Algorithms

Lecture Topic: NP-Completeness (Part 3)

Anxiao (Andrew) Jiang

Roadmap of this lecture:

1. NP Completeness

    1.1 Prove the "Clique Problem" is NPC.

    1.2 Prove the "Vertex Cover Problem" is NPC.

NP-Completeness

How to prove a problem L is NP-complete (NPC):

1) Show that $L \in NP$ (by showing a "certificate" and polynomial-time verification for YES-instances).

2) Pick a known NPC problem A and show $A \leq_p L$

   2.1) Show mapping from A to L

   2.2) Show the mapping preserves the "YES/NO" answer

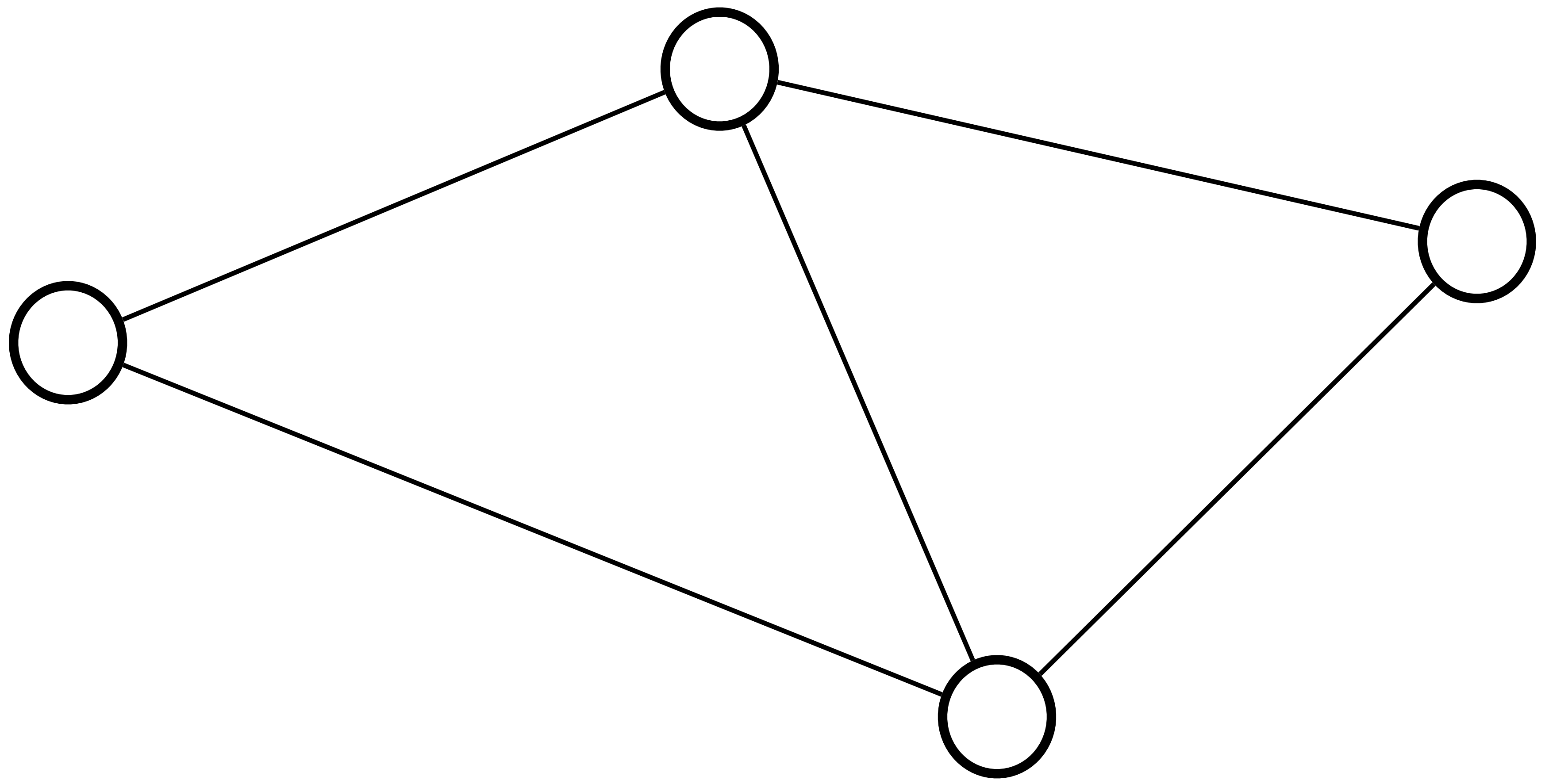
   2.3) Show the mapping takes polynomial time

NP-Completeness

Clique Problem

Clique: Given a graph G=(V,E), a clique in G is a subgraph of G that is a complete graph.

Size of Clique: its number of nodes.



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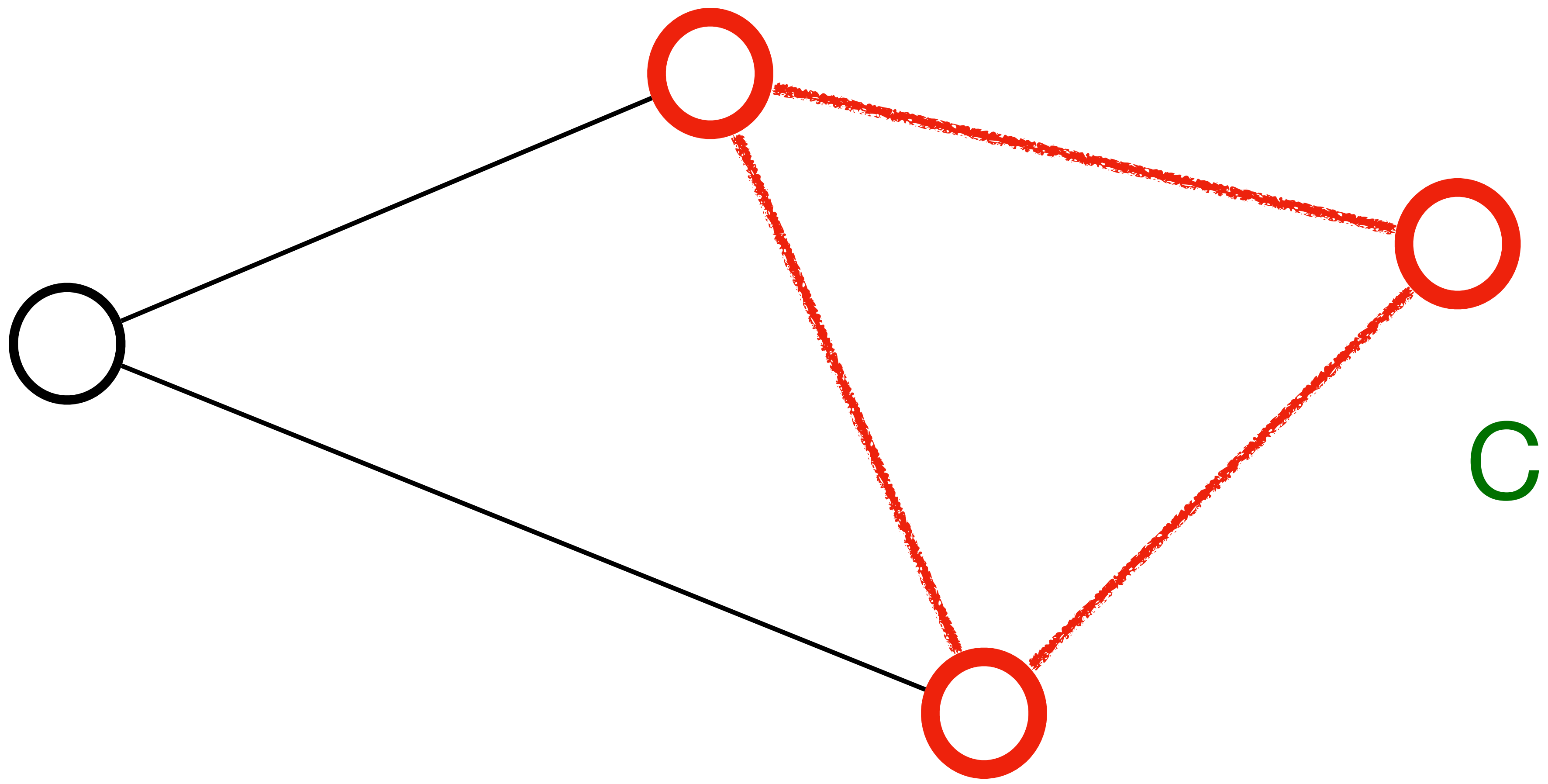

Clique of size 3

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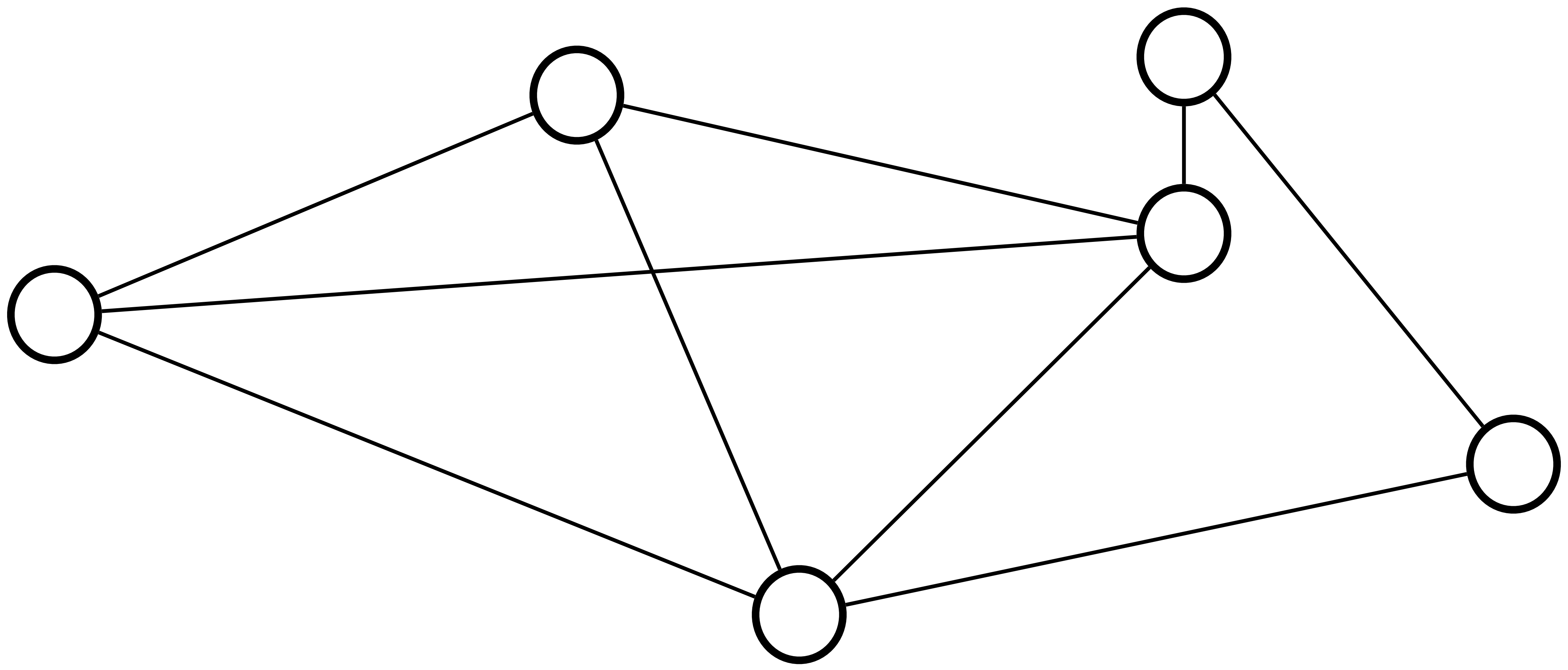

NP-Completeness

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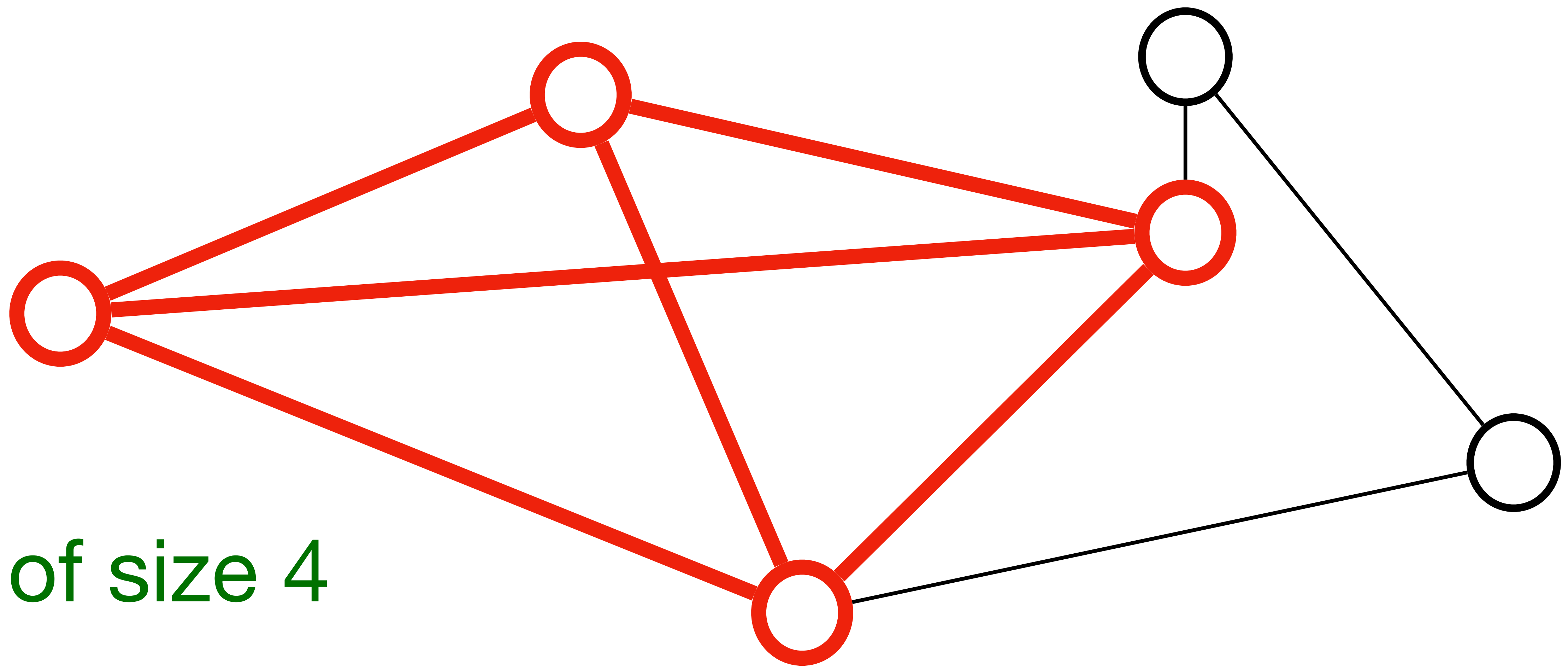

Clique of size 4

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NP-Completeness

Clique Problem:

Input: An undirected graph G=(V,E).

A positive integer k.

Question: Does G have a clique of size k?

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Theorem: The Clique Problem is NPC.

NP-Completeness

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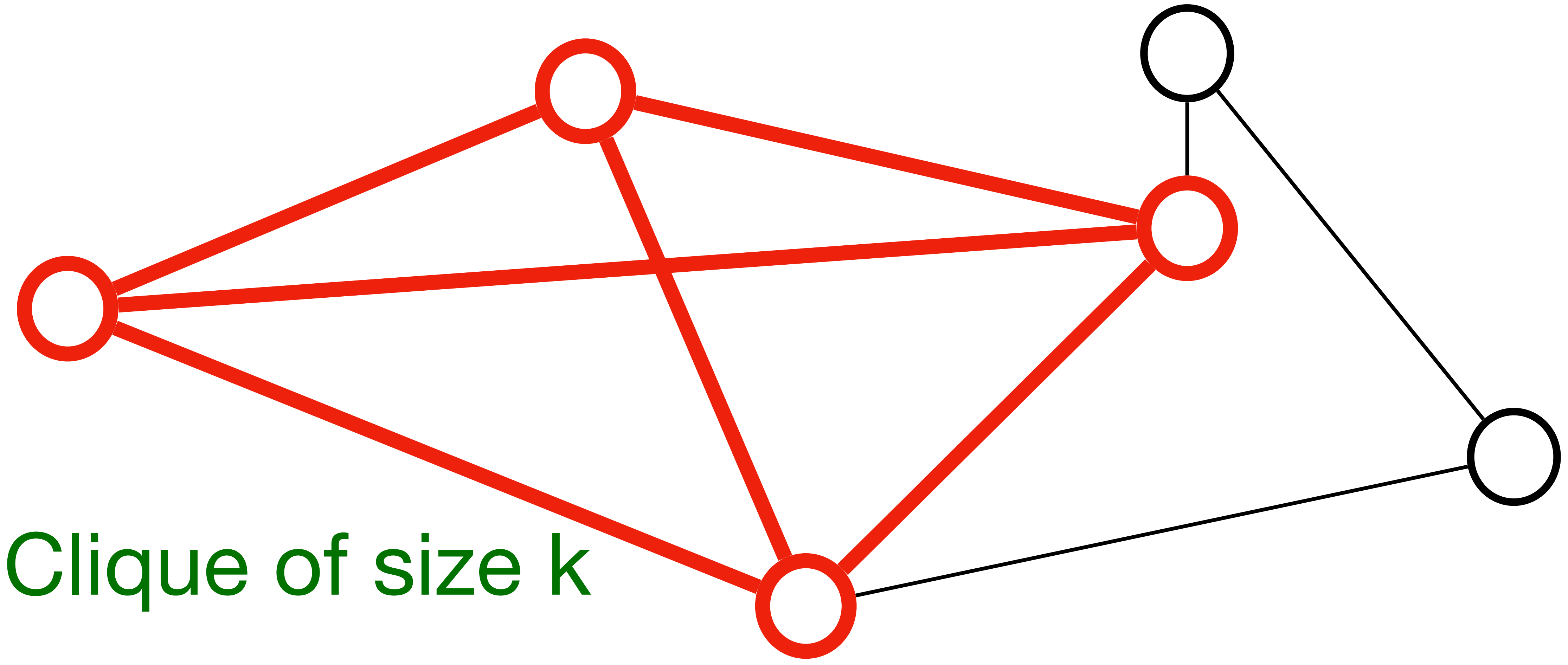
Theorem: The Clique Problem is NPC.

Proof: 1) Clique Problem $\in NP$.



Certificate: a Clique of size k

NP-Completeness

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Theorem: The Clique Problem is NPC.

Proof: 1) Clique Problem $\in NP$.

2) What known NPC problem shall we reduce to the Clique Problem?

NP-Completeness

3-CNF SAT Problem: a known NPC problem.

Boolean logic: AND operation : $0 \wedge 0 = 0, \quad 0 \wedge 1 = 0, \quad 1 \wedge 0 = 0, \quad 1 \wedge 1 = 1$

OR operation : $0 \vee 0 = 0, \quad 0 \vee 1 = 1, \quad 1 \vee 0 = 1, \quad 1 \vee 1 = 1$

NOT operation : $\bar{0} = 1, \quad \bar{1} = 0$

Boolean variables: $x_1, x_2, \cdots, x_n \in \{0,1\}$

Boolean literal: $x_i, \bar{x}_i$

Boolean formula: $\underbrace{(x_1 \vee x_2 \vee x_3)}_{\text{clause}} \wedge \underbrace{(\bar{x}_1 \vee \bar{x}_2 \vee x_4)}_{\text{clause}} \wedge \underbrace{(x_1 \vee \bar{x}_4 \vee x_5)}_{\text{clause}} \wedge \underbrace{(x_2 \vee \bar{x}_3 \vee \bar{x}_5)}_{\text{clause}}$

CNF: Conjunctive Normal Form

NP-Completeness

3-CNF SAT Problem: a known NPC problem.

3-CNF SAT Problem:

Input: A CNF formula with n variables and k clauses,
where each clause is the "OR" of 3 literals.

Question: Does there exist a solution to the variables that make the formula be true?

Instance: $(x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_4) \wedge (x_1 \vee \bar{x}_4 \vee x_5) \wedge (x_2 \vee \bar{x}_3 \vee \bar{x}_5)$

n=5 variables k=4 clauses

NP-Completeness

3-CNF SAT Problem: a known NPC problem.

3-CNF SAT Problem:

Input: A CNF formula with n variables and k clauses, where each clause is the "OR" of 3 literals.

Question: Does there exist a solution to the variables that make the formula be true?

We now show a polynomial-time reduction from the 3-CNF SAT Problem to Clique Problem.

Instance: $(x_1 \vee x_2 \vee x_3) \wedge (\bar{x_1} \vee \bar{x_2} \vee x_4) \wedge (x_1 \vee \bar{x_4} \vee x_5) \wedge (x_2 \vee \bar{x_3} \vee \bar{x_5})$

n=5 variables k=4 clauses

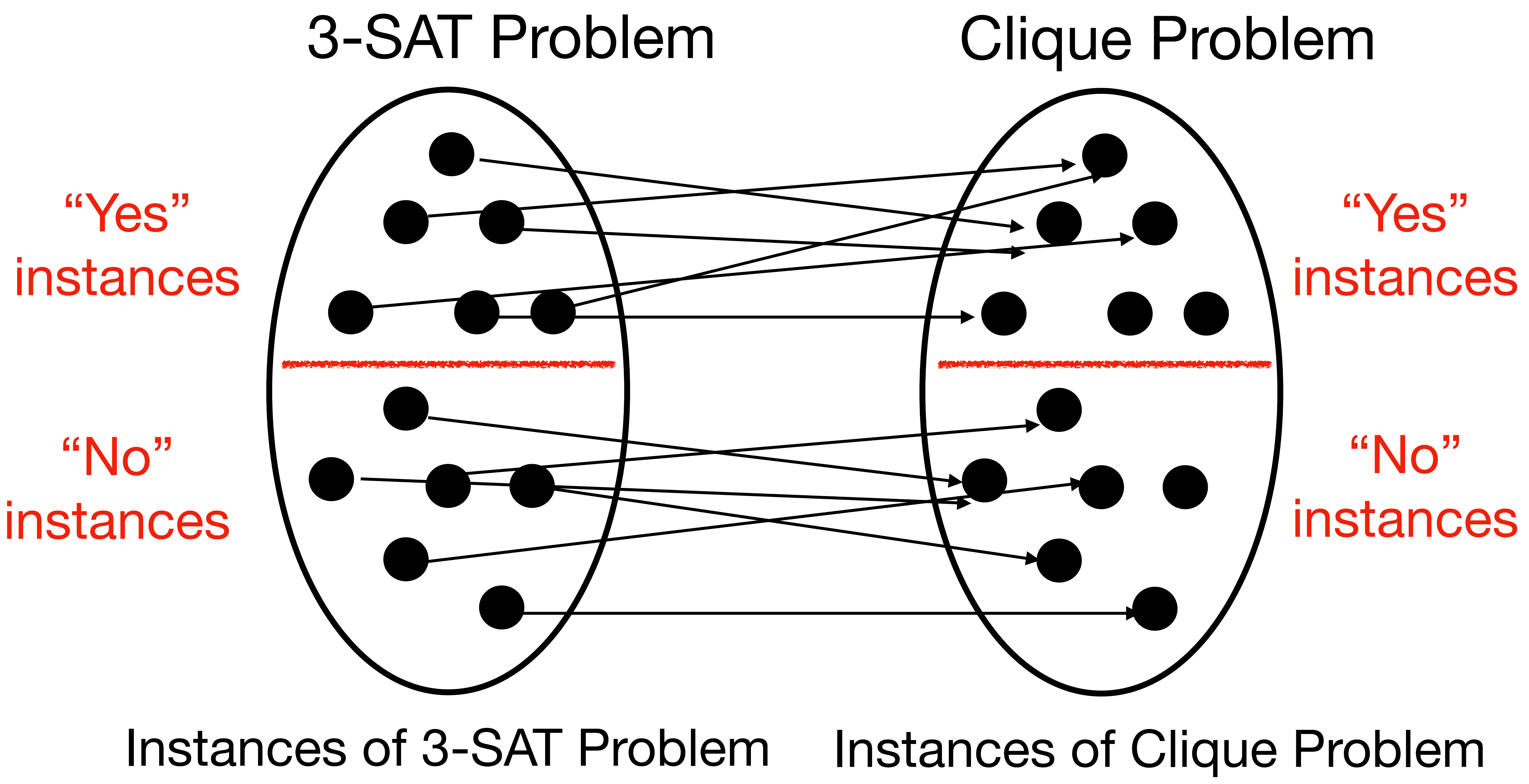
3-CNF SAT Problem:

Example of Instance: $(x_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$

3-CNF SAT Problem:

Example of Instance: $\underbrace{(x_1 \vee \bar{x}_2 \vee \bar{x}_3)}_{C_1} \wedge \underbrace{(\bar{x}_1 \vee x_2 \vee x_3)}_{C_2} \wedge \underbrace{(x_1 \vee x_2 \vee x_3)}_{C_3}$

Polynomial-time reduction:



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x_1

C_1 $\bar{x}_2$

$\bar{x}_3$

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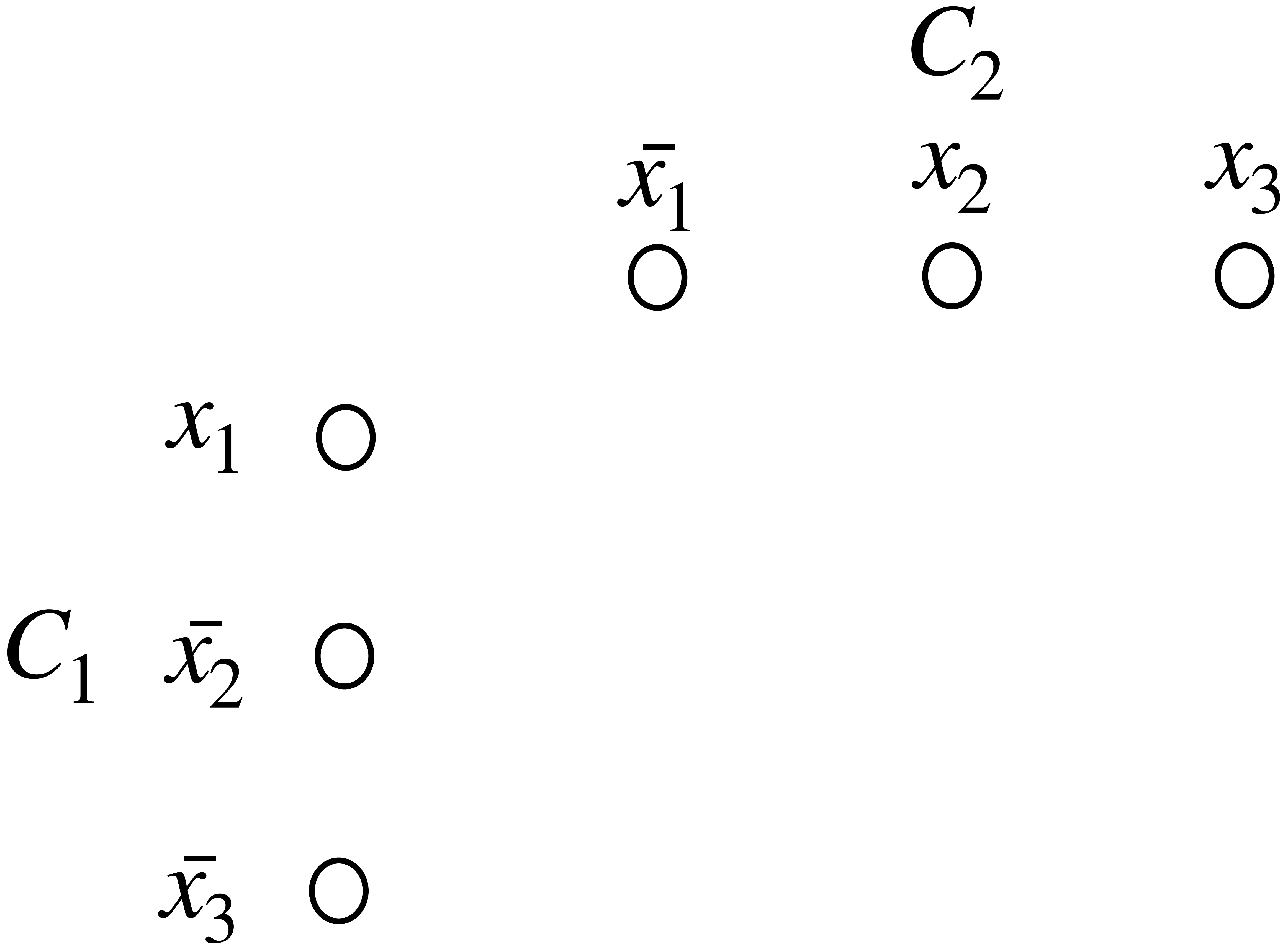

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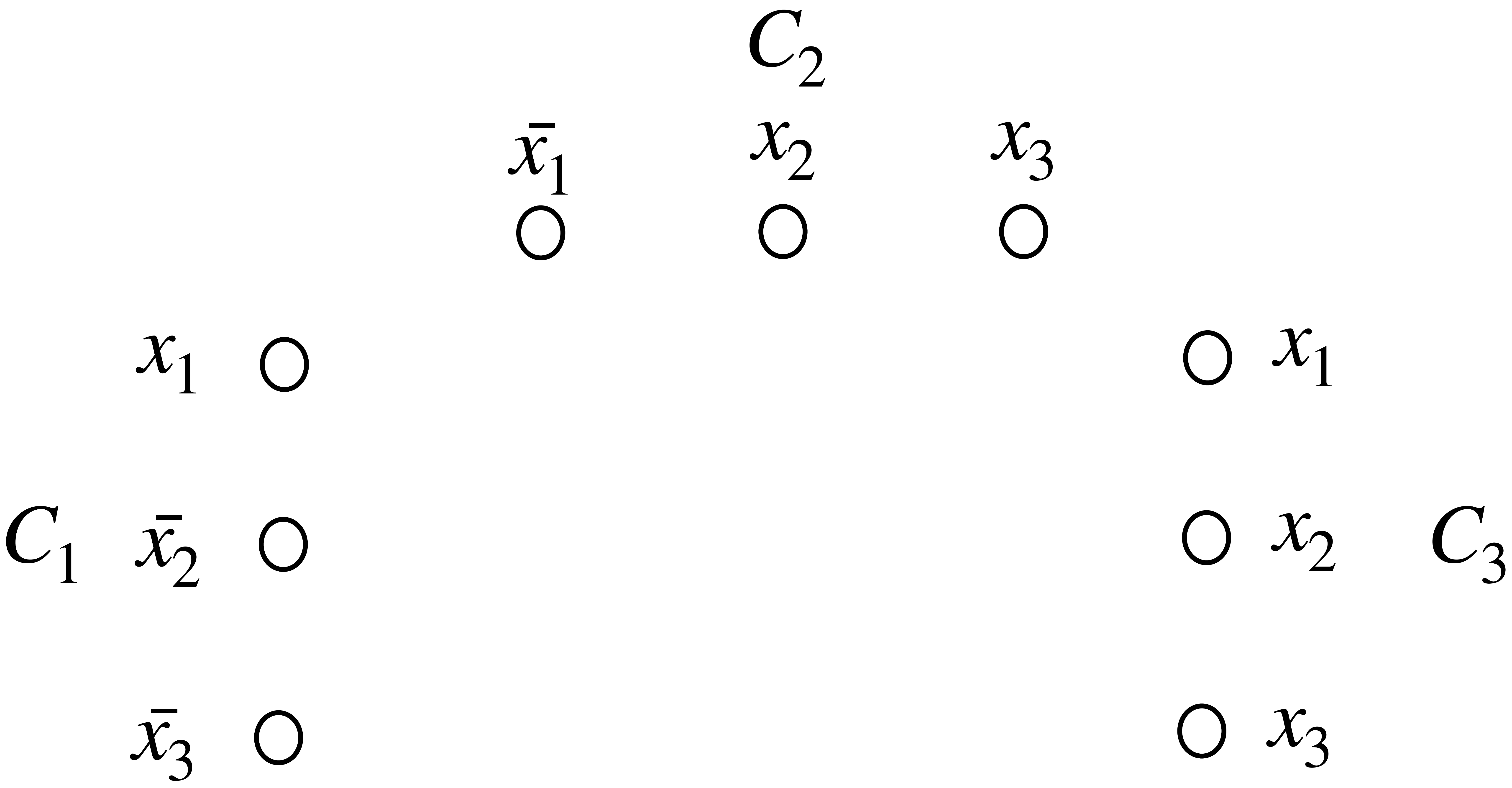

In general, k clauses will lead to 3k nodes.

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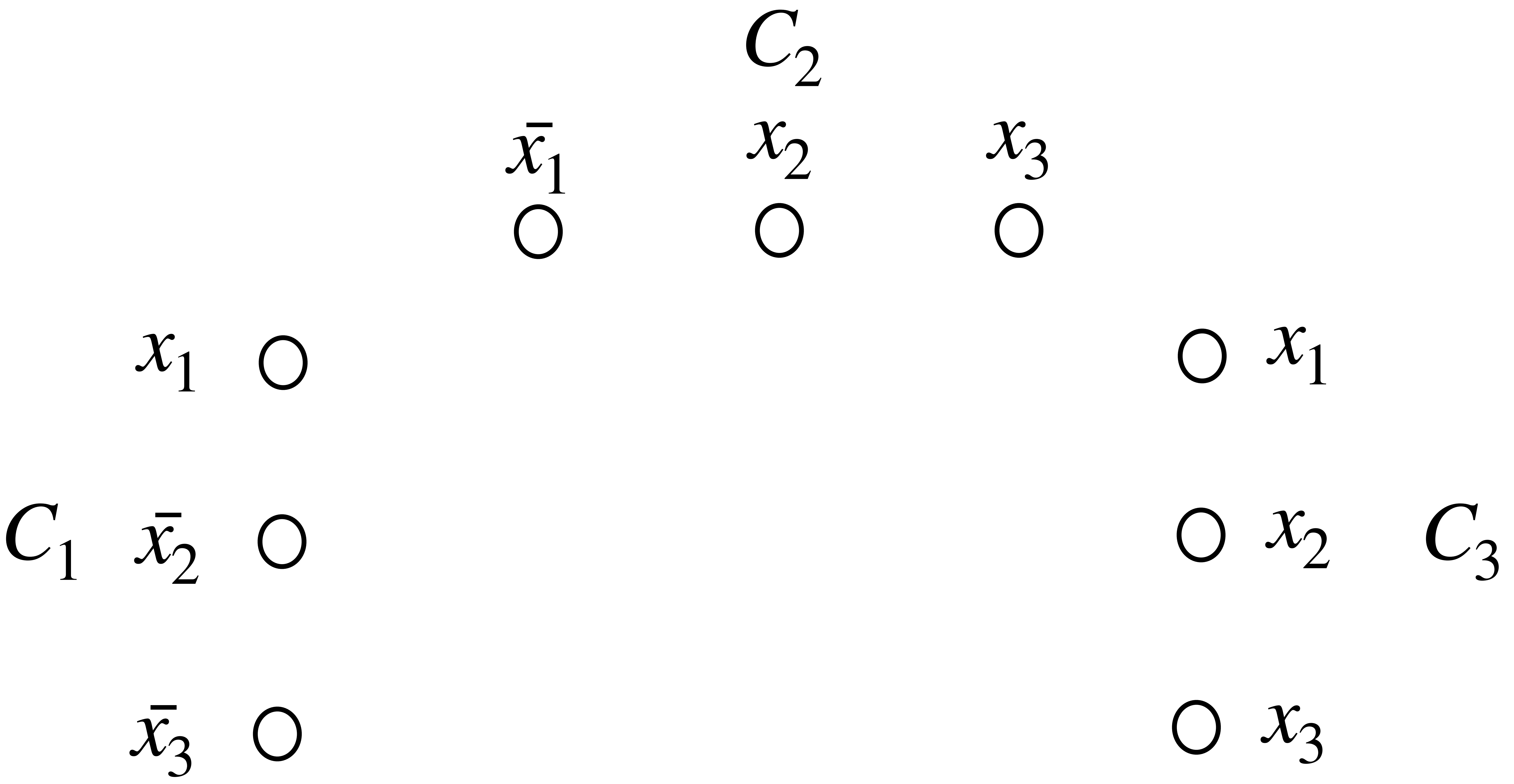

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Two nodes u and v have an edge if:

1) u and v are in two different clauses.
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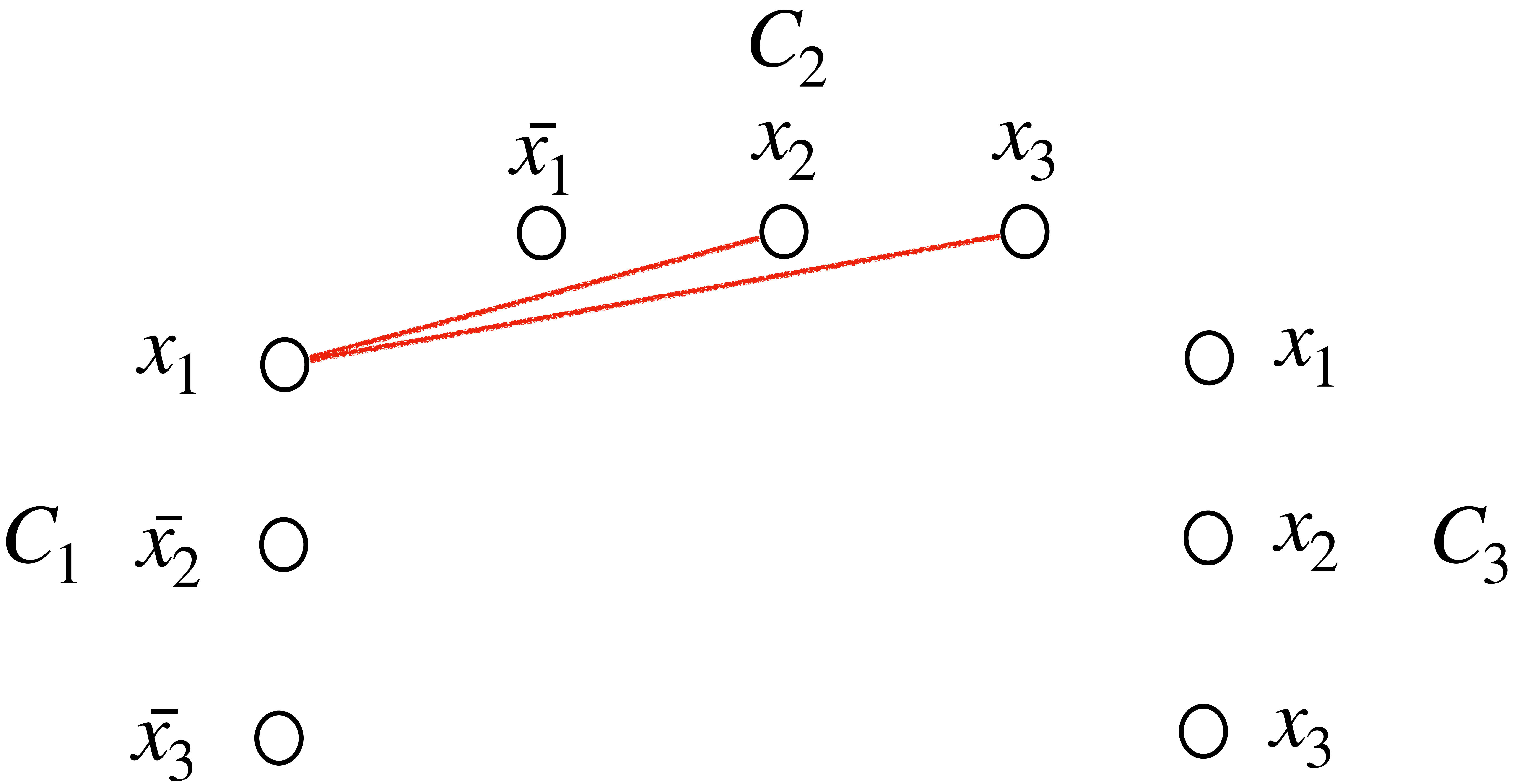

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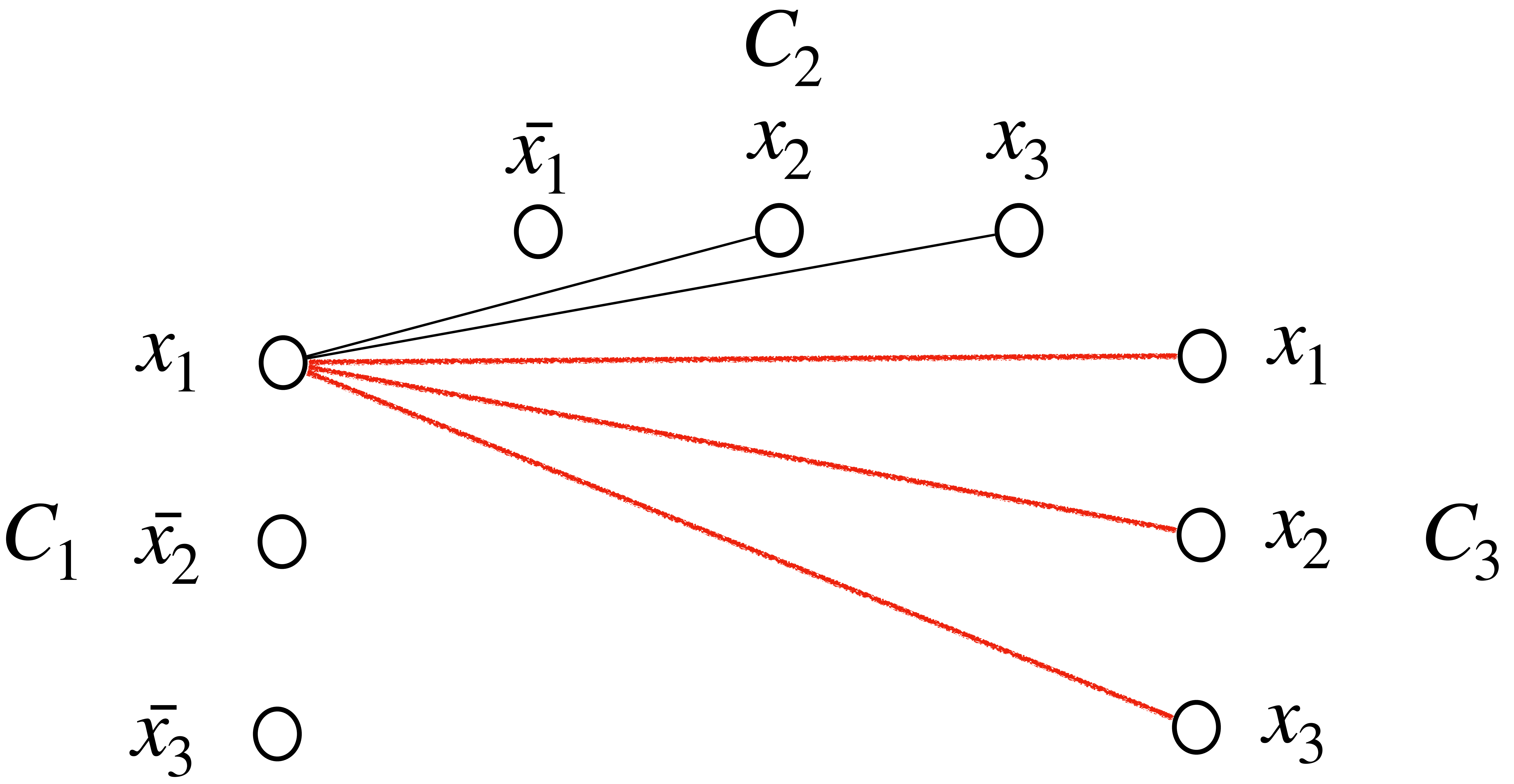

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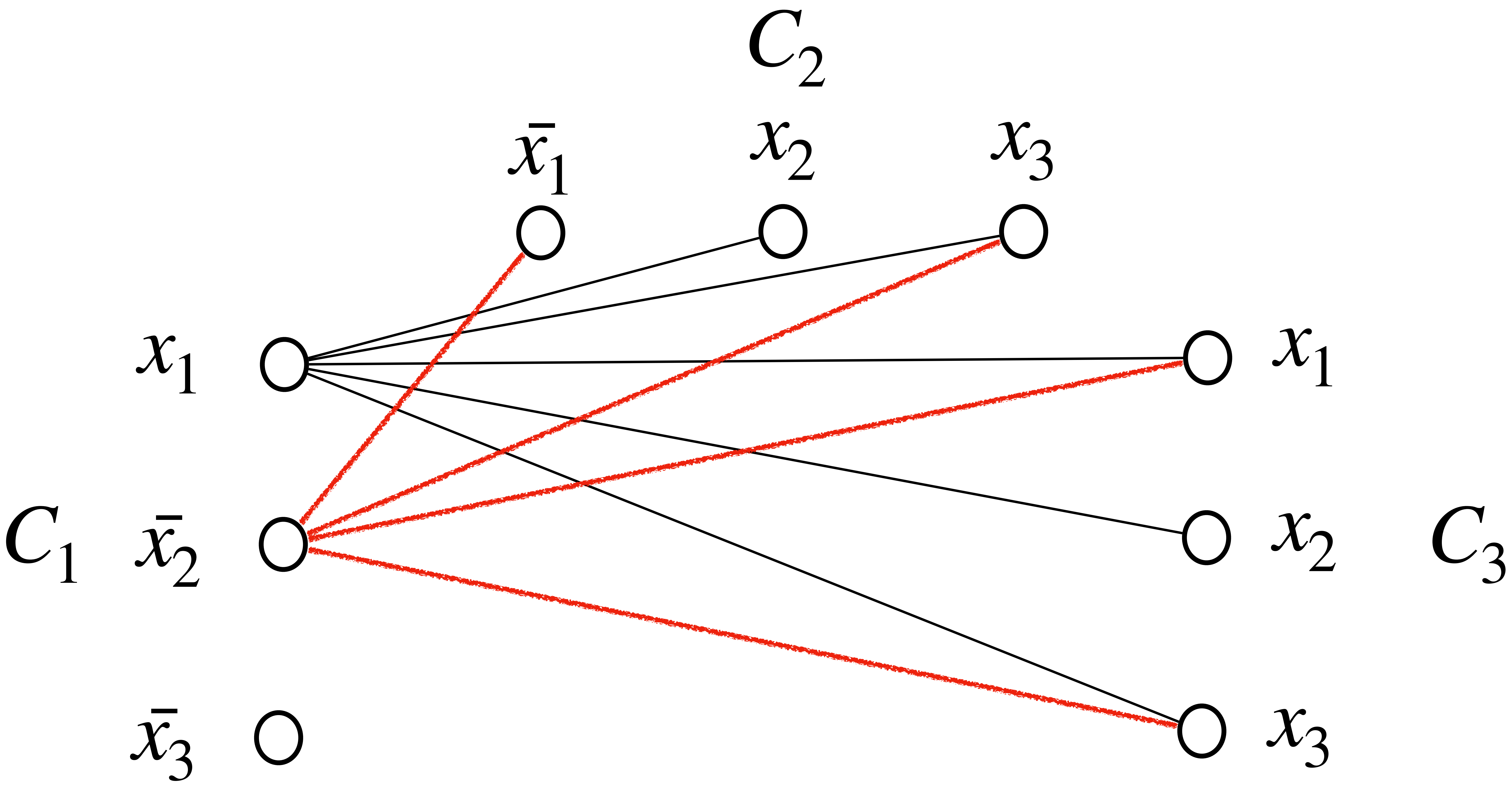

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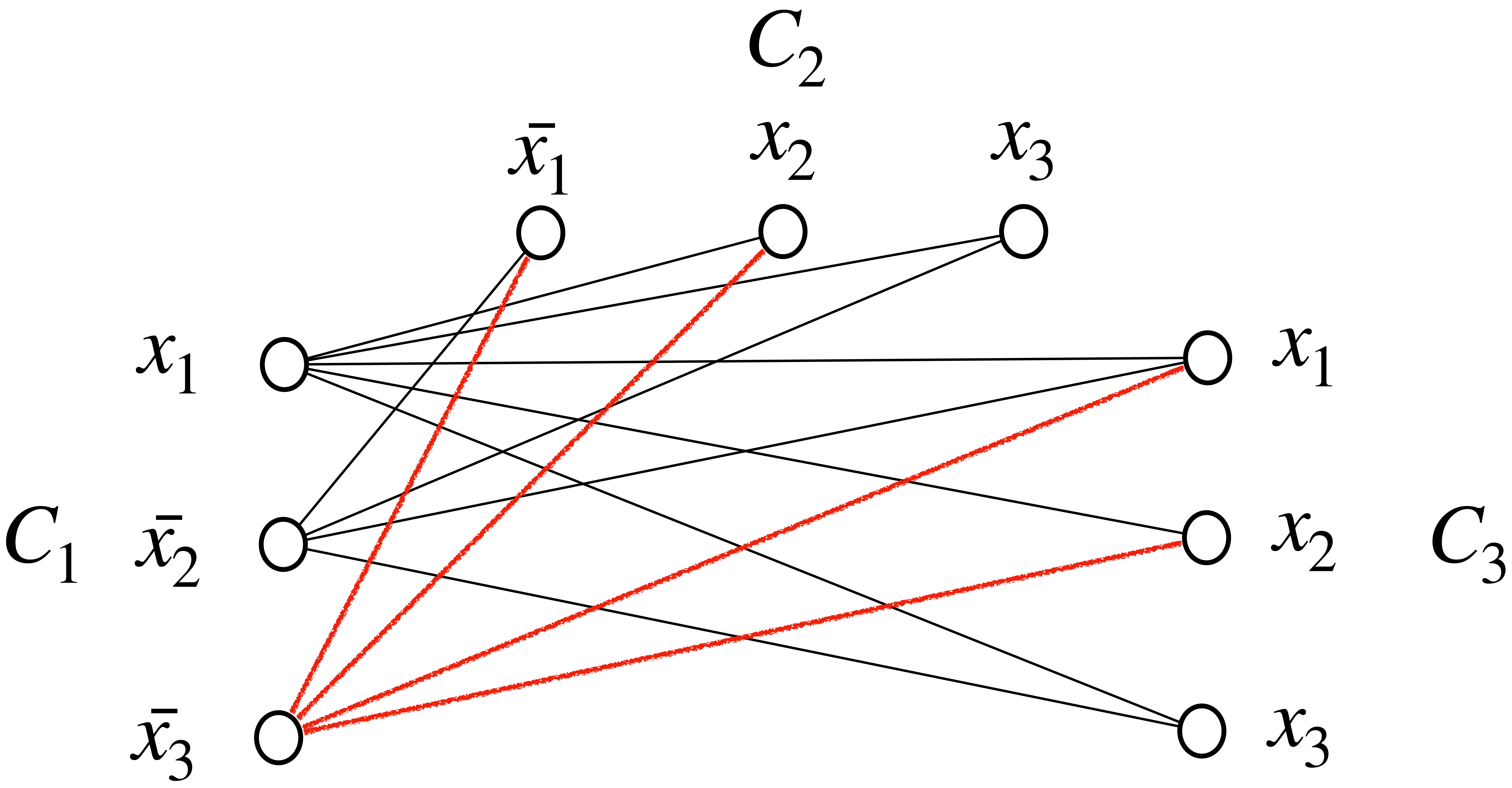

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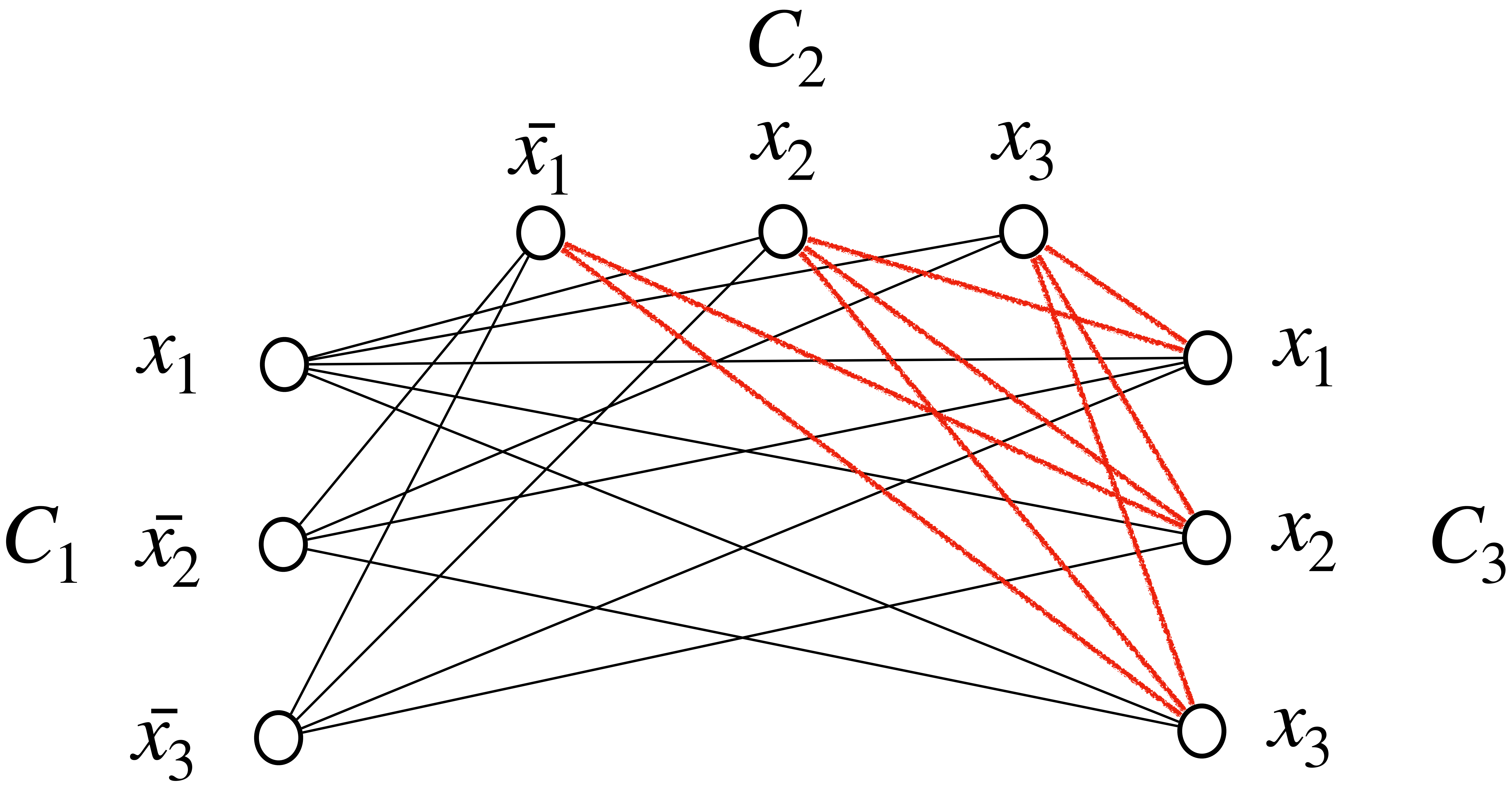

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Question: Can this formula (of k clauses) be satisfied?

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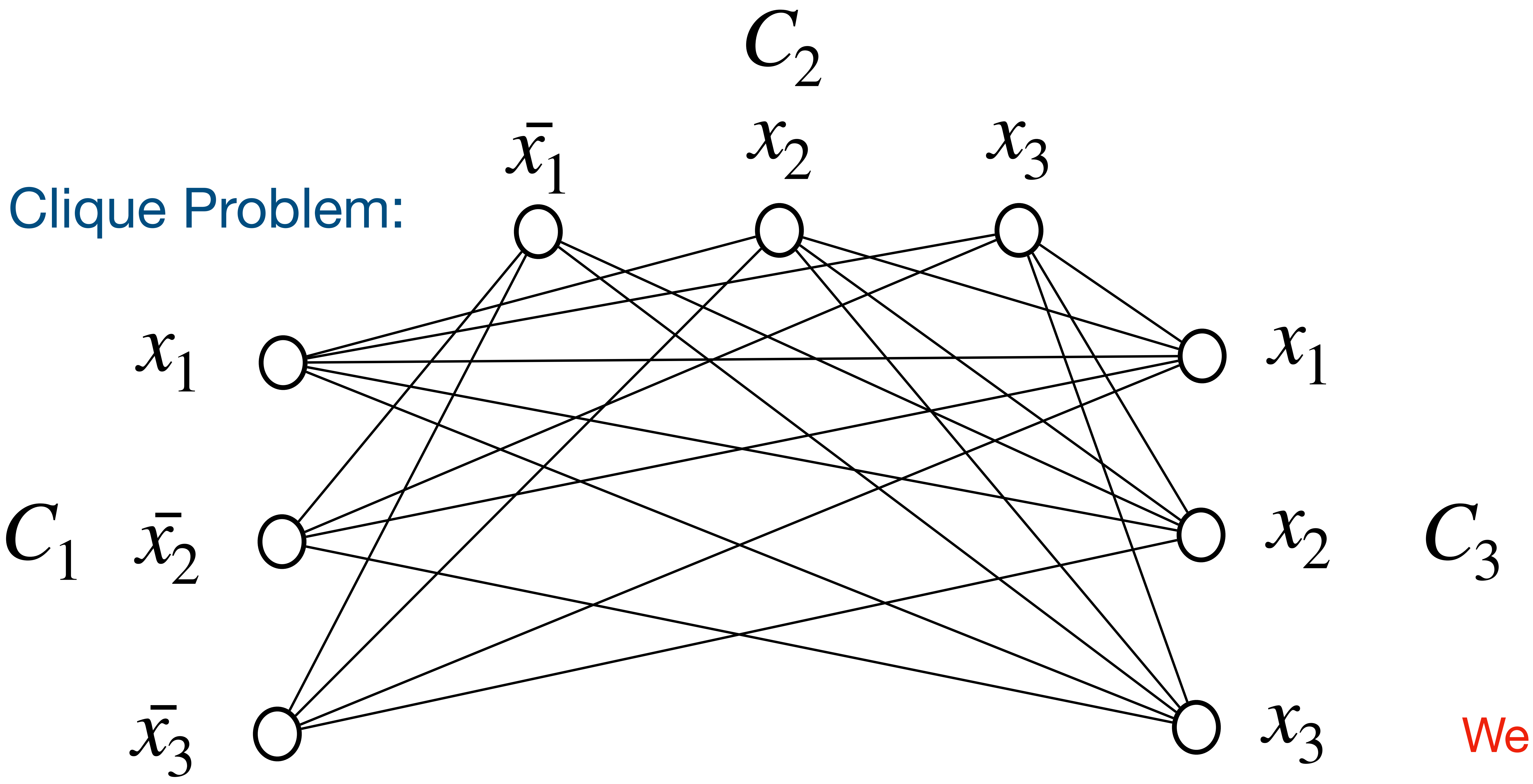

Question: Does this graph have a clique of size k?

In general, k clauses will lead to 3k nodes.

Two nodes u and v have an edge if:

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We get a polynomial-time mapping.

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We now prove the mapping preserves “YES/NO” answers.

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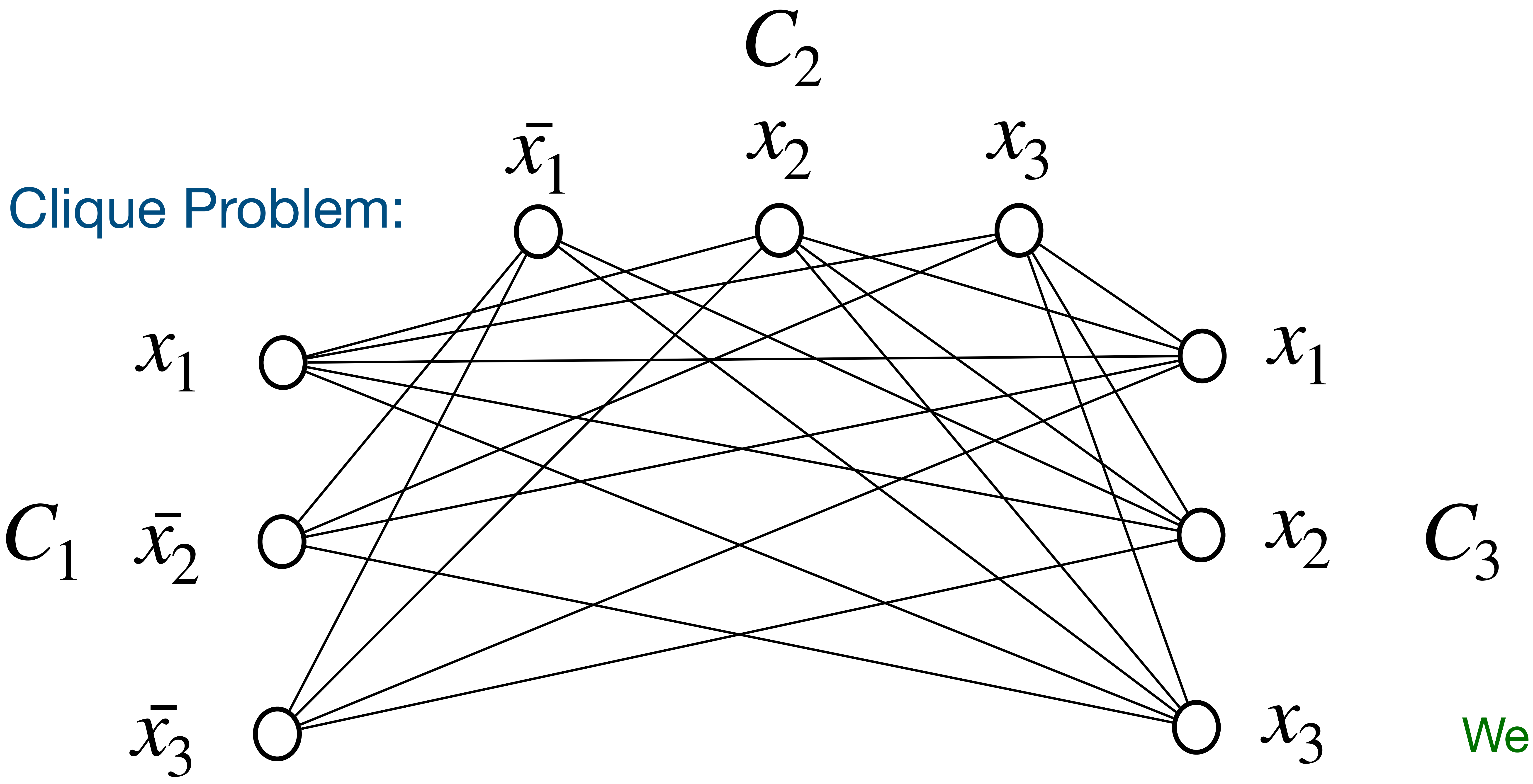

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We now prove: "YES for 3-SAT" implies "YES for Clique Problem".

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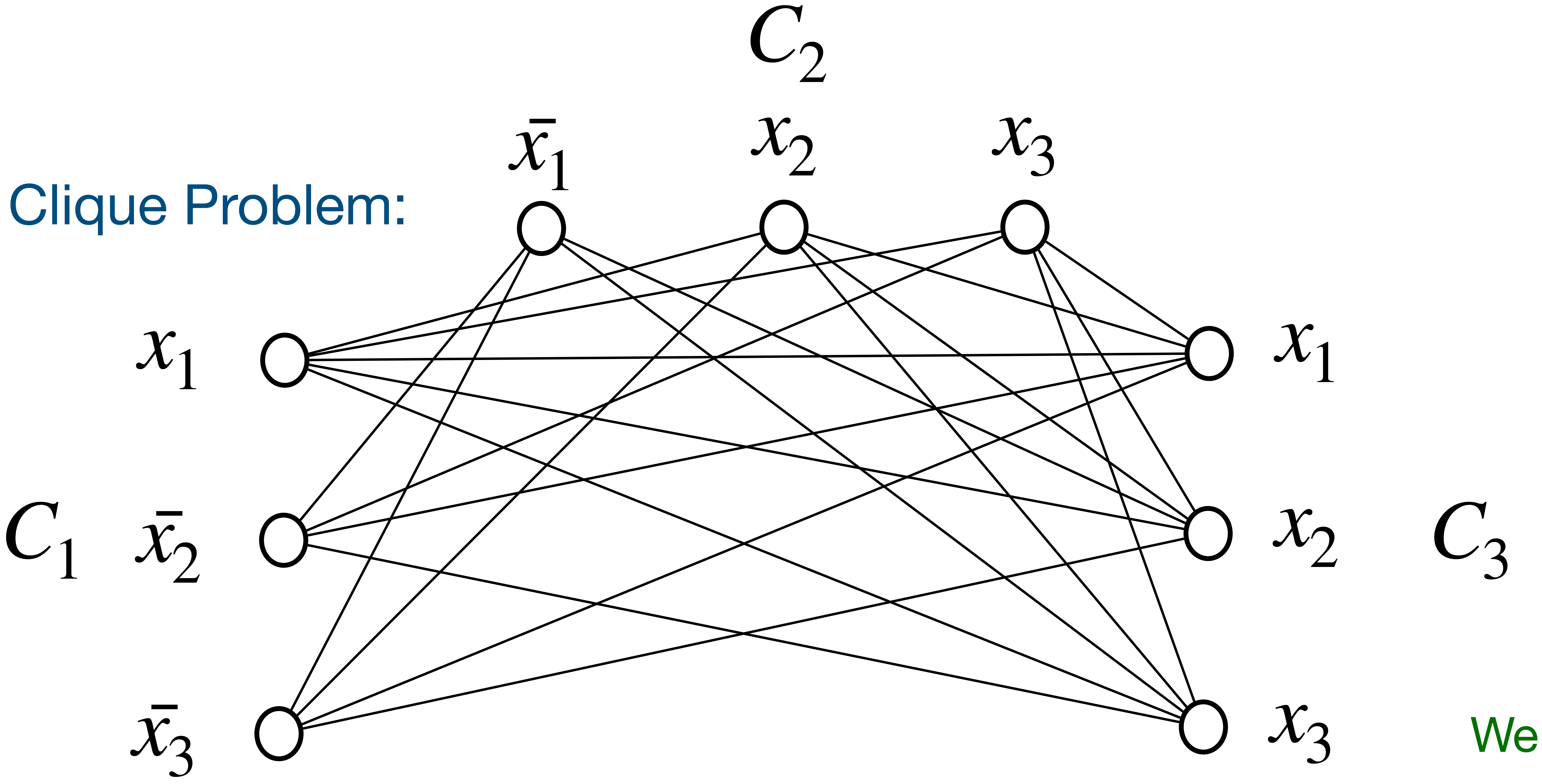

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Question: Can this formula (of k clauses) be satisfied?

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A solution to 3-SAT:

$x_1 = 1,\ x_2 = 1,\ x_3 = 1$

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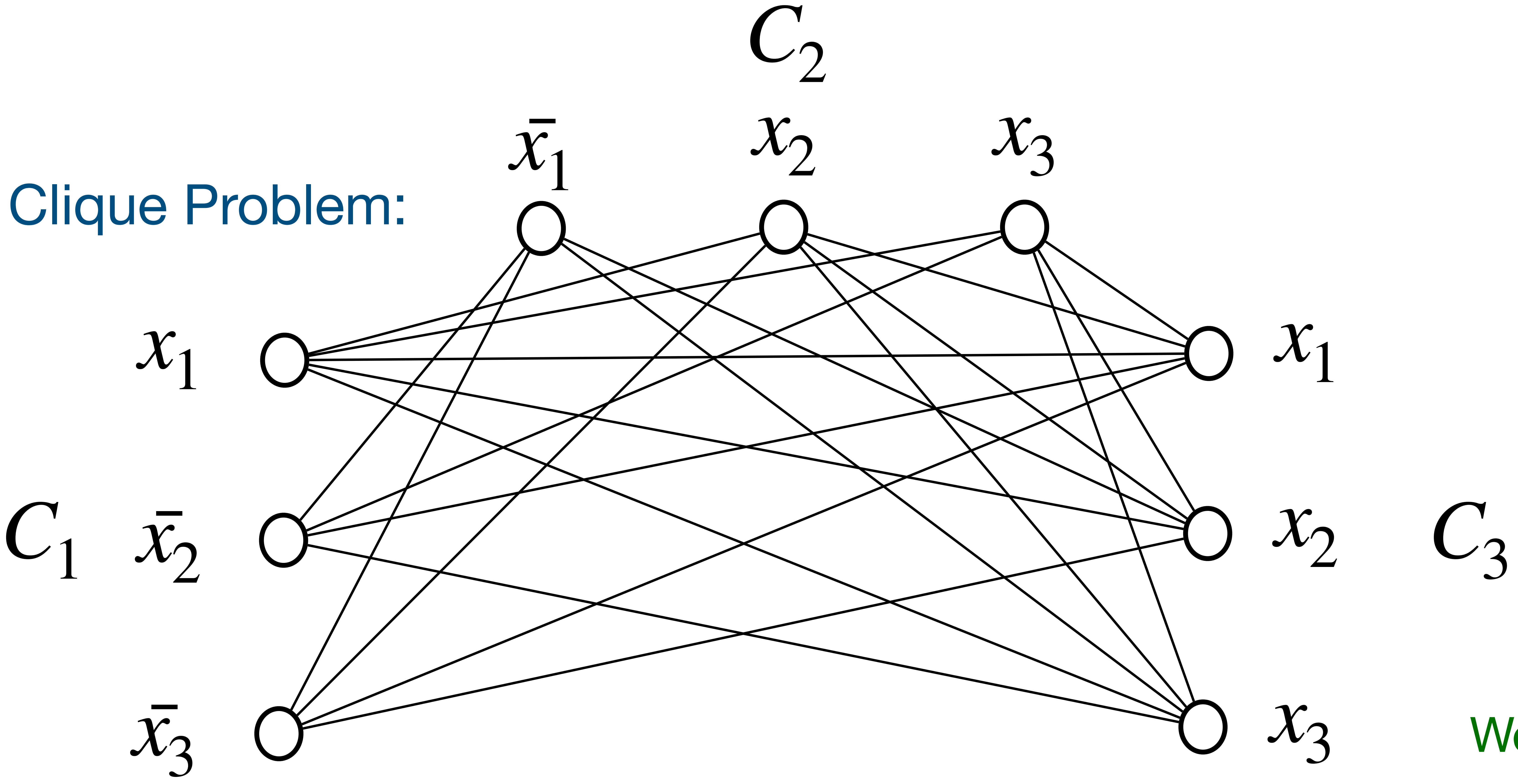

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: 1

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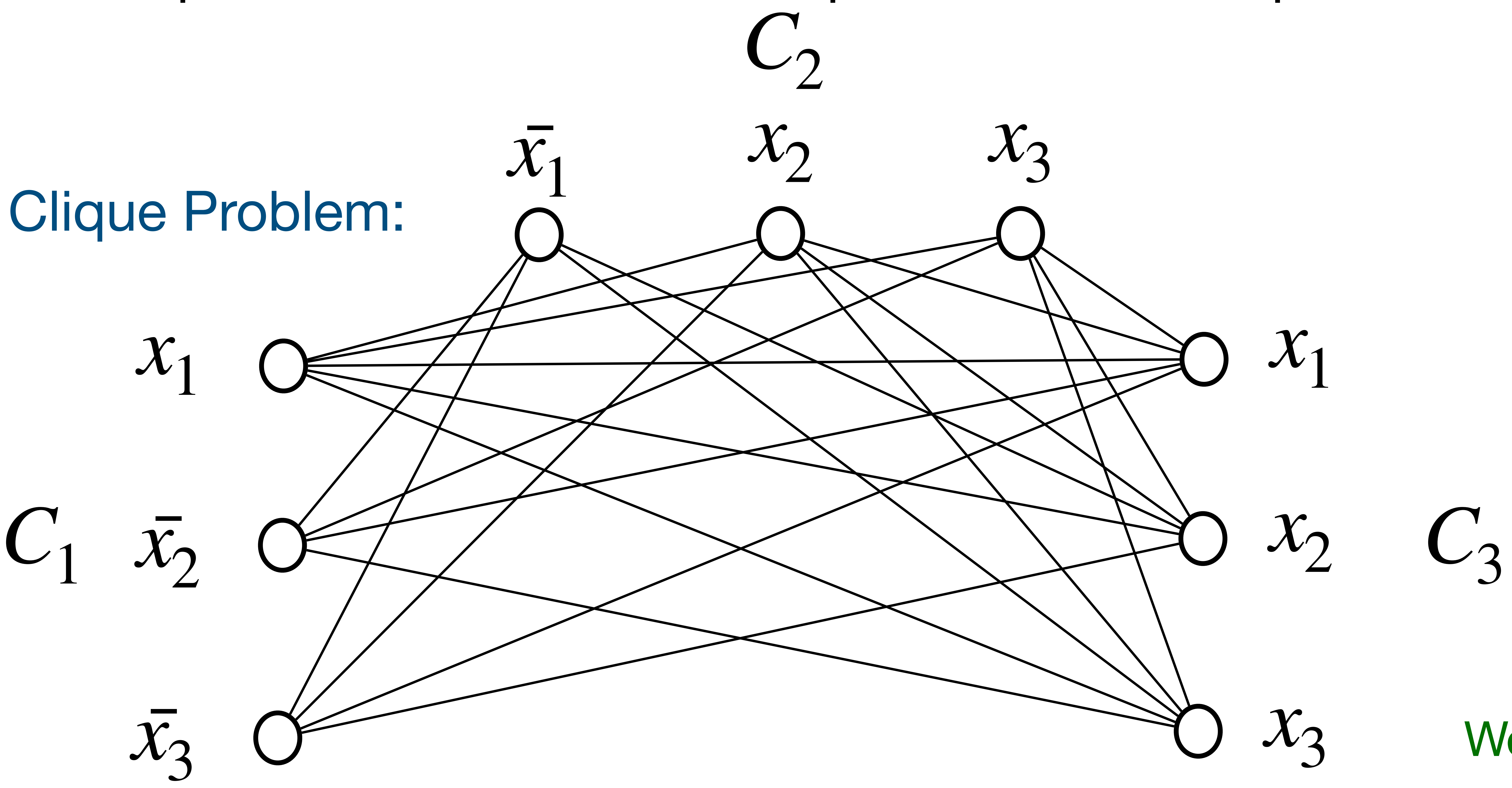

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Pick one satisfied literal from each clause

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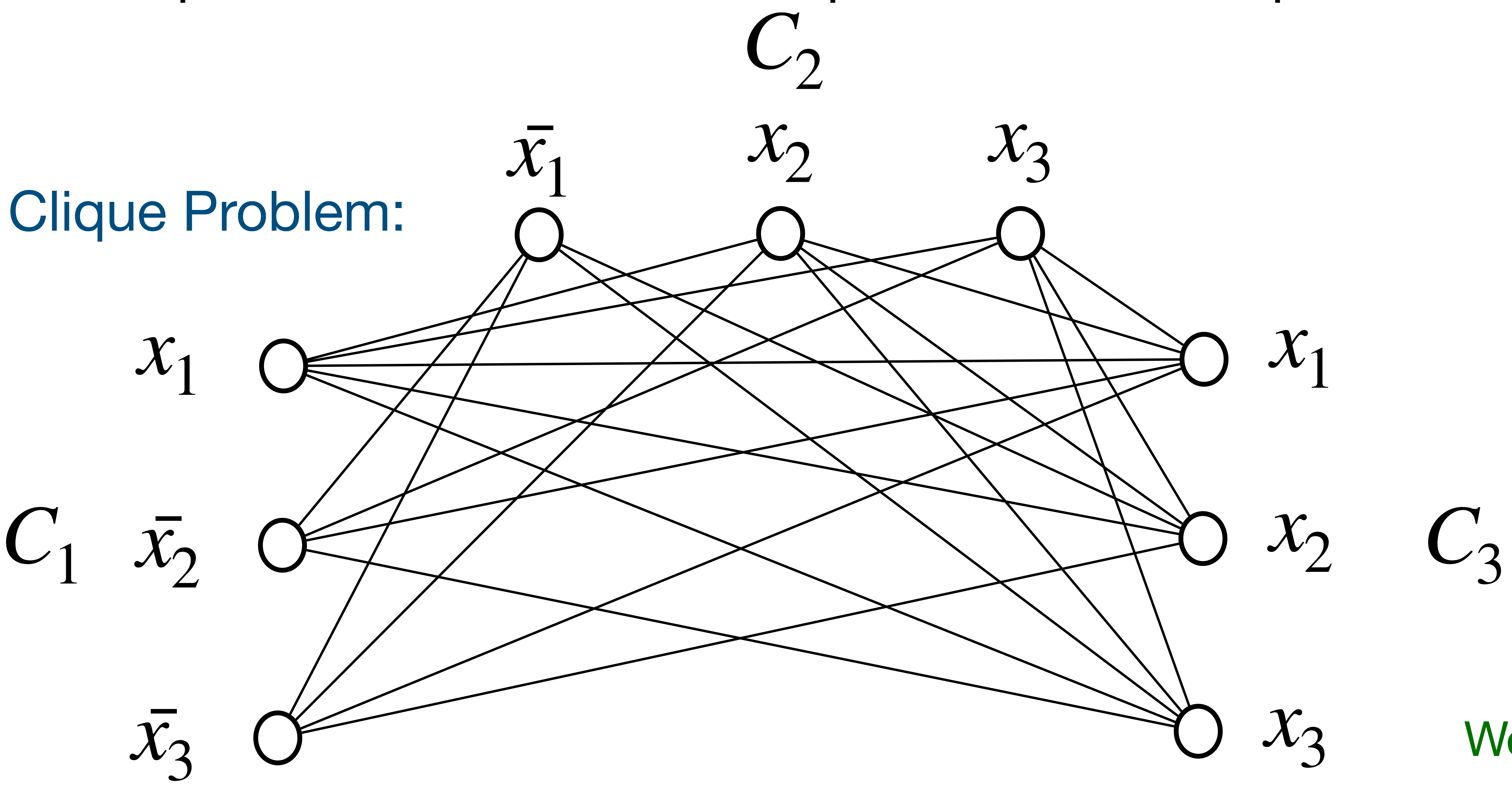

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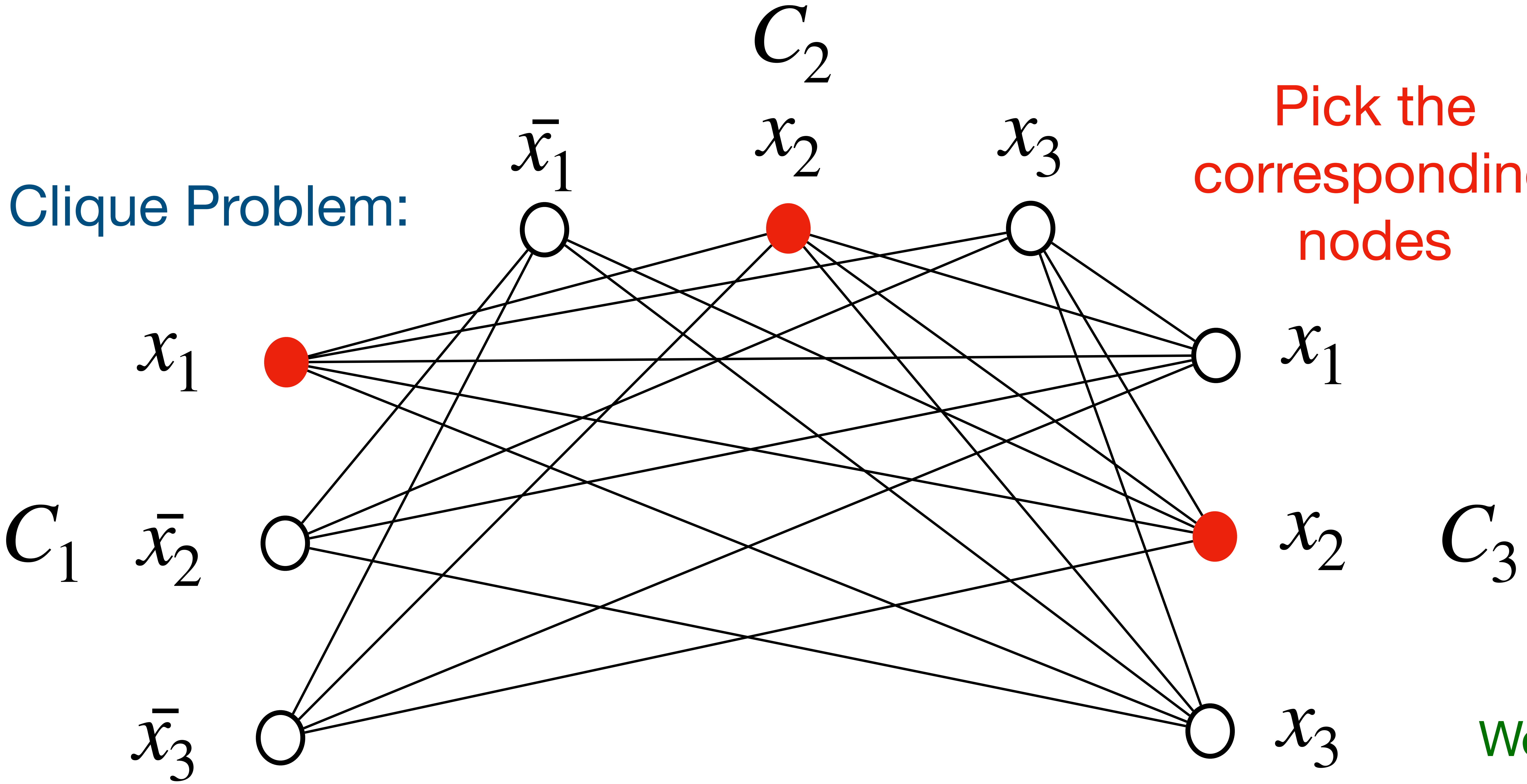

Pick the corresponding nodes

Question: Does this graph have a clique of size k?

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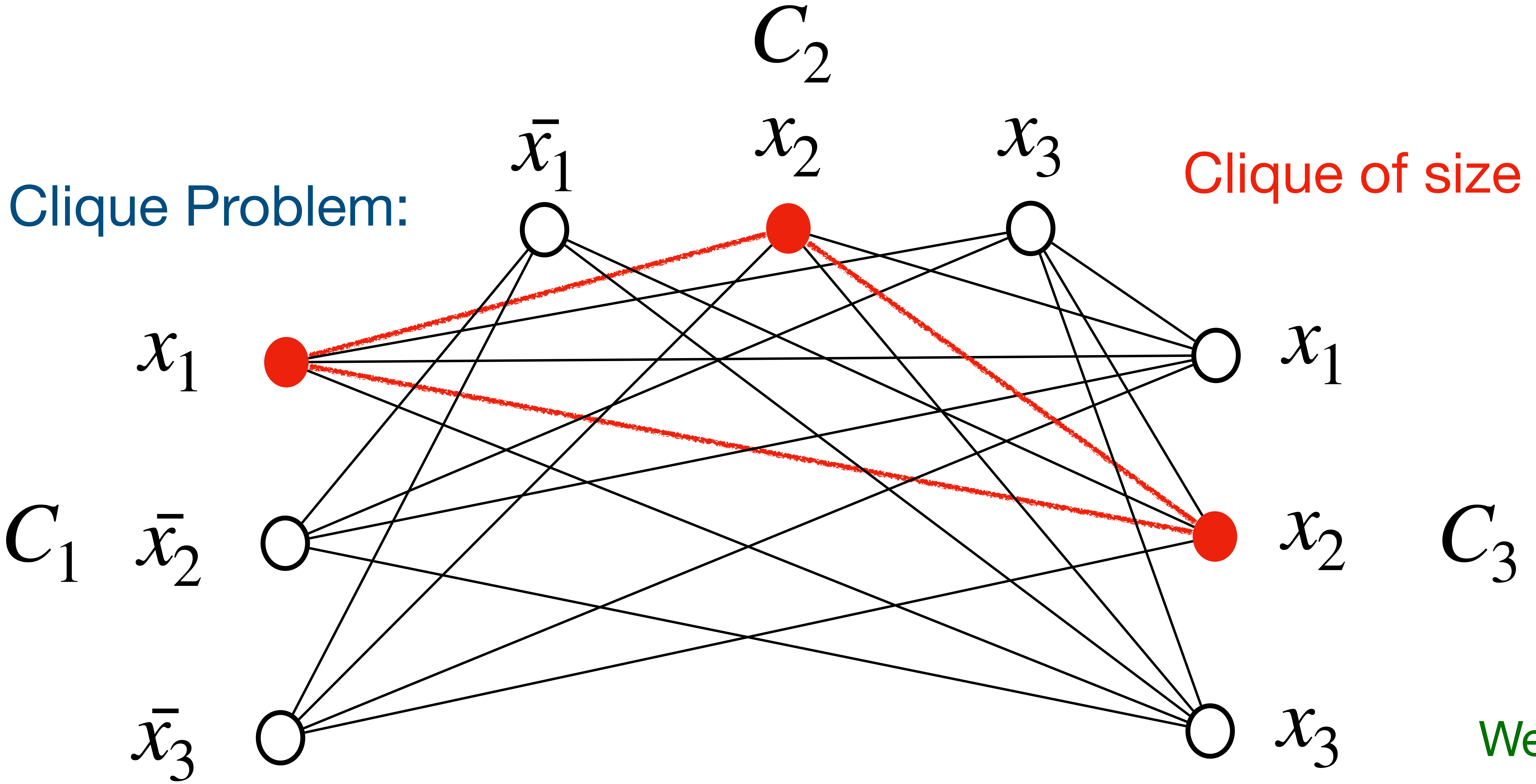

Clique of size k

Question: Does this graph have a clique of size k?

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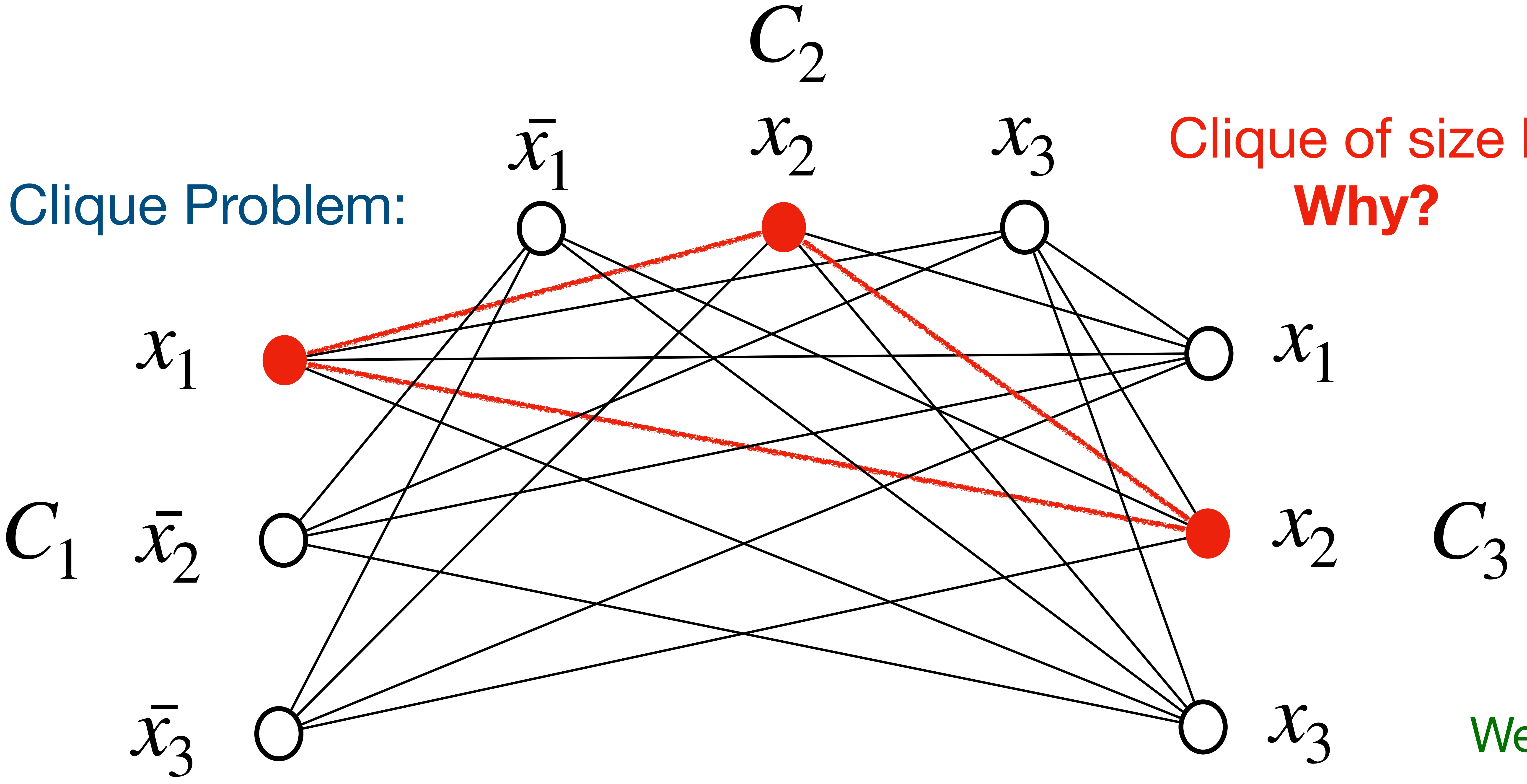

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Clique of size k.

Why?

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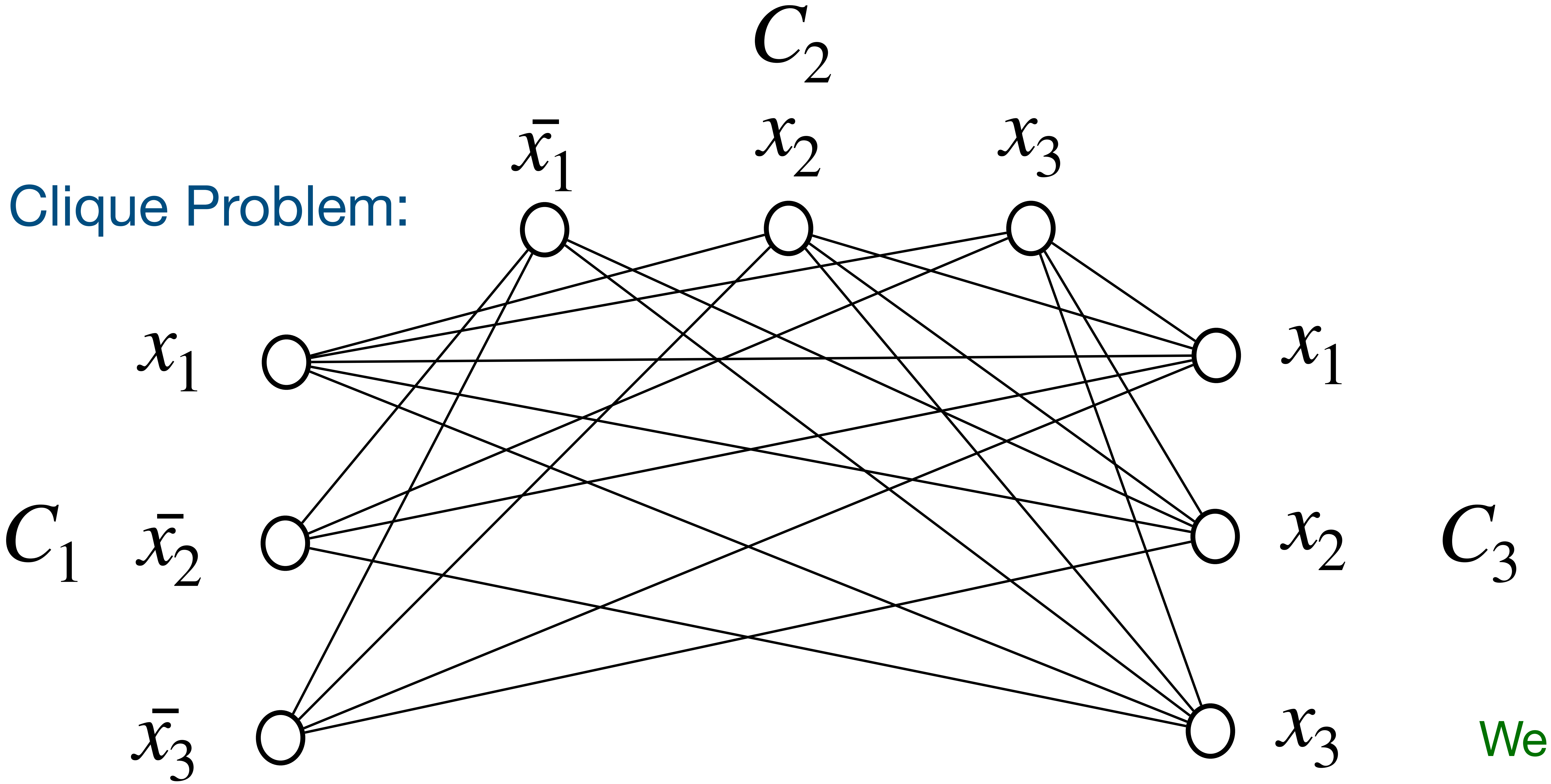
We now prove: “YES for Clique Problem” implies “YES for 3-SAT”.

Clique Problem:



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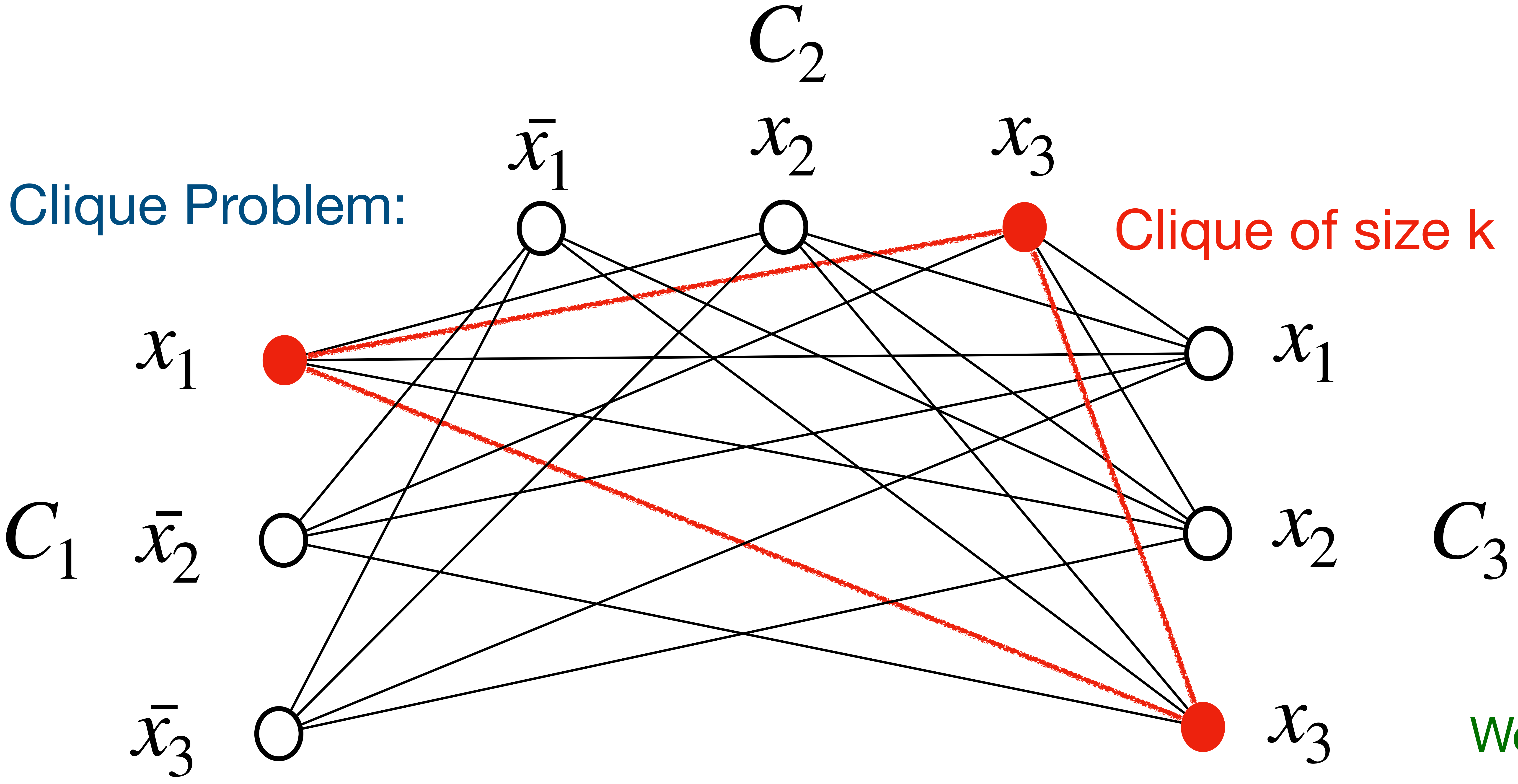

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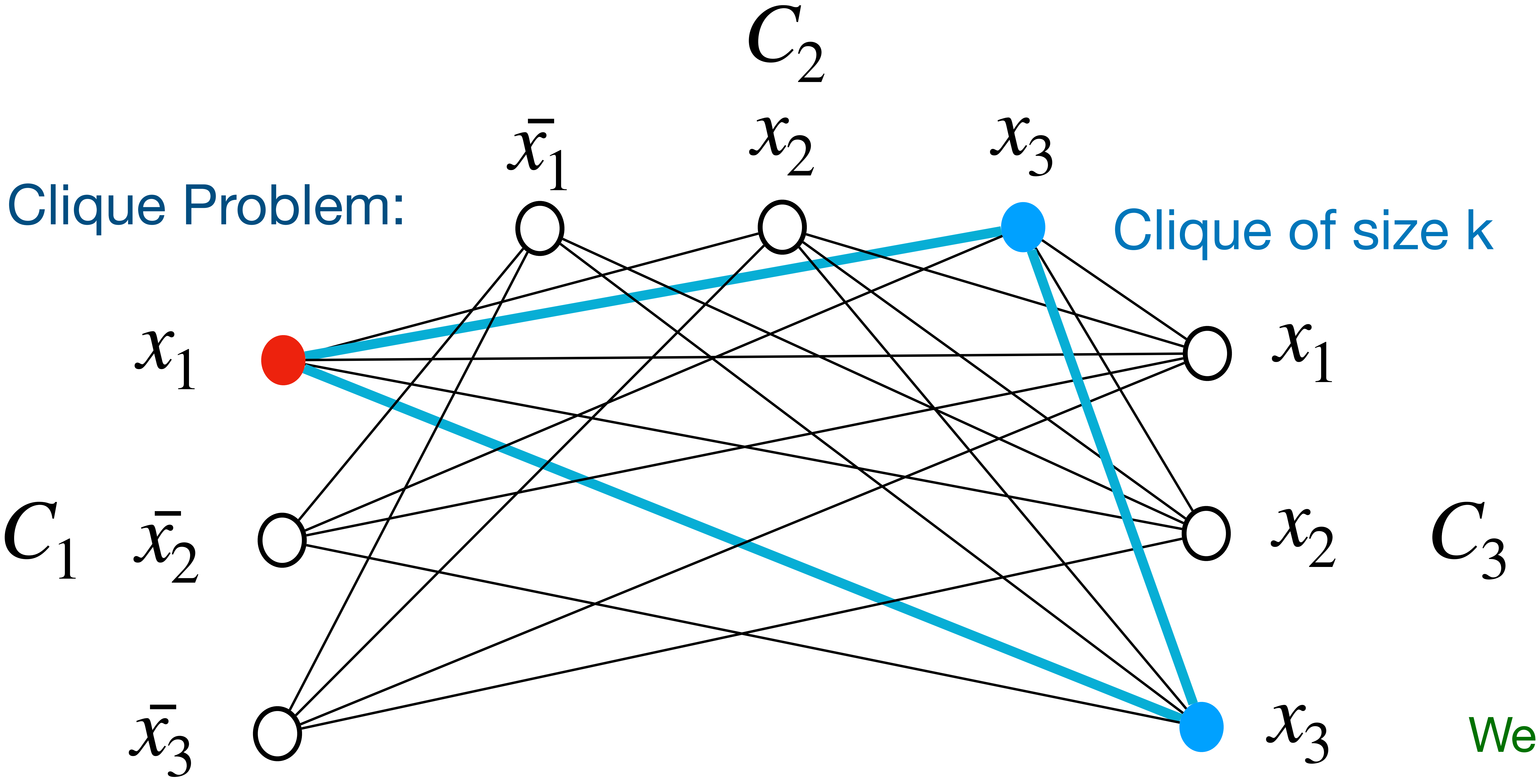

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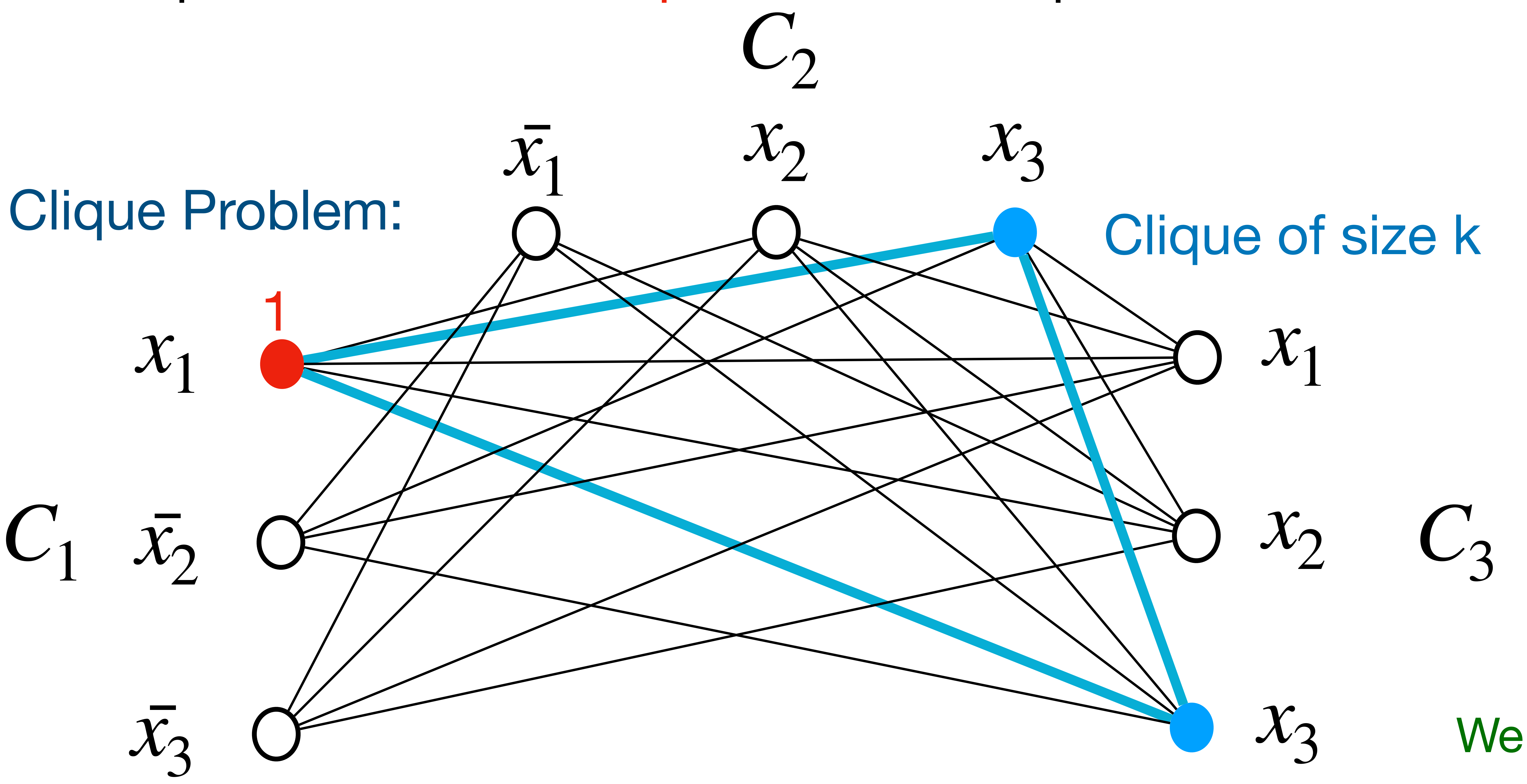

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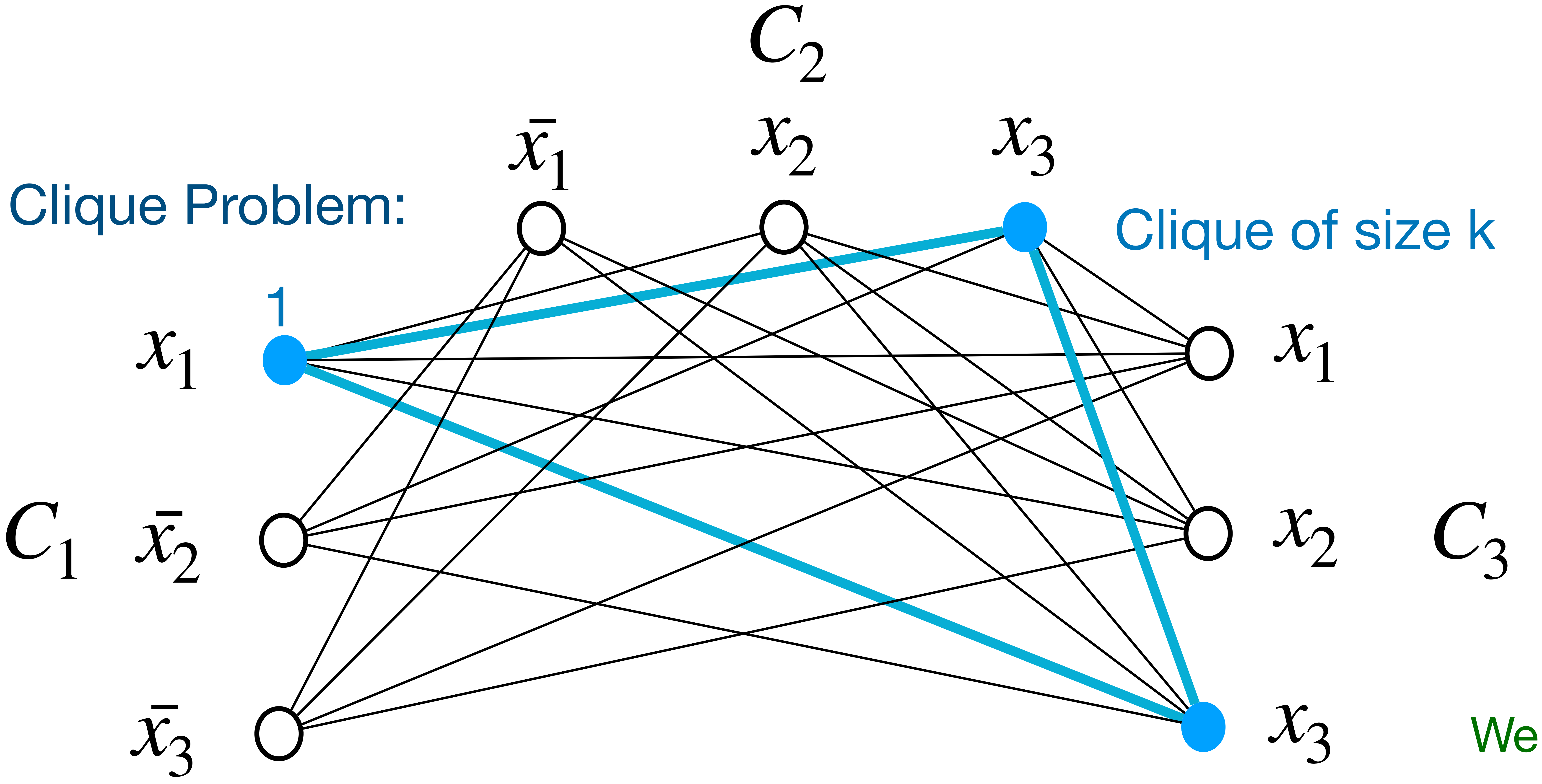

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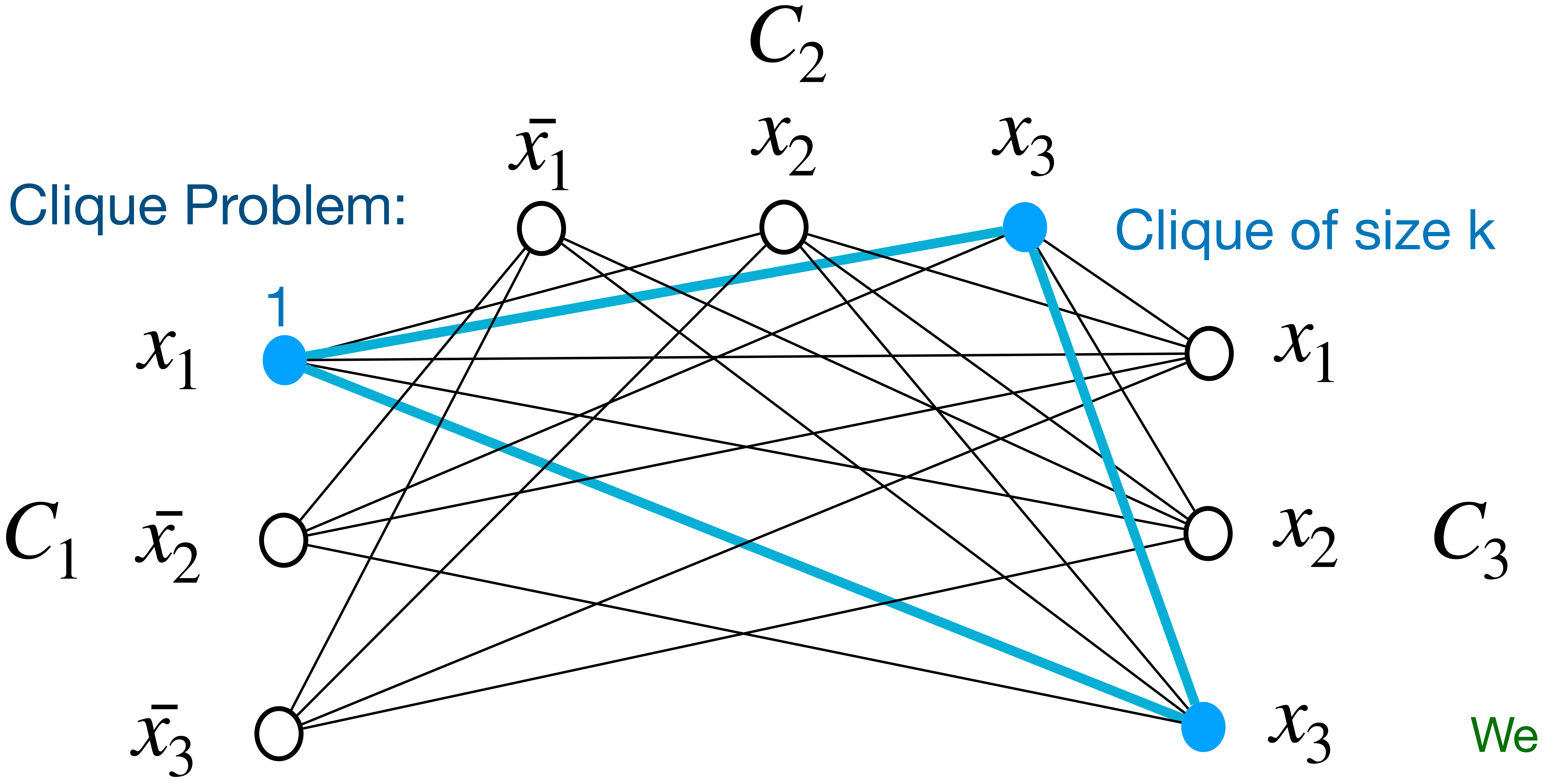

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$x_1 = 1$
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$x_3 = ?$

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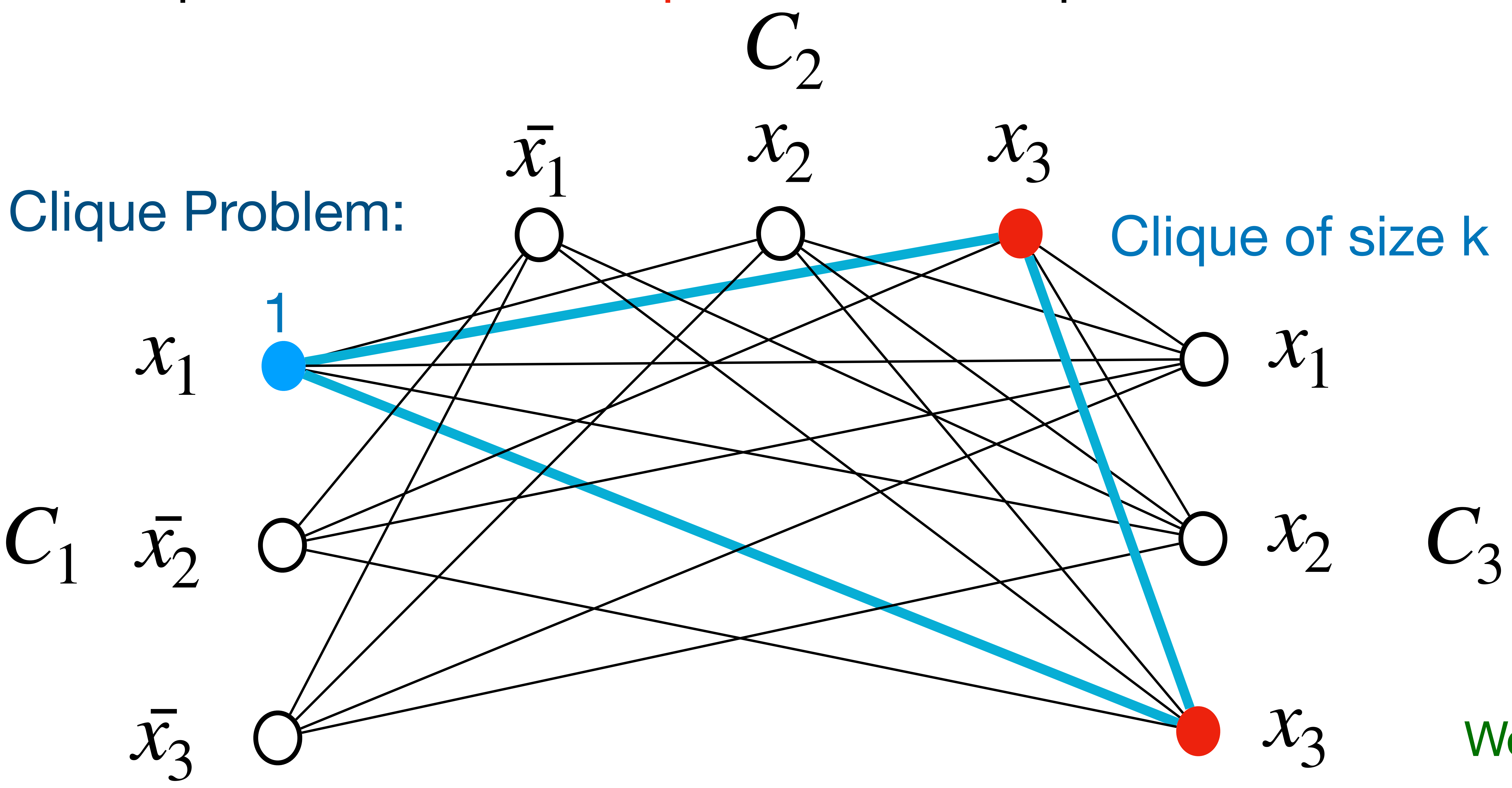

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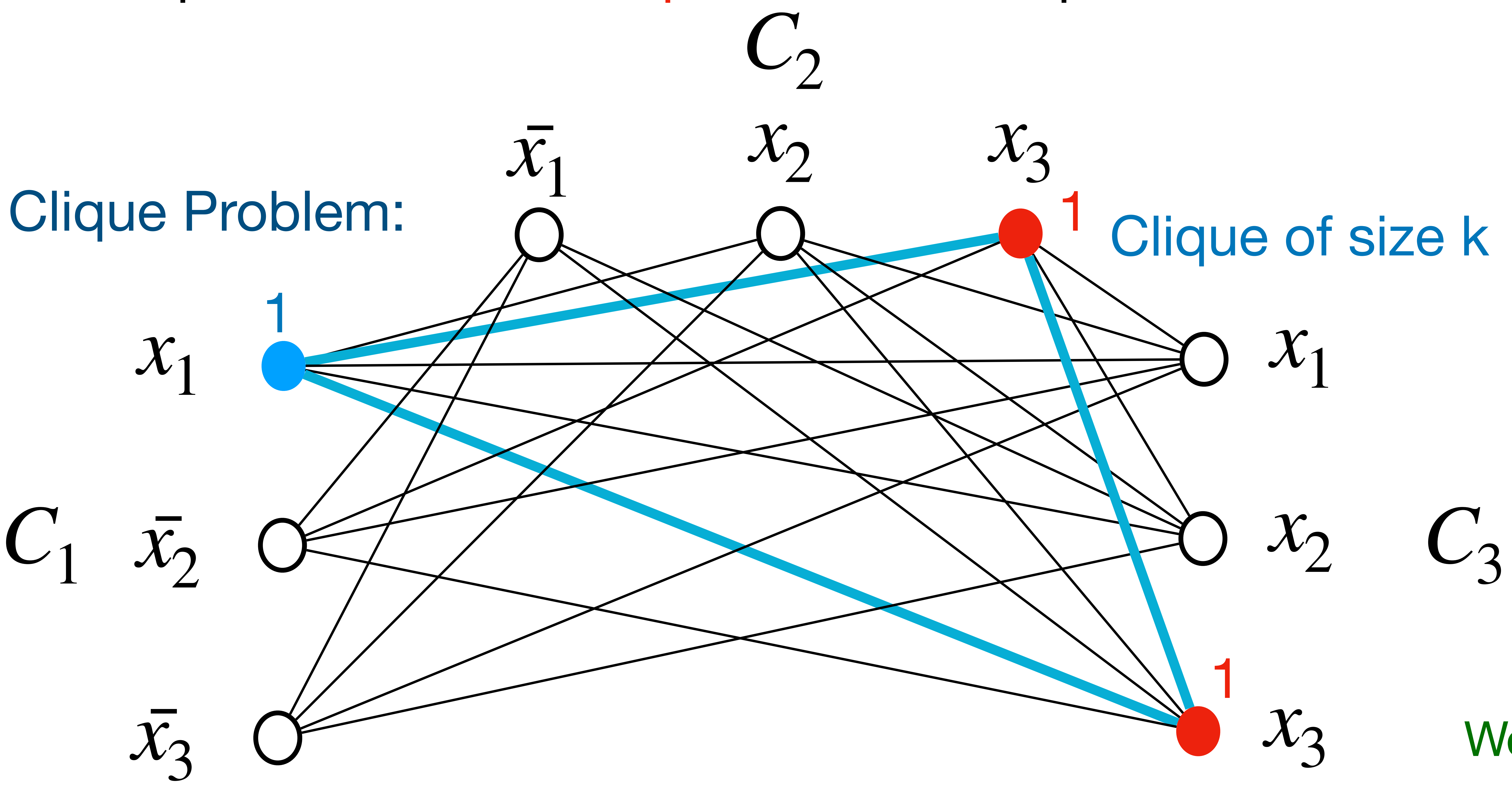

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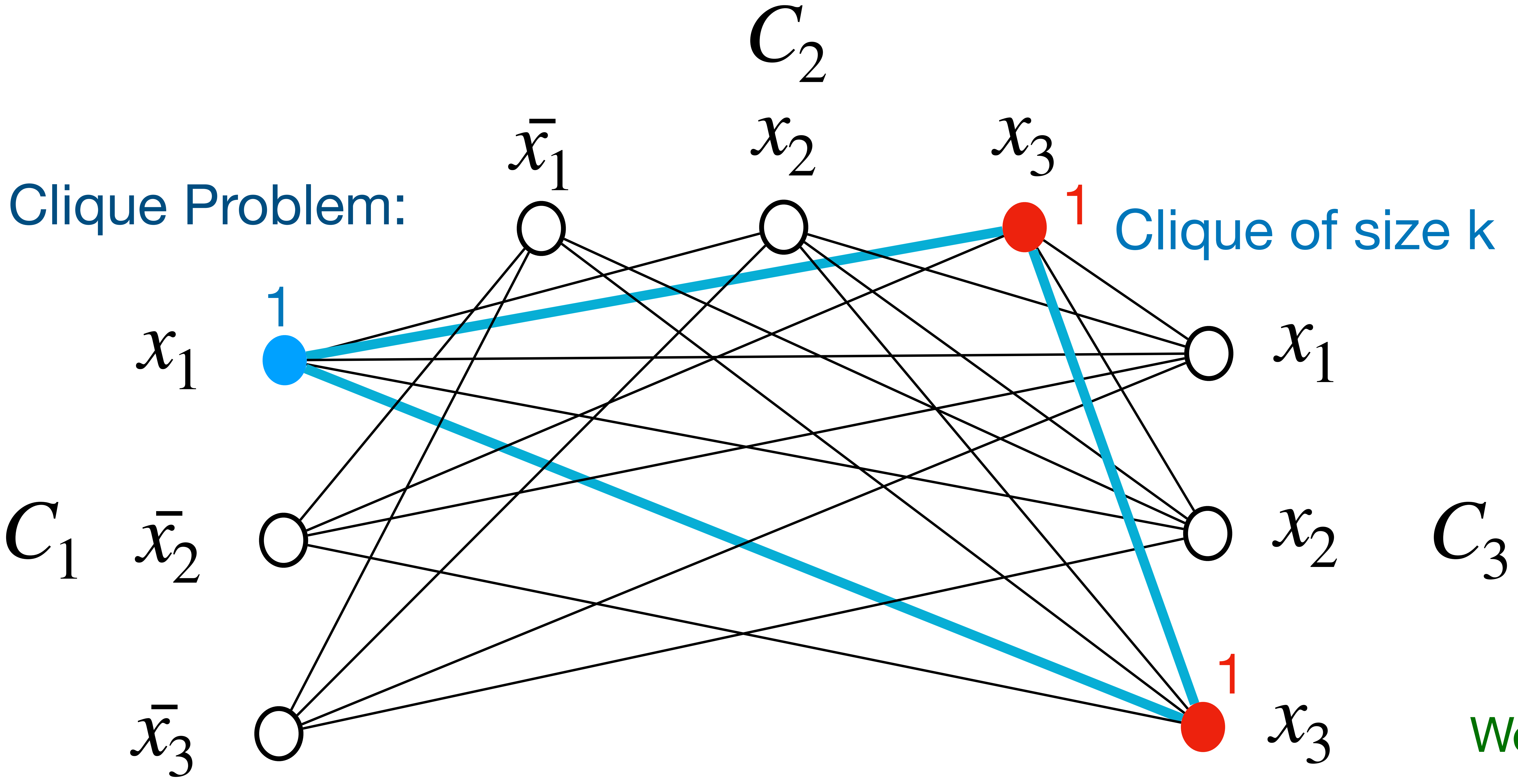
3-SAT formula is satisfied!

We now prove: "YES for Clique Problem" implies "YES for 3-SAT".

Clique Problem:



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How about we try again?

3-CNF SAT Problem:

Question: Can this formula (of k clauses) be satisfied?

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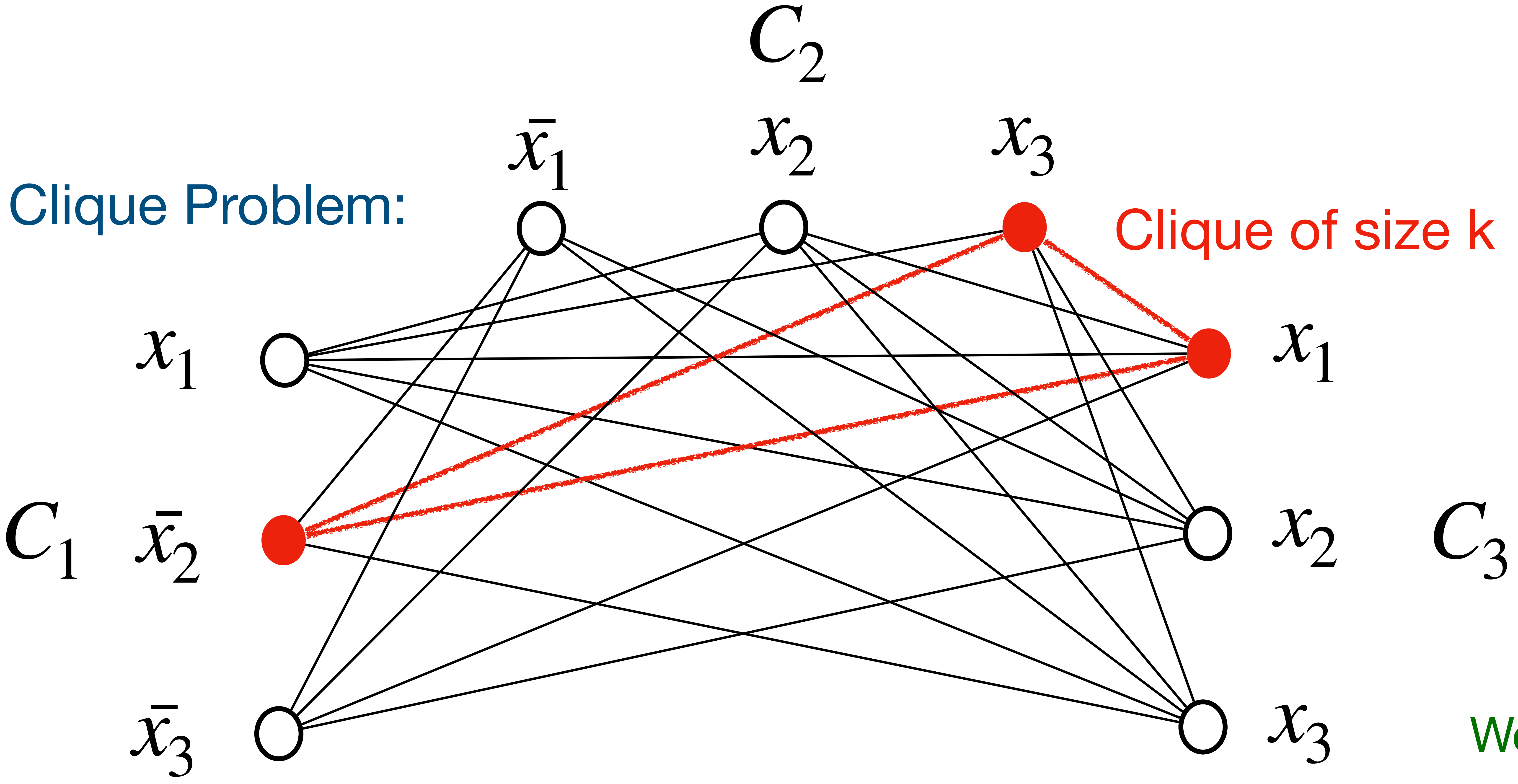

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$\boxed{x_1 \;=\; ?}$

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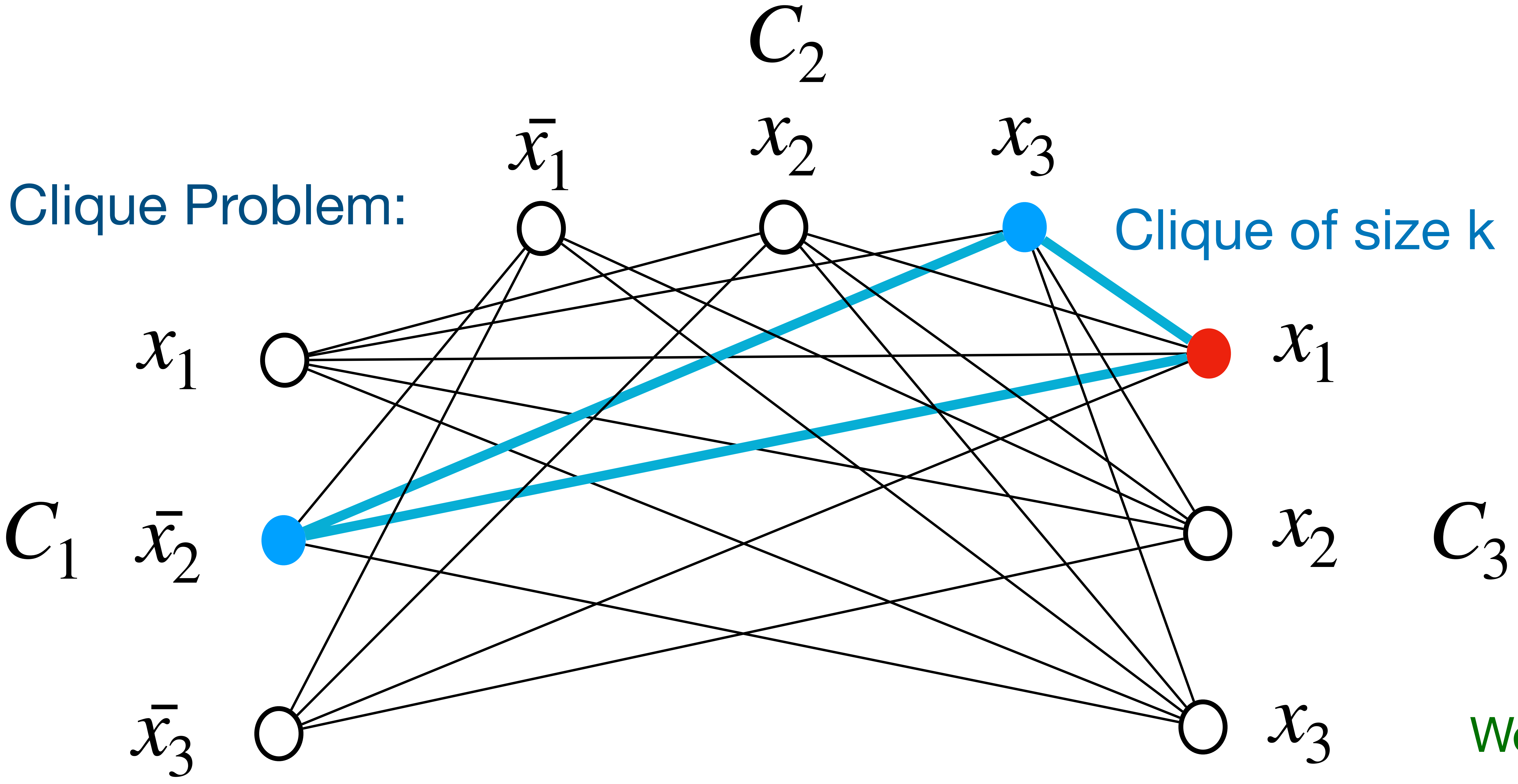

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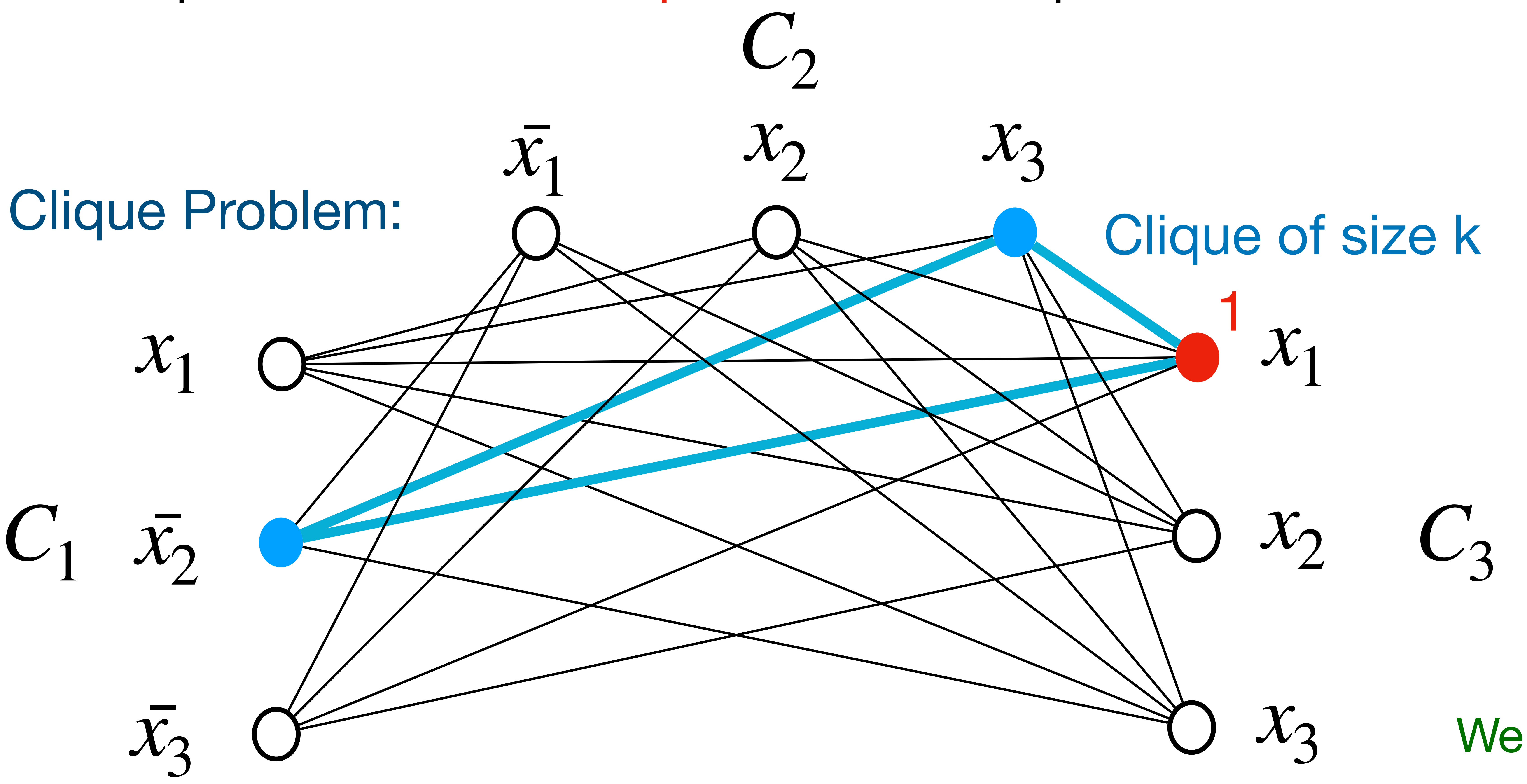

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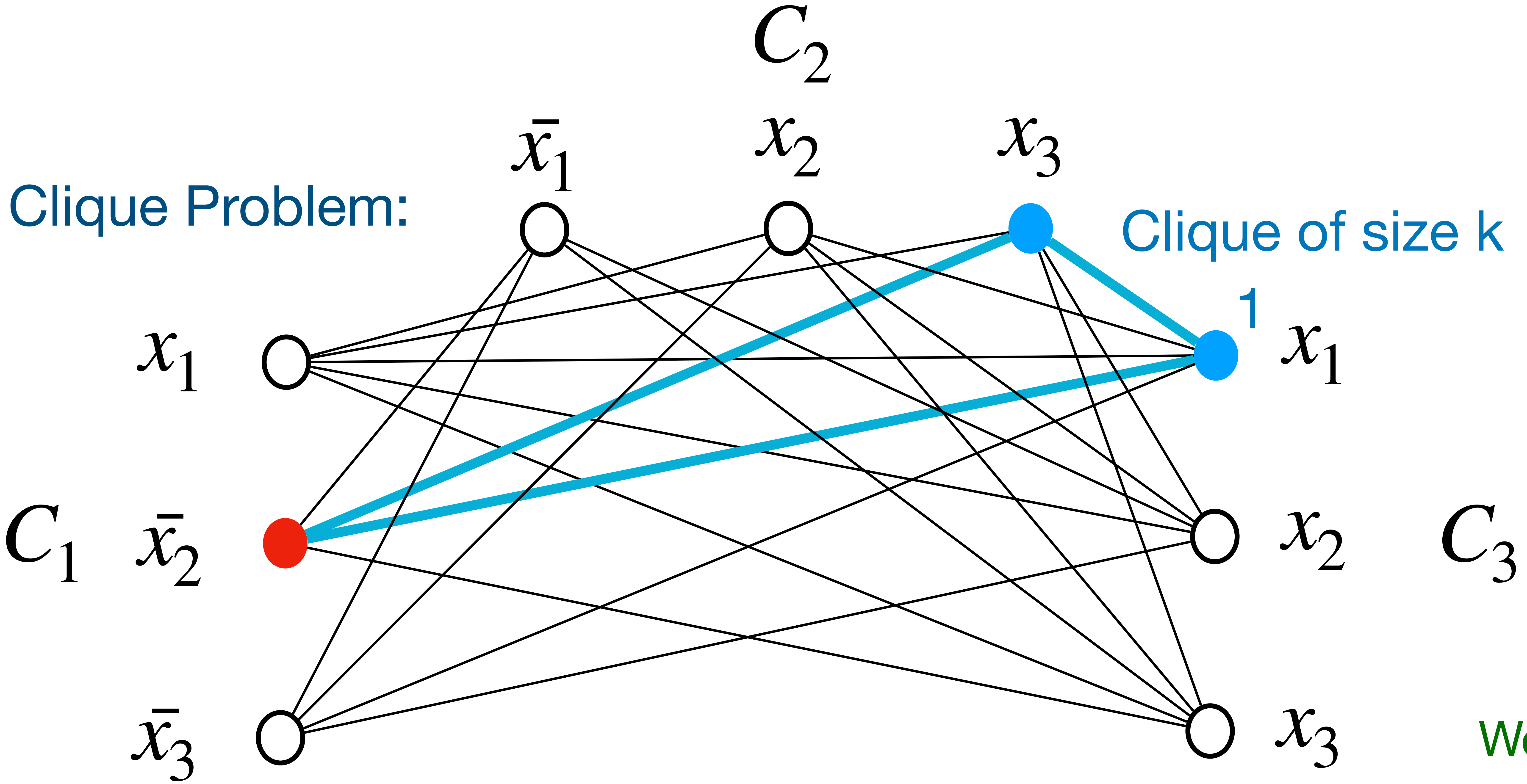

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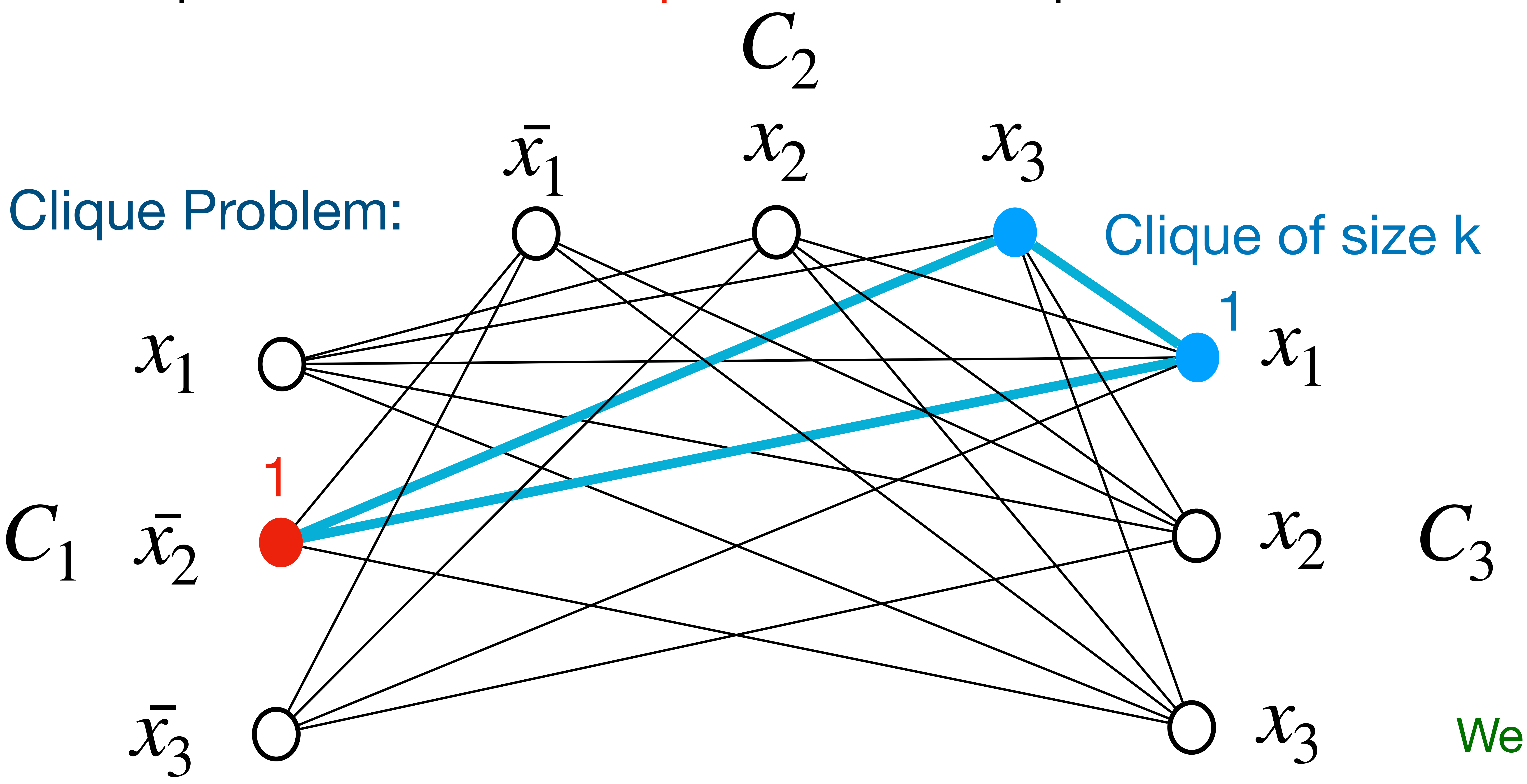

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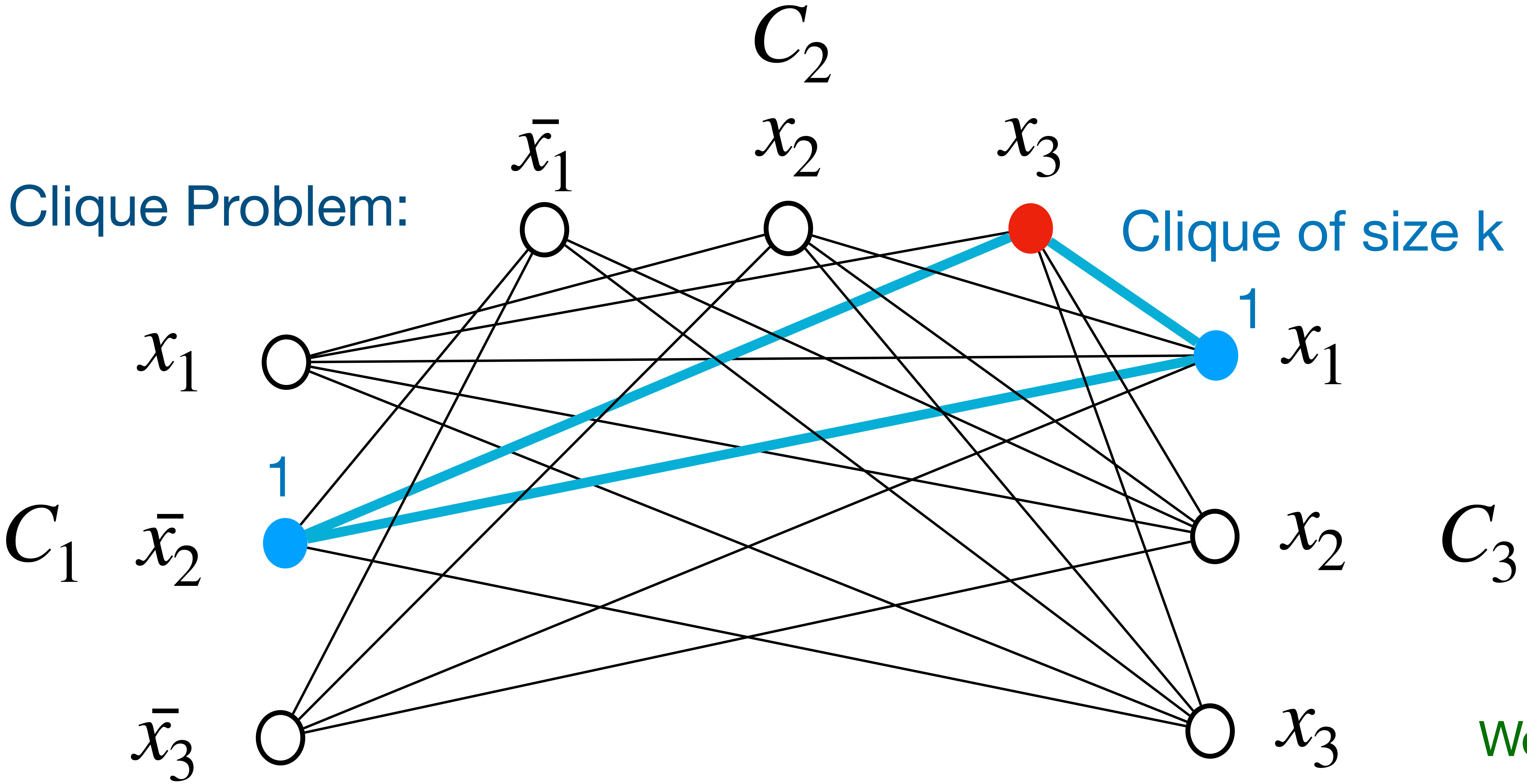

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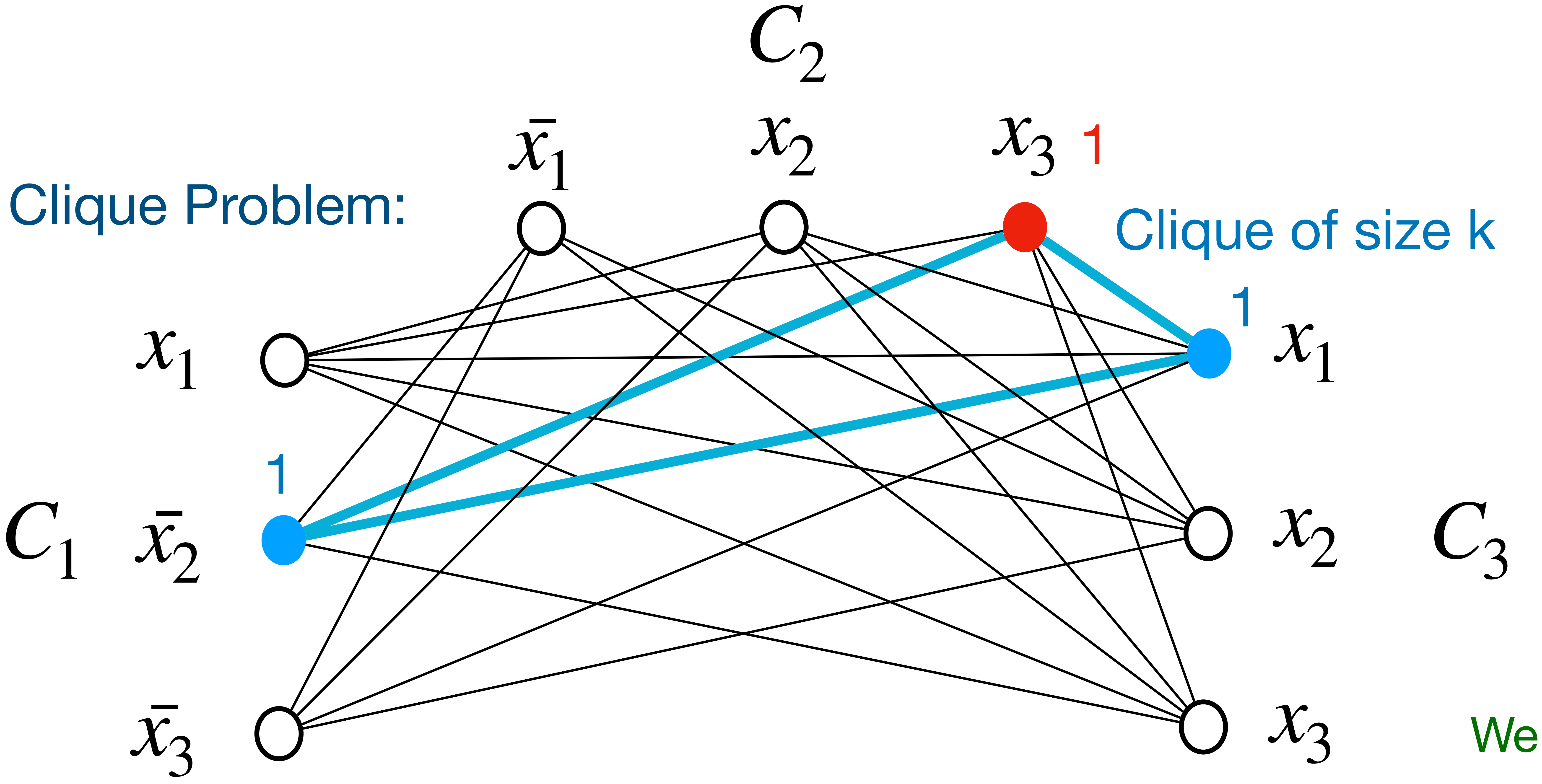

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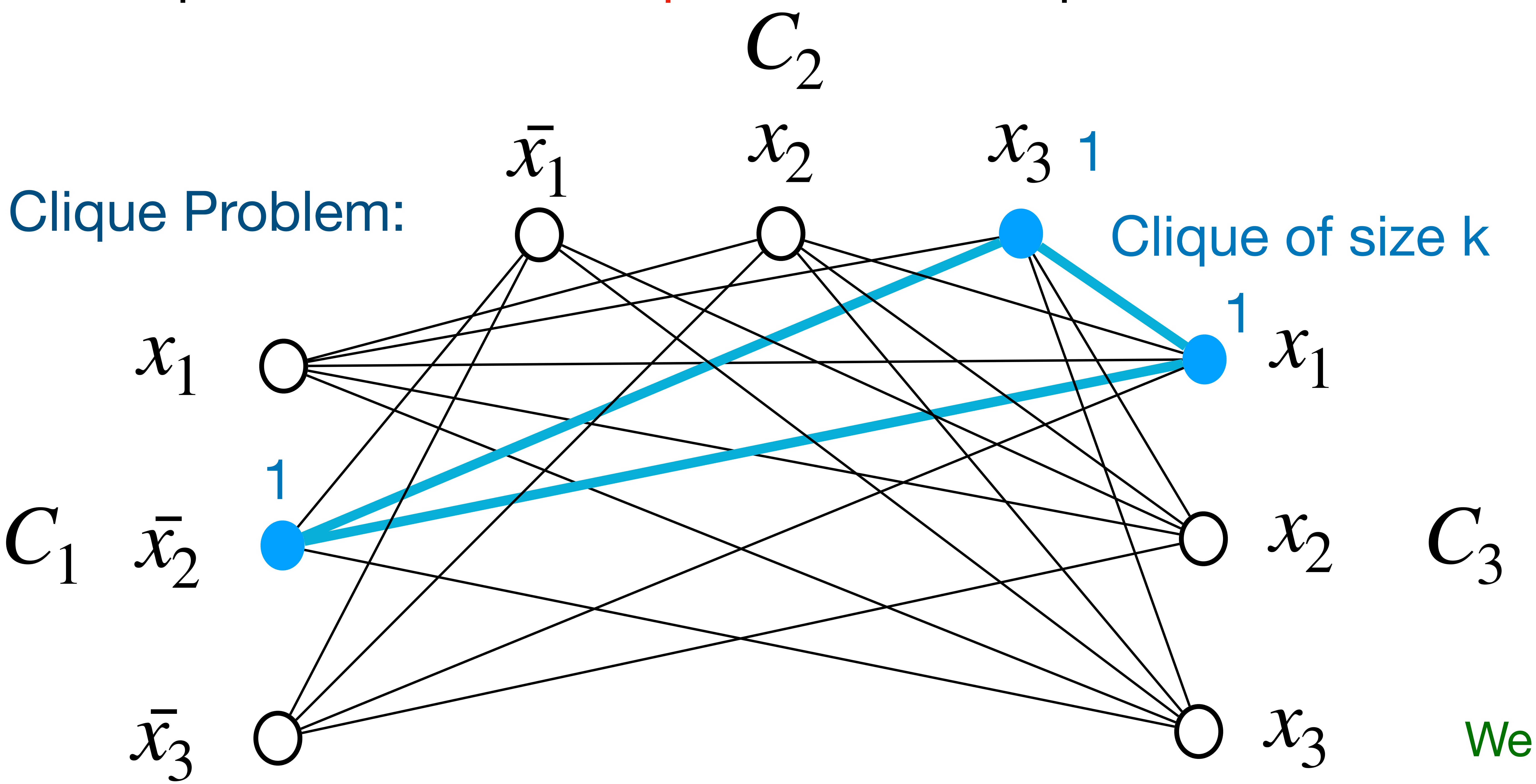

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So the reduction preserves the "YES/NO" answer.

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It automatically implies:

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So the reduction preserves the “YES/NO” answer.

3-CNF SAT Problem $\leq_p$ Clique Problem

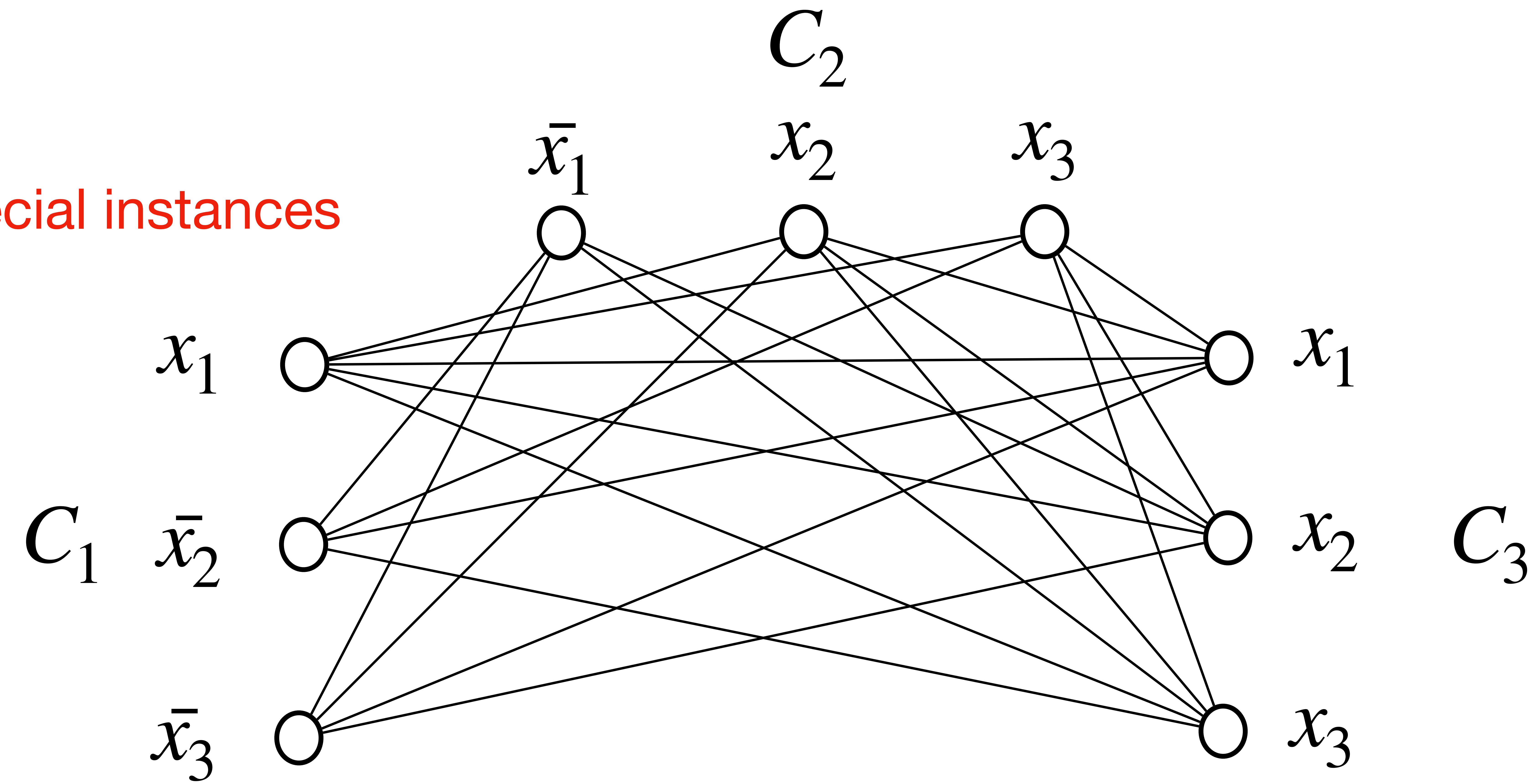
Note: the reduction we showed is from “3-CNF SAT Problem” to “Clique Problem”, not vice versa.

3-CNF SAT Problem: $\underbrace{(x_1 \vee \bar{x}_2 \vee \bar{x}_3)}_{C_1} \wedge \underbrace{(\bar{x}_1 \vee x_2 \vee x_3)}_{C_2} \wedge \underbrace{(x_1 \vee x_2 \vee x_3)}_{C_3}$

General, any instance is OK

Clique Problem:

Only special instances



Note: the reduction we showed is from "3-CNF SAT Problem" to "Clique Problem", not vice versa.

$$\text{3-CNF SAT Problem} \leq_p \text{Clique Problem}$$

All the instances of "3-CNF SAT Problem" are mapped to some instances of the "Clique Problem".

But since 3-CNF SAT Problem $\in NPC$ and Clique Problem $\in NP$,

we also have

$$\text{Clique Problem} \leq_p \text{3-CNF SAT Problem}$$

So there is a reduction from "Clique Problem" to the "3-CNF SAT Problem". It is a different reduction.

Here all the instances of "Clique Problem" are mapped to some instances of the "3-CNF SAT Problem".

But then, who has more instances, “3-CNF SAT” or “Clique Problem”?

Answer: both have infinitely many instances.

Who has more numbers?

$$0,\ 2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14,\ 16,\ 18,\ 20,\ \cdots$$

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ \cdots$$

But then, who has more instances, “3-CNF SAT” or “Clique Problem”?

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Quiz questions:

1. What is the main idea for proving the NP-completeness of the "Clique Problem"?

2. In the above proof, did we use a reduction from the "3-SAT Problem" to the "Clique Problem", or from the "Clique Problem" to the "3-SAT Problem"?

Roadmap of this lecture:

1. NP Completeness

    1.1 Prove the "Clique Problem" is NPC.

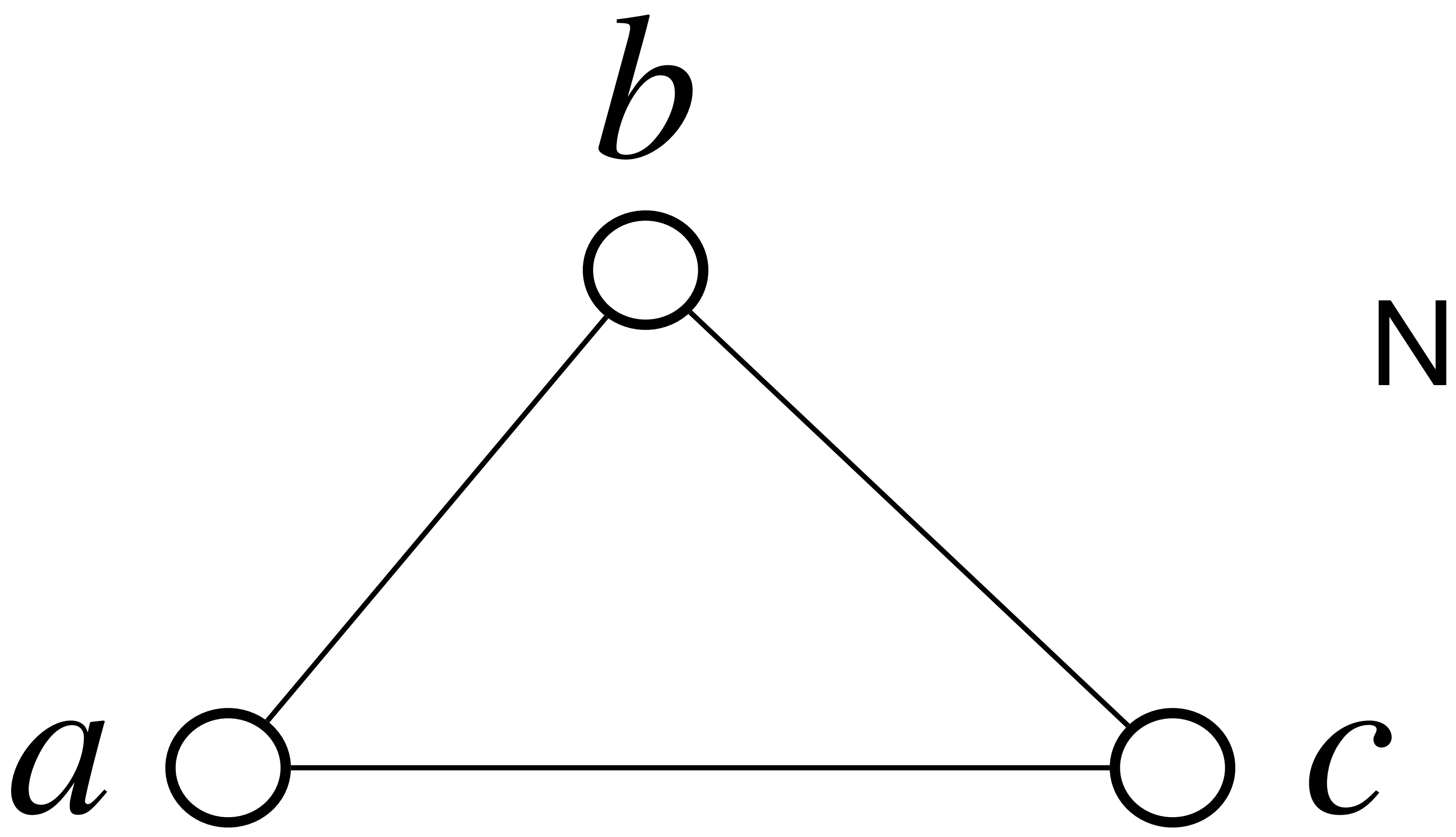
    1.2 Prove the "Vertex Cover Problem" is NPC.

NP-Completeness

Vertex Cover Problem

Known NPC Problems:
1) 3-CNF SAT Problem
2) Clique Problem

Vertex Cover: Given an undirected graph G=(V,E), a vertex cover of G is a subset $S \subseteq V$ of vertices such that for every edge $(u, v) \in E$, either $u \in S$ or $v \in S$.



Size of vertex cover:
Number of vertices in the vertex cover, namely, $|S|$.

NP-Completeness

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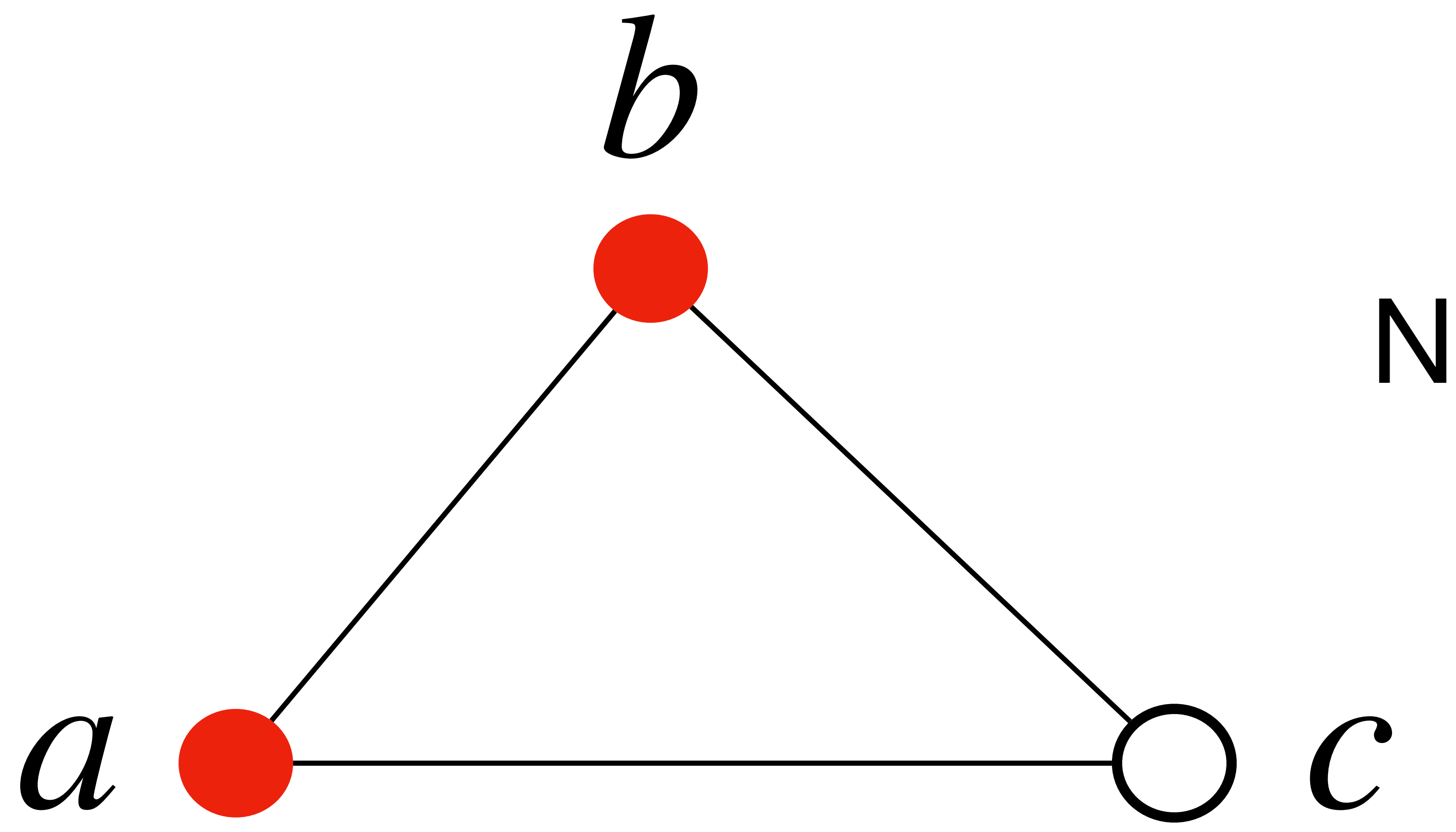

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Number of vertices in the vertex cover, namely, $|S|$.

$\{a, b\}$ is a Vertex Cover of size 2.

NP-Completeness

Vertex Cover Problem

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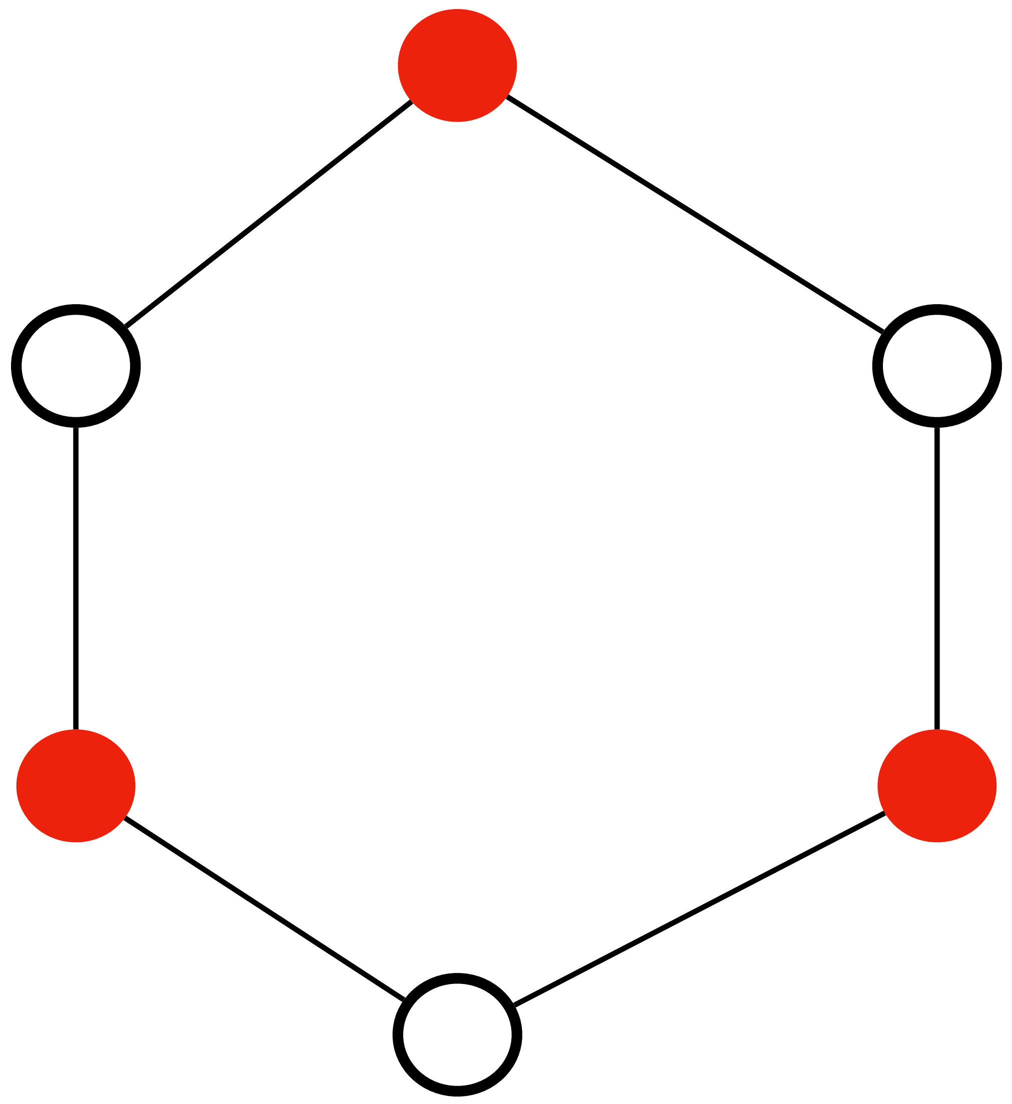

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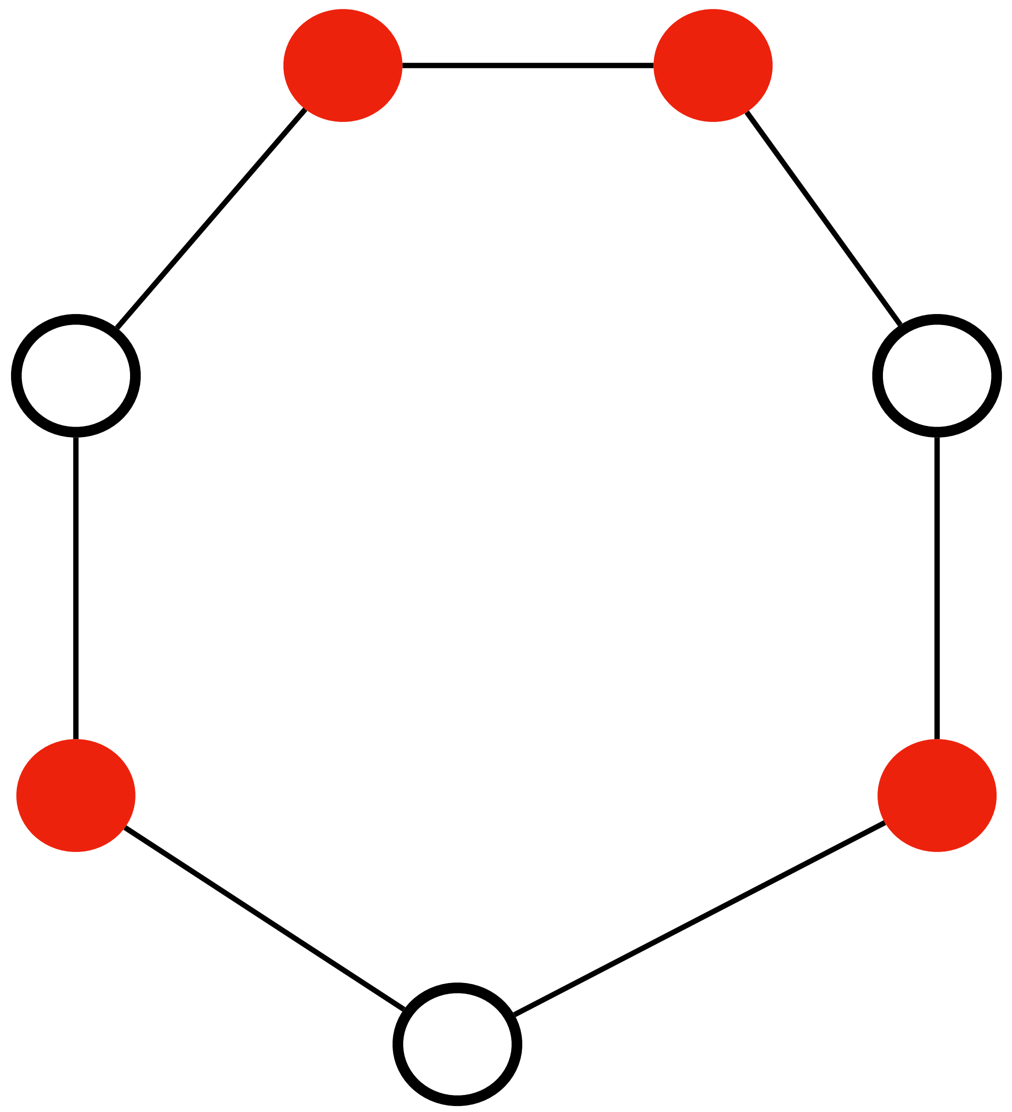

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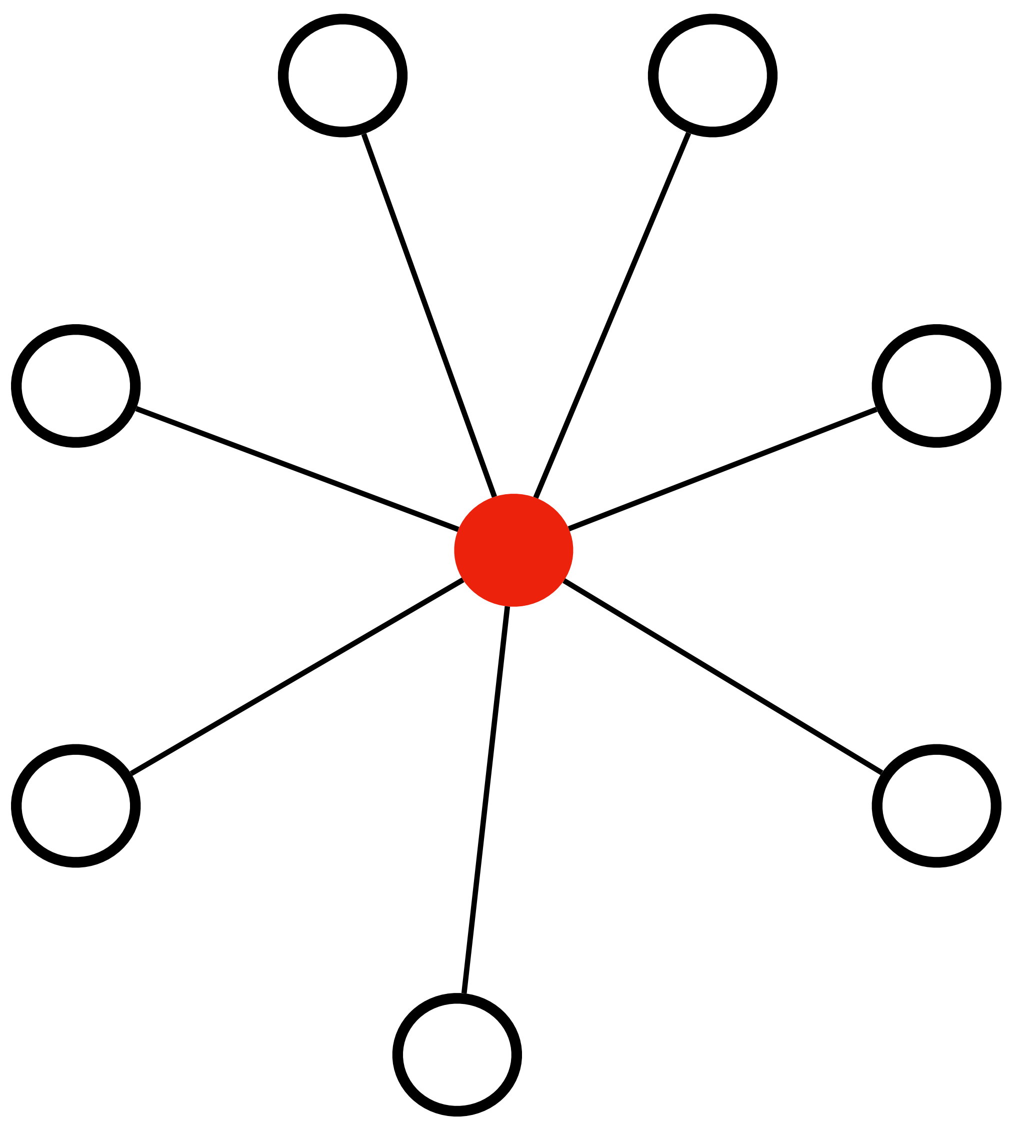

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NP-Completeness

Known NPC Problems:

1) 3-CNF SAT Problem

2) Clique Problem

Vertex Cover Problem

Input: An undirected graph G=(V,E).

An integer k.

Question: Does G have a vertex cover of size k?

Theorem: Vertex Cover Problem $\in NPC$.

NP-Completeness

Known NPC Problems:
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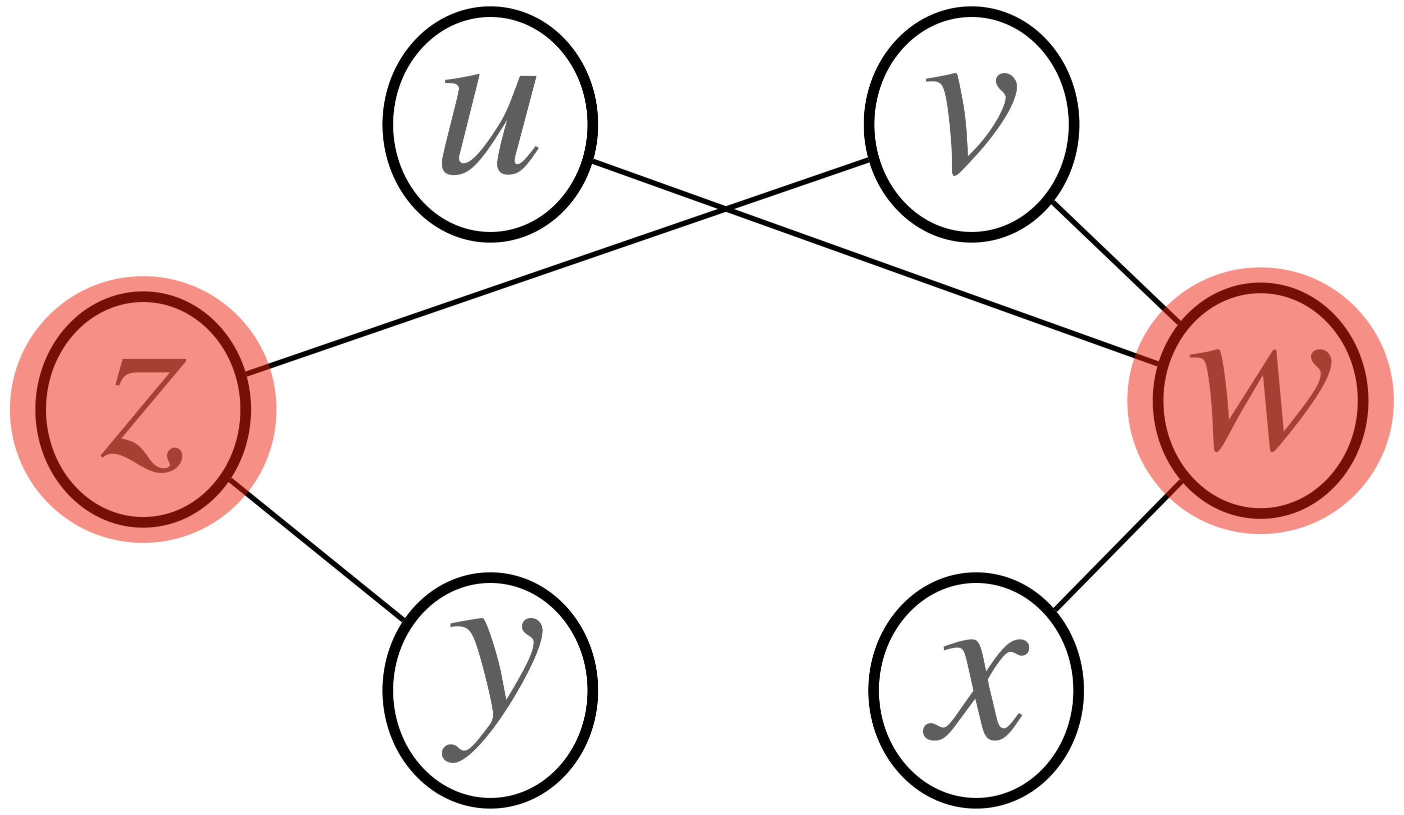
Vertex Cover Problem

Input: An undirected graph G=(V,E).
An integer k.

Question: Does G have a vertex cover of size k?

Example: k=2



Theorem: Vertex Cover Problem $\in NPC$.

Proof: 1) Vertex Cover Problem $\in NP$.

Certificate: a vertex cover of size k.

Polynomial-time verification.

NP-Completeness

Known NPC Problems:

1) 3-CNF SAT Problem
2) Clique Problem

Vertex Cover Problem

Input: An undirected graph G=(V,E).

An integer k.

Question: Does G have a vertex cover of size k?

Theorem: Vertex Cover Problem $\in NPC$.

Proof: 1) Vertex Cover Problem $\in NP$.

Certificate: a vertex cover of size k.

Polynomial-time verification.

2) Clique Problem $\leq_p$ Vertex Cover Problem

Clique Problem:

Input: An undirected graph G=(V,E).
A positive integer k.

Question: Does G have a clique of size k?

Vertex Cover Problem

Input: An undirected graph $G' = (V', E')$.
An integer k'.

Question: Does G' have a vertex cover of size k'?

Clique Problem:

Input: An undirected graph G=(V,E).
A positive integer k.

Question: Does G have a clique of size k?

Vertex Cover Problem

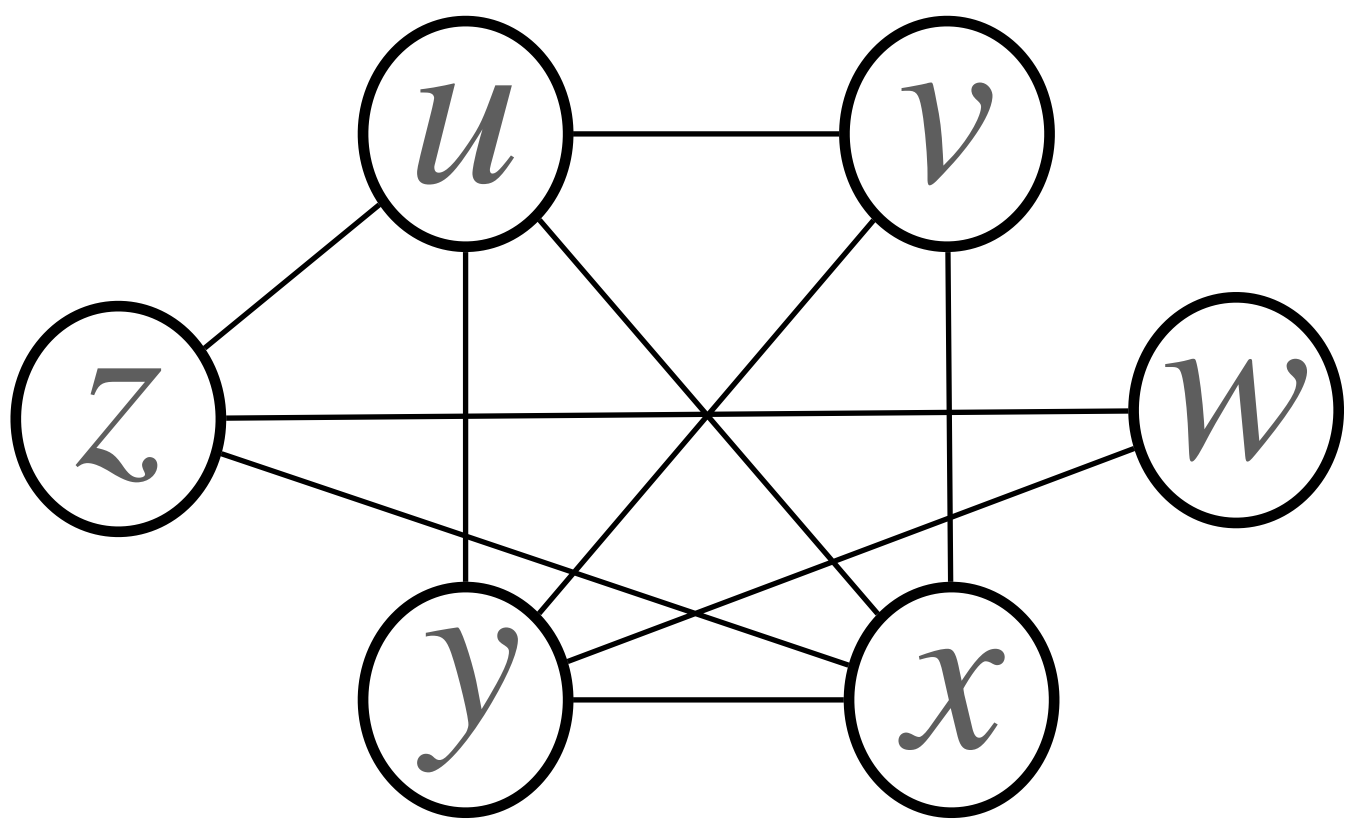
Input: An undirected graph $G' = (V', E')$.
An integer k'.

Question: Does G' have a vertex cover of size k'?

Example of instance:

$G = (V, E)$

$k = 4$



Clique Problem:

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Question: Does G have a clique of size k?

Vertex Cover Problem

Input: An undirected graph $G' = (V', E')$.
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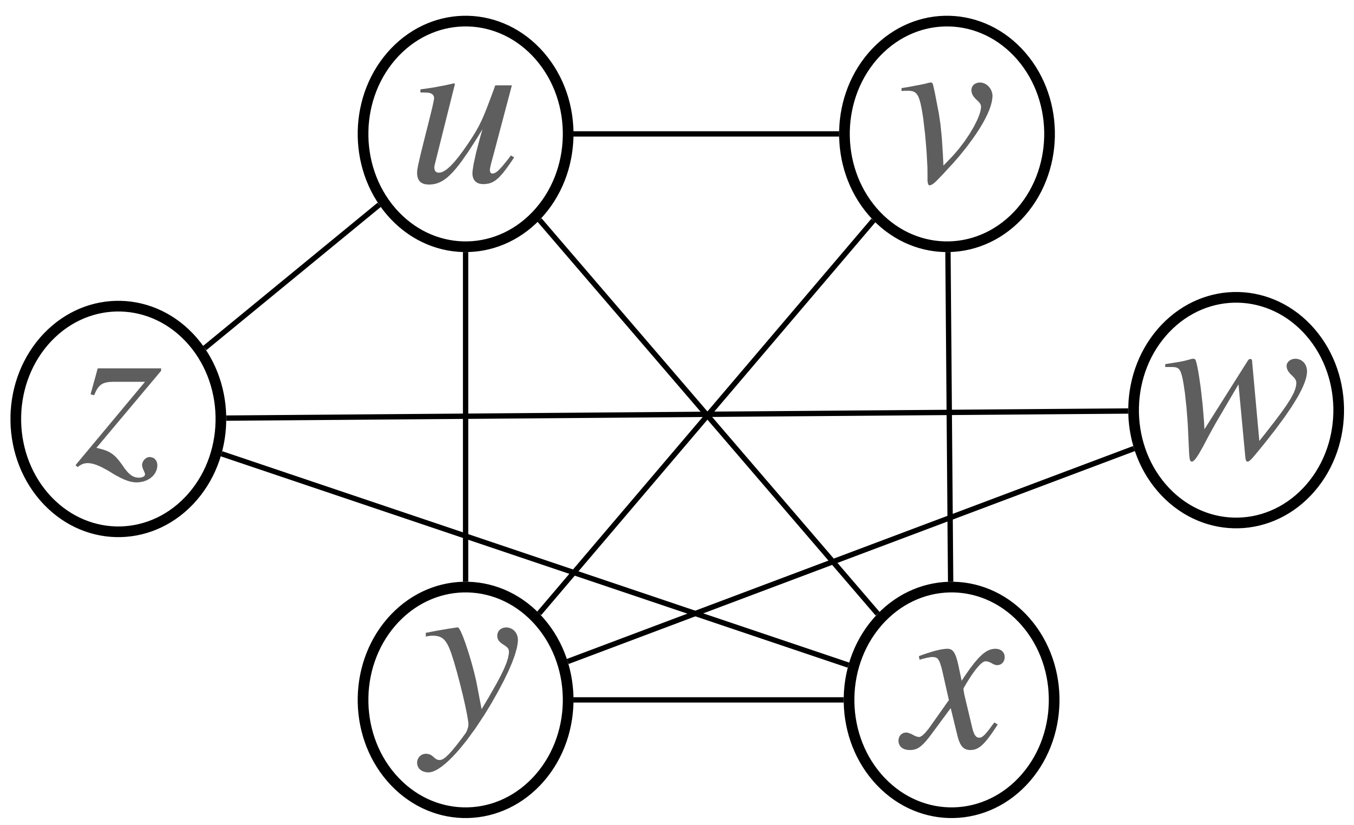
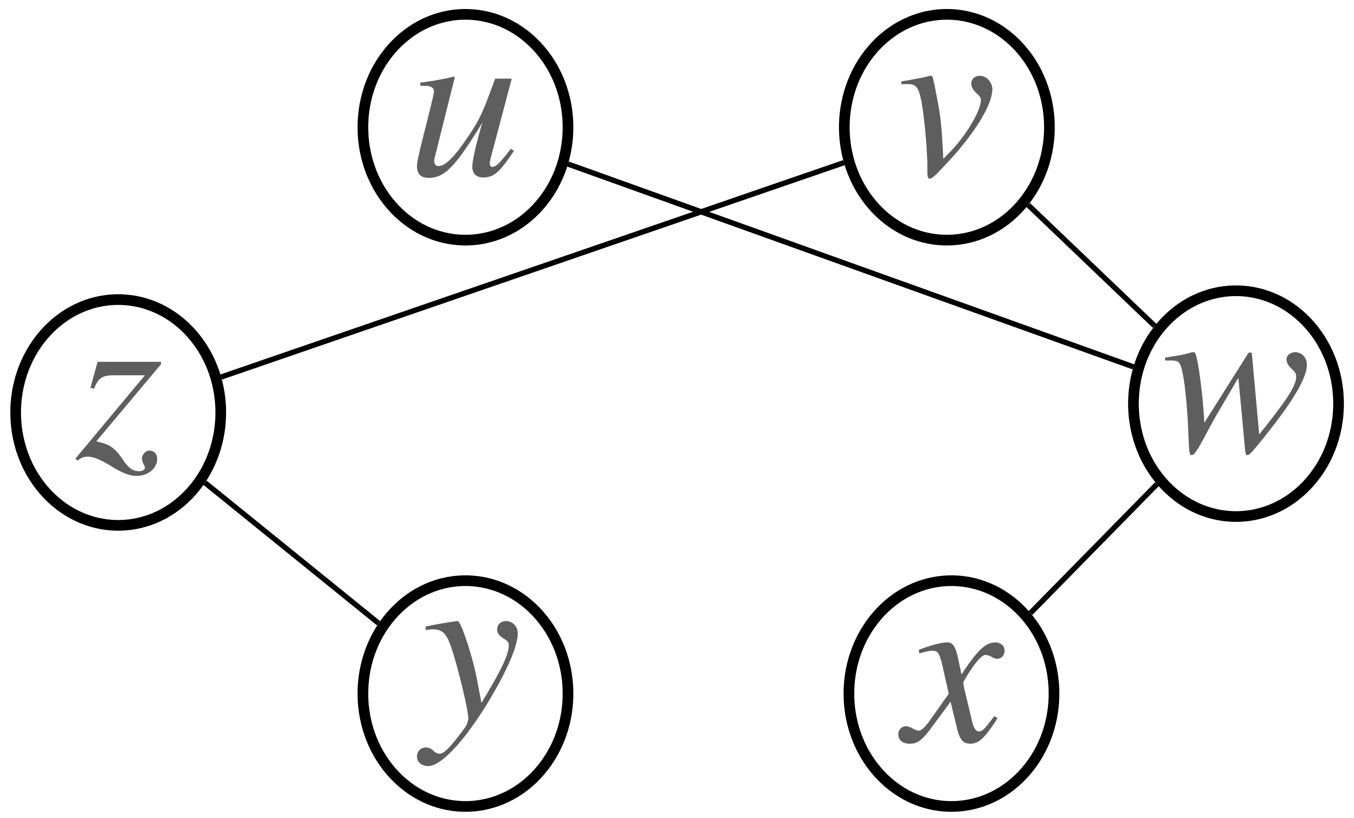


Corresponding instance:

$G' = (V', E')$. Complement graph of G.



$k' = |V| - k$

=6-4=2

Polynomial-time mapping.

Is the “YES/NO” answer preserved?

Clique Problem:

Input: An undirected graph G=(V,E).
A positive integer k.

Question: Does G have a clique of size k?

Vertex Cover Problem

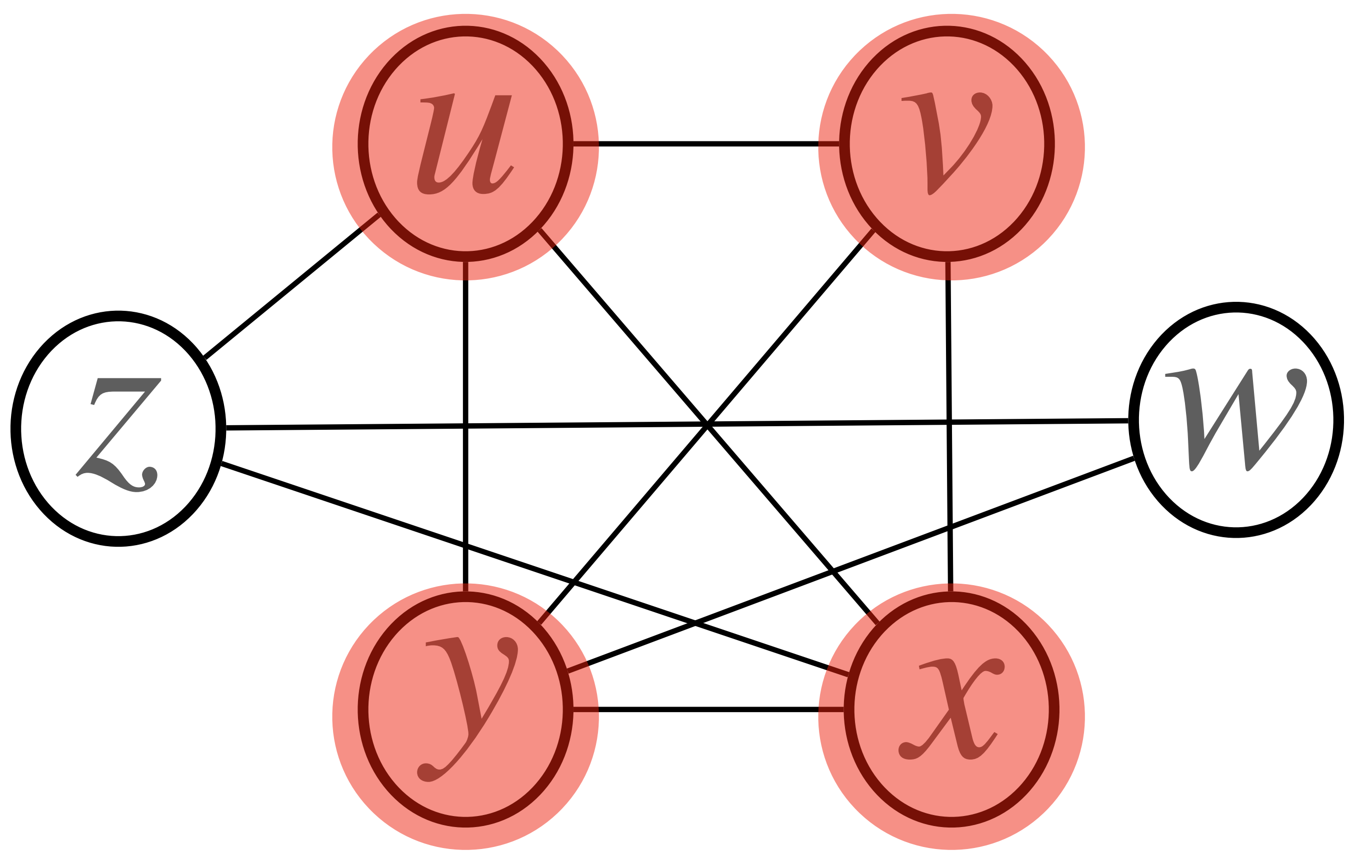
Input: An undirected graph $G' = (V', E')$.
An integer k'.

Question: Does G' have a vertex cover of size k'?

Example of instance:

$G = (V, E)$

$k = 4$



Corresponding instance:

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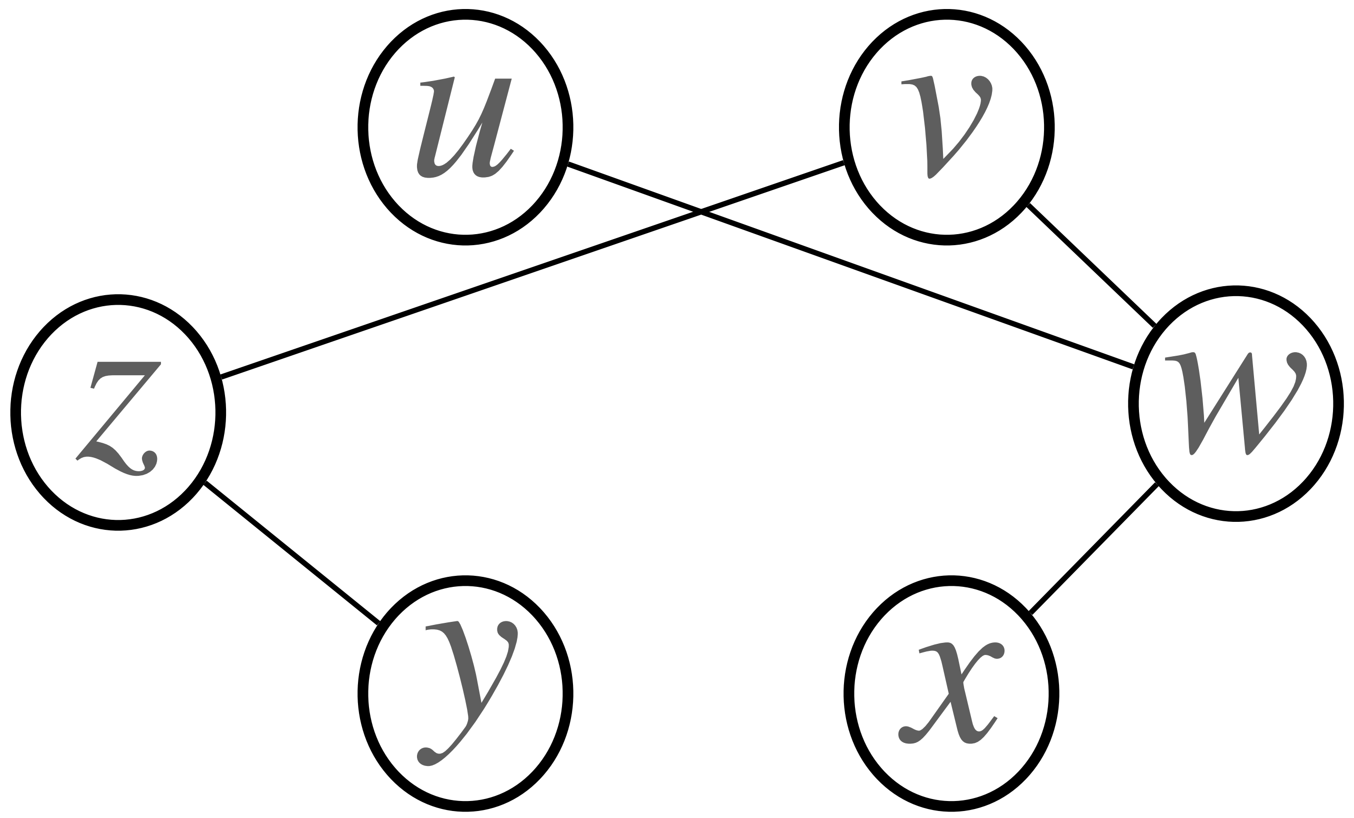


$k' = |V| - k$

=6-4=2

Assume "YES" for Clique Problem.

Clique Problem:

Input: An undirected graph G=(V,E).
A positive integer k.

Question: Does G have a clique of size k?

Vertex Cover Problem

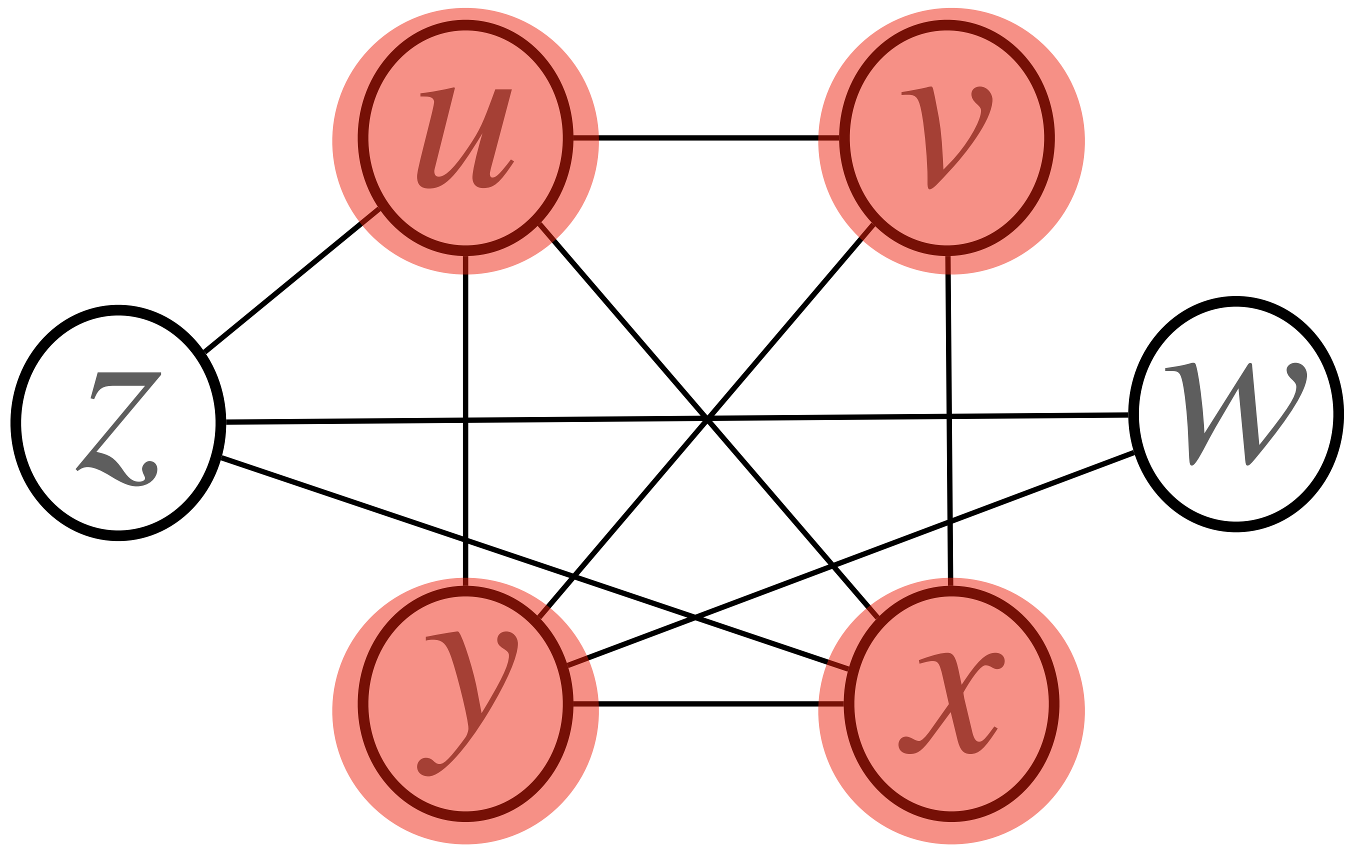
Input: An undirected graph $G' = (V', E')$.
An integer k'.

Question: Does G' have a vertex cover of size k'?

Example of instance:

$G = (V, E)$

$k = 4$



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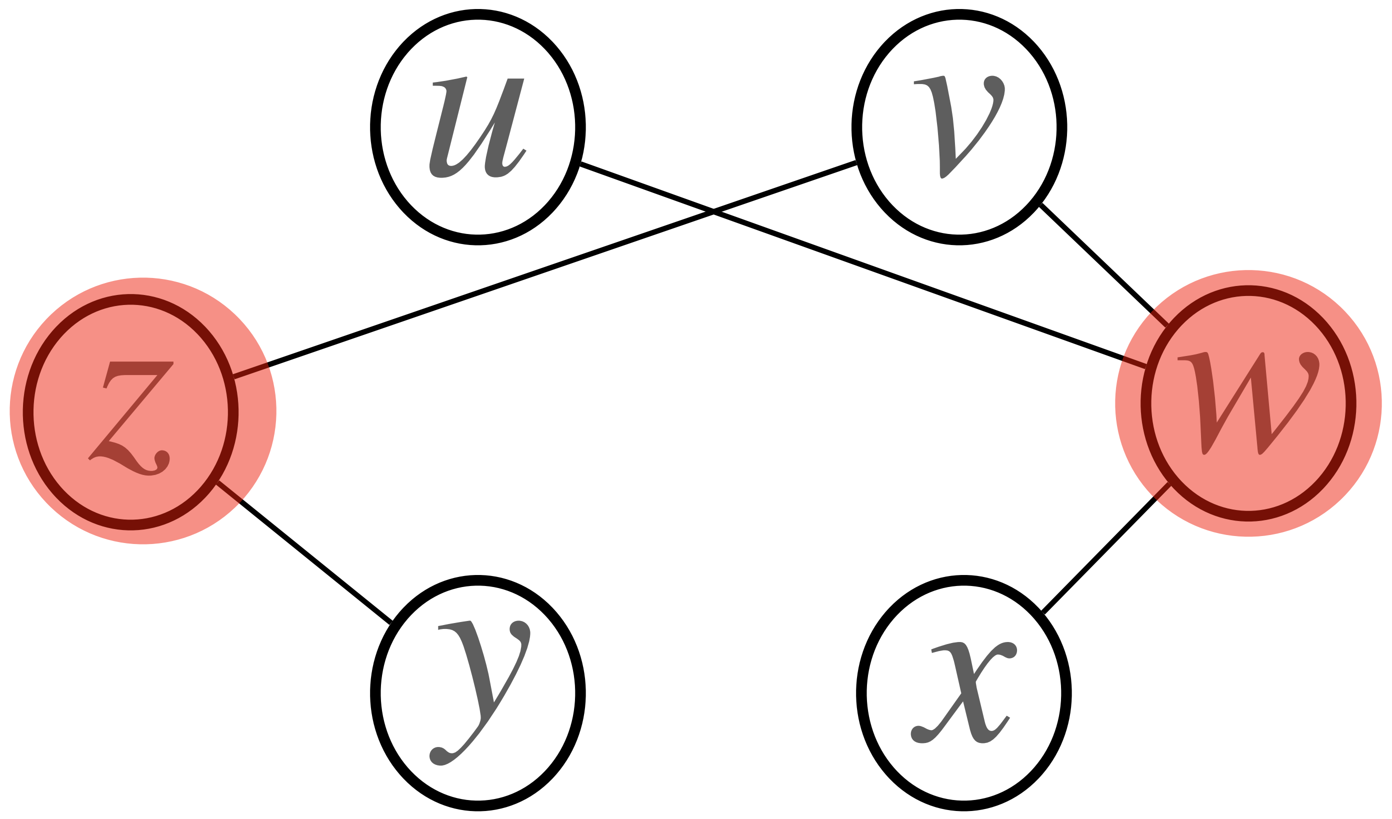

$k' = |V| - k$

=6-4=2

Assume "YES" for Clique Problem.

"YES" for Vertex Cover Problem.

Why?

Clique Problem:

Input: An undirected graph G=(V,E).
A positive integer k.

Question: Does G have a clique of size k?

Vertex Cover Problem

Input: An undirected graph $G' = (V', E')$.
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Example of instance:

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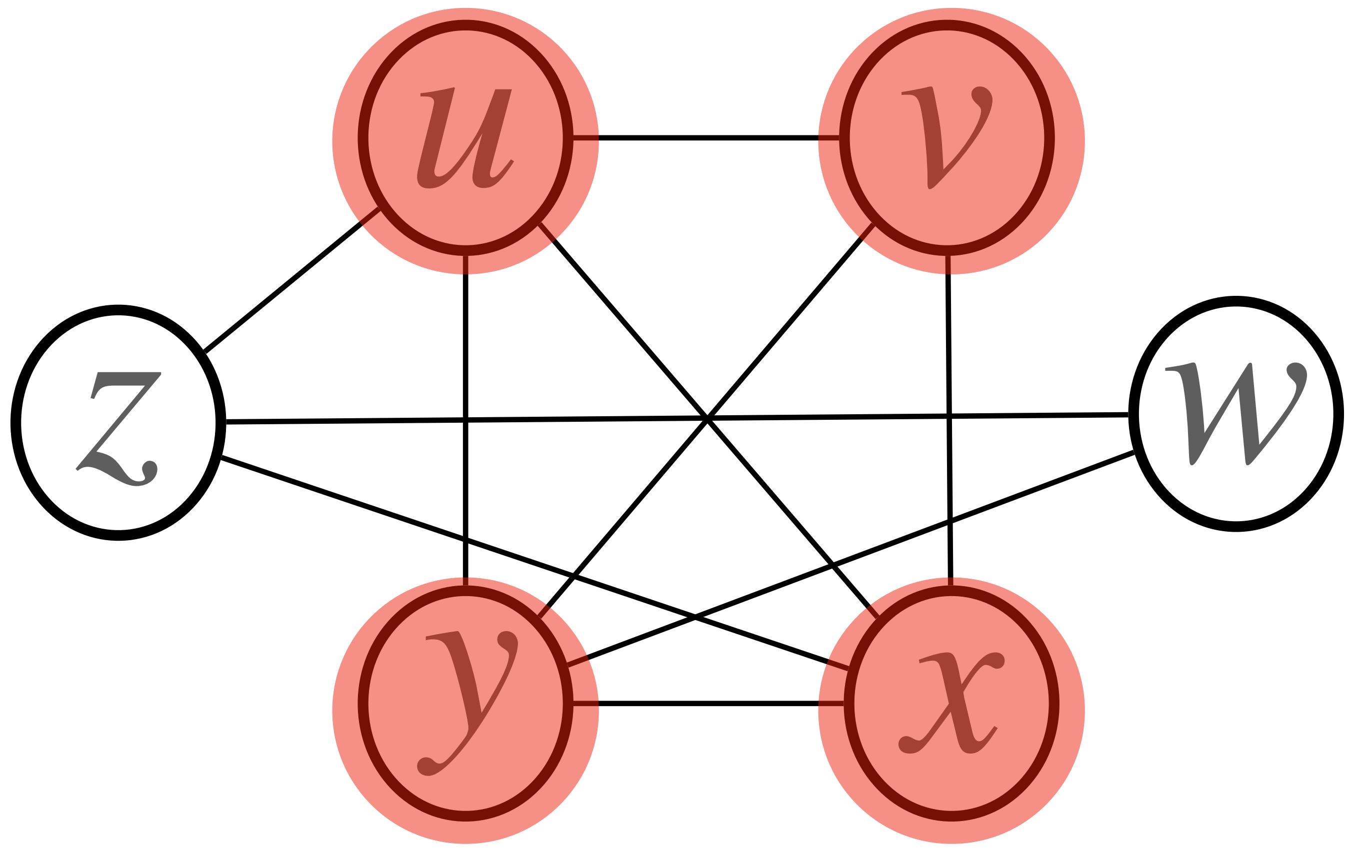
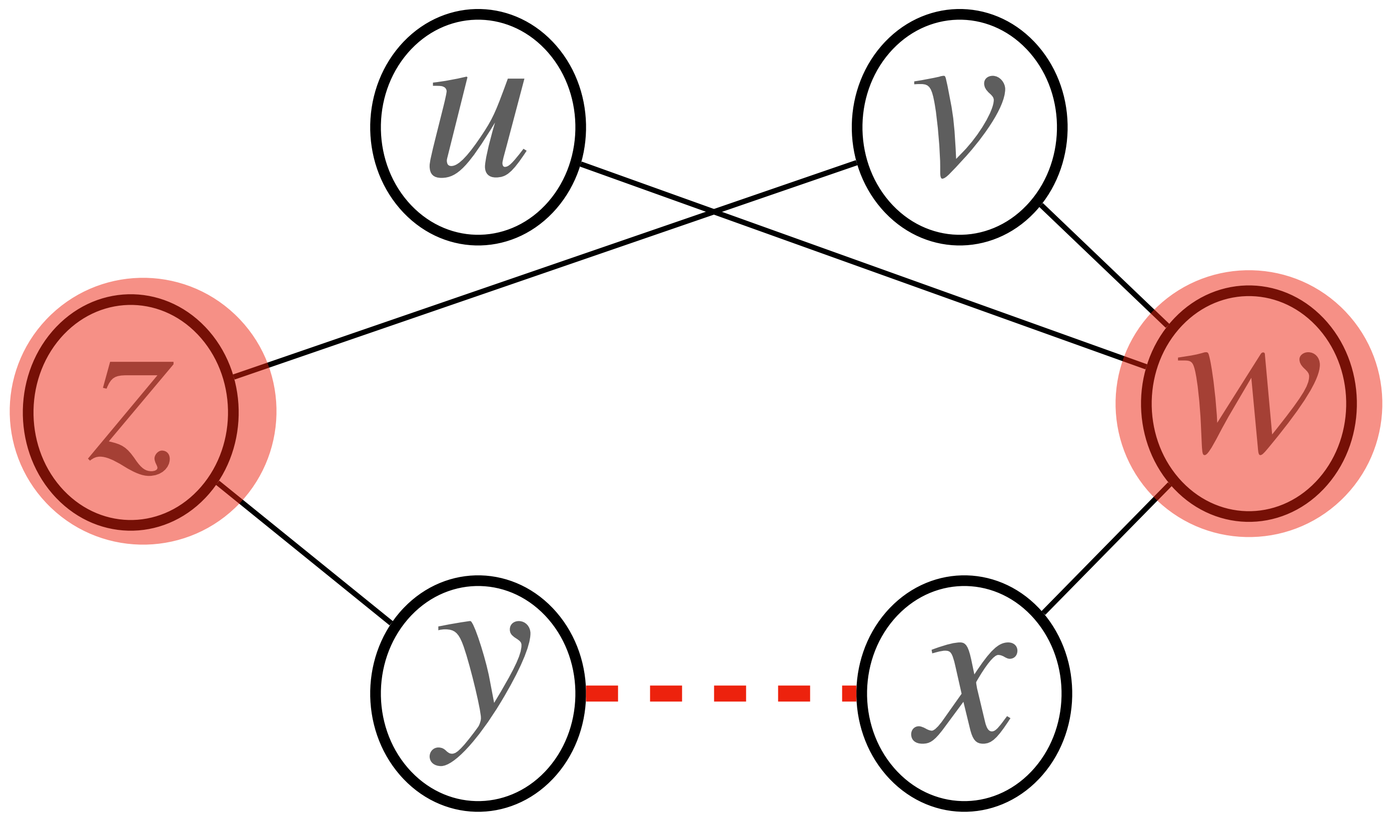
$k = 4$



Corresponding instance:

$G' = (V', E')$. Complement graph of G.



$k' = |V| - k$

=6-4=2

There cannot be an edge between two un-selected vertices.

Assume “YES” for Clique Problem.

“YES” for Vertex Cover Problem.

Why?

Clique Problem:

Input: An undirected graph G=(V,E).
A positive integer k.

Question: Does G have a clique of size k?

Vertex Cover Problem

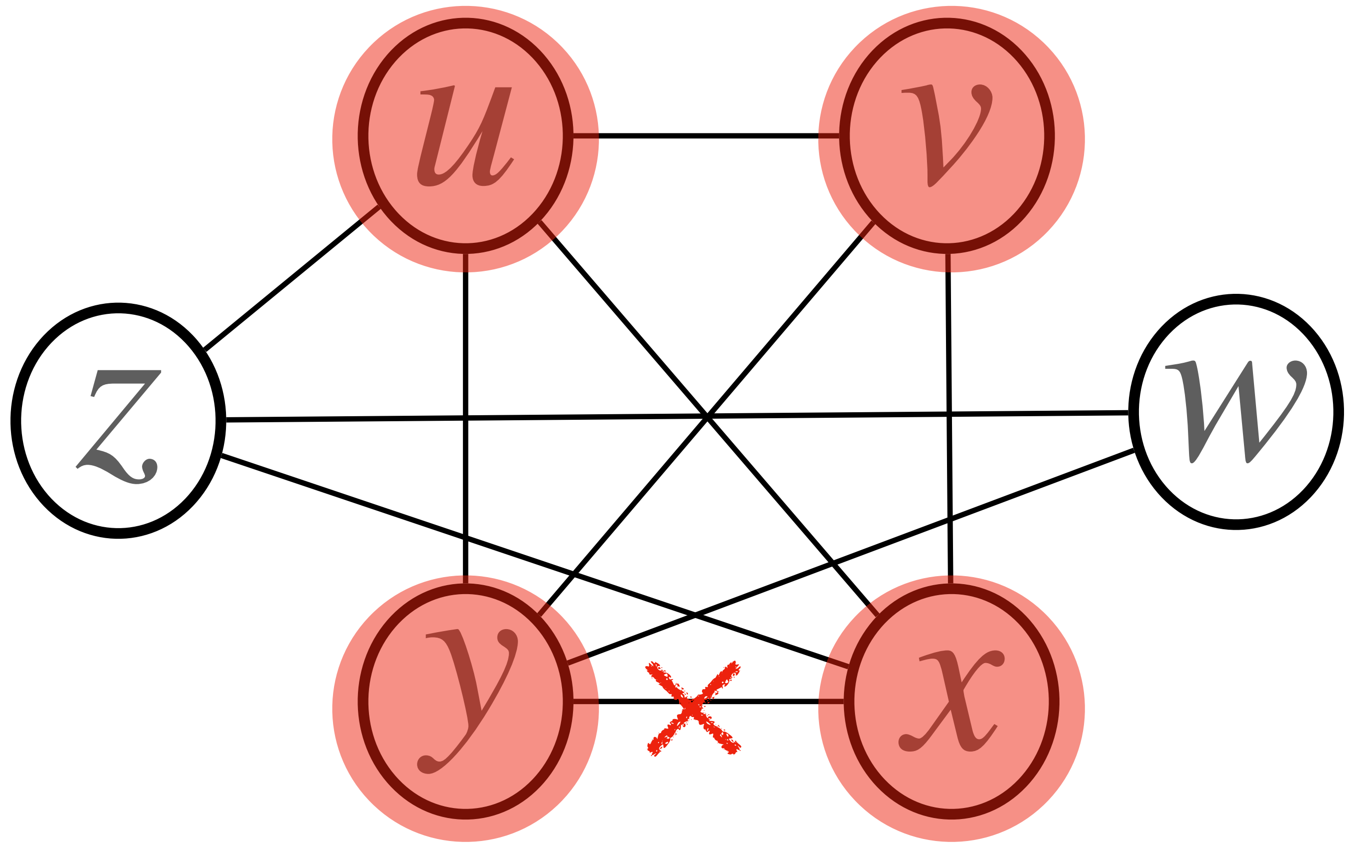
Input: An undirected graph $G' = (V', E')$.
An integer k'.

Question: Does G' have a vertex cover of size k'?

Example of instance:

$G = (V, E)$

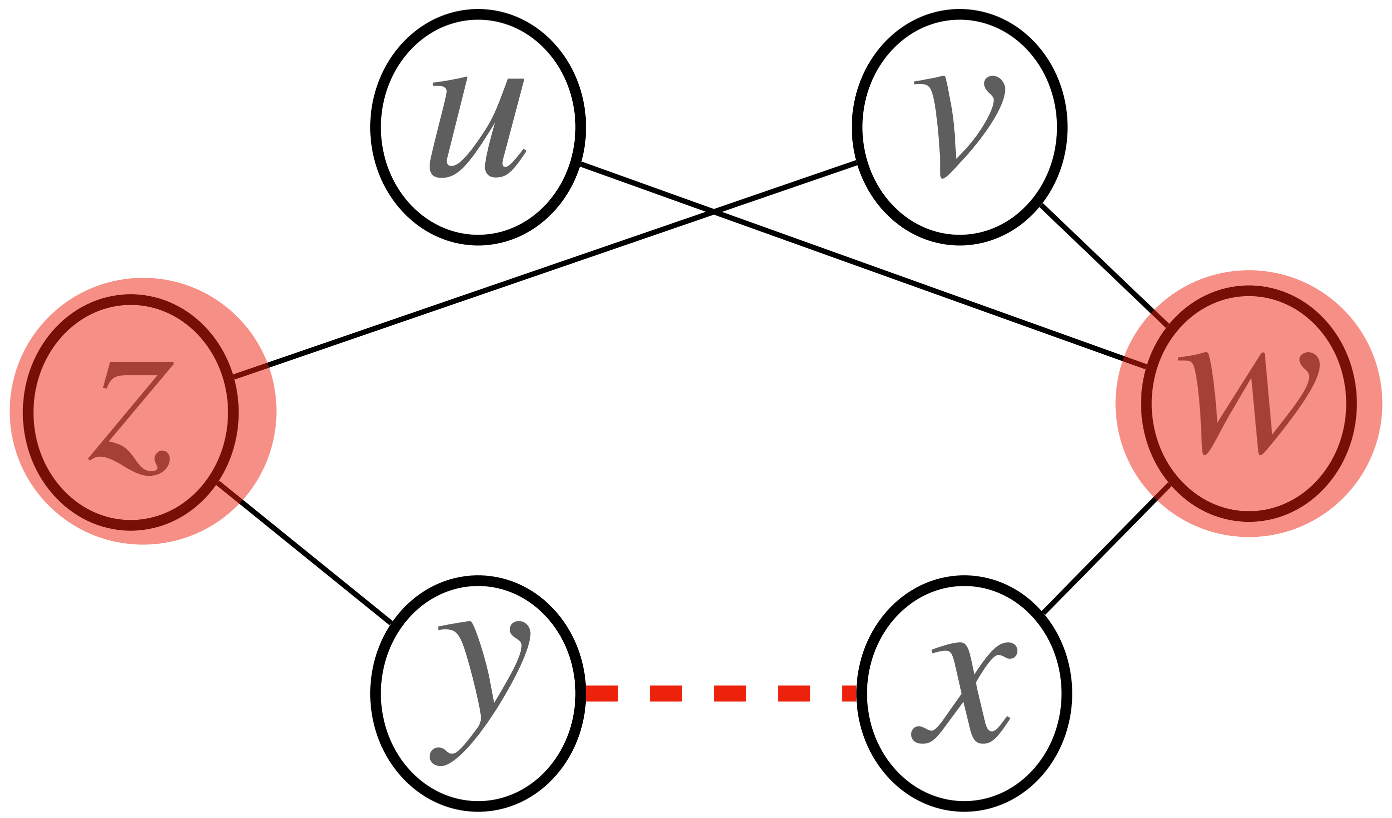
$k = 4$



Otherwise it will not be a clique.

Corresponding instance:

$G' = (V', E')$. Complement graph of G.



$k' = |V| - k$

=6-4=2

There cannot be an edge between two un-selected vertices.

Assume "YES" for Clique Problem. ⇒ "YES" for Vertex Cover Problem.

Clique Problem:

Input: An undirected graph G=(V,E).
A positive integer k.

Question: Does G have a clique of size k?

Vertex Cover Problem

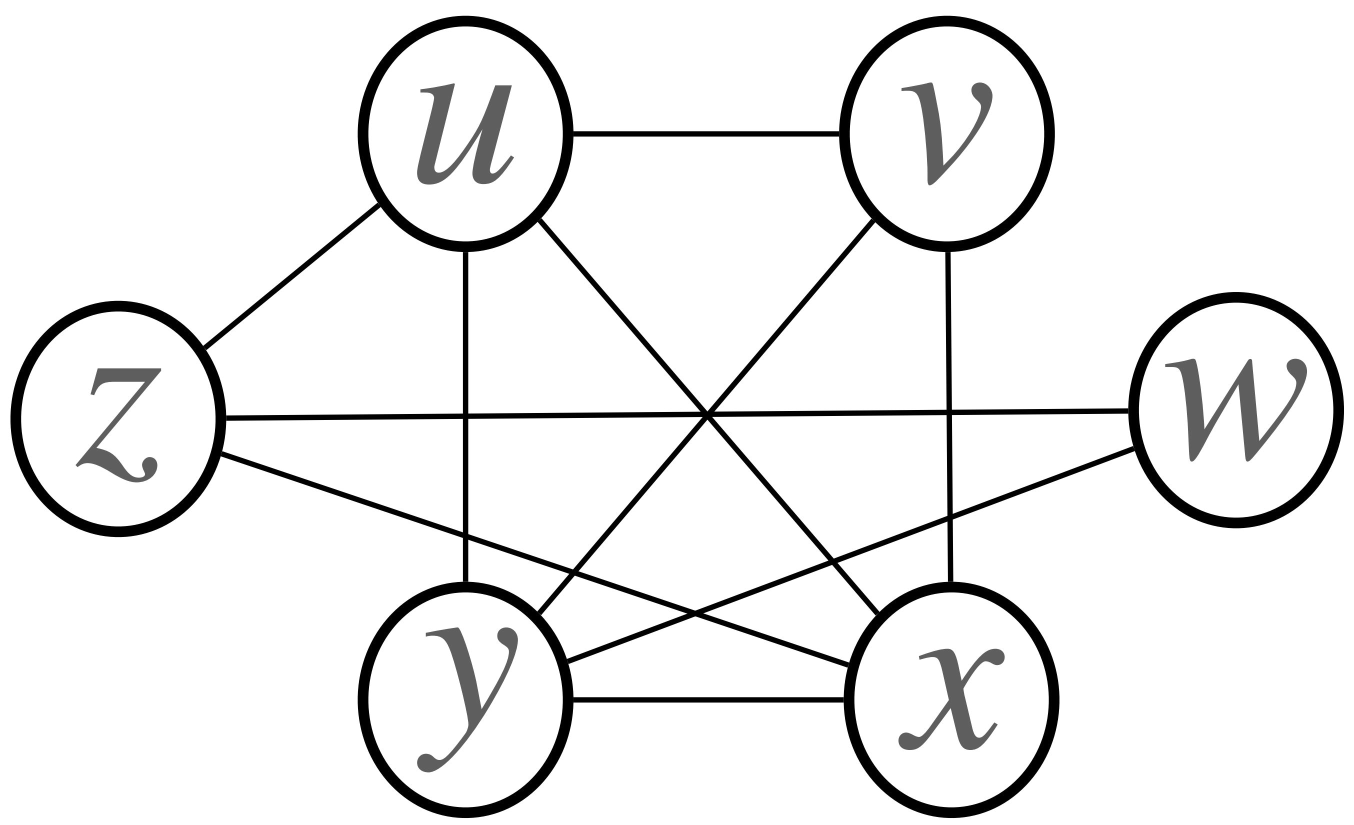
Input: An undirected graph $G' = (V', E')$.
An integer k'.

Question: Does G' have a vertex cover of size k'?

Example of instance:

$G = (V, E)$

$k = 4$



Corresponding instance:

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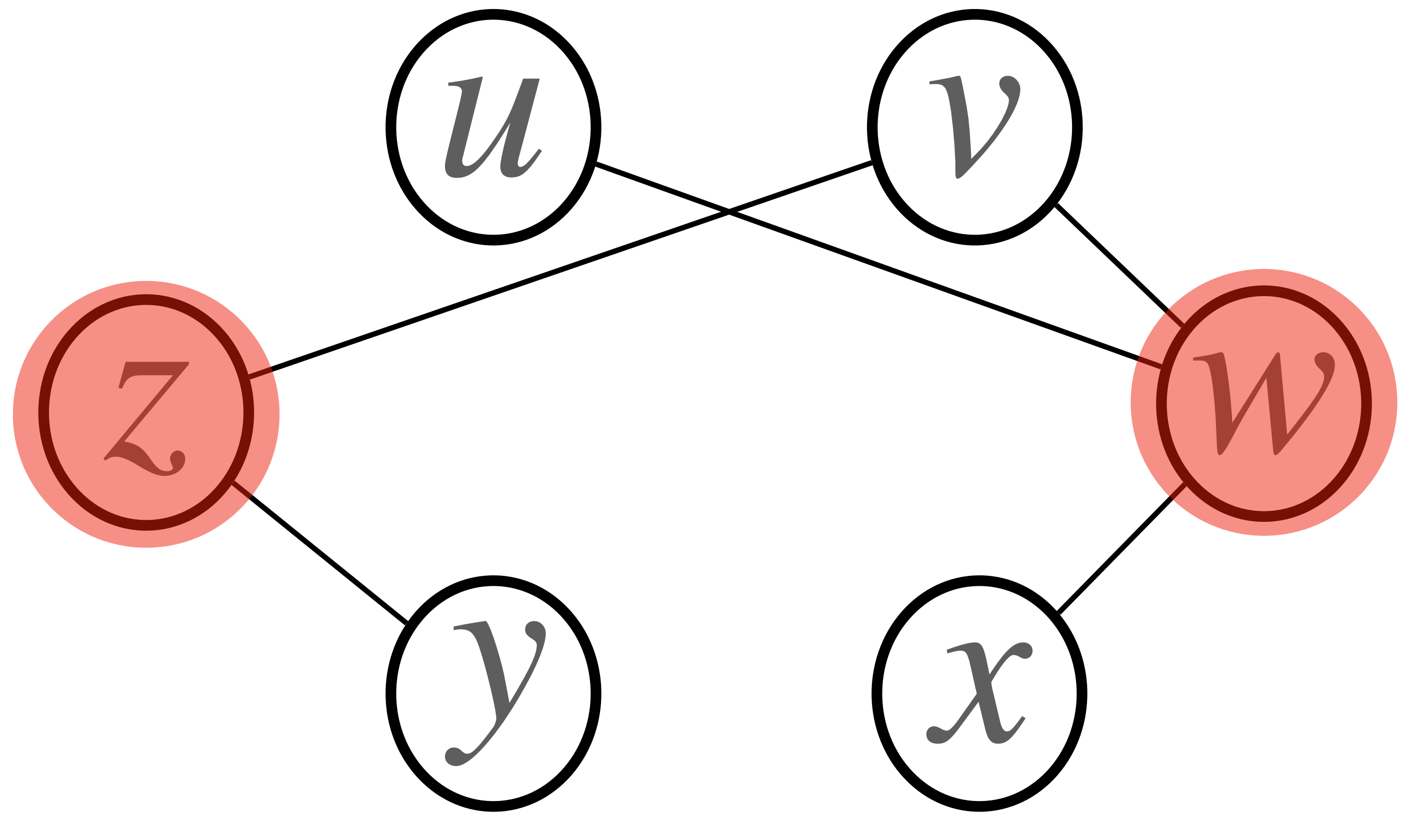


$k' = |V| - k$

=6-4=2

Assume “YES” for Vertex Cover Problem.

Clique Problem:

Input: An undirected graph G=(V,E).
A positive integer k.

Question: Does G have a clique of size k?

Vertex Cover Problem

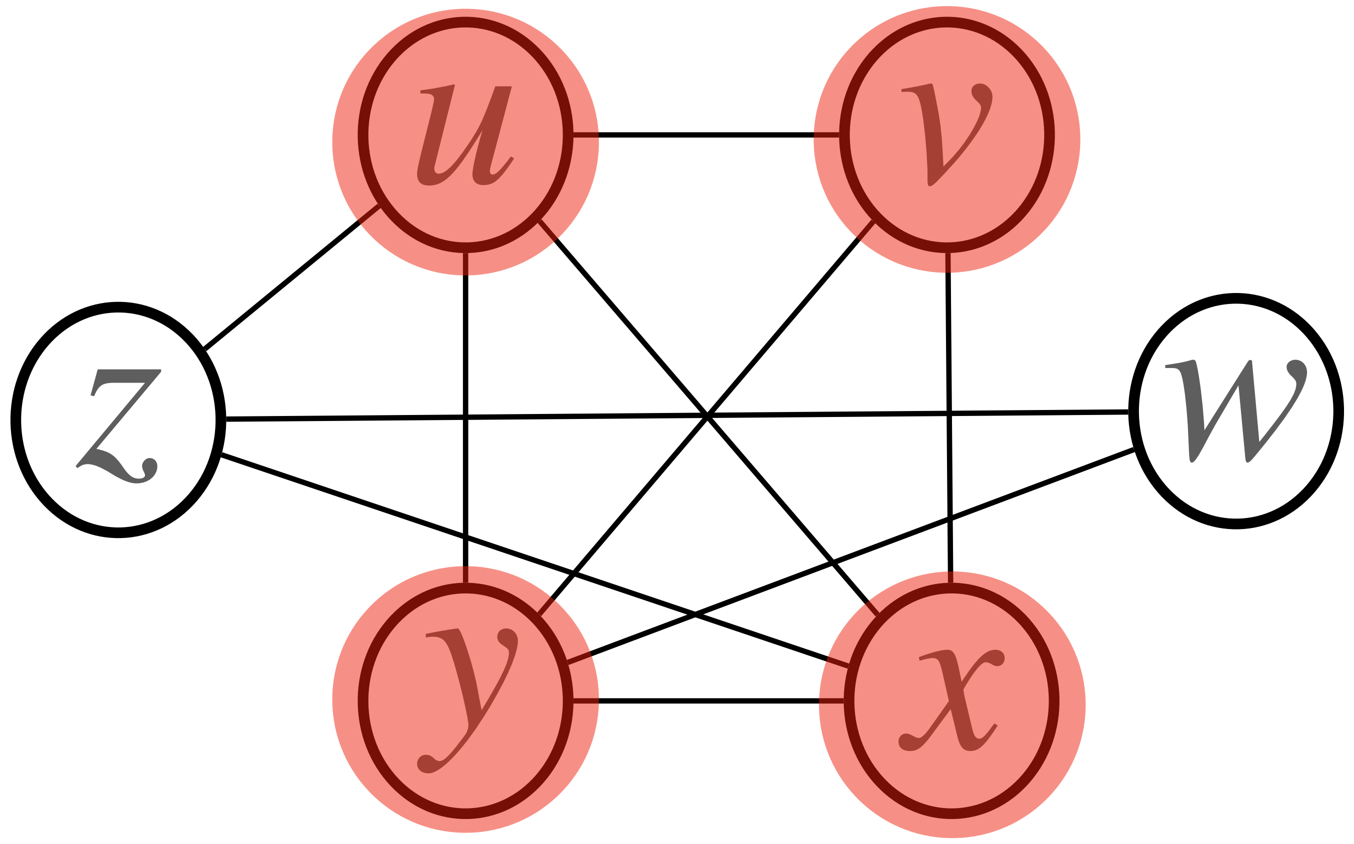
Input: An undirected graph $G' = (V', E')$.
An integer k'.

Question: Does G' have a vertex cover of size k'?

Example of instance:

$G = (V, E)$

$k = 4$



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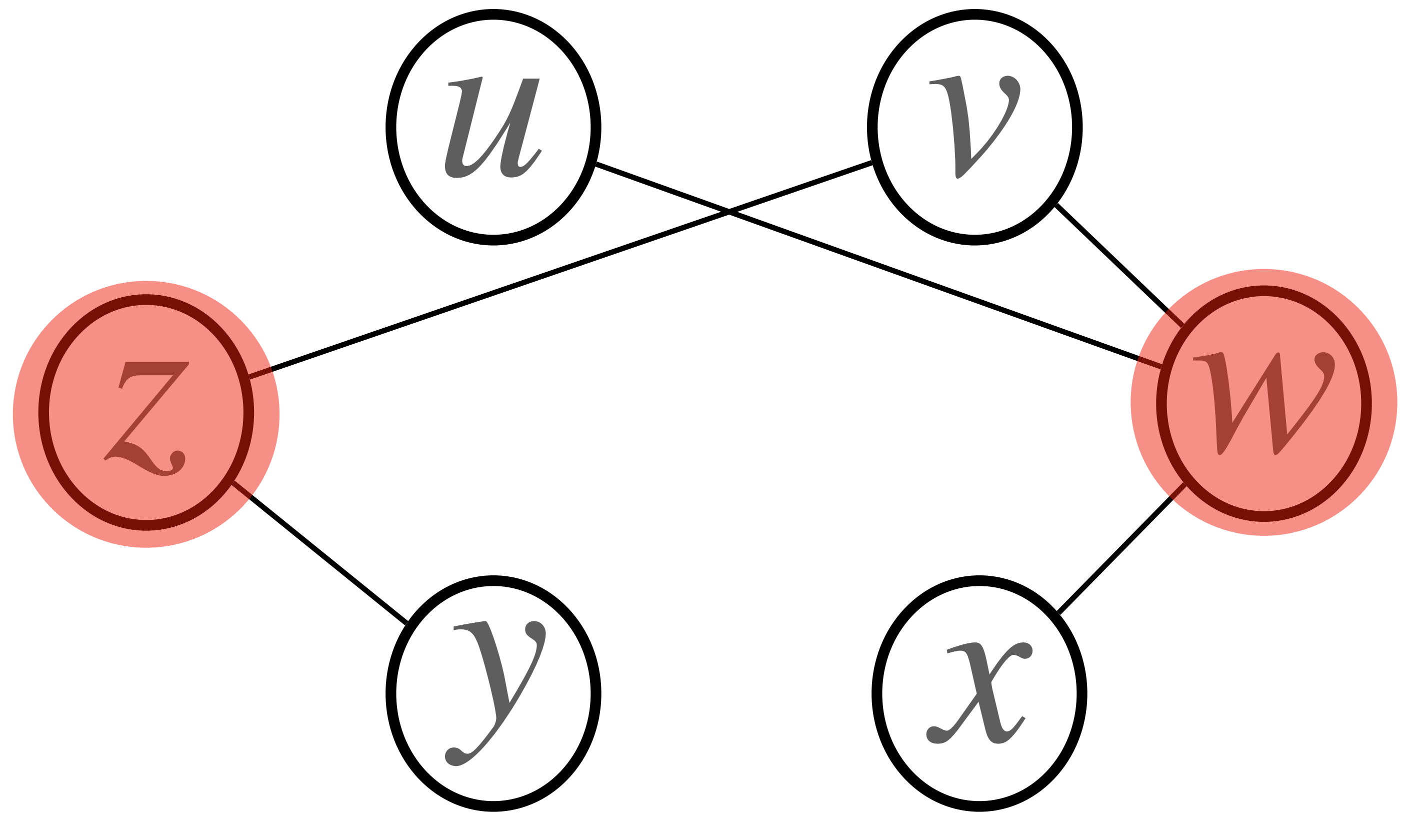


$k' = |V| - k$

=6-4=2

"YES" for Clique Problem.

Why?

Assume "YES" for Vertex Cover Problem.

Clique Problem:

Input: An undirected graph G=(V,E).
A positive integer k.

Question: Does G have a clique of size k?

Vertex Cover Problem

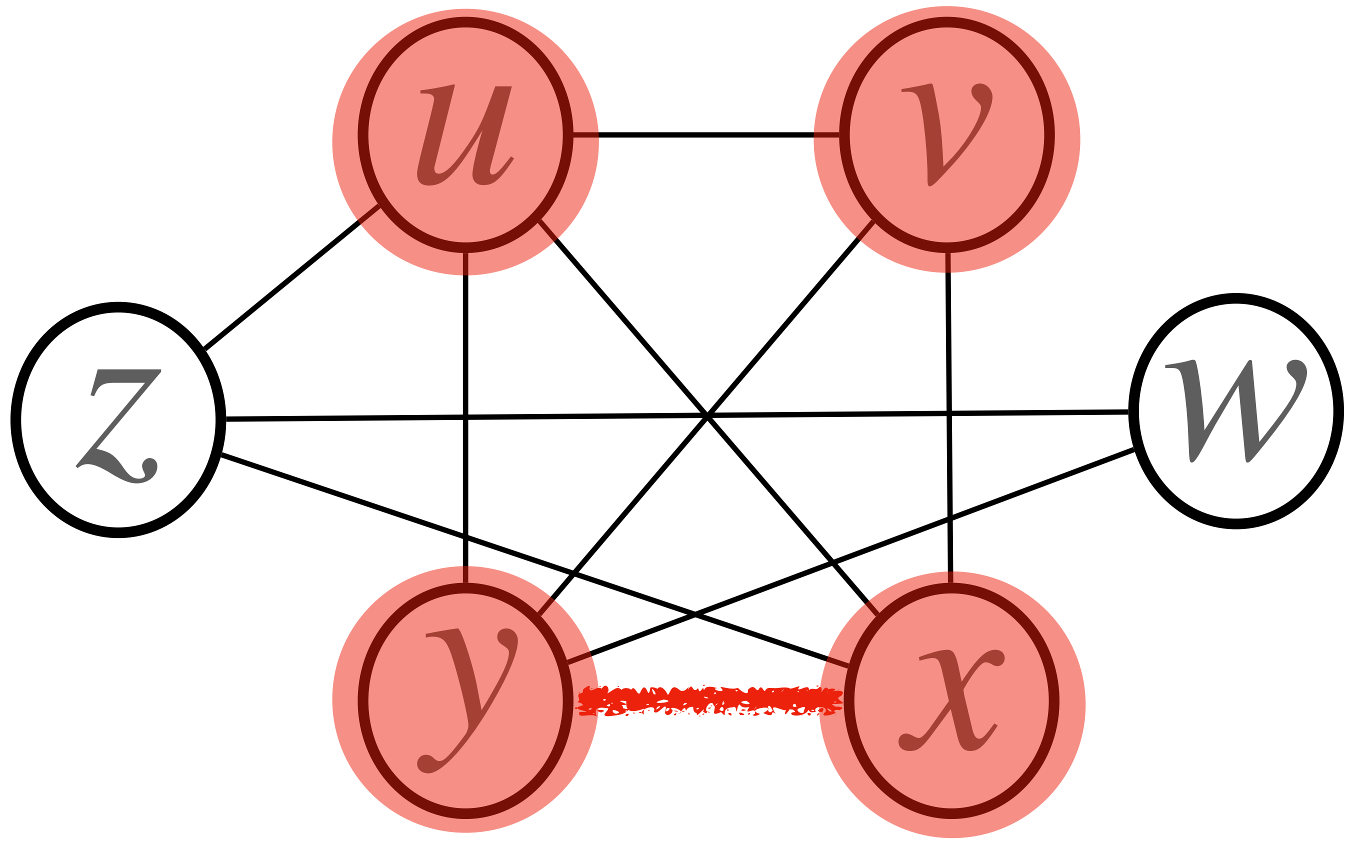
Input: An undirected graph $G' = (V', E')$.
An integer k'.

Question: Does G' have a vertex cover of size k'?

Example of instance:

$G = (V, E)$

$k = 4$



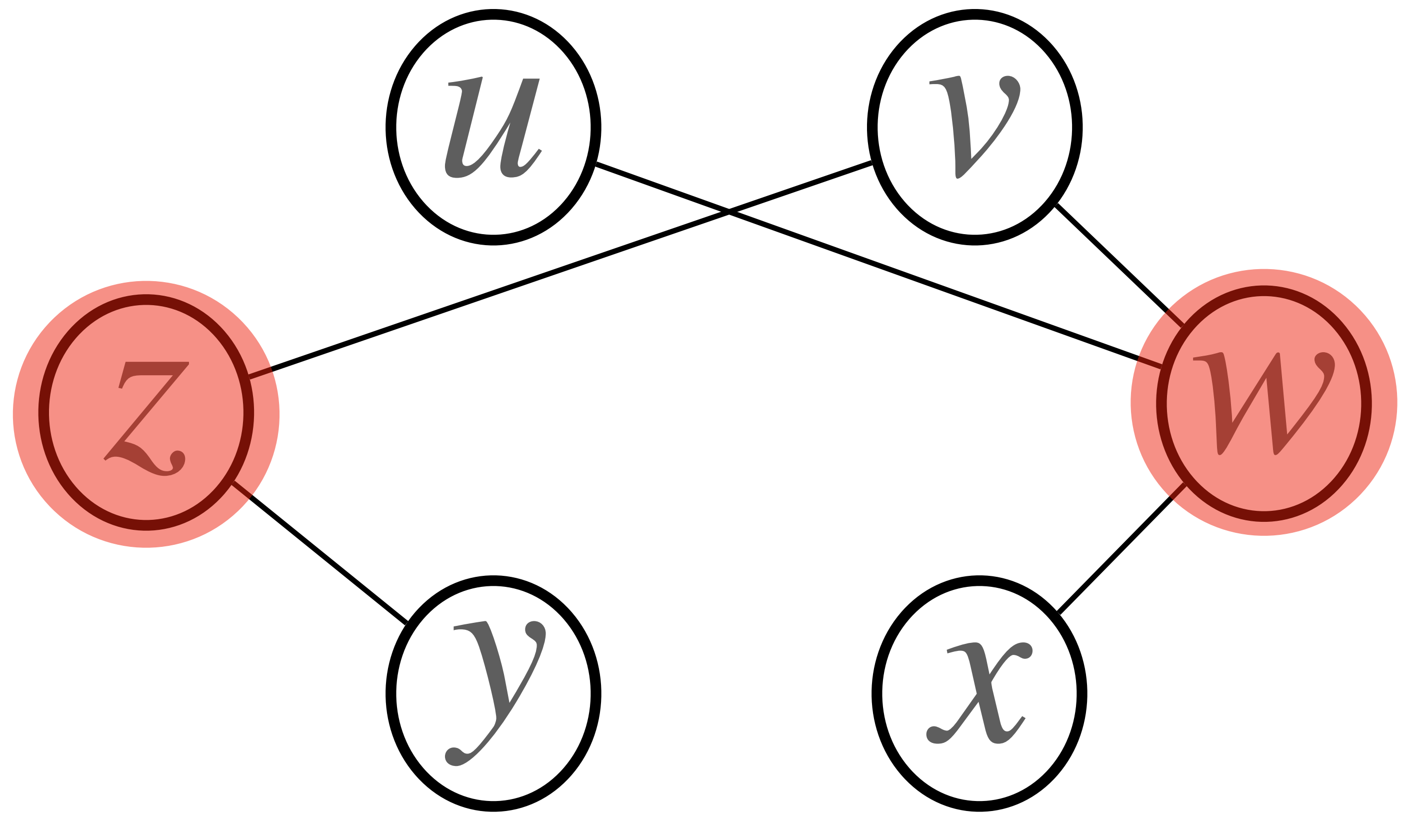
There must be an edge between every two clique vertices.

"YES" for Clique Problem.

Why?

Corresponding instance:

$G' = (V', E')$. Complement graph of G.



$k' = |V| - k$

=6-4=2

Assume "YES" for Vertex Cover Problem.

Clique Problem:

Input: An undirected graph G=(V,E).
A positive integer k.

Question: Does G have a clique of size k?

Vertex Cover Problem

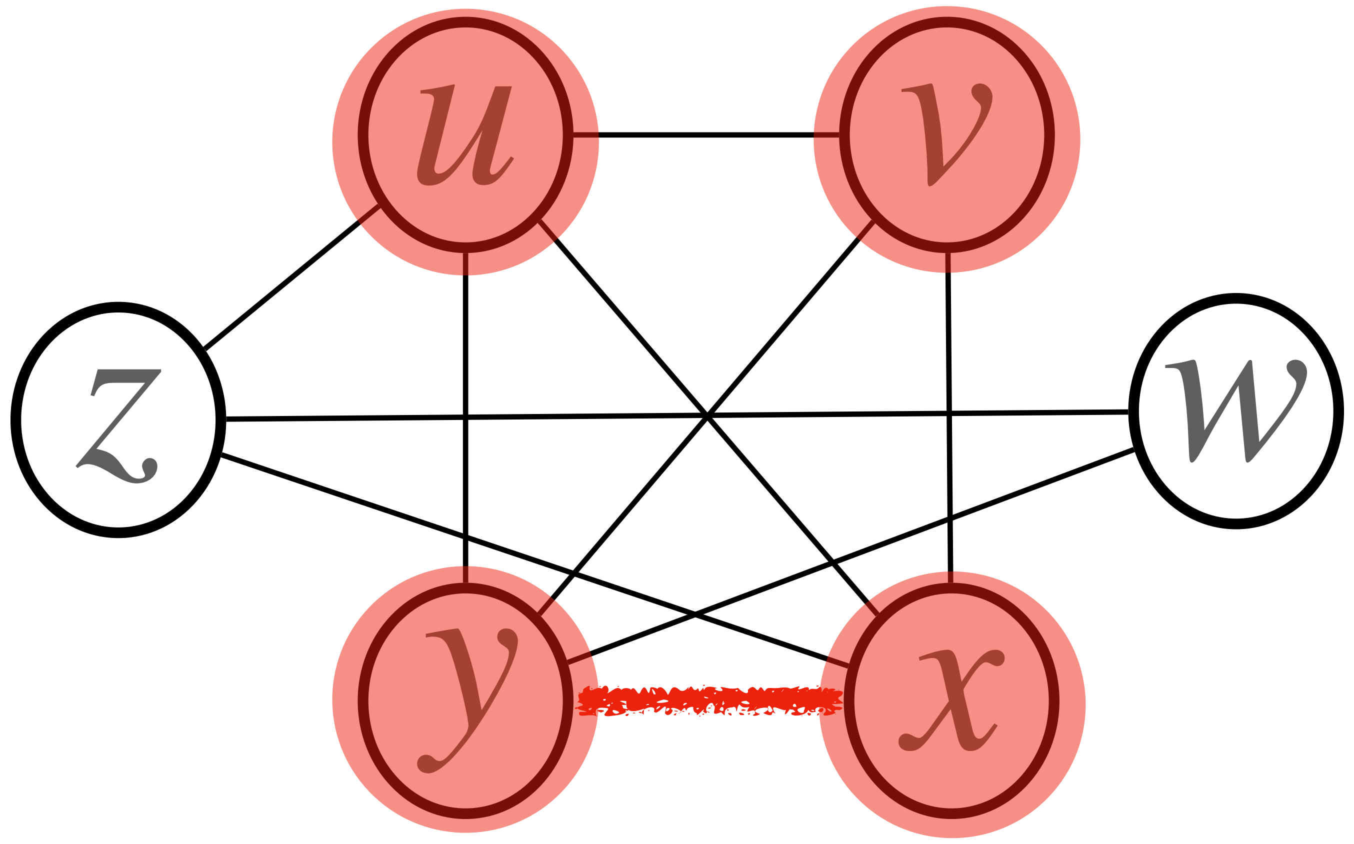
Input: An undirected graph $G' = (V', E')$.
An integer k'.

Question: Does G' have a vertex cover of size k'?

Example of instance:

$G = (V, E)$

$k = 4$

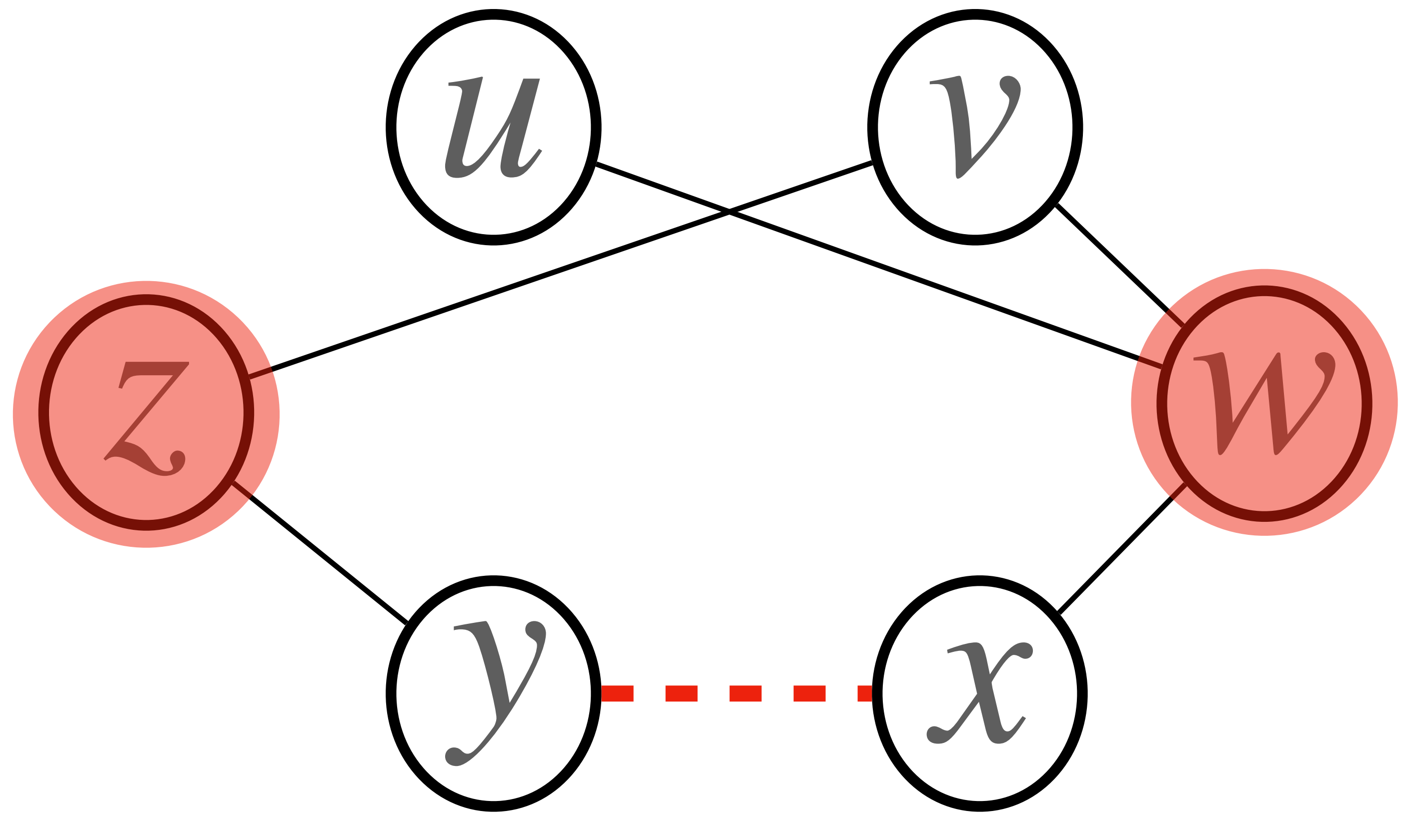


There must be an edge between every two clique vertices.

"YES" for Clique Problem.

Corresponding instance:

$G' = (V', E')$. Complement graph of G.



$k' = |V| - k$

=6-4=2

Otherwise this edge will not be covered.

Assume "YES" for Vertex Cover Problem.

Clique Problem:

Input: An undirected graph G=(V,E).

A positive integer k.

Question: Does G have a clique of size k?

Vertex Cover Problem

Input: An undirected graph $G' = (V', E')$.

An integer k'.

Question: Does G' have a vertex cover of size k'?

Clique Problem $\leq_p$ Vertex Cover Problem

Theorem: Vertex Cover Problem $\in NPC$.

Quiz questions:

1. What is the main idea for proving the NP-completeness of the “Vertex Cover Problem”?

2. What is a “complement graph”?