Algorithms

Lecture Topic: Maximum Flow (Part 1)

Anxiao (Andrew) Jiang

Roadmap of this lecture:

1. Maximum Flow.

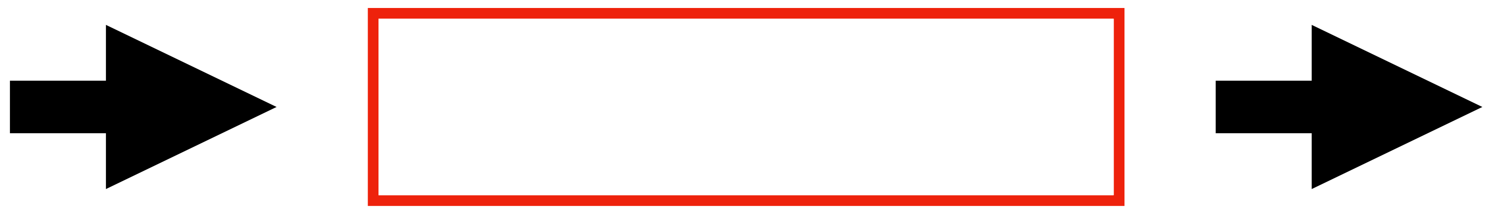
   1.1 Define "maximum flow".

   1.2 Ford-Fulkerson Method for maximum flow.

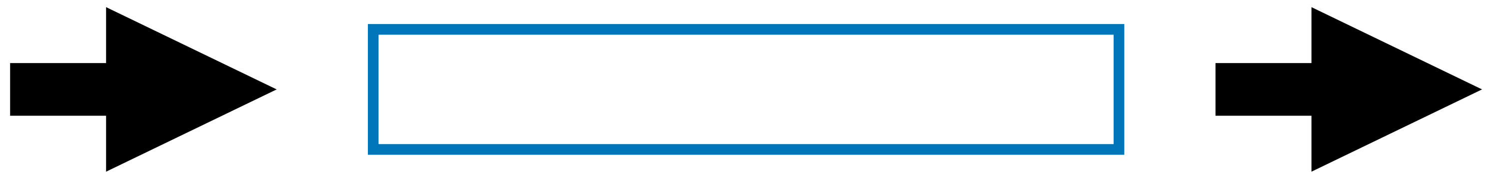
Maximum Flow

Capacity of a flow “pipe"

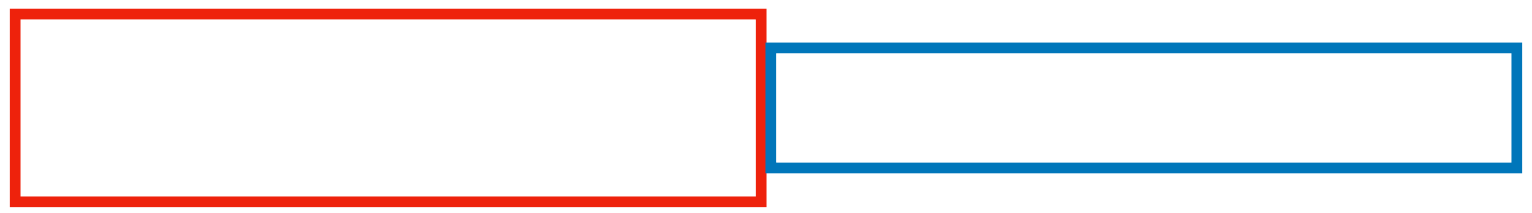
Capacity: 3 units of flow / second



Capacity: 1 unit of flow / second



What is its overall capacity?



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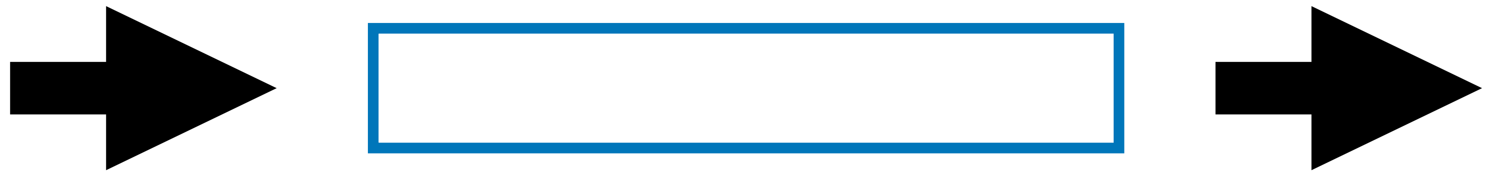

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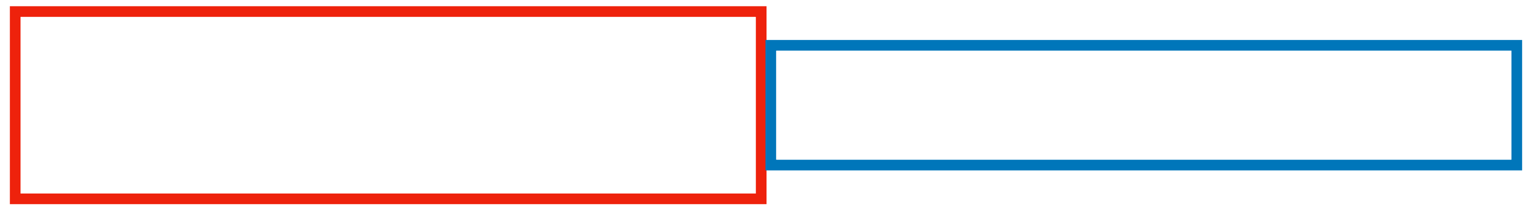
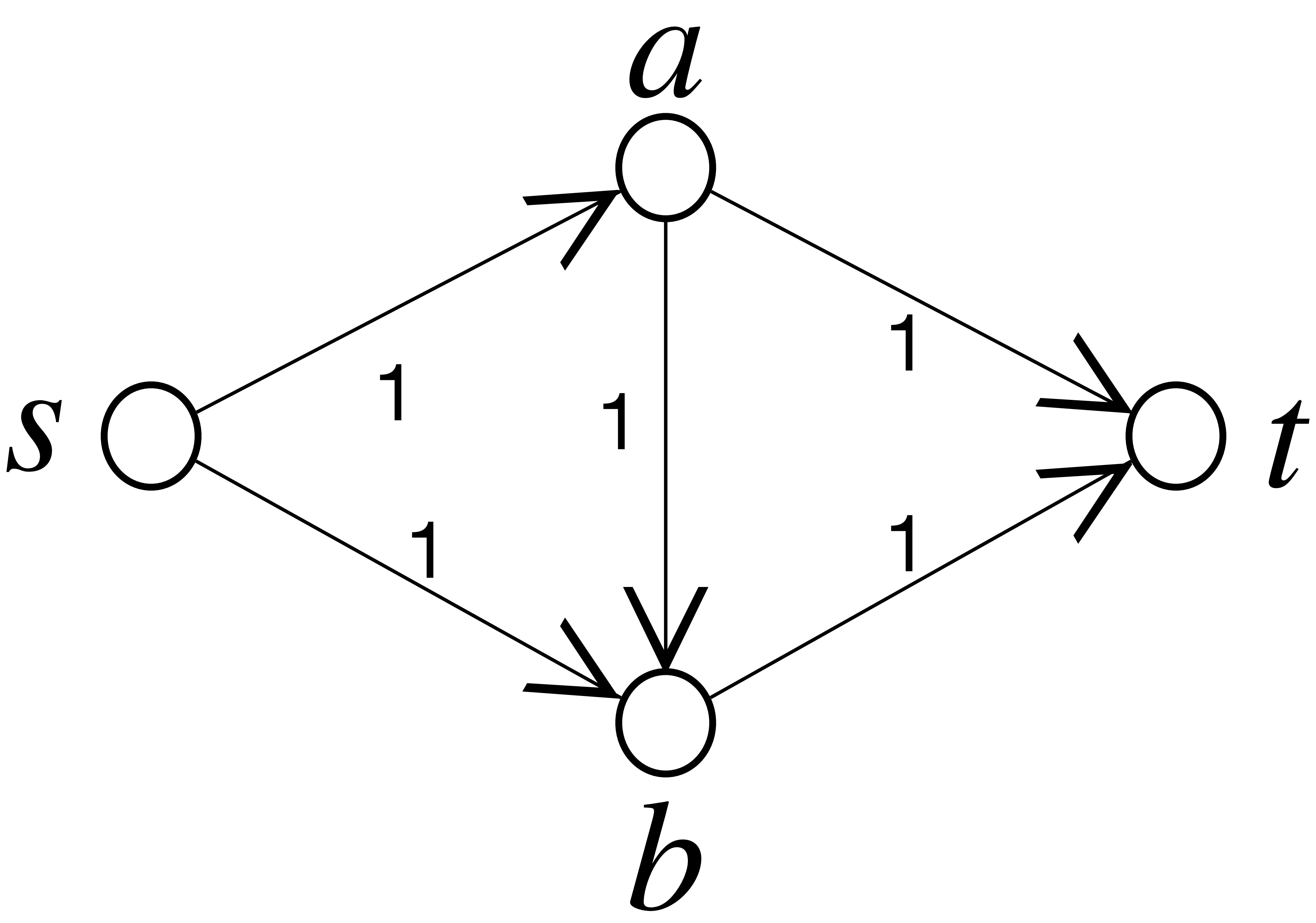


$$min\{3,1\} = 1$$

Maximum Flow

How much flow can be transmitted from s to t,
assuming every edge has capacity 1?



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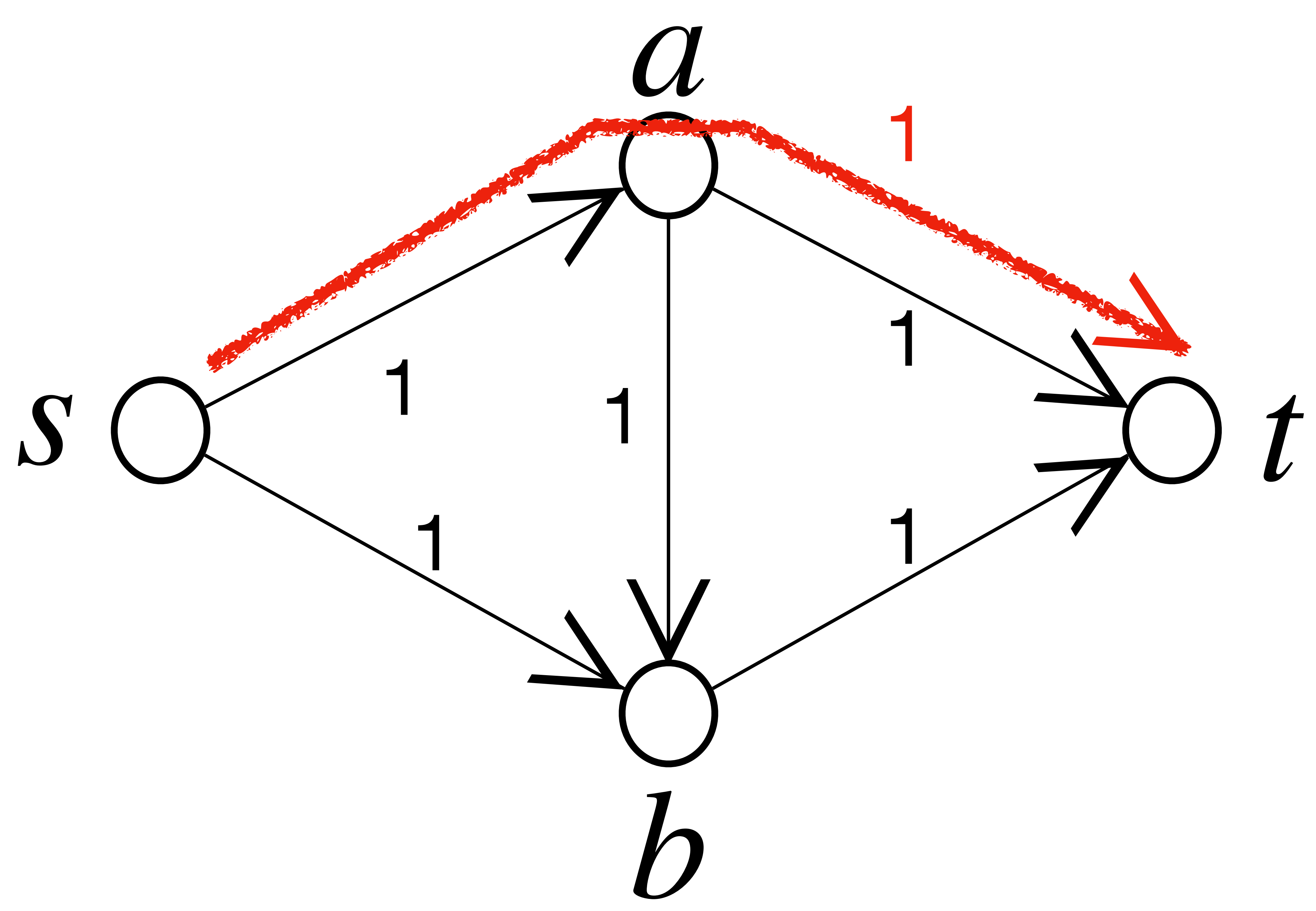

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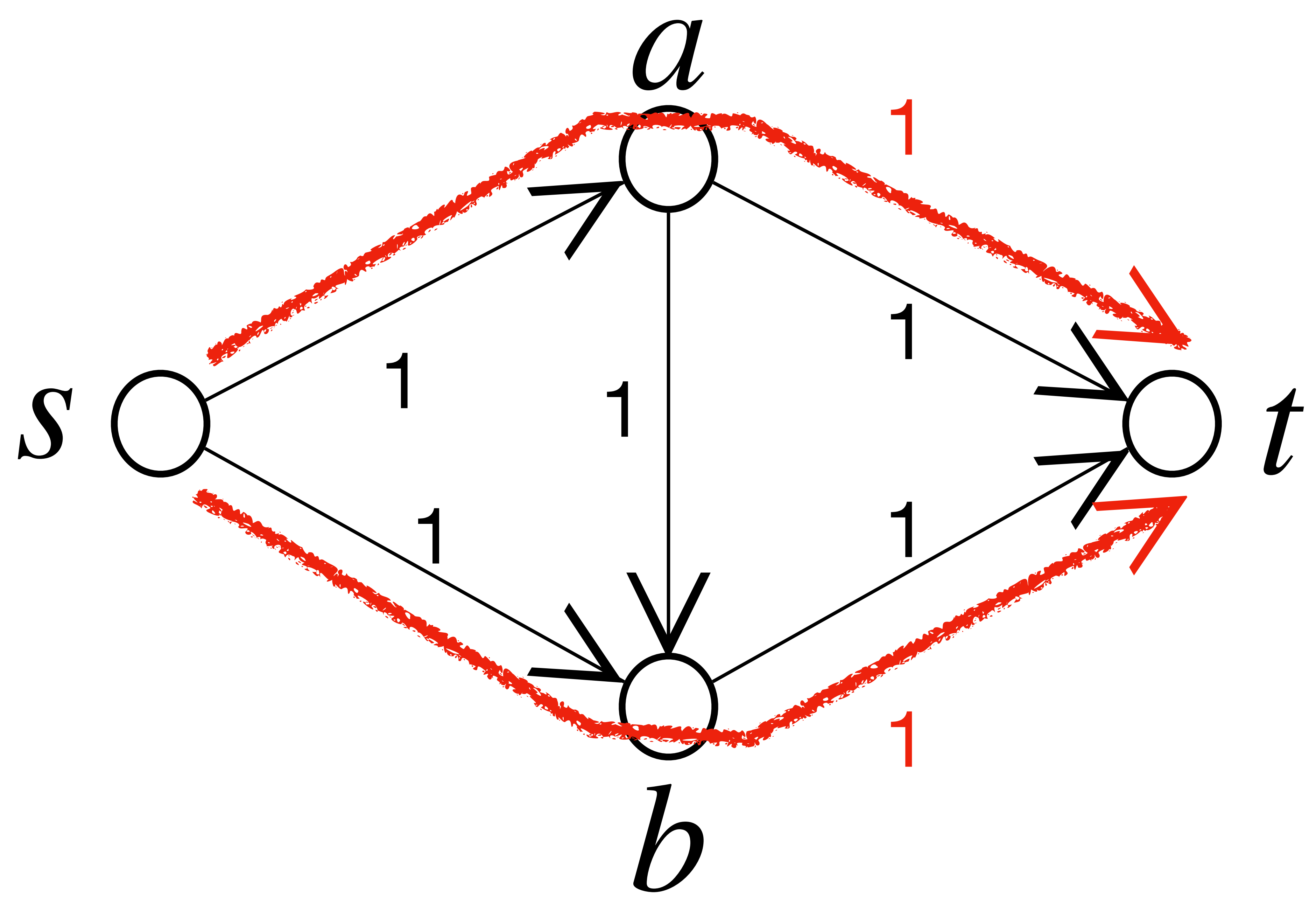
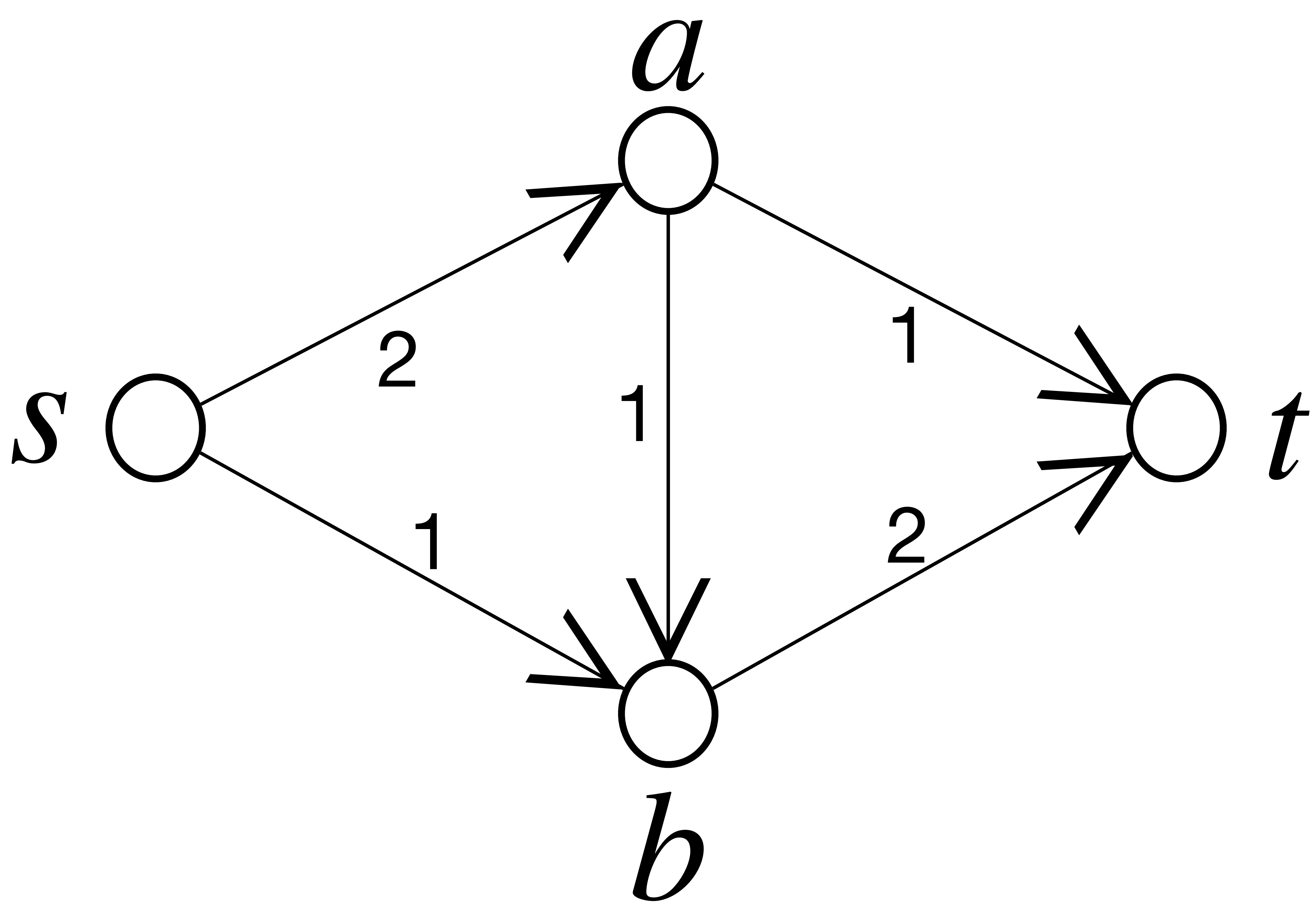


Maximum flow from s to t :

2

Maximum Flow

How much flow can be transmitted from s to t,
assuming the capacities of edges are as shown?



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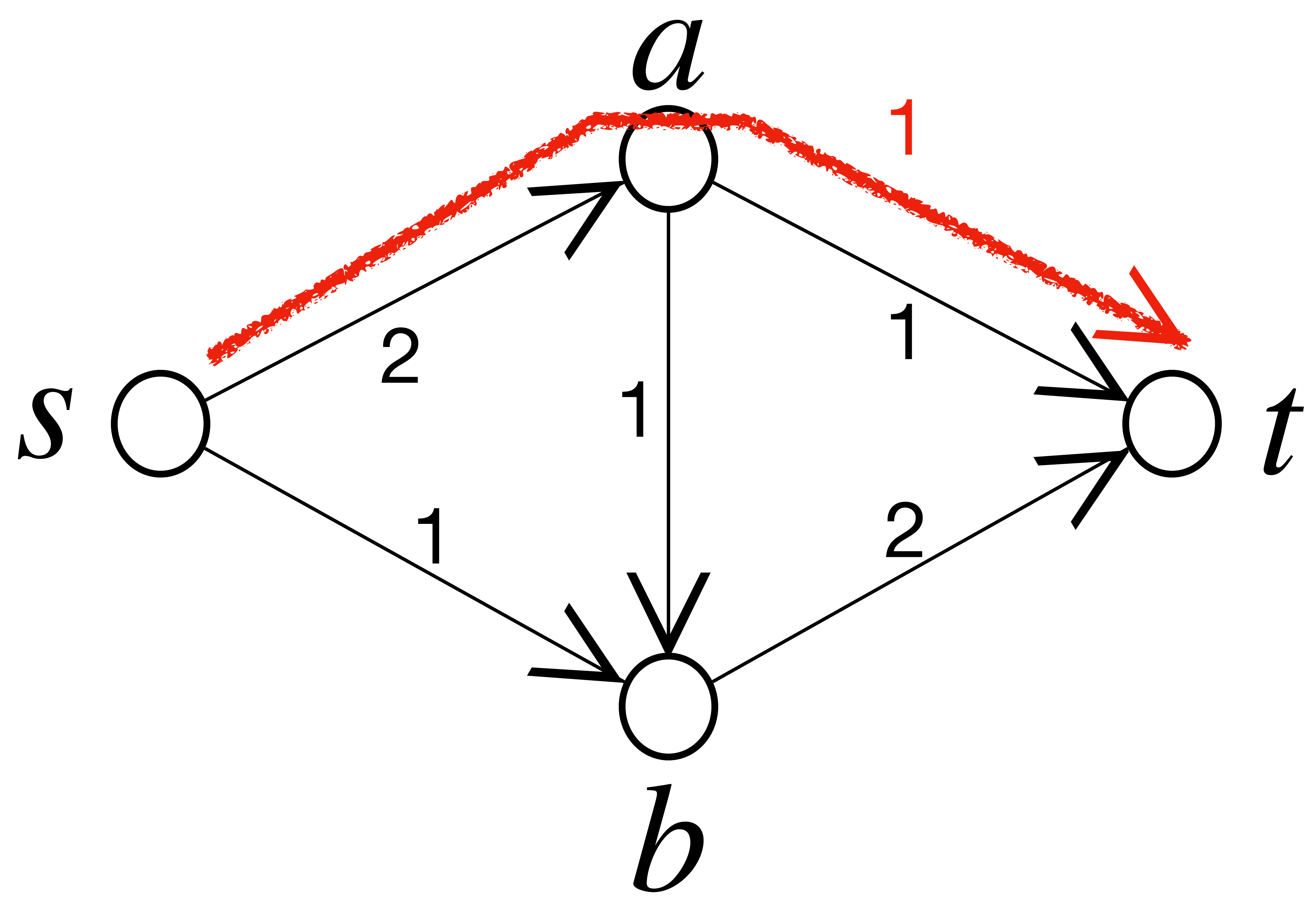

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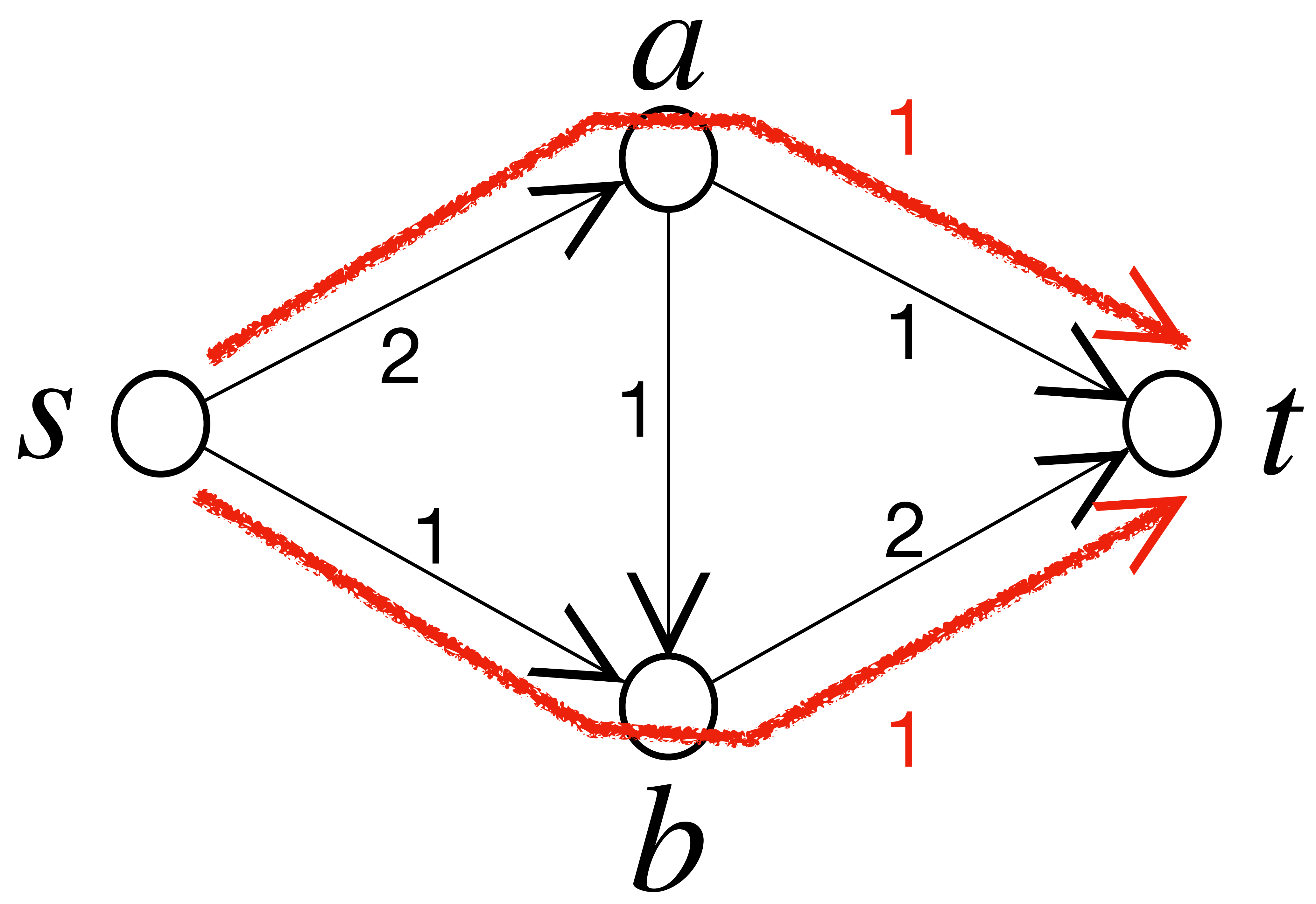

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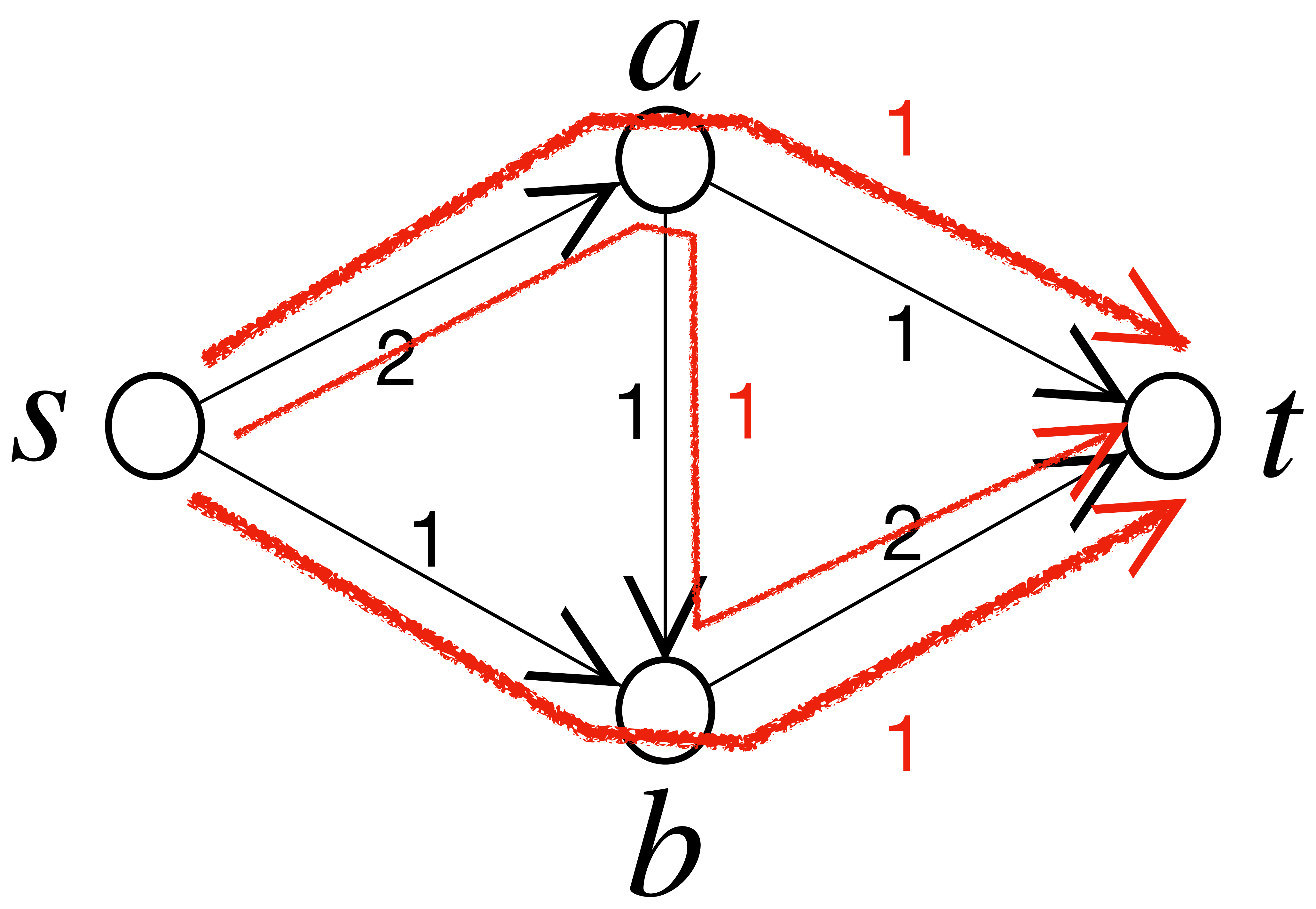


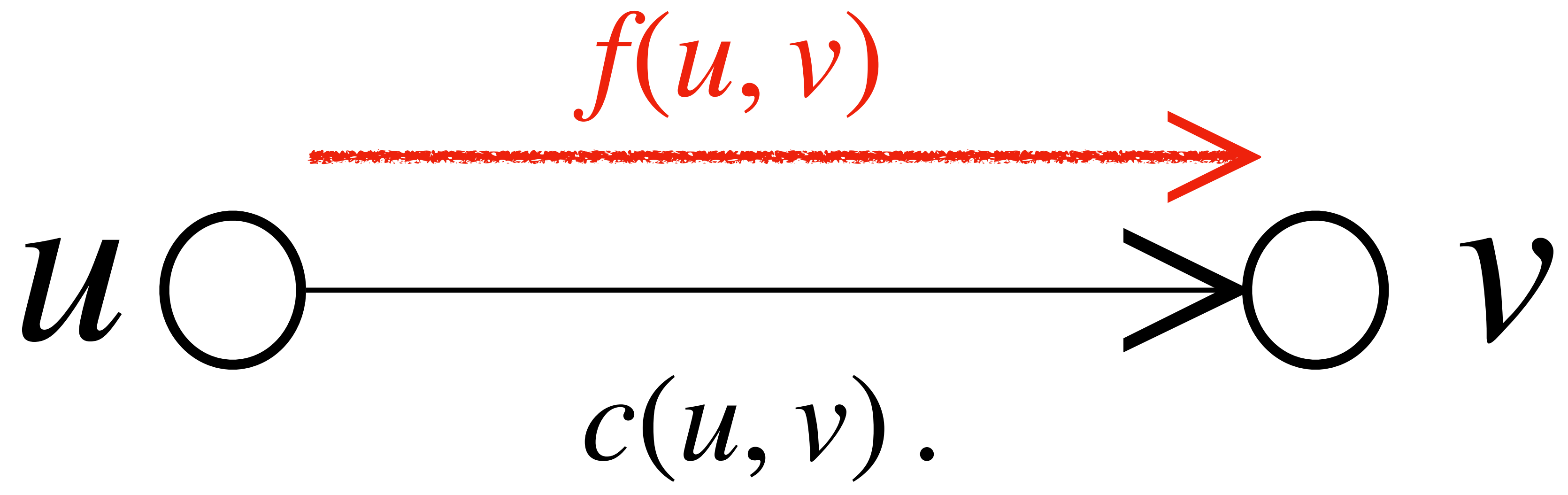
Maximum flow from s to t :

3

Maximum Flow

Input: A directed graph G=(V,E), where every edge $(u, v) \in E$ has a non-negative capacity $c(u, v)$. Let $s \in V$ be a “source" node, and let $t \in V$ be a “sink" node.

Flow: $\forall\ (u, v) \in E$, let $f(u, v)$ be the flow from u to v in the edge (u, v).



Capacity constraint: $0 \leq f(u, v) \leq c(u, v)$

Maximum Flow

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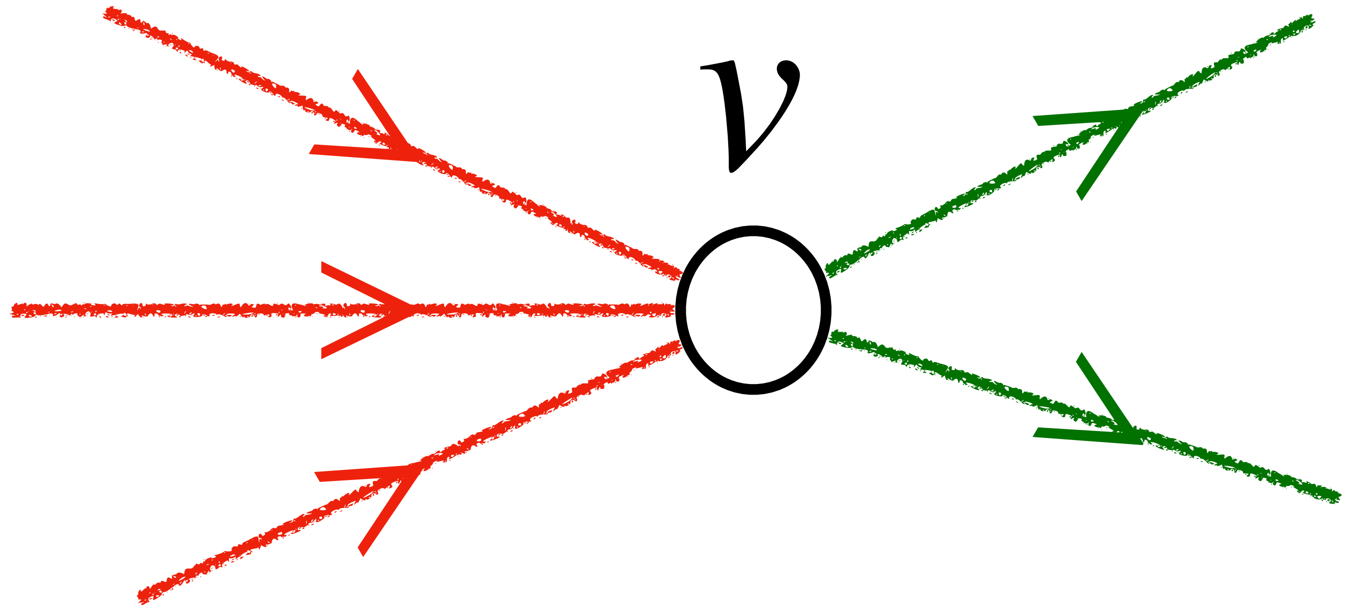
Constraints for flow f:

1) Capacity constraint: $\forall\ (u, v) \in E, \qquad 0 \leq f(u, v) \leq c(u, v)$

2) Flow-conservation constraint (also called “in-flow equals out-flow” constraint):

For every node that is not source s or sink t,
its total incoming flow equals its total outgoing flow.



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$$\forall\ v \in V - \{s, t\}, \qquad \sum_{(u,v) \in E} f(u, v) = \sum_{(v,u) \in E} f(v, u)$$

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Size of flow f: The net amount of flow leaving the source s, which is also the net amount of flow entering the sink t.

Maximum Flow

Input: A directed graph G=(V,E), where every edge $(u, v) \in E$ has a non-negative capacity $c(u, v)$. Let $s \in V$ be a “source" node, and let $t \in V$ be a “sink" node.

Output: A maximum flow from s to t.

Constraints for flow f:

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Quiz questions:

1. What are the constraints for a network flow?
2. What are the applications of the "Maximum Flow Problem"?

Roadmap of this lecture:

1. Maximum Flow.

    1.1 Define "maximum flow".

    1.2 Ford-Fulkerson Method for maximum flow.

Ford-Fulkerson Method:

Gradually add flow to the network G=(V,E) as follows:

Each time, look at the network with the residual capacities (i.e., “remaining capacities”), and find a path from s to t using only residual capacities, then add as much flow as possible along that path.

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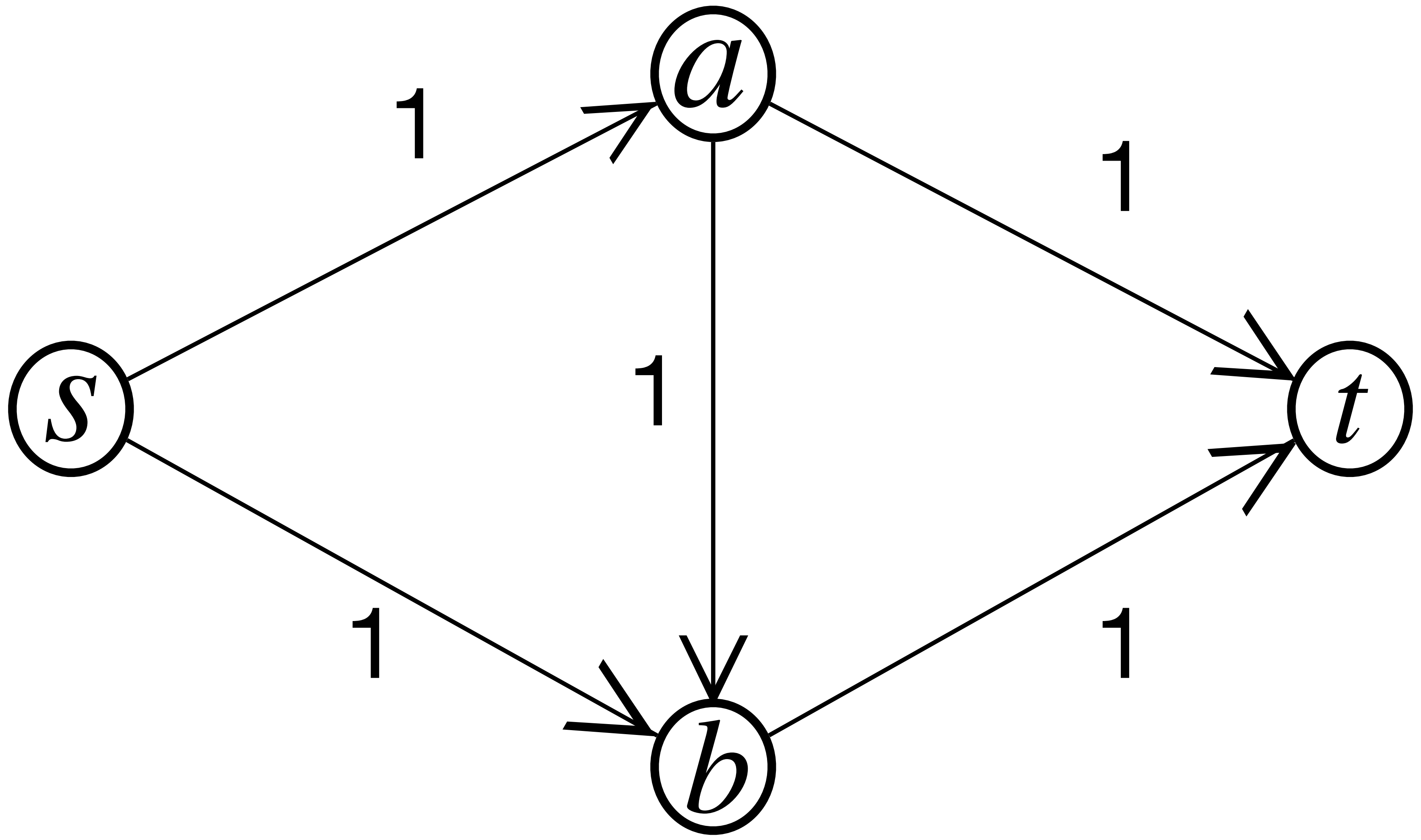

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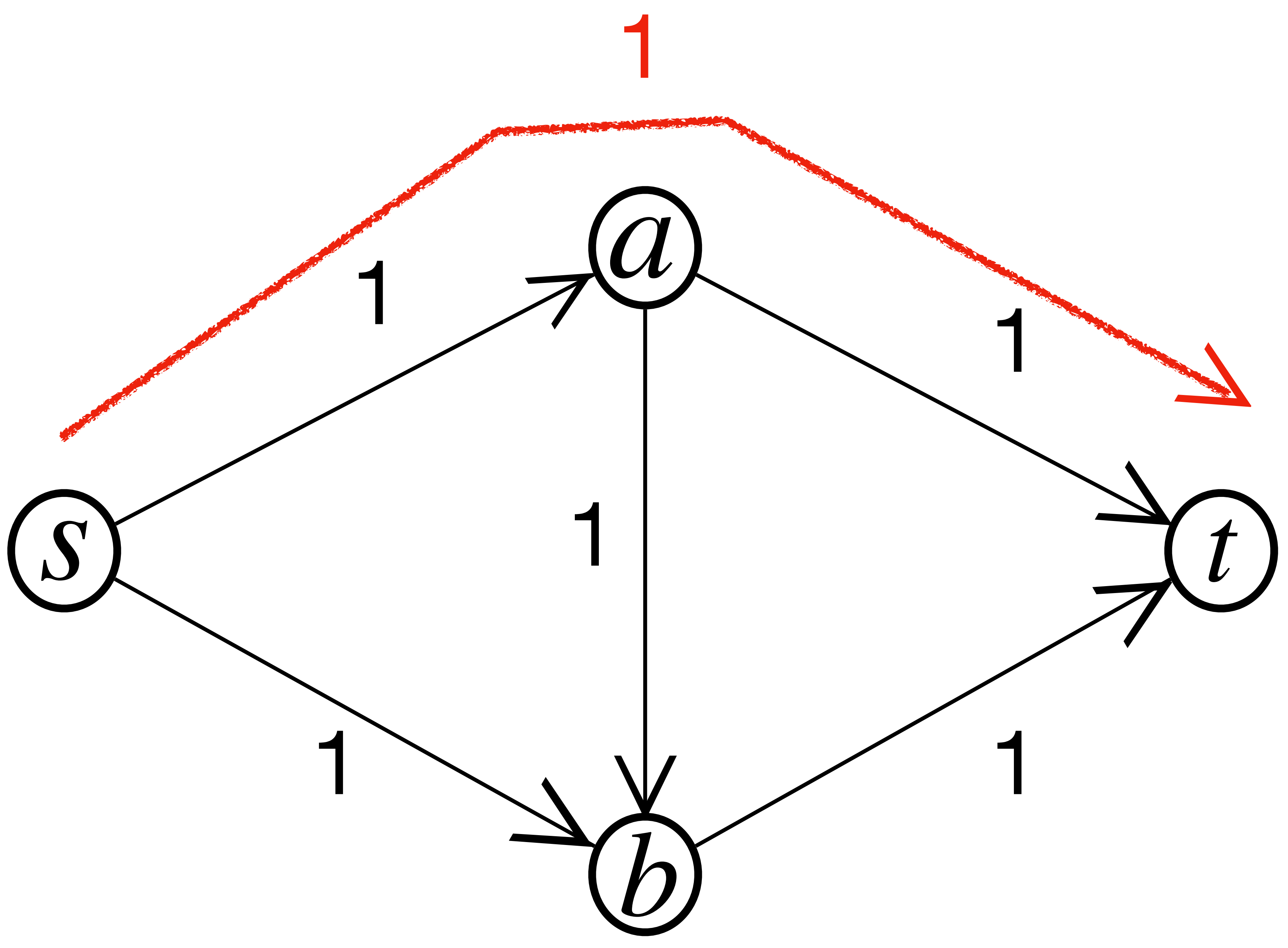

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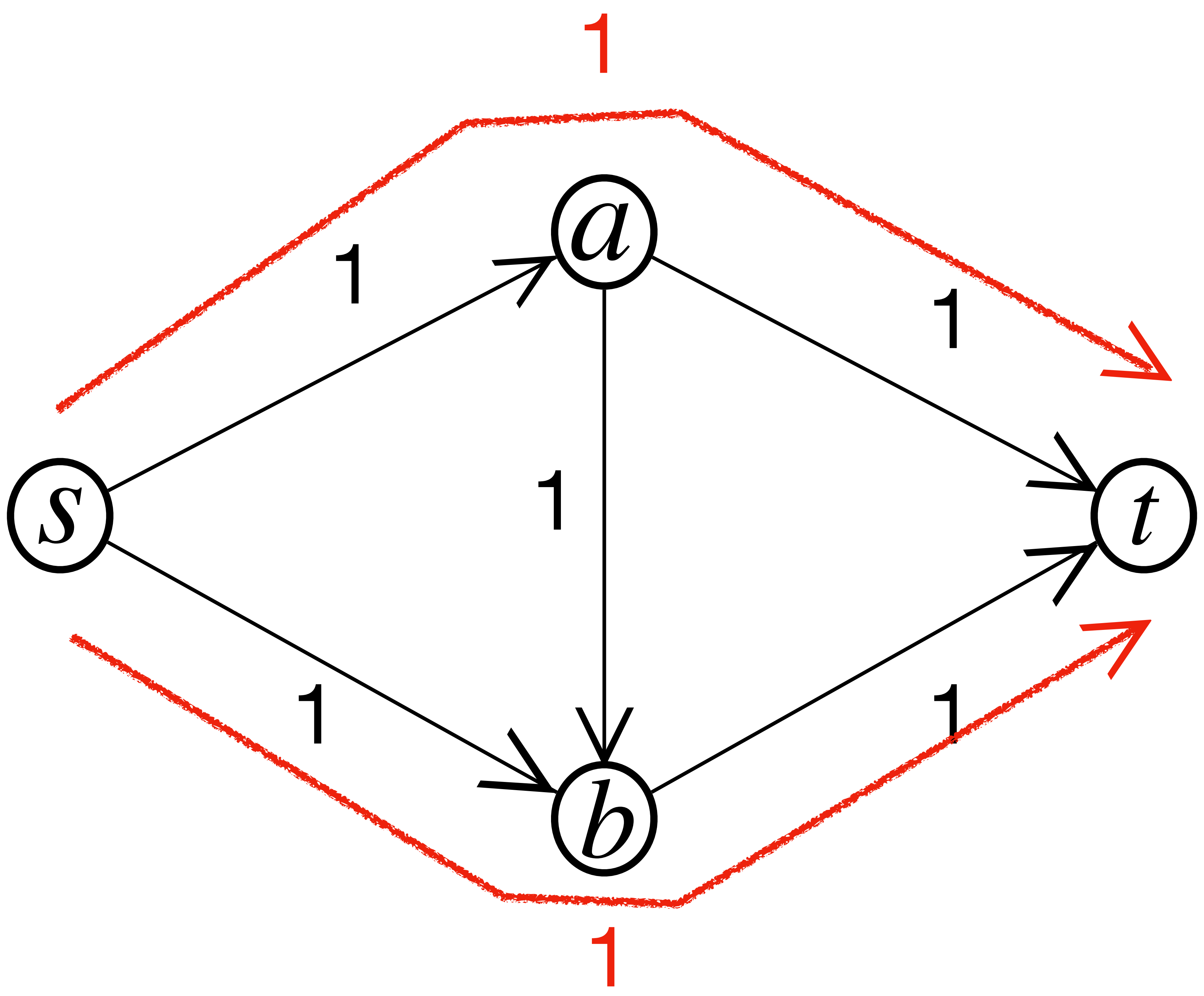

size of maximum flow = 2

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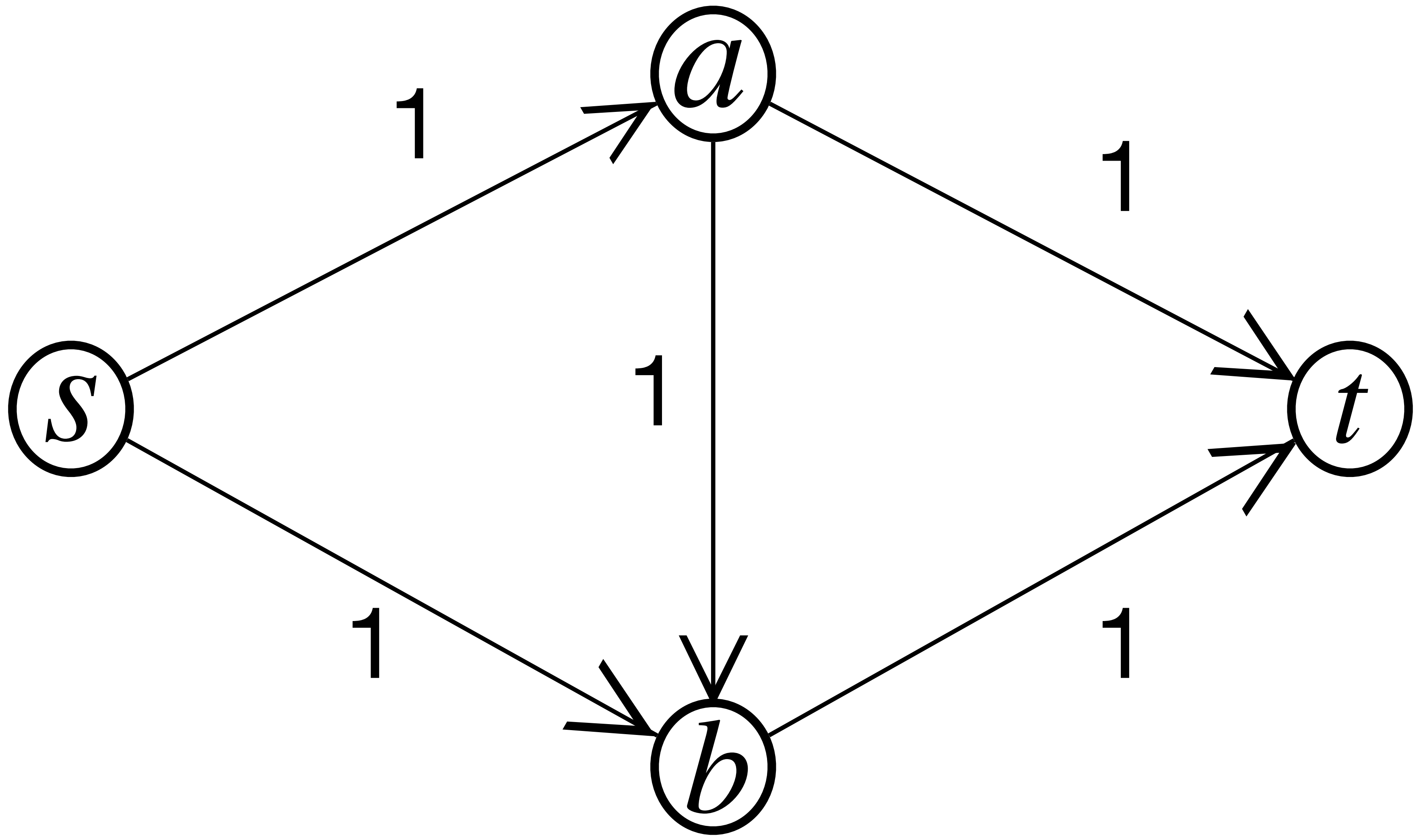
But what if …



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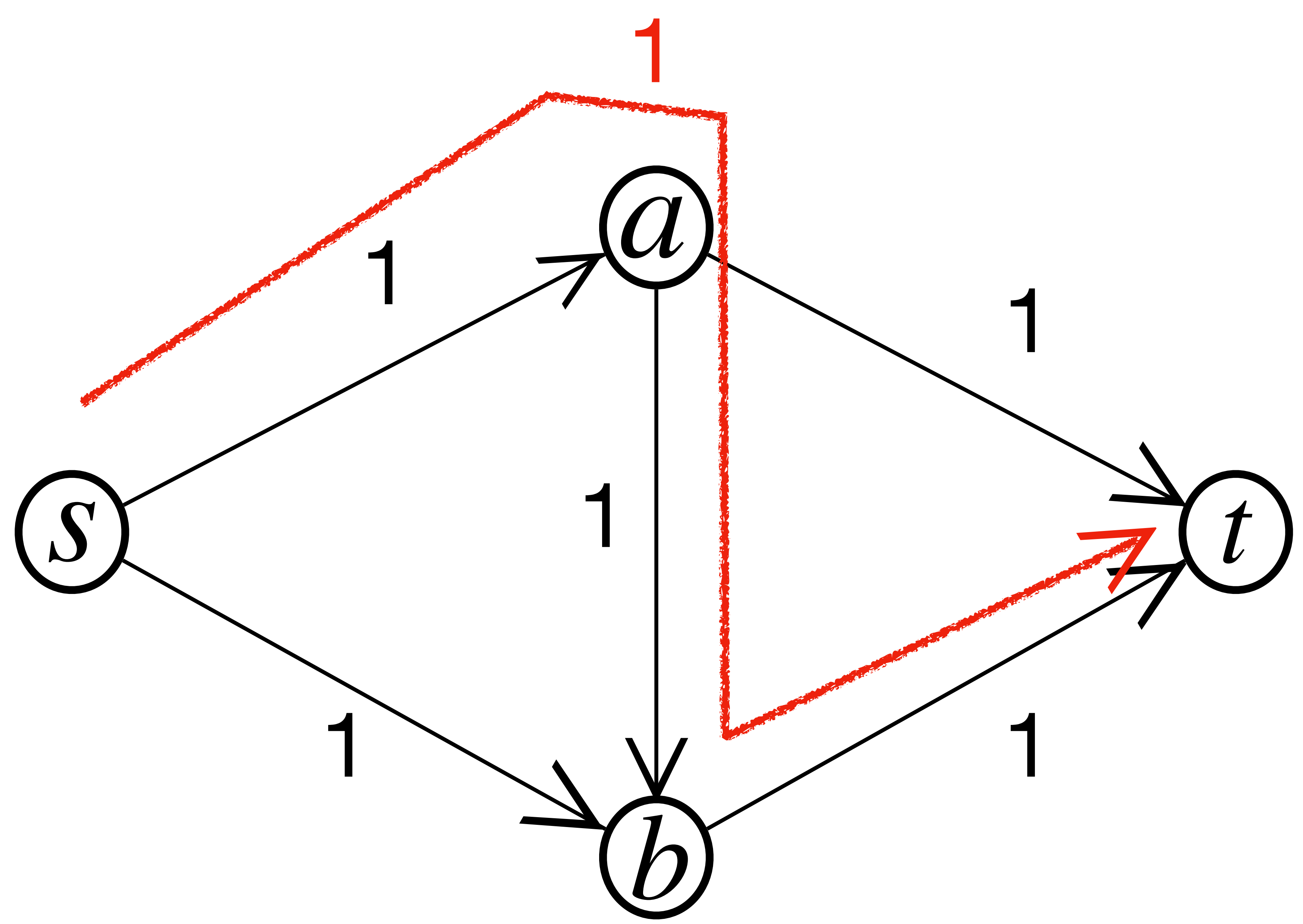

Does the Ford-Fulkerson Method still work?

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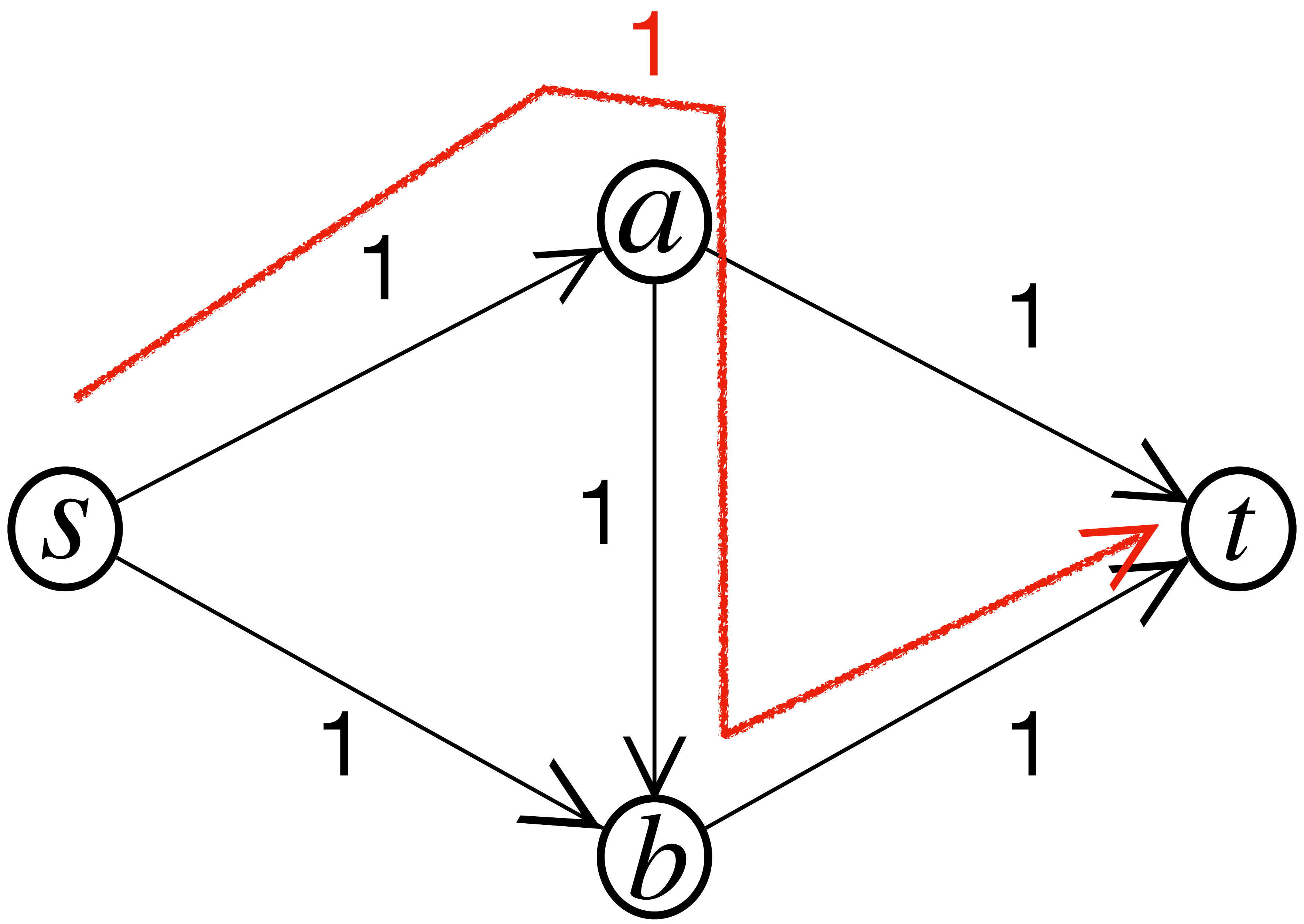
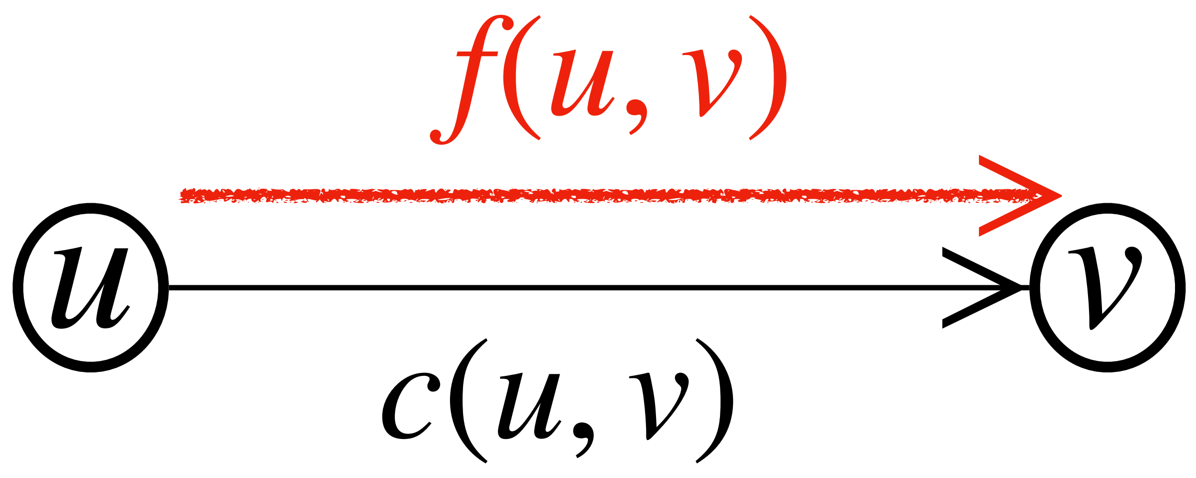


Does the Ford-Fulkerson Method still work?

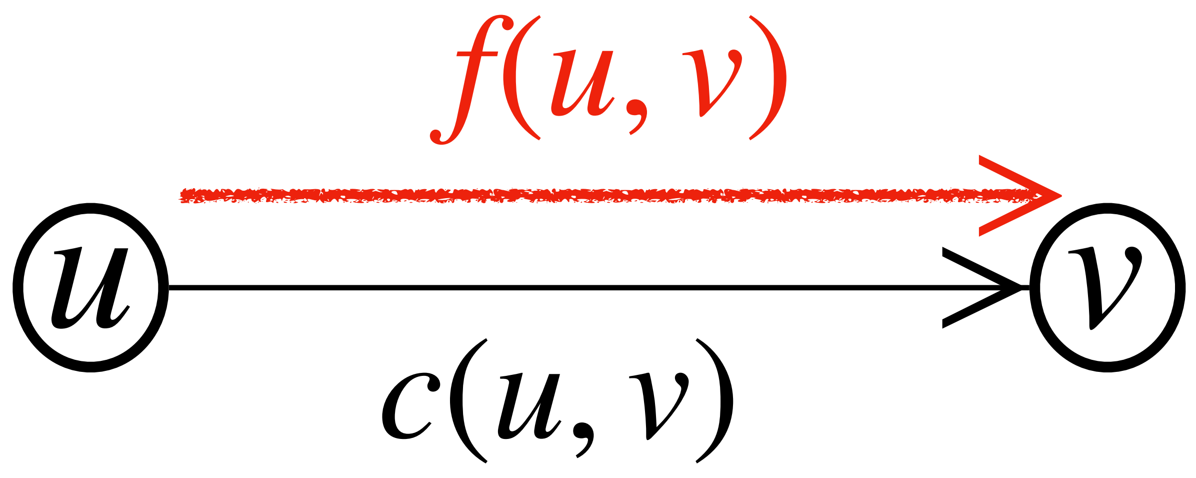
Yes, but we need to understand “residual capacity" correctly.

Residual capacity:

1) How much can the flow from u to v be increased by?



Residual capacity:



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$$c(u,v) - f(u,v)$$

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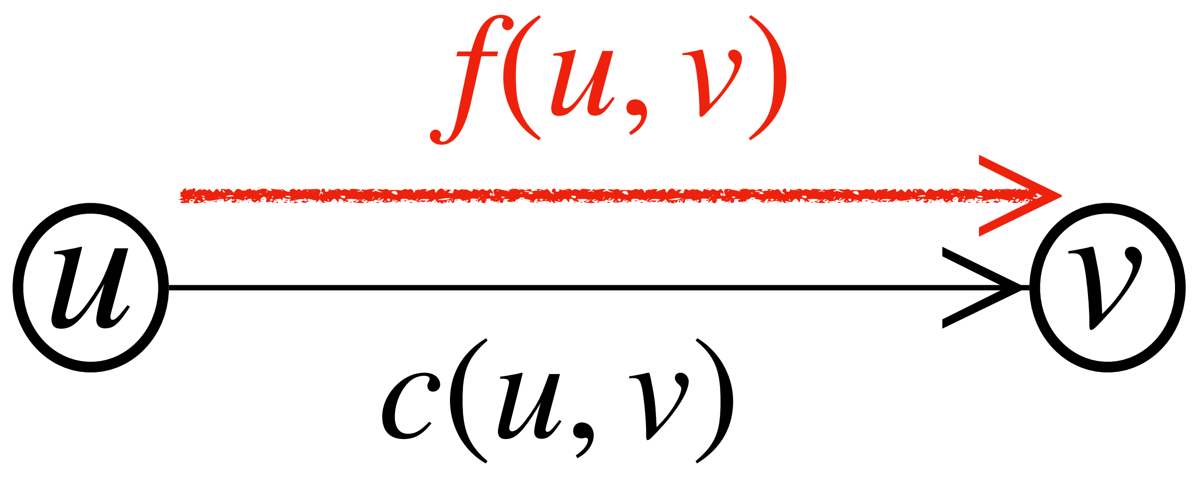

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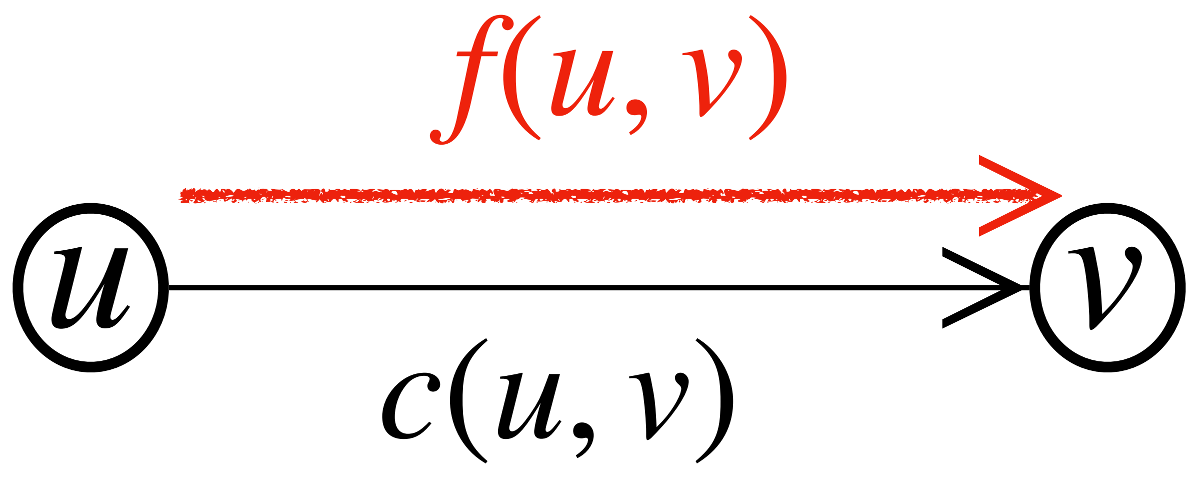

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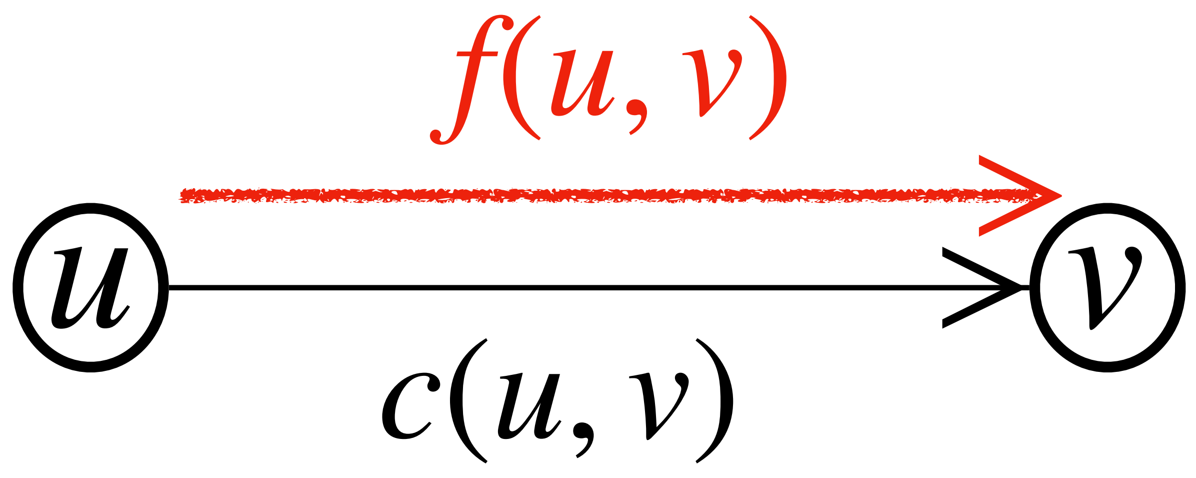


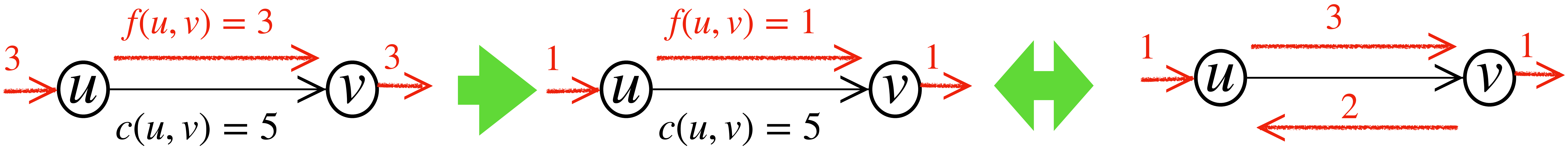
1) How much can the flow from u to v be increased by?

$$c(u, v) - f(u, v)$$

2) How much can the flow from u to v be decreased by?

$$f(u, v)$$

3) Decreasing the flow from u to v is equivalent to increasing the flow from v to u.



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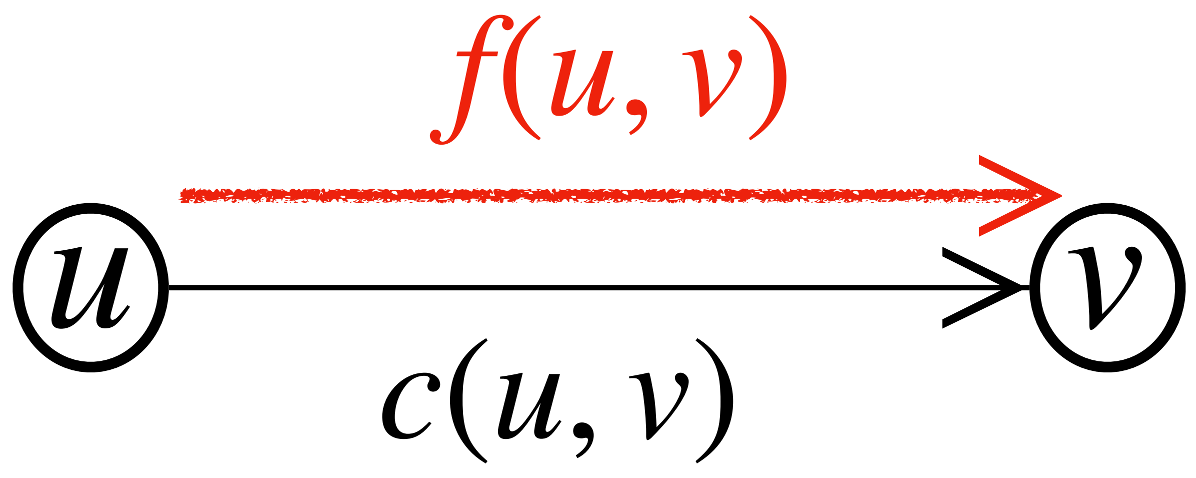

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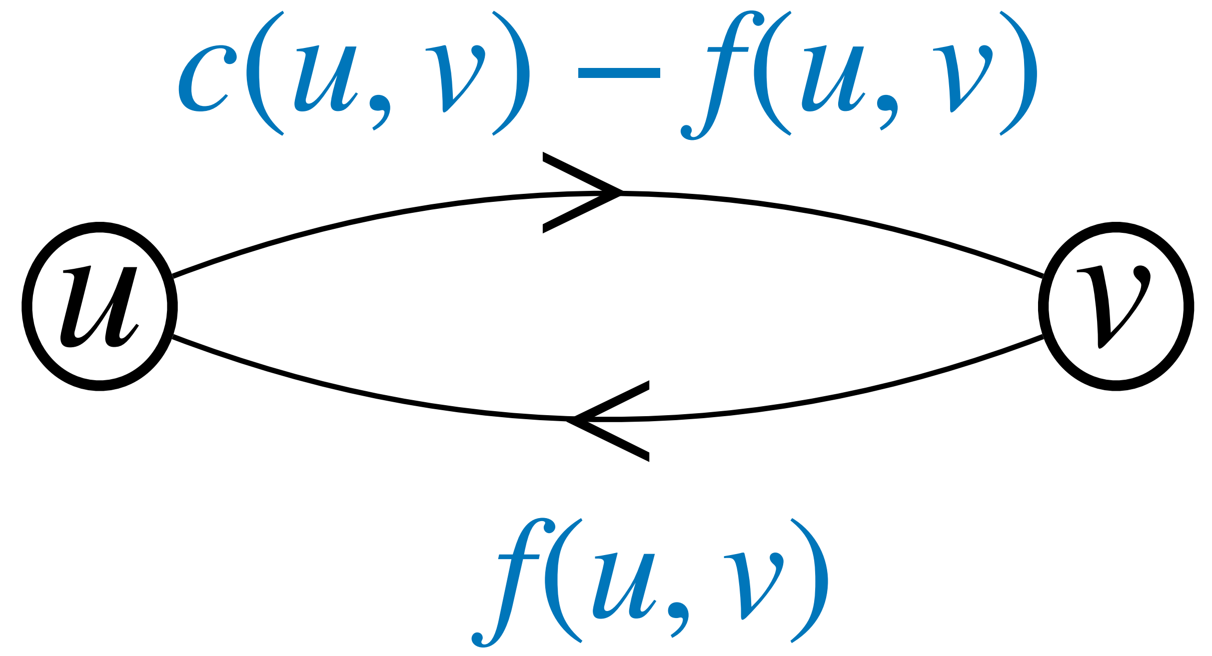

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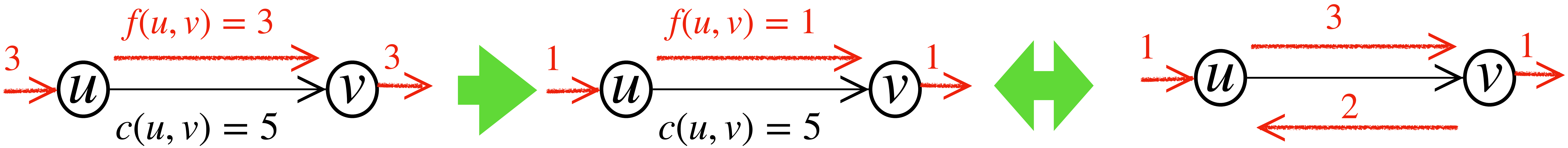

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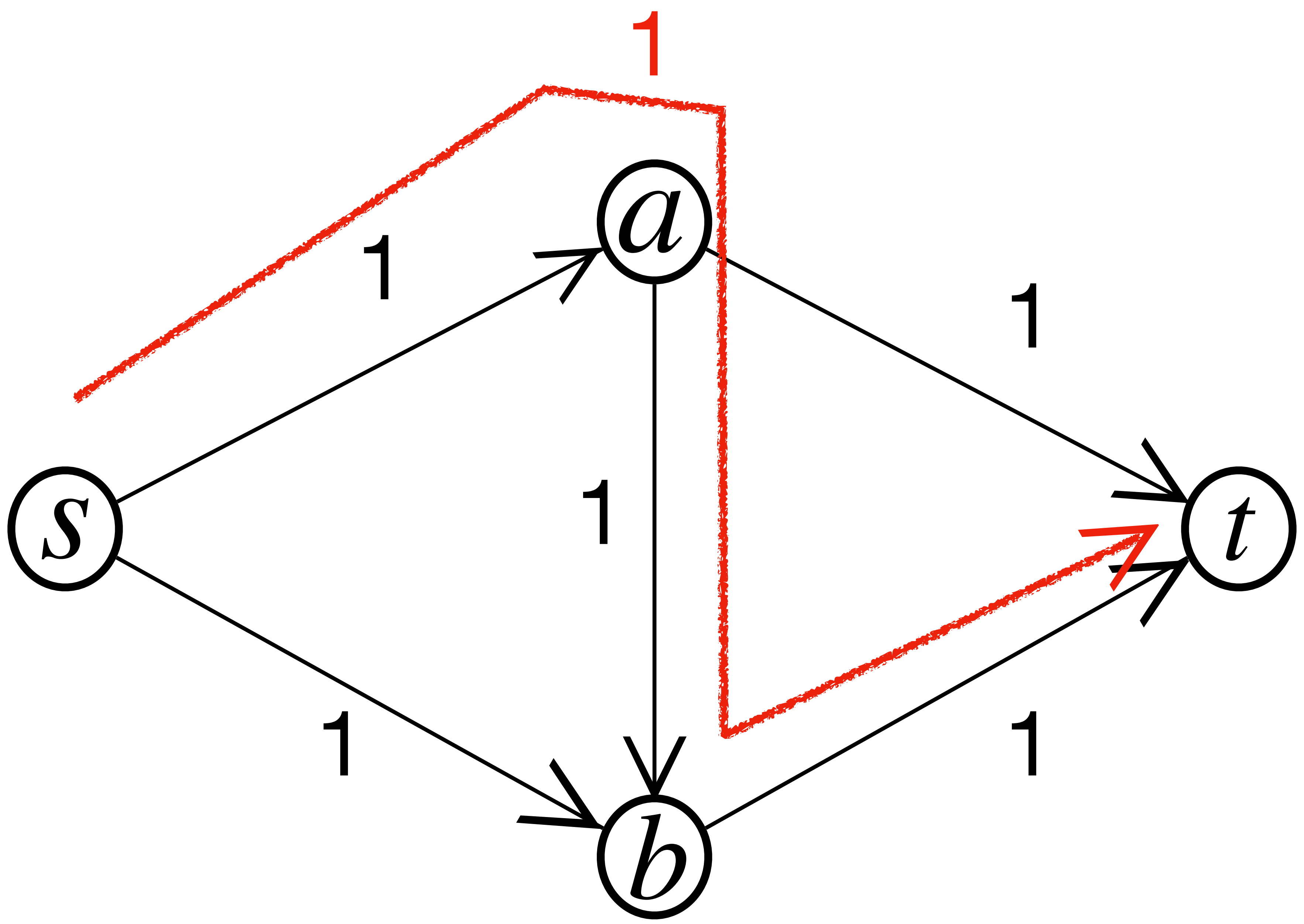

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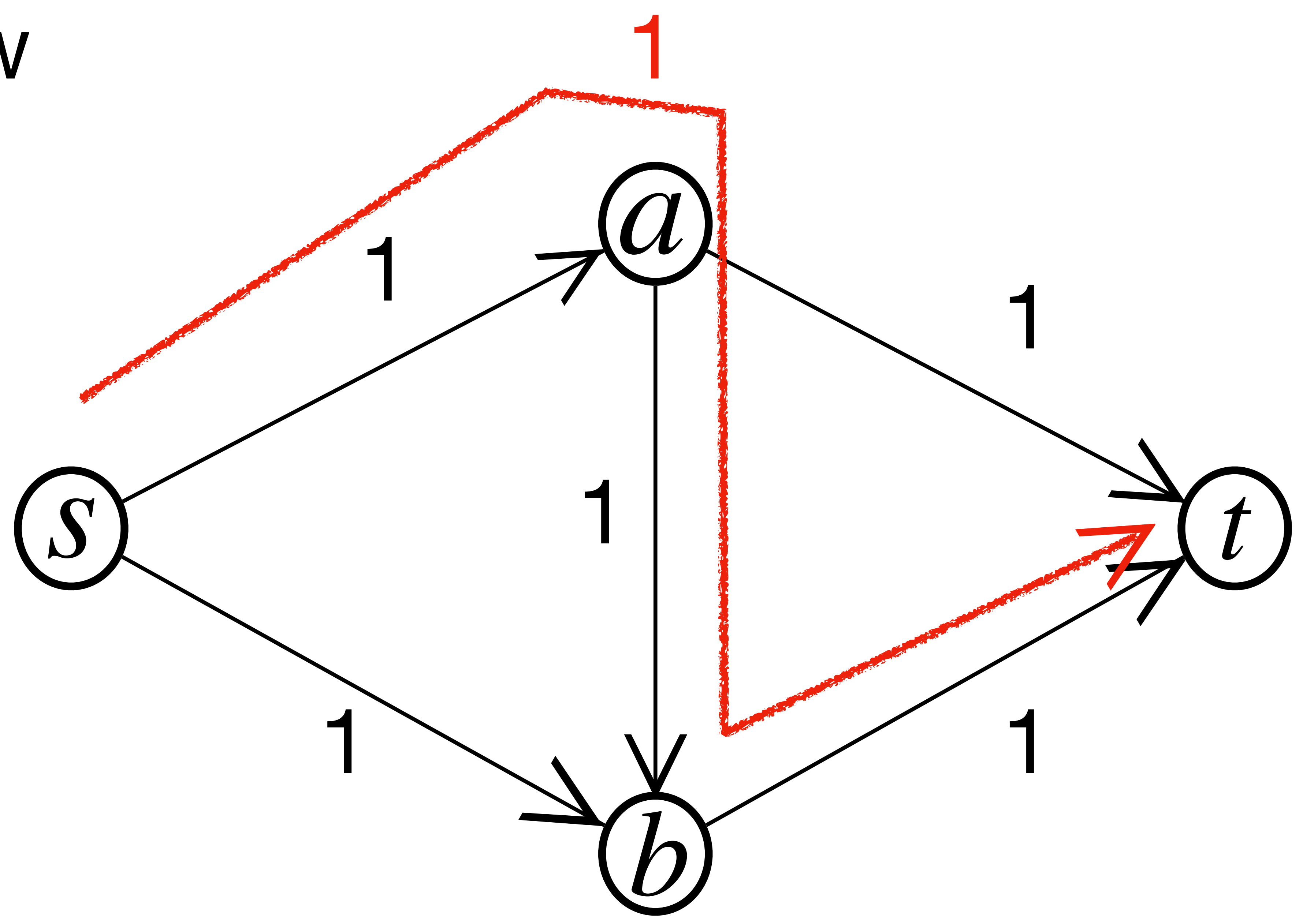
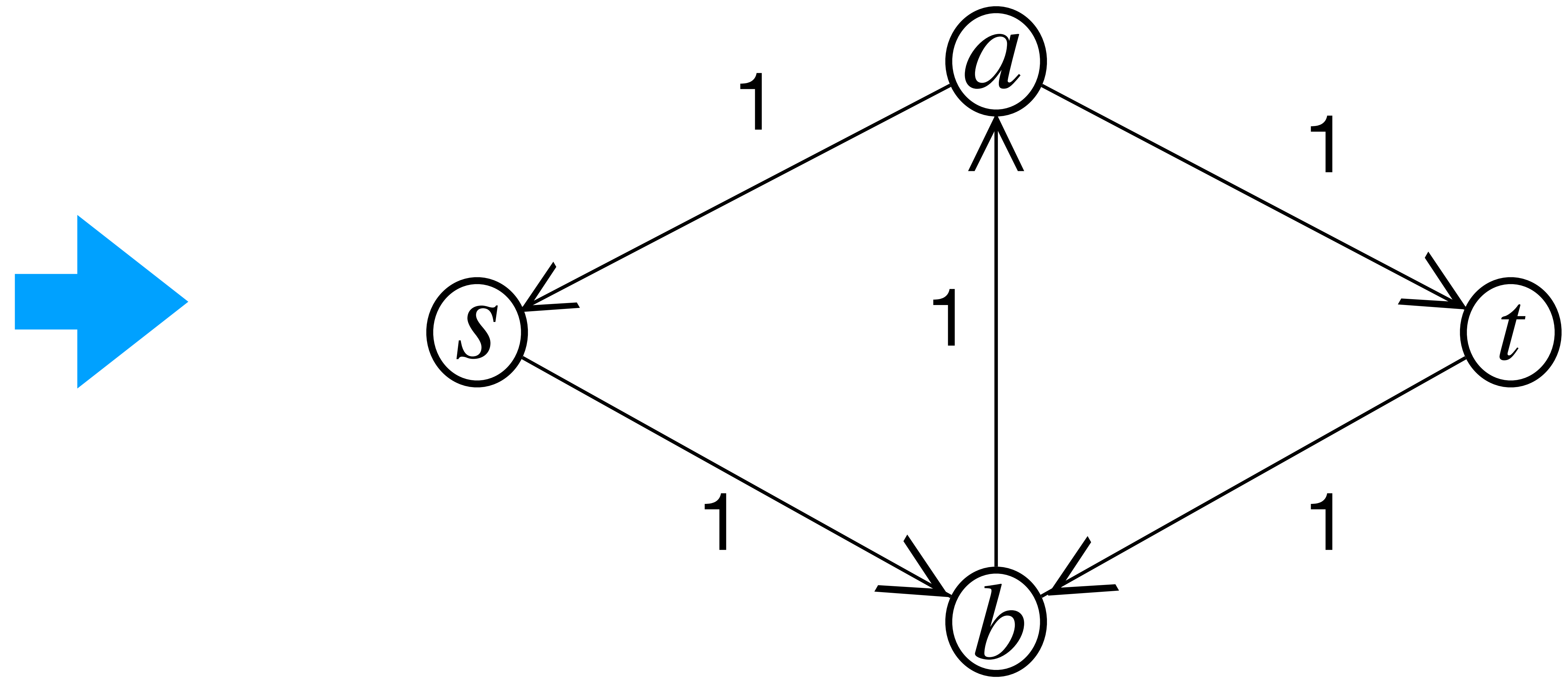


Residual capacity



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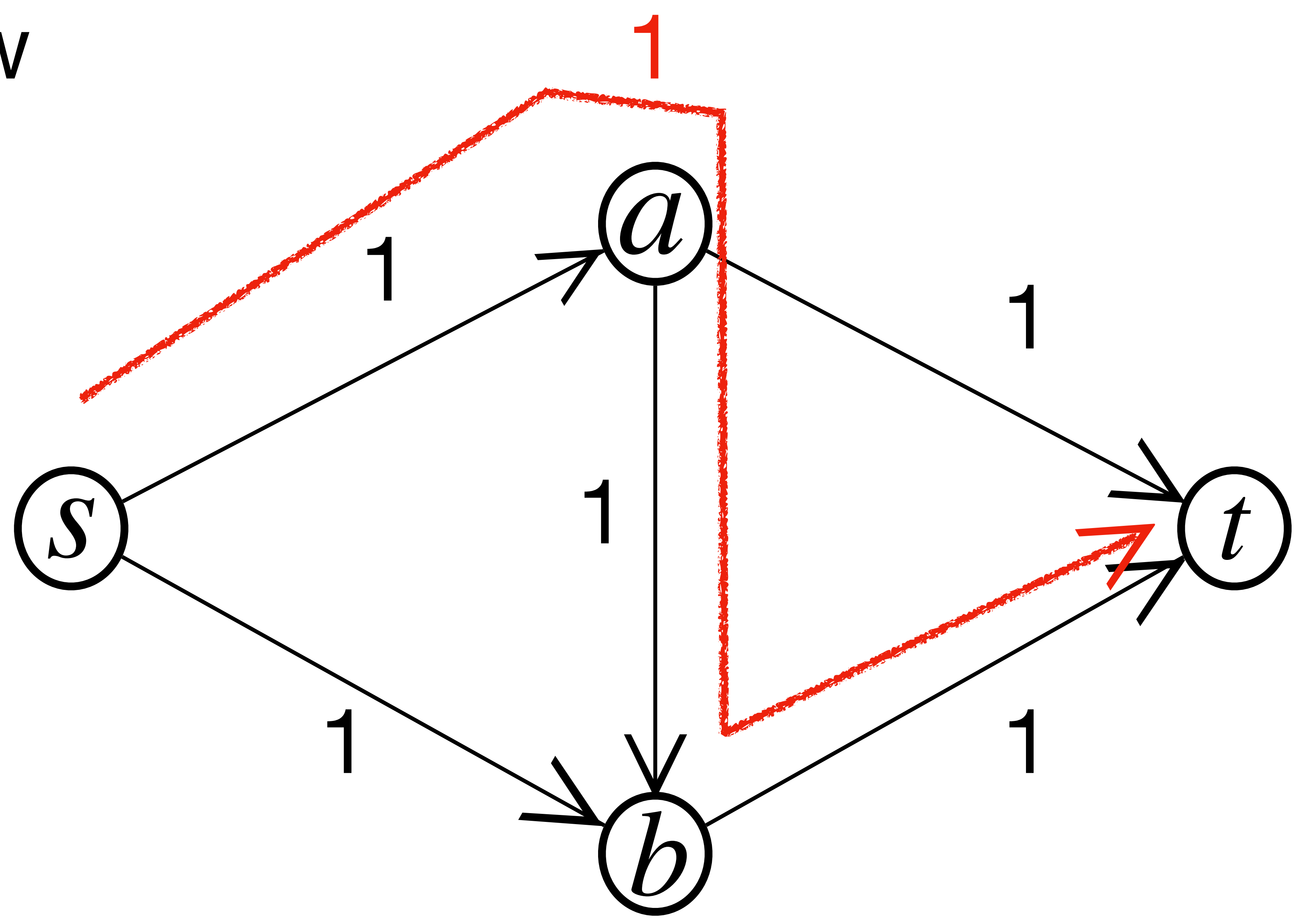

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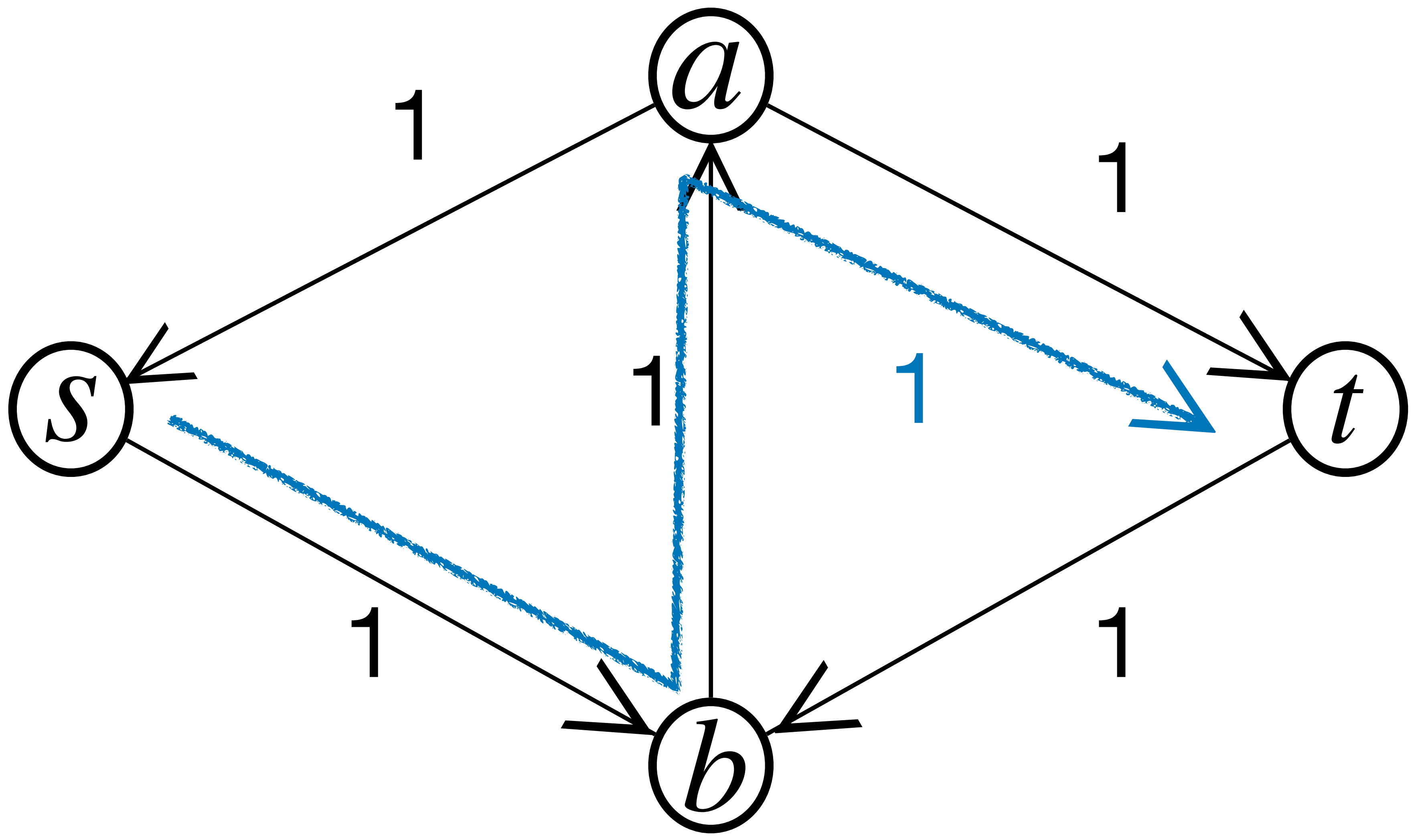

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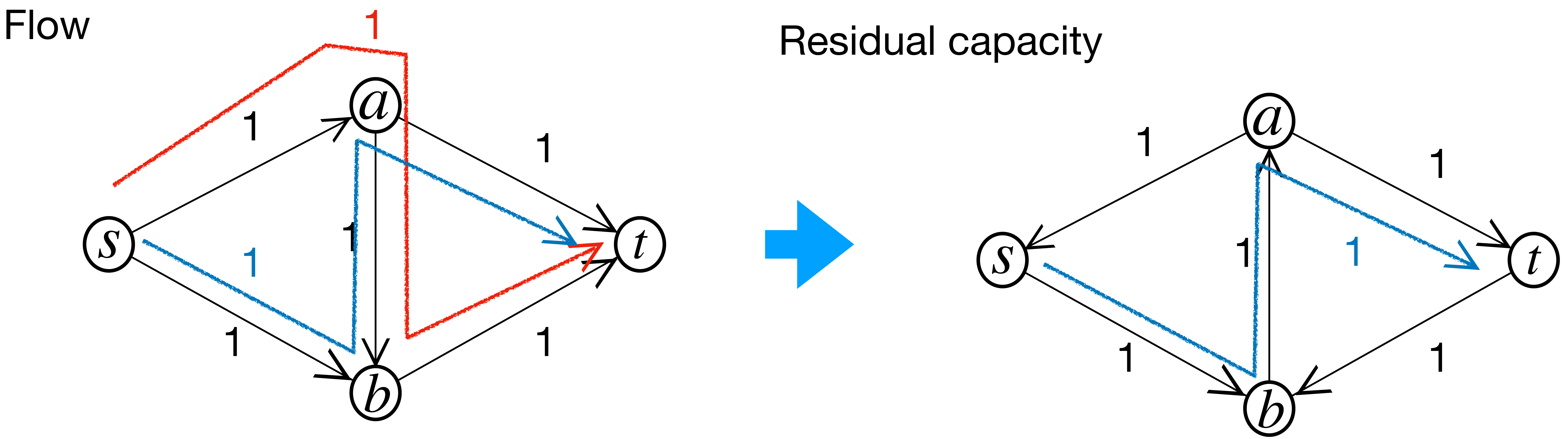

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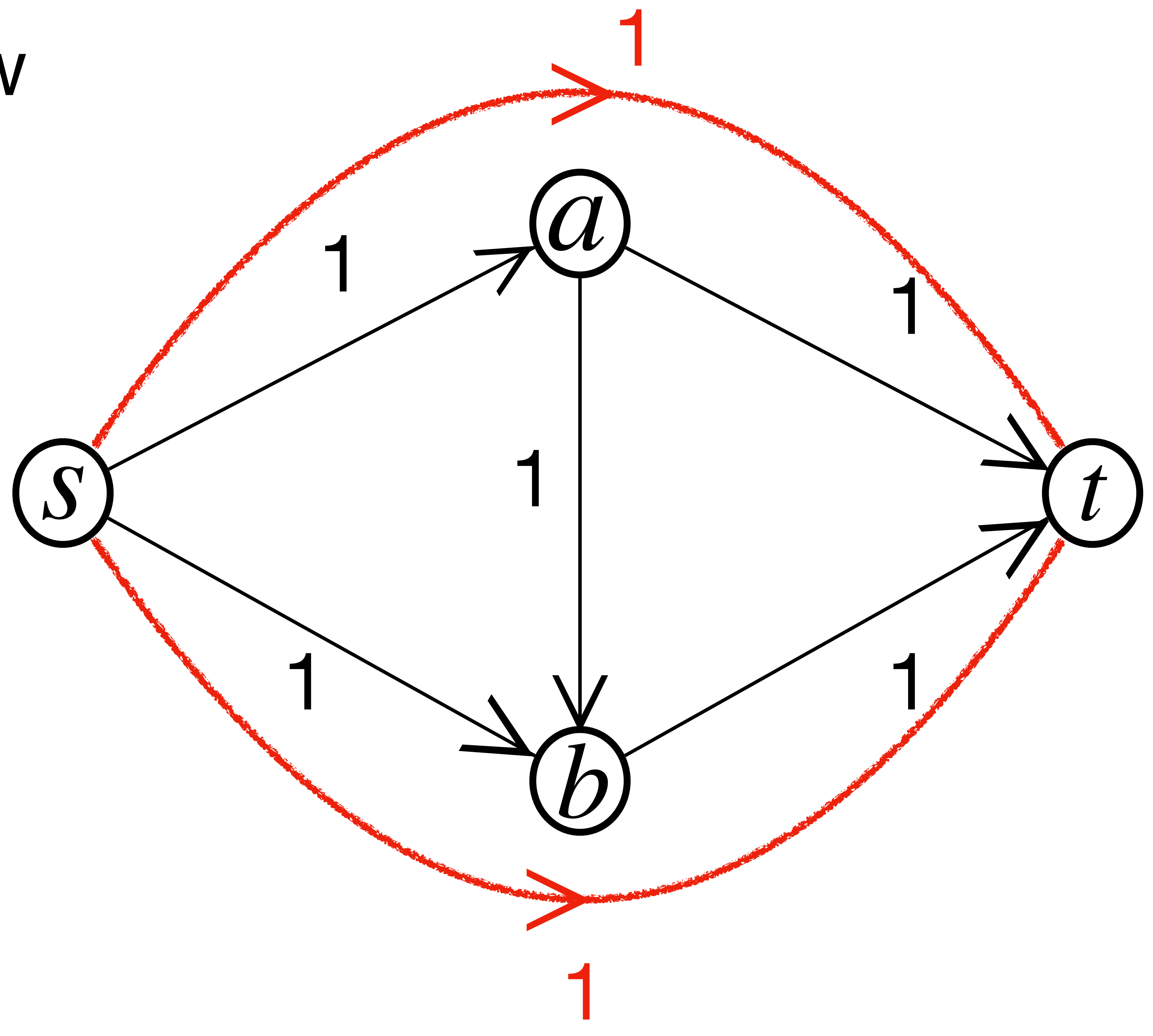
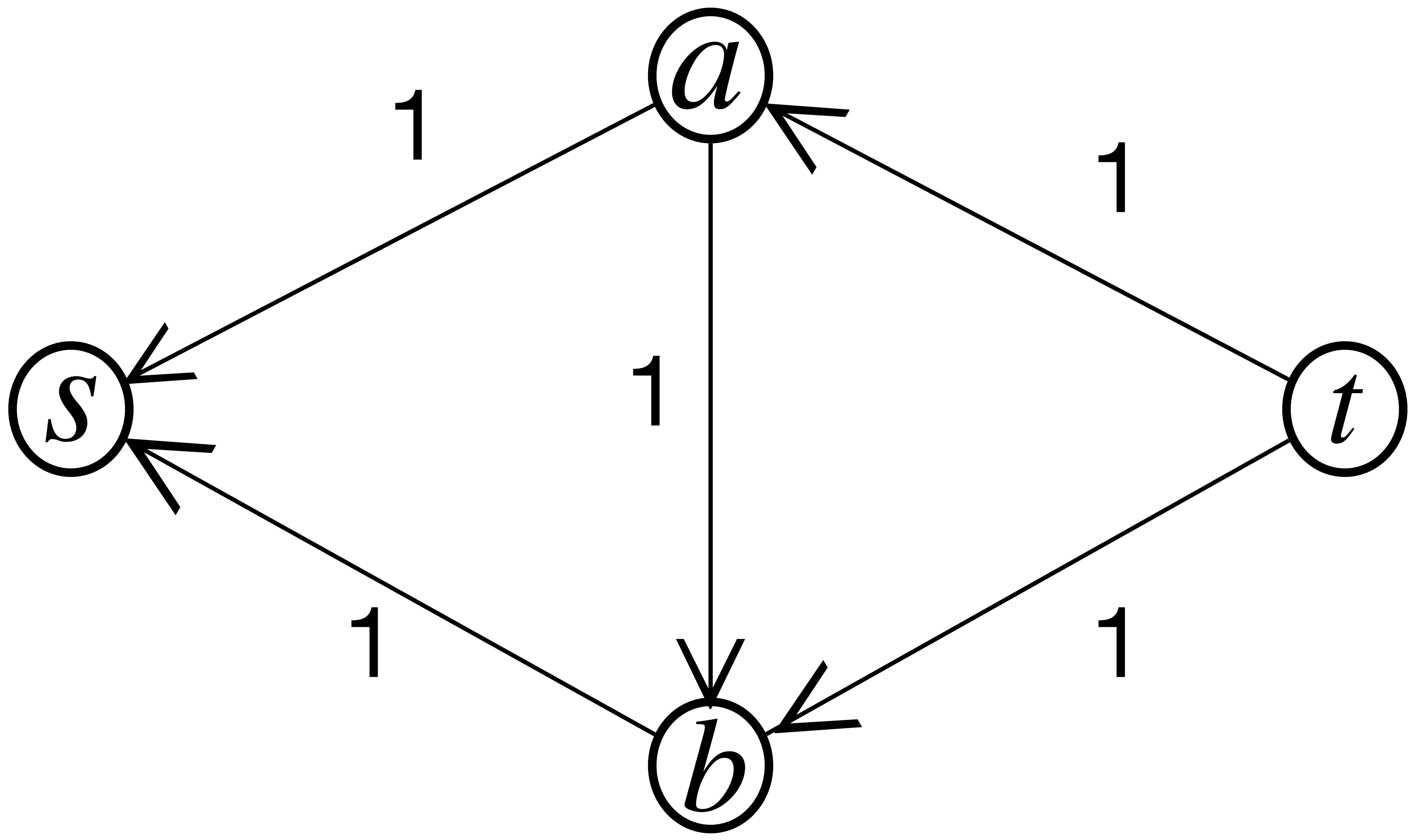
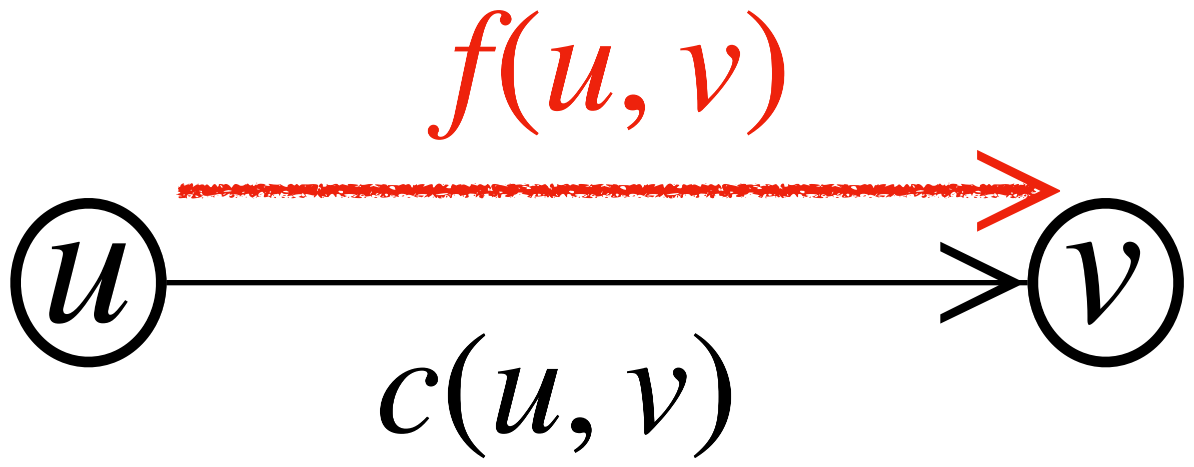


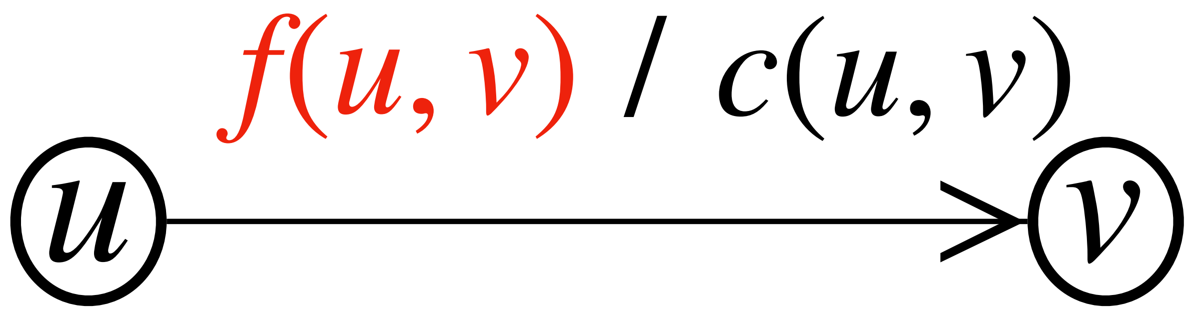
Residual capacity



No more path from s to t

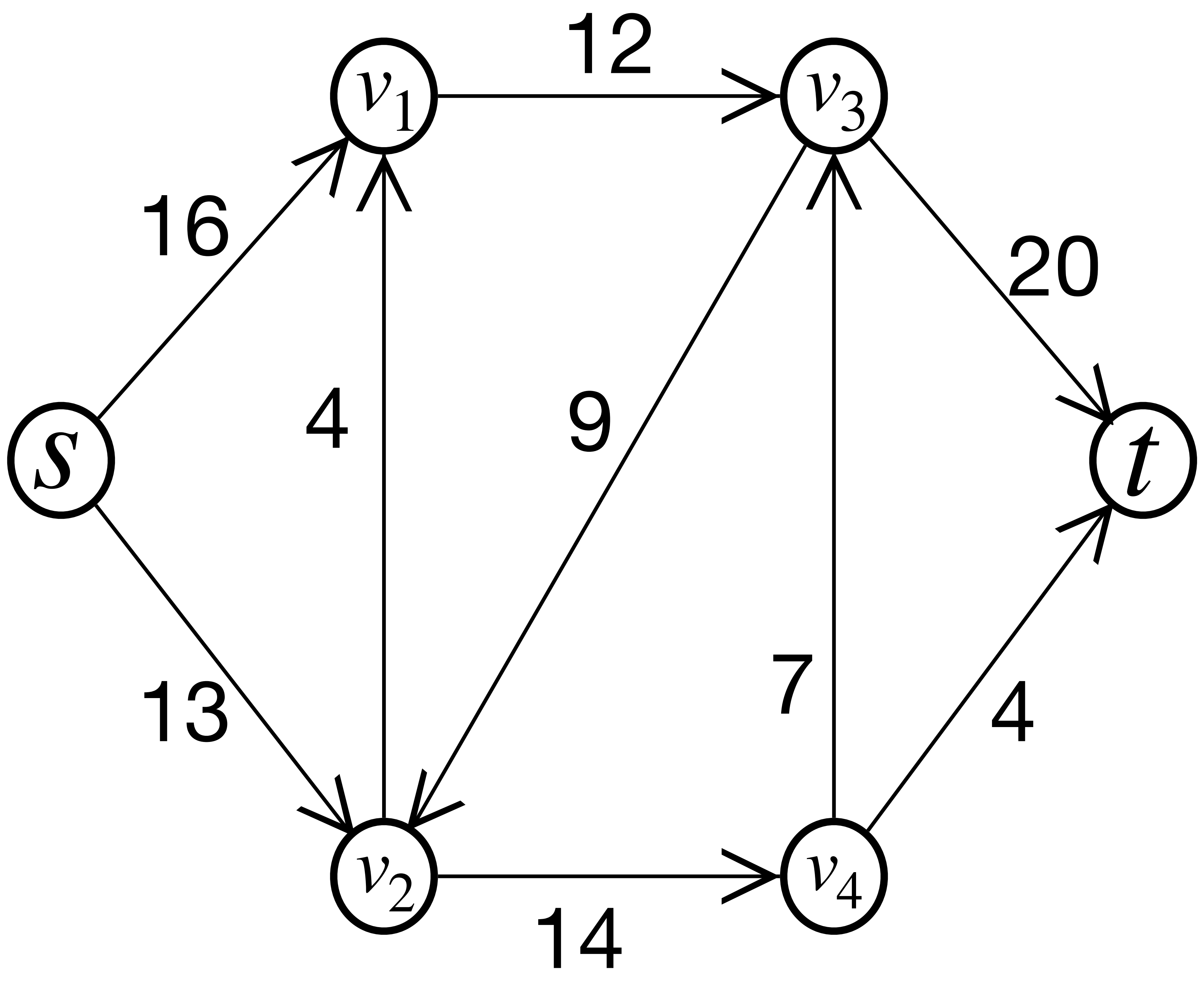


Simpler notation:



Ford-Fulkerson Method

Original network



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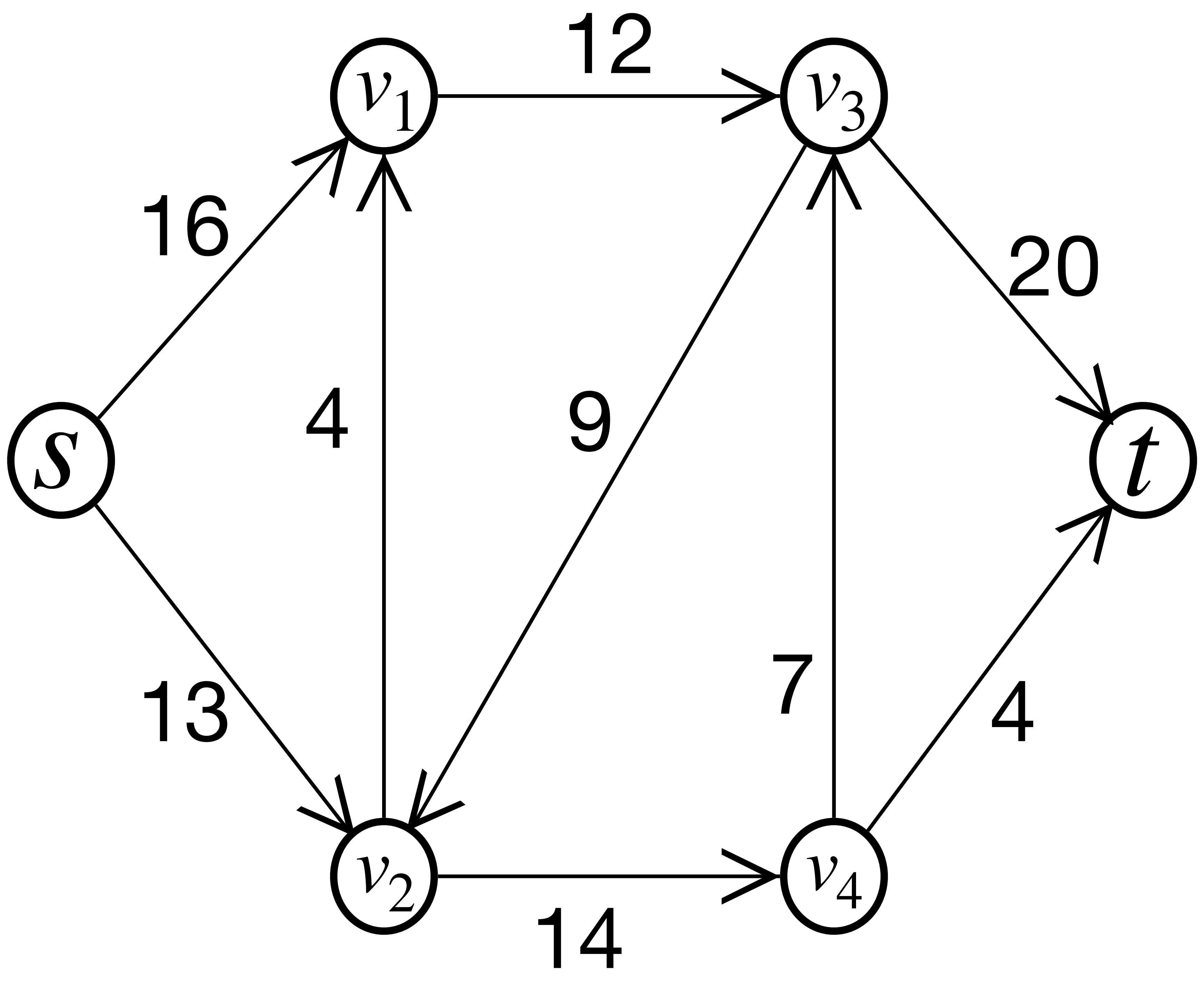
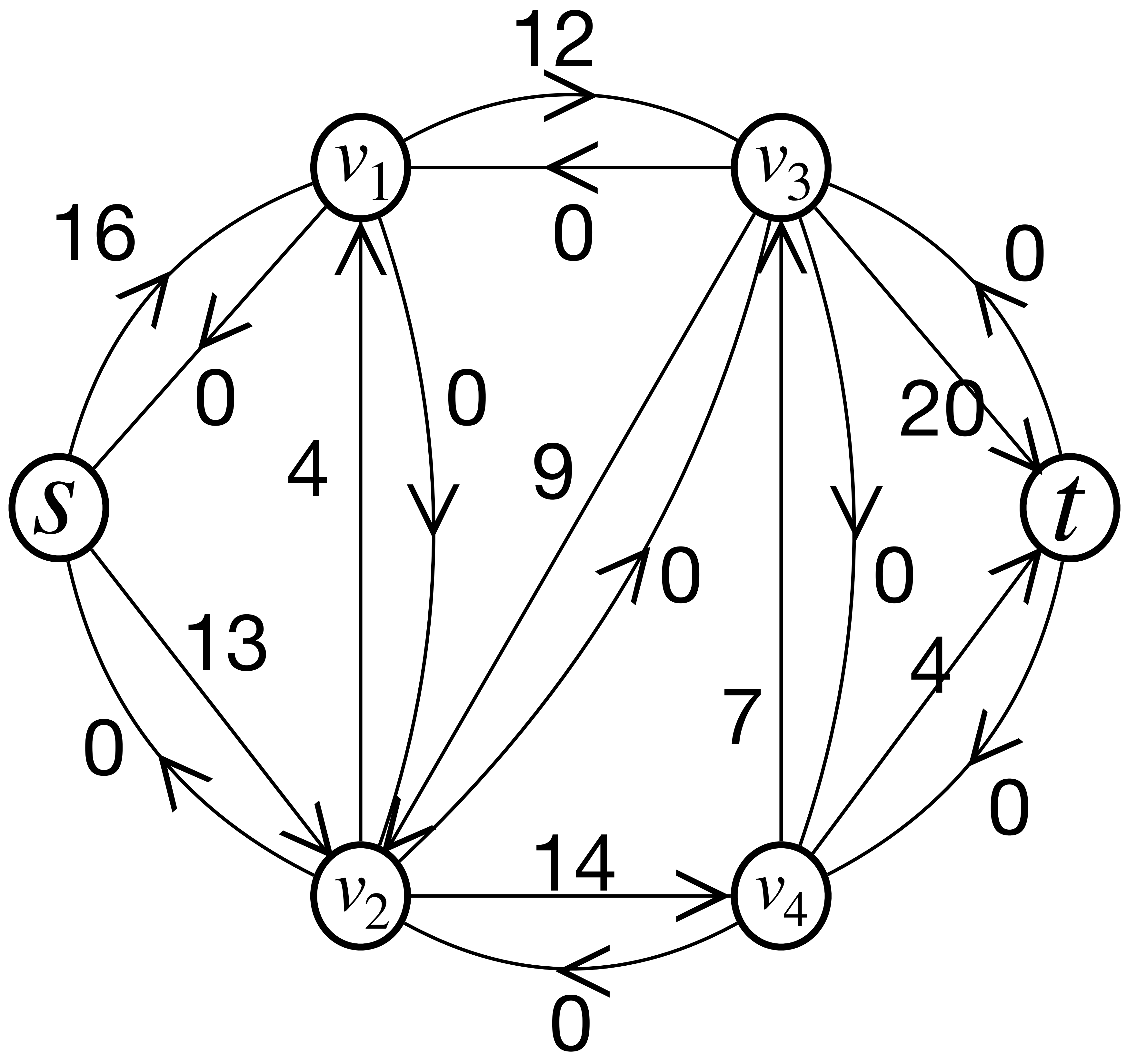


Residual network



Ford-Fulkerson Method

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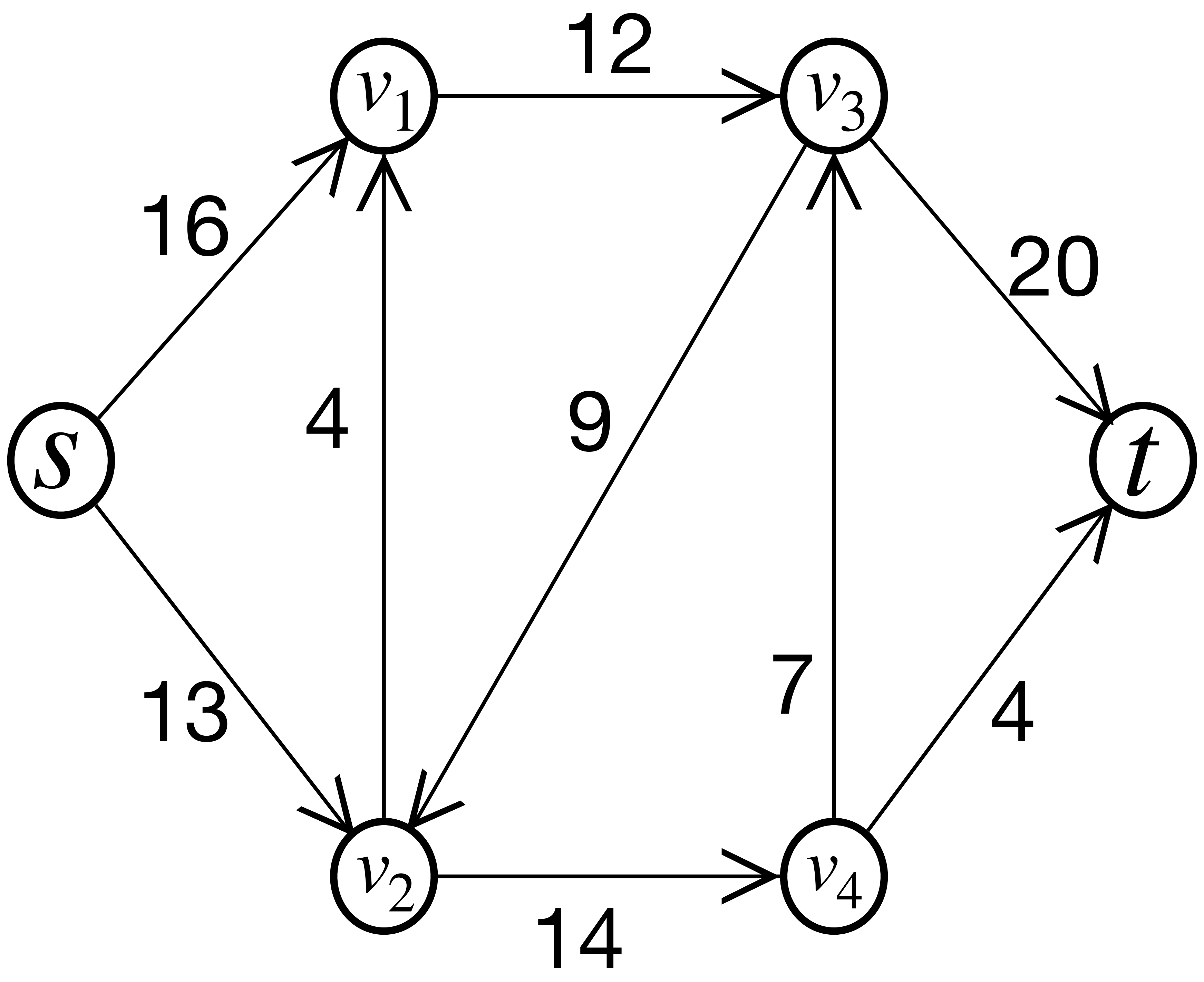

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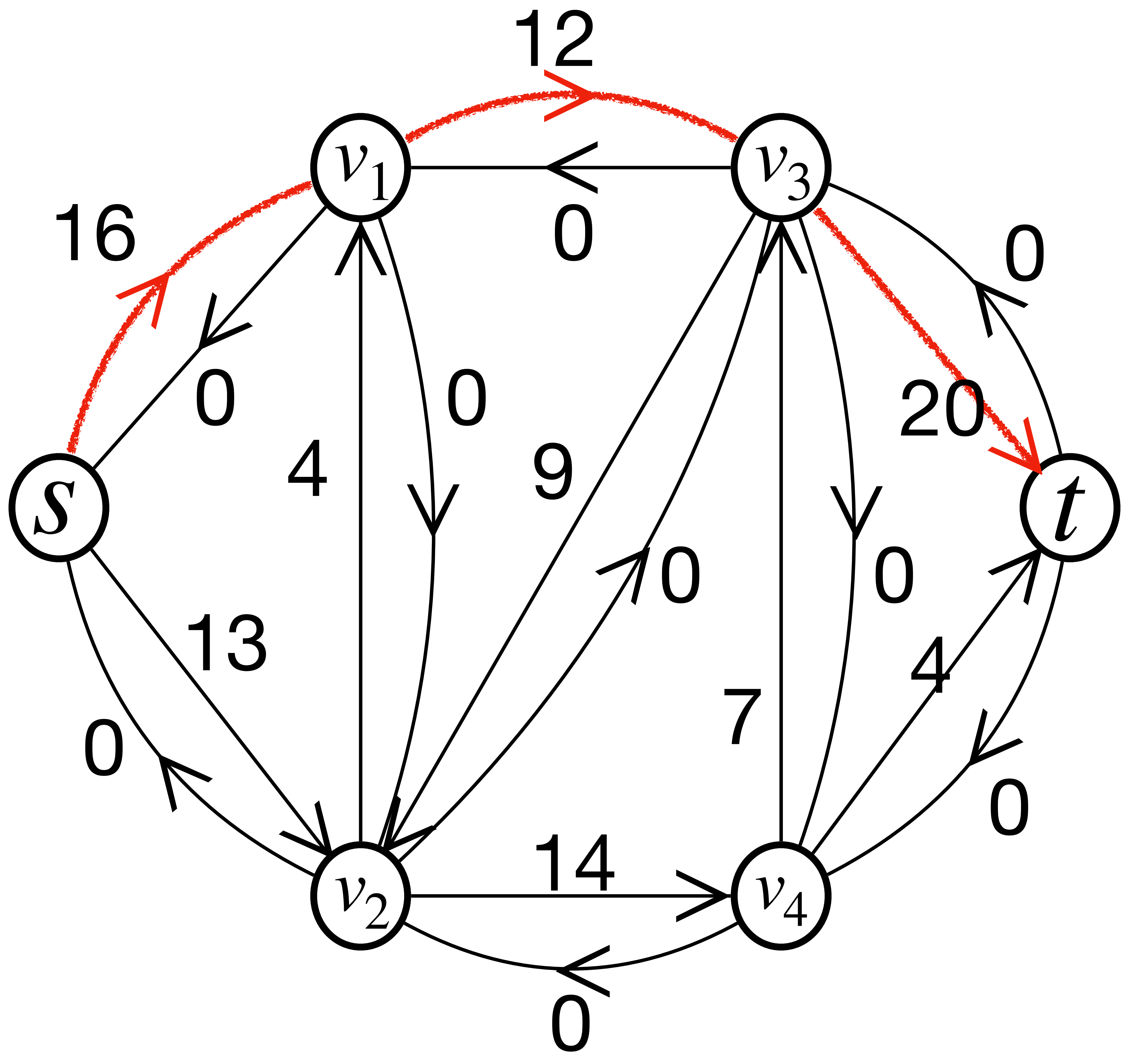

Ford-Fulkerson Method

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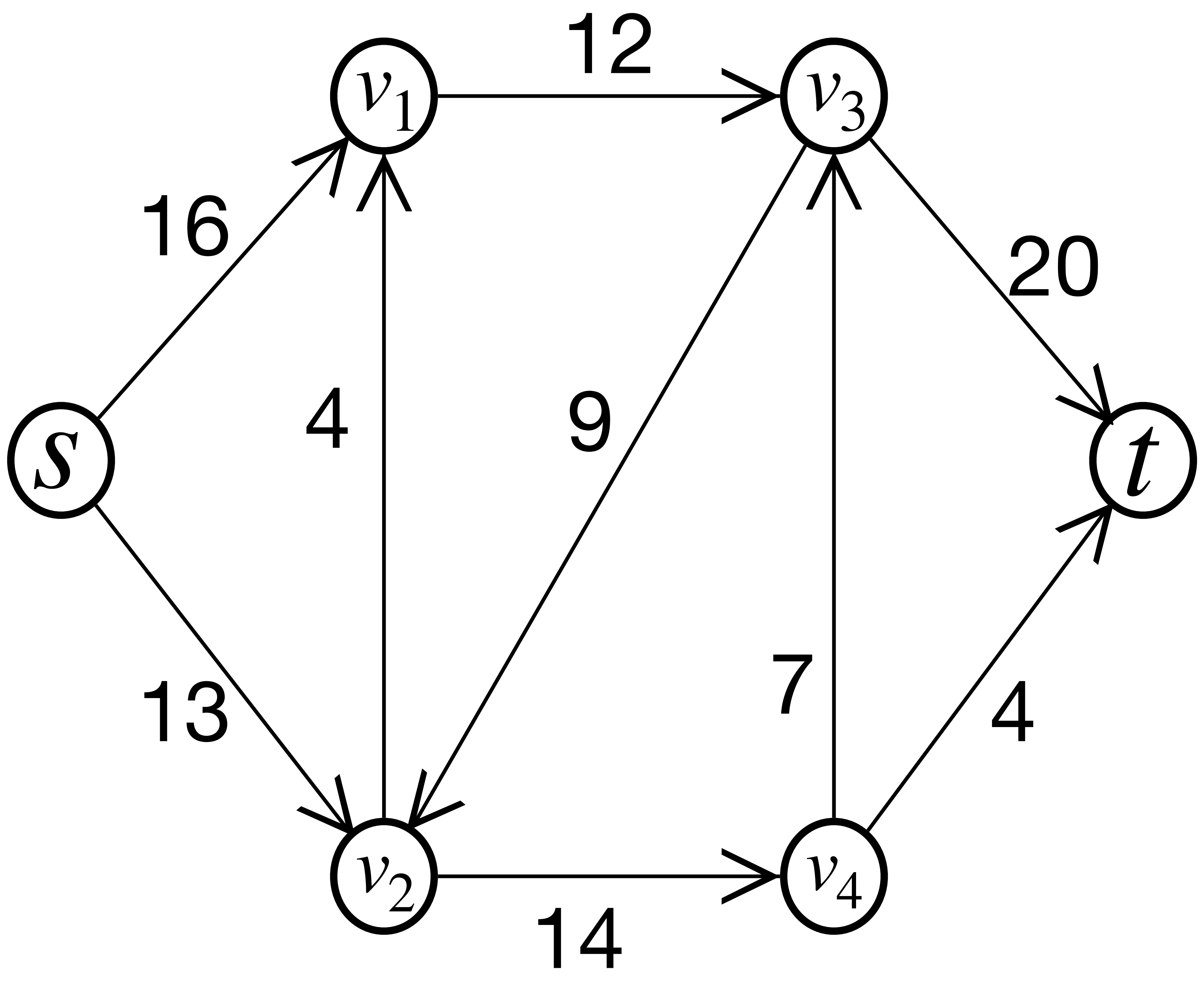

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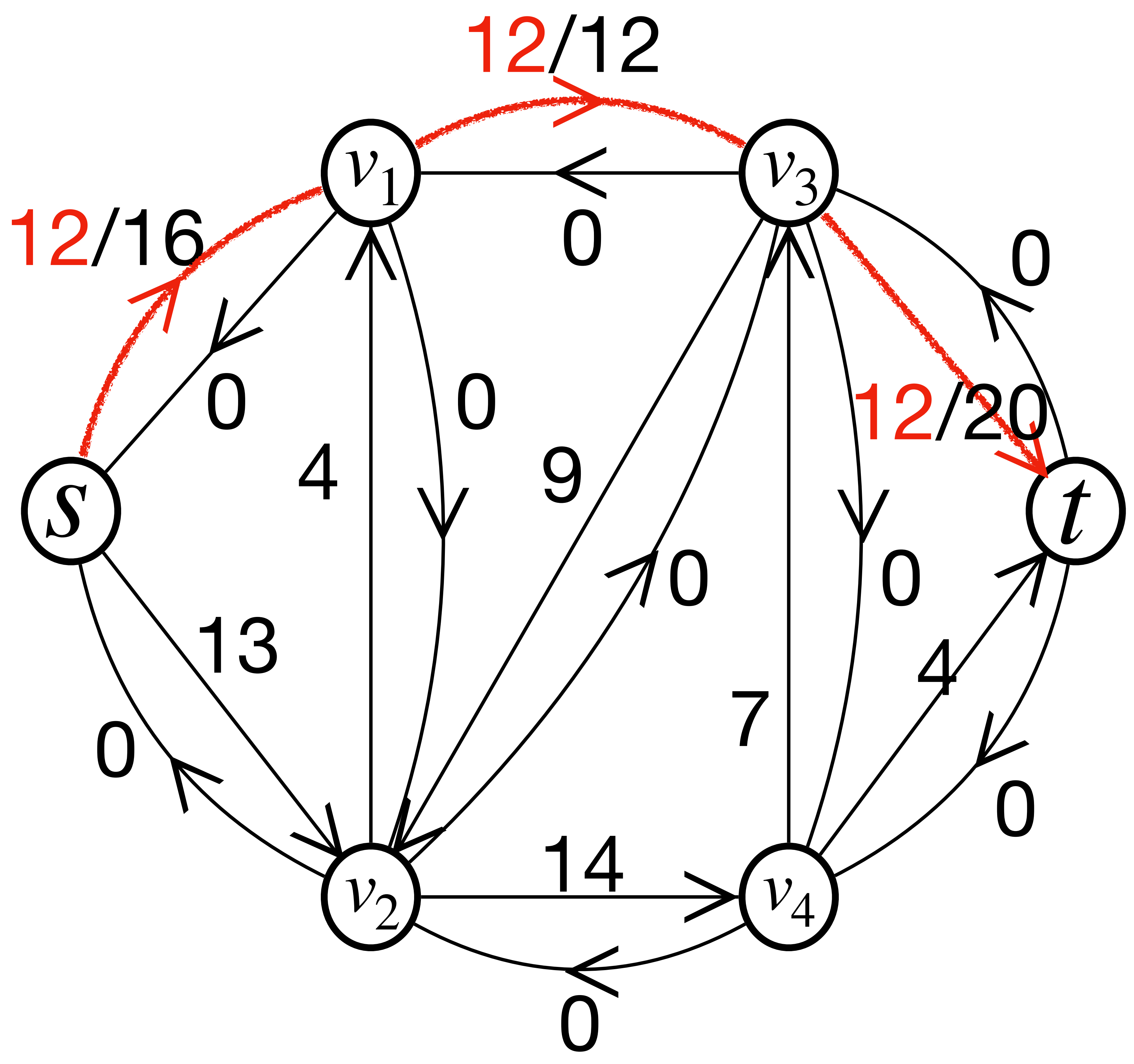

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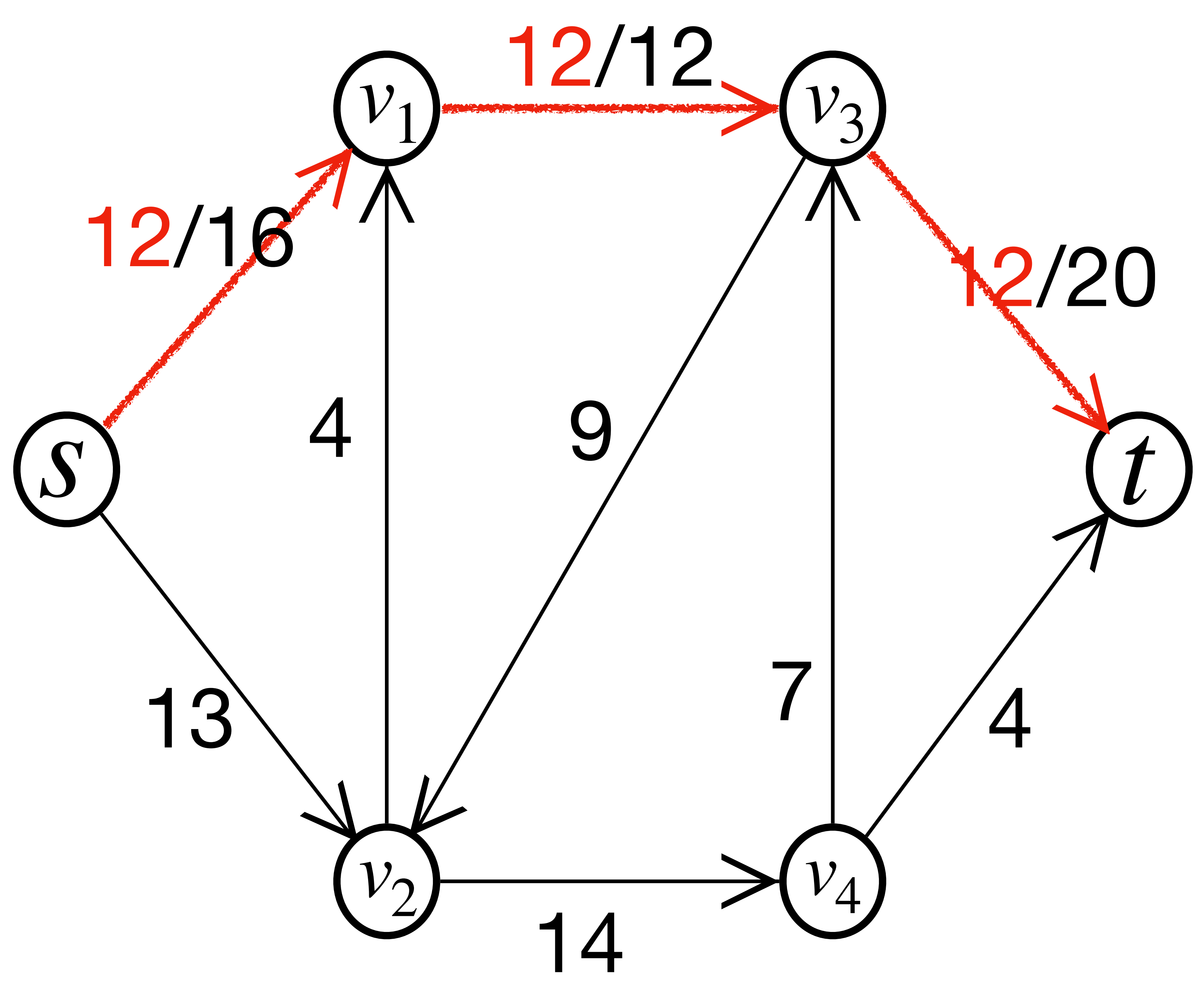

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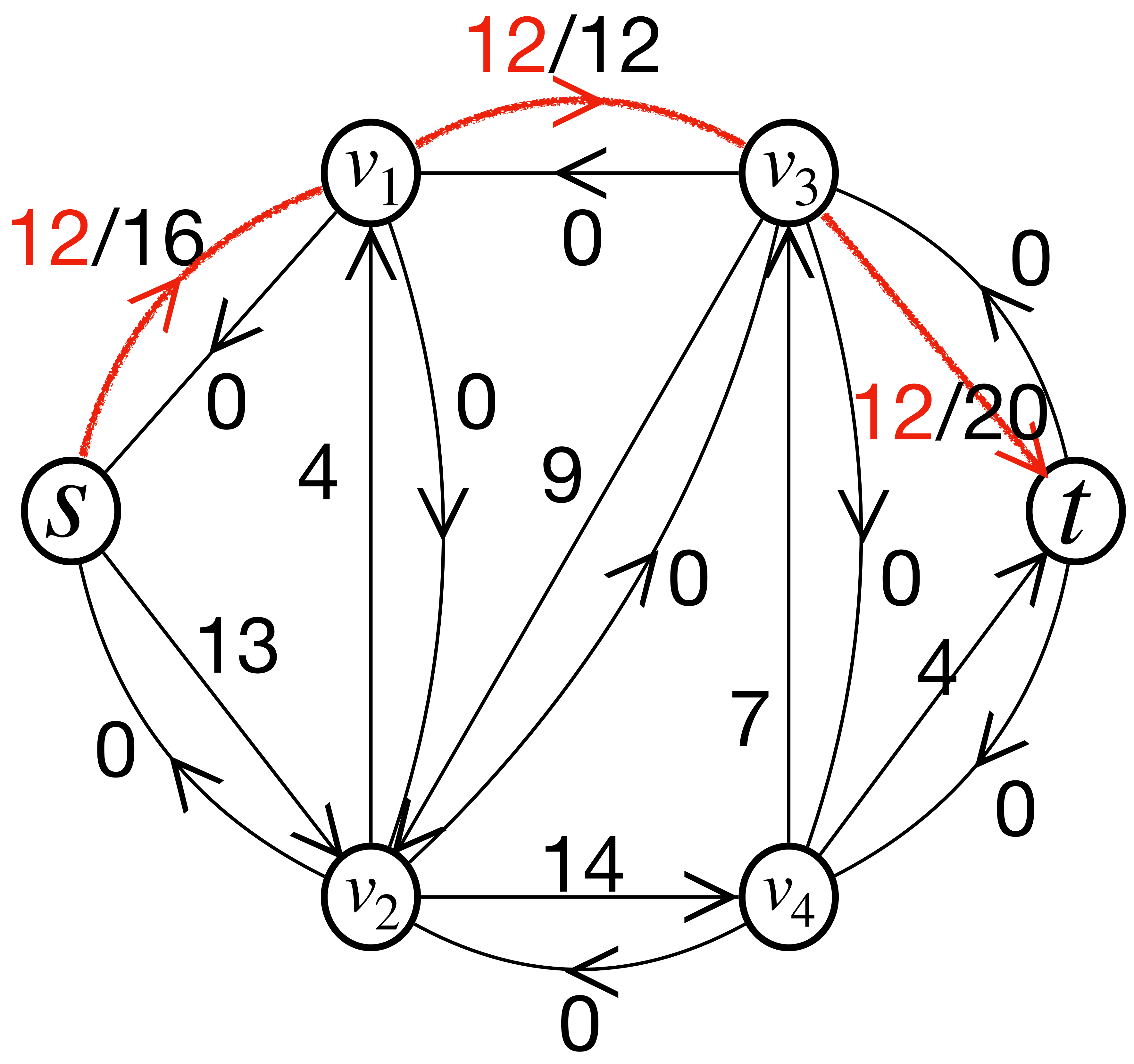

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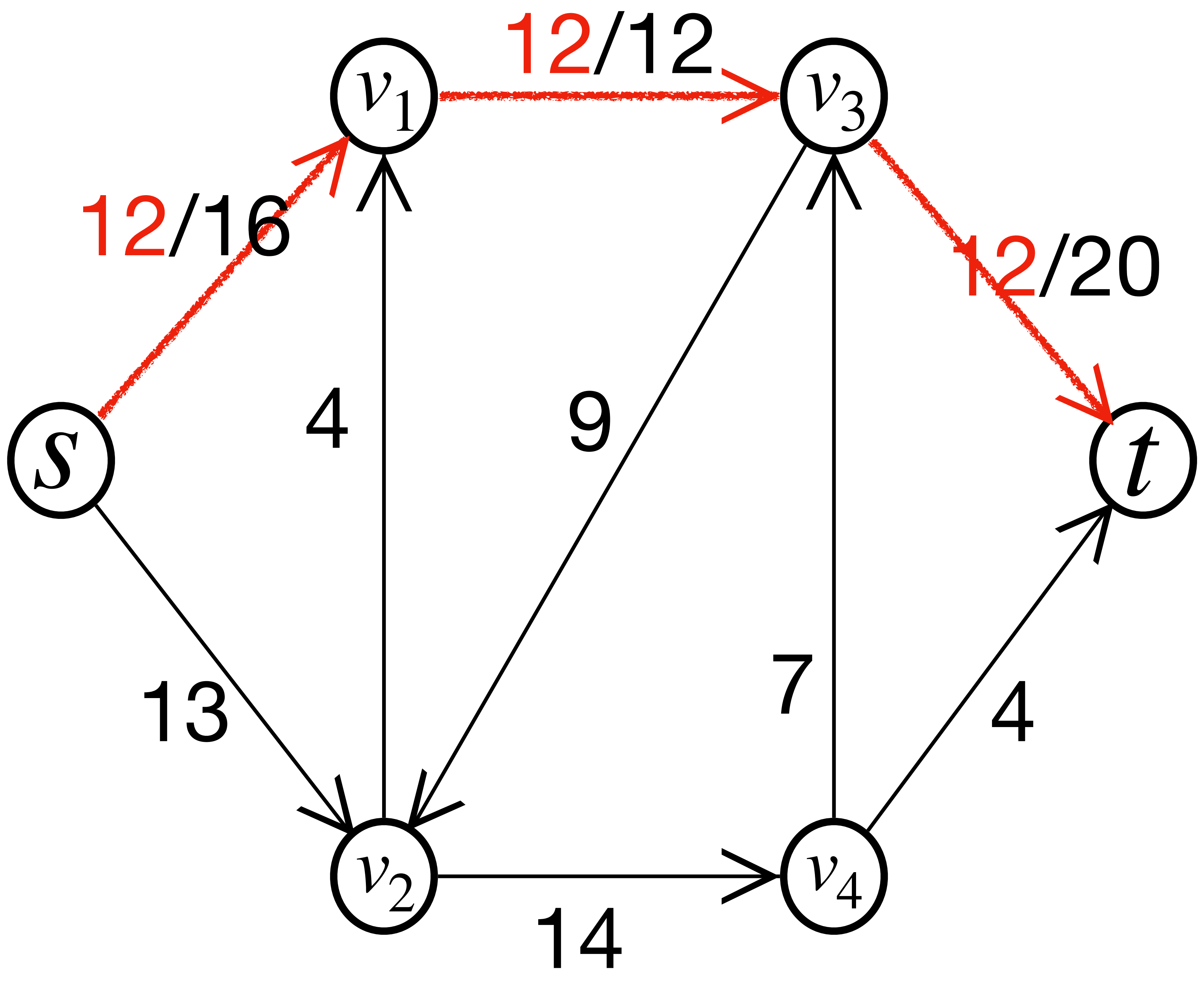

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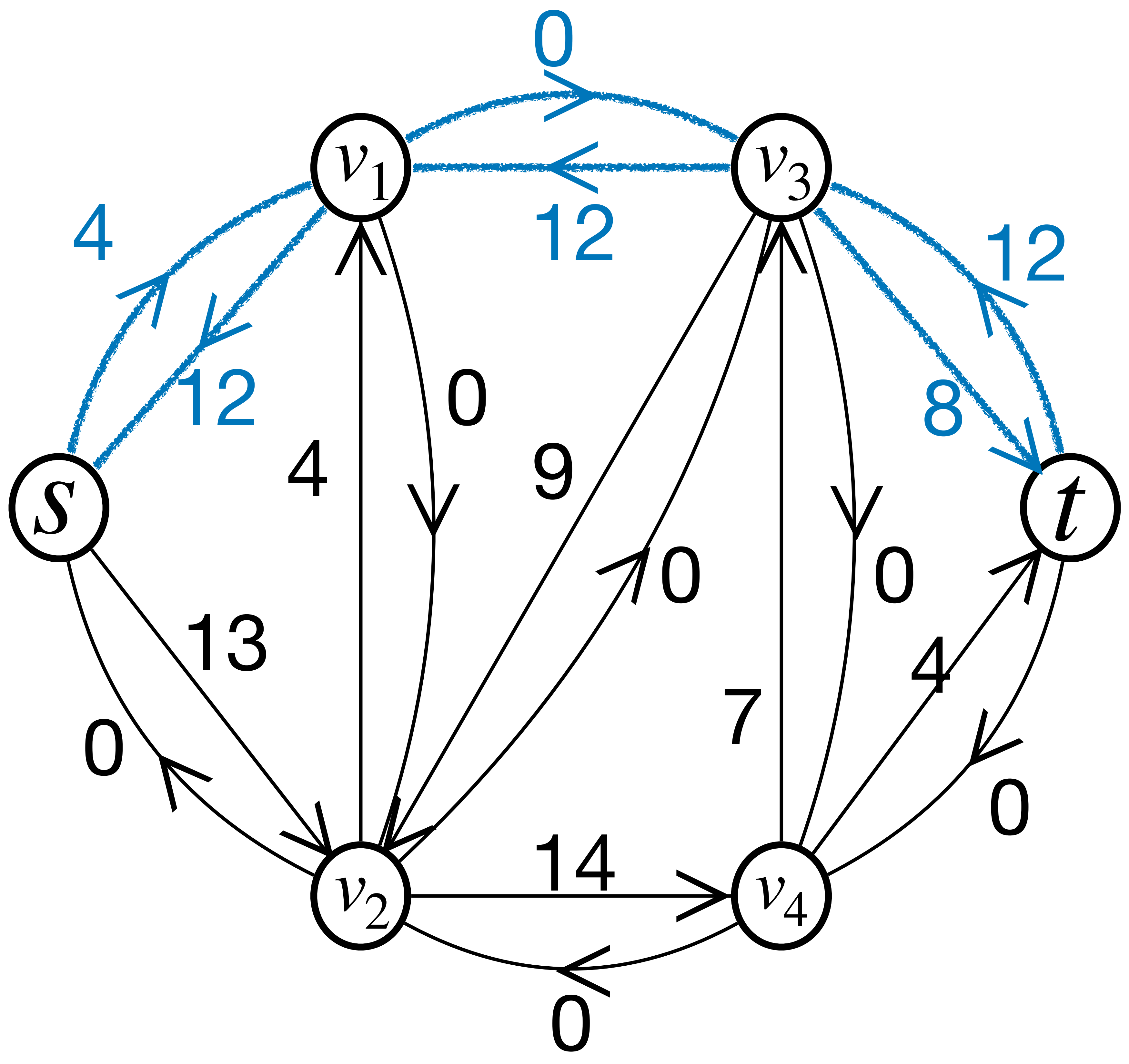
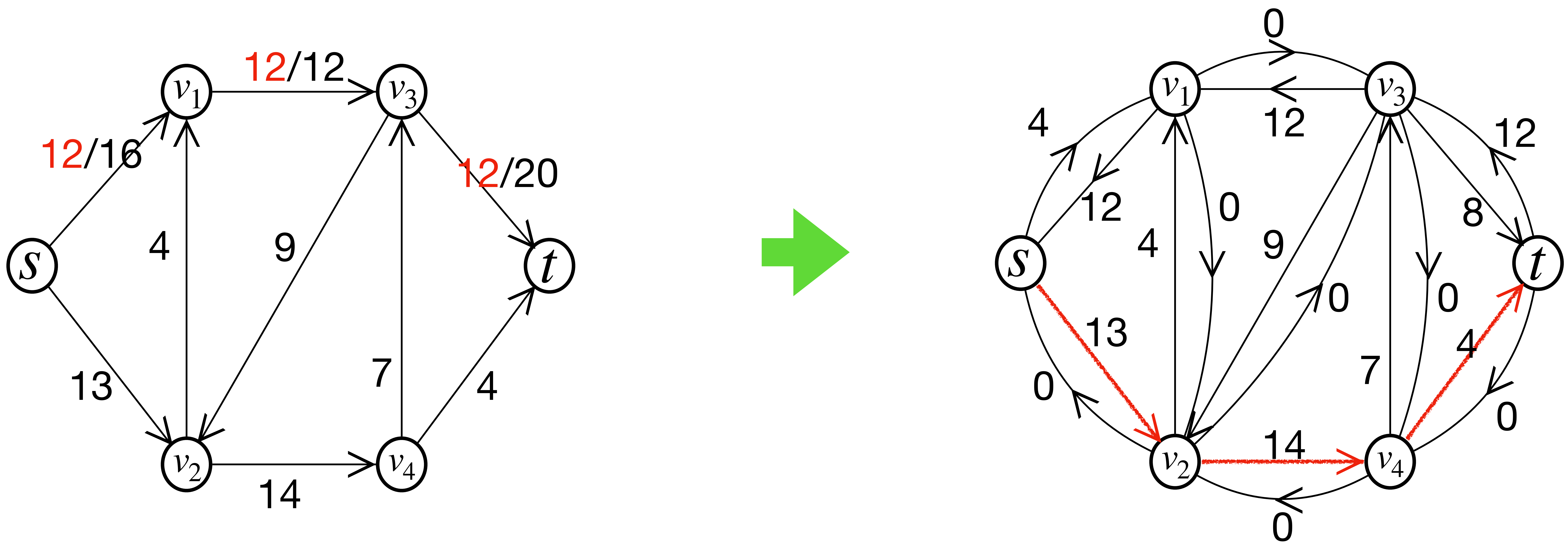


Ford-Fulkerson Method

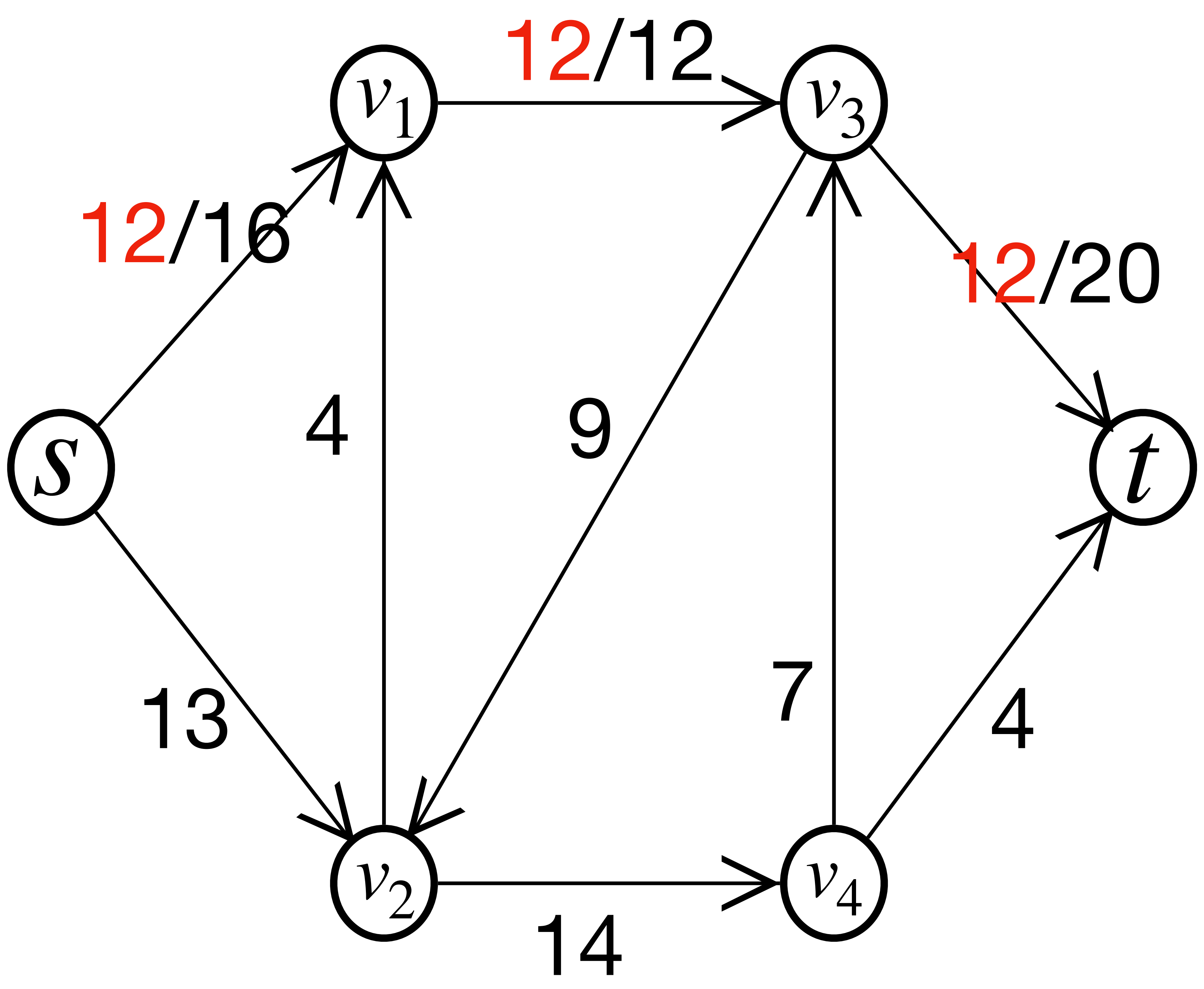
Original network

Residual network

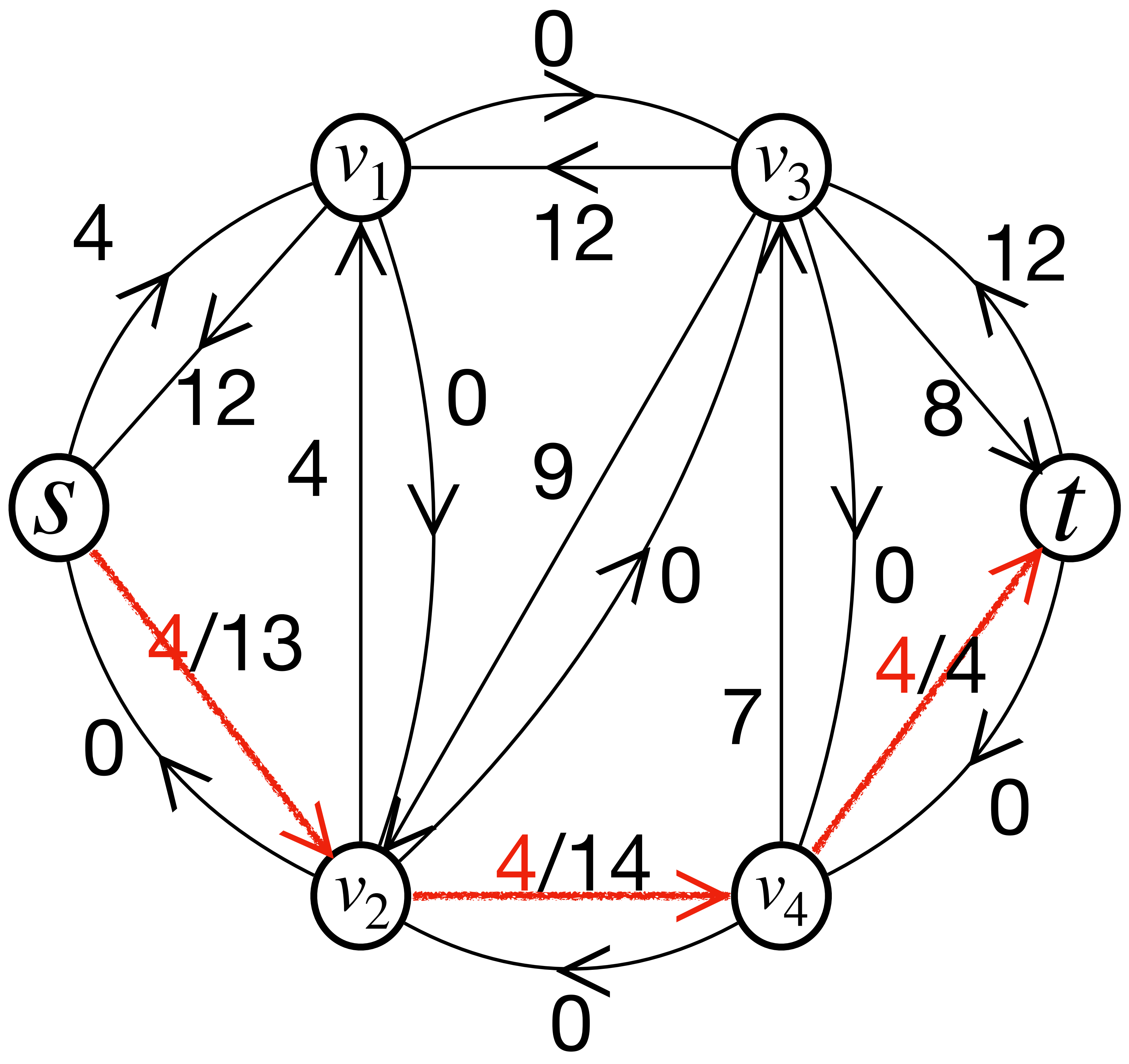


Ford-Fulkerson Method

Original network

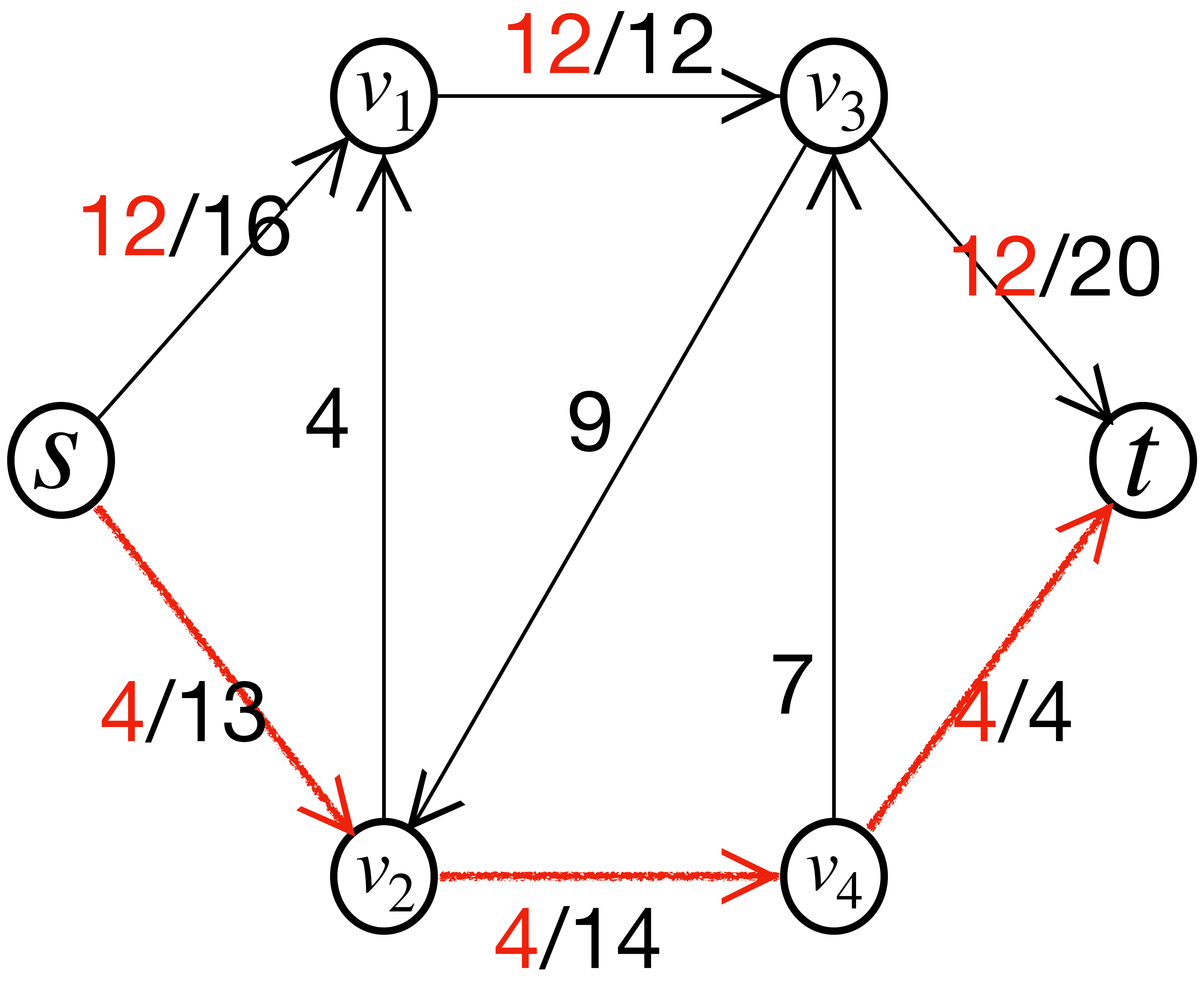


Residual network

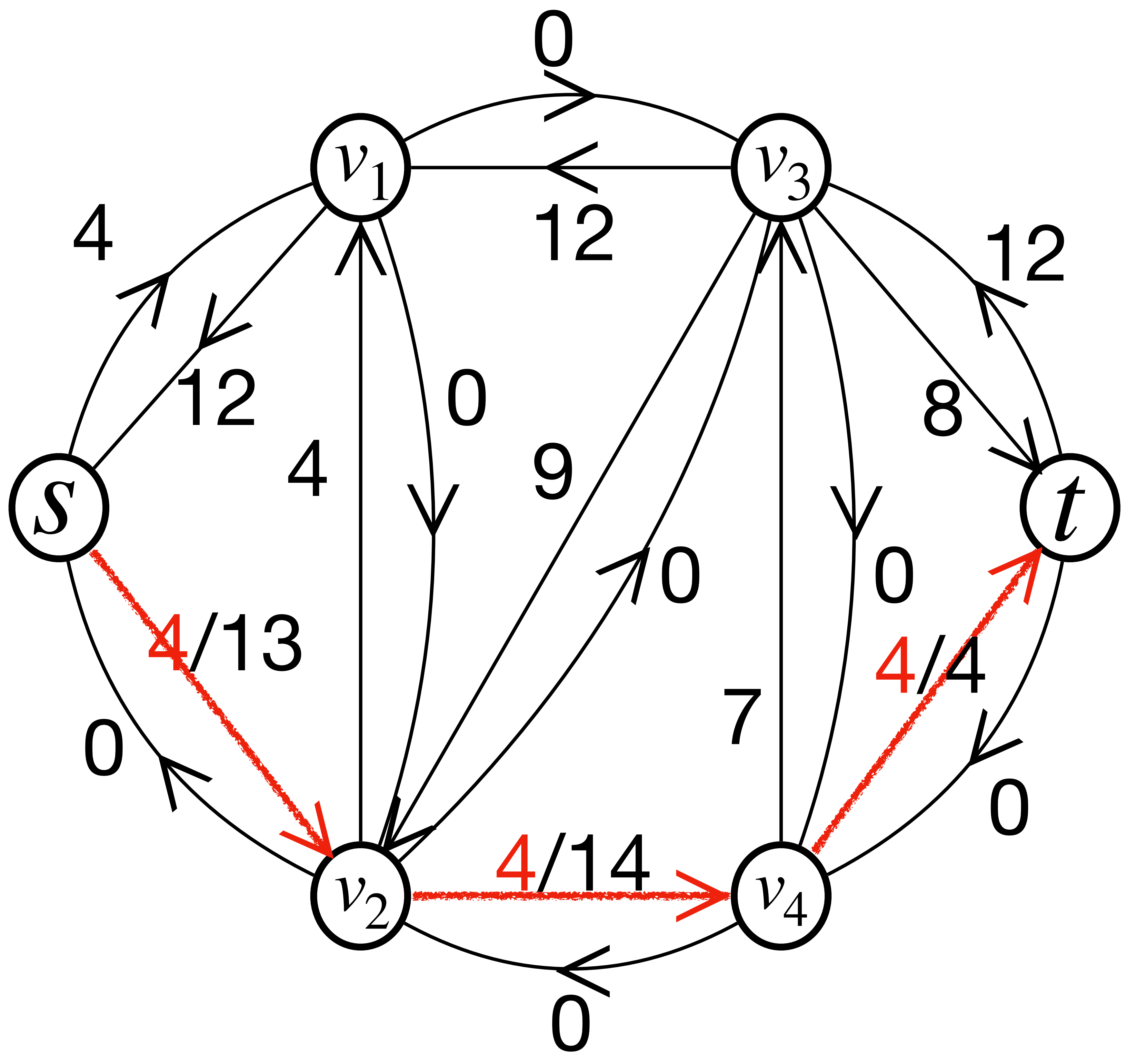


Ford-Fulkerson Method

Original network

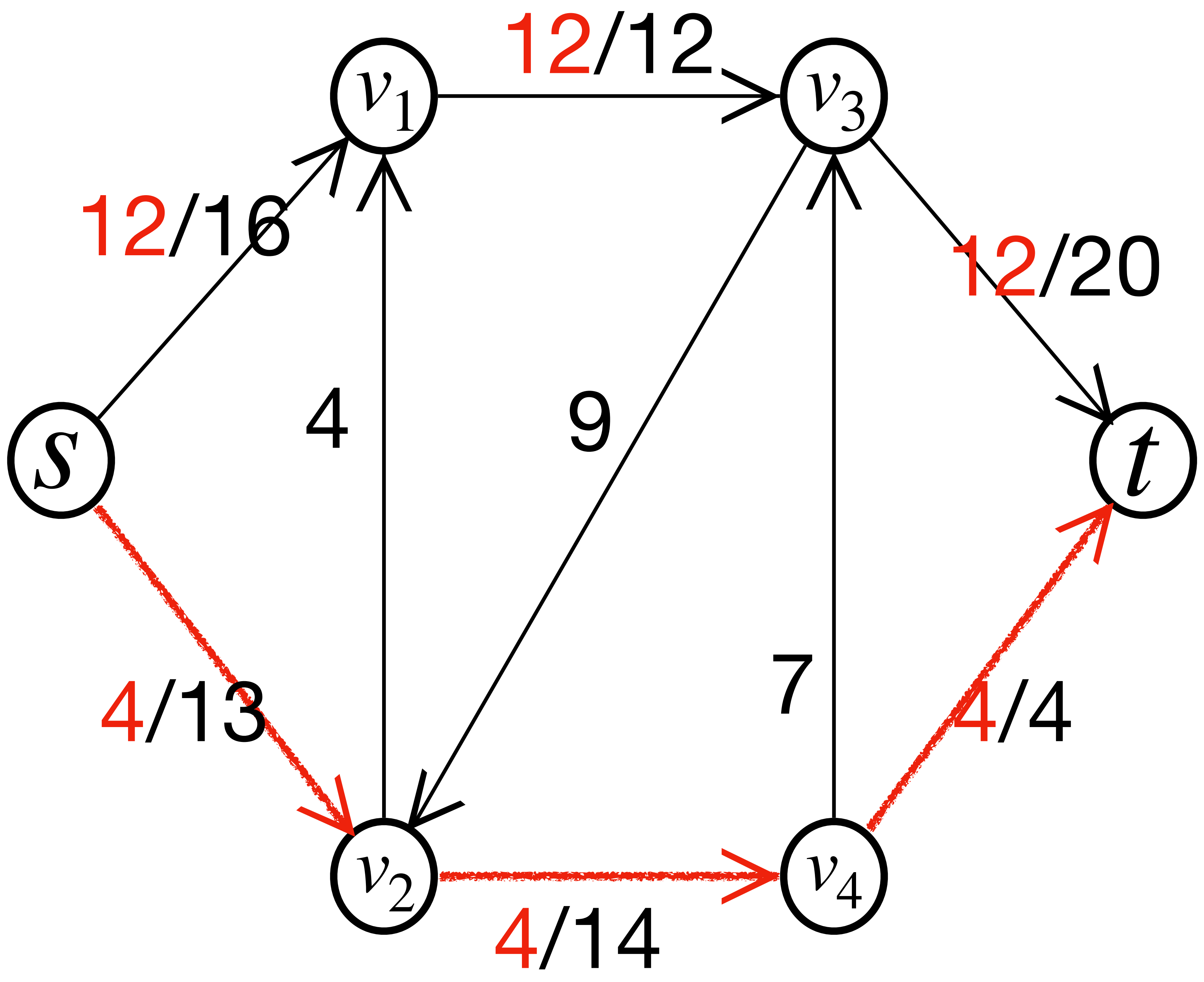


Residual network

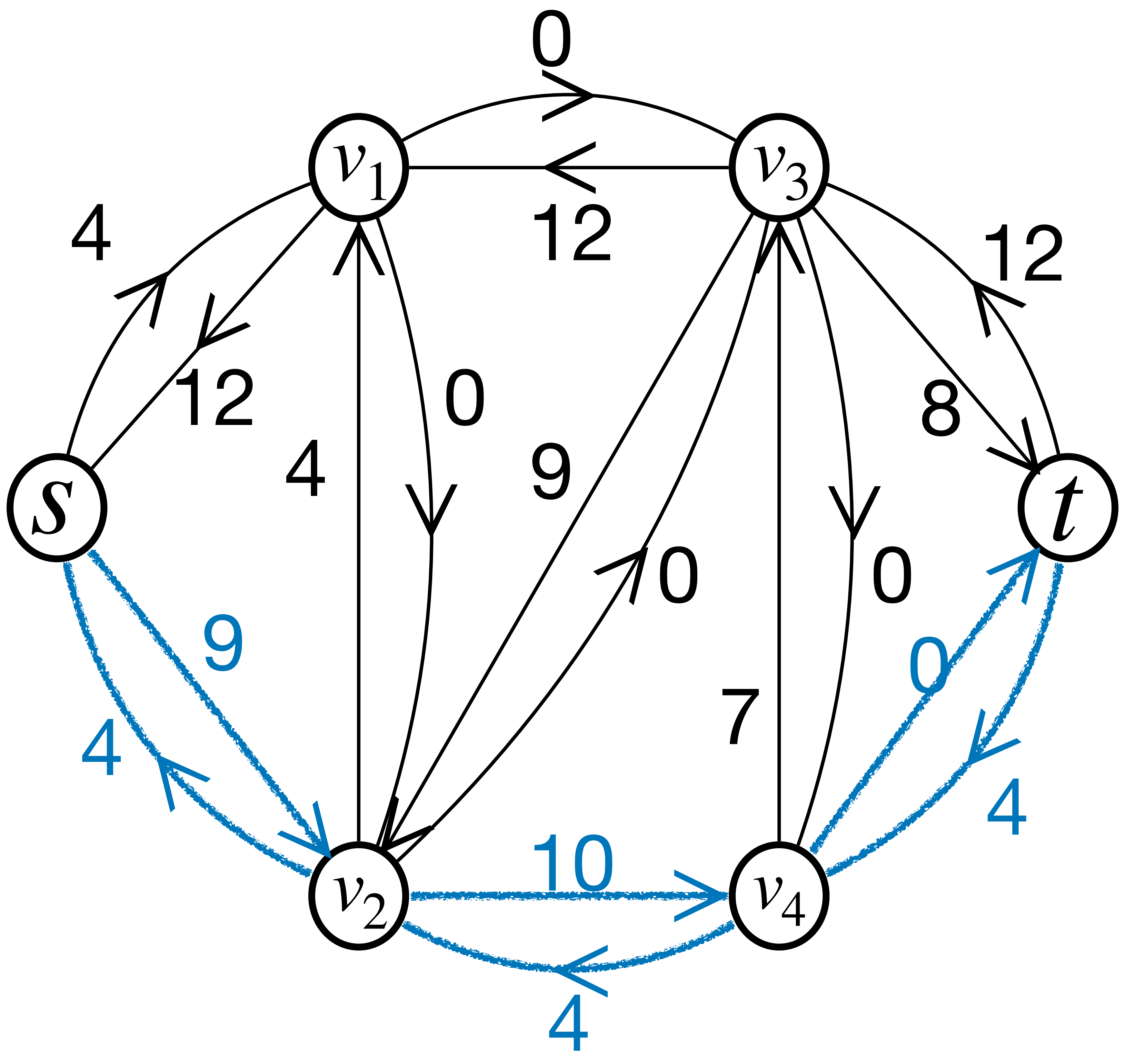


Ford-Fulkerson Method

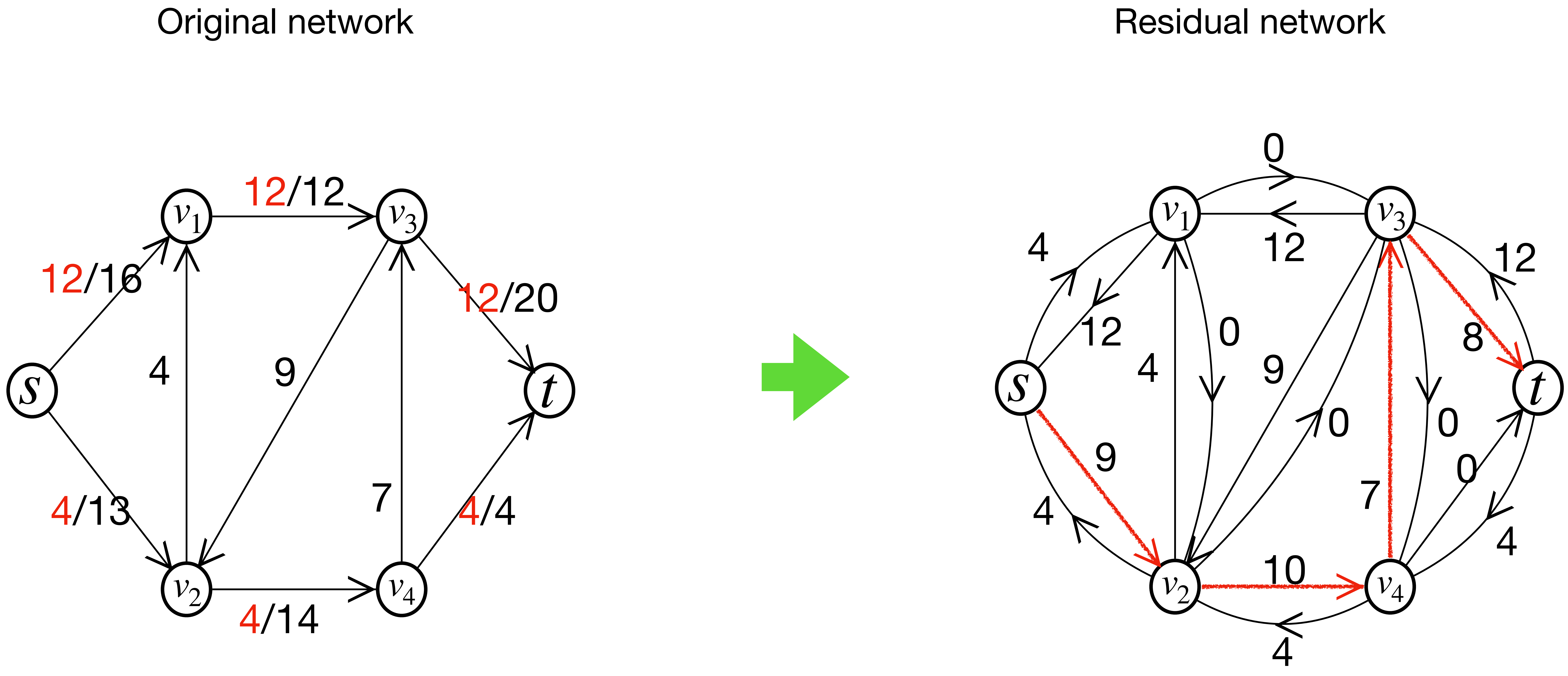
Original network



Residual network

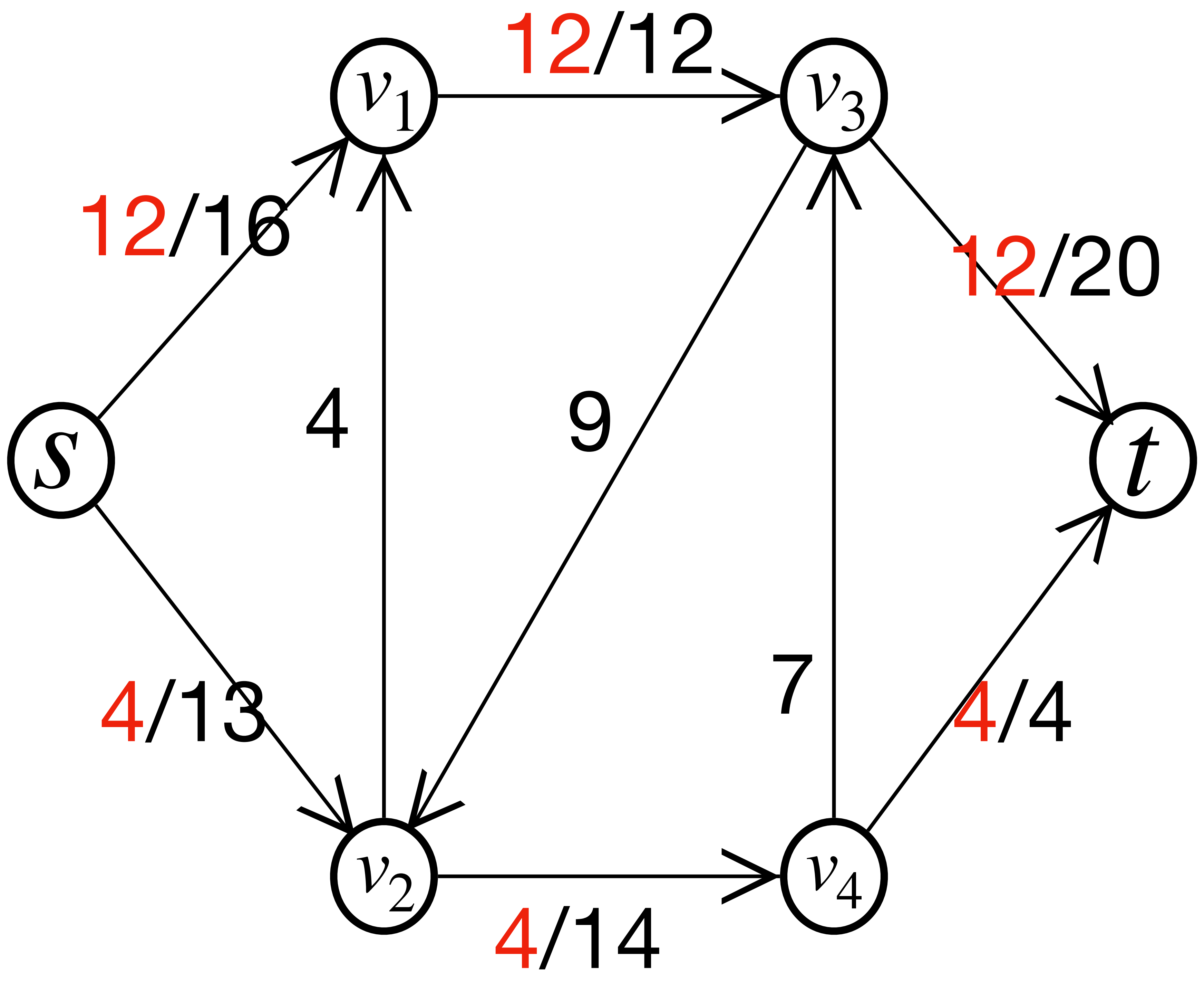


Ford-Fulkerson Method

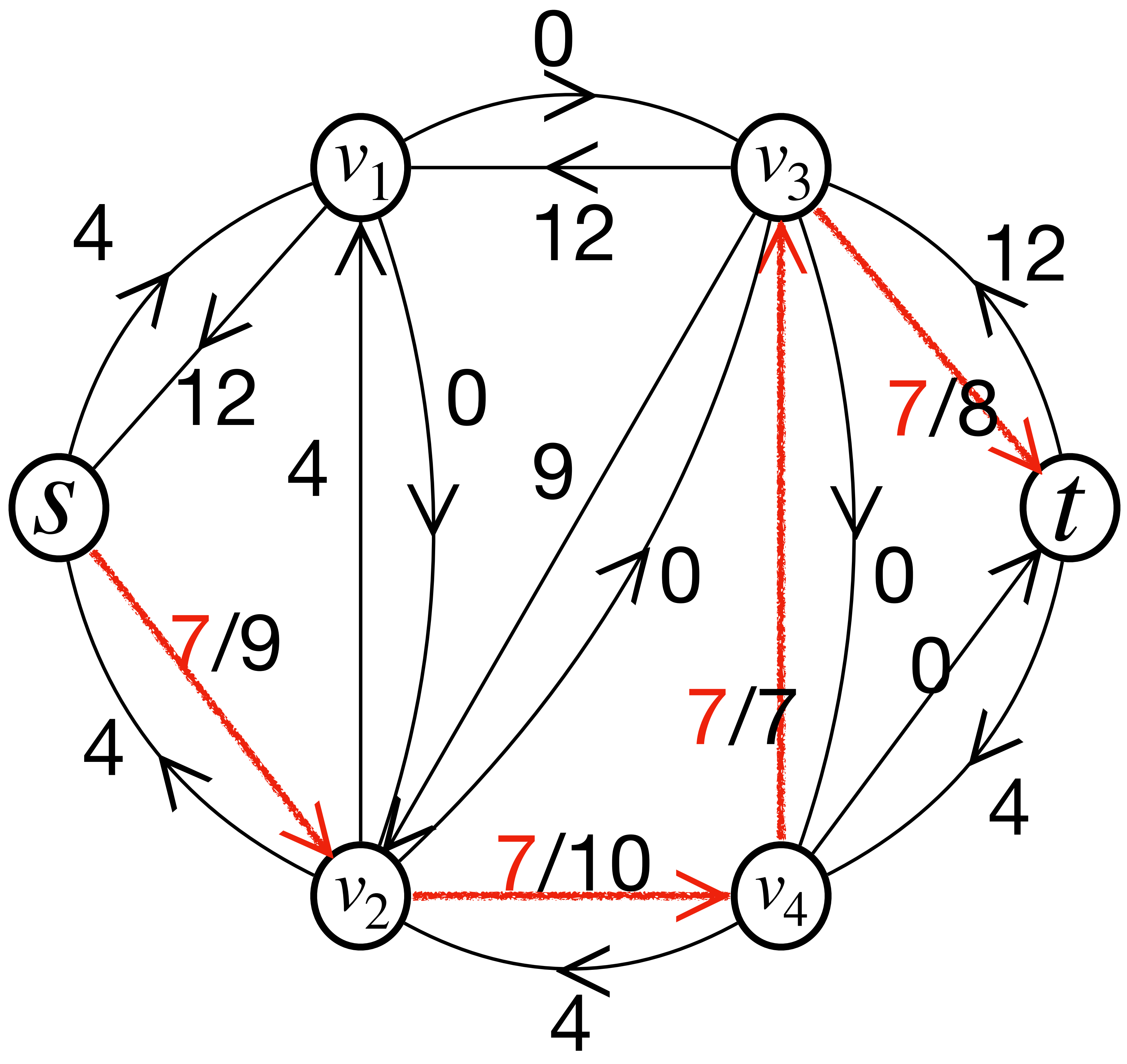


Ford-Fulkerson Method

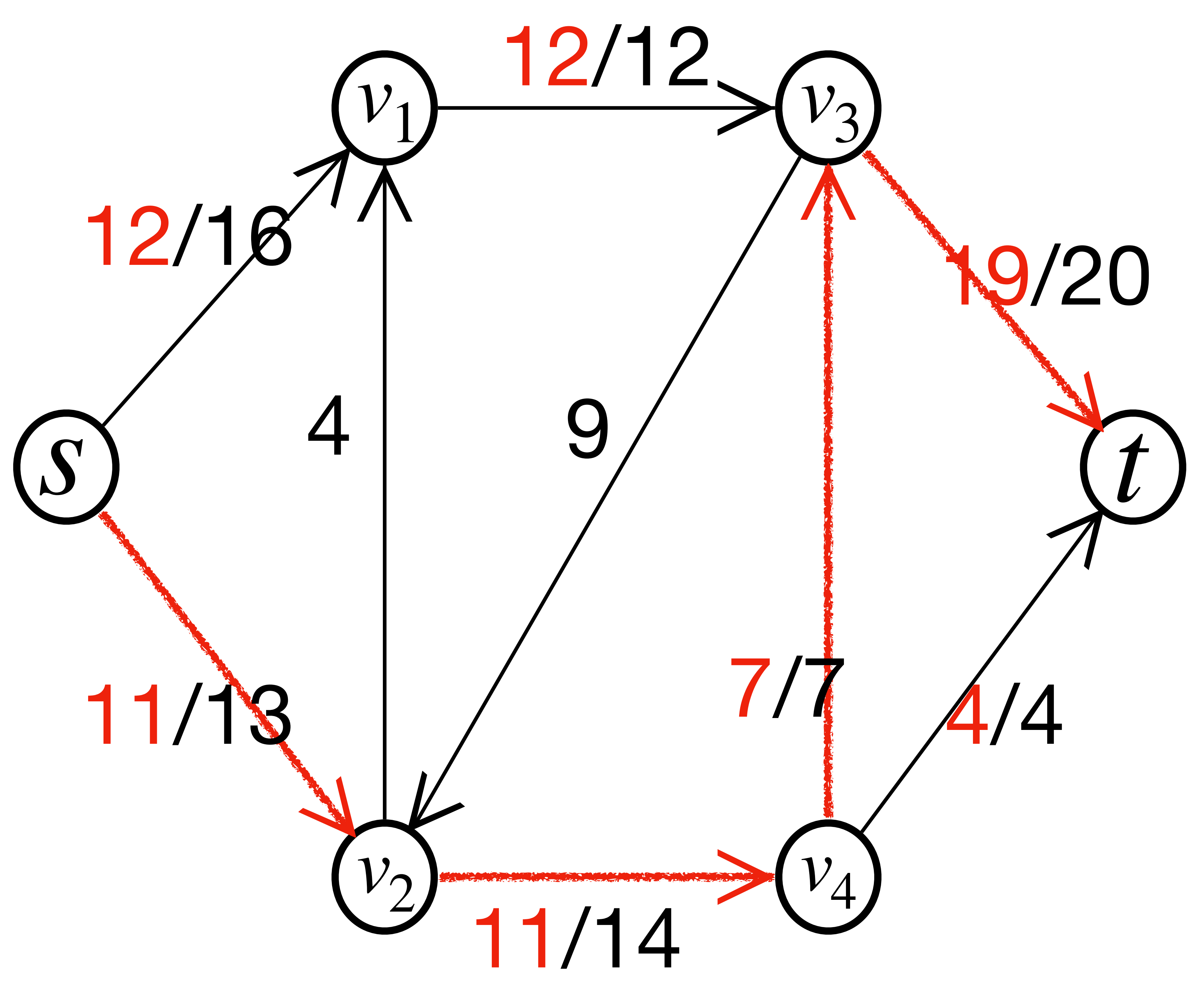
Original network

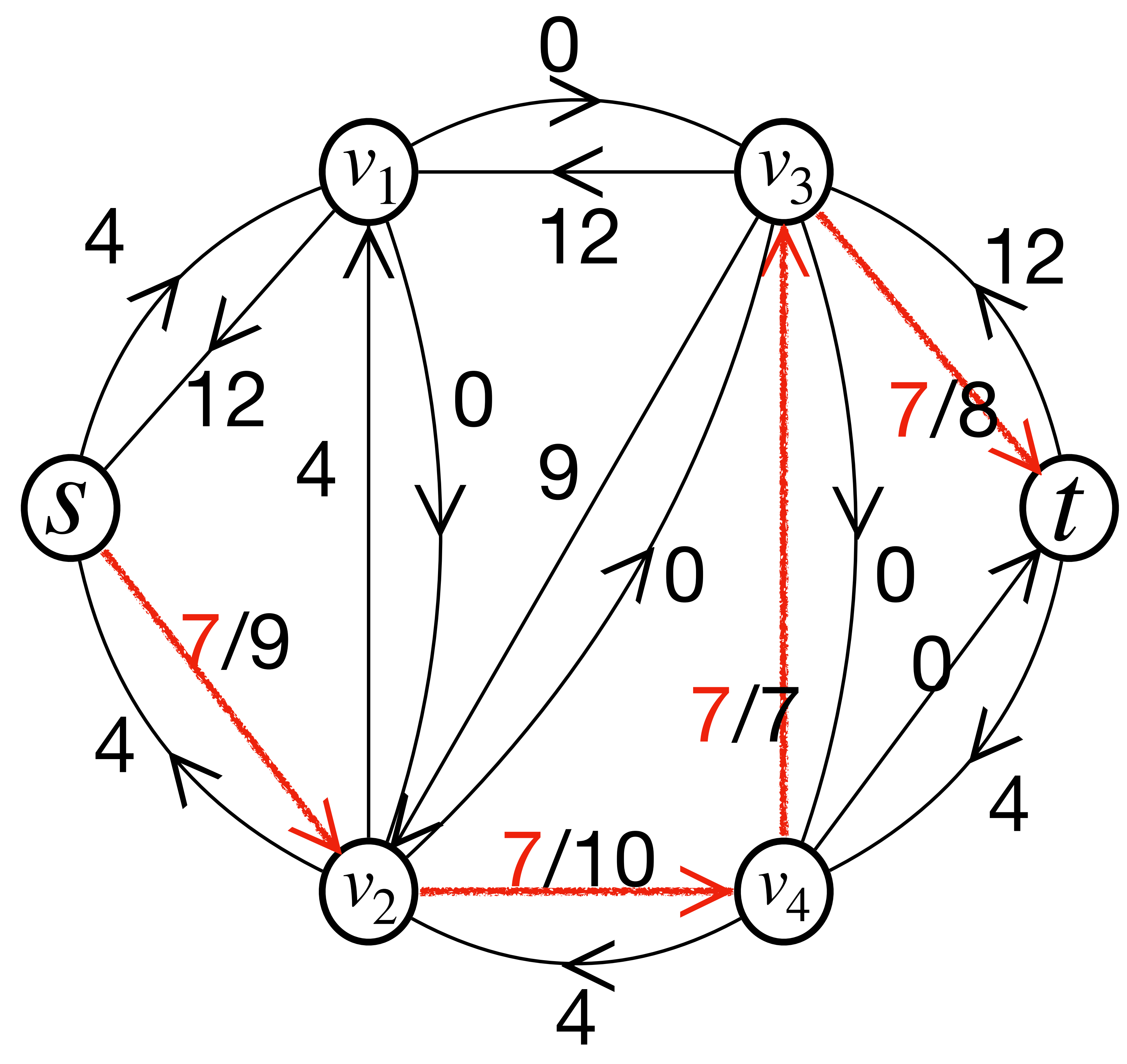


Residual network



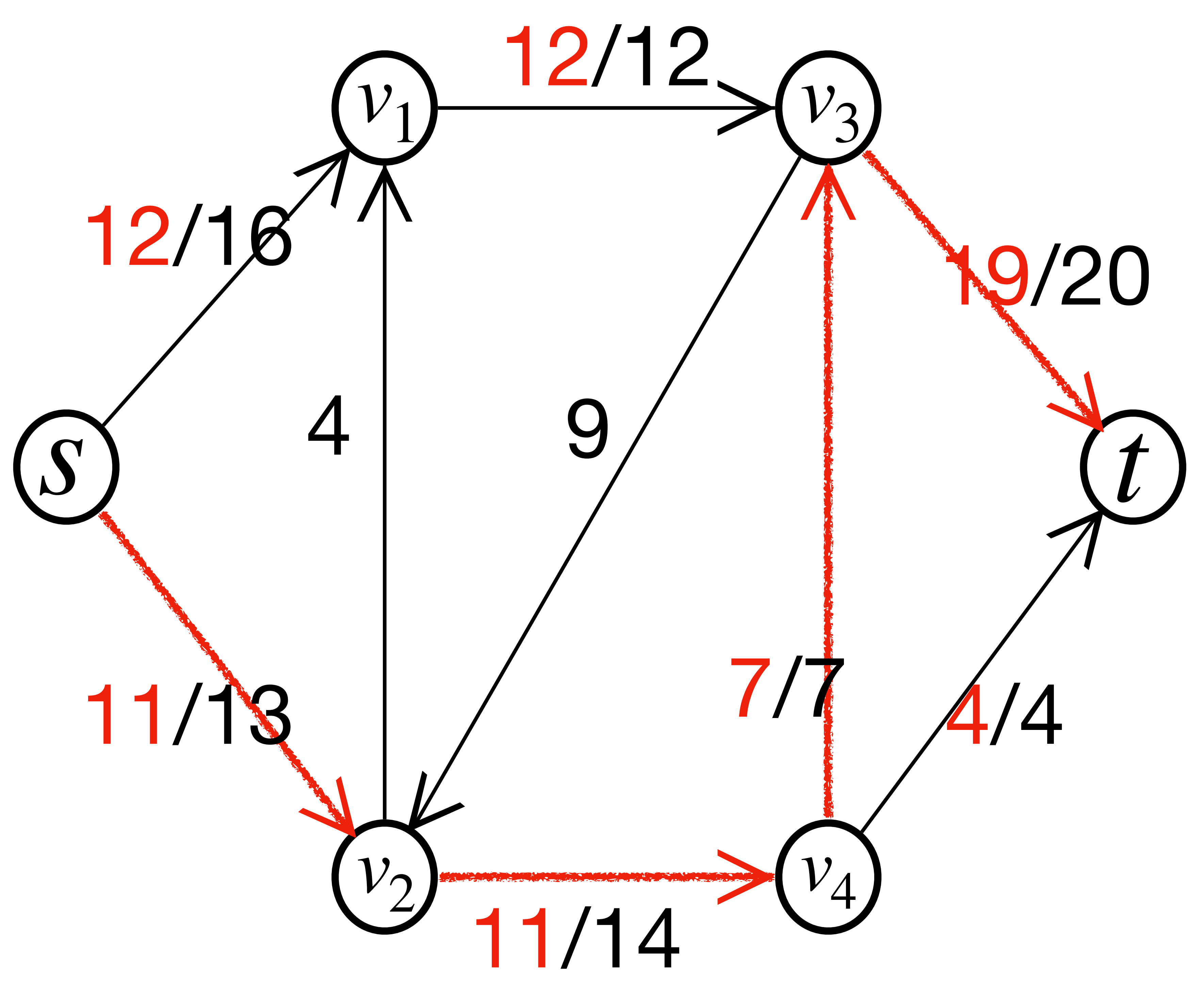
Ford-Fulkerson Method



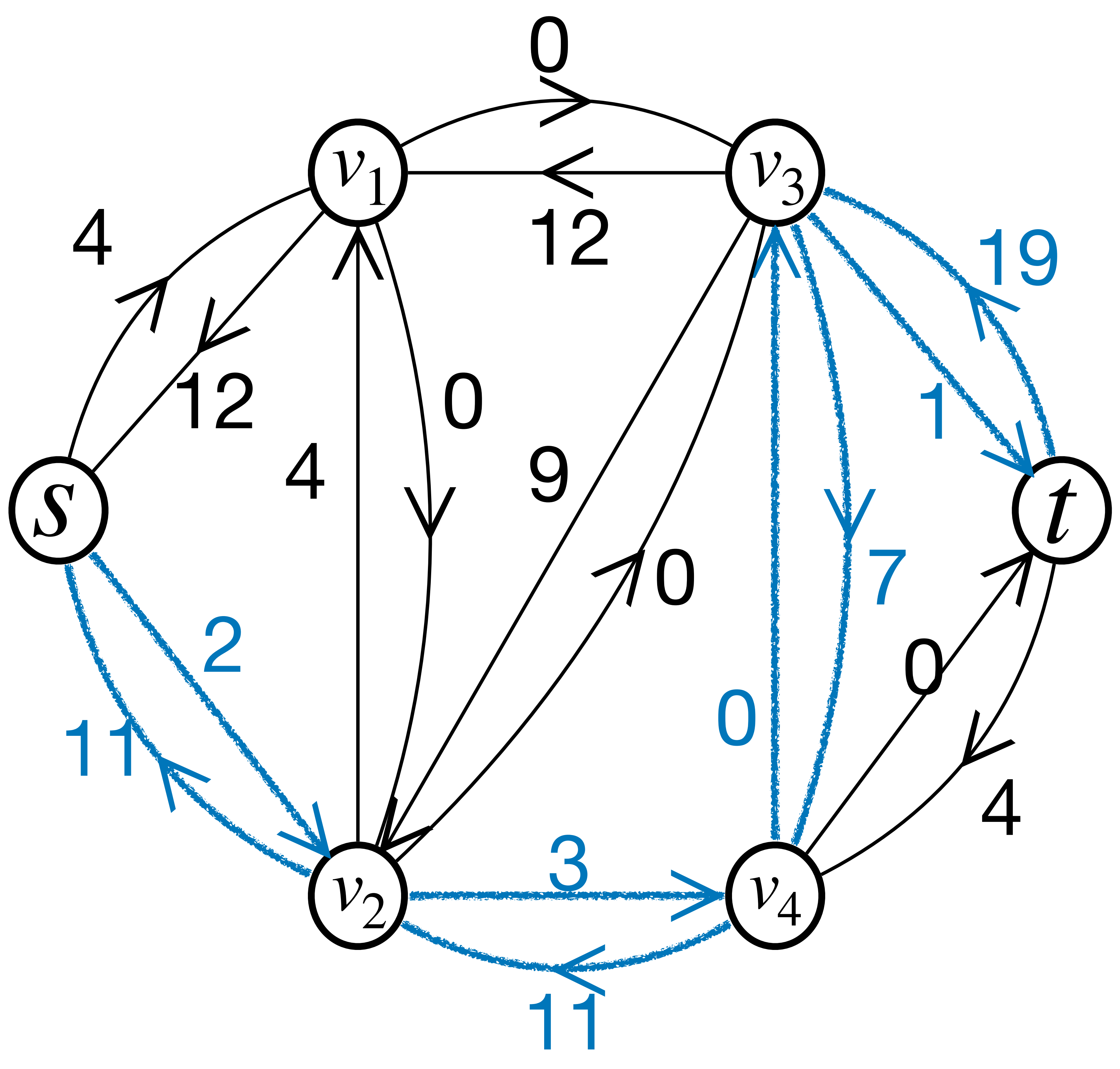


Ford-Fulkerson Method

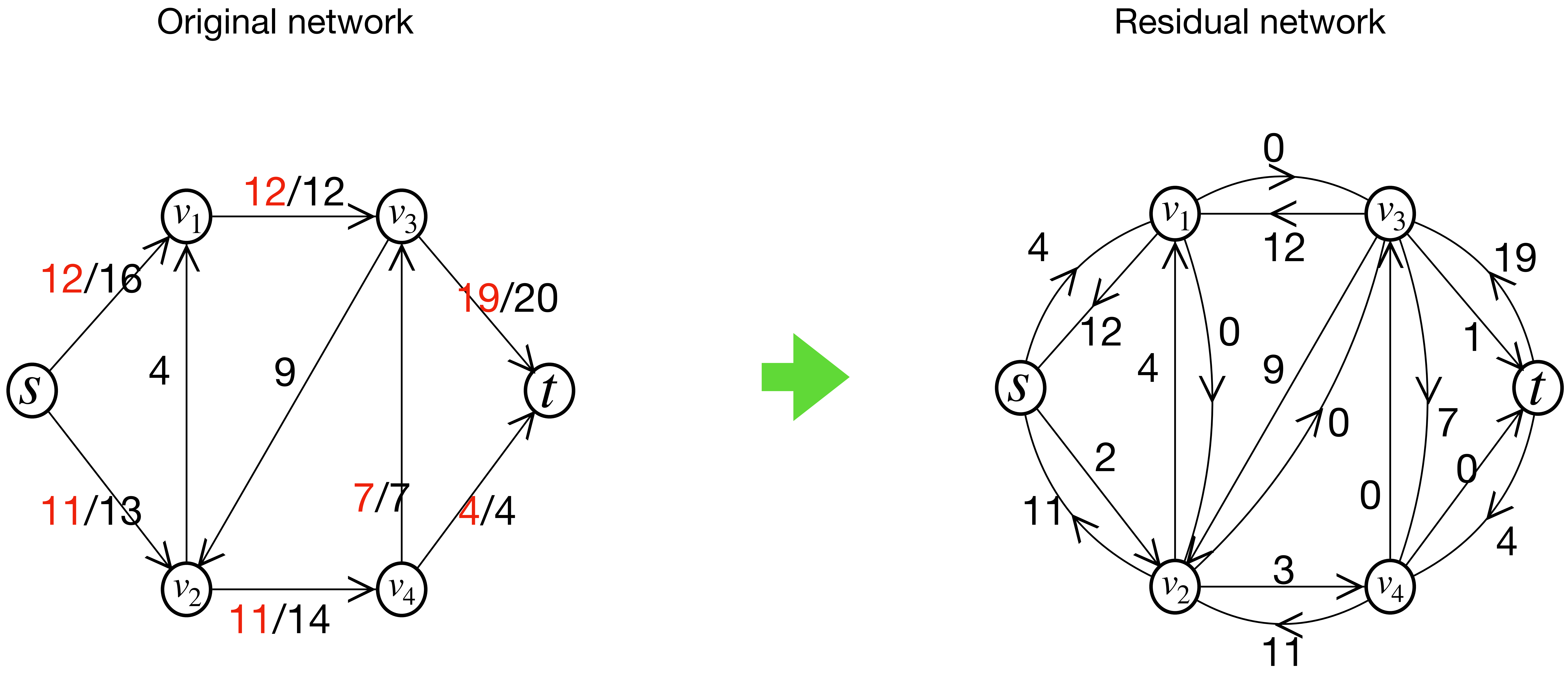
Original network



Residual network



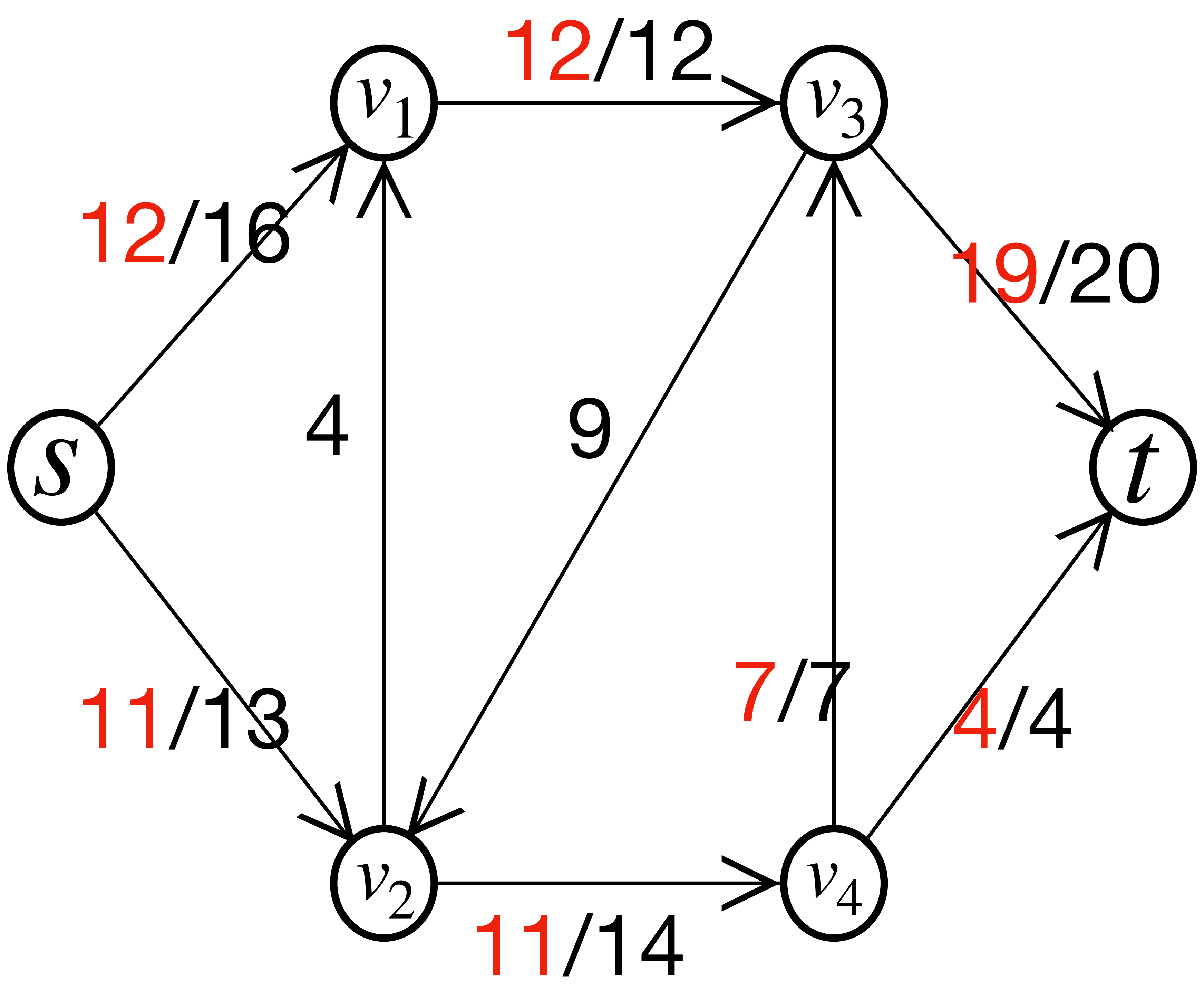
Ford-Fulkerson Method



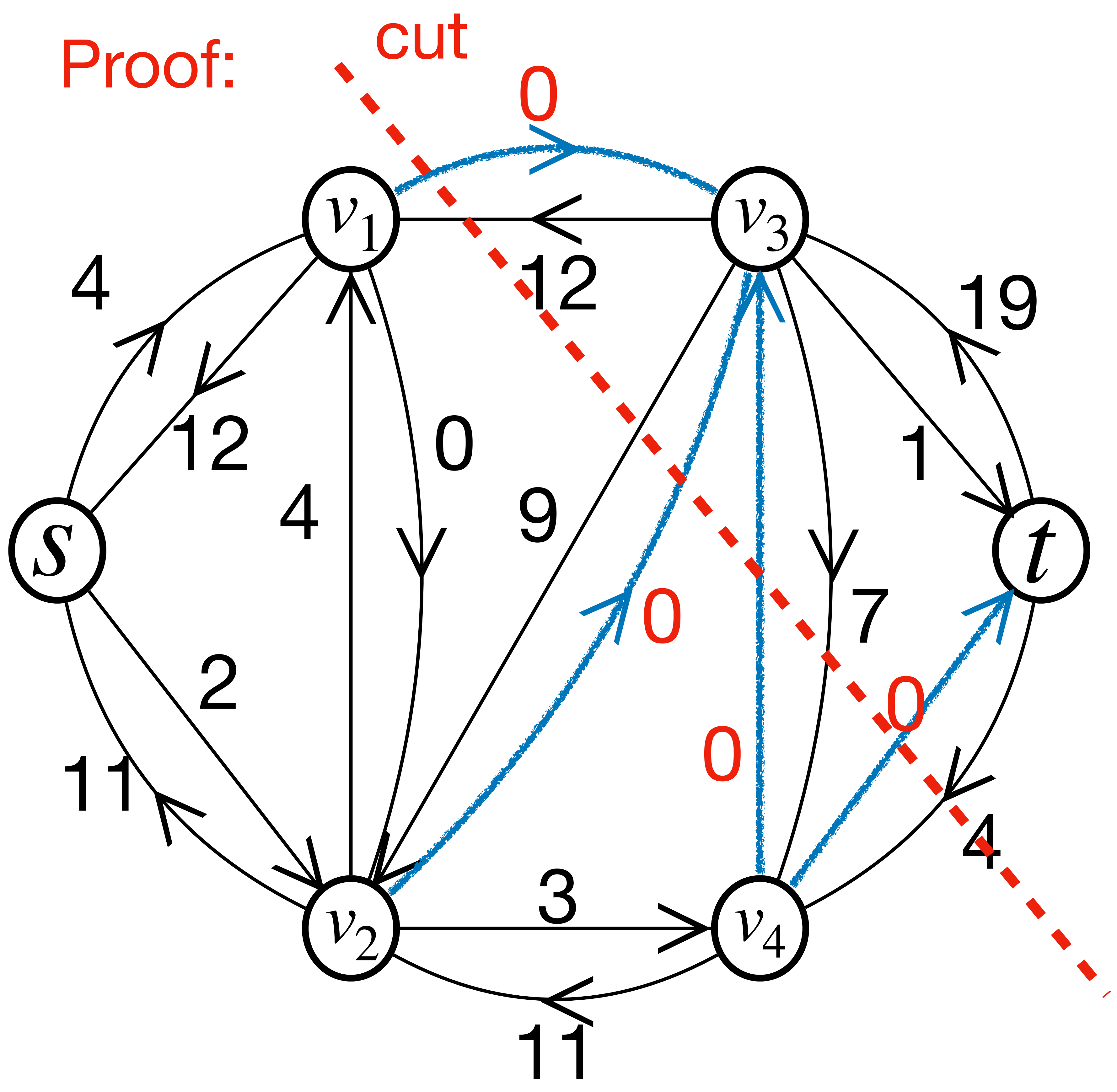
Is there still a path from s to t of residual capacity > 0?

Ford-Fulkerson Method

Original network



Residual network

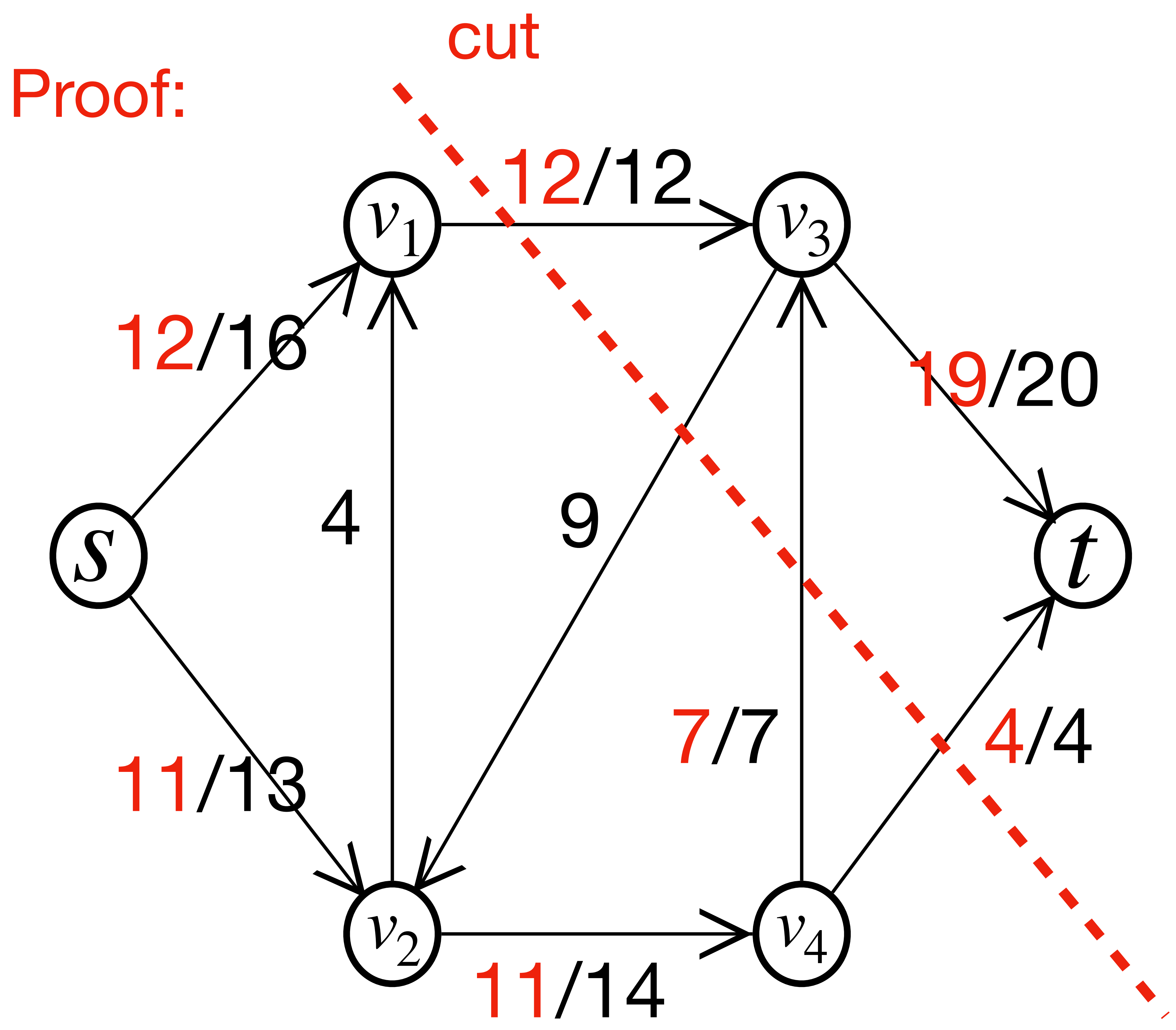


Is there still a path from s to t of residual capacity > 0?

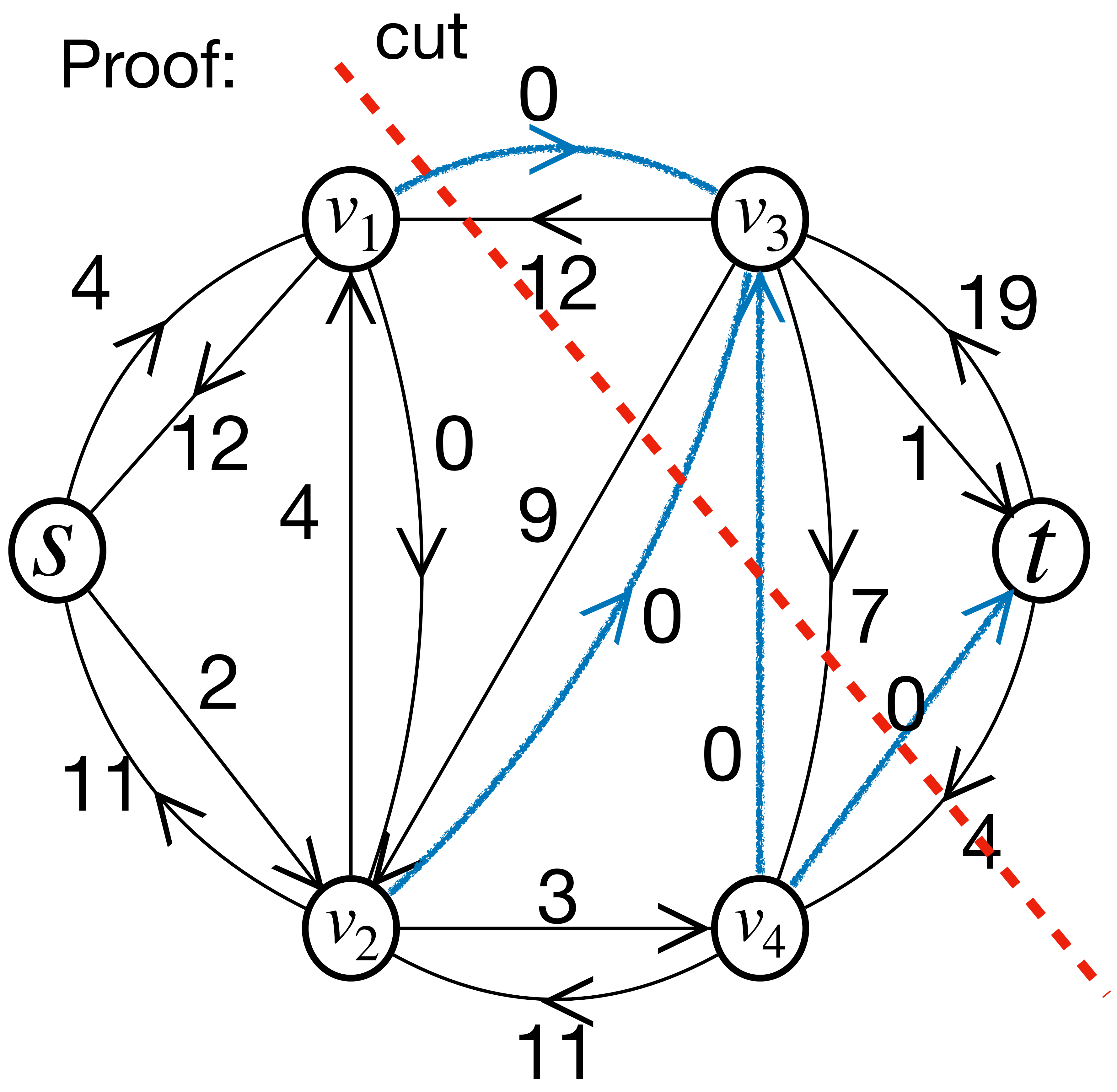
NO

Ford-Fulkerson Method

Original network



Residual network



Size of flow: 12+11=19+4=23

Is it optimal? YES.

Is there still a path from s to t of residual capacity > 0?

NO

Quiz questions:

1. What is “residual capacity”?

2. How does the Ford-Fulkerson Method use residual capacity to augment the flow?