Algorithms

Lecture Topic: Amortized Analysis

Anxiao (Andrew) Jiang

Roadmap of this lecture:

1. Amortized analysis by the "Potential Method" technique.

   1.1 Define "Potential Method".

   1.2 Understand "Potential Method" through the example of "Stack Operations".

   1.3 Understand "Potential Method" through the example of "Counter Incrementation".

Technique 3: Potential Method

Starting with an initial data structure D_0, a sequence of n operations occurs.

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A potential function Φ maps each data structure D_i to a real number $\Phi(D_i)$, which is the potential associated with D_i.

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Consider the cost of n operations:

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} c_i + \Phi(D_i) - \Phi(D_{i-1}) = \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0)$$

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Technique 3: Potential Method

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So if $\Phi(D_n) \geq \Phi(D_0)$, then

$$\sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i$$

Quiz question:

1. How is the "Potential Method" different from the "Accounting Method"?

2. What property does the "potential function" need to have?

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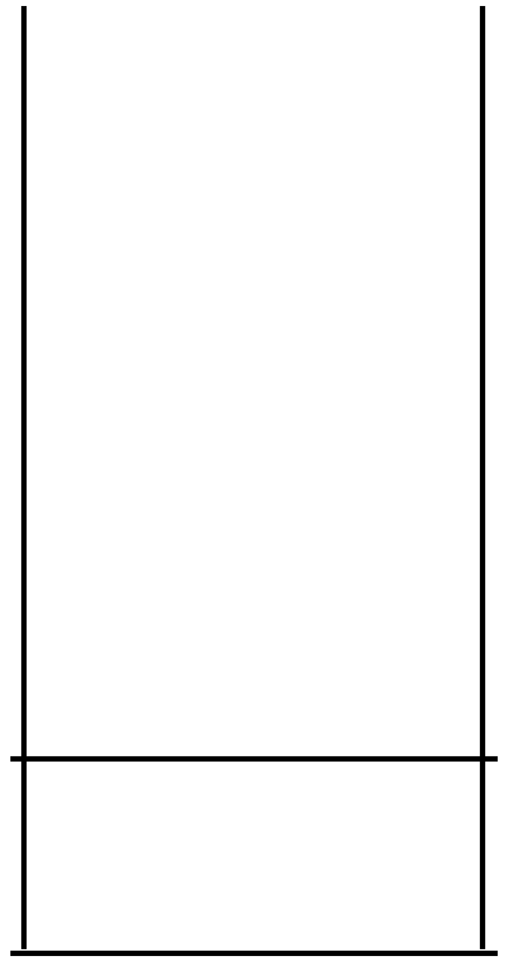
Technique 3: Potential Method

Example: Stack Operations



Size of Stack: $|S| = 0$

Consider a sequence of n stack operations.
What is a tight upper bound on its total cost ?

Operations:

1) PUSH: push a number into stack

   Real cost: 1

2) POP(k): pop out the top k numbers from stack. If the stack has fewer than k numbers, we pop out all numbers in stack.

   Real cost: $\min\{k, |S|\}$

Technique 3: Potential Method

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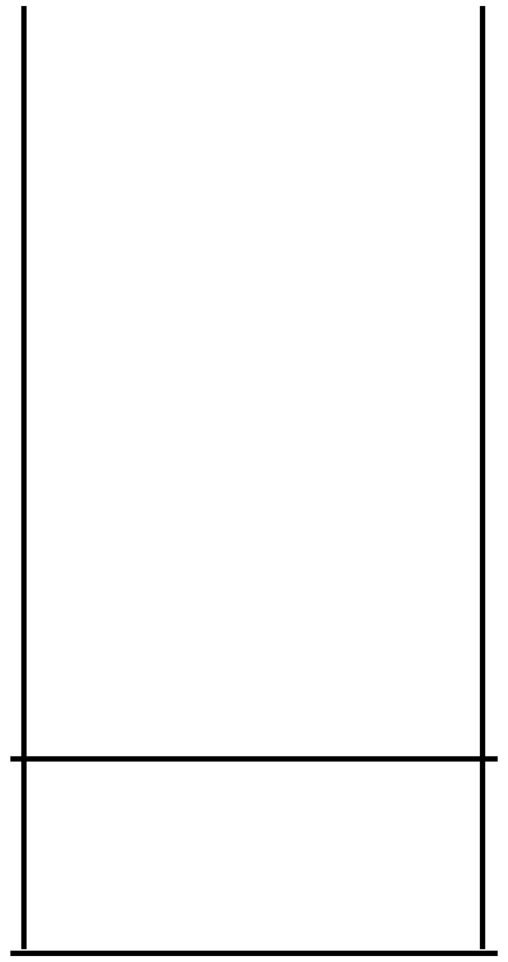

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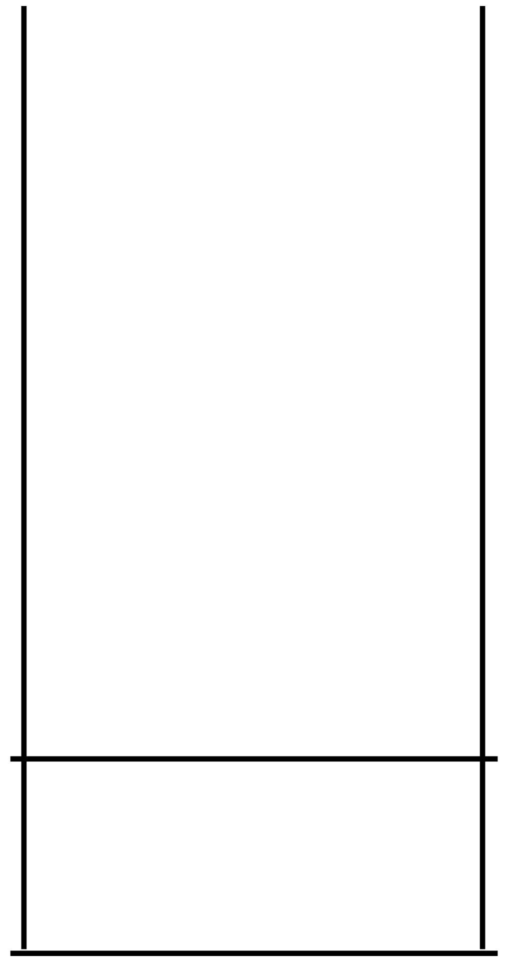

Size of Stack: $|S| = 0$

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$\Phi_i \geq \Phi_0 = 0$

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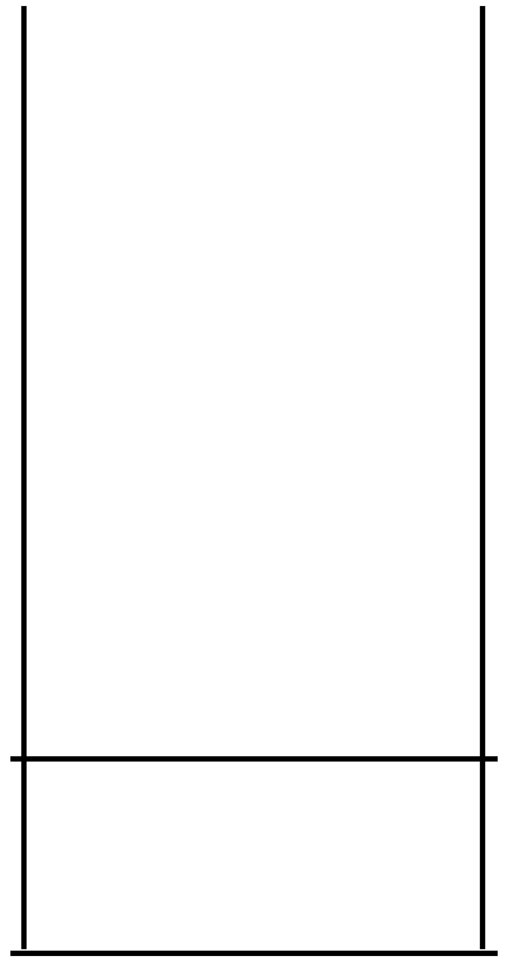

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Consider a sequence of n stack operations.
What is a tight upper bound on its total cost ?

Operations:

1) PUSH: push a number into stack

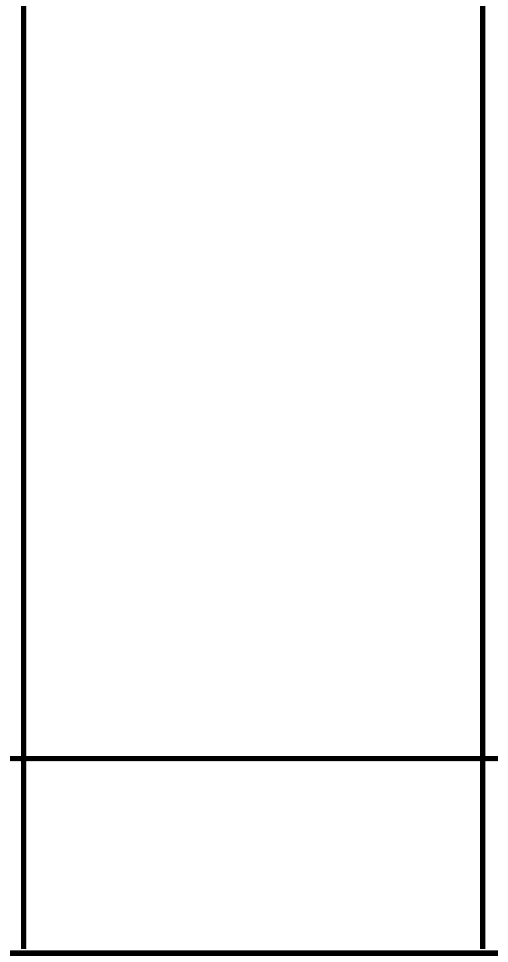
Real cost: 1 Amortized cost: 2

2) POP(k): pop out the top k numbers from stack. If the stack has fewer than k numbers, we pop out all numbers in stack.

Real cost: $\min\{k, |S|\}$

Technique 3: Potential Method

Example: Stack Operations



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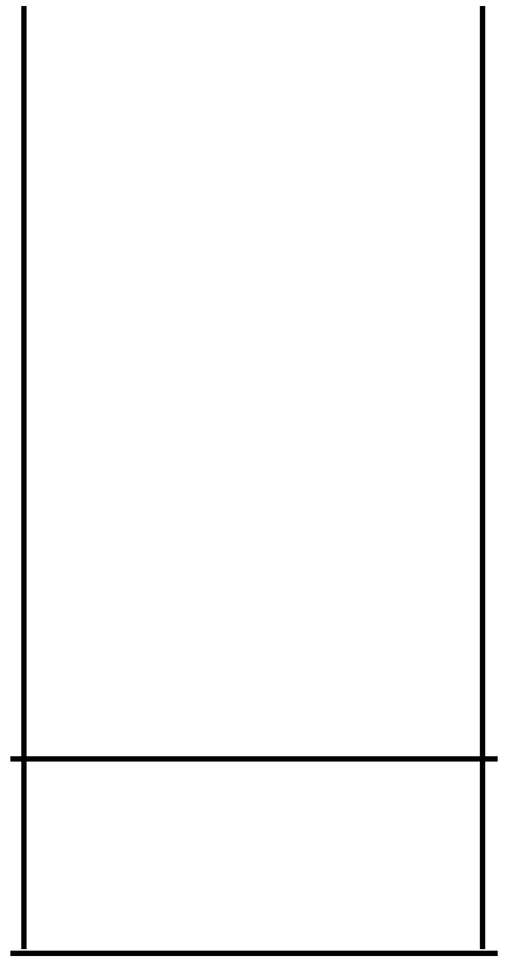

Size of Stack: $|S| = 0$

Φ_i : number of objects in the stack after the i-th operation

$\Phi_i \geq \Phi_0 = 0$

$$\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i \leq 2n \qquad O(n)$$

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Real cost: $\min\{k, |S|\}$

Amortized cost: 0

Quiz question:

1. What was the "potential function" defined in the above example of "stack operations"?

2. Why can the above potential function help us analyze the total cost?

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Large Binary Counter

0 0 0 0 0 0 0 0 0 0 0

cost: 1

1 0 0 0 0 0 0 0 0 0 1

cost: 2

2 0 0 0 0 0 0 0 0 1 0

cost: 1

3 0 0 0 0 0 0 0 0 1 1

cost: 3

4 0 0 0 0 0 0 0 1 0 0

cost: 1

5 0 0 0 0 0 0 0 1 0 1

cost: 2

6 0 0 0 0 0 0 0 1 1 0

cost: 1

7 0 0 0 0 0 0 0 1 1 1

cost: 4

8 0 0 0 0 0 0 1 0 0 0

cost: 1

9 0 0 0 0 0 0 1 0 0 1

cost: 2

10 0 0 0 0 0 0 1 0 1 0

cost: 1

11 0 0 0 0 0 0 1 0 1 1

Cost of incrementing counter:

Number of bits that are changed.

What is the total cost of n increments?

Technique 3: Potential Method

Φ_i : number of 1s in the counter after the i-th operation

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0	
		cost: 1
1	0 0 0 0 0 0 0 0 0 1	
		cost: 2
2	0 0 0 0 0 0 0 0 1 0	
		cost: 1
3	0 0 0 0 0 0 0 0 1 1	
		cost: 3
4	0 0 0 0 0 0 0 1 0 0	
		cost: 1
5	0 0 0 0 0 0 0 1 0 1	
		cost: 2
6	0 0 0 0 0 0 0 1 1 0	
		cost: 1
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Consider the i-th operation:

? ? ?⋯0 1 1 ⋯1

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? ? ?⋯1 0 0 ⋯0

Large Binary Counter

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Consider the i-th operation:

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$\Downarrow$

? ? ?⋯1 0 0 ⋯0

Amortized cost: 2

Large Binary Counter

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↓

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2. Why can the above potential function help us analyze the total cost?