Algorithms

Lecture Topic: Minimum Spanning Tree

Anxiao (Andrew) Jiang

Roadmap of this lecture:

1. Minimum Spanning Tree.

    1.1 Define "Minimum Spanning Tree (MST)".

    1.2 Kruskal's Algorithm for MST.

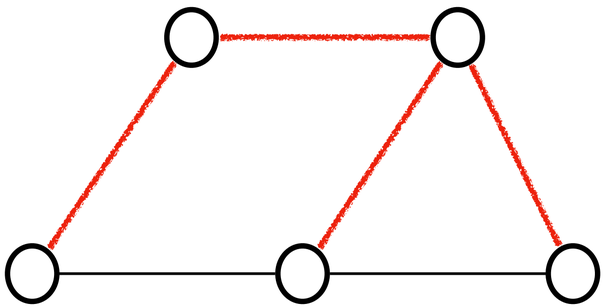
    1.3 Prim's Algorithm for MST.

    1.4 Correctness of Kruskal's Algorithm and Prim's Algorithm.

Minimum Spanning Tree

Spanning Tree of graph G=(V,E)
is a sub-graph of G that:

1) is a tree
2) contains all nodes of G



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Minimum Spanning Tree (MST) of graph G=(V,E)
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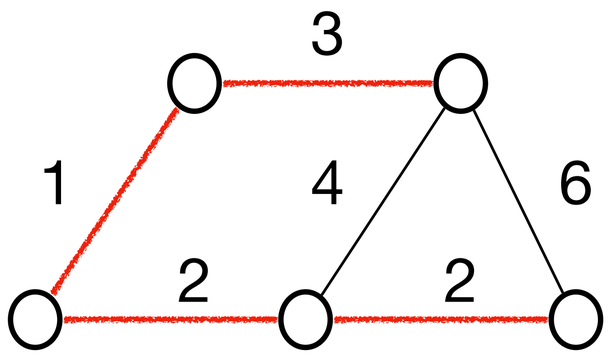
3) Minimizes its total edge weight



MST

Total weight = 1 + 2 + 3 + 2 = 8

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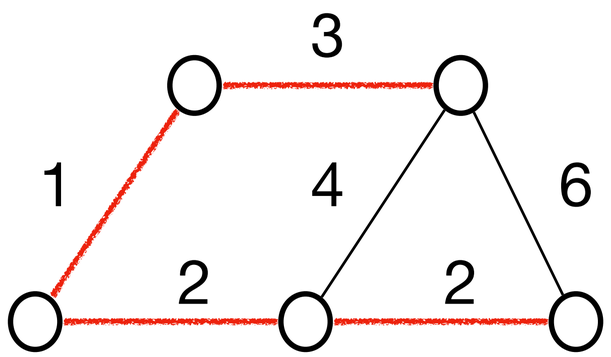

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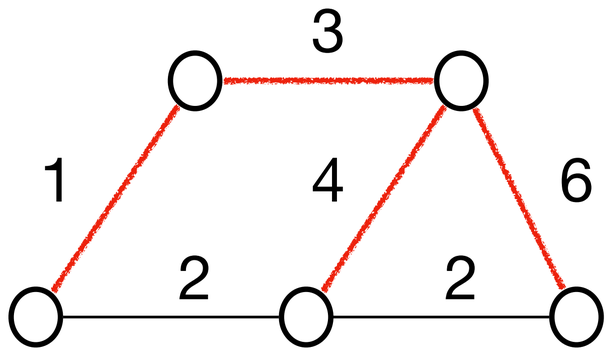

Not MST

Total weight = 1 + 3 + 4 + 6 = 14

Minimum Spanning Tree

Input: A connected, undirected graph G=(V,E),

where every edge $e \in E$ has a weight $w(e)$.

Output: Find an MST of G.

Quiz questions:

1. Given a graph, is its MST uniquely determined?

2. What are the applications of finding an MST in a graph?

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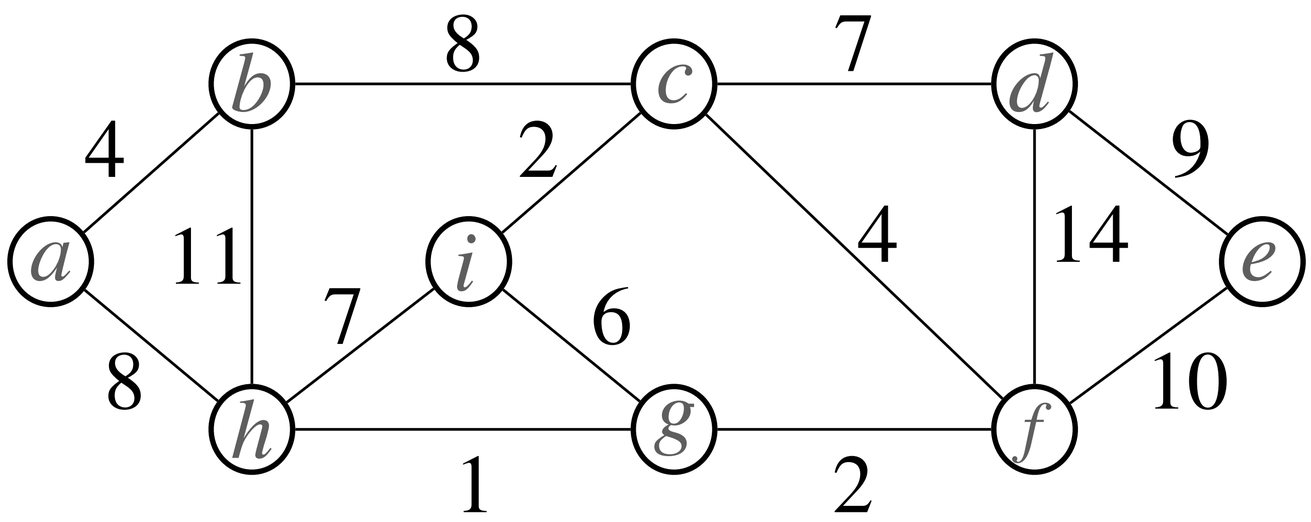

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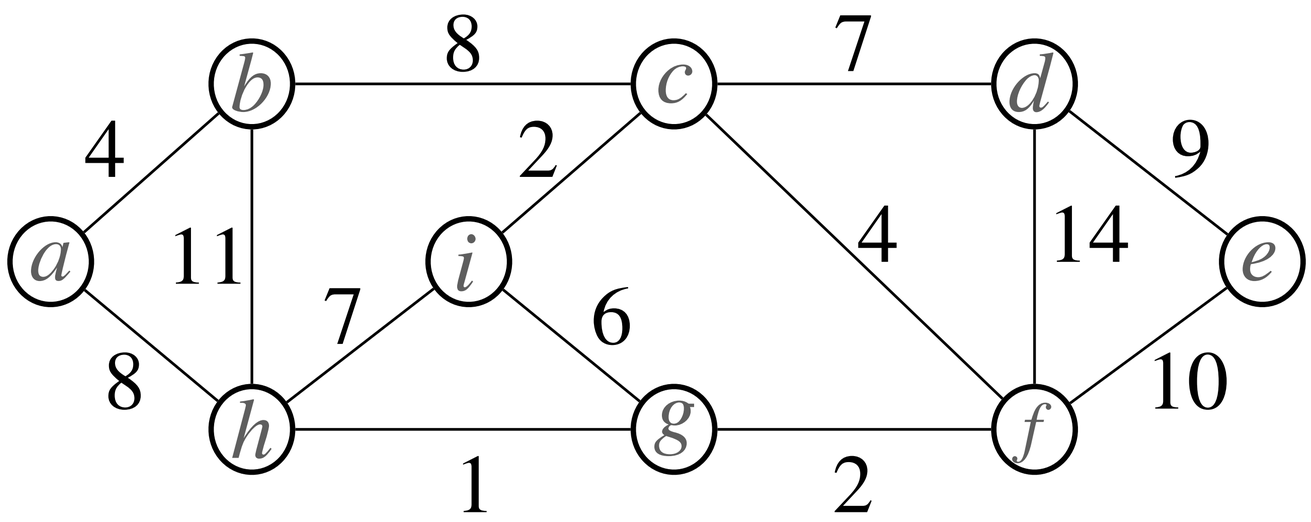

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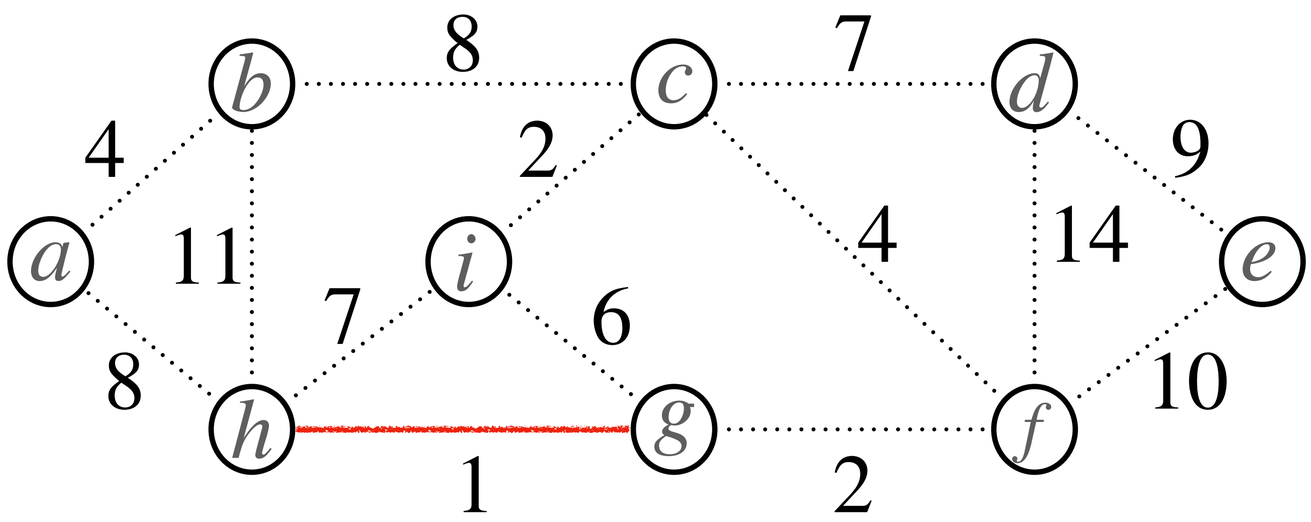

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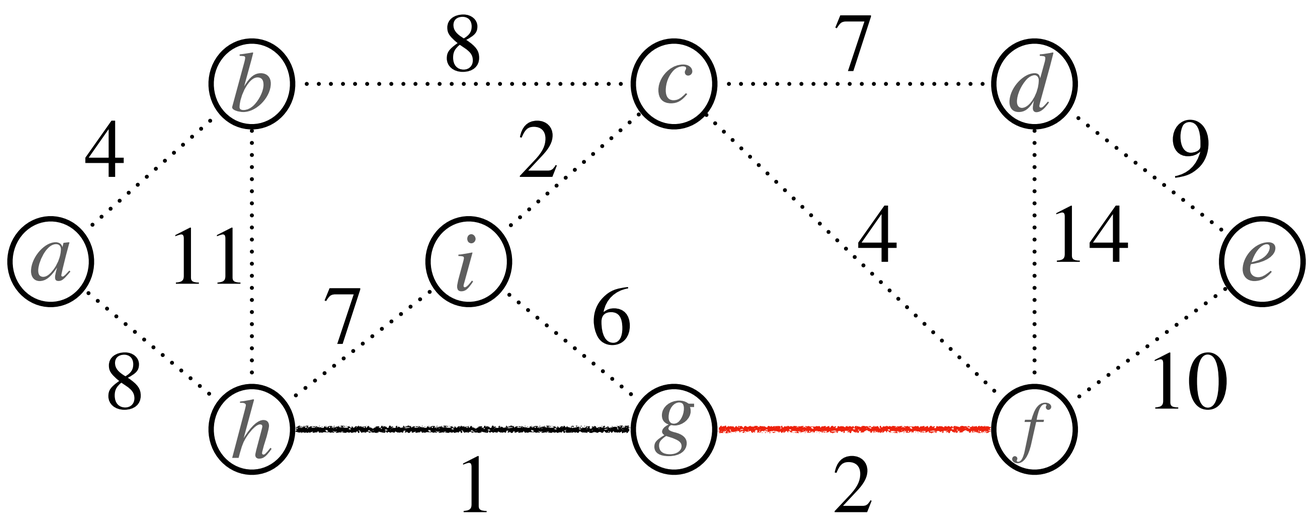

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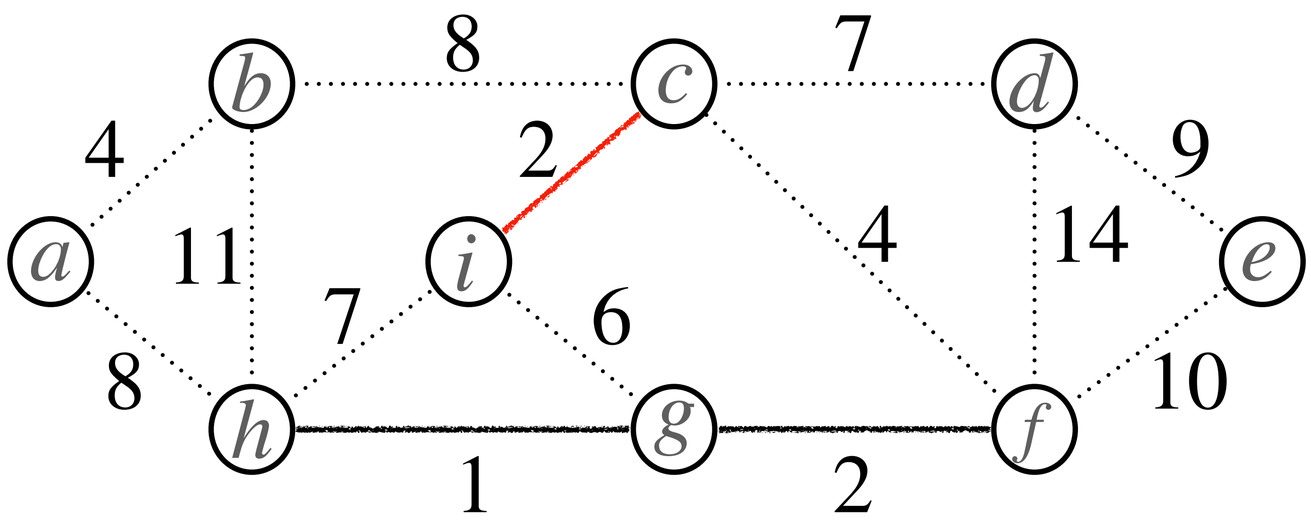

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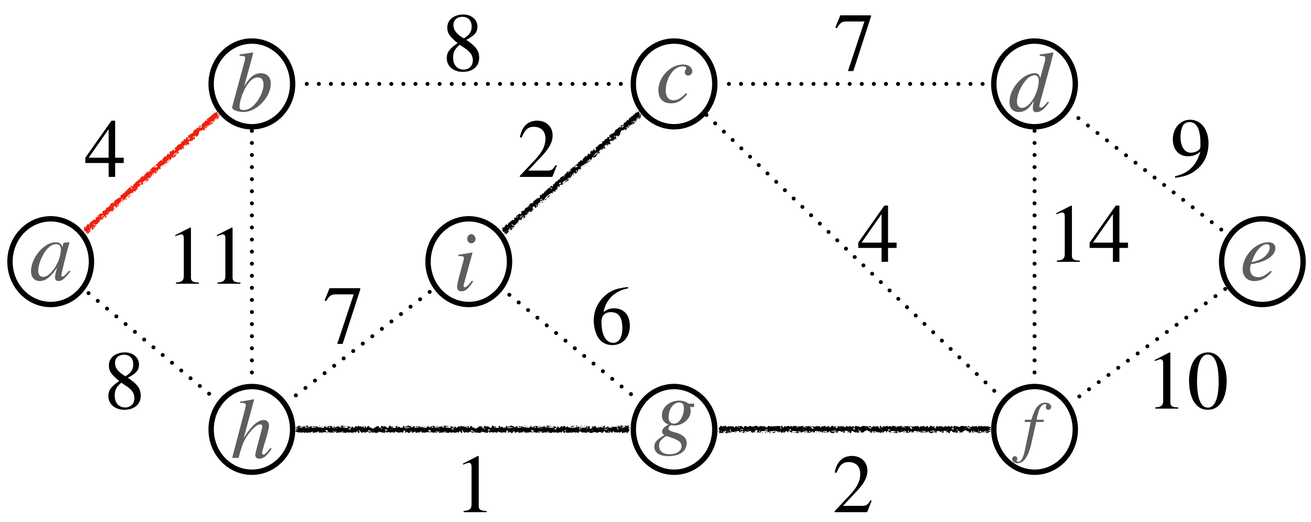

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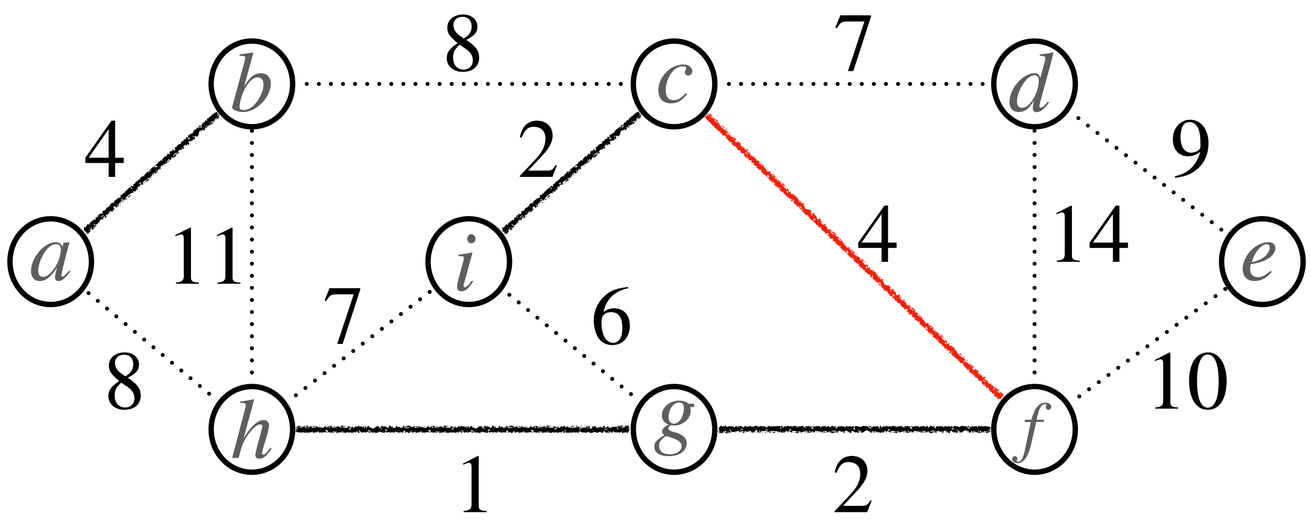

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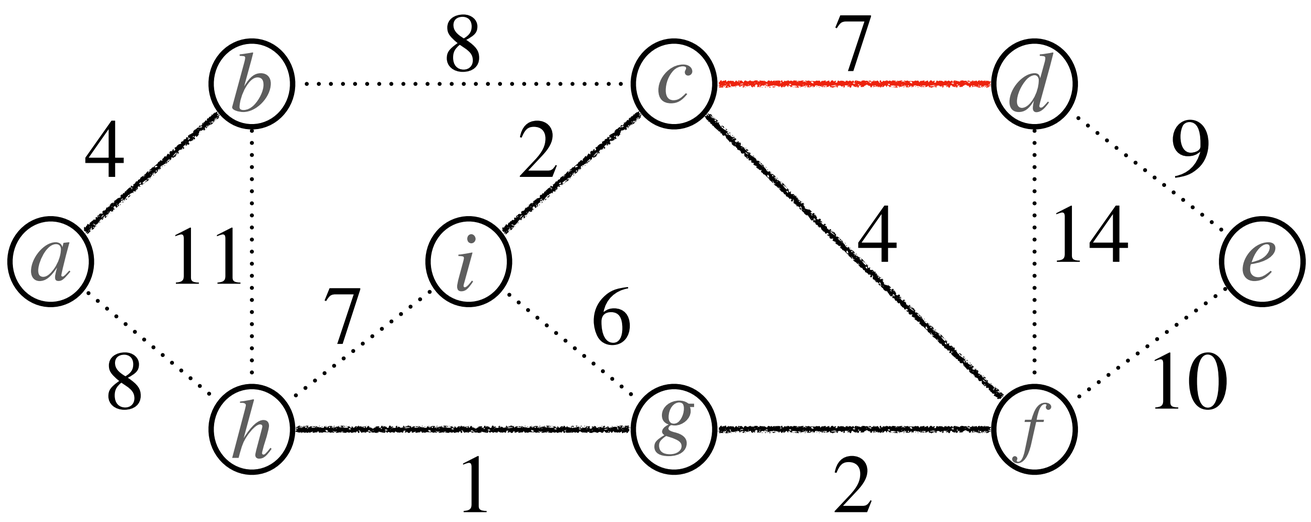

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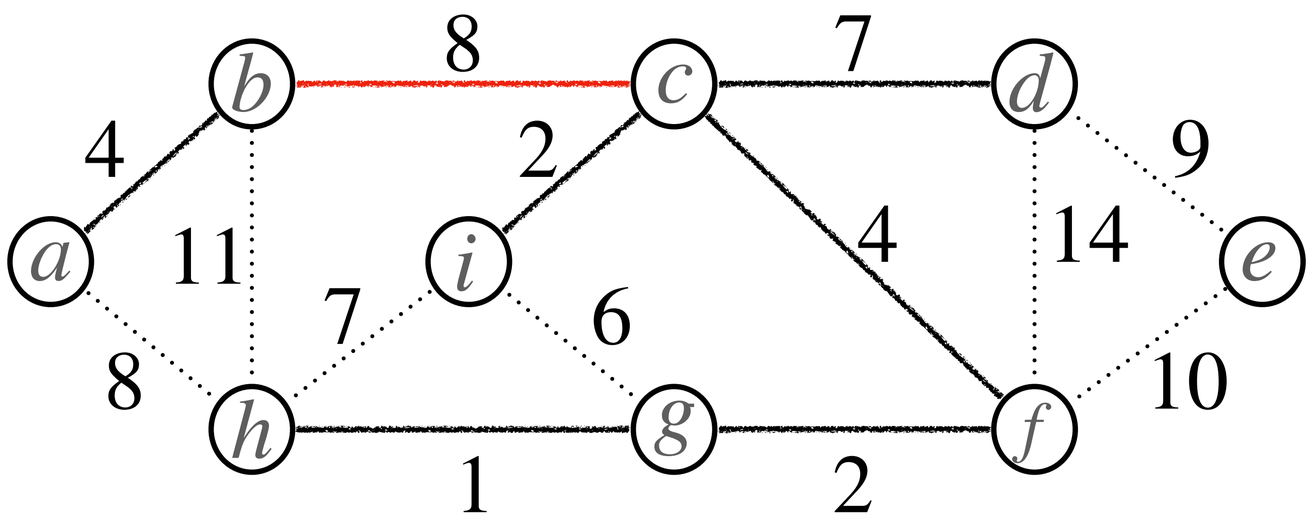

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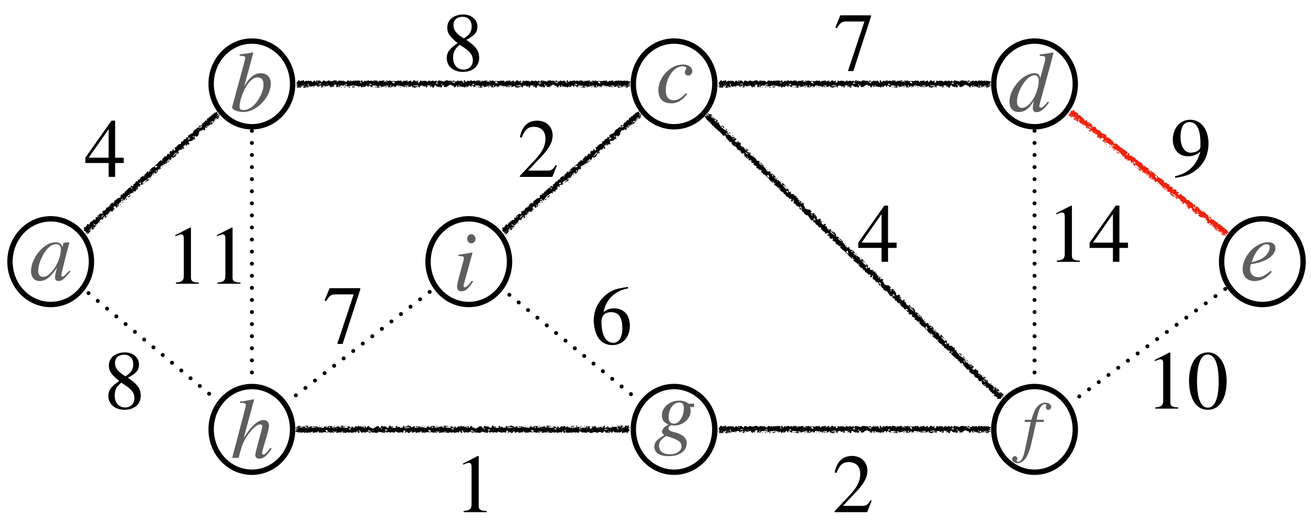

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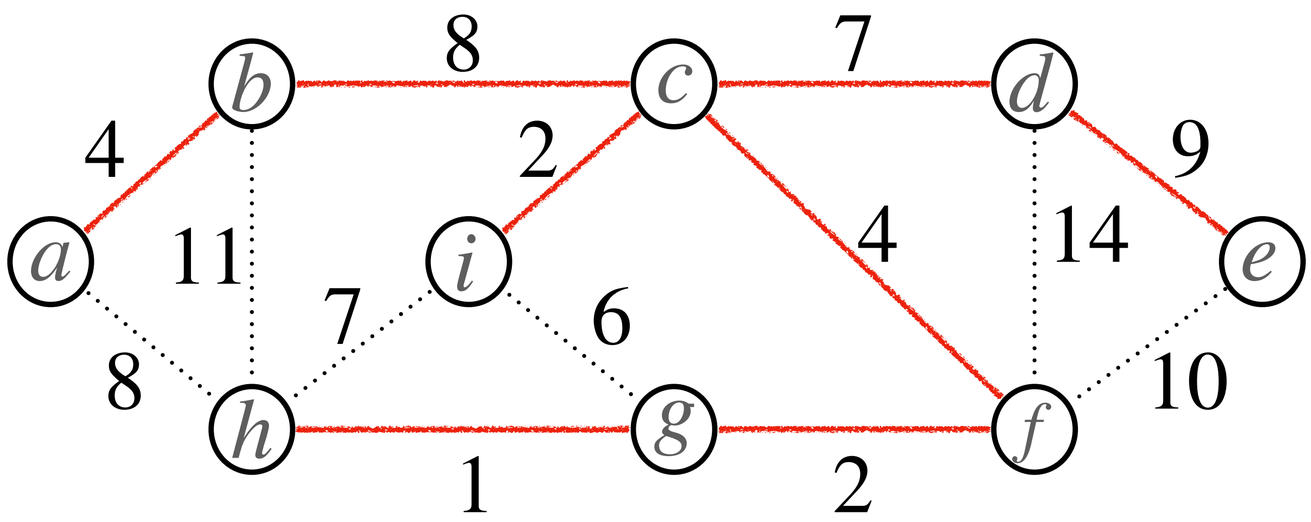

Weight = 1+2+2+4+4+7+8+9
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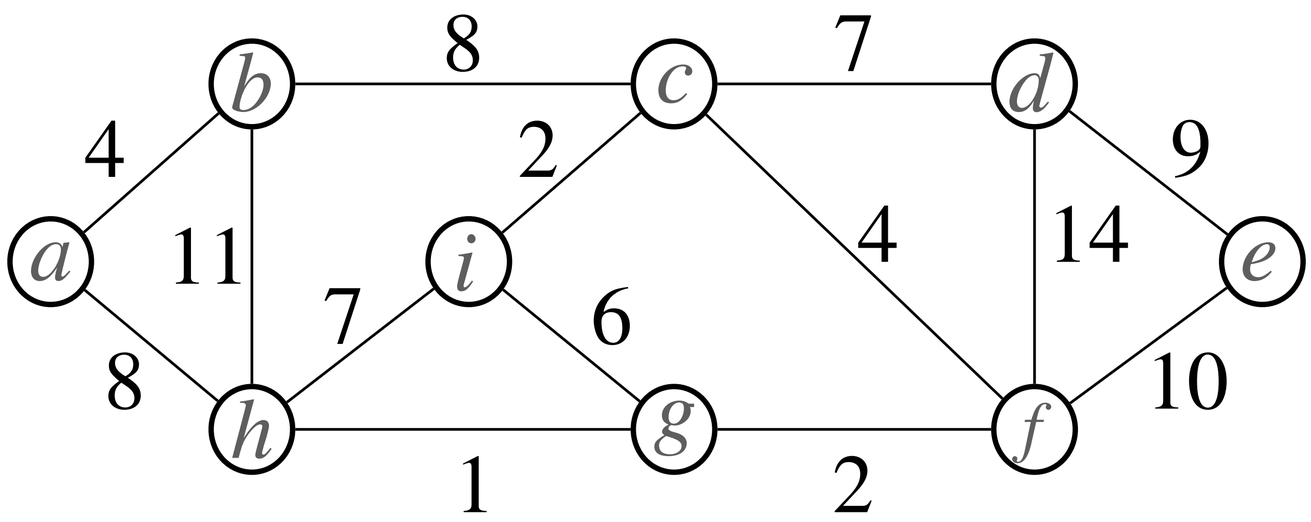

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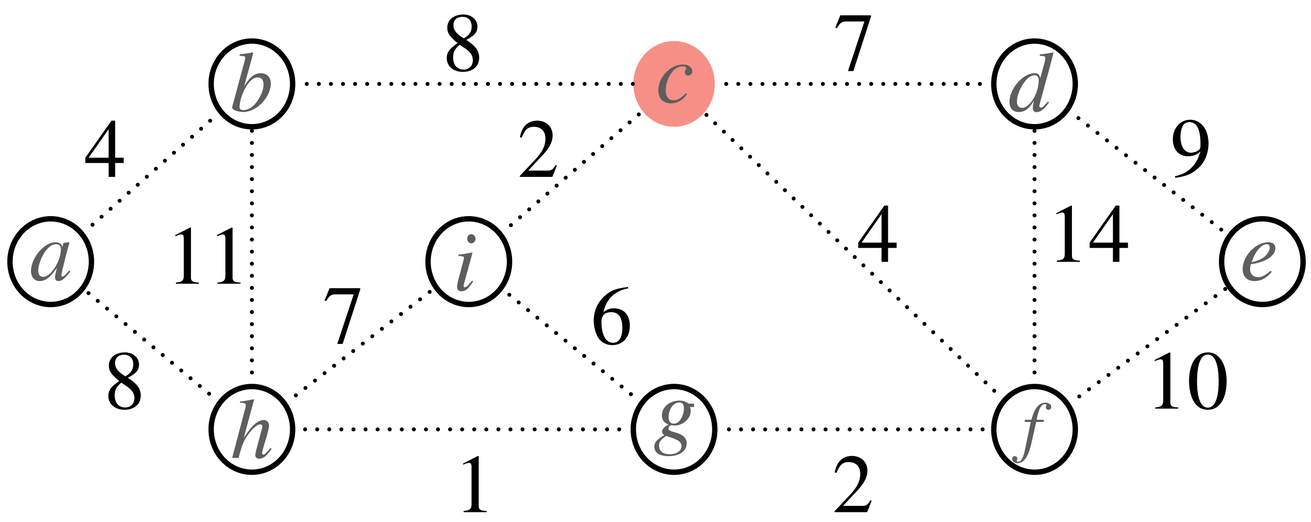

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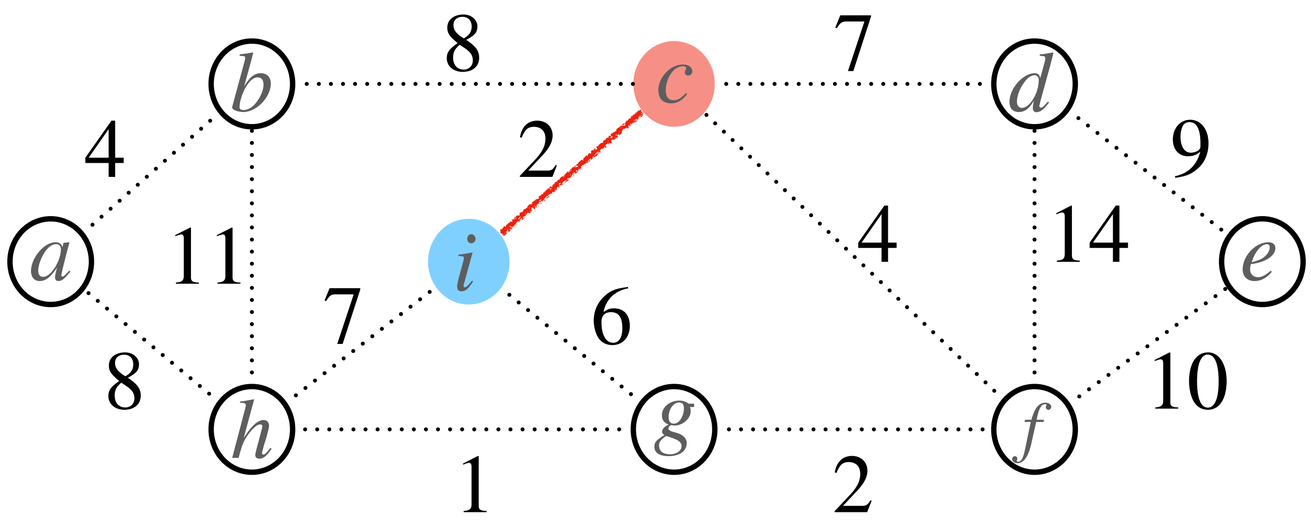

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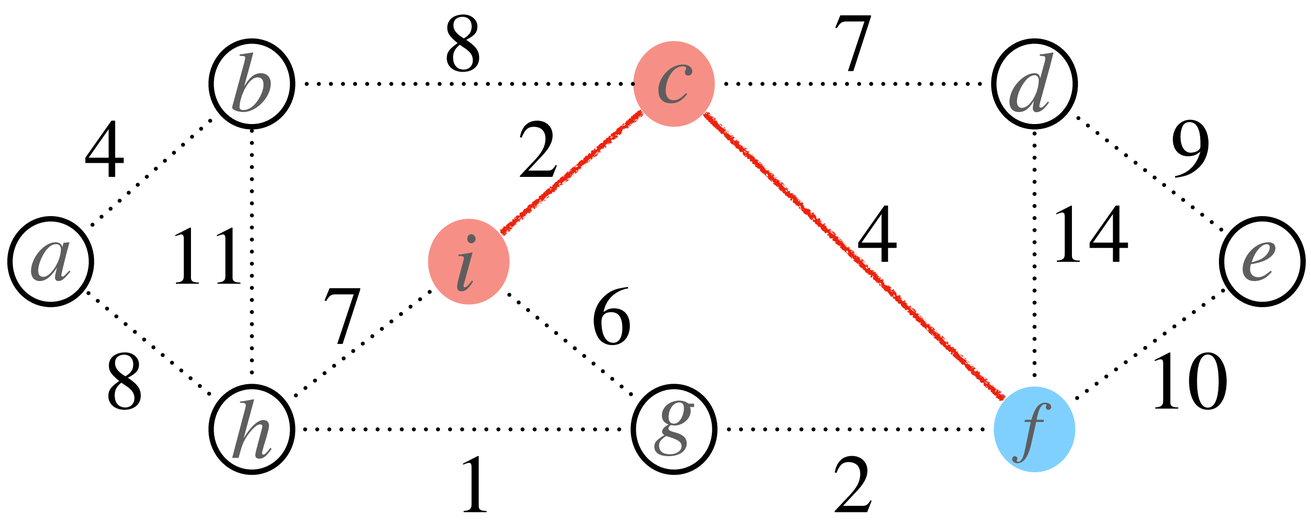

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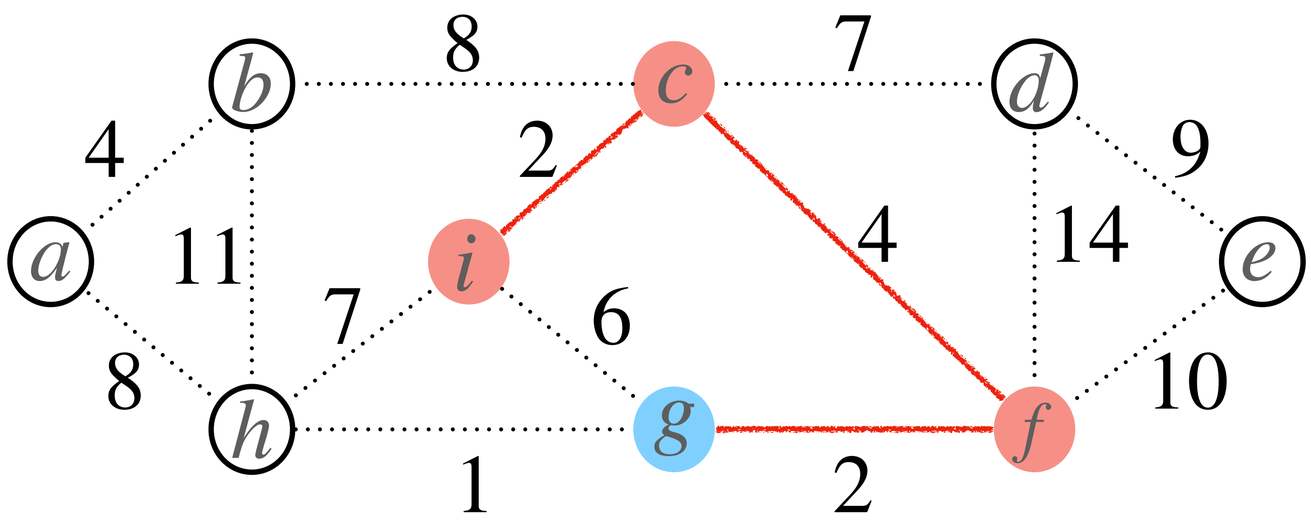

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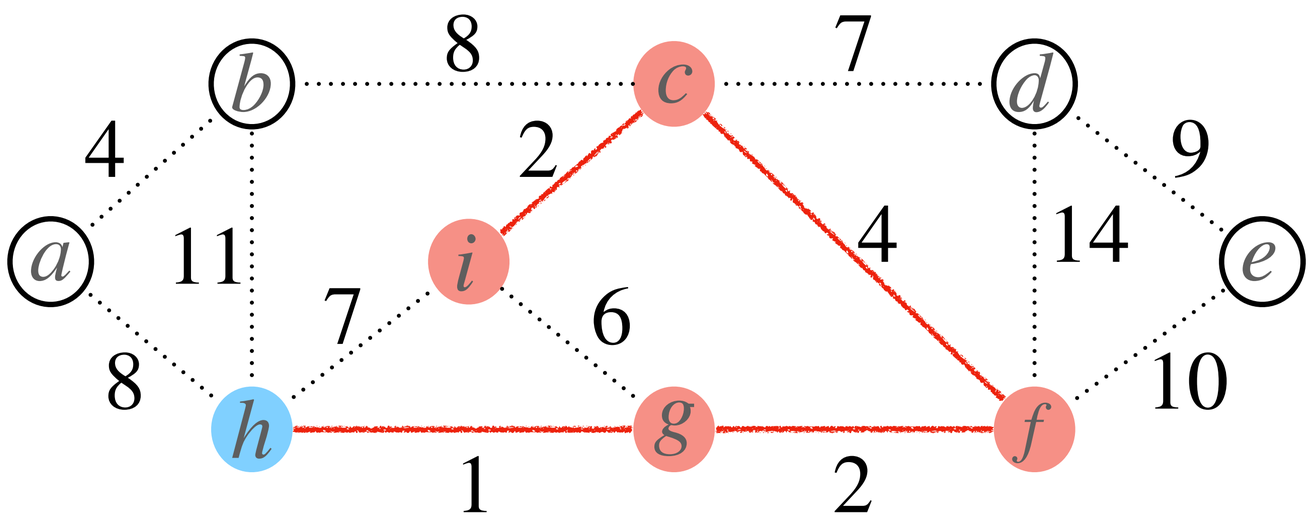

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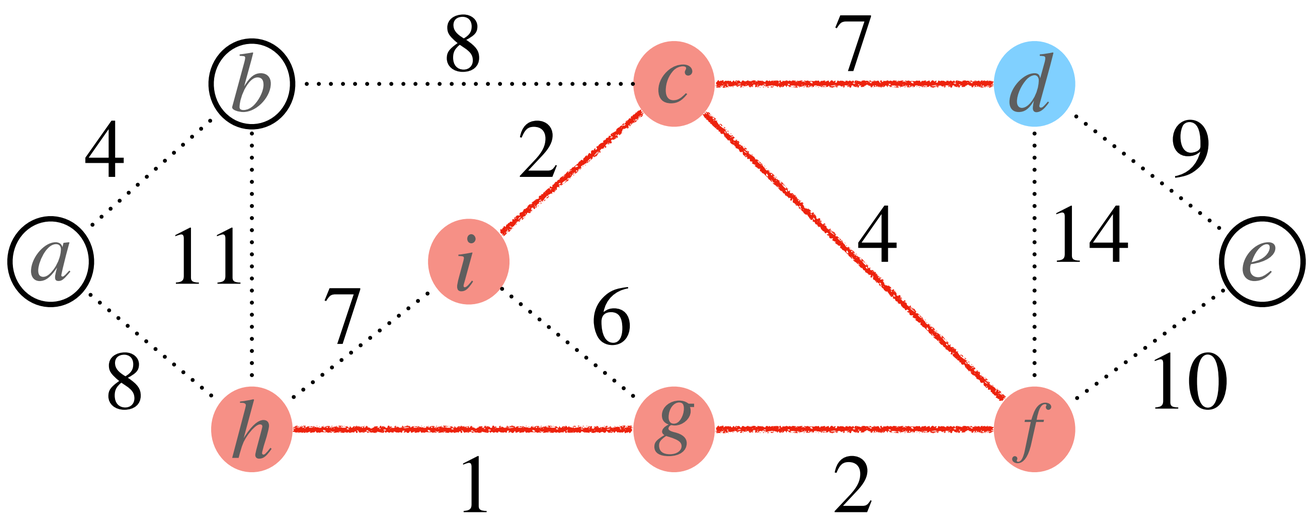

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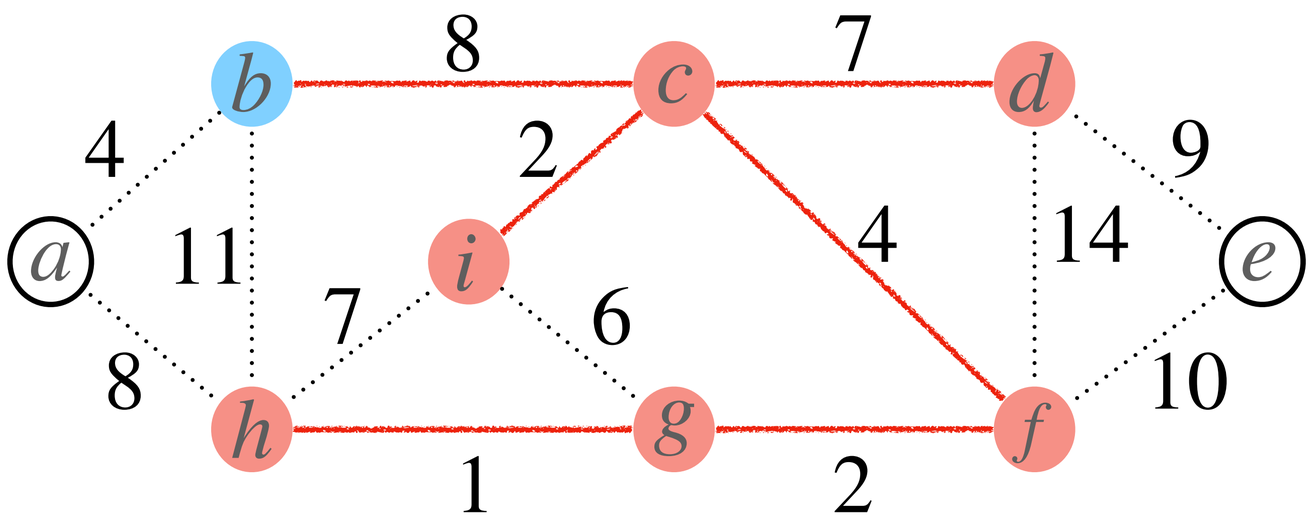

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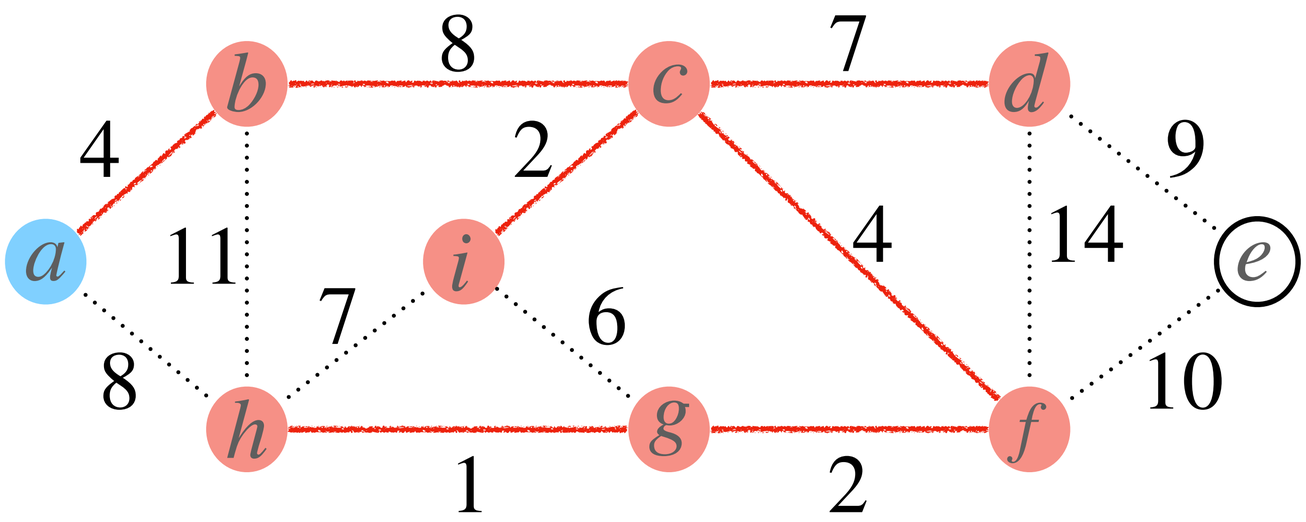

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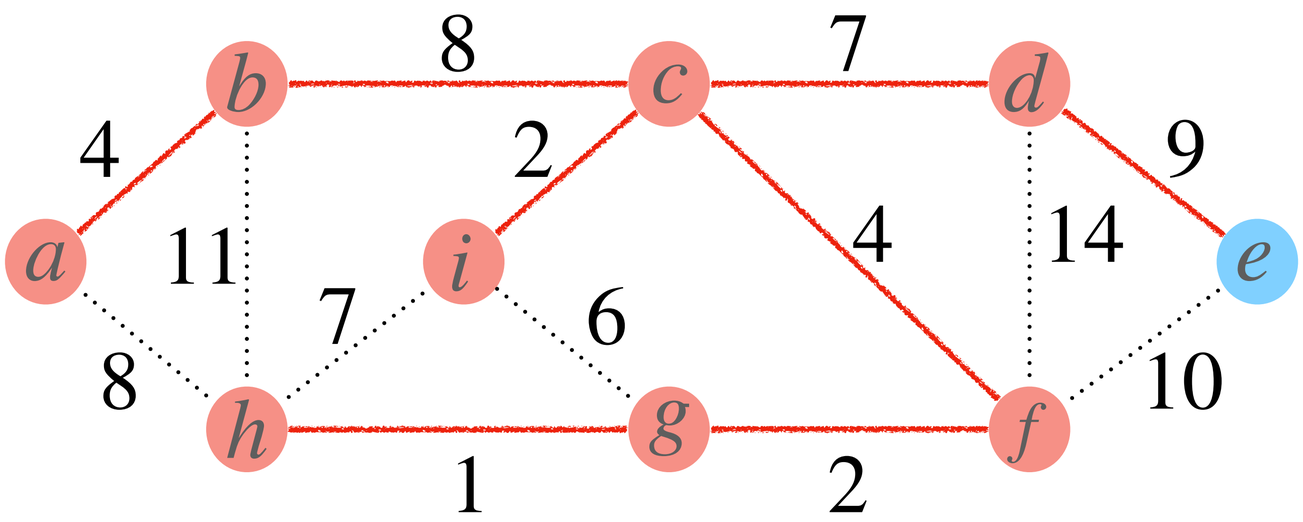

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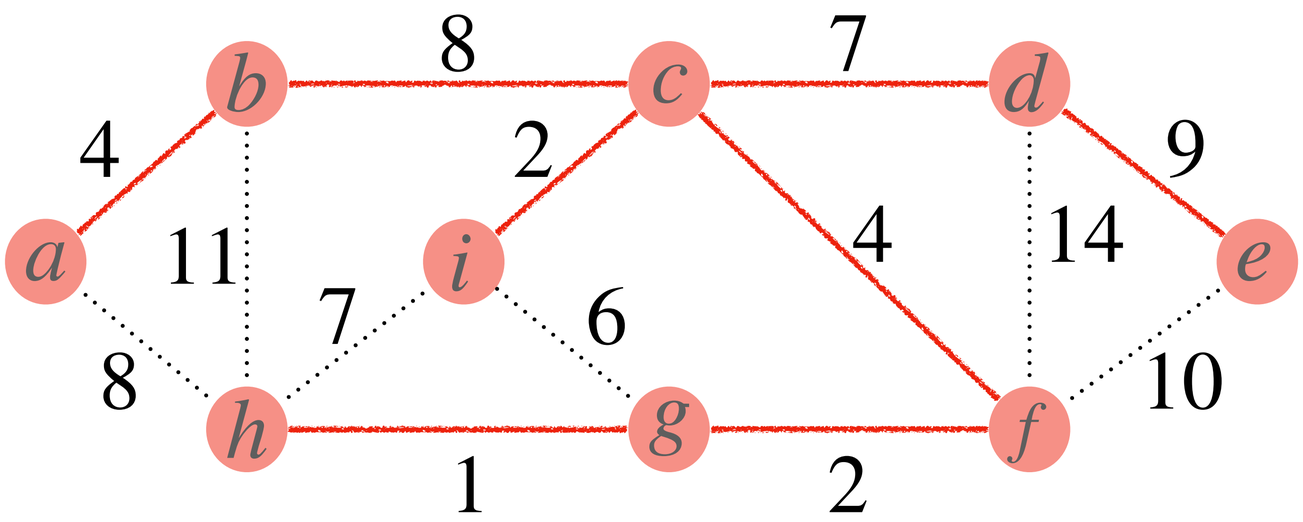

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Minimum Spanning Tree

Both Kruskal's Algorithm and Prim's Algorithm give optimal solutions (i.e, MST).
Why?

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A more general algorithm:

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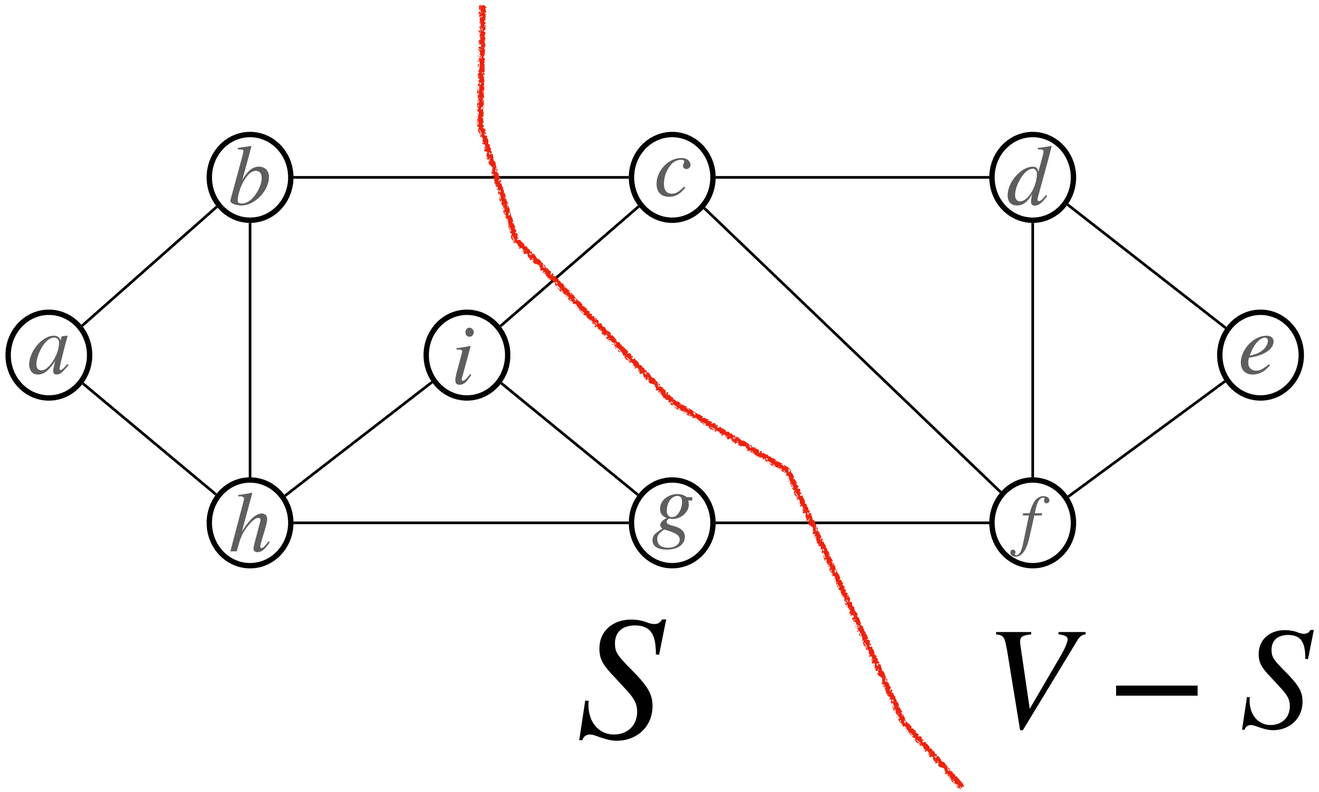

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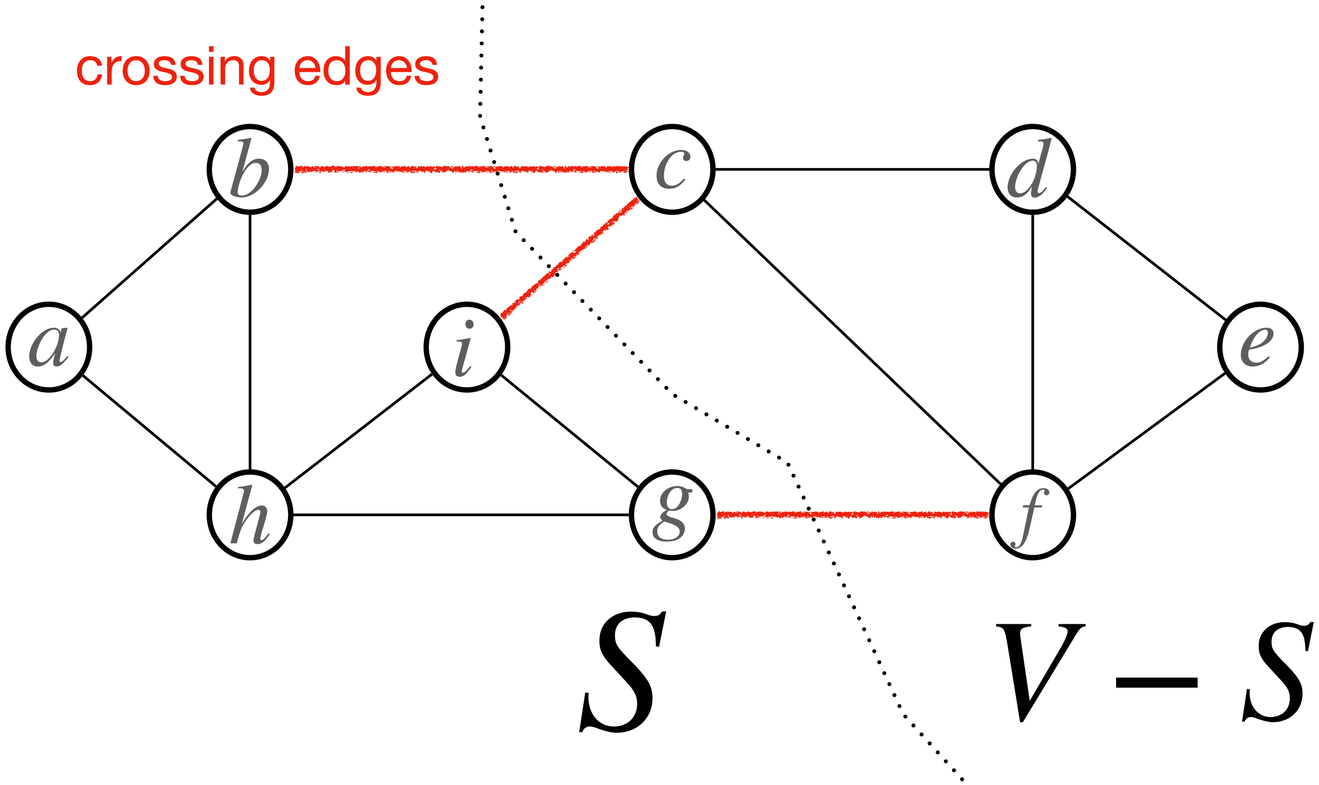

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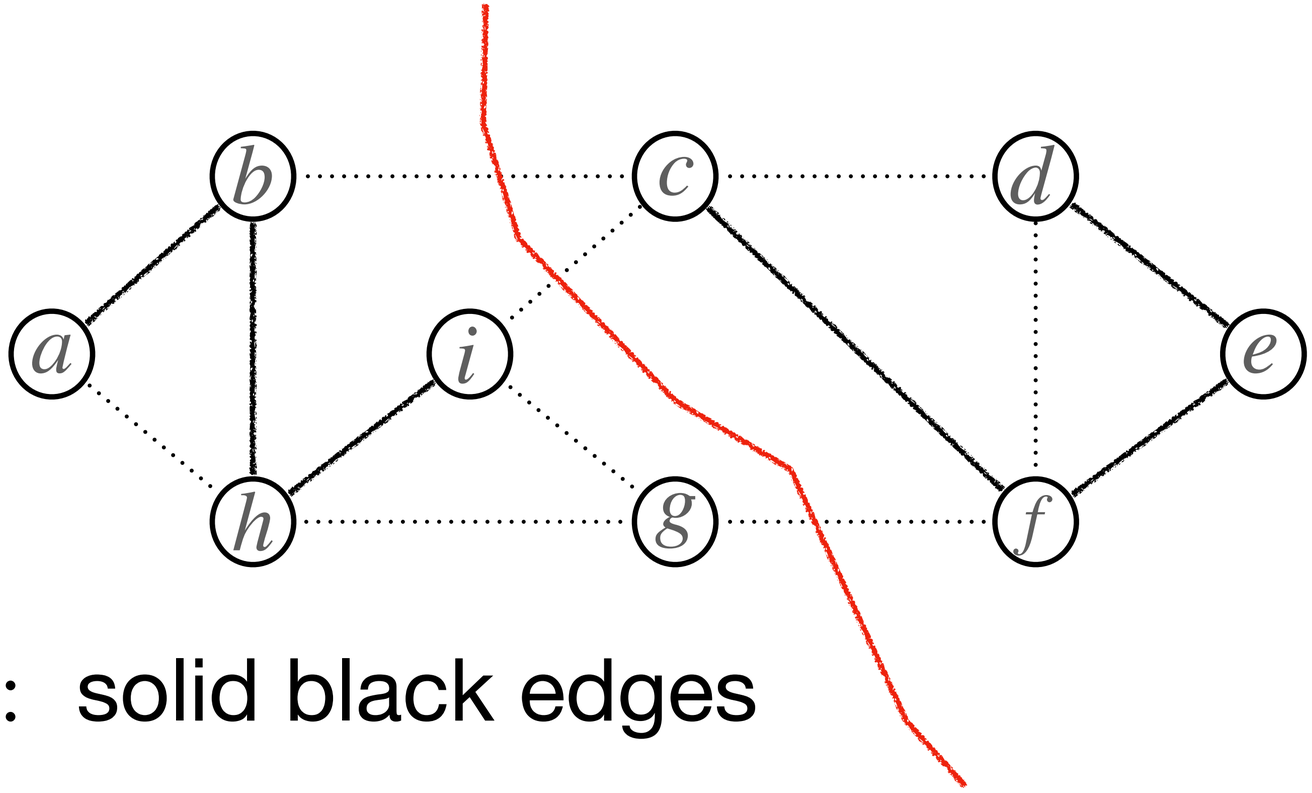

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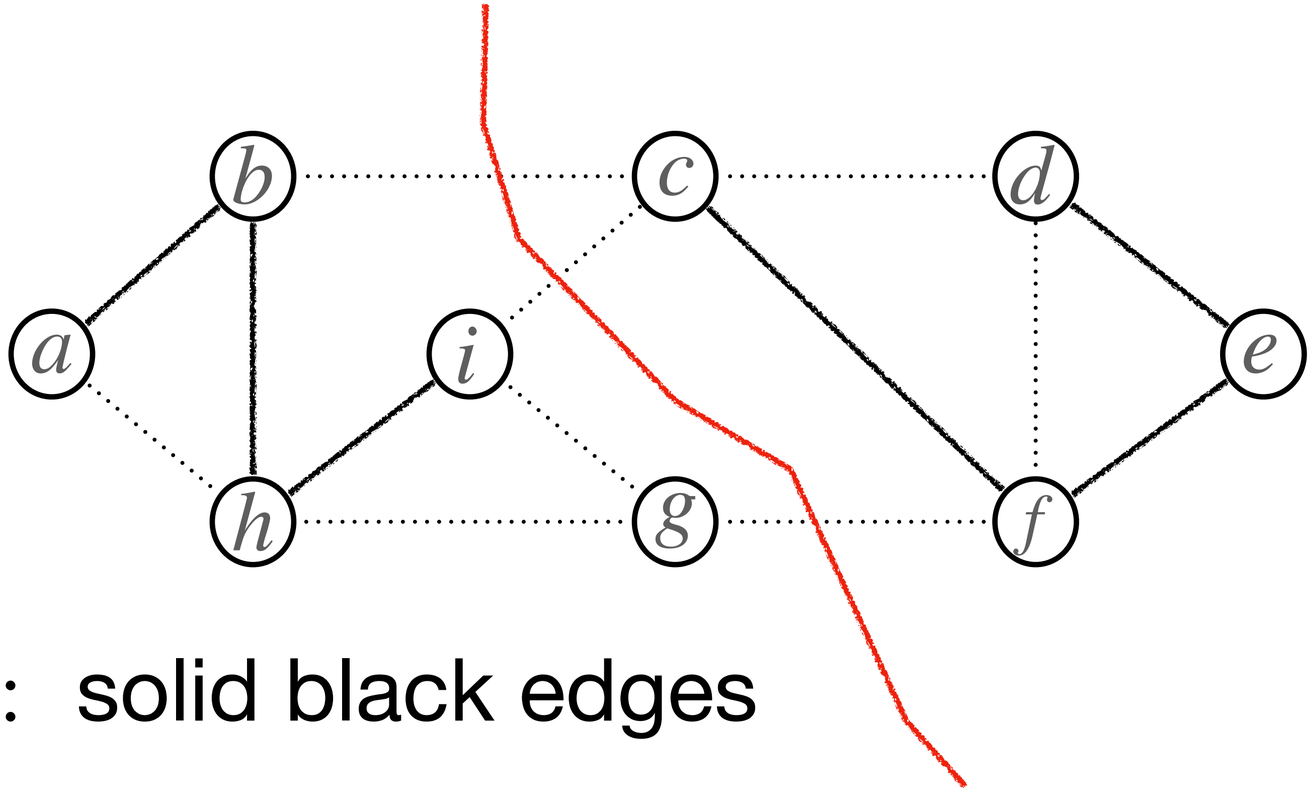

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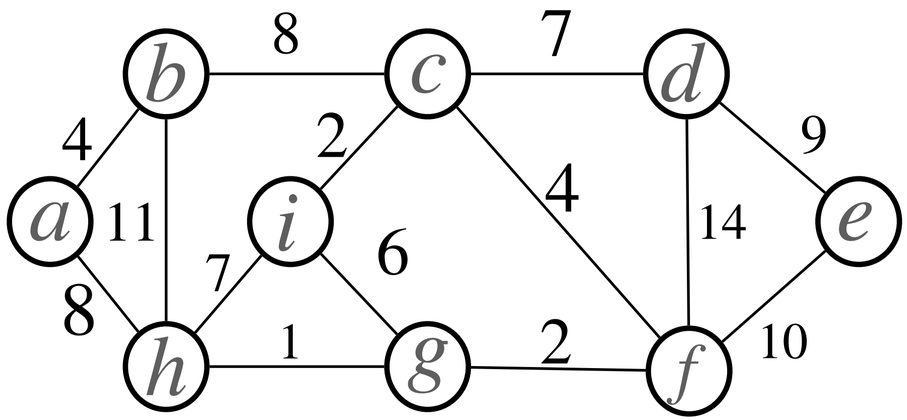

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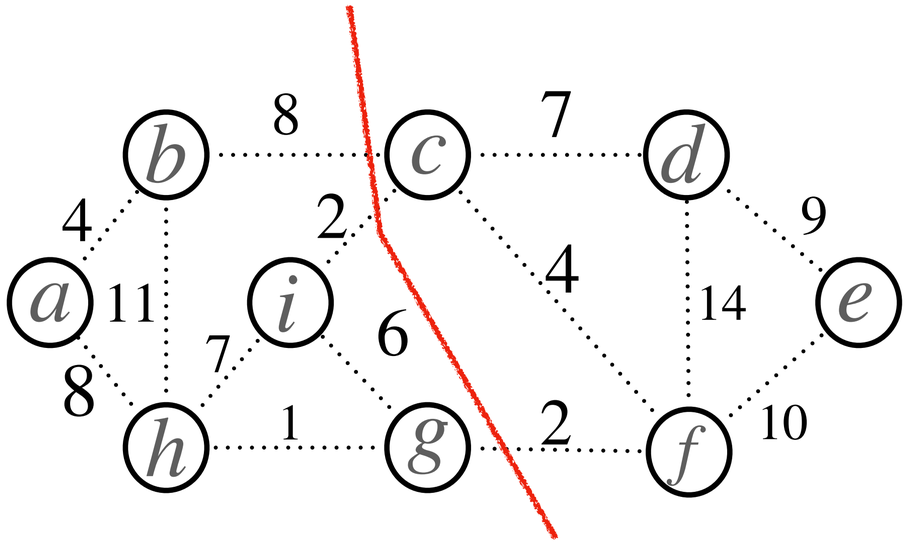

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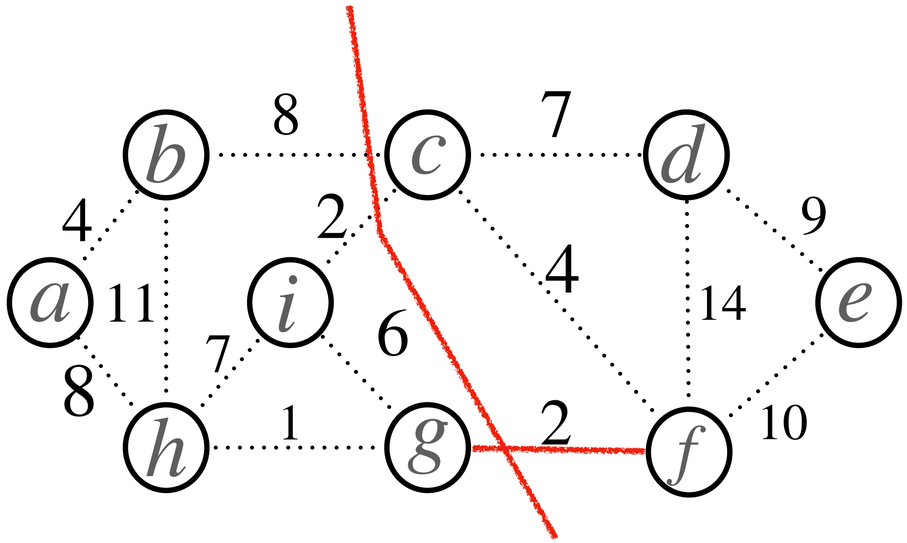

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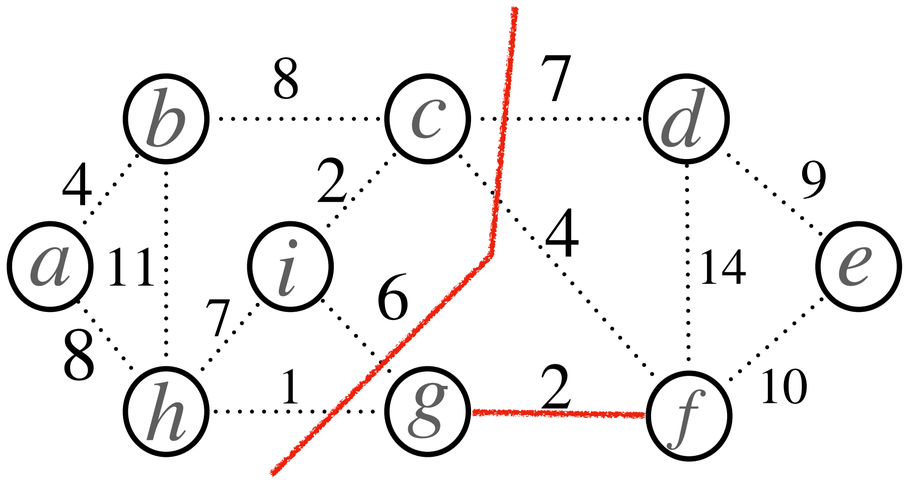

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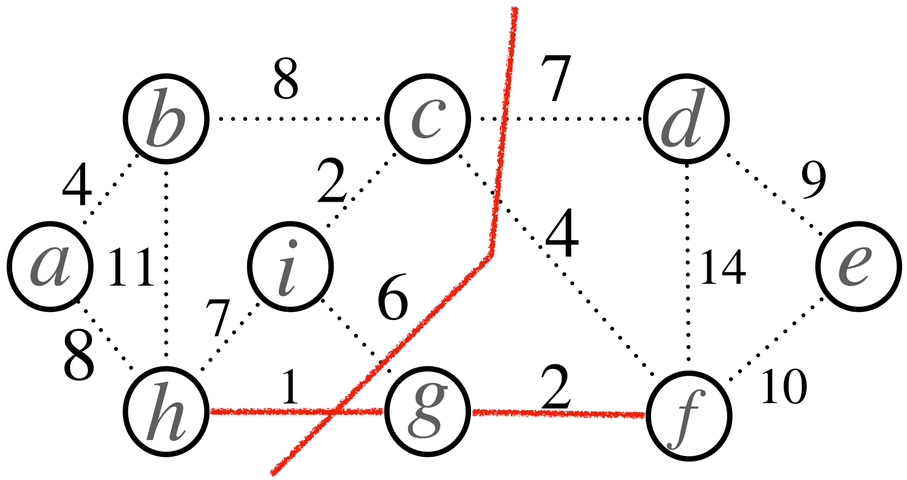

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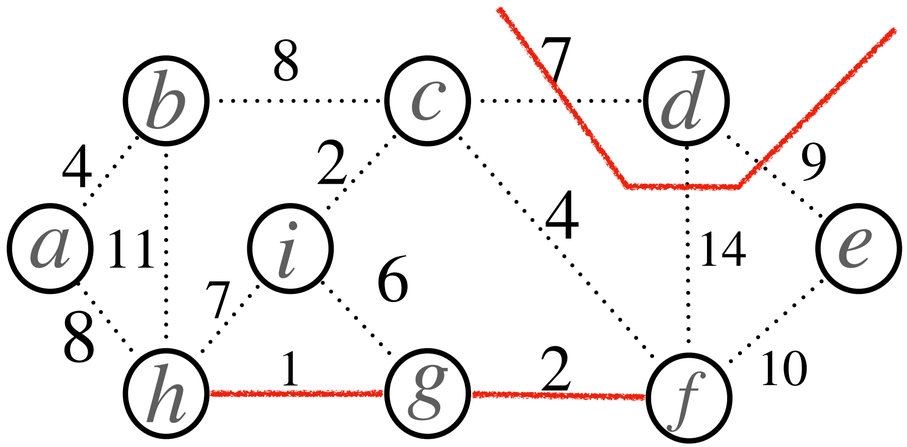

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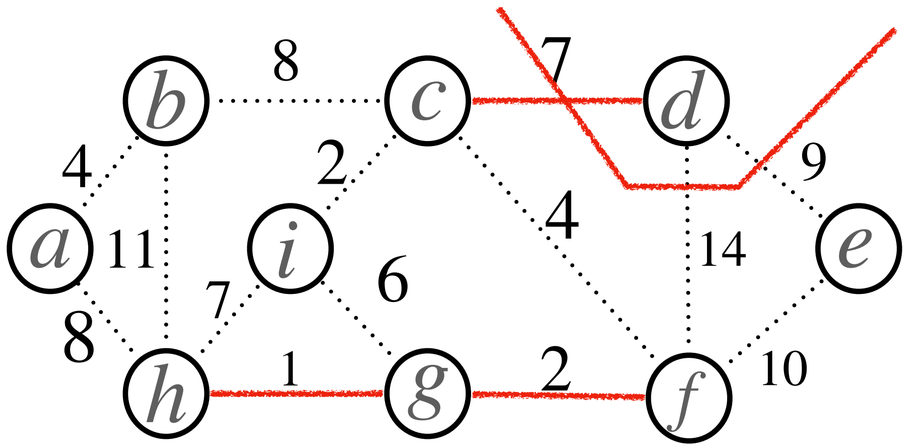

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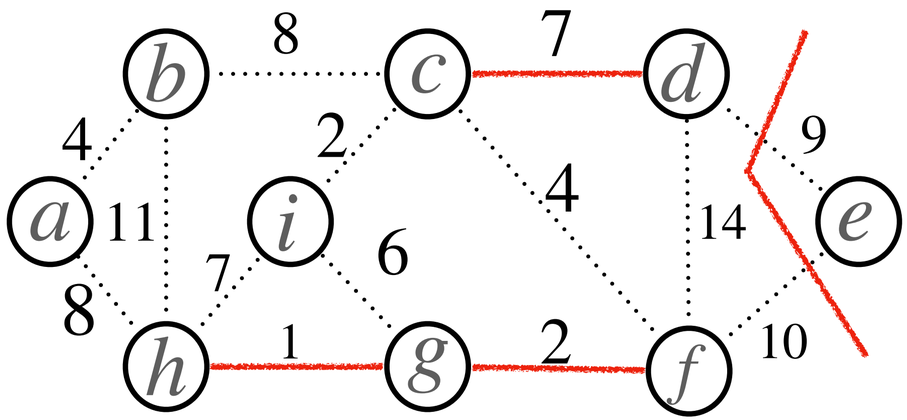

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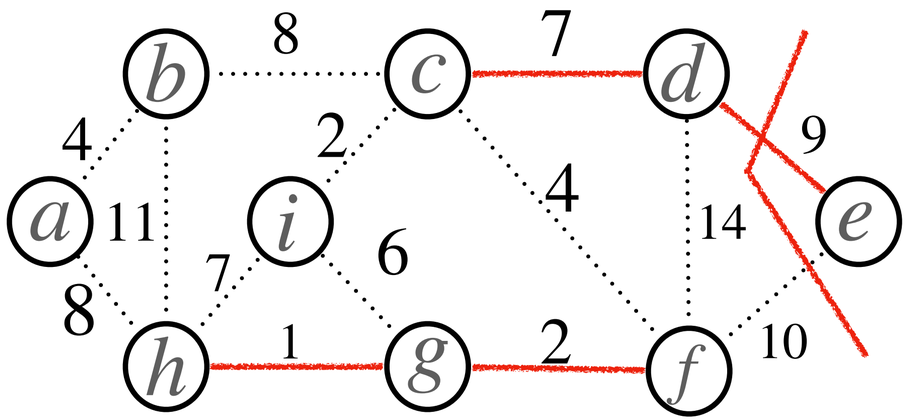

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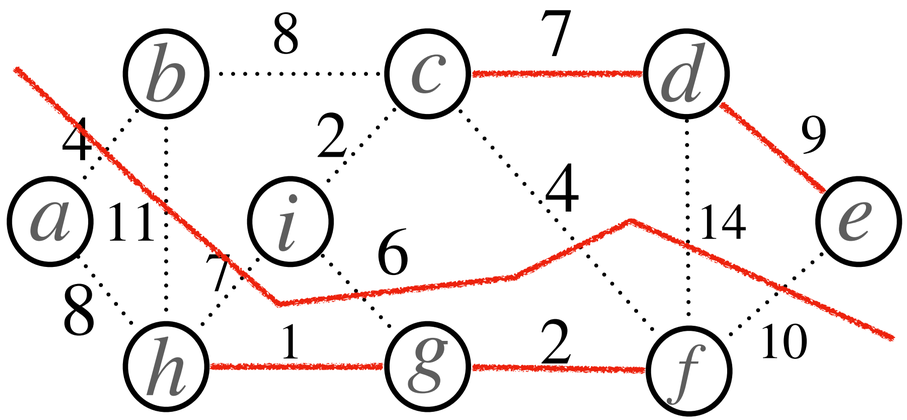

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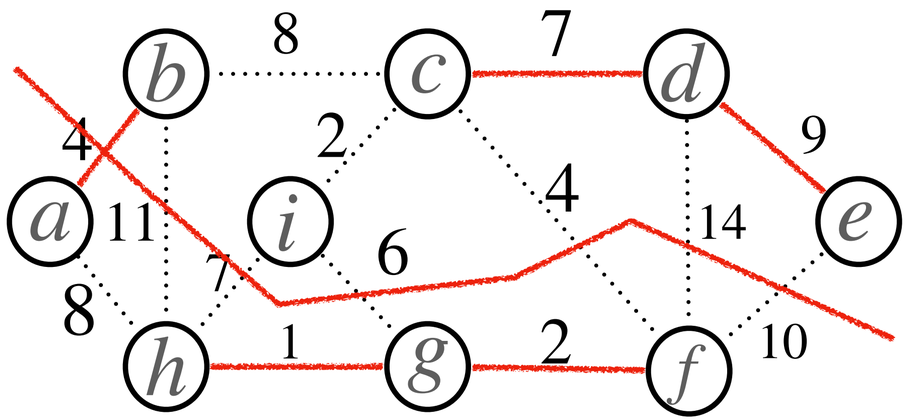

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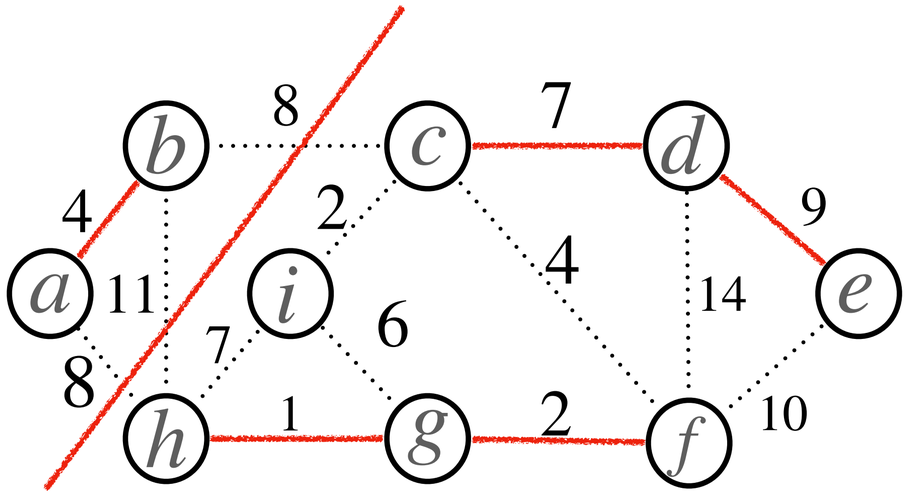

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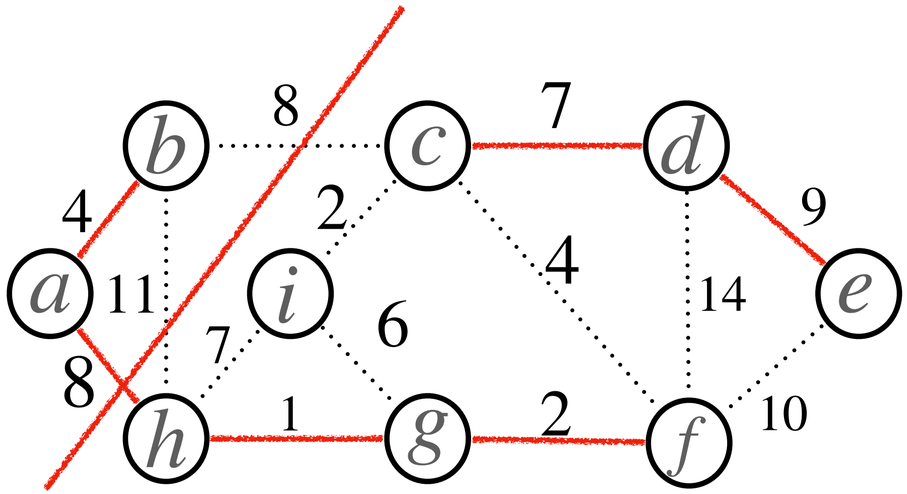

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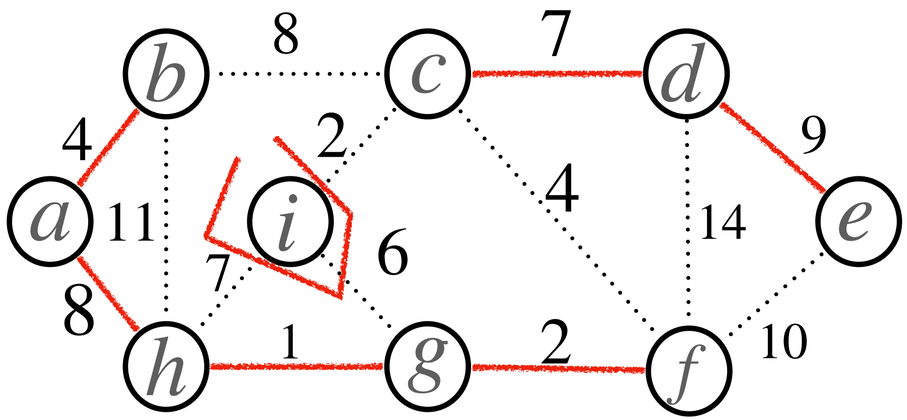

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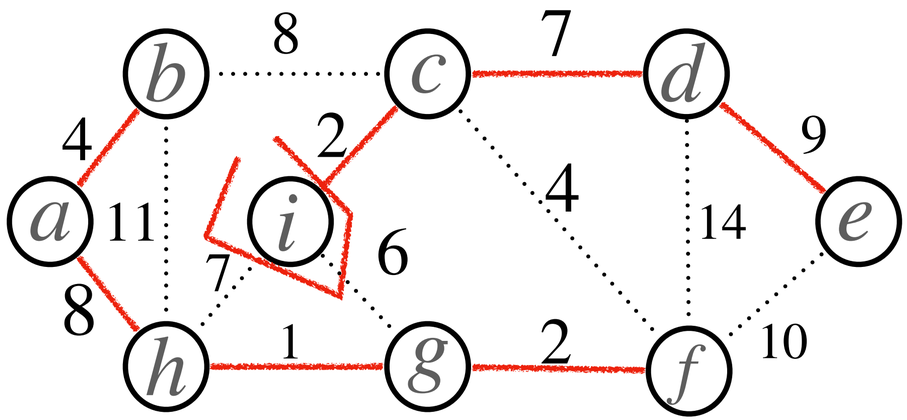

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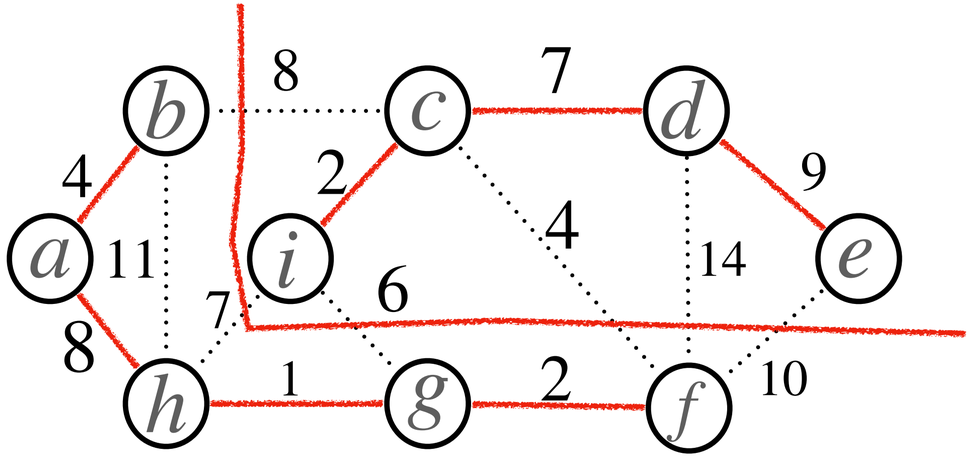

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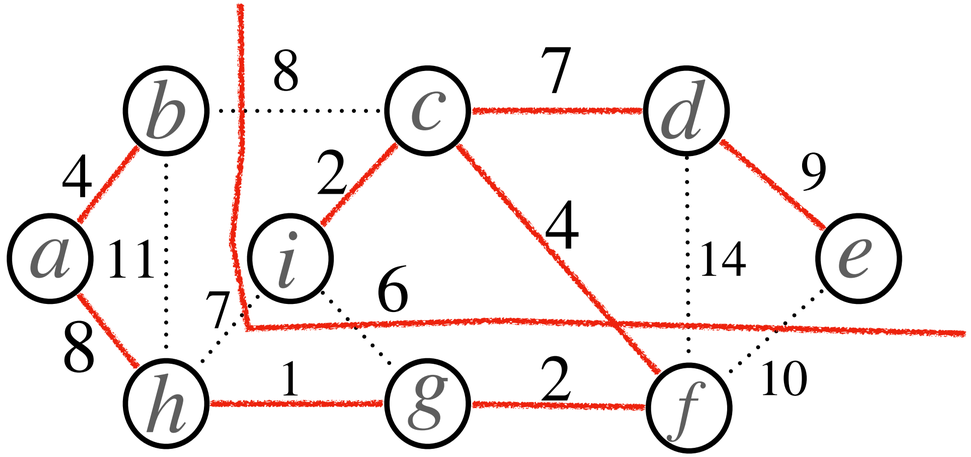

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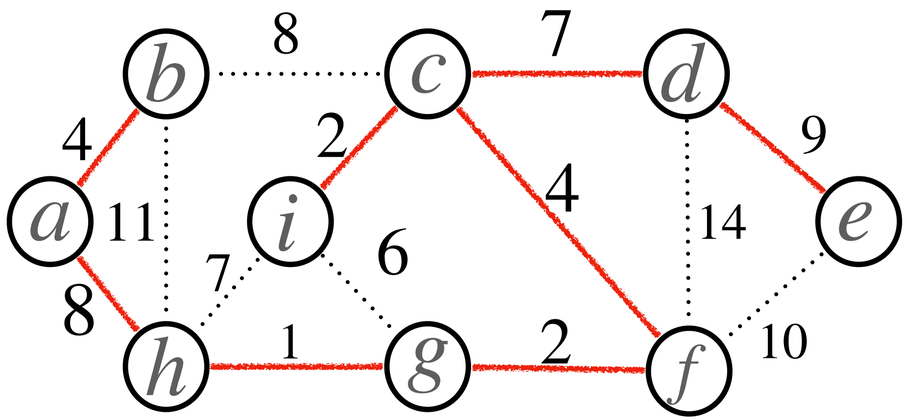

Weight = 4+8+1+2+4+2+7+9 = 37

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Lemma: Assume there exists an MST that contains all the edges chosen so far. Then, given a cut that respects those chosen edges, there also exists an MST that contains not only the chosen edges, but also the minimum-weight crossing edge for the cut.

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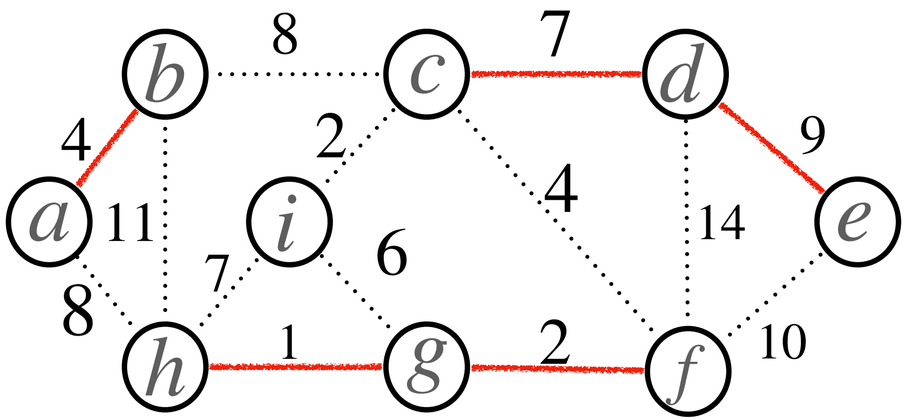
Proof:

Red edges: edges chosen so far.



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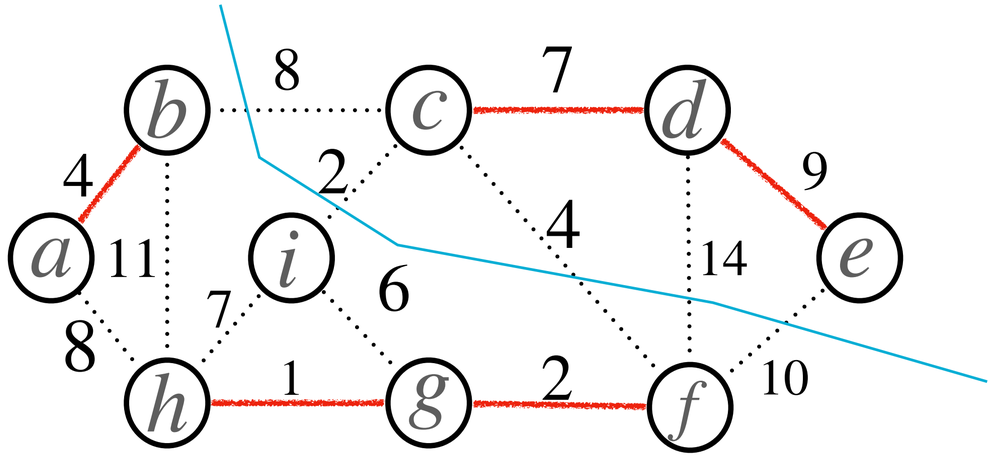
Lemma: Assume there exists an MST that contains all the edges chosen so far. Then, given a cut that respects those chosen edges, there also exists an MST that contains not only the chosen edges, but also the minimum-weight crossing edge for the cut.

Proof:

A cut that respects the chosen edges.



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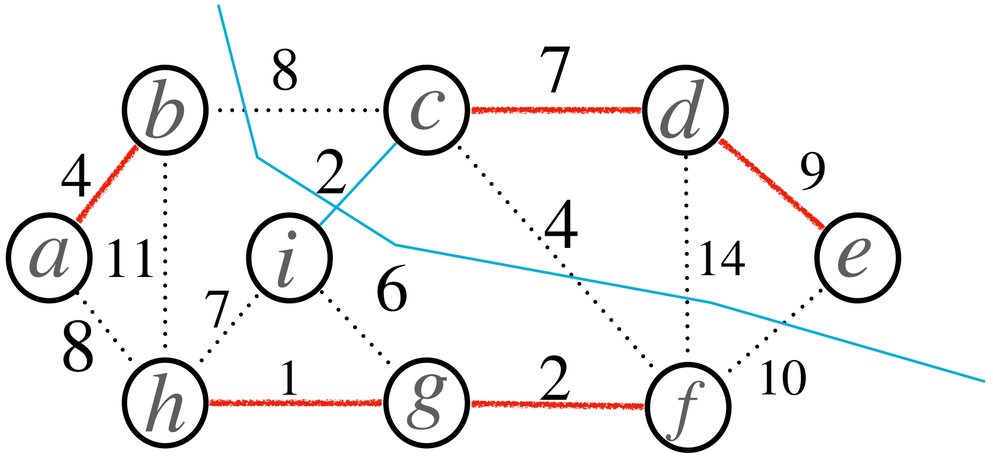
Lemma: Assume there exists an MST that contains all the edges chosen so far. Then, given a cut that respects those chosen edges, there also exists an MST that contains not only the chosen edges, but also the minimum-weight crossing edge for the cut.

Proof:

The minimum-weight crossing edge



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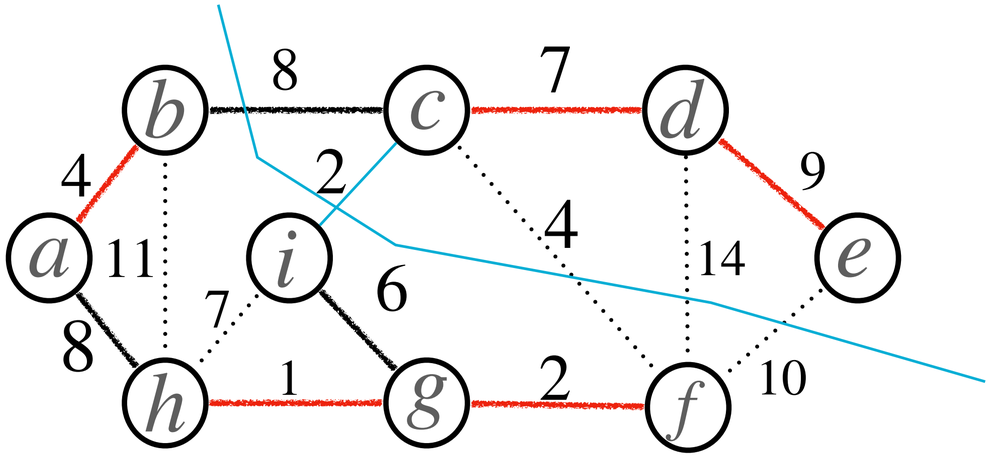
Lemma: Assume there exists an MST that contains all the edges chosen so far. Then, given a cut that respects those chosen edges, there also exists an MST that contains not only the chosen edges, but also the minimum-weight crossing edge for the cut.

Proof: Consider an MST that contains the chosen edges. Assume it does not contain the blue edge.



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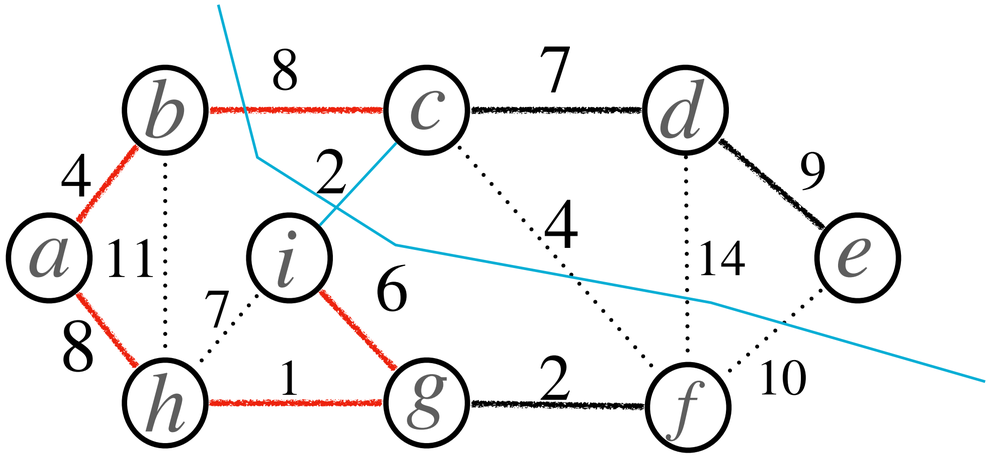
Proof:

There is a path in MST connecting i and c



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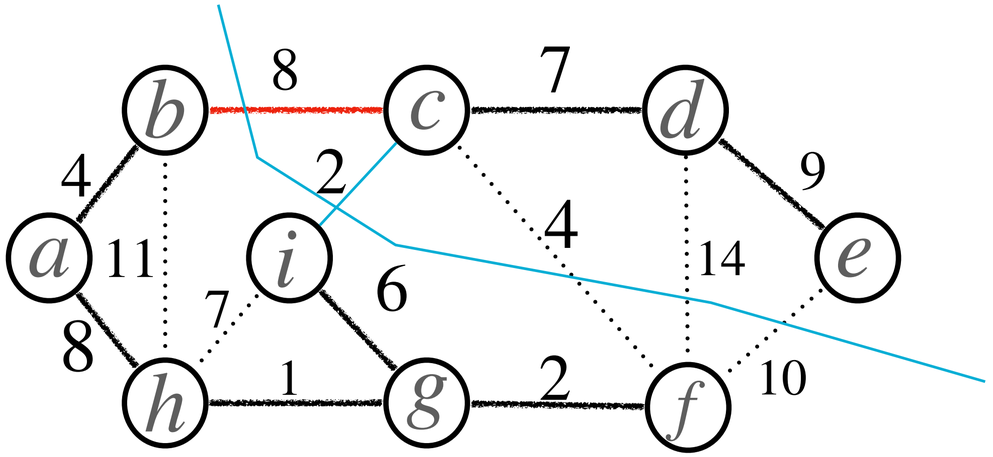
Lemma: Assume there exists an MST that contains all the edges chosen so far. Then, given a cut that respects those chosen edges, there also exists an MST that contains not only the chosen edges, but also the minimum-weight crossing edge for the cut.

Proof:

This path must contain a crossing edge



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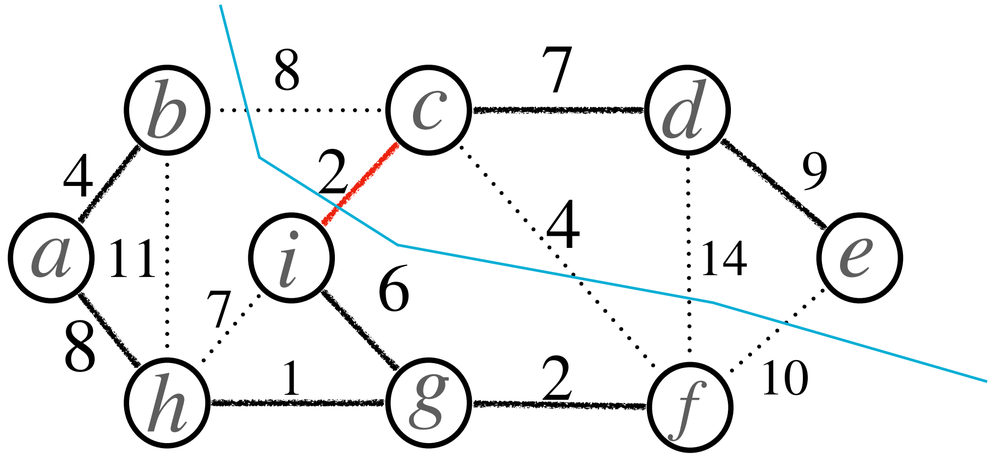
Proof:

Replace it by the minimum-weight crossing edge, we get a better MST (but that's a contradiction).



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This is GREAT…

Lemma: Assume there exists an MST that contains all the edges chosen so far. Then, given a cut that respects those chosen edges, there also exists an MST that contains not only the chosen edges, but also the minimum-weight crossing edge for the cut.

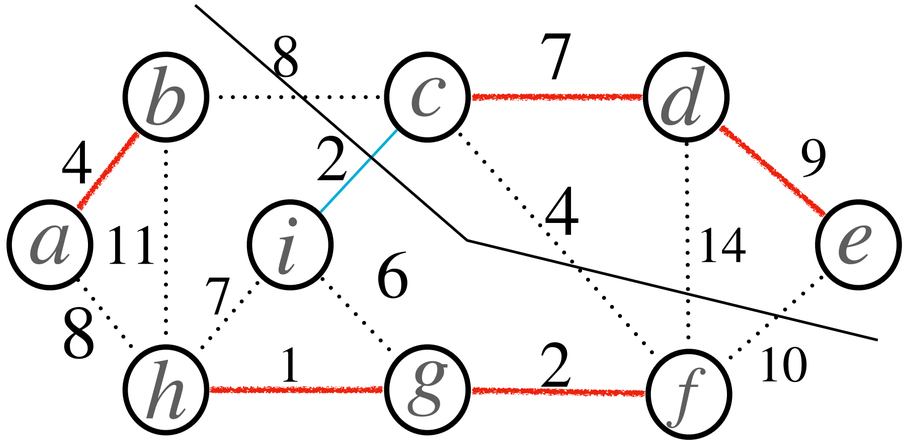
This "more general algorithm" must produce an MST.

But why is it more general than Kruskal's algorithm and Prim's algorithm?
(If that is the case, then Kruskal's algorithm and Prim's algorithm also produce MST.)

Minimum Spanning Tree

Kruskal's Algorithm:

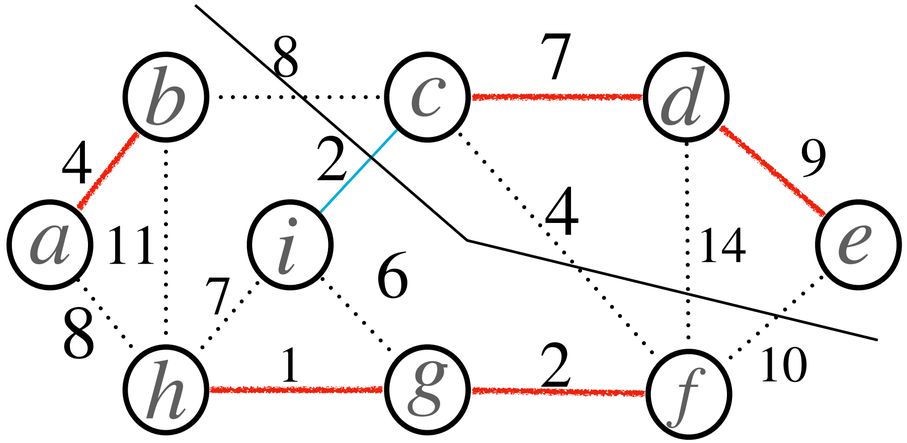
Select the cut that minimizes the weight of the minimum-weight crossing edge.



Minimum Spanning Tree

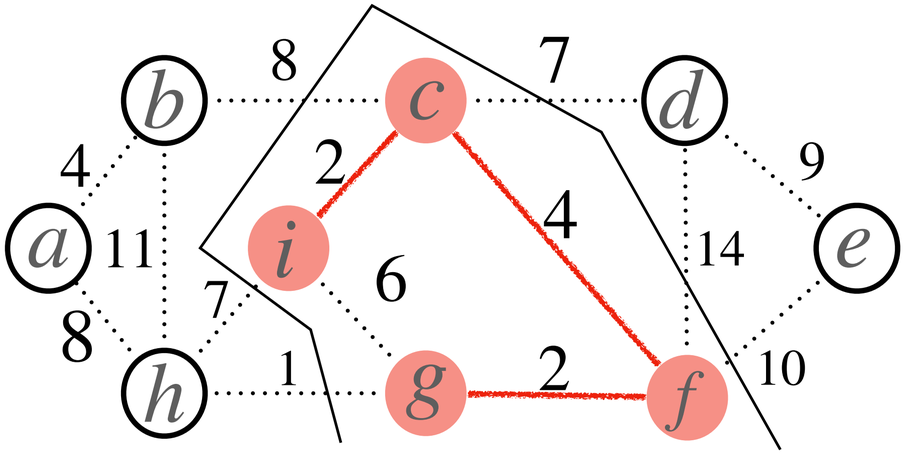
Kruskal's Algorithm:

Select the cut that minimizes the weight of the minimum-weight crossing edge.



Prim's Algorithm:

Select the cut that separates the connected rooted tree from the remaining nodes.



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Kruskal's Algorithm:

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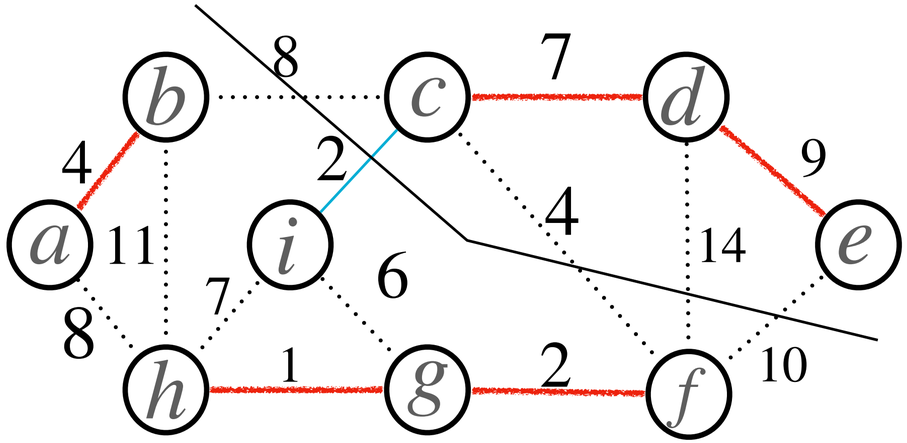

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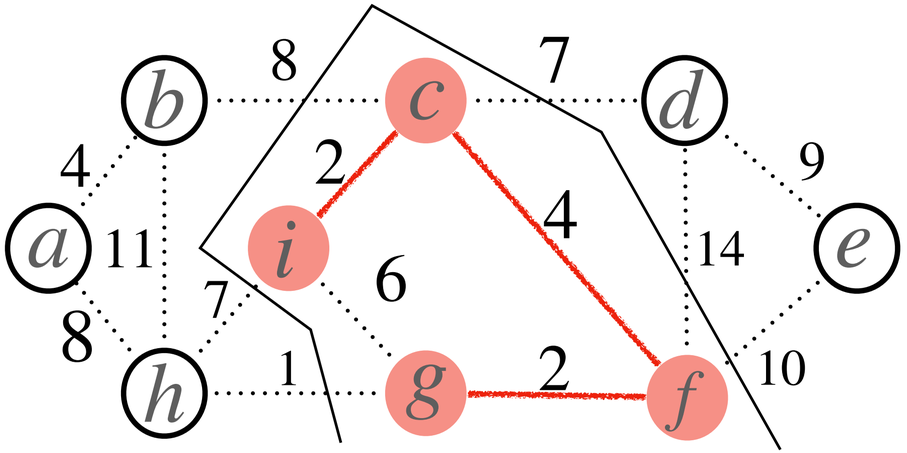

Then why use the above two algorithms instead of the "more general algorithm"?
Lower time complexity can be achieved when the algorithm is more specific.

Quiz questions:

1. What is the main idea of the more general algorithm for finding an MST?

2. Why are Kruskal's Algorithm and Prim's Algorithm both special cases of the more general algorithm?