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Analysis of Algorithm (HW-3)

16.3-1. Suppose you have a potential function ϕ such that $\phi(D_i) \geq \phi(D_0)$ for all i, but $\phi(D_0) \neq 0$. Show that there exists a potential function ϕ' such that $\phi'(D_0) = 0$, $\phi'(D_i) \geq 0$ for all $i \geq 1$, and the amortized cost using ϕ' are the same as the amortized costs using ϕ.

16.3-1.(i) <u>To prove</u>: there exists ϕ' such that $\phi'(D_0) = 0$, $\phi'(D_i) \geq 0$ for all $i \geq 1$

Let $\phi'(D_i) = \phi(D_i) - \phi(D_0)$ for all $i \geq 0$ — (1)

$\Rightarrow$ for $i = 0$, $\phi'(D_0) = \phi(D_0) - \phi(D_0) = 0$

$\Rightarrow$ for $i \geq 1$, $\phi'(D_i) = \phi(D_i) - \phi(D_0)$ — (2)

we know that, $\phi(D_i) \geq \phi(D_0)$. — (3)

Using (2) and (3)

$\phi'(D_i) \geq 0$.

Hence proved. There exist a potential function such that $\phi'(D_0) = 0$, $\phi'(D_i) \geq 0$ for $(i \geq 1)$.

(ii) To prove :- amortized cost of ϕ' = amortized cost of ϕ

$\Rightarrow$ The amortized cost of ϕ' is for i^{th} step is :-

$$\hat{c}_i = c_i + \phi'(D_i) - \phi'(D_i - 1)$$

(adding & sub $\phi(D_0)$ in above eqn^x)

$$= c_i + (\phi(D_i) - \phi(D_0)) - (\phi(D_i - 1) - \phi(D_0)) \quad \text{— (4)}$$

from eqn (1) and (4)

$$= c_i + \phi(D_i) - \phi(D_i - 1) \quad \text{— (5)}$$

$\Rightarrow$ Amotized cost of ϕ for i^{th} step is :-

$$\hat{c}_i = c_i + \phi(D_i) - \phi(D_{i-1}) \quad \text{— (6)}$$

since eqn (5) is same as eqn^x (6)
we can say that;
amortized cost of ϕ' = amortized cost of ϕ

27.1-1. Suppose that when hedging your bets, you wait for b minutes, instead of for k minutes, before taking the stairs. What is the competitive ratio as a function of b and k? How should you choose b to minimize the competitive ratio?

27.1-1. Main Idea:

Let's say we are waiting for b minutes instead of k. There can be 3 possible cases for this i.e $b=k$, $b<k$ and $b>k$. We will calculate competitive ratio for all three cases and then use the minimum ratio for predicting b value such that ~~overall~~ we get minimum competitive ratio.

Let m be the time when lift arrives.

m:	0	1	2	------	$k-1$	k	$k+1$	------	$B-1$.
optimal cost:	1	2	3	------	k	k	k	------	k
Case-1: ($b=k$).	1	2	3	------	k	$k+1$	$2k$	------	$2k$.

Case-1 cost ($b=k$) : $\begin{cases} m+1 & ; \quad m \leq k \\ 2k & ; \quad m > k \end{cases}$

$$\text{competitive ratio} = \frac{2K}{K} = 2.$$

m: ~~(p<K)~~	0	1	2	... $p-1$	p	$p+1$	.... $K-1$	K	----	$B-1$
optimal:	1	2	3	p	$p+1$	$p+2$	.... K	K	---	K
Case 2. ($p<K$)	1	2	3	(p)	($p+1$)	($p+K$)	($p+K$)	($p+K$)		($p+K$)

$$\text{Case 2: cost} \begin{cases} m+1. & ; \ m \leq p \\ p+K & ; \ m > p. \end{cases}$$

$$\text{competitive ratio} = \max\left(\frac{p+K}{K}, \frac{p+K}{p+2}\right).$$

$$= \begin{cases} \dfrac{p+K}{K} & ; \ K-2 < p < K \\[2ex] \dfrac{p+K}{p+2}. & ; \ p \leq K+2 \end{cases}$$

m:	0	1	2	-- $K-1$	K	$K+1$	--- $p-1$	p	$p+1$	--	$B-1$
optimal:	1	2	3 ...	K	K	K	K	K	K		K
Case 3: ($p>K$)	1	2	3	K	$K+1$	$K+2$	p	$p+1$	$p+K$		$p+K$

Case 3 : cost $\begin{cases} m+1 & ; m \leq \beta \\ \beta + K & ; m > \beta \end{cases}$

competitive ratio $= \dfrac{\beta + K}{K}$

$\Rightarrow$ $\therefore$ competitive ratio as a function of β and K is

$$= \begin{cases} \dfrac{\beta+K}{K} & ; \beta > K \\ 2 & ; \beta = K \\ \dfrac{\beta+K}{K} & ; K \div 2 < \beta < K \\ \dfrac{\beta+K}{\beta+2} & ; \beta \leq K \div 2 \end{cases}$$

$\Rightarrow$ Now we want to find β such that competitive ratio is minimum.

Case: 1 : $\beta > K \Rightarrow C.P = \dfrac{\beta+K}{K} > 2$

Case 2 : $\beta = K \Rightarrow C.P = 2$

Case.3: $\cancel{p}$ $K-2 < p < K$ i.e $p = K-1$

$$C.P = \frac{p+K}{K} = \frac{(K-1)+K}{K}$$

$$= 2 - \frac{1}{K} \quad \text{— ⓐ}$$

Case.4 $p \leq K-2.$ — (i)

$$CP = \frac{p+K}{p+2}.$$

for $CP < 2$,

$$\frac{p+K}{p+2} < 2$$

$$K-4 < p \quad \text{— (ii)}$$

from (i) & (ii); p can take following values: $K-2$ & $K-3$ ~~and K-4~~.

at $K-2$, $C.P = \frac{2K-2}{K} = 2 - \frac{2}{K}$ — ⓑ

at $K-3$, $C.P = \frac{2K-3}{K-1} = 2 - \left(\frac{1}{K-1}\right)$ — ⓒ

~~at $K-4$, $CP = \frac{2K-4}{K-2}$~~

From ⓐ, ⓑ and ⓒ we can say that we will get min comp. ratio at $\boxed{p = K-2}$

27.1-2. Given: rent $= r$/day.
buy $= b$.
Let d be the no. of days we ski.
Give and analyze an algorithm that has a competitive ratio of 2.

⇒ Main Idea:- This is a online algorithm problem as we do not know future data. Let say d is the no. of days we ski. We will see that if we buy ski on day 1 then the competitive ratio will be b/r. Whereas if we will never buy skii and always rent it then the competitive ratio will be $\left(\frac{Br}{b}\right)$, where $B \gg b$. Both are ≥ 2. These two approaches of always ~~taking the stairs~~ renting and always buying are extreme solutions. We can instead guard even better against a worst-case future. We can wait for buying for a while and if number of days exceeds that limit, buy the ski. From below solution we will notice that when we wait for $\left(\frac{b}{r}\right)$ no. of days before buying we will get competitive ratio of 2.

d :	1	2	$\cdots$	$\left(\frac{b}{r}\right)$	$\left(\frac{b}{r}\right)+1$	$\cdots$	$b \cdots B$ (large no. of days)
optimal :	r	$2r$	$\cdots$	b	b	$\cdots$	$b \cdots B$
Method 1 : (always rent)	r	$2r$	$\cdots$	b	$\left(\frac{b}{r}+1\right)r$	$\cdots$	$br \cdots Br$
Method 2 : (always buy)	b	b	$\cdots$	b	b	$\cdots$	$b \cdots b$
Method 3 : (wait for $\left(\frac{b}{r}\right)$ days)	r	$2r$	$\cdots$	b	$2b$	$\cdots$	$2b \cdots 2b$

- optimal cost $= \begin{cases} dr & ; \ d \leq \frac{b}{r} \\ b & ; \ d > \frac{b}{r} \end{cases}$

- Method 1, cost $= \begin{cases} dr & ; \ d \in I. \end{cases}$

  competitive ratio $= \frac{Br}{b}$. (where $B \gg b$).

- Method 2 cost $= \{ \ b \ ; \ d \in I.$

  competitive ratio $= \frac{b}{r}$ (where $b \gg r$).

- Method 3 cost $= \begin{cases} dr & ; d \leqslant \frac{b}{r} \\ 2b & ; d > b \end{cases}$

competitive ratio $= \frac{2b}{b} = 2$. (always).

Hence, we can say that if we will wait for $\left(\frac{b}{r}\right)$ days before buying, we can make our competitive ratio 2 and limit the upper cap of cost.