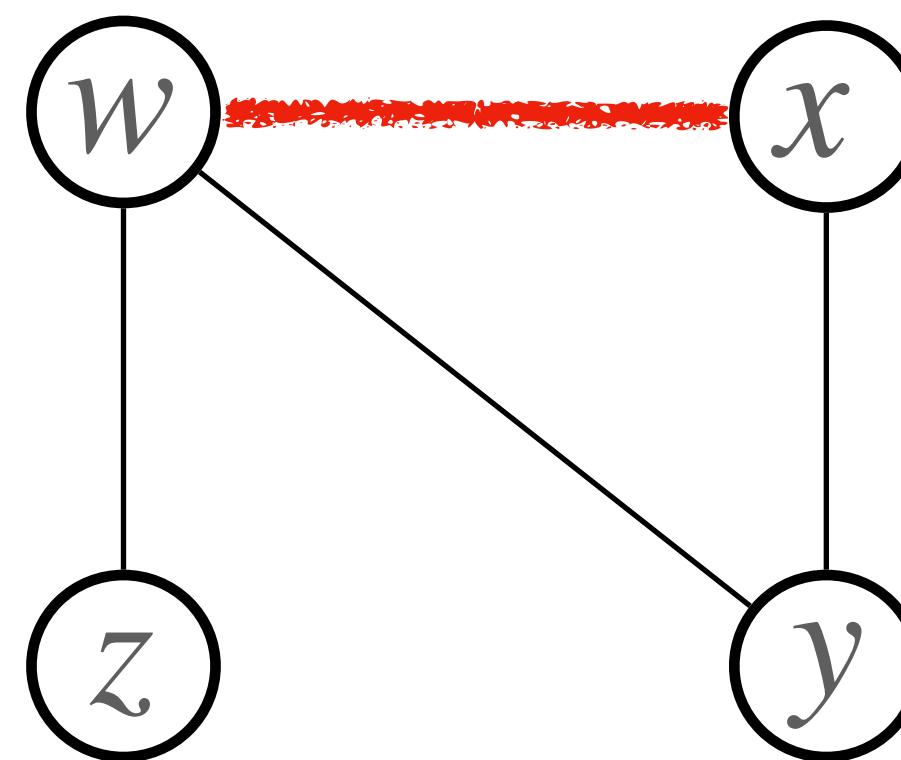
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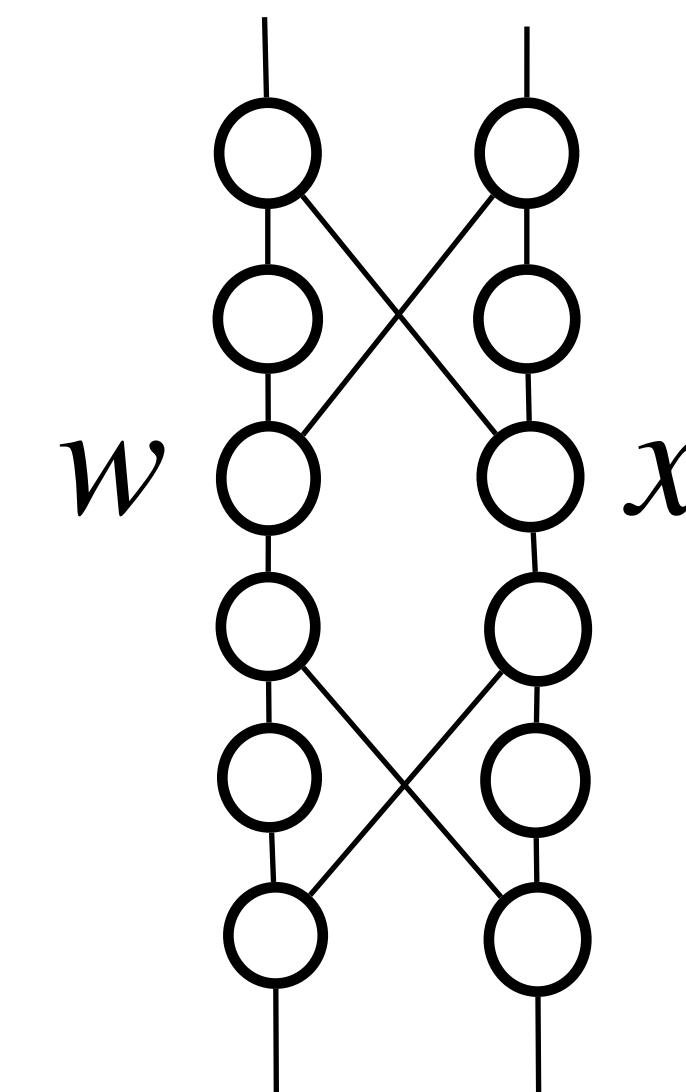
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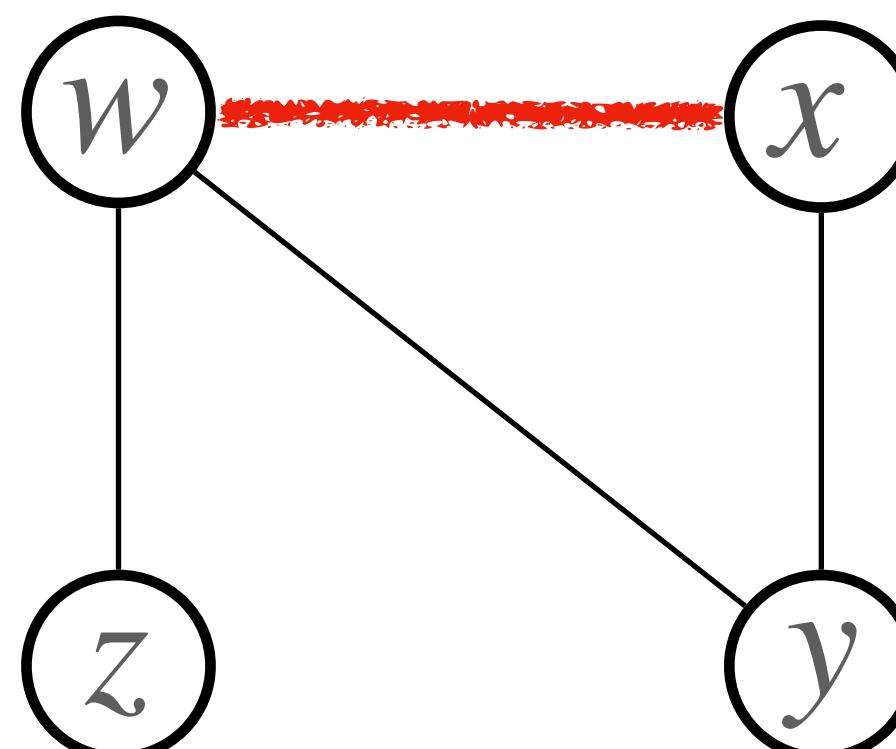
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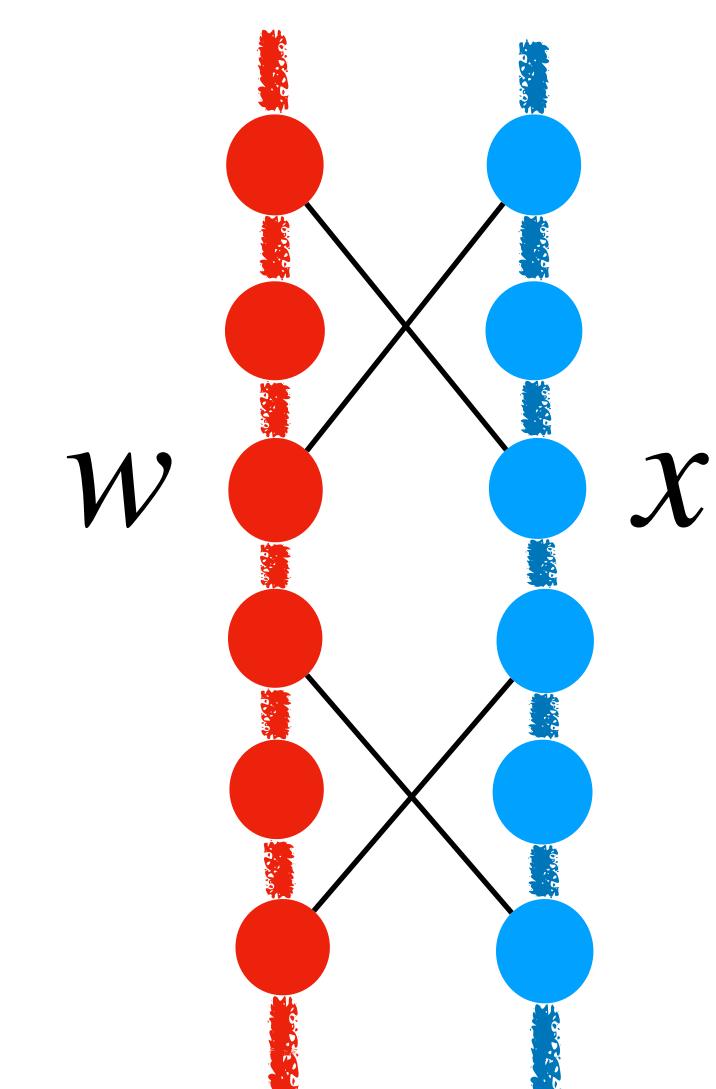
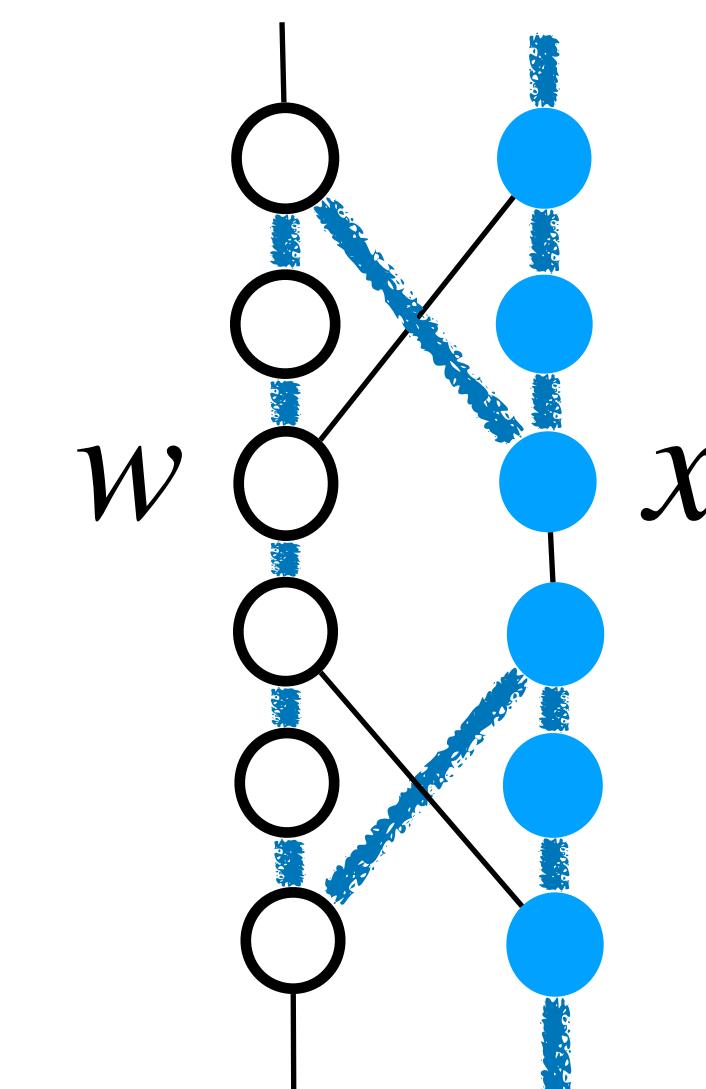
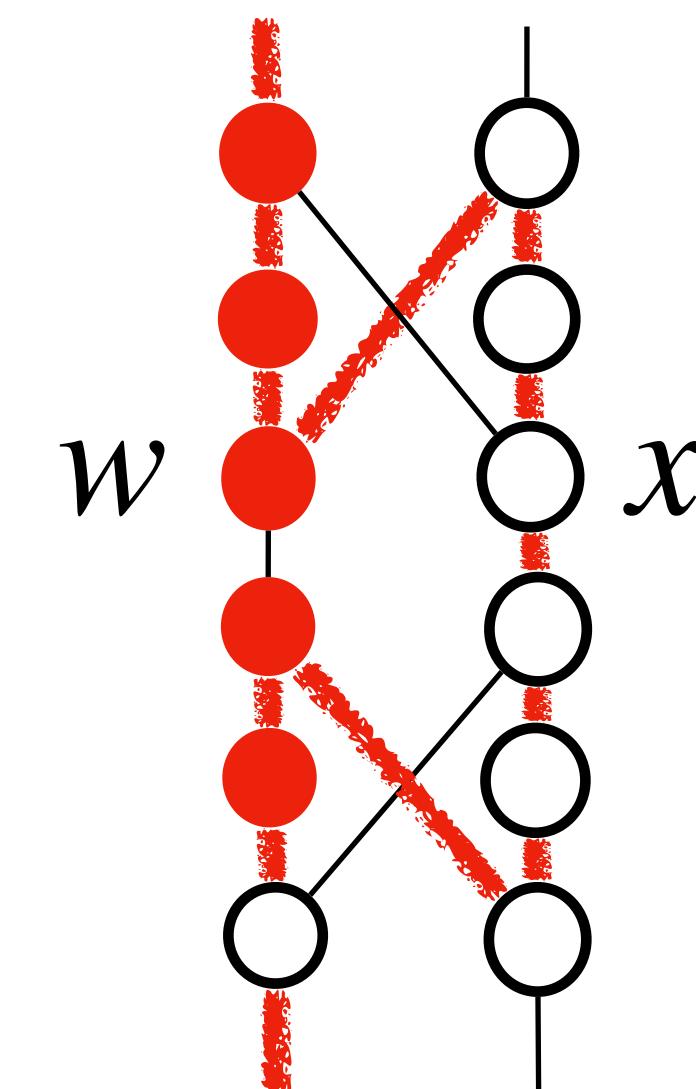
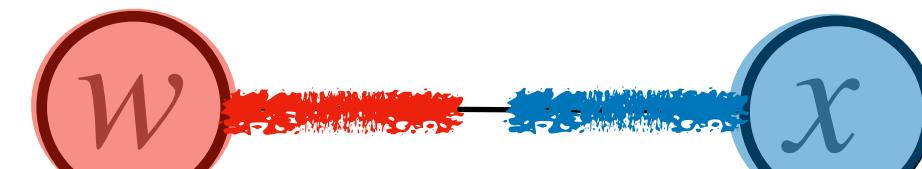
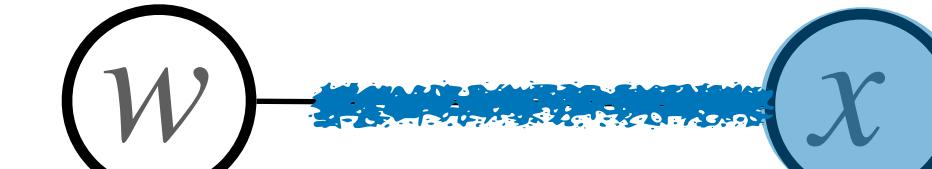
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3 ways to cover an edge
3 ways to traverse the “gadget”

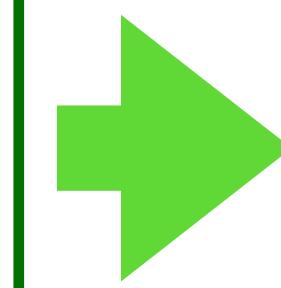


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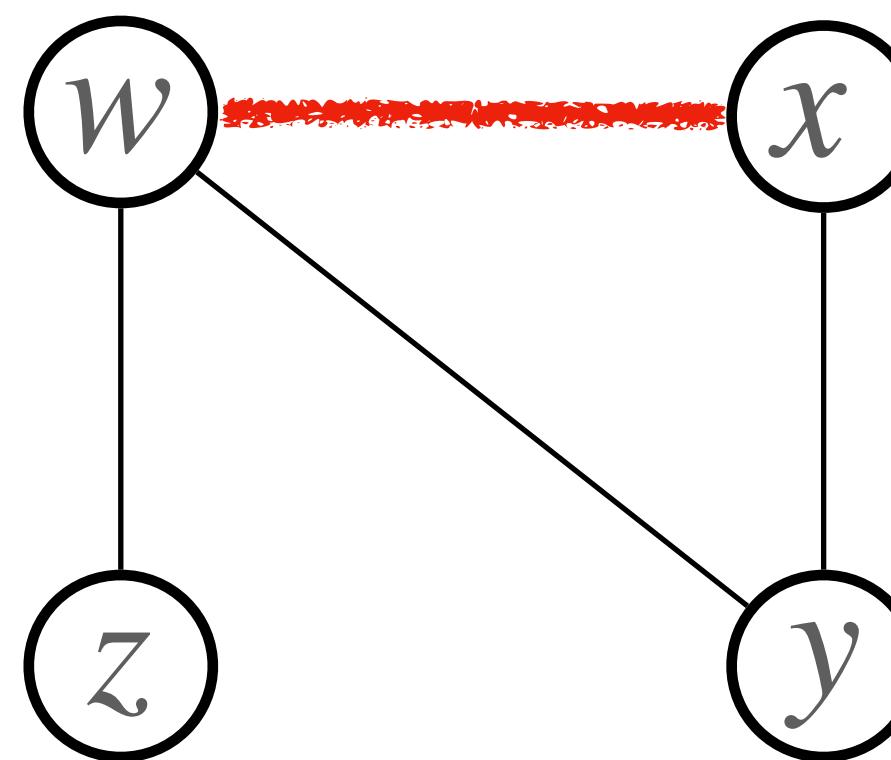
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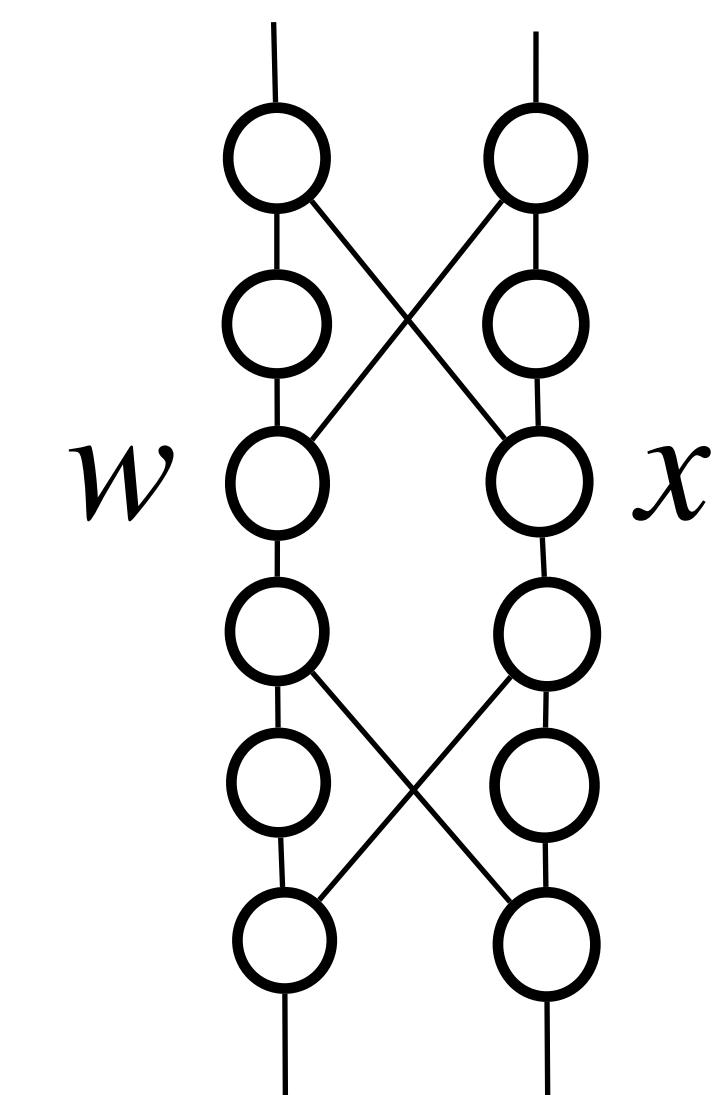
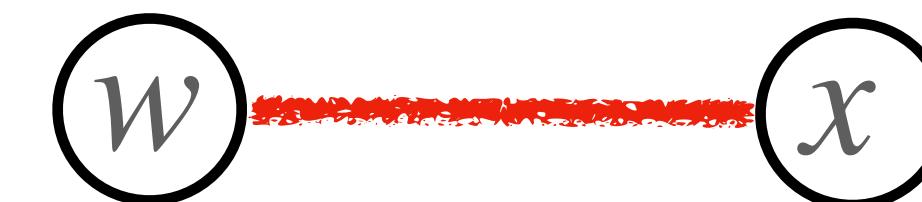
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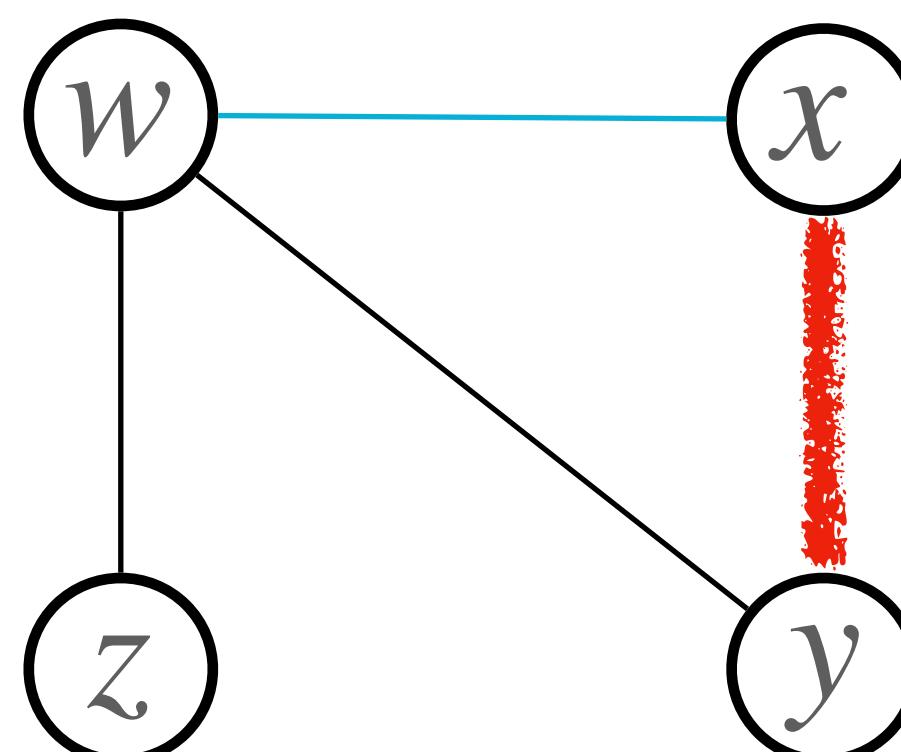
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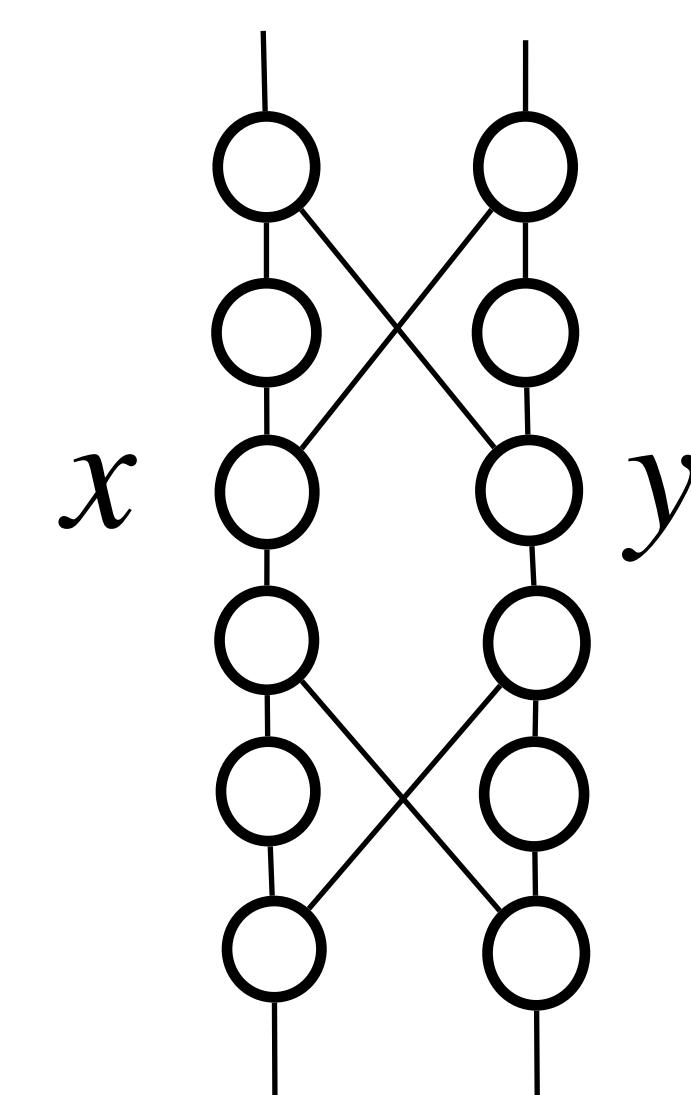
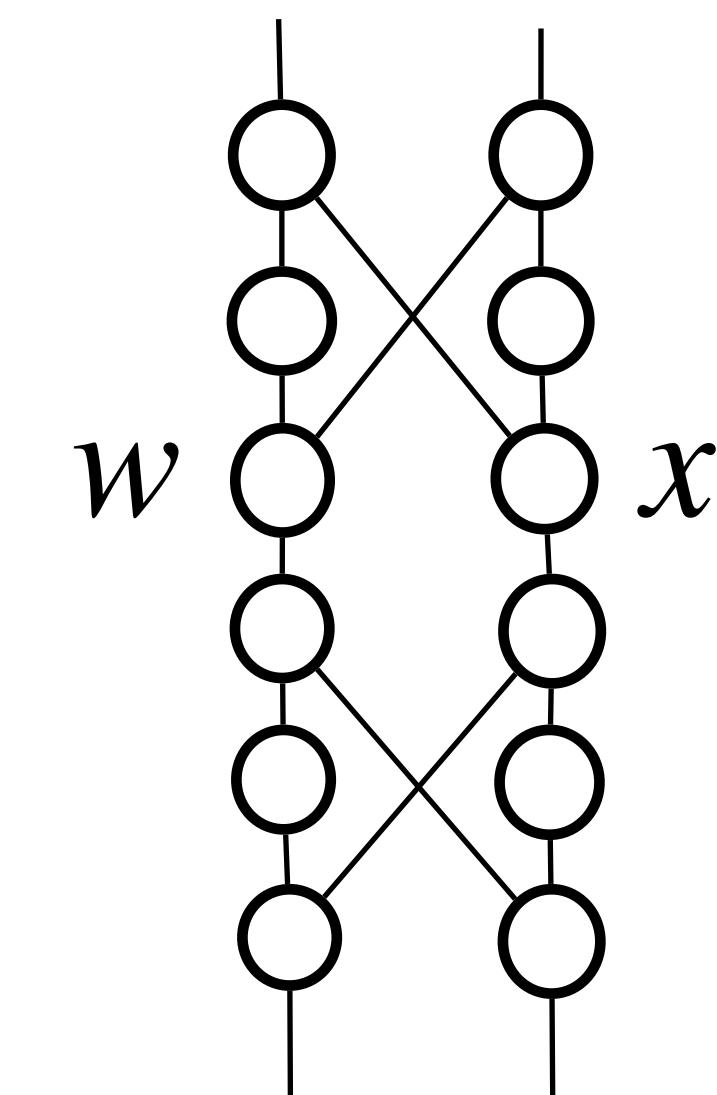
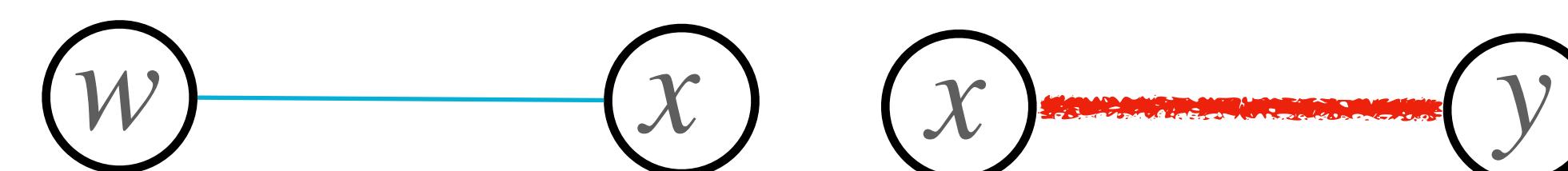
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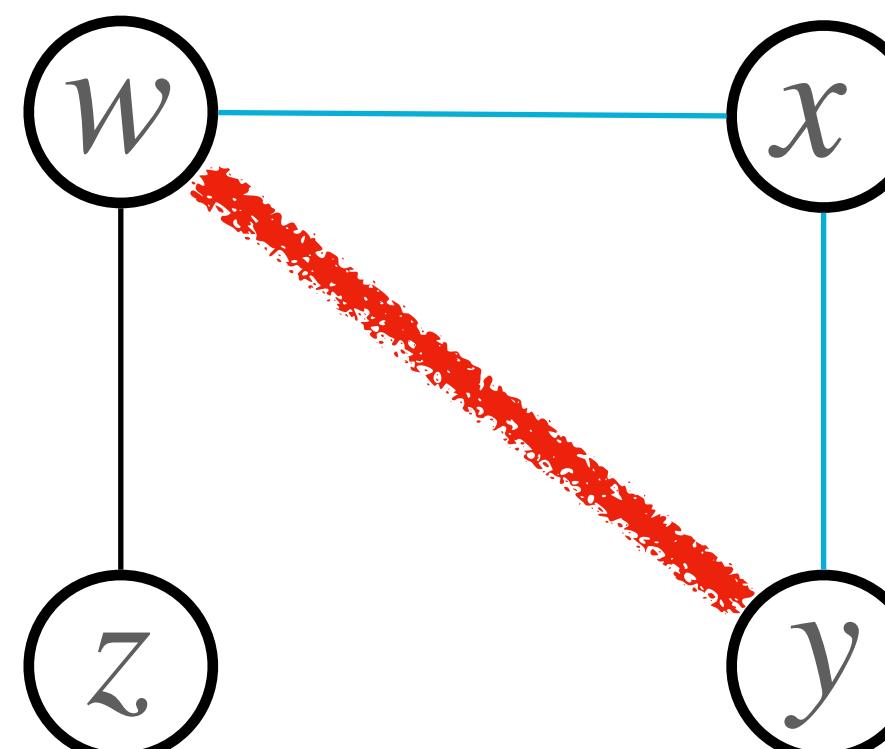
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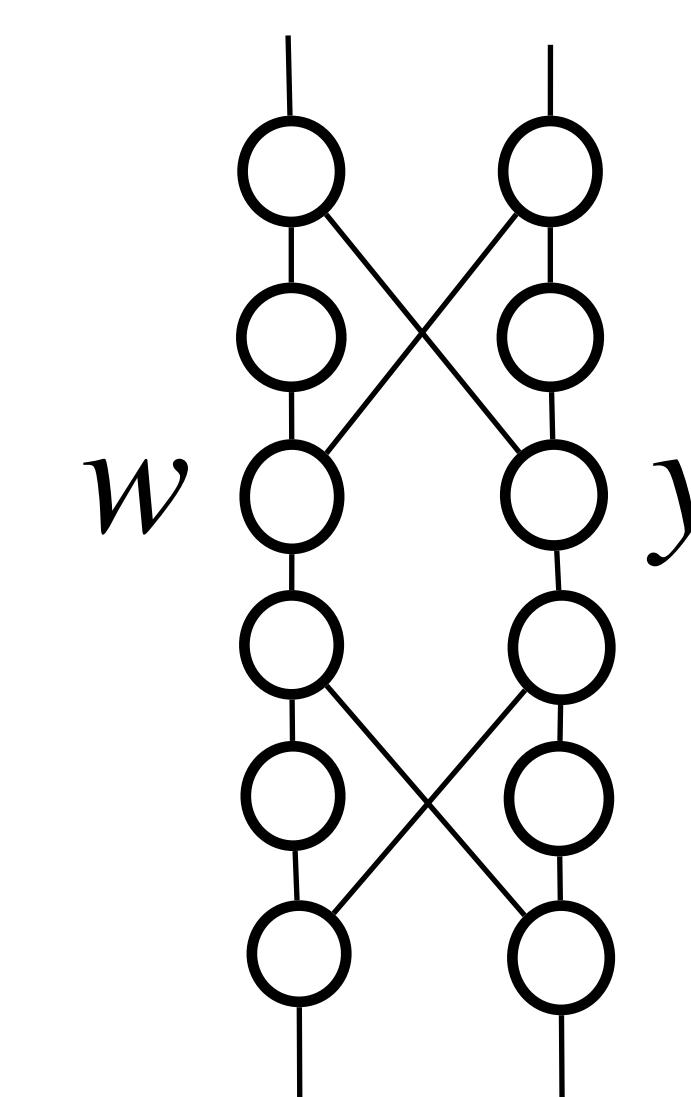
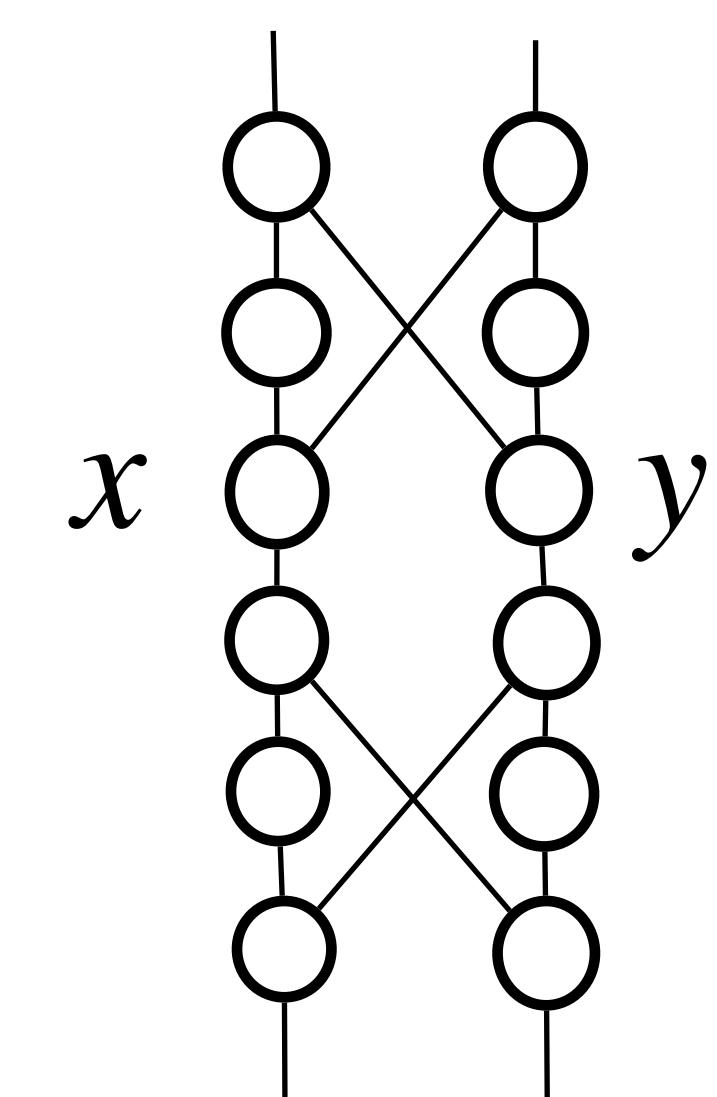
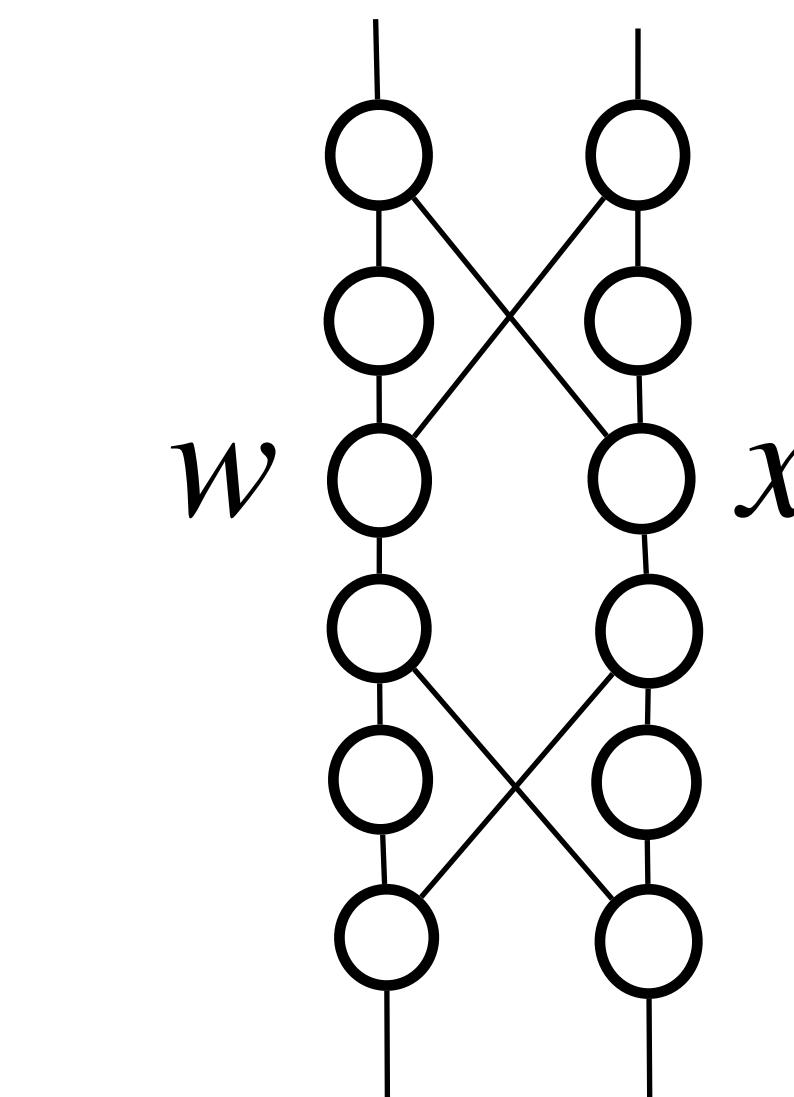
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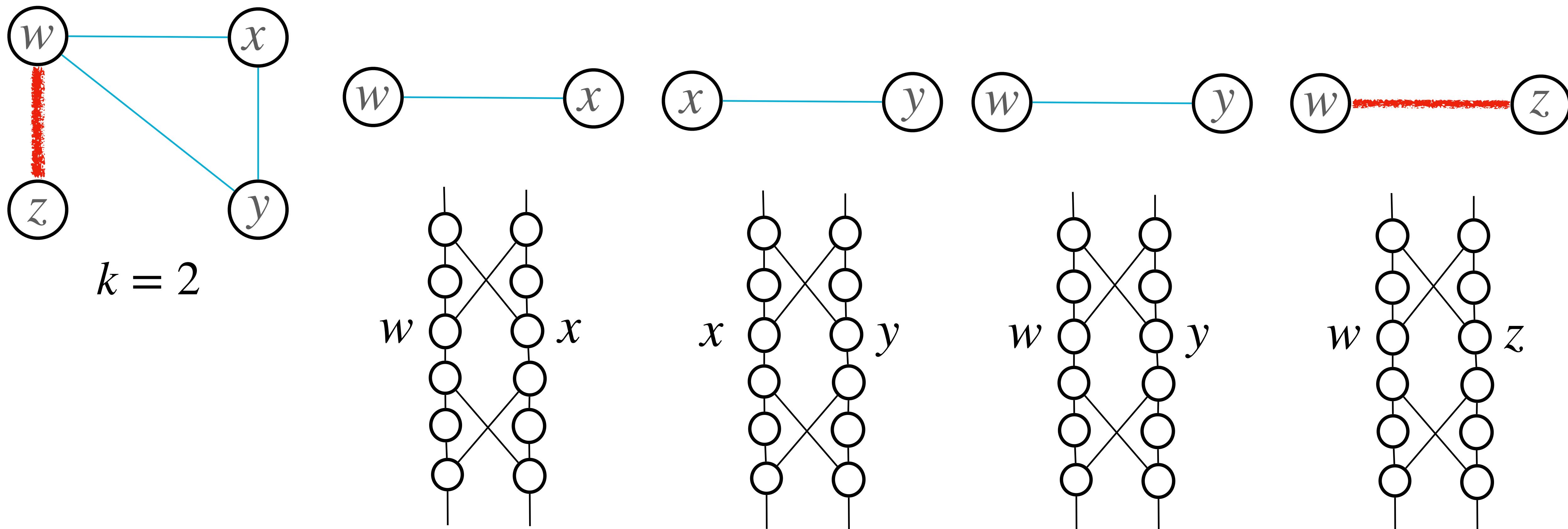
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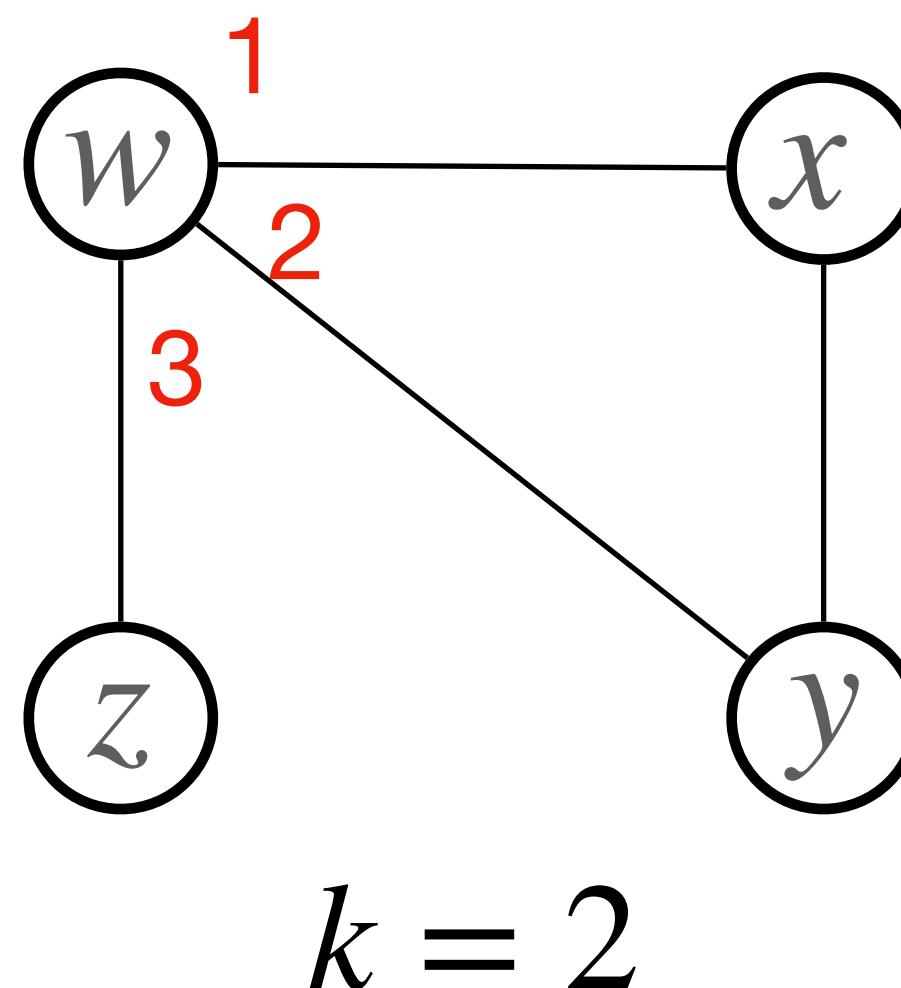
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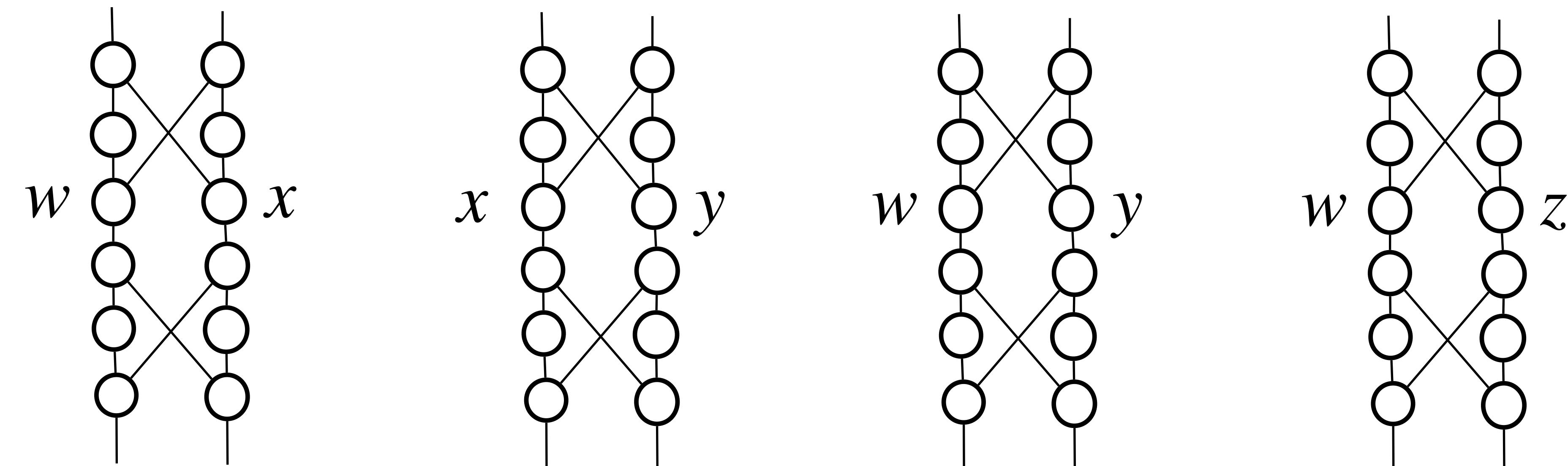
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Now let's look at the neighbors of each node.
Let's order the neighbors in an arbitrary order.



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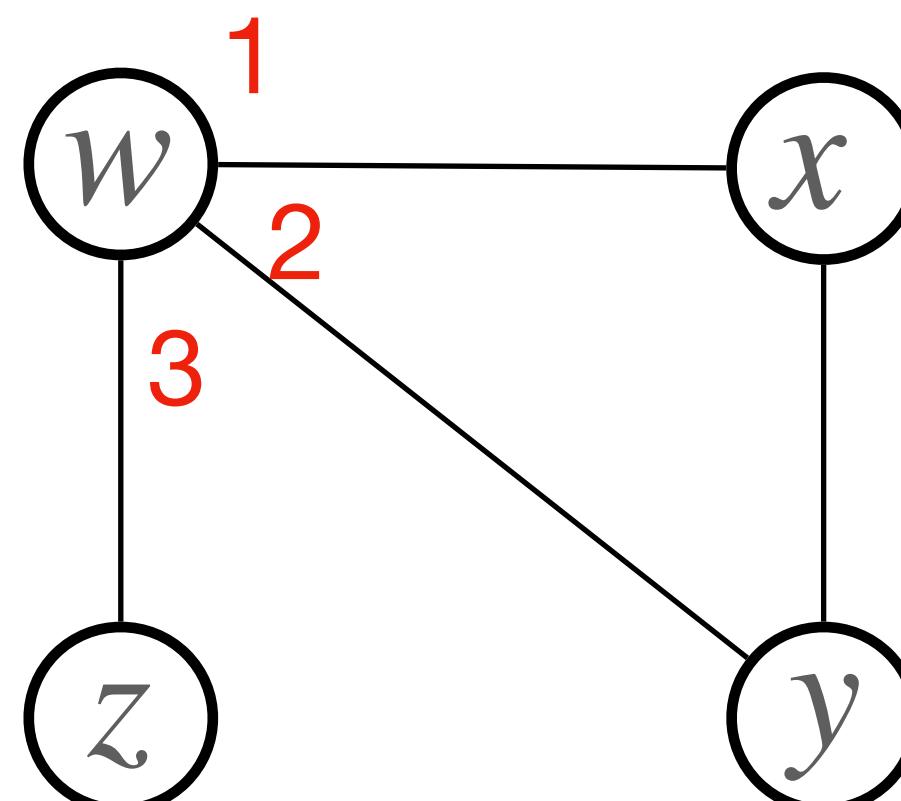
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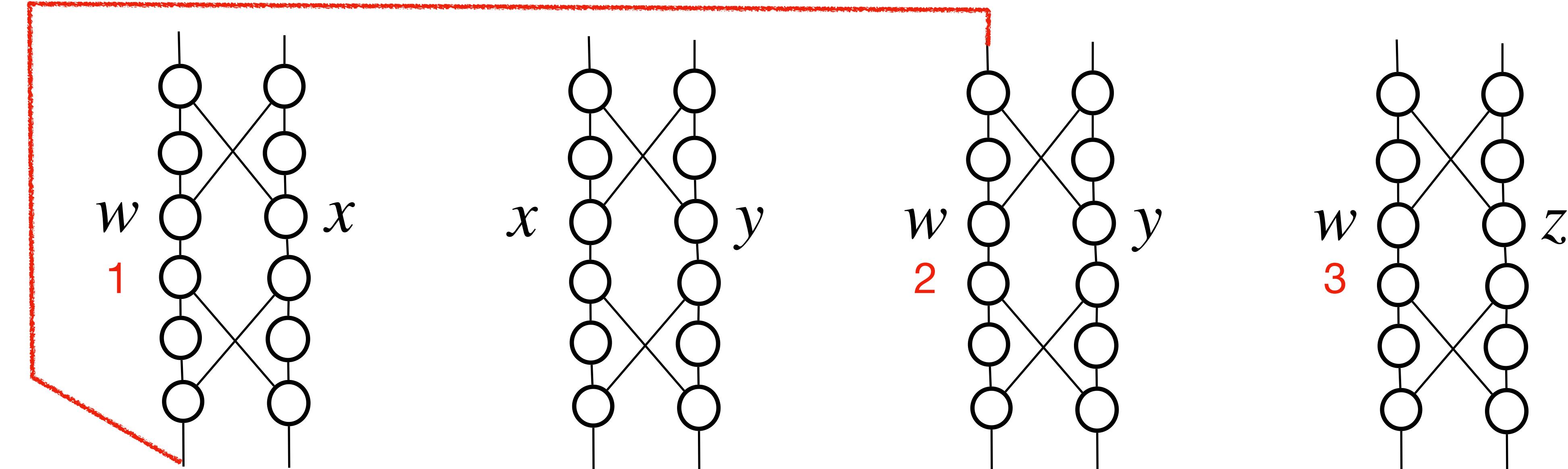
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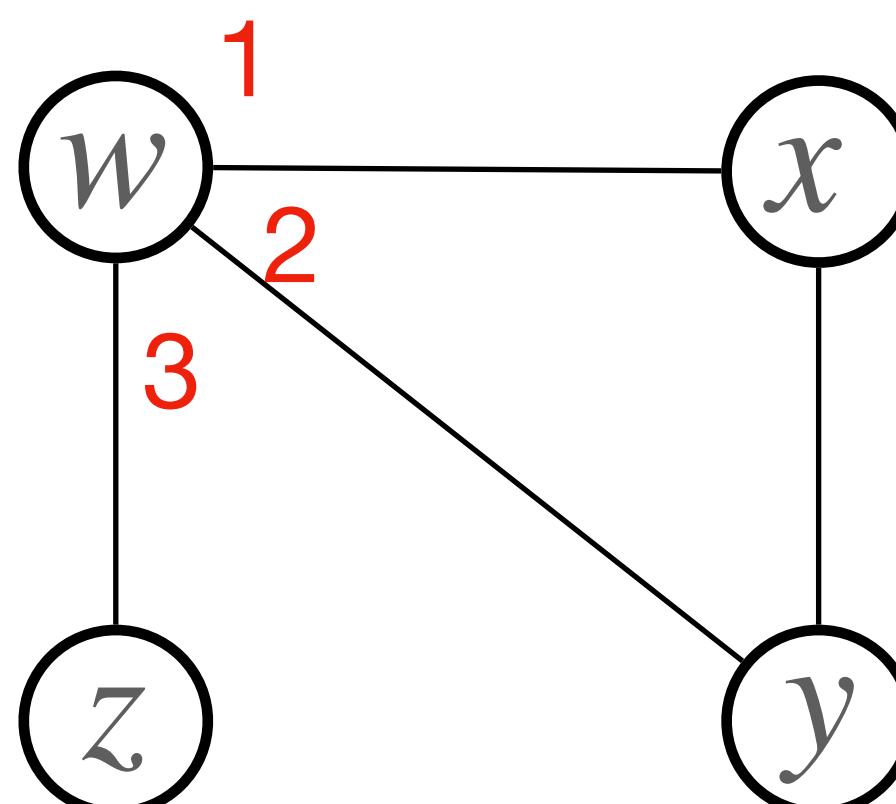
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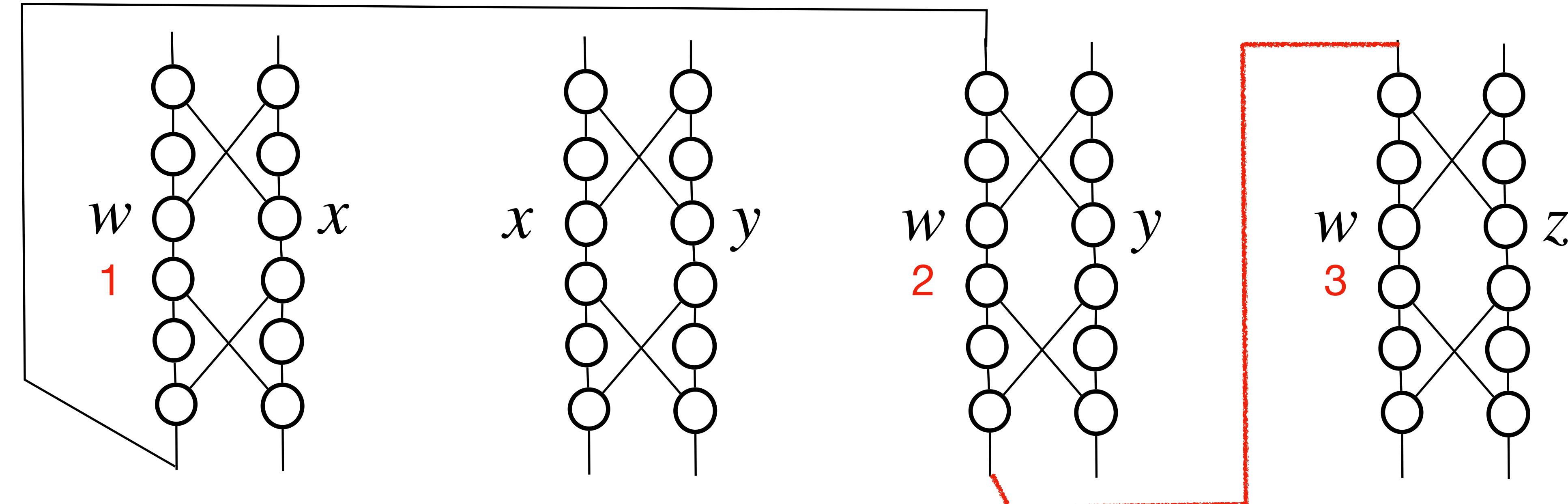
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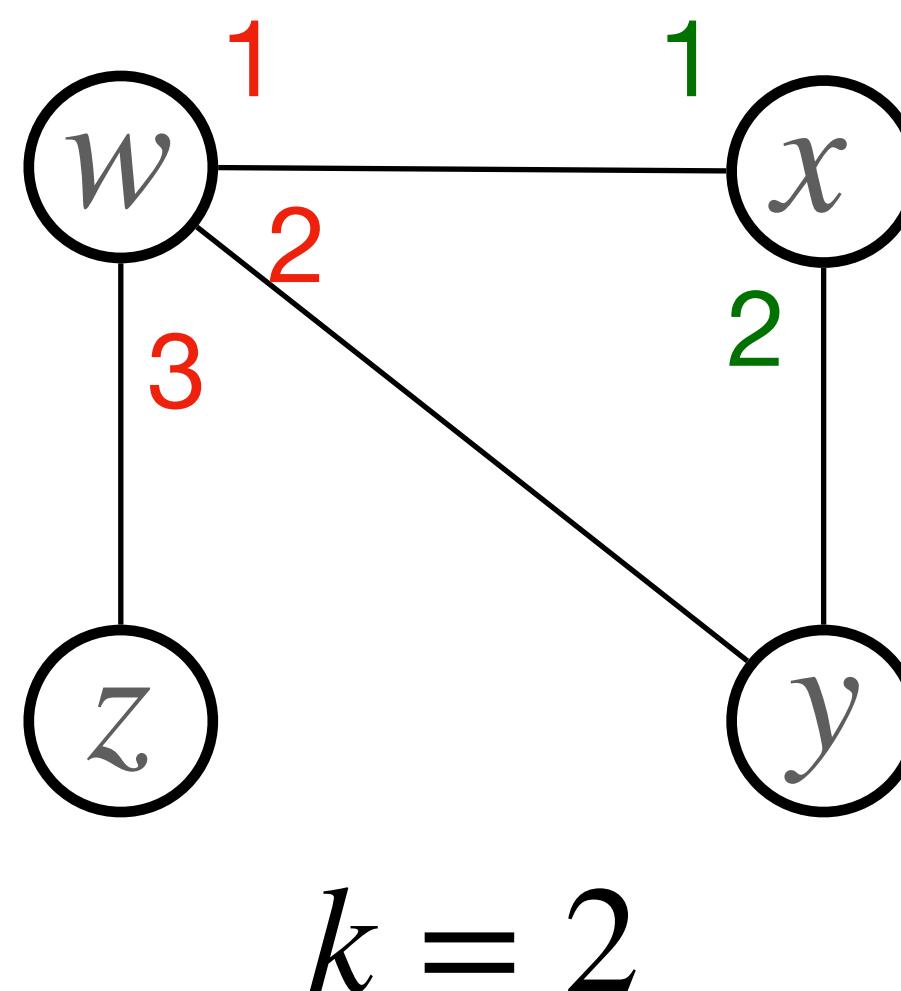
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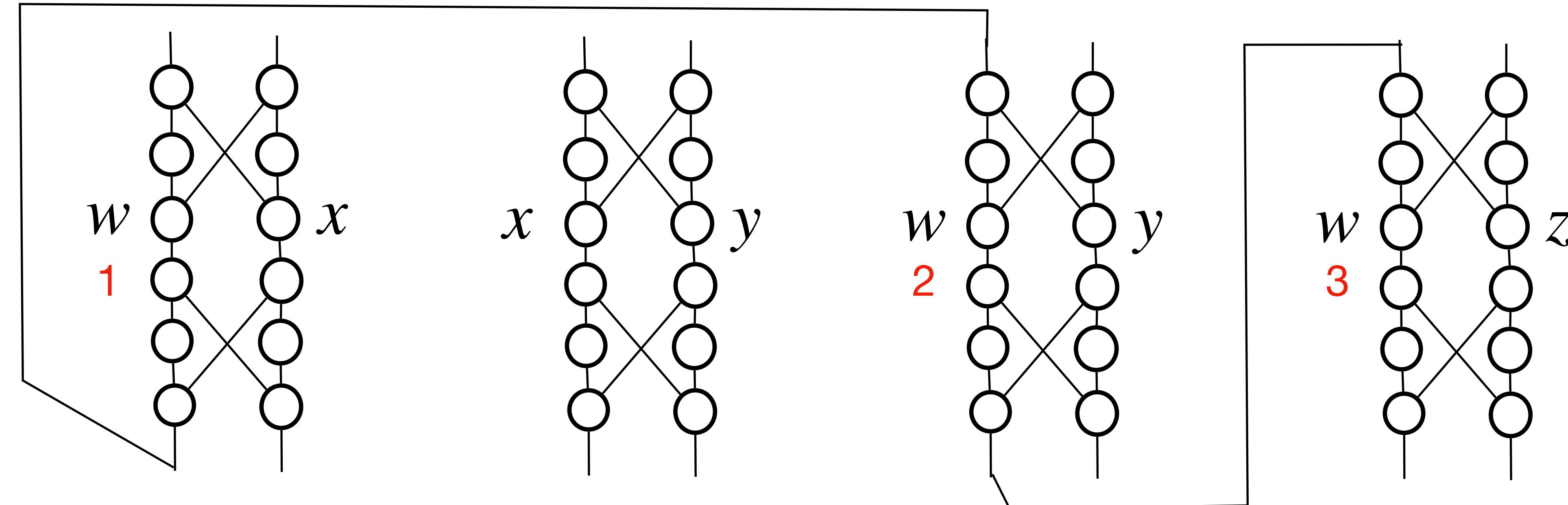
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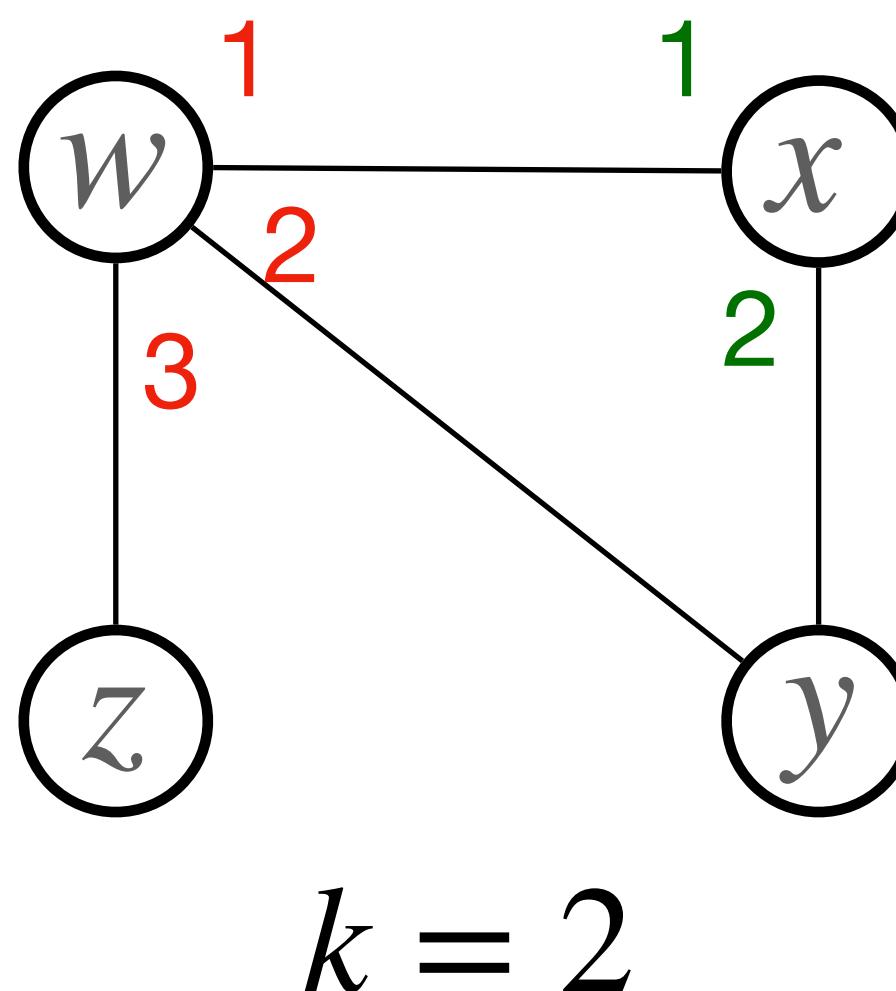
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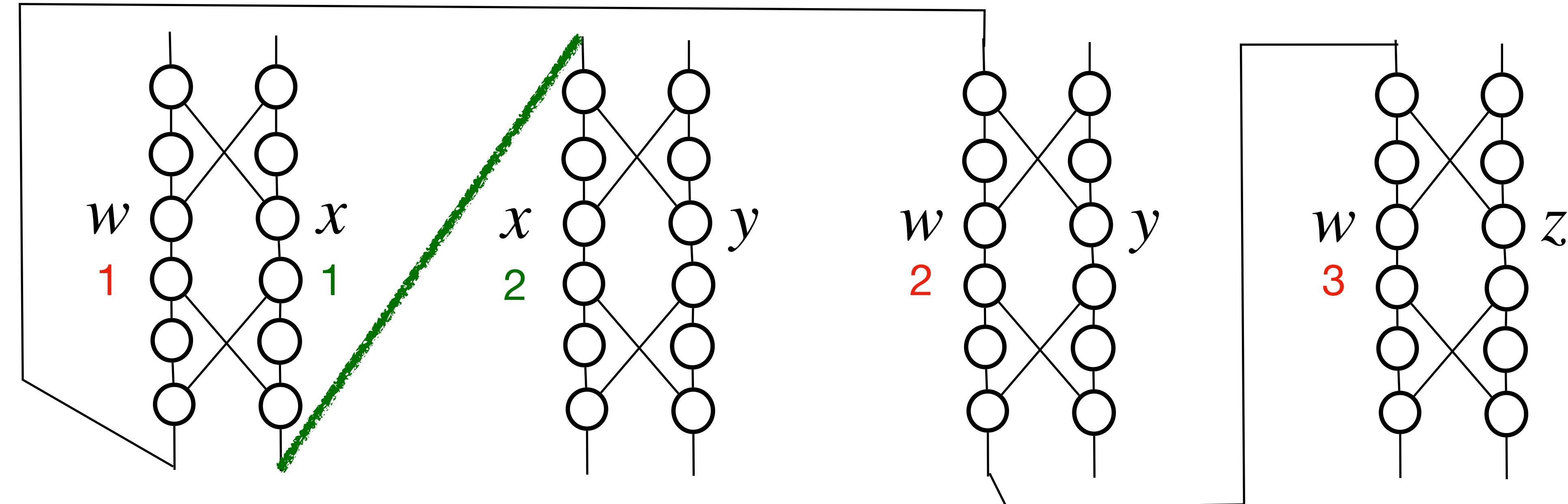
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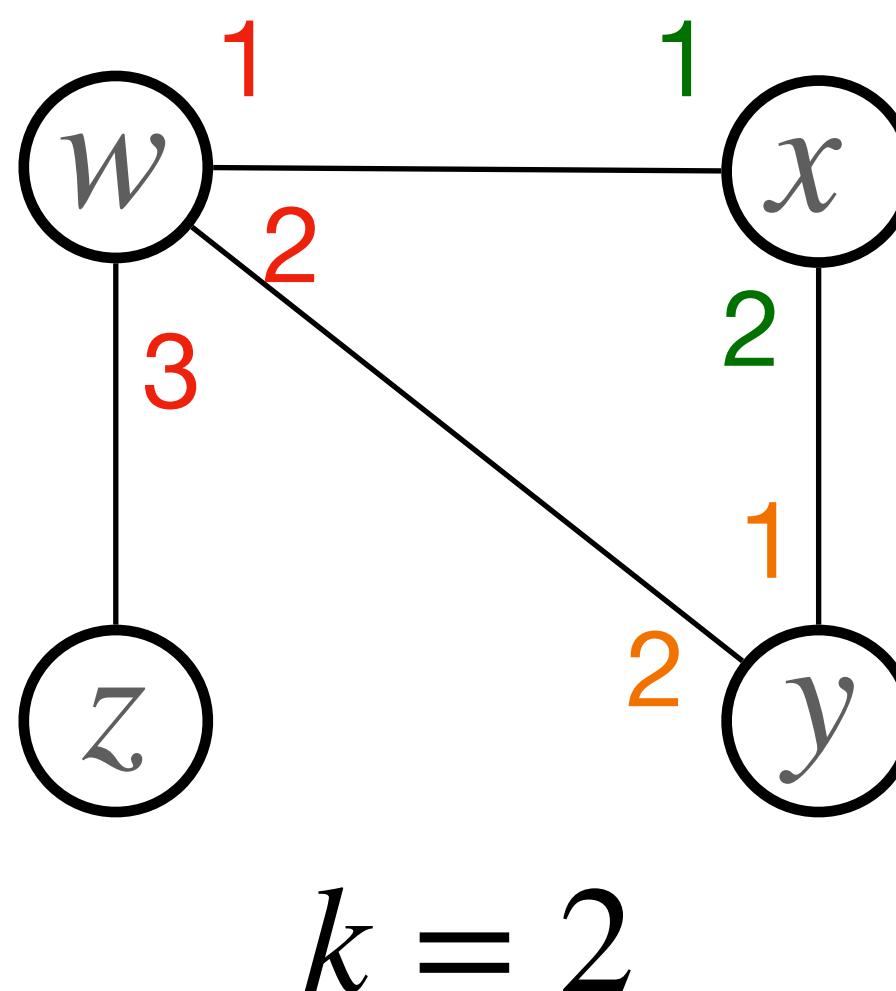
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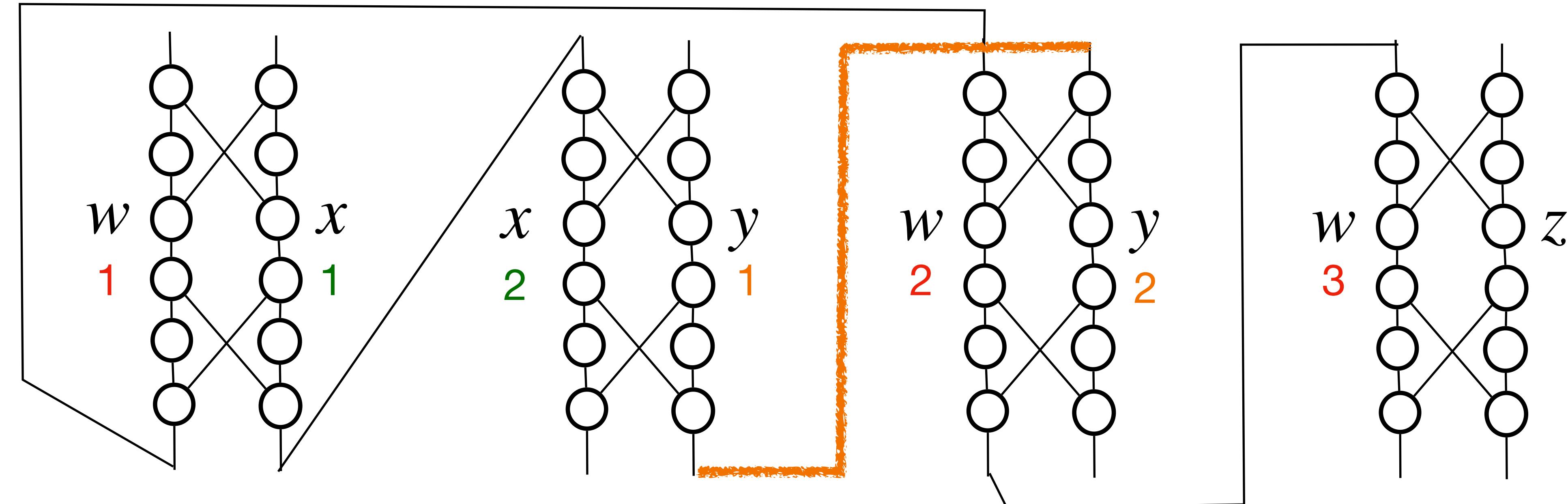
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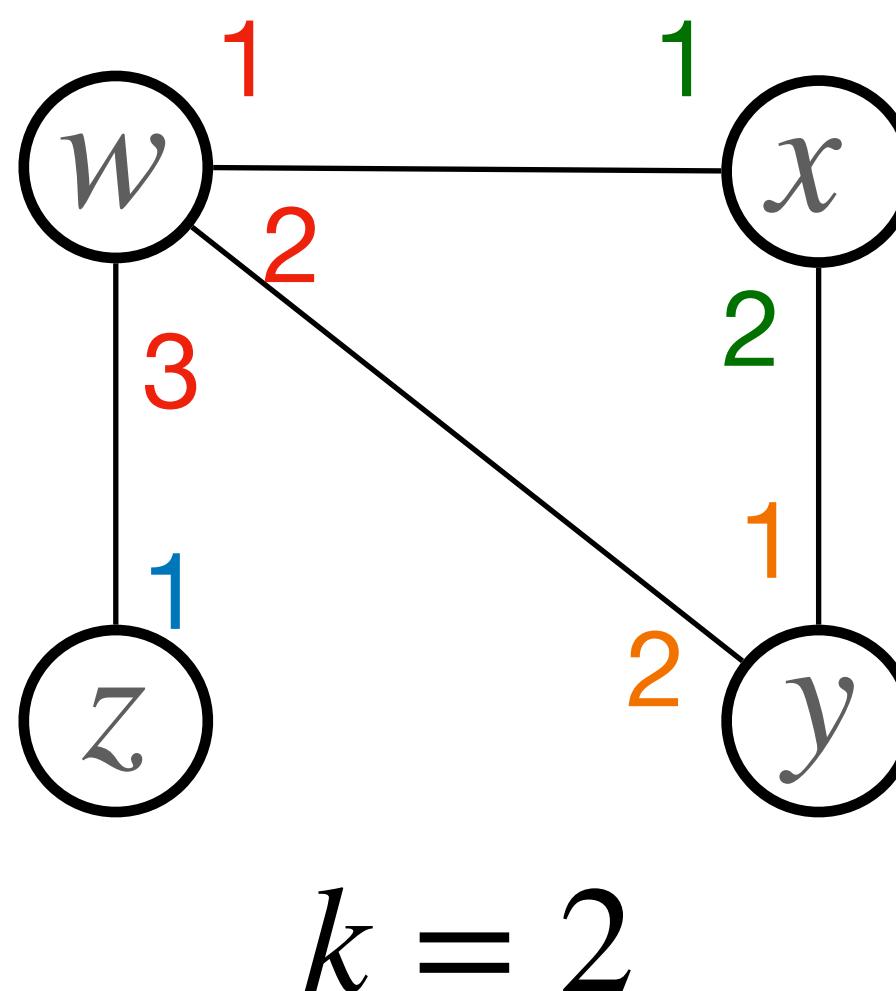
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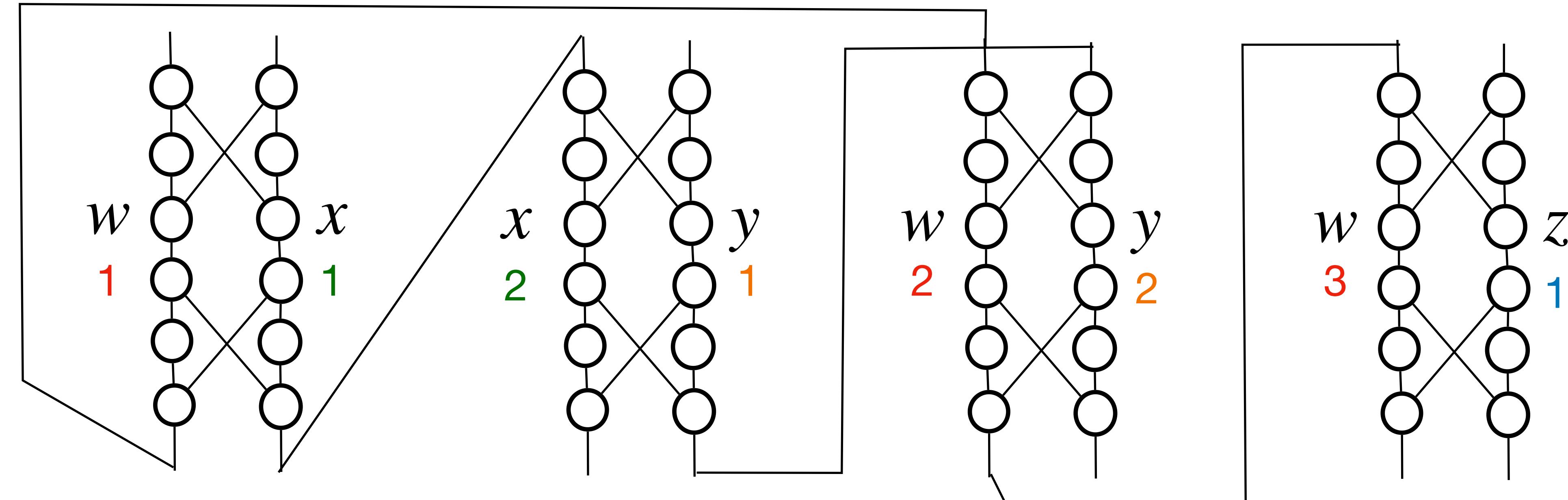
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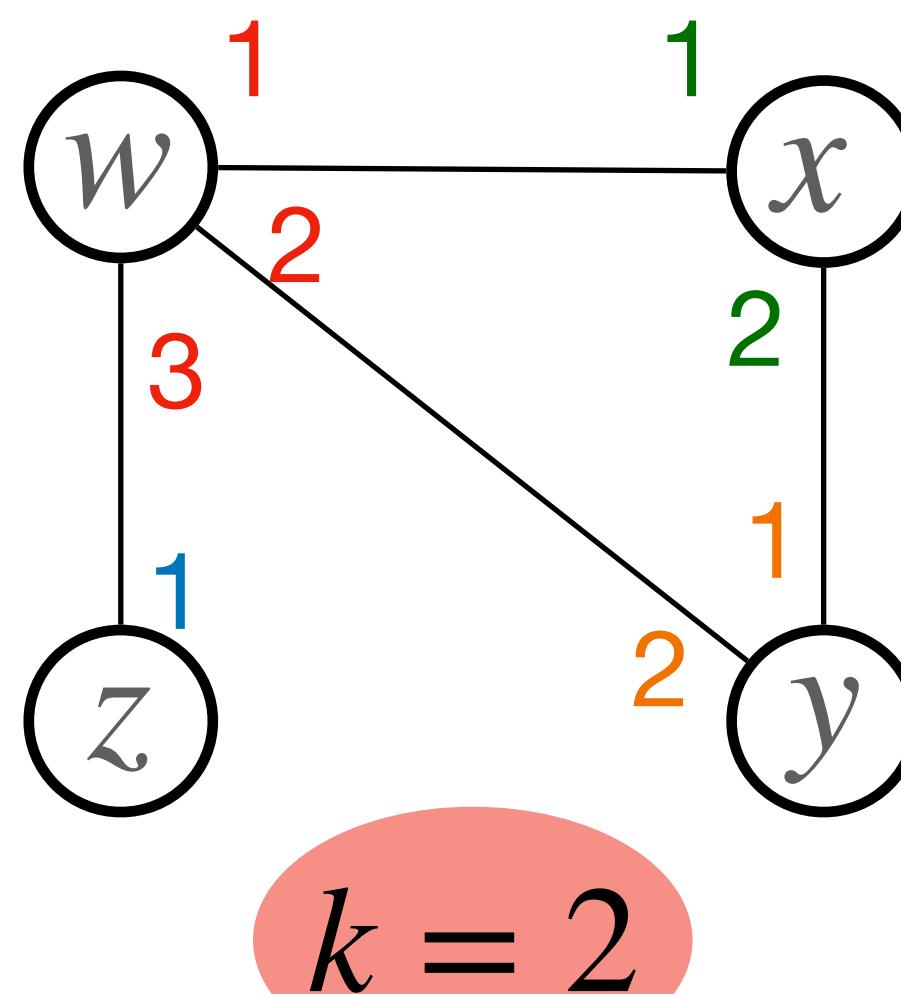
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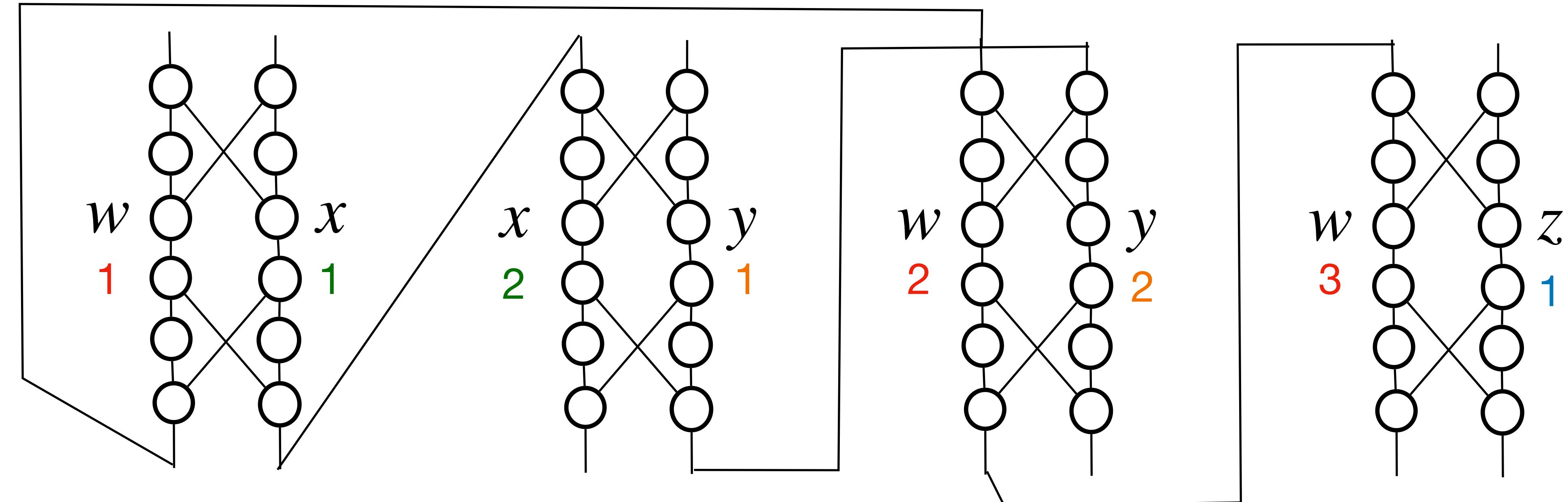
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Now consider the k node selections for the vertex cover.



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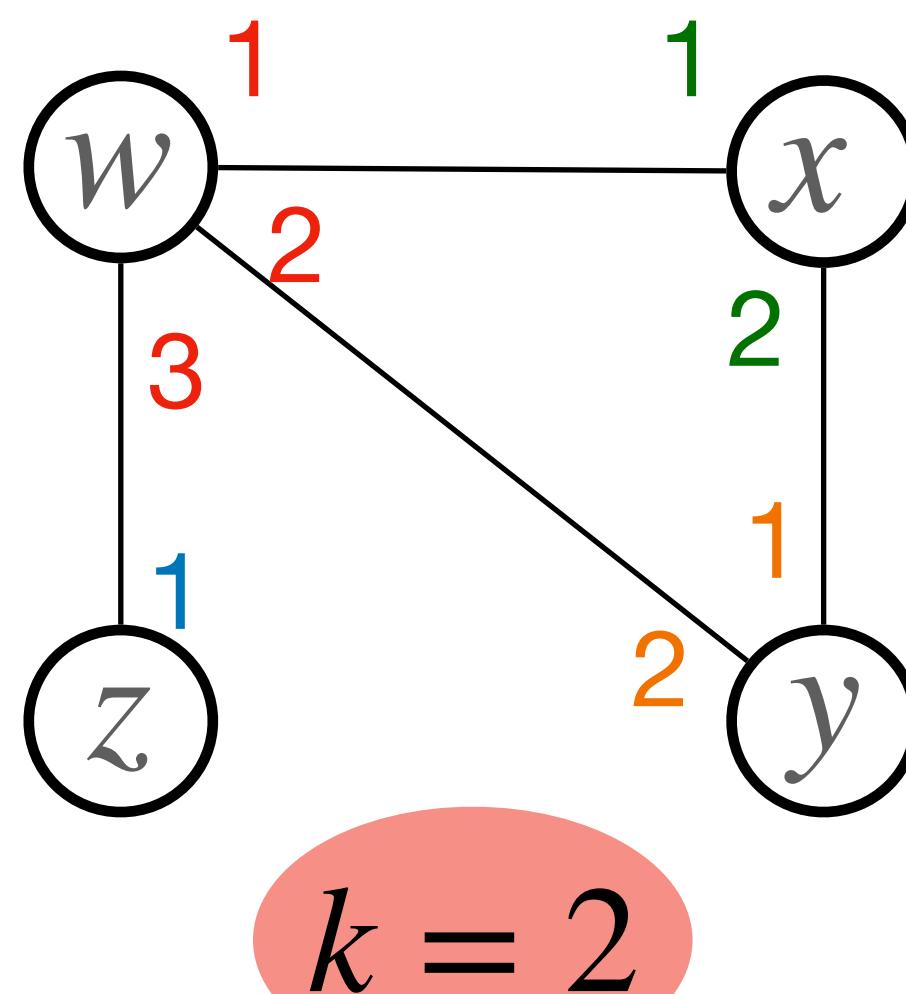
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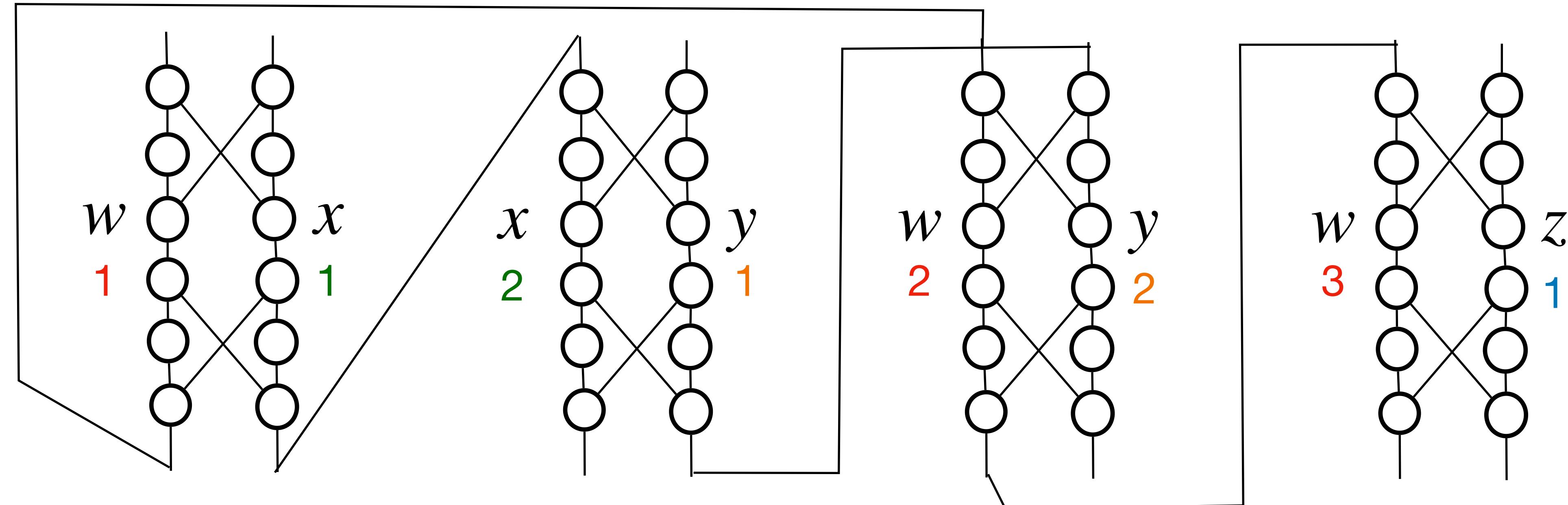


s_1

1st node in vertex cover

s_2

2nd node in vertex cover



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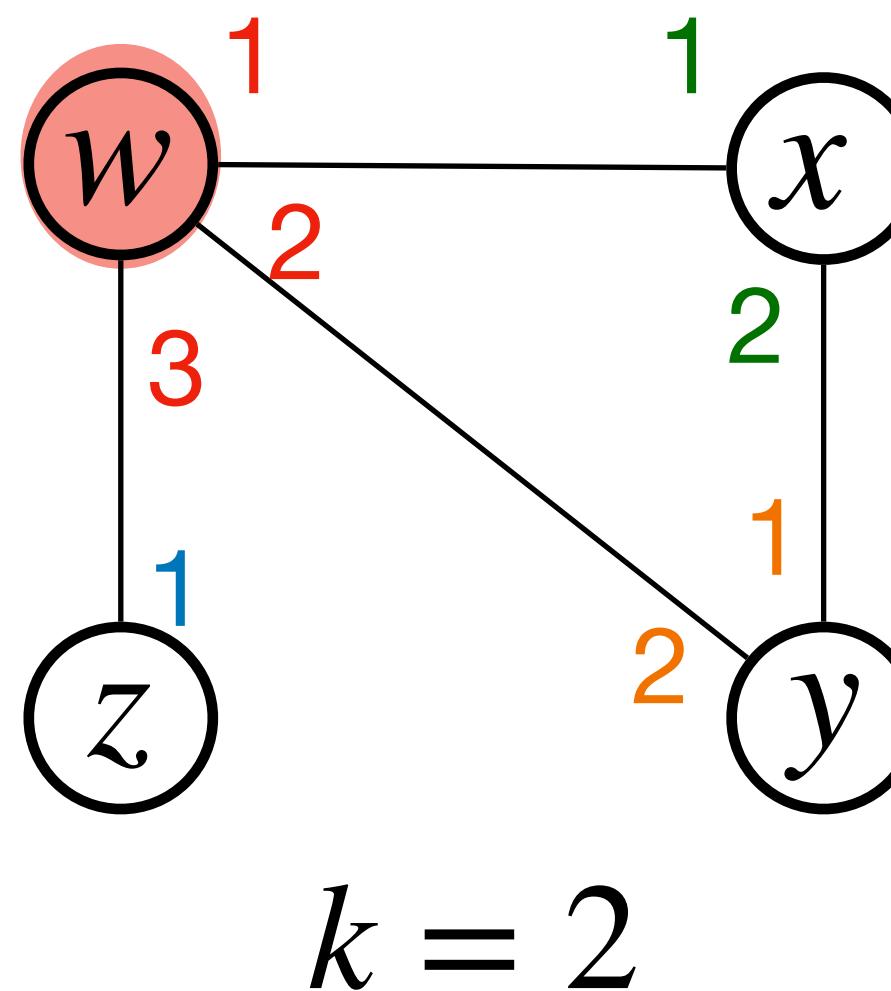
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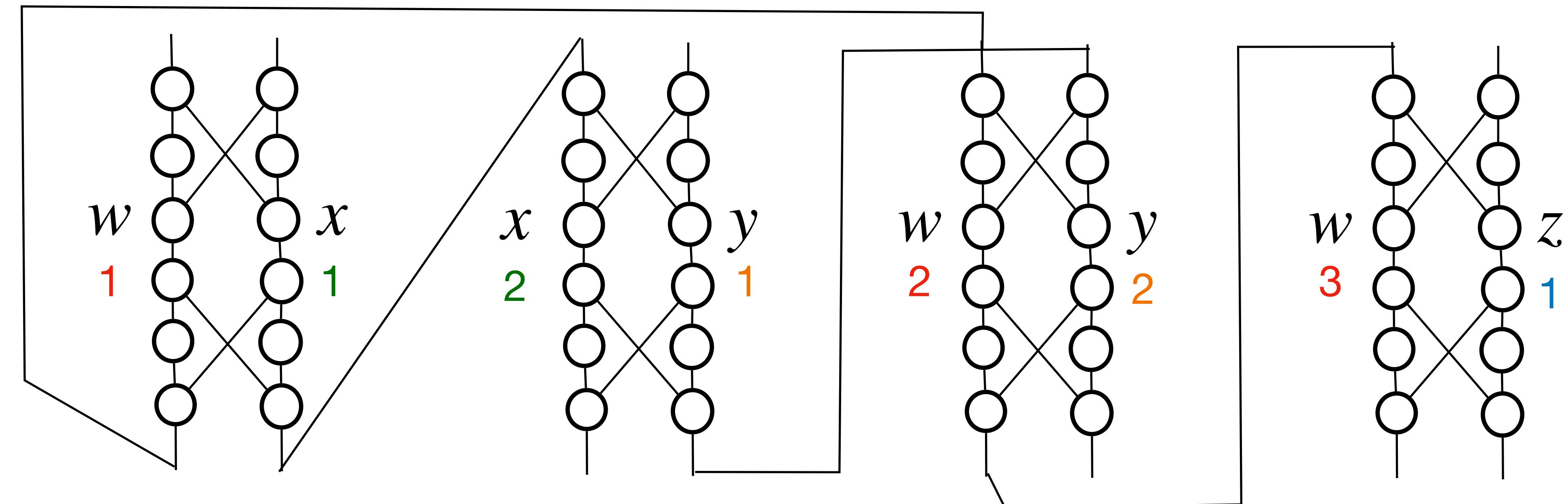
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1st node in vertex cover



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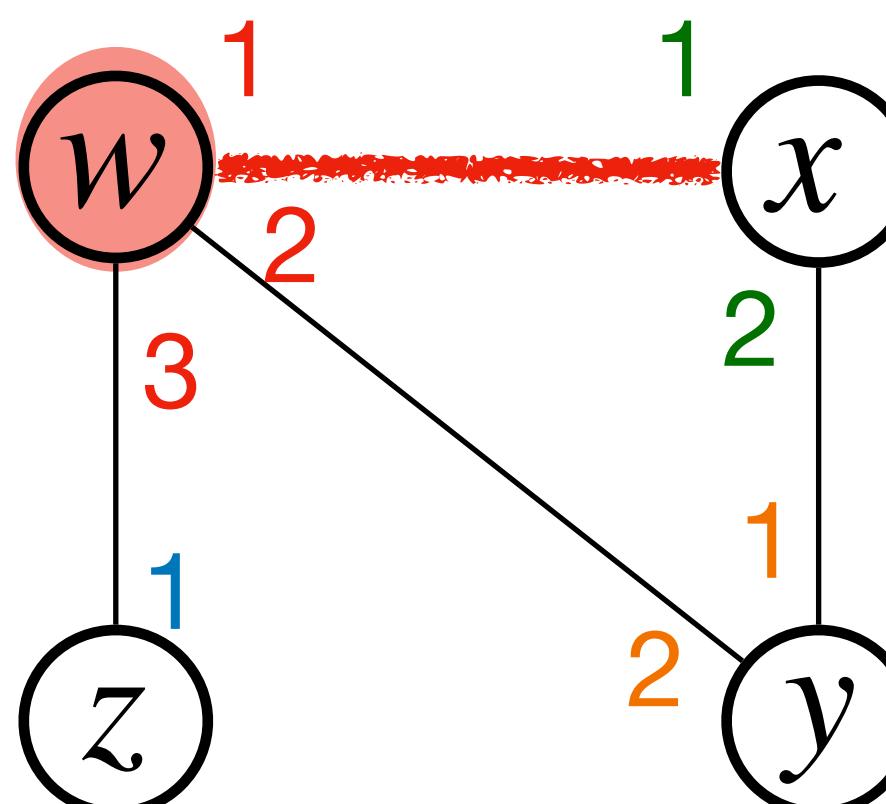
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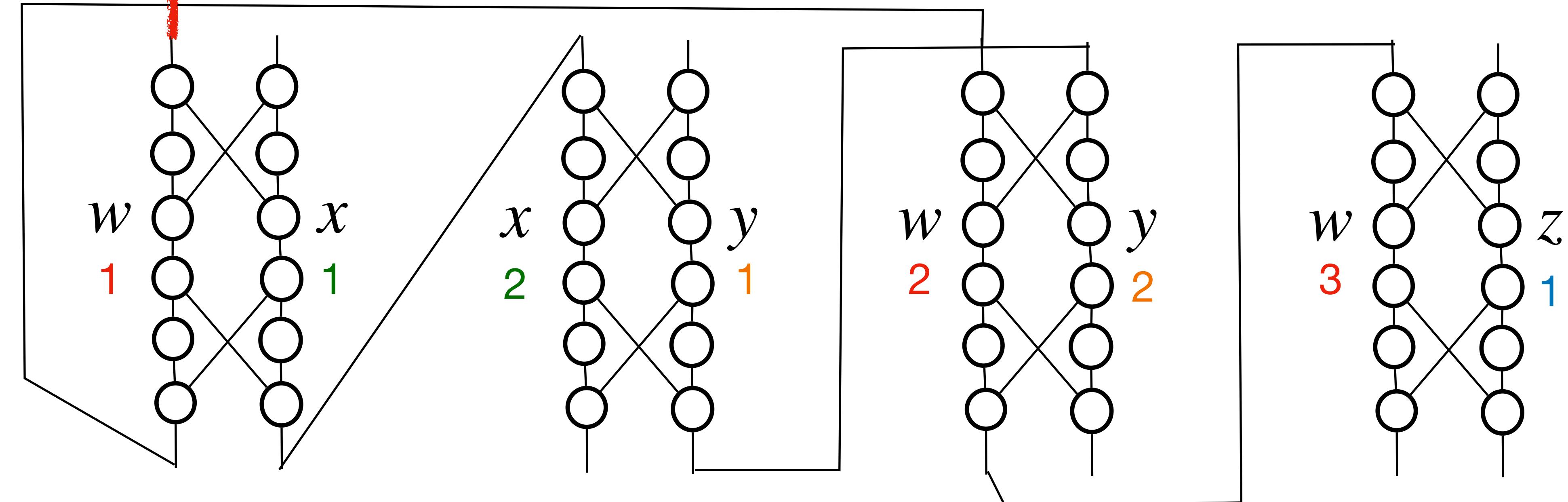
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$k = 2$

1st node in vertex cover

2nd node in vertex cover



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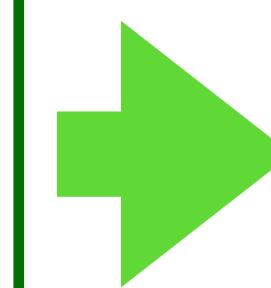
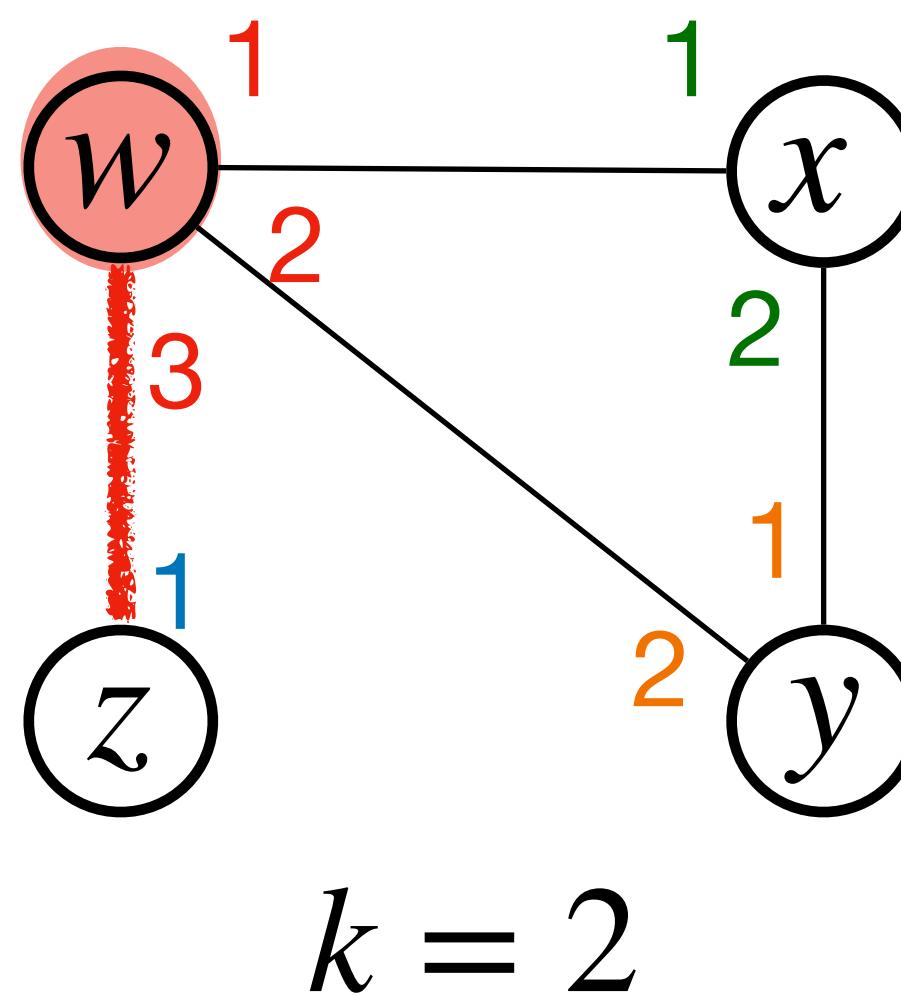
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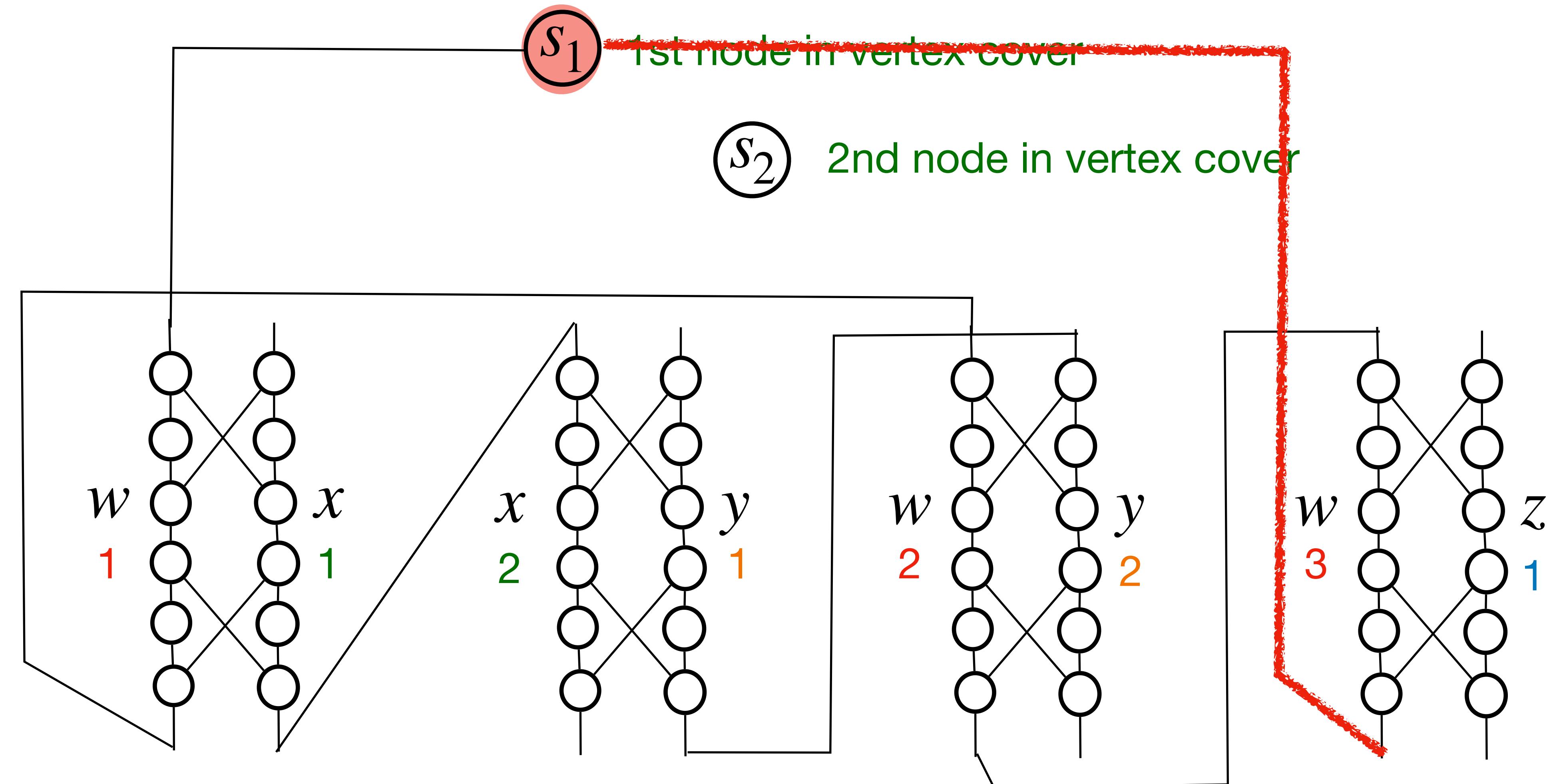
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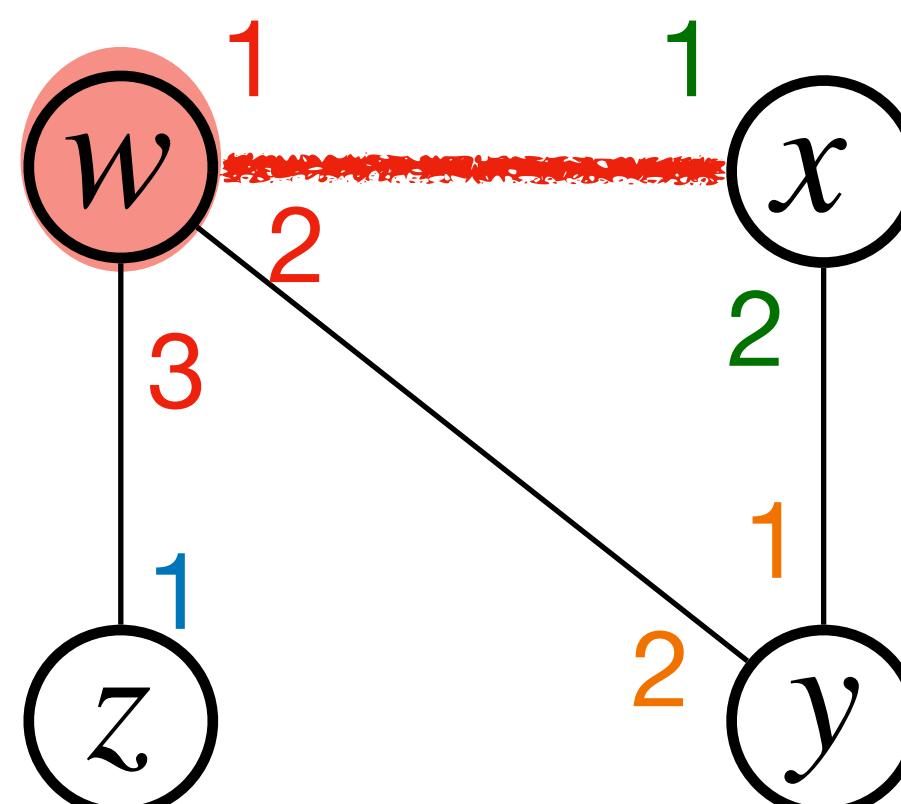
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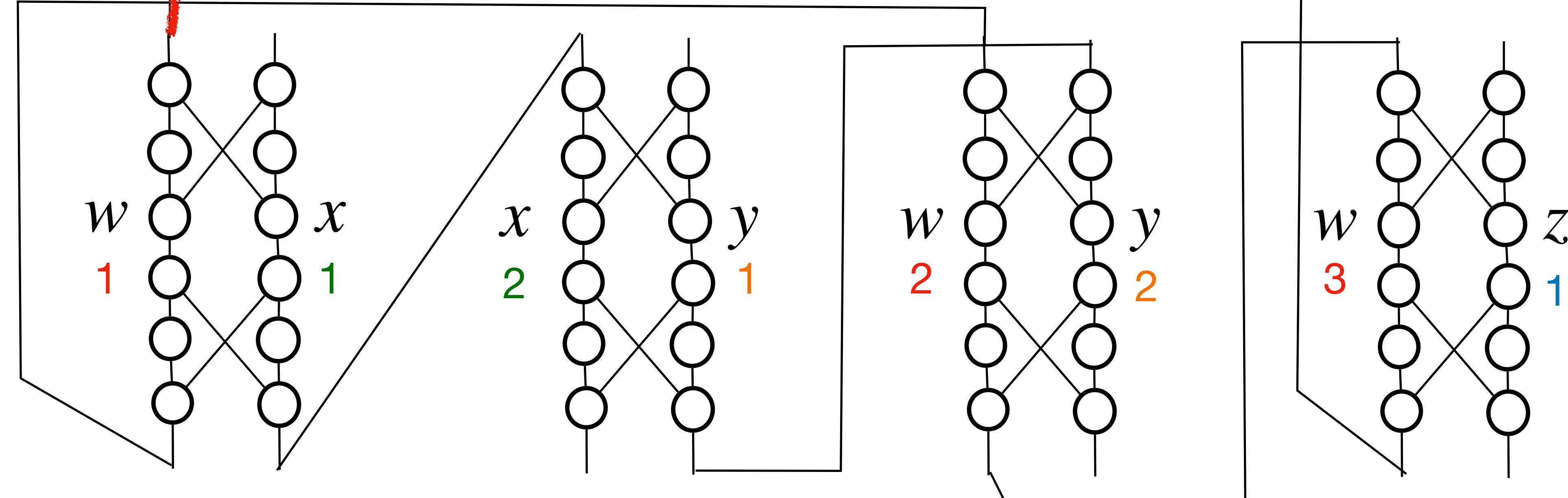
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2nd node in vertex cover

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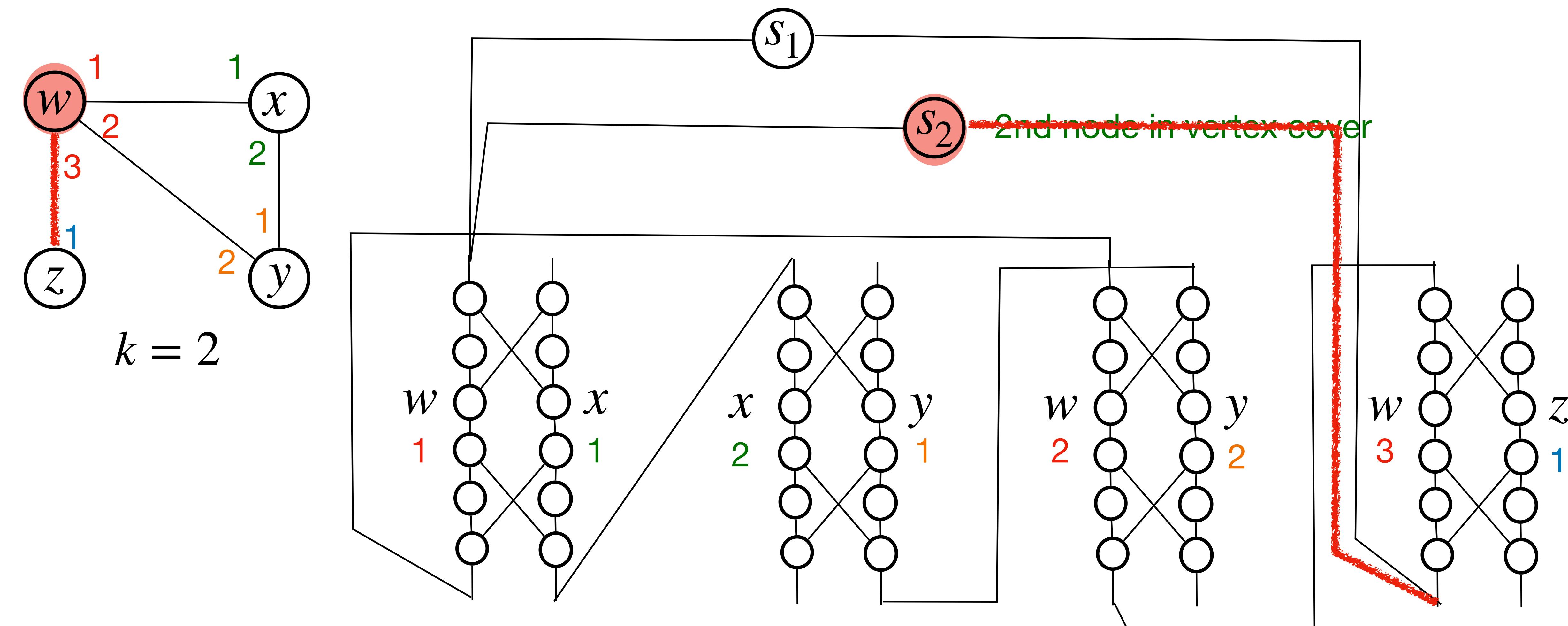
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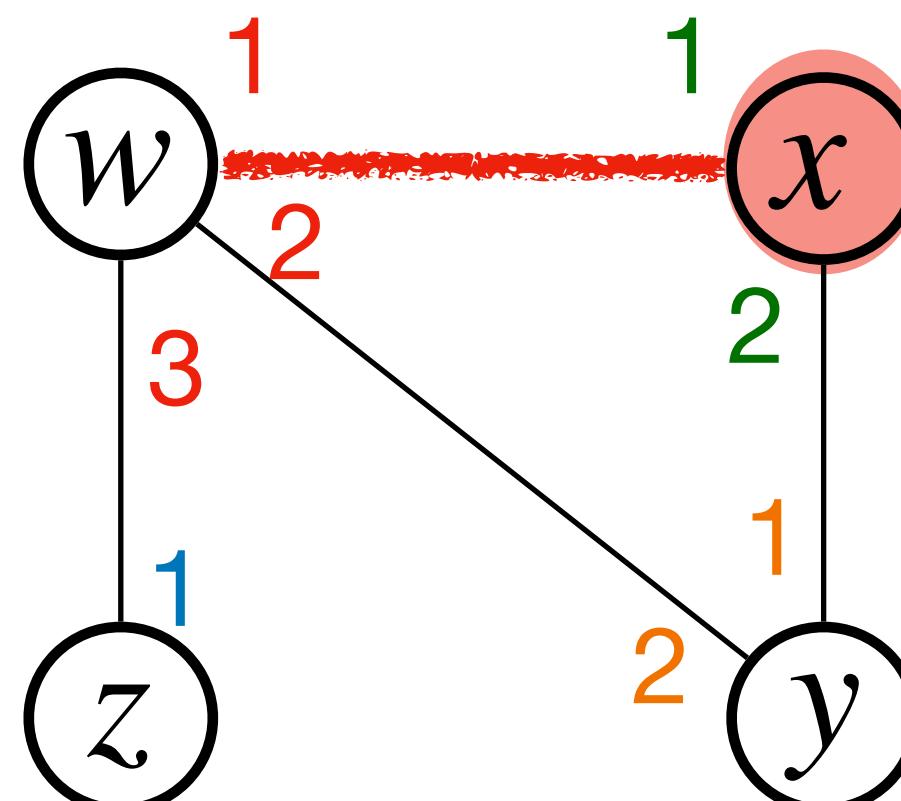
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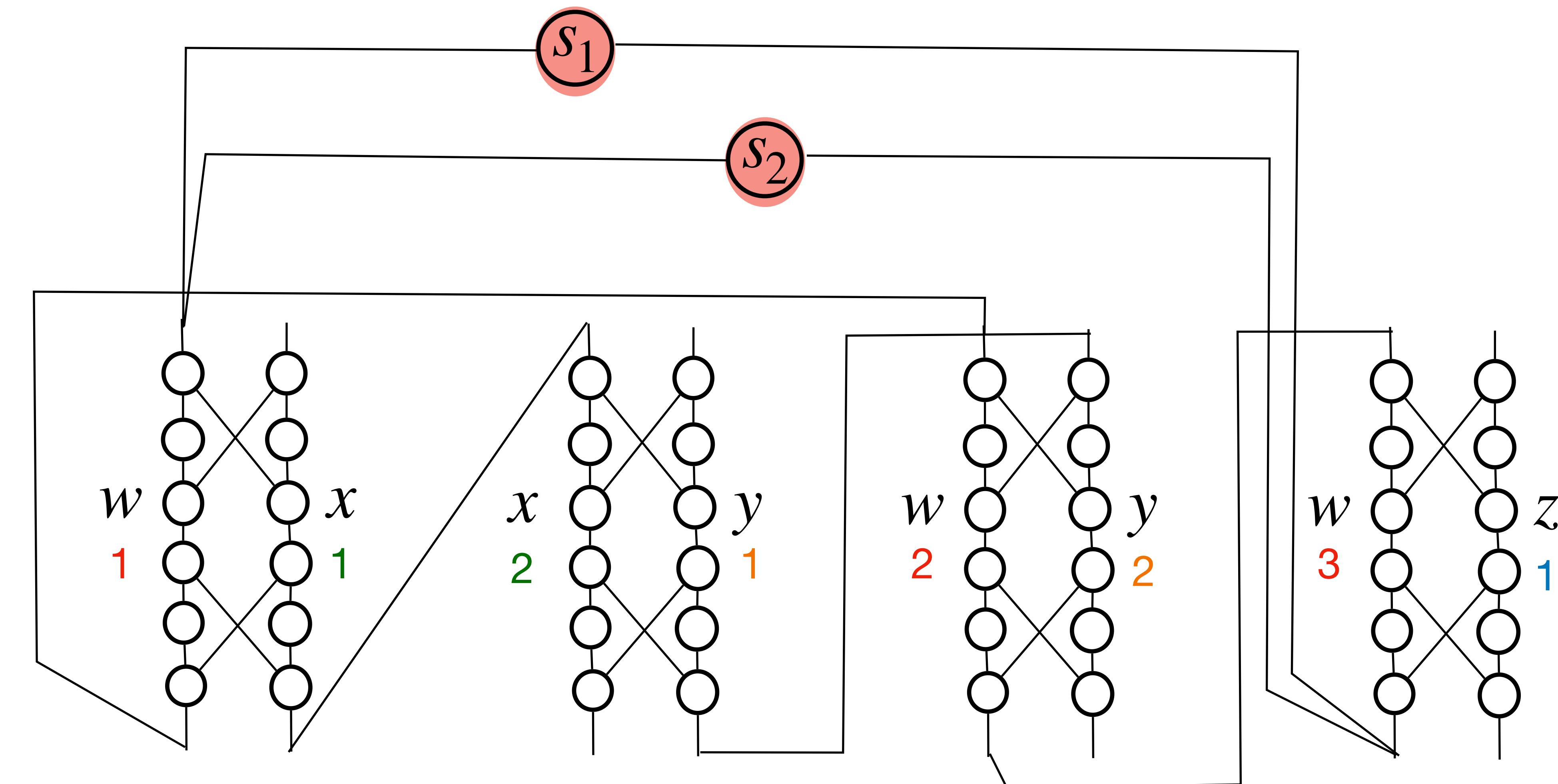
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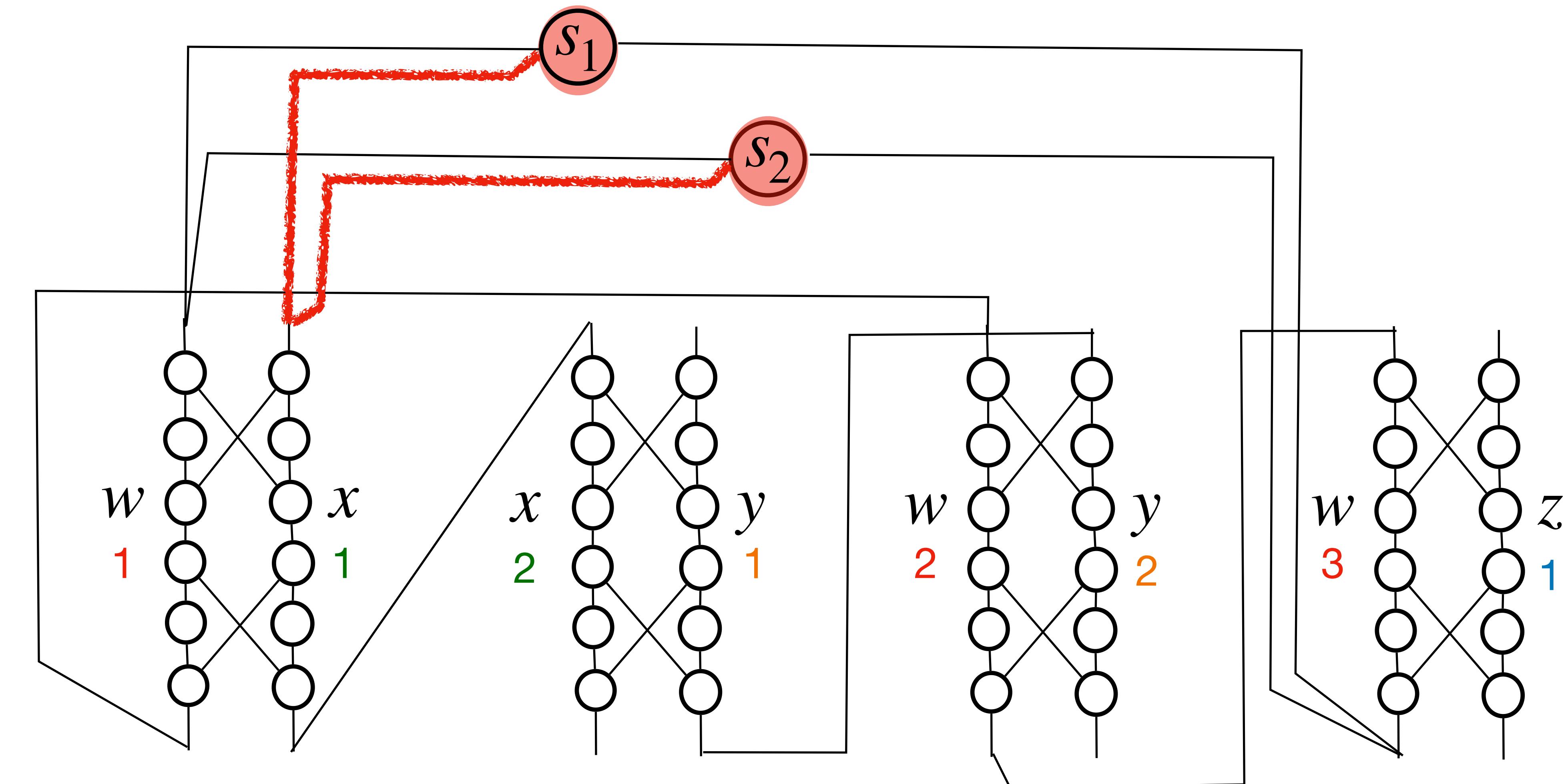
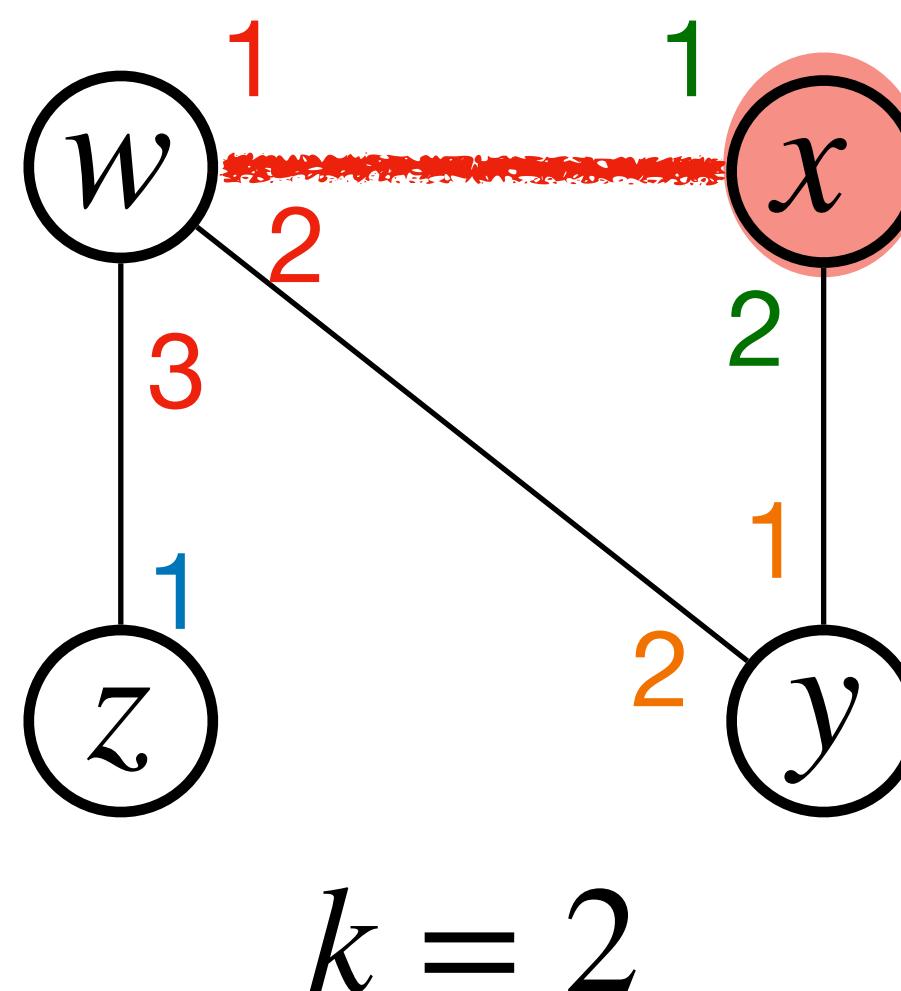
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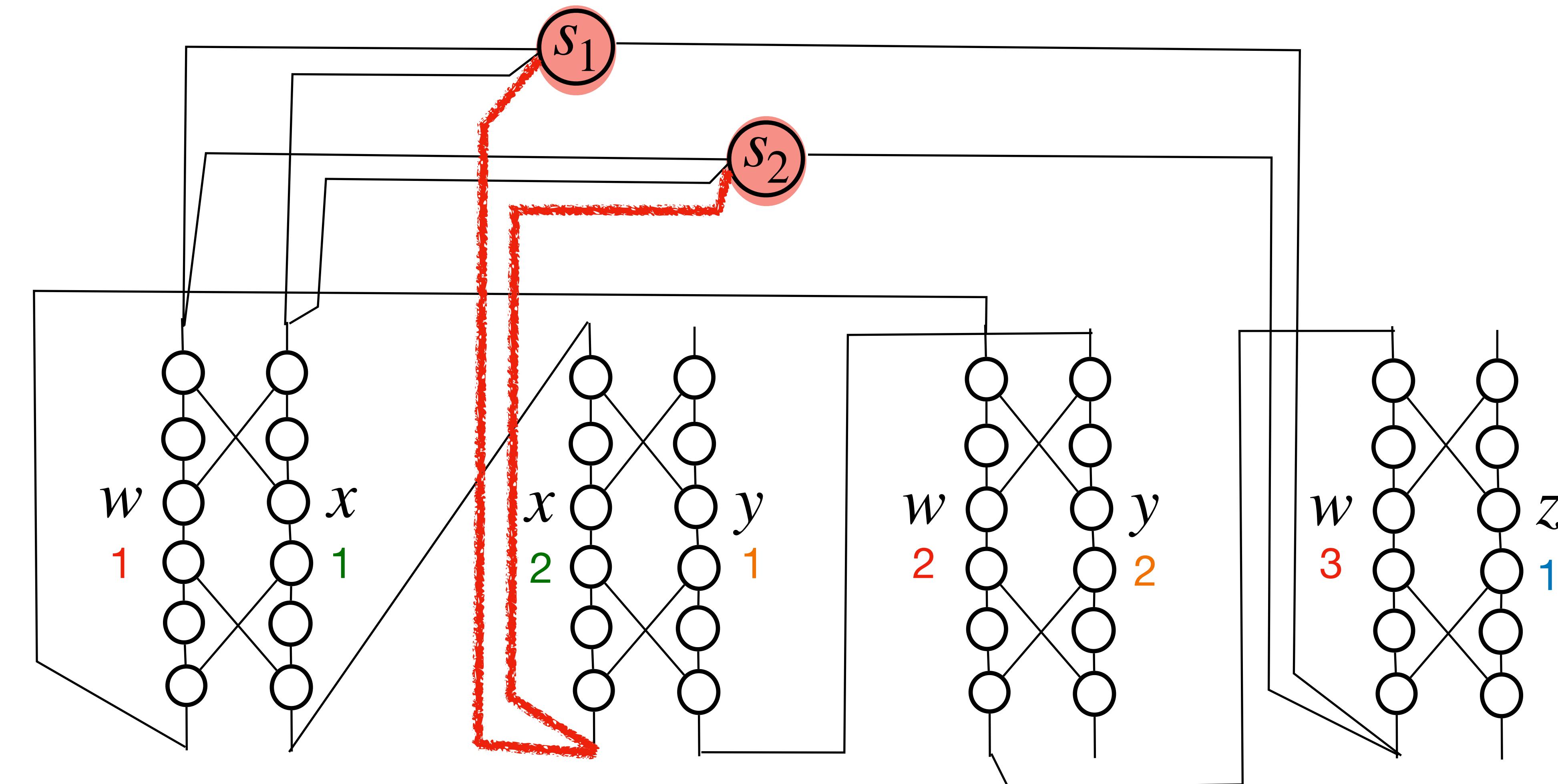
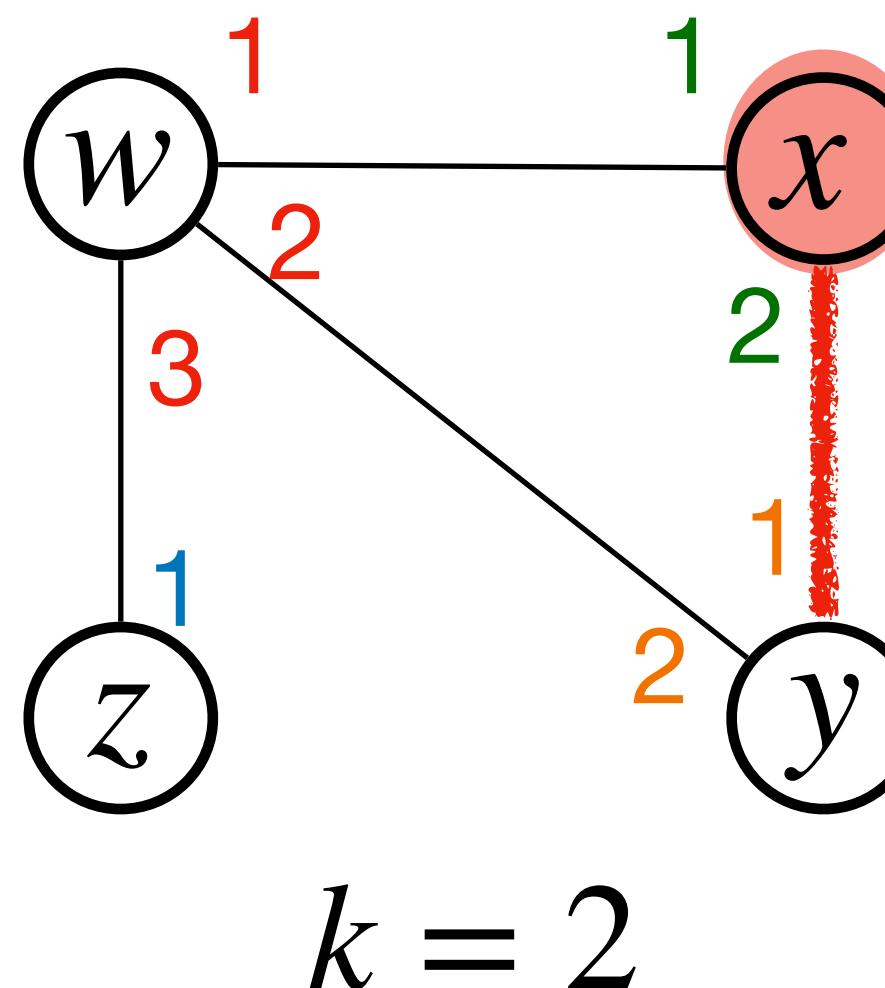
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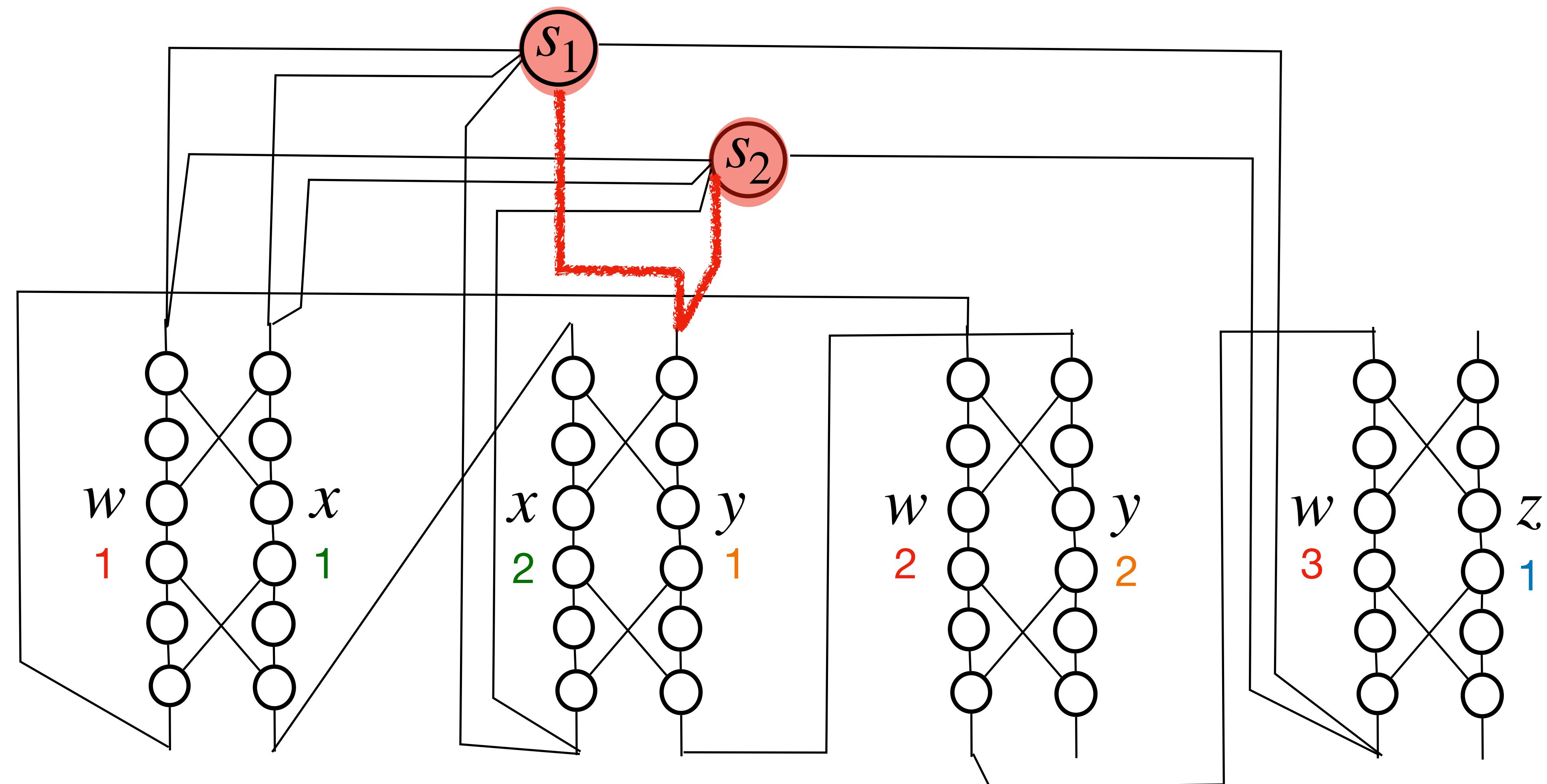
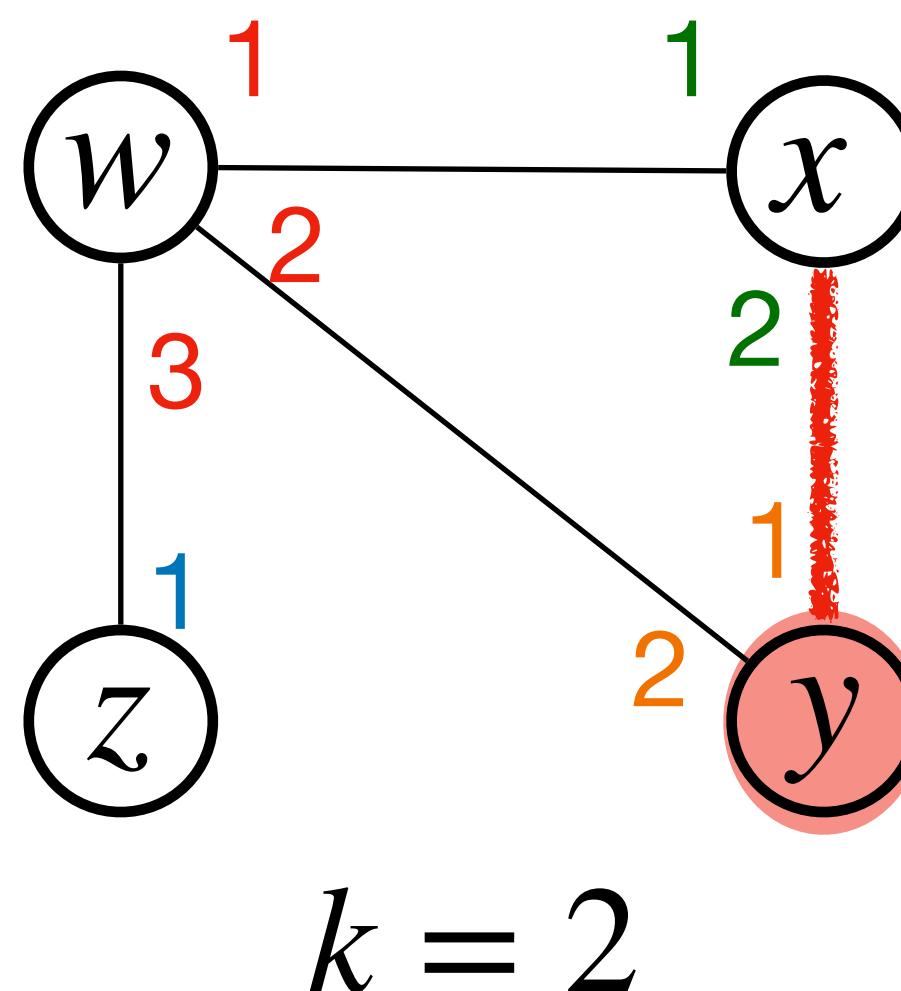
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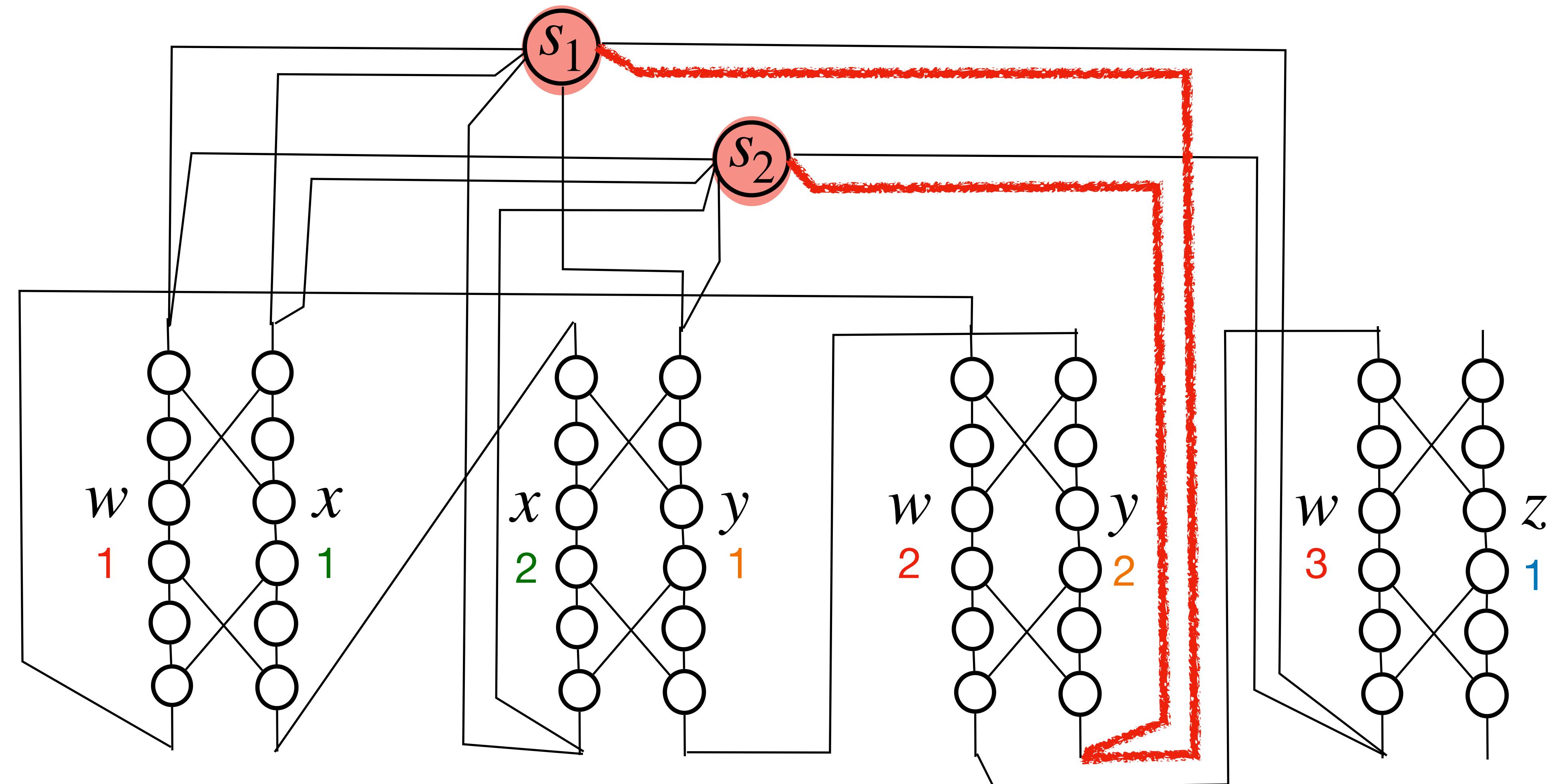
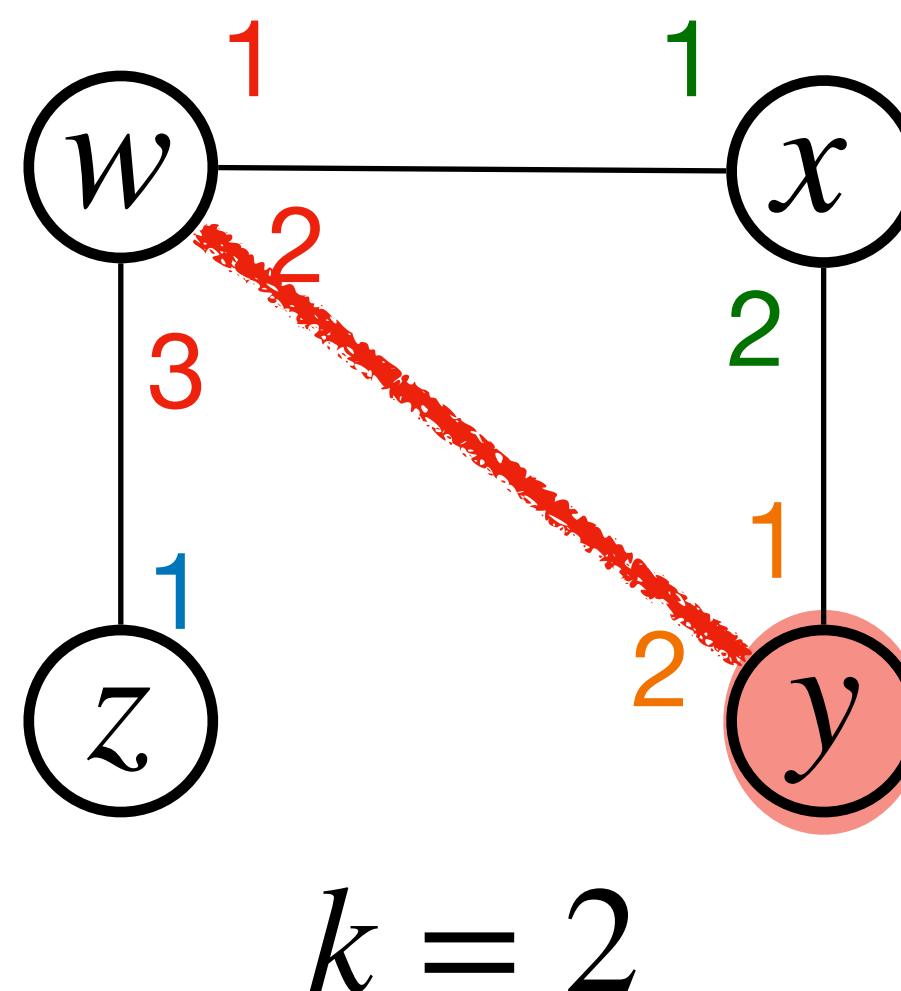
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Vertex Cover Problem

Input: An undirected graph $G = (V, E)$.

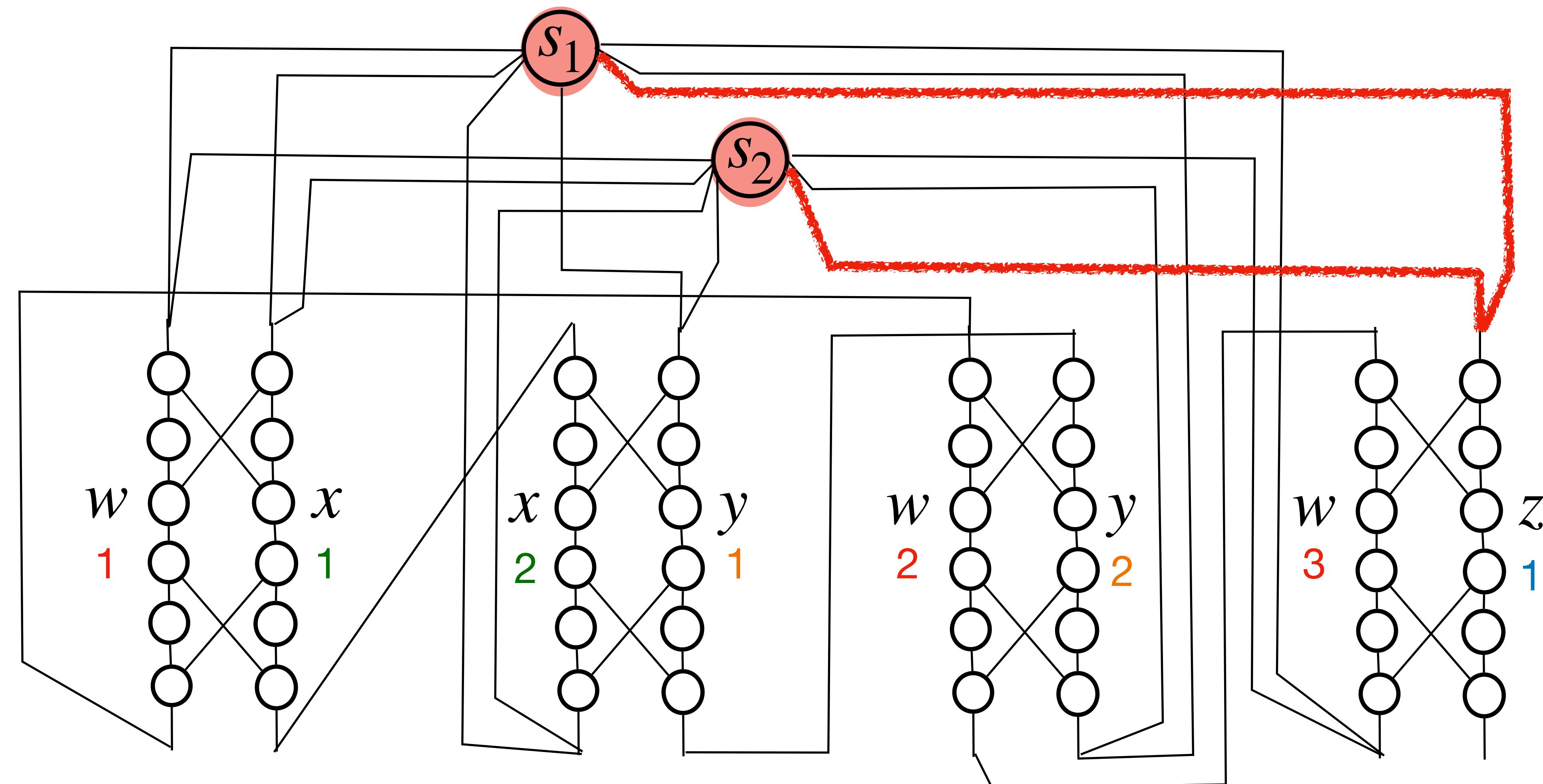
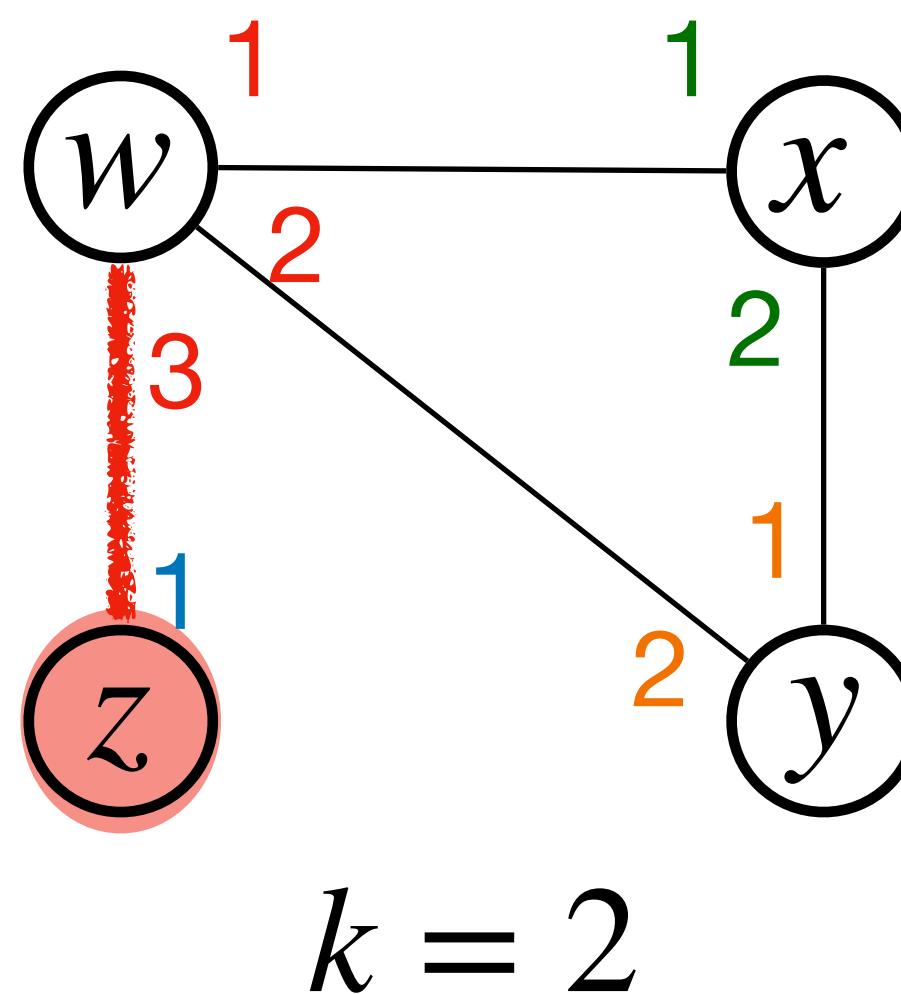
An integer k .

Question: Does G have a vertex cover of size k ?

Hamiltonian cycle Problem:

Input: A graph $G' = (V', E')$.

Question: Does G' have a Hamiltonian cycle?



Vertex Cover Problem

Input: An undirected graph $G = (V, E)$.

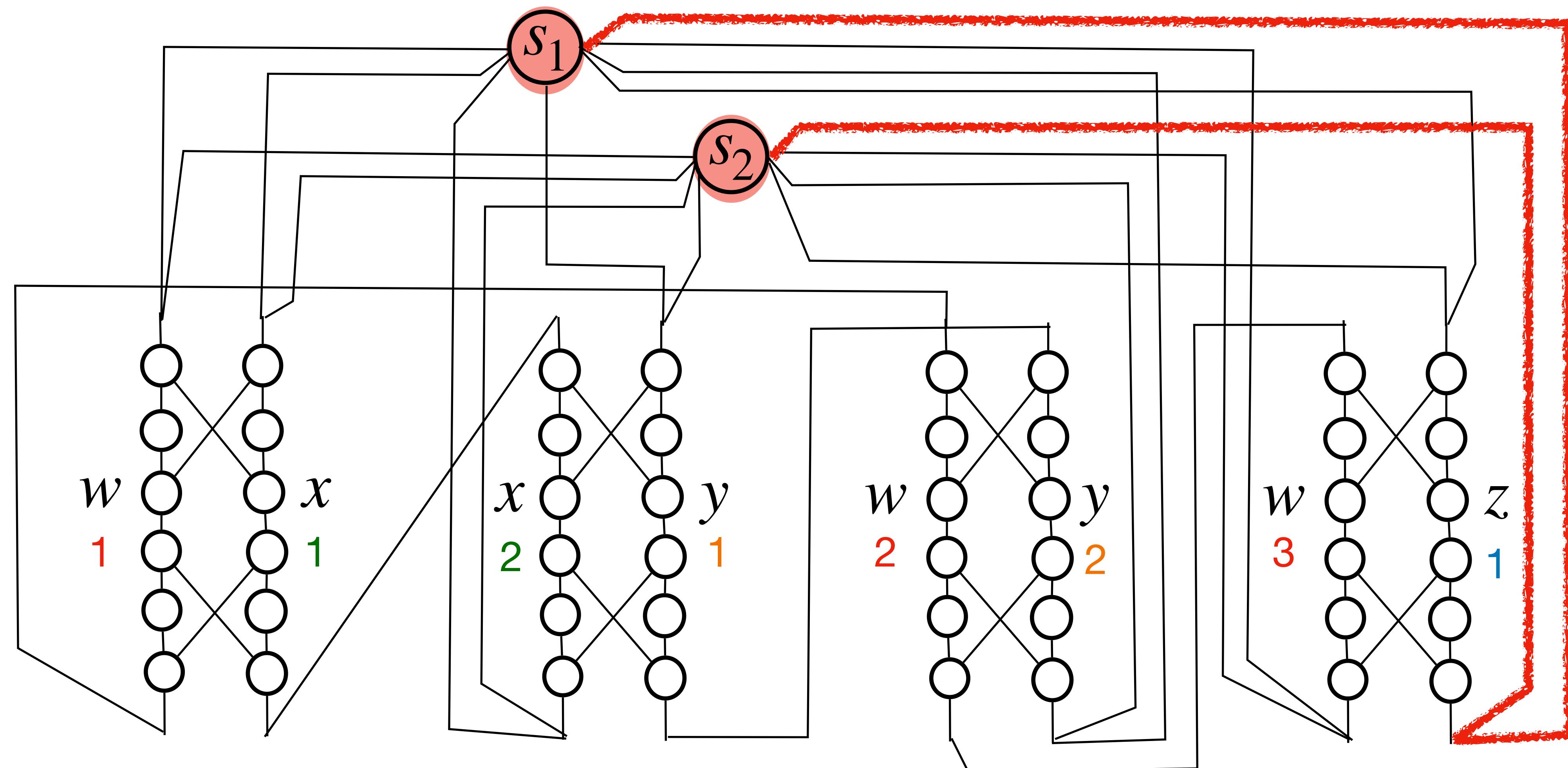
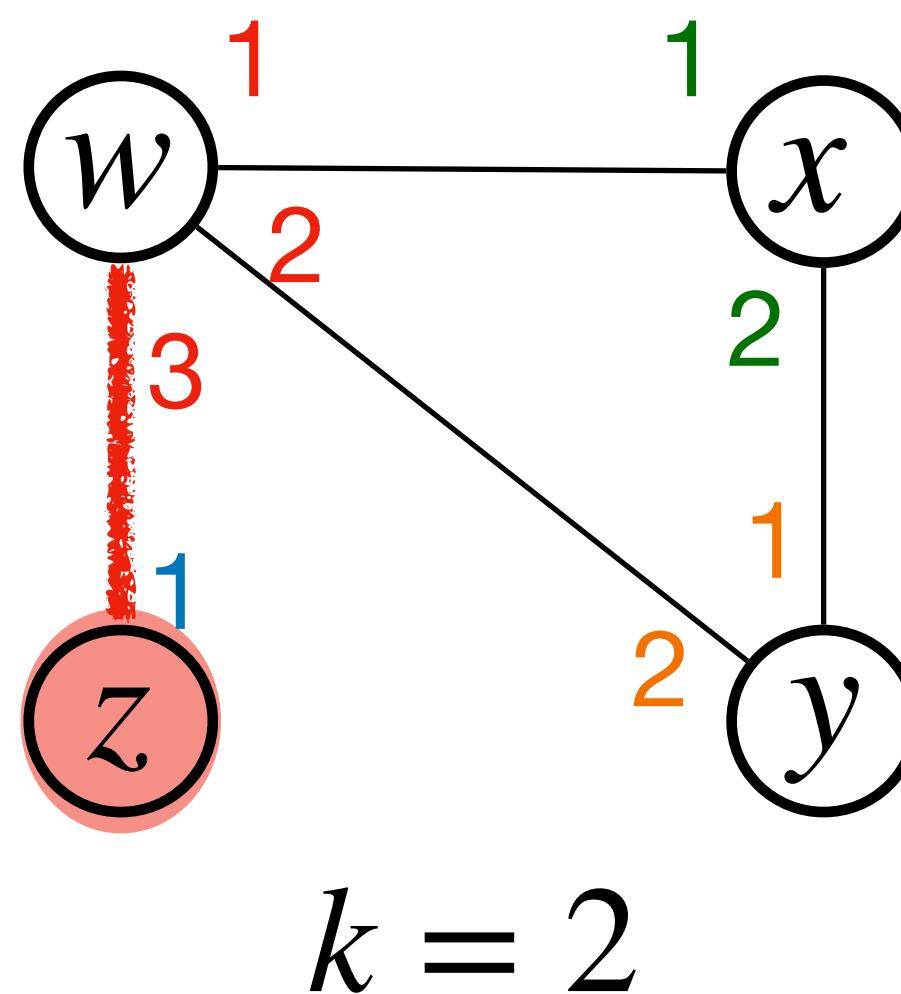
An integer k .

Question: Does G have a vertex cover of size k ?

Hamiltonian cycle Problem:

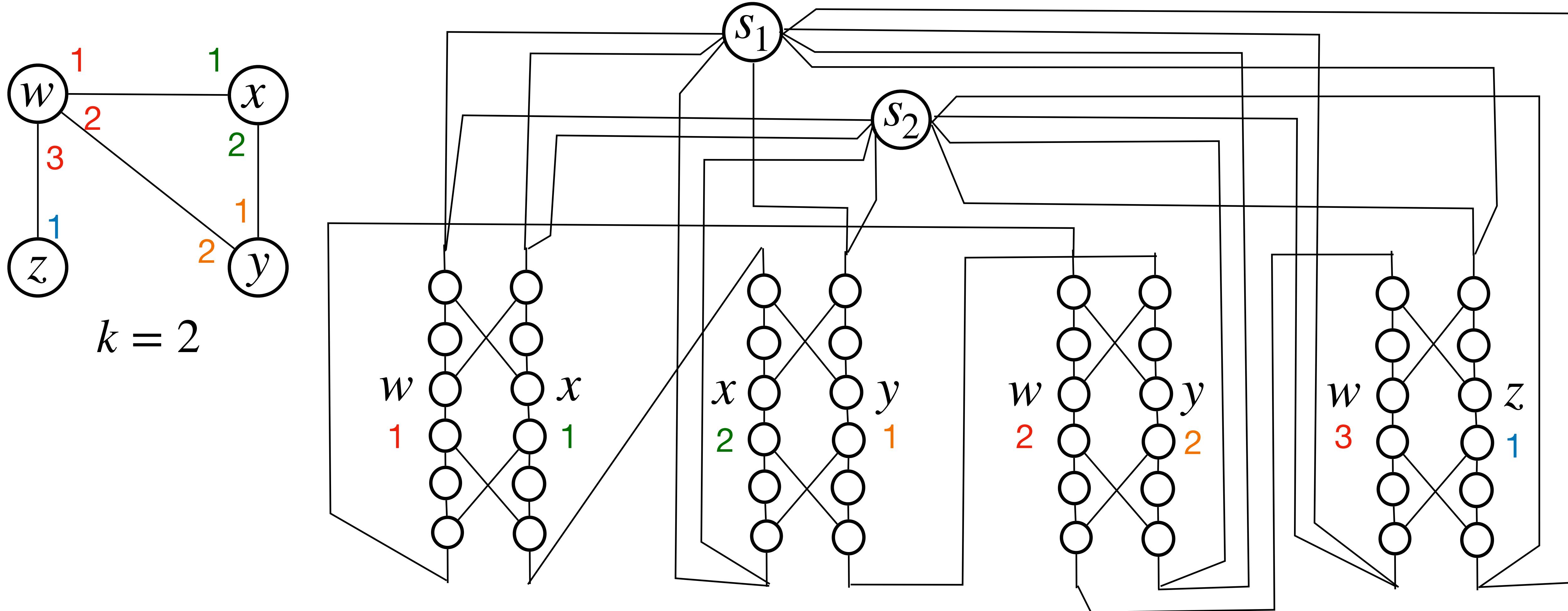
Input: A graph $G' = (V', E')$.

Question: Does G' have a Hamiltonian cycle?



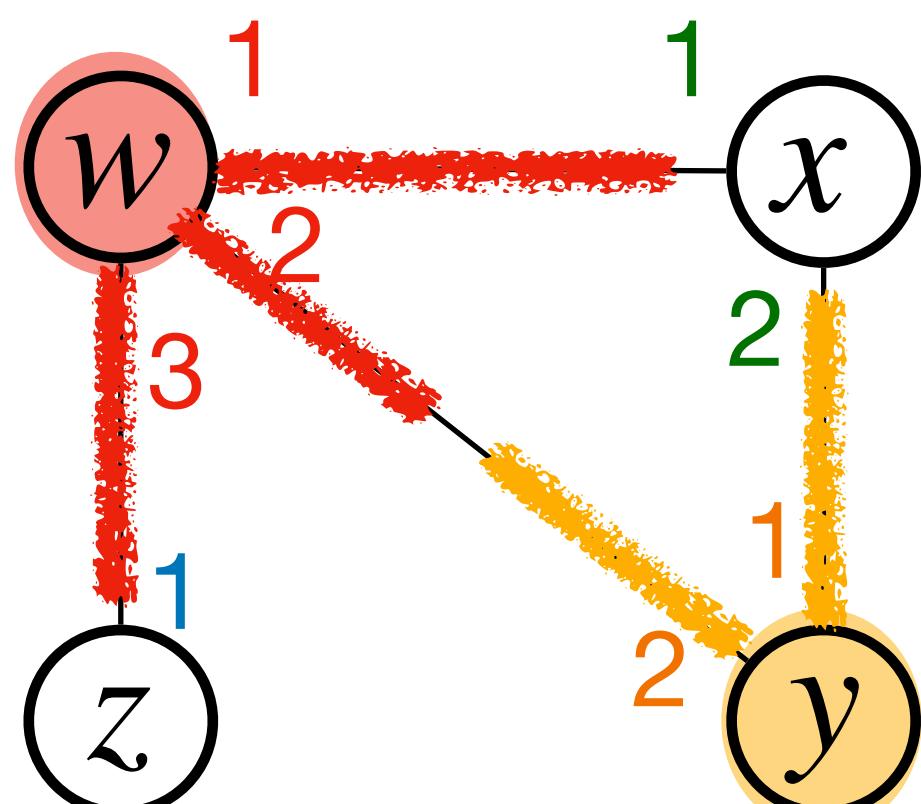
The mapping from the “Vertex Cover Problem” to the “Hamiltonian Cycle Problem” is a polynomial-time mapping.

Now let's show that it preserves the YES/NO answer.

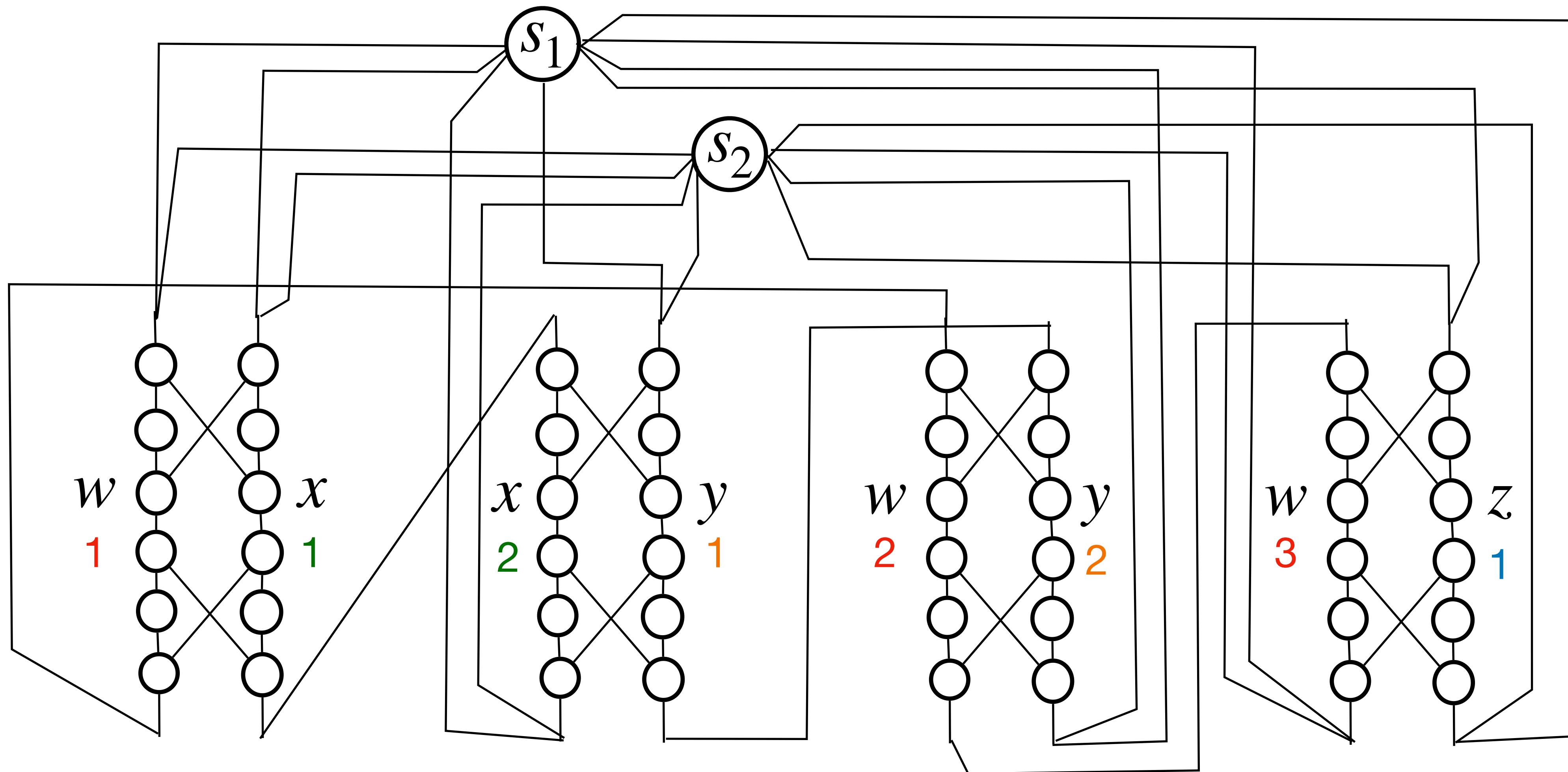


Assume “YES” for the Vertex Cover Problem.

Assume the vertex cover is $\{w, y\}$.



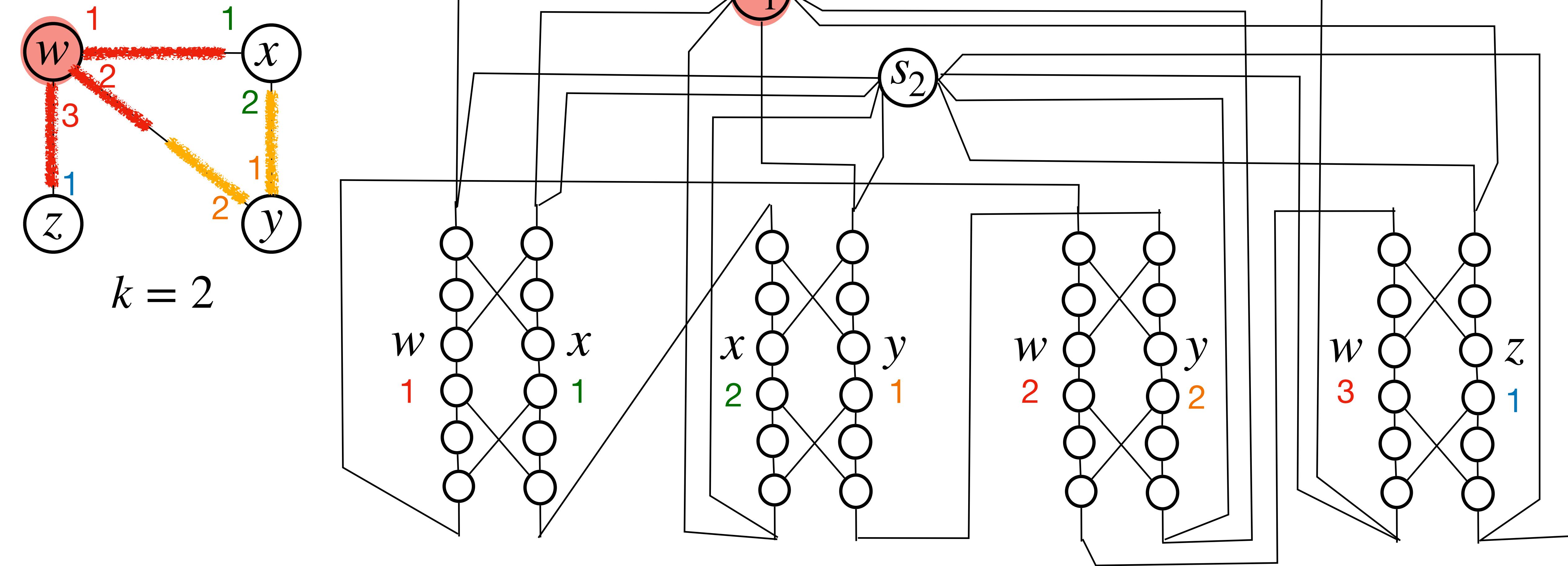
$$k = 2$$



Assume “YES” for the Vertex Cover Problem.

Assume the vertex cover is $\{w, y\}$.

Select 1st node in the vertex cover: w

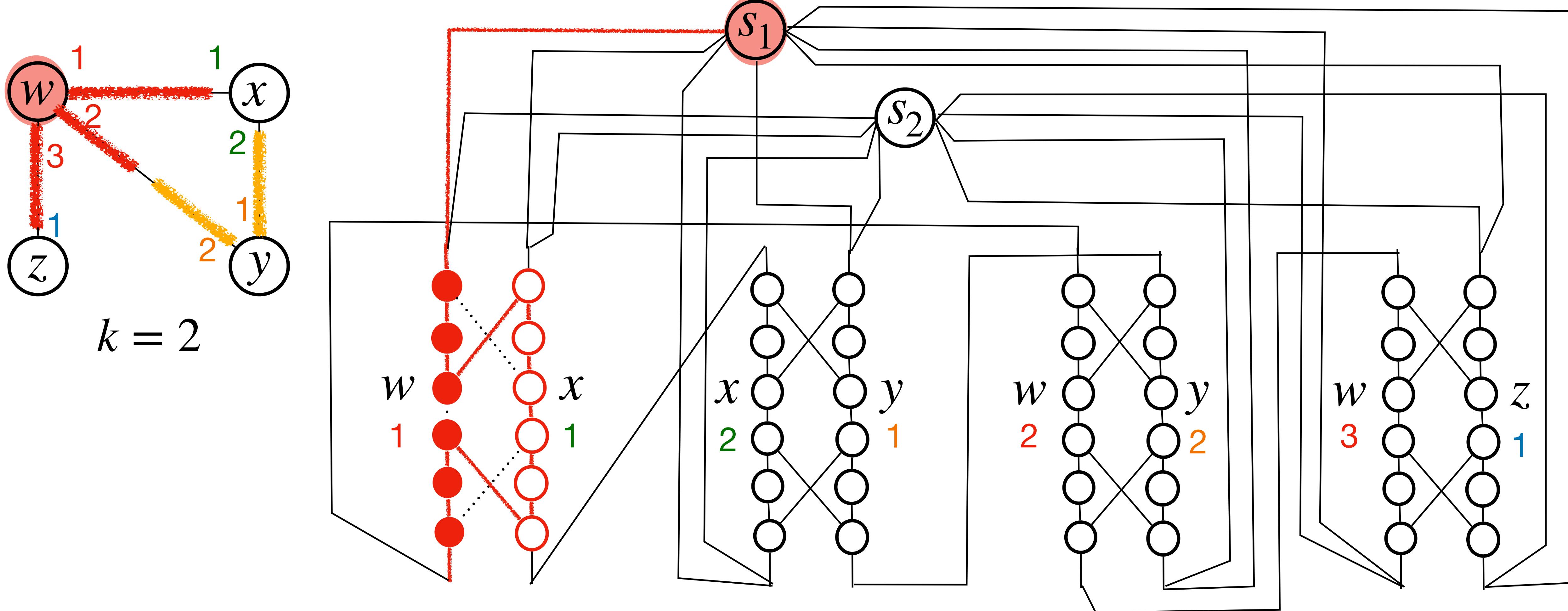


Assume “YES” for the Vertex Cover Problem.

Assume the vertex cover is $\{w,y\}$.

Select 1st node in the vertex cover: w

How does w cover its 1st edge (w,x) ?

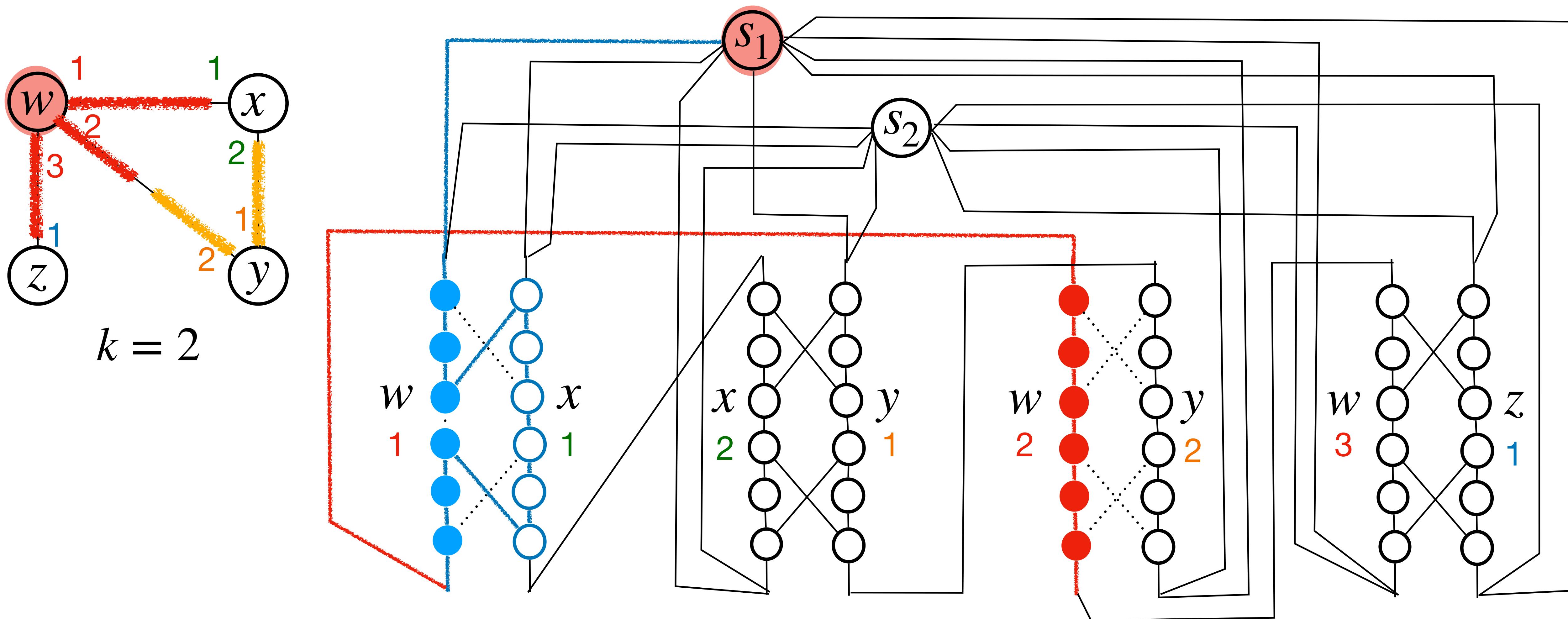


Assume “YES” for the Vertex Cover Problem.

Assume the vertex cover is $\{w,y\}$.

Select 1st node in the vertex cover: w

How does w cover its 2nd edge (w,y) ?

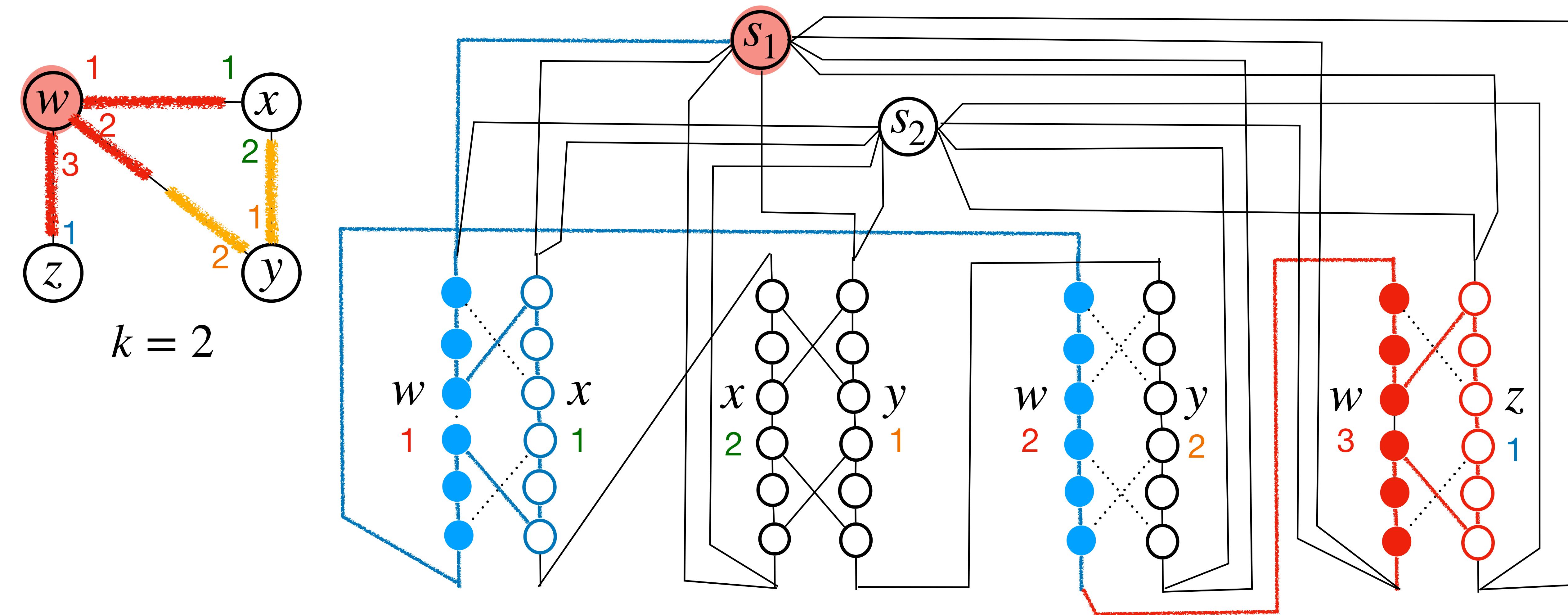


Assume “YES” for the Vertex Cover Problem.

Assume the vertex cover is $\{w,y\}$.

Select 1st node in the vertex cover: w

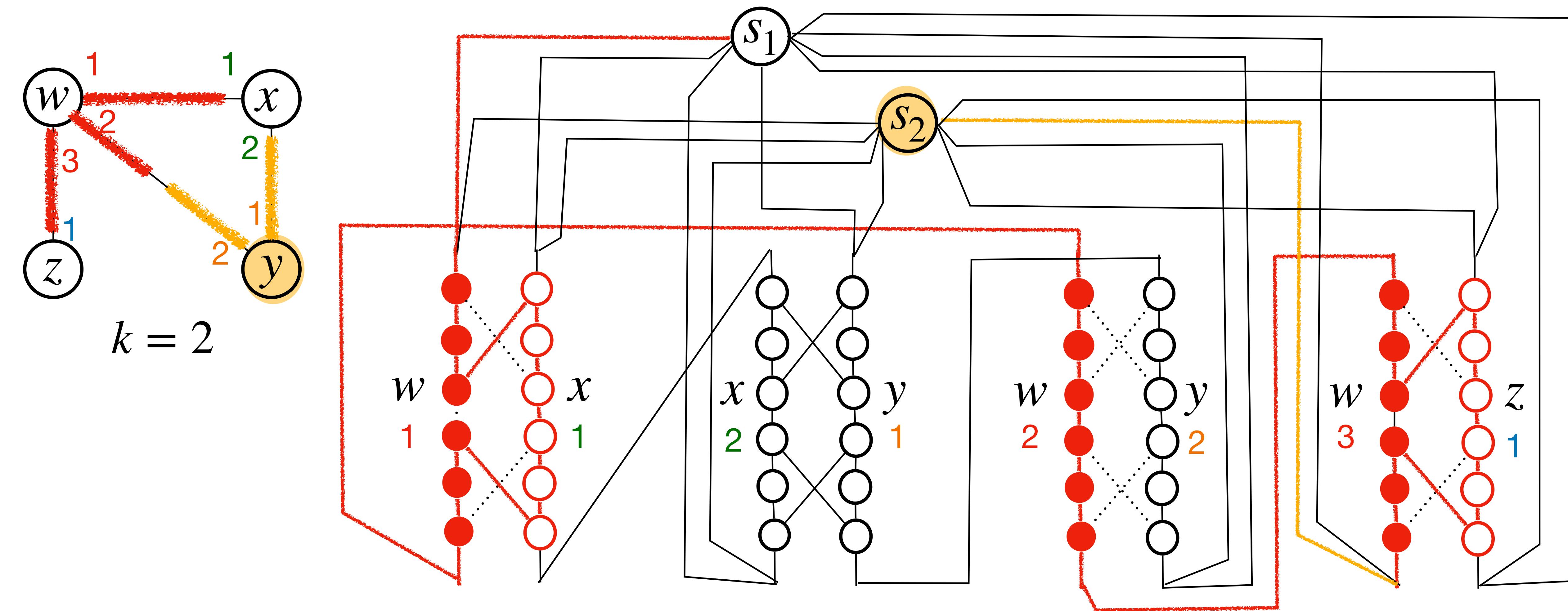
How does w cover its 3rd edge (w,z) ?



Assume “YES” for the Vertex Cover Problem.

Assume the vertex cover is $\{w, y\}$.

Select 2nd node in the vertex cover: y

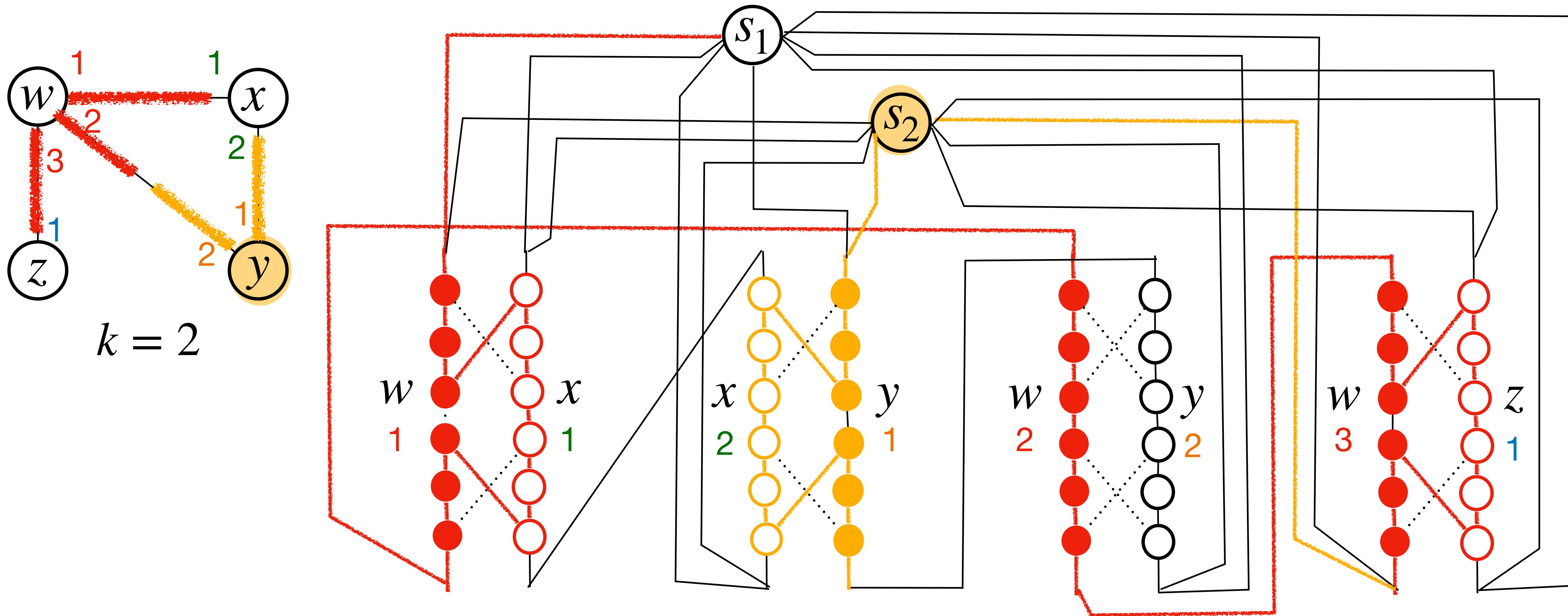


Assume “YES” for the Vertex Cover Problem.

Assume the vertex cover is {w,y}.

Select 2nd node in the vertex cover: y

How does y cover its 1st edge (y,x)?

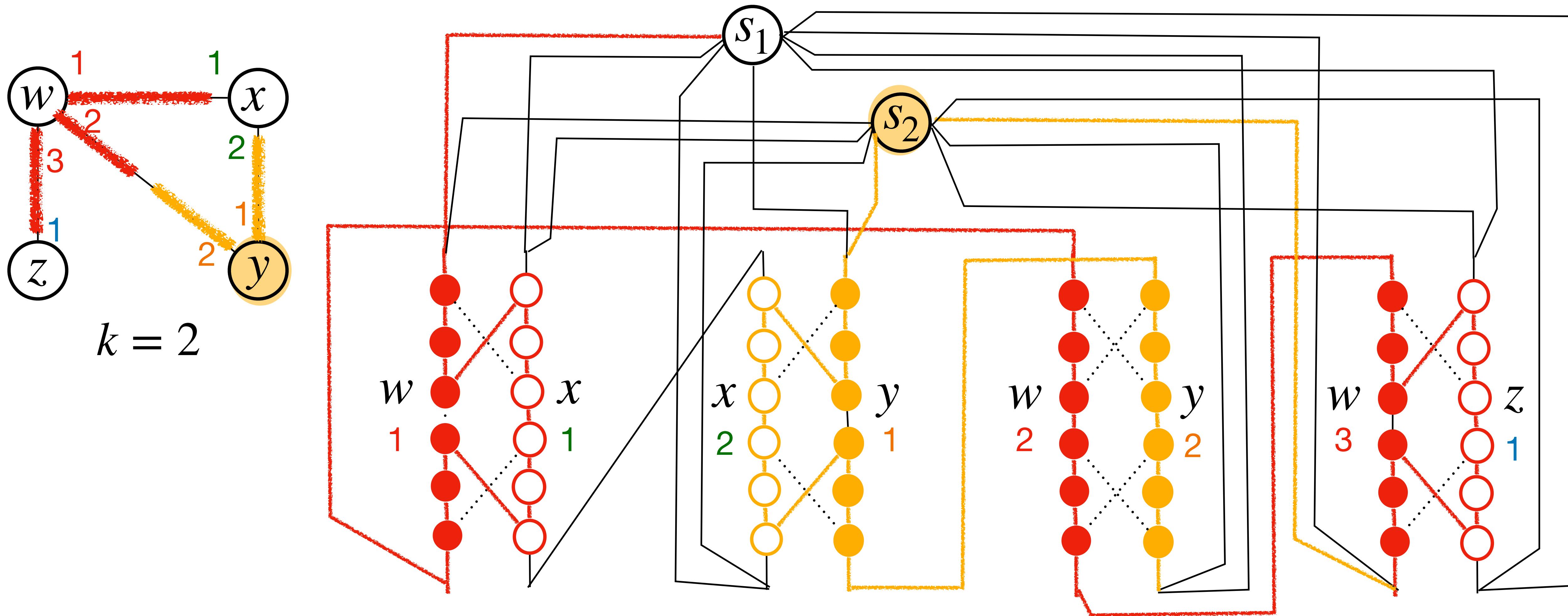


Assume “YES” for the Vertex Cover Problem.

Assume the vertex cover is {w,y}.

Select 2nd node in the vertex cover: y

How does y cover its 2nd edge (y,w)?



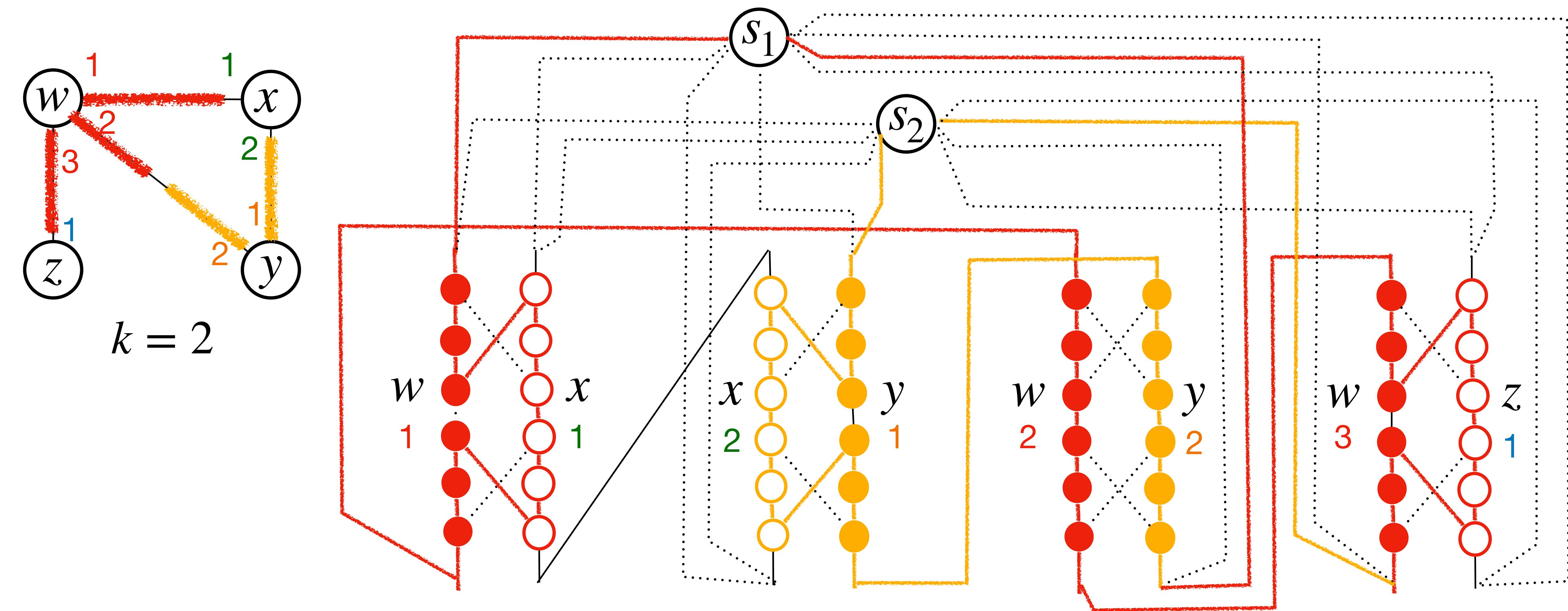
Assume “YES” for the Vertex Cover Problem.

Assume the vertex cover is $\{w,y\}$.

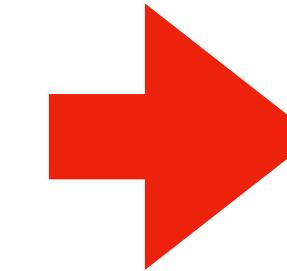
All edges are covered.

All nodes are passed through exactly once.

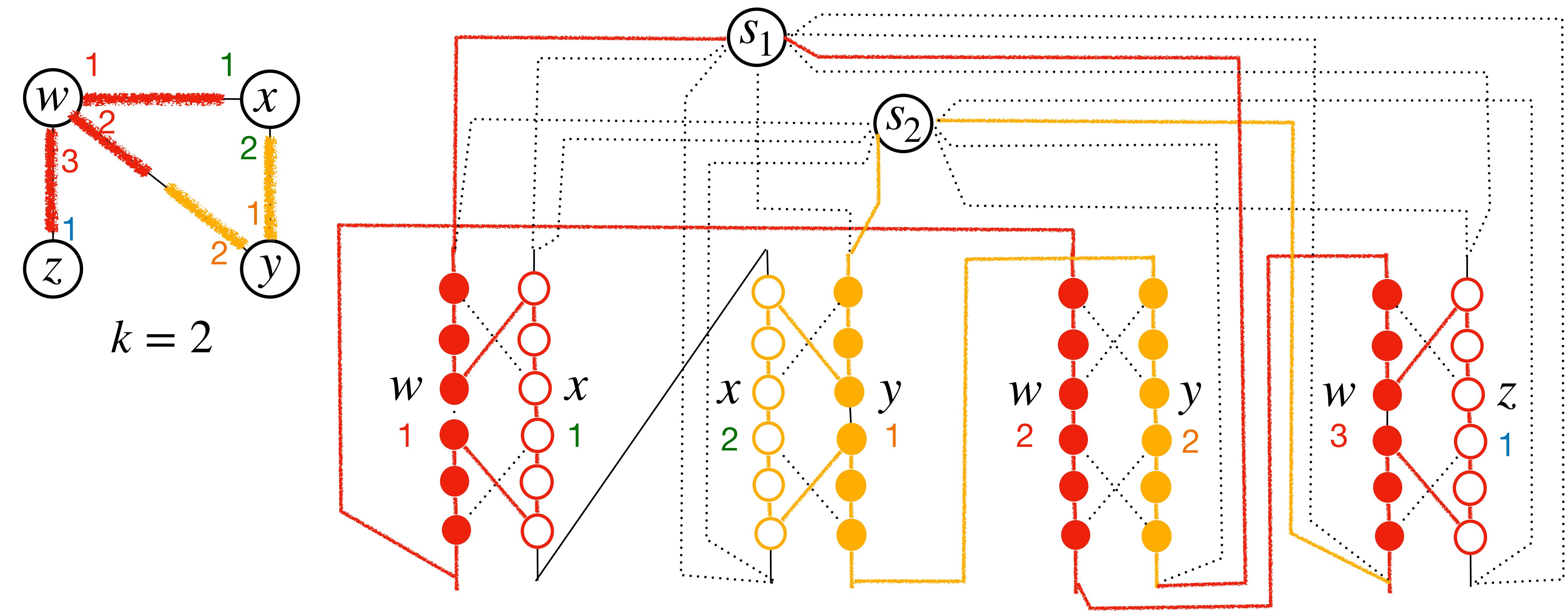
We got a Hamiltonian cycle!



“YES” for the Vertex Cover Problem

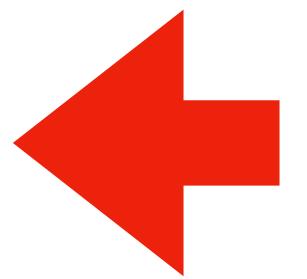


“YES” for Hamiltonian Cycle Problem.



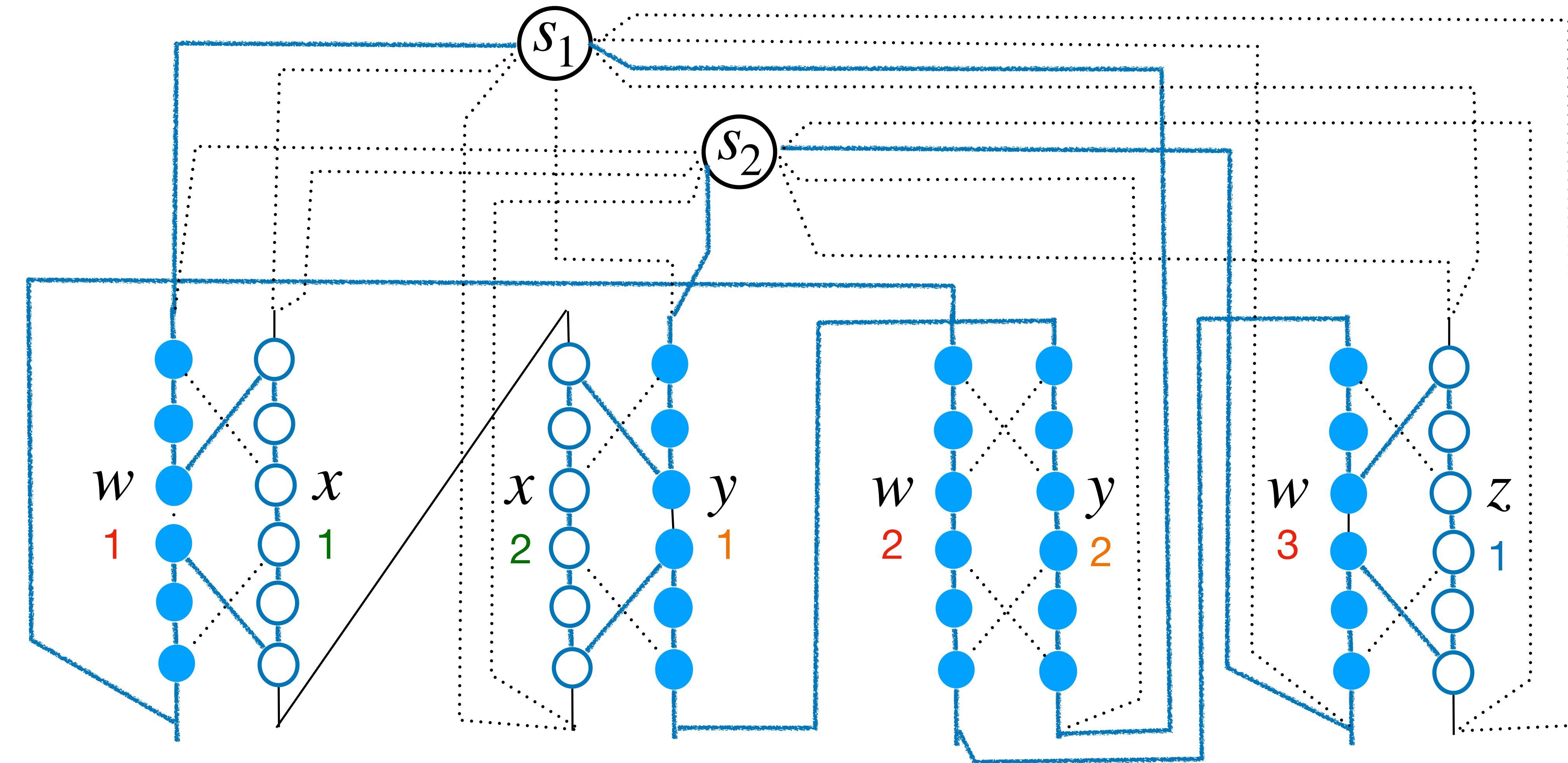
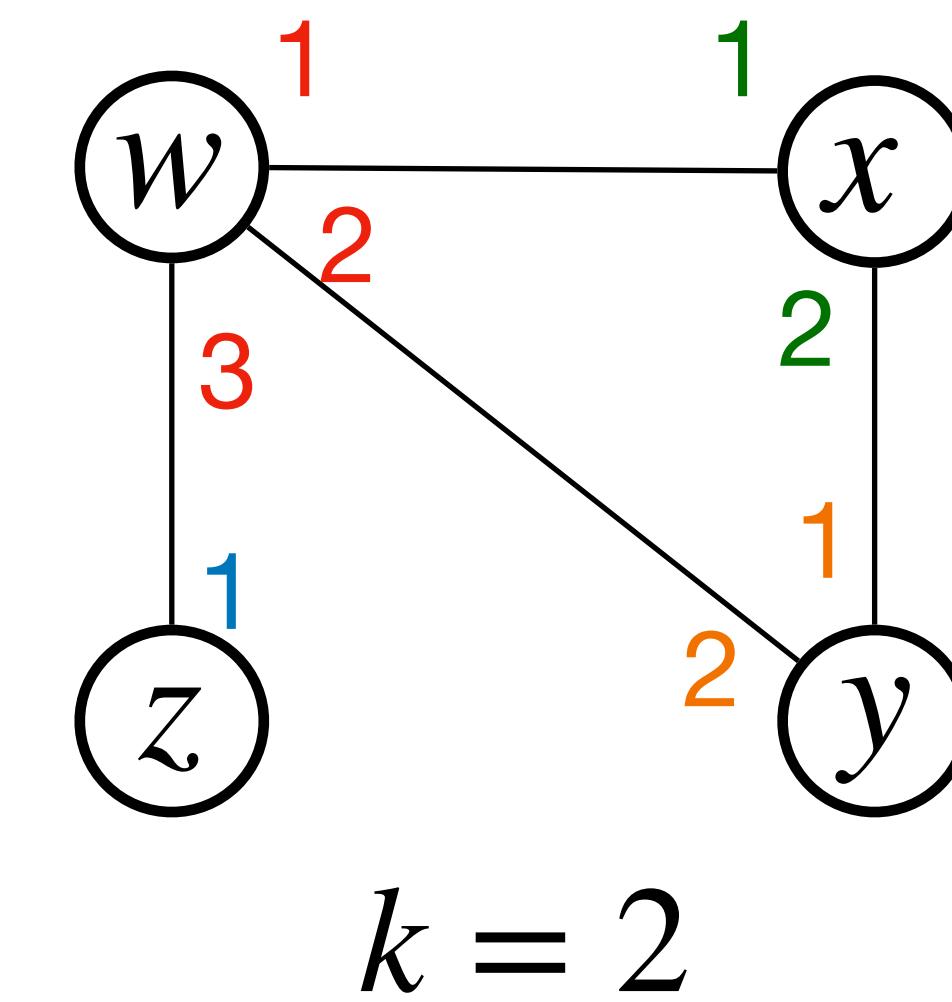
We now show:

“YES” for the Vertex Cover Problem.



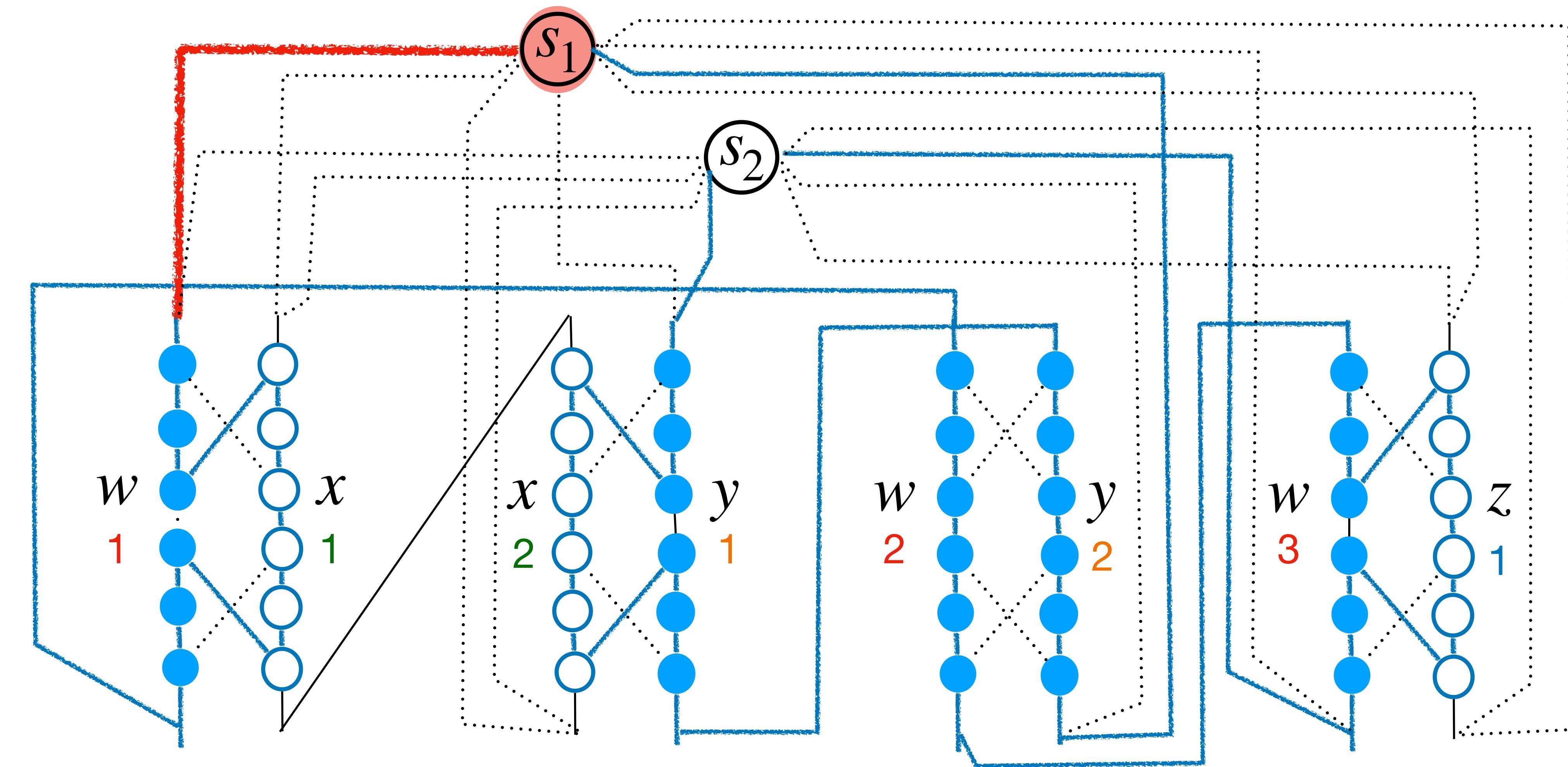
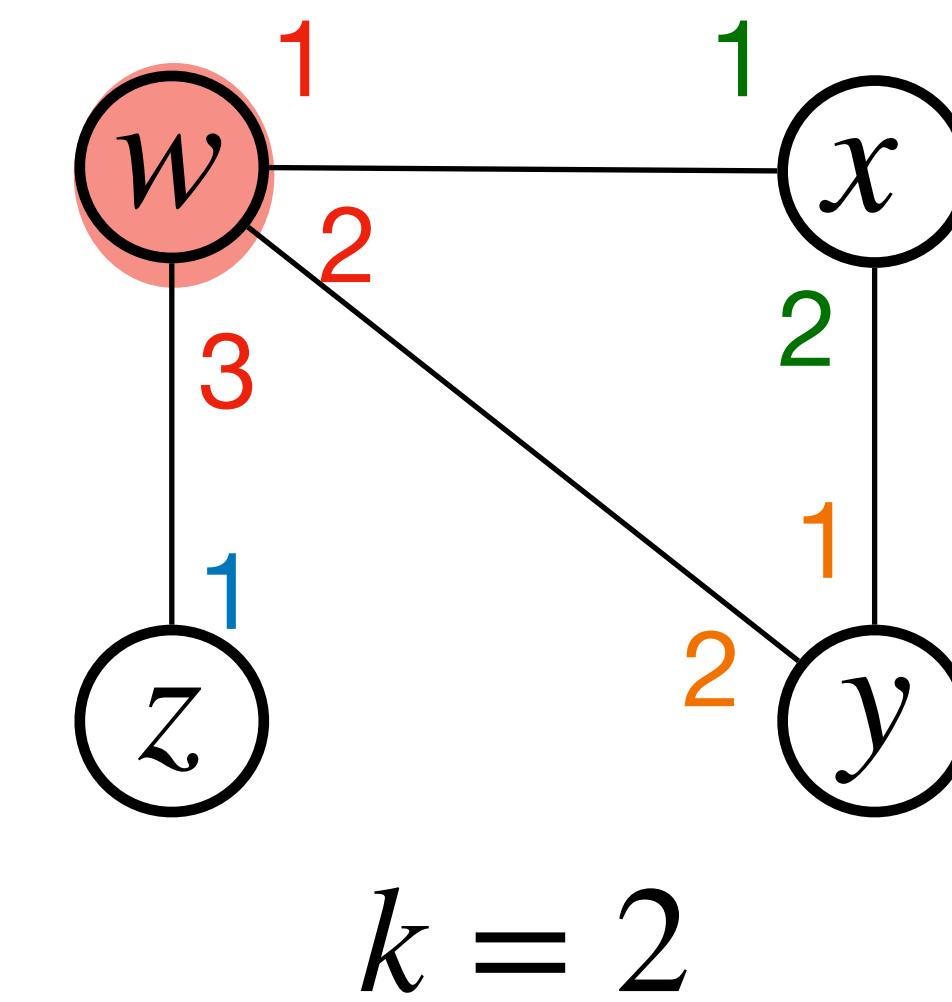
“YES” for Hamiltonian Cycle Problem.

Assume “YES” for Hamiltonian Cycle Problem.



Assume “YES” for Hamiltonian Cycle Problem.

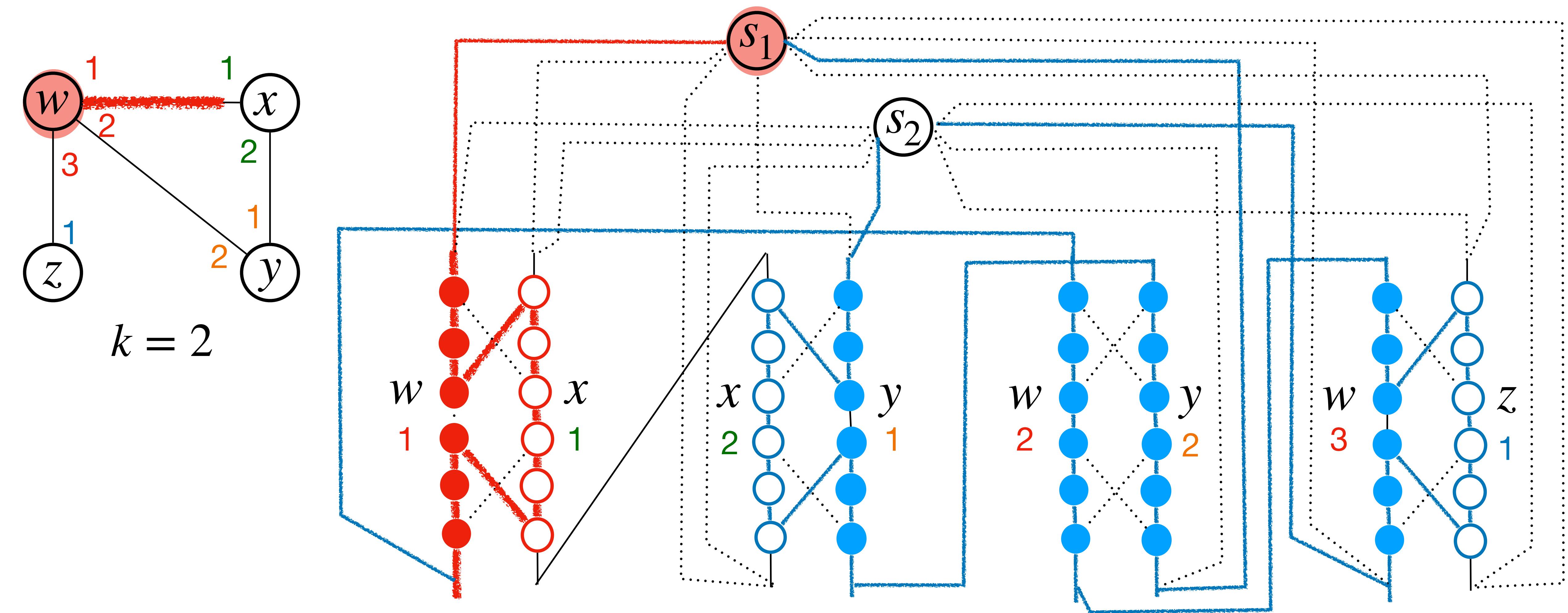
1st selected node is: **w**



Assume “YES” for Hamiltonian Cycle Problem.

1st selected node is: w

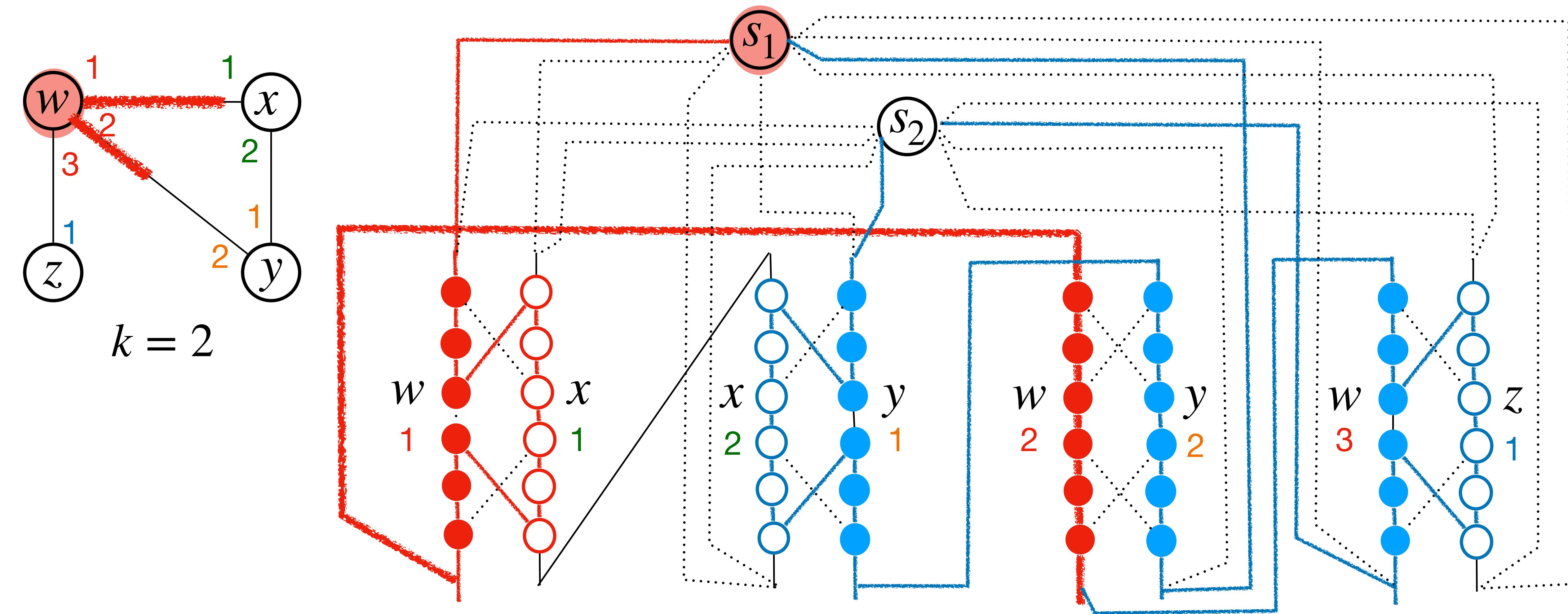
How should w cover its 1st edge (w,x)?



Assume “YES” for Hamiltonian Cycle Problem.

1st selected node is: w

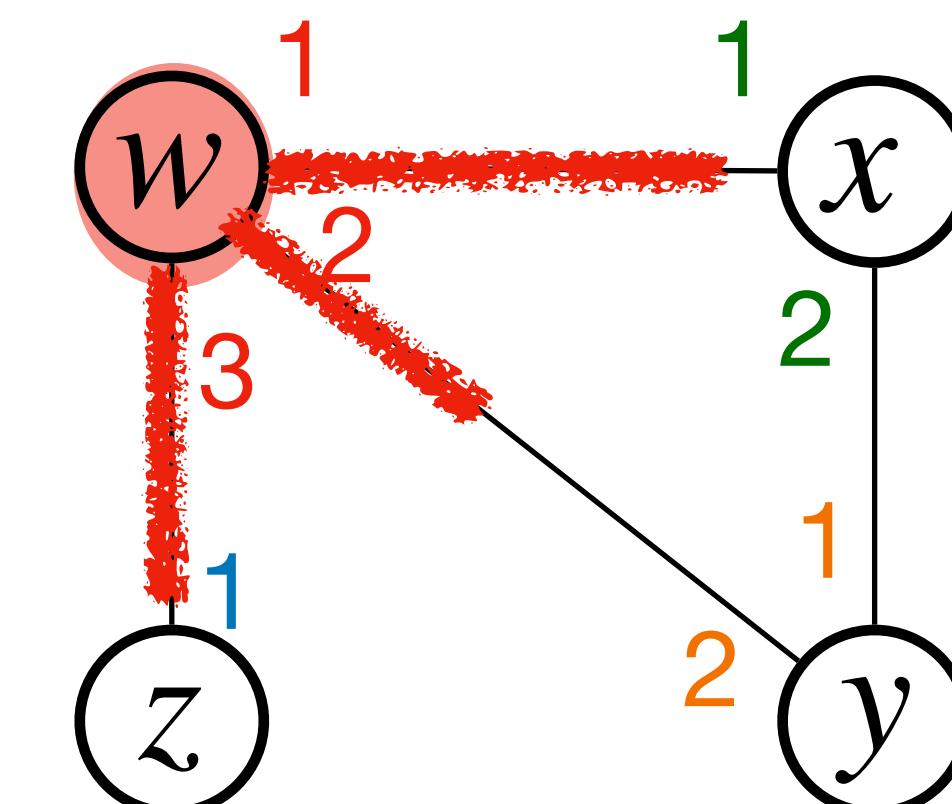
How should w cover its 2nd edge (w,y) ?



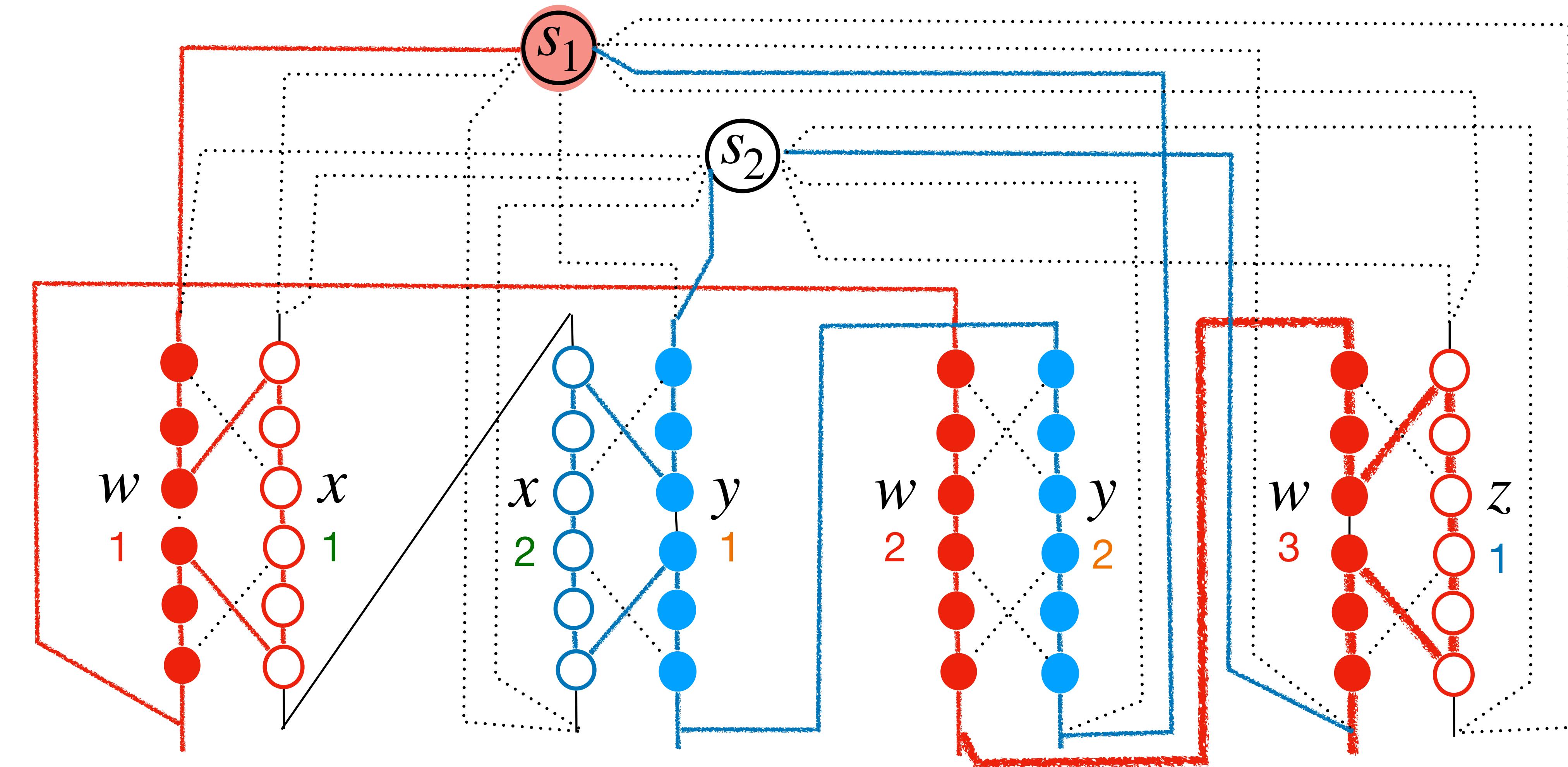
Assume “YES” for Hamiltonian Cycle Problem.

1st selected node is: w

How should w cover its 3rd edge (w,z) ?

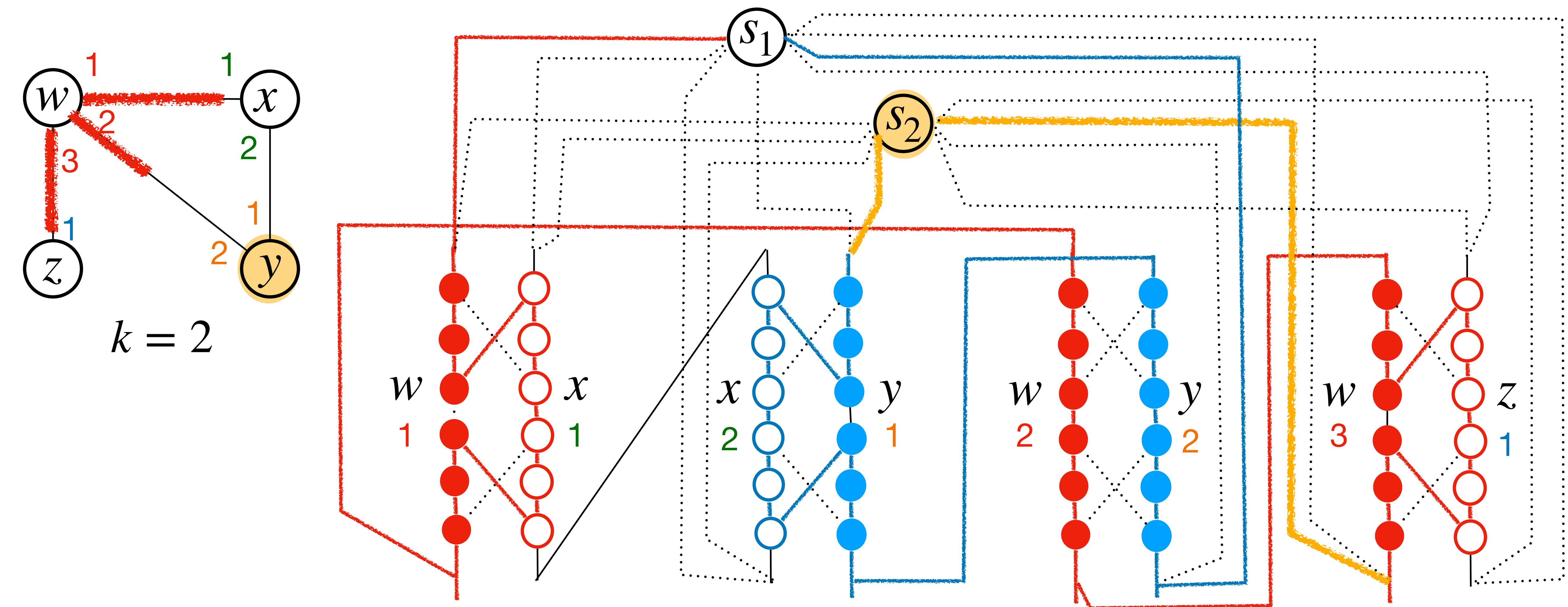


$k = 2$



Assume “YES” for Hamiltonian Cycle Problem.

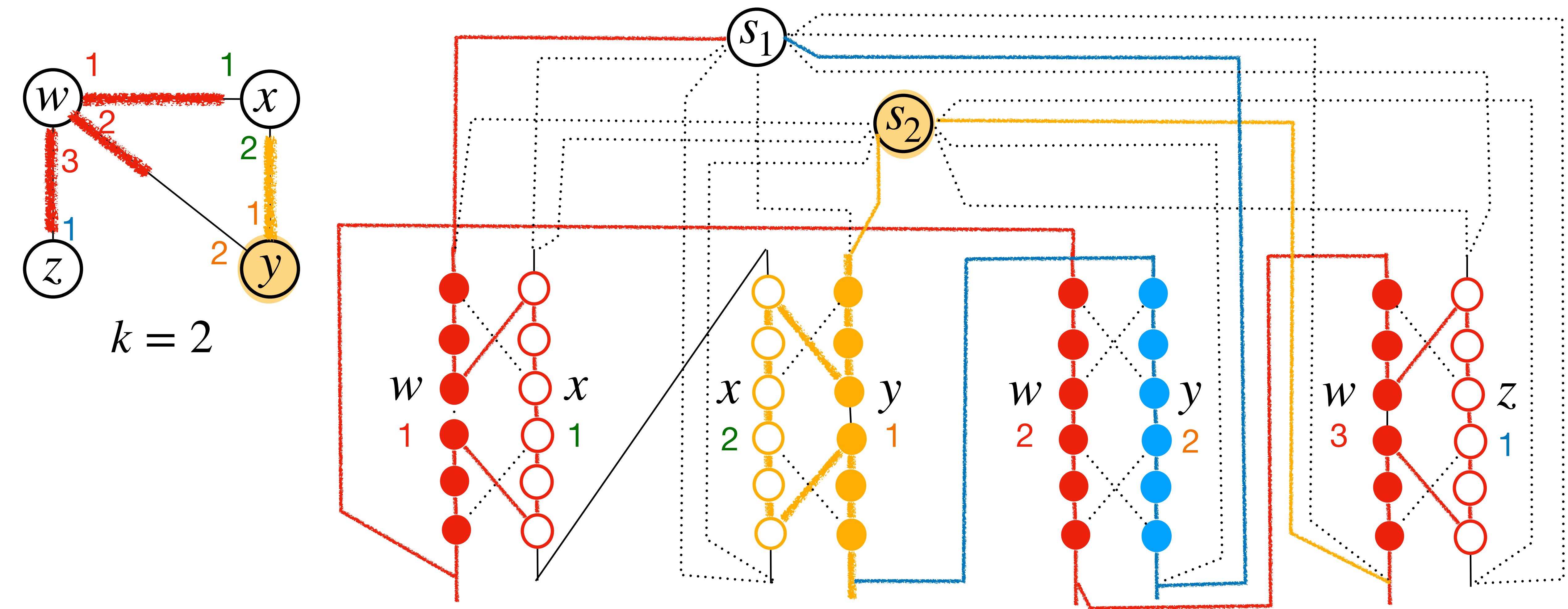
2nd selected node is: y



Assume “YES” for Hamiltonian Cycle Problem.

2nd selected node is: y

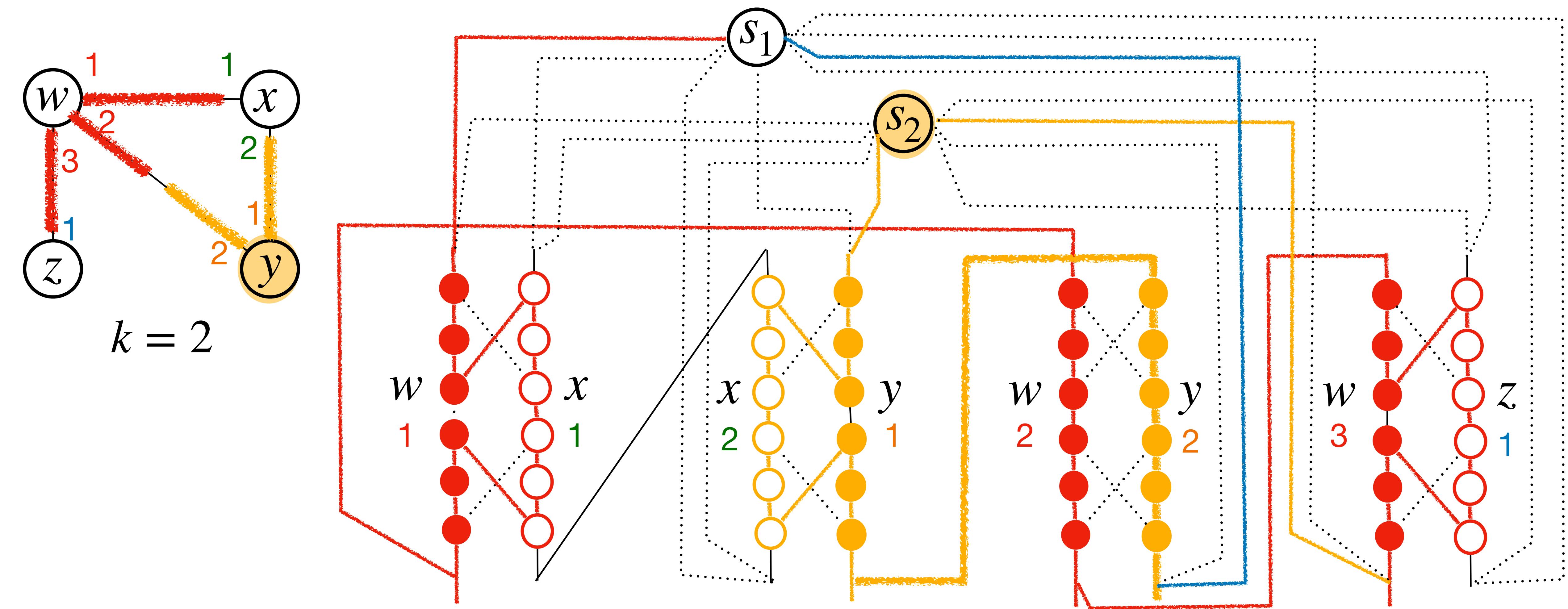
How should y cover its 1st edge (y,x) ?



Assume “YES” for Hamiltonian Cycle Problem.

2nd selected node is: y

How should y cover its 2nd edge (y,w) ?

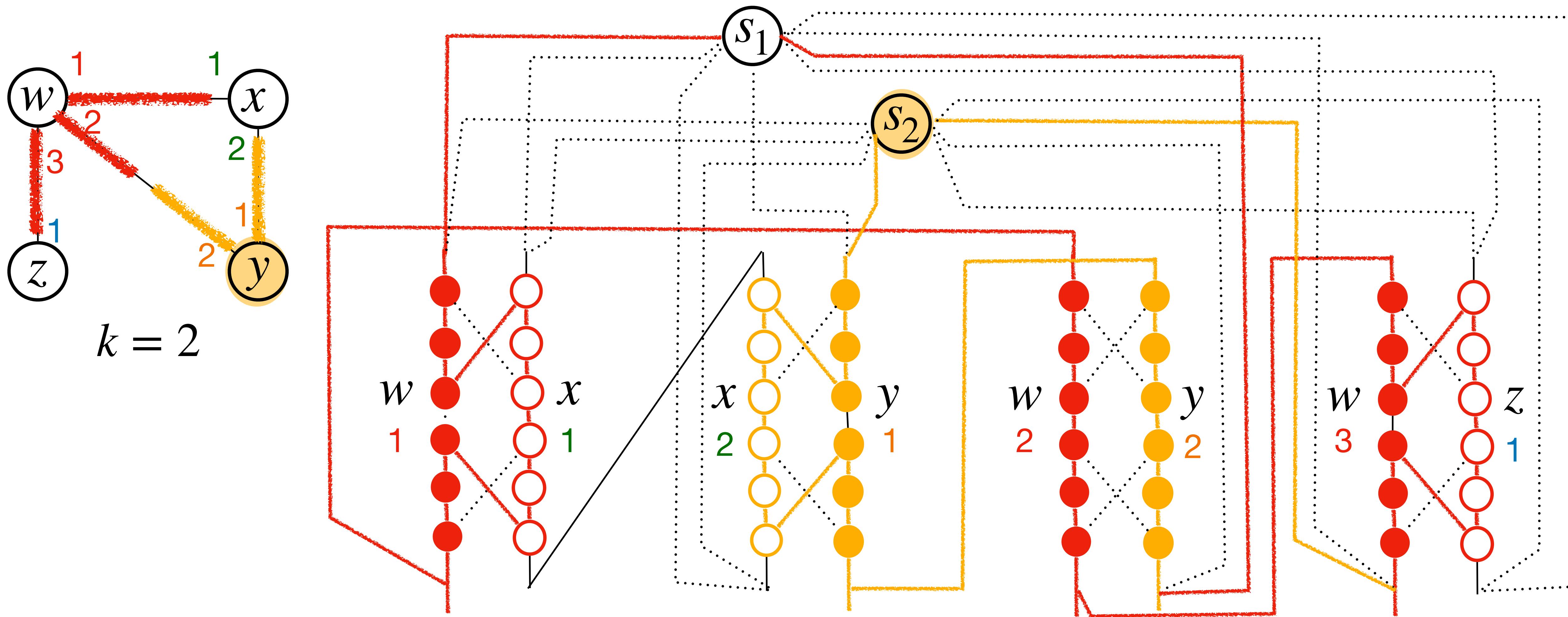


“YES” for Vertex Cover Problem.

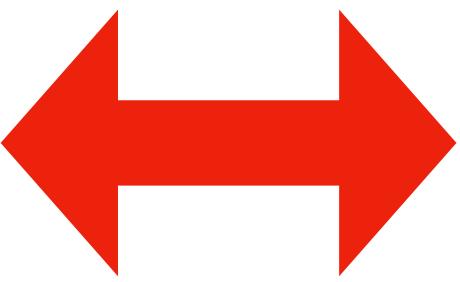
“YES” for Hamiltonian Cycle Problem.

All edges are covered.

G has a vertex cover of size 2.



“YES” for the Vertex Cover Problem.



“YES” for Hamiltonian Cycle Problem.

There is a polynomial-time reduction from “Vertex Cover Problem” to “Hamiltonian Cycle Problem”.

Hamiltonian Cycle Problem $\in NPC$

Quiz questions:

1. What is the main idea for proving the NP-completeness of the “Hamiltonian Cycle Problem”?
2. In the above proof, how does “covering an edge” for vertex cover correspond to “traversing a gadget” for the Hamiltonian cycle?