Algorithms

Lecture Topic: Linear Programming (Part 1)

Anxiao (Andrew) Jiang

Roadmap of this lecture:

1. Linear Programming (LP)

   1.1 Define "Linear Programming".

   1.2 A basic example of linear programming.

   1.3 Standard-Form LP and Slack-Form LP.

Linear Programming

maximize $3x_1 + x_2 + 2x_3$

subject to:

$$
\begin{aligned}
x_1 + x_2 + 3x_3 &\le 30 \\
2x_1 + 2x_2 + 5x_3 &\le 24 \\
4x_1 + x_2 + 2x_3 &\le 36
\end{aligned}
$$

$$x_1,\ x_2,\ x_3 \ge 0$$

Linear Programming

maximize $\quad 3x_1 + x_2 + 2x_3$ Linear Objective function

subject to:

$$\boxed{\begin{aligned} x_1 + x_2 + 3x_3 &\le 30 \\ 2x_1 + 2x_2 + 5x_3 &\le 24 \\ 4x_1 + x_2 + 2x_3 &\le 36 \\ x_1,\ x_2,\ x_3 &\ge 0 \end{aligned}}$$

Linear Constraints

Variables: $x_1,\ x_2,\ x_3$

Linear Programming

minimize

maximize $\quad 3x_1 + x_2 + 2x_3$

Linear
Objective function

subject to:

$$\boxed{\begin{aligned} x_1 + \quad x_2 + 3x_3 &\leq 30 \\ 2x_1 + 2x_2 + 5x_3 &\leq 24 \\ 4x_1 + \quad x_2 + 2x_3 &\leq 36 \\ x_1,\ x_2,\ x_3 &\geq 0 \end{aligned}}$$

Linear
Constraints

Variables: $\quad x_1,\ x_2,\ x_3$

Linear Programming

minimize

maximize $3x_1 + x_2 + 2x_3$

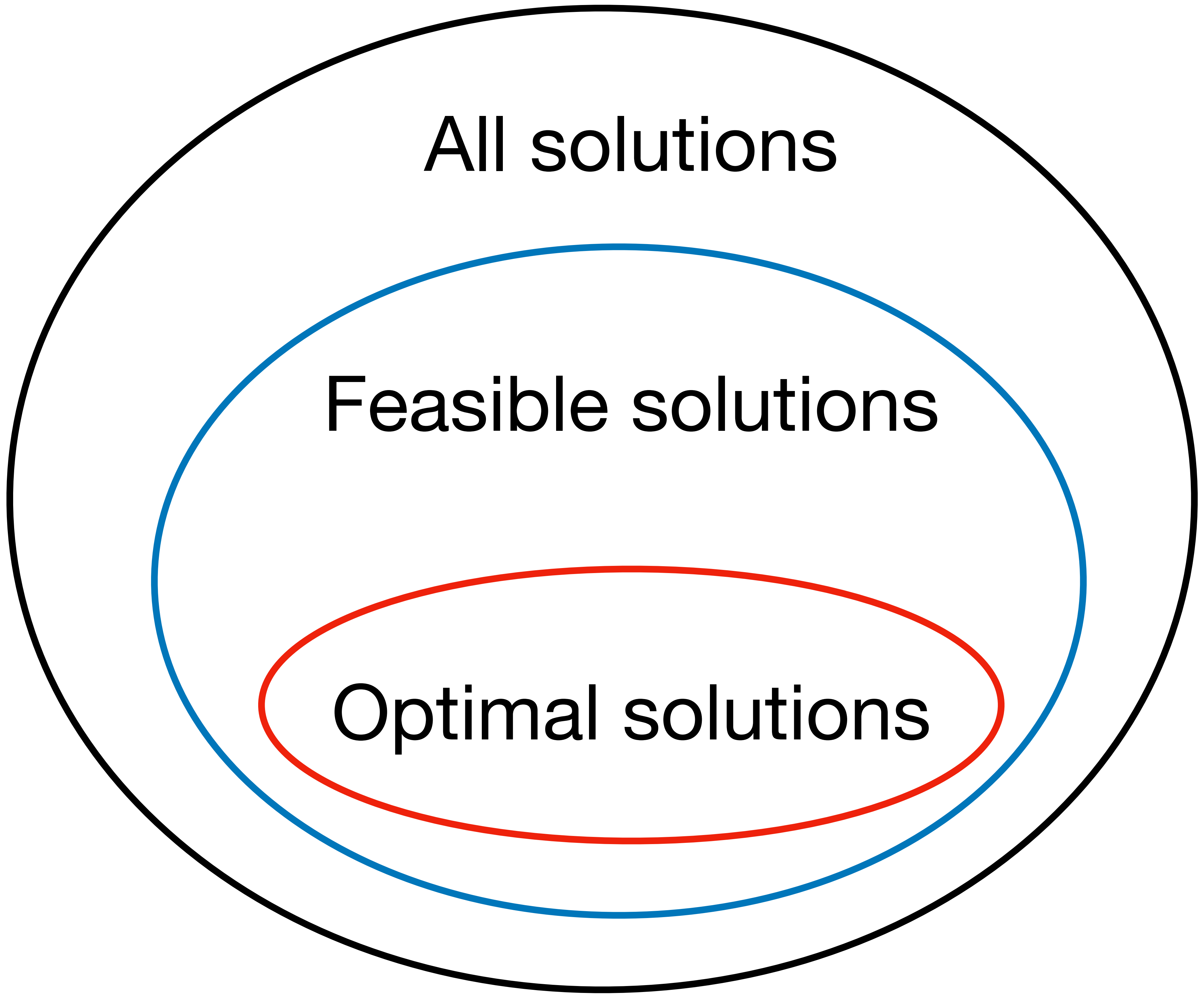
Linear Objective function

subject to:

$$\boxed{\begin{aligned} x_1 + x_2 + 3x_3 &\leq 30 \\ 2x_1 + 2x_2 + 5x_3 &\leq 24 \\ 4x_1 + x_2 + 2x_3 &\leq 36 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}}$$

Linear Constraints

Variables: x_1, x_2, x_3



Quiz questions:

1. What is a linear program?

2. What are the applications of linear programming?

Roadmap of this lecture:

1. Linear Programming (LP)

   1.1 Define "Linear Programming".

   1.2 A basic example of linear programming.

   1.3 Standard-Form LP and Slack-Form LP.

Linear Programming

maximize $\quad x_1 + x_2$

s.t.

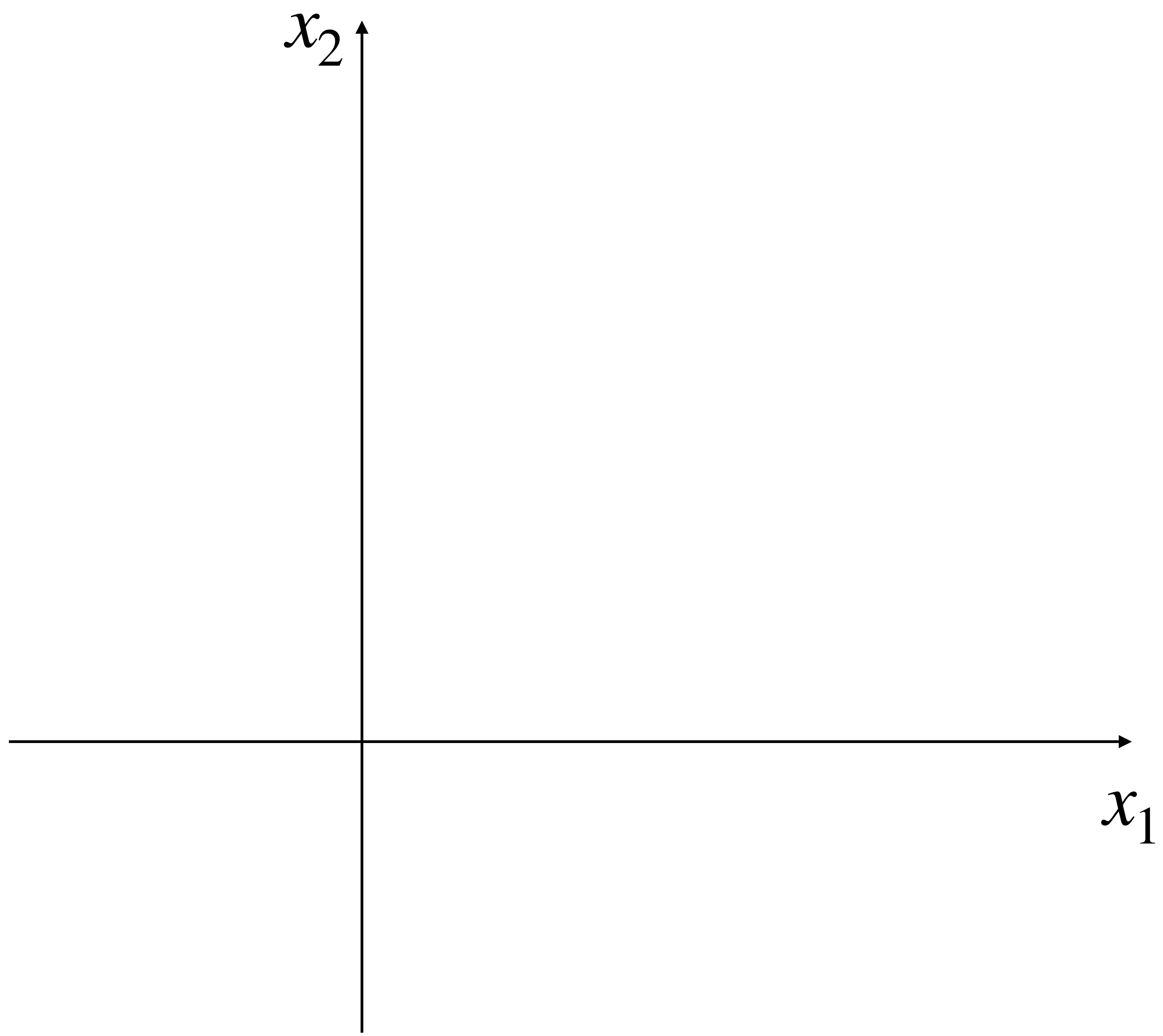
$$
\begin{aligned}
4x_1 - \ \ x_2 &\leq 8 \\
2x_1 + \ \ x_2 &\leq 10 \\
5x_1 - 2x_2 &\geq -2 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Linear Programming

$$\text{maximize} \quad x_1 + x_2$$

s.t.

$$\begin{aligned} 4x_1 - \ x_2 &\le 8 \\ 2x_1 + \ x_2 &\le 10 \\ 5x_1 - 2x_2 &\ge -2 \\ x_1, x_2 &\ge 0 \end{aligned}$$

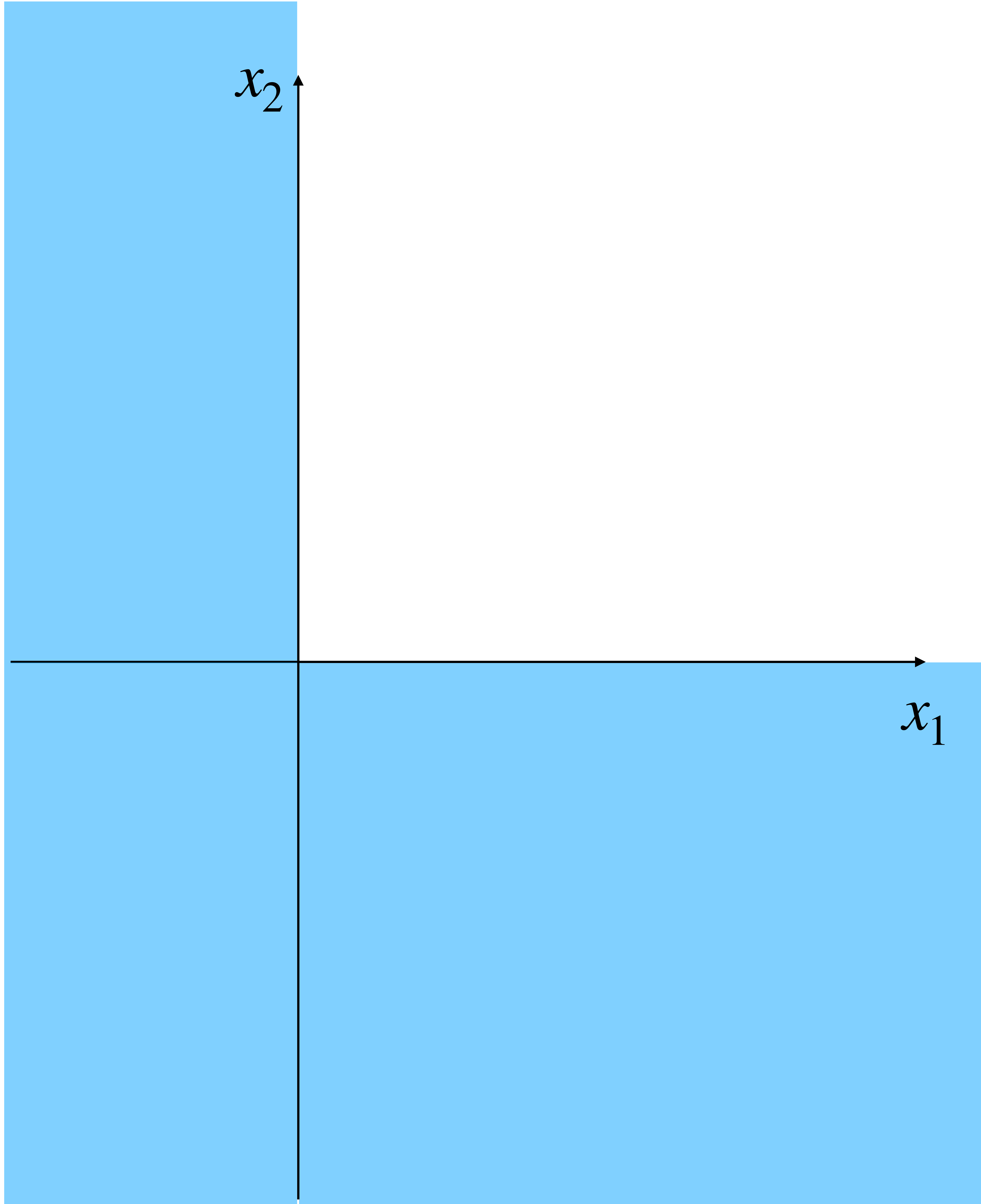


Linear Programming

maximize $\quad x_1 + x_2$

s.t.

$$
\begin{aligned}
4x_1 - x_2 &\leq 8 \\
2x_1 + x_2 &\leq 10 \\
5x_1 - 2x_2 &\geq -2 \\
\boxed{x_1, x_2 \geq 0}&
\end{aligned}
$$

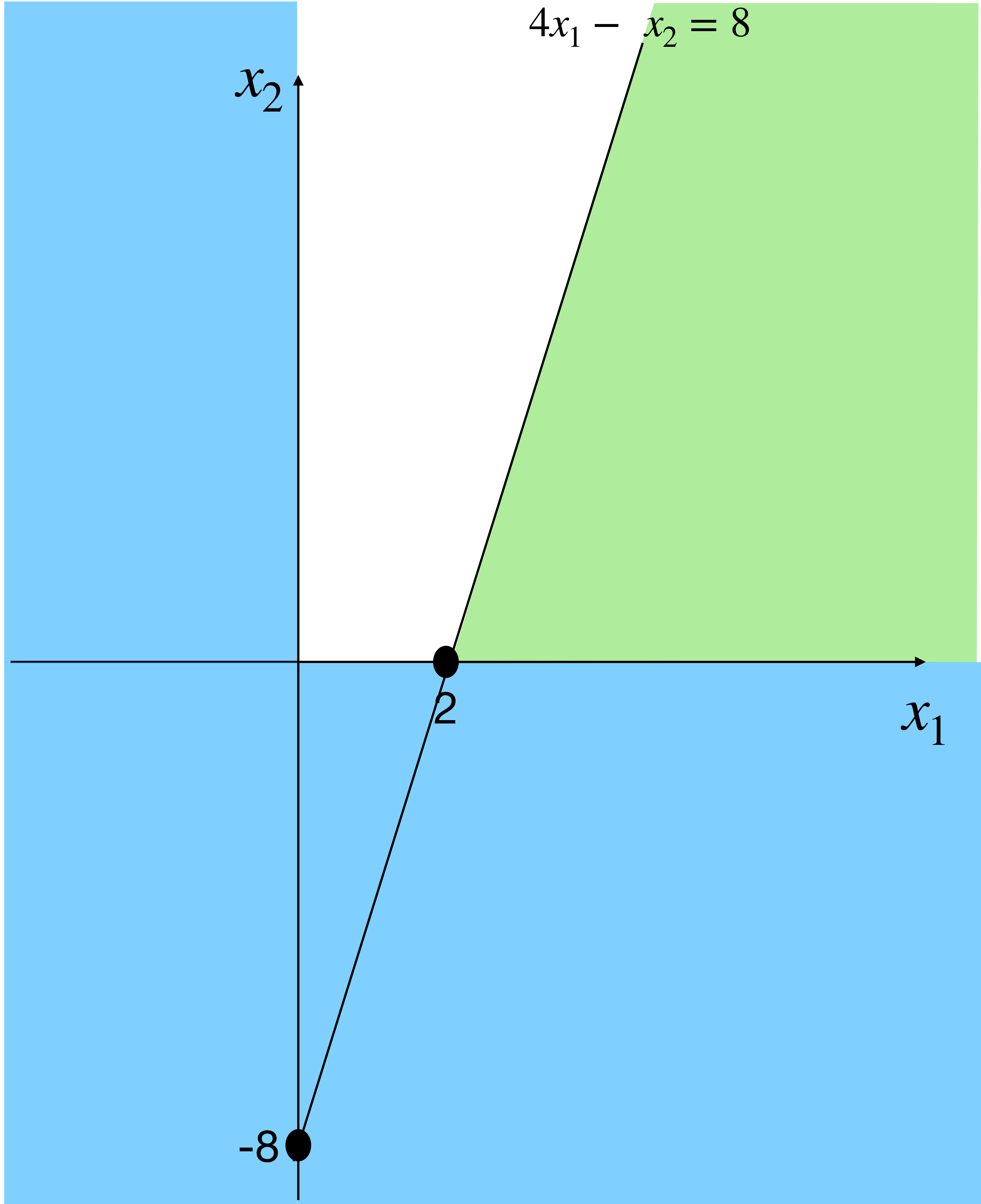


Linear Programming

maximize $\quad x_1 + x_2$

s.t.

$$4x_1 - x_2 \leq 8$$
$$2x_1 + x_2 \leq 10$$
$$5x_1 - 2x_2 \geq -2$$
$$x_1, x_2 \geq 0$$

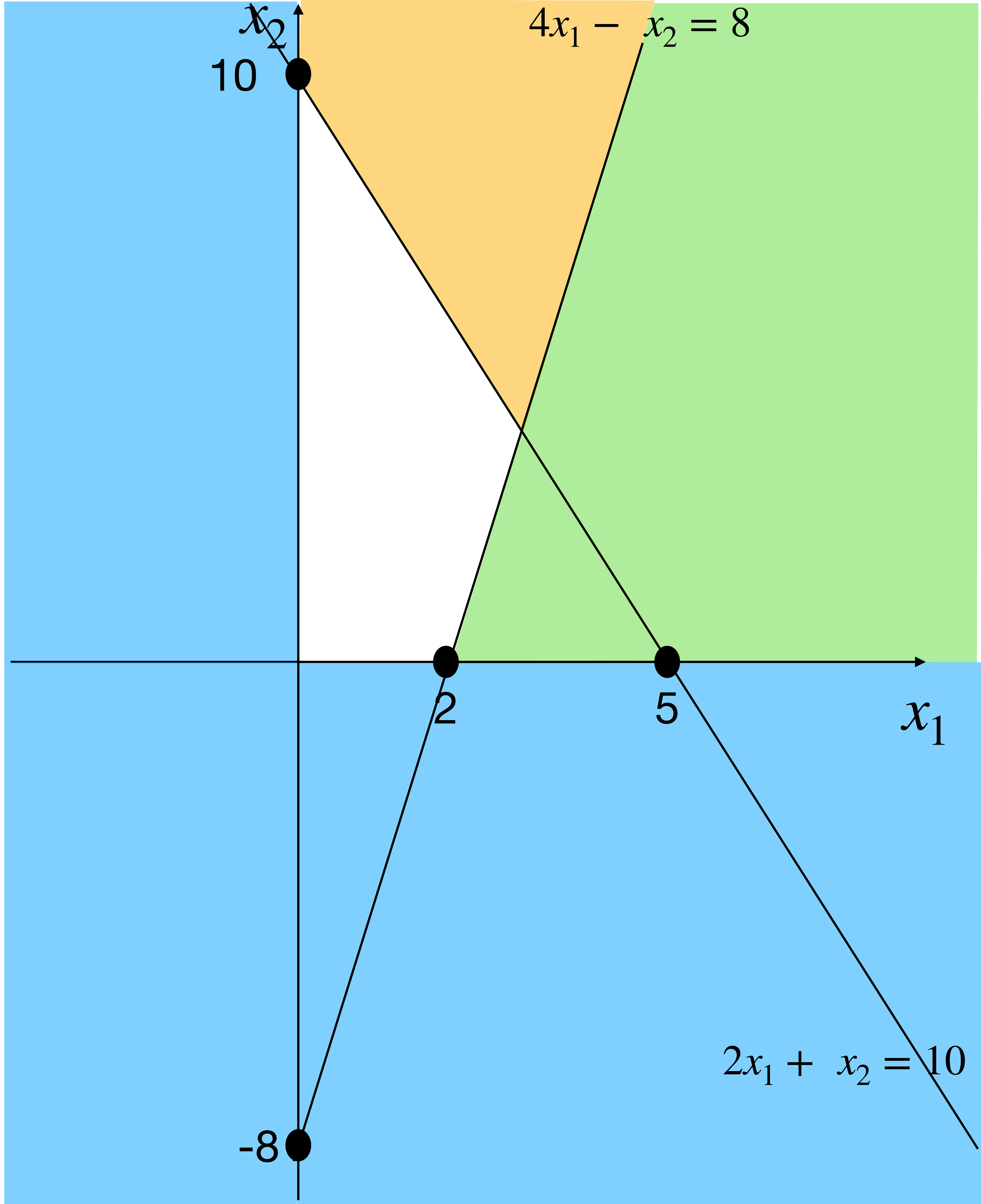


Linear Programming

maximize $x_1 + x_2$

s.t.

$$4x_1 - x_2 \leq 8$$
$$2x_1 + x_2 \leq 10$$
$$5x_1 - 2x_2 \geq -2$$
$$x_1, x_2 \geq 0$$

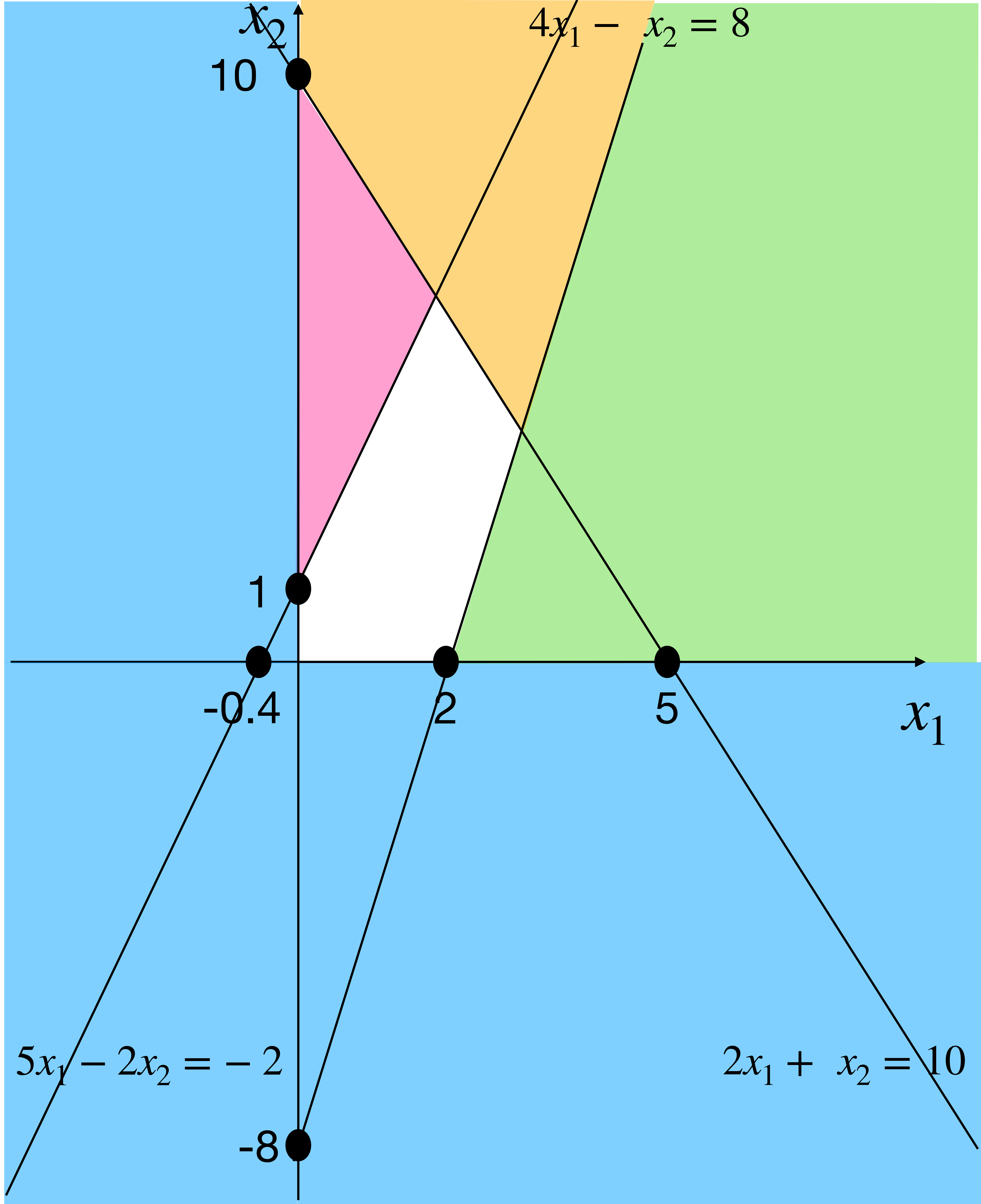


Linear Programming

maximize $\quad x_1 + x_2$

s.t.

$$4x_1 - x_2 \leq 8$$
$$2x_1 + x_2 \leq 10$$
$$5x_1 - 2x_2 \geq -2$$
$$x_1, x_2 \geq 0$$

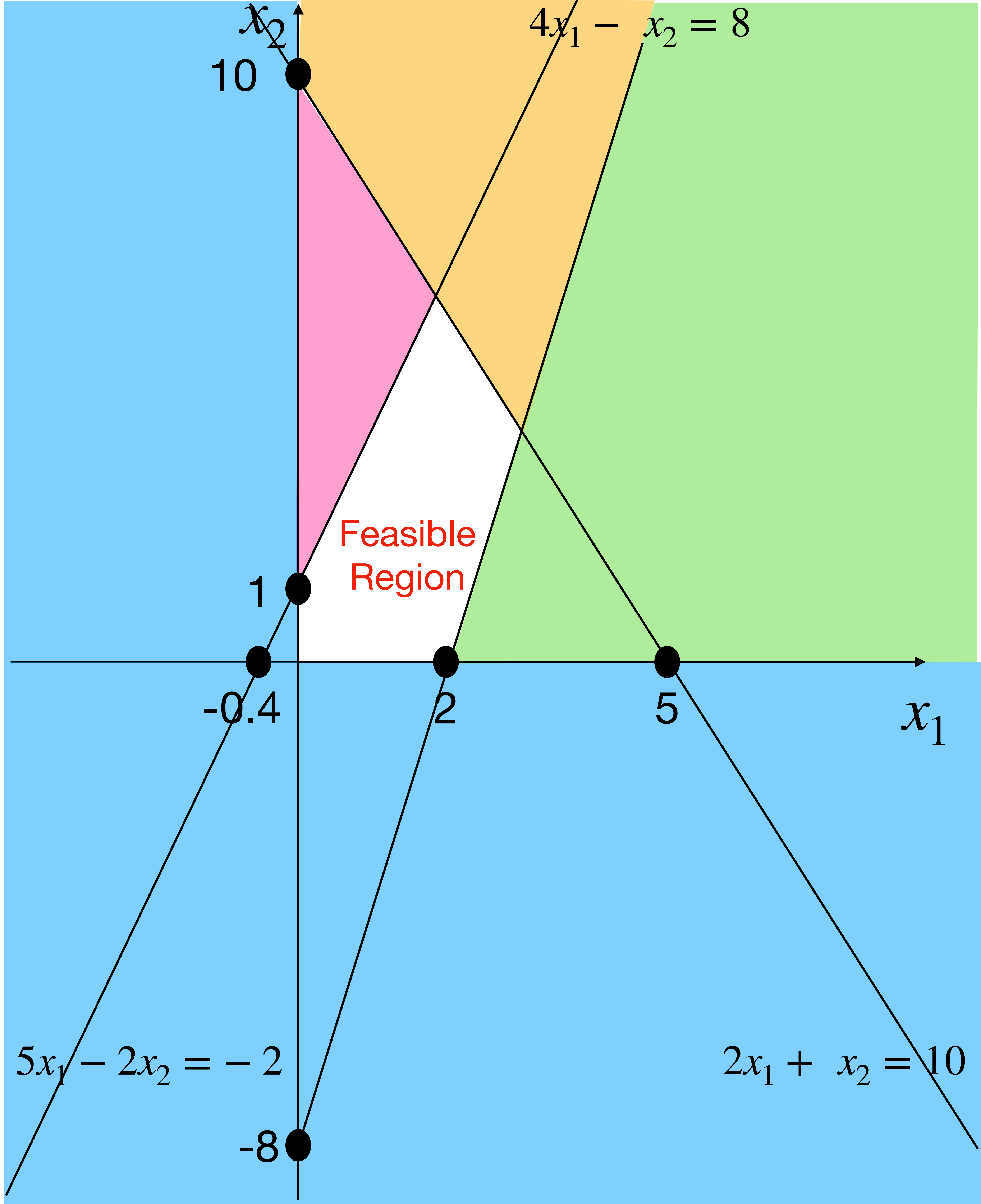


Linear Programming

maximize $\quad x_1 + x_2$

s.t.

$$4x_1 - x_2 \leq 8$$
$$2x_1 + x_2 \leq 10$$
$$5x_1 - 2x_2 \geq -2$$
$$x_1, x_2 \geq 0$$

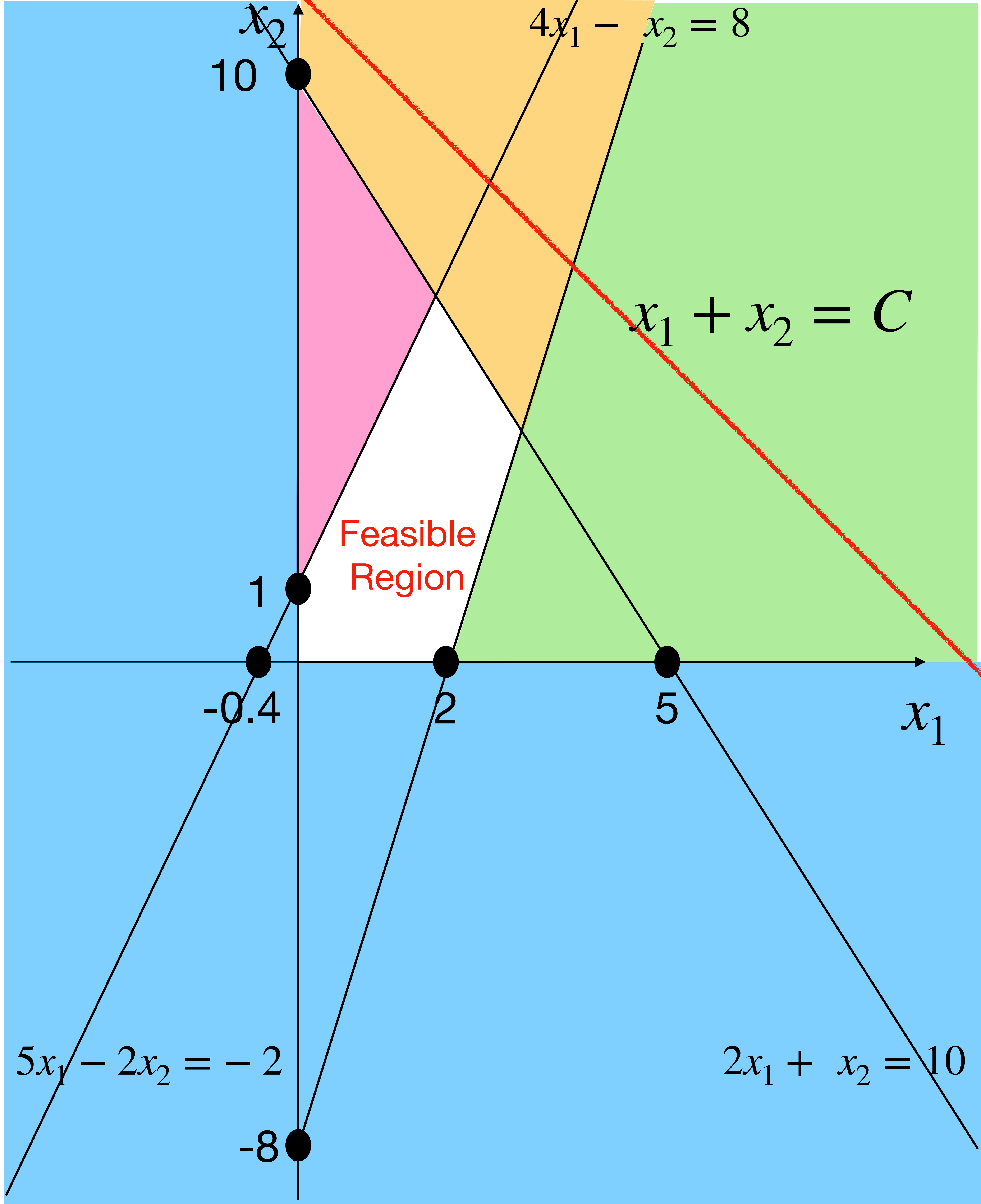


Linear Programming

maximize $x_1 + x_2$

s.t.

$$4x_1 - x_2 \leq 8$$
$$2x_1 + x_2 \leq 10$$
$$5x_1 - 2x_2 \geq -2$$
$$x_1, x_2 \geq 0$$

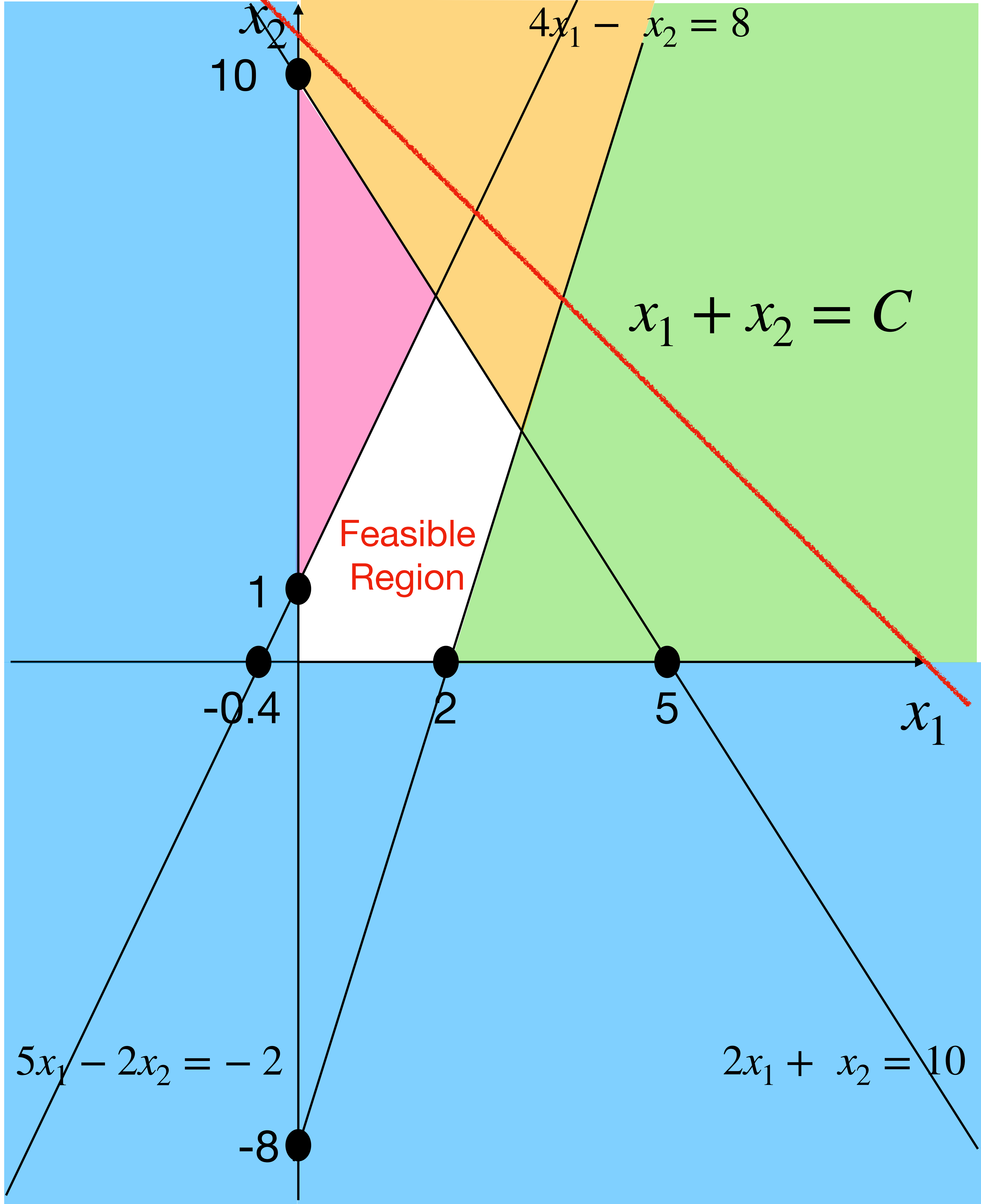


Linear Programming

maximize $x_1 + x_2$

s.t.

$$4x_1 - x_2 \leq 8$$
$$2x_1 + x_2 \leq 10$$
$$5x_1 - 2x_2 \geq -2$$
$$x_1, x_2 \geq 0$$



As we move the red line to the left,
C gets smaller.

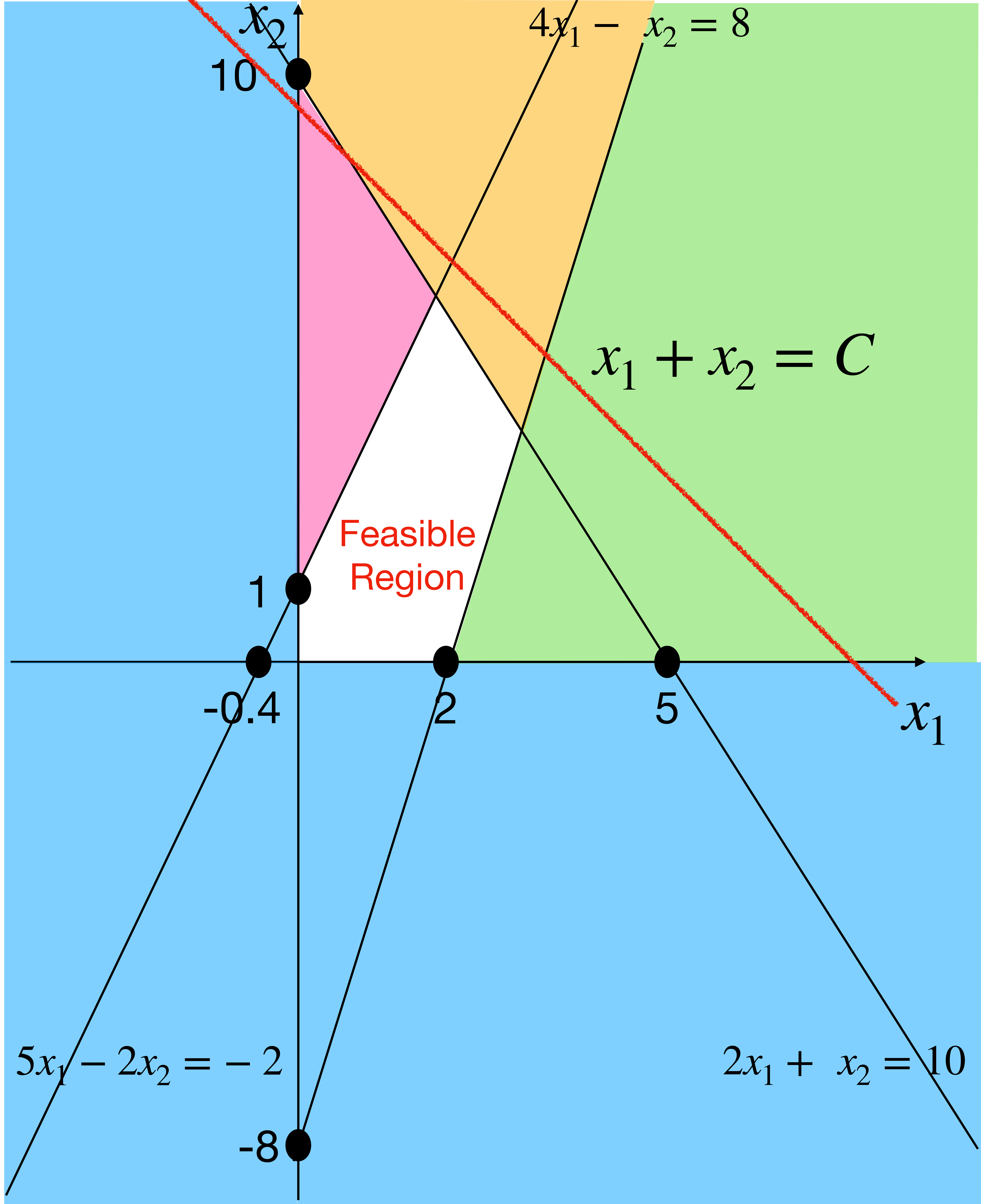
Linear Programming

maximize $x_1 + x_2$

s.t.

$$4x_1 - x_2 \leq 8$$
$$2x_1 + x_2 \leq 10$$
$$5x_1 - 2x_2 \geq -2$$
$$x_1, x_2 \geq 0$$

As we move the red line to the left,
C gets smaller.



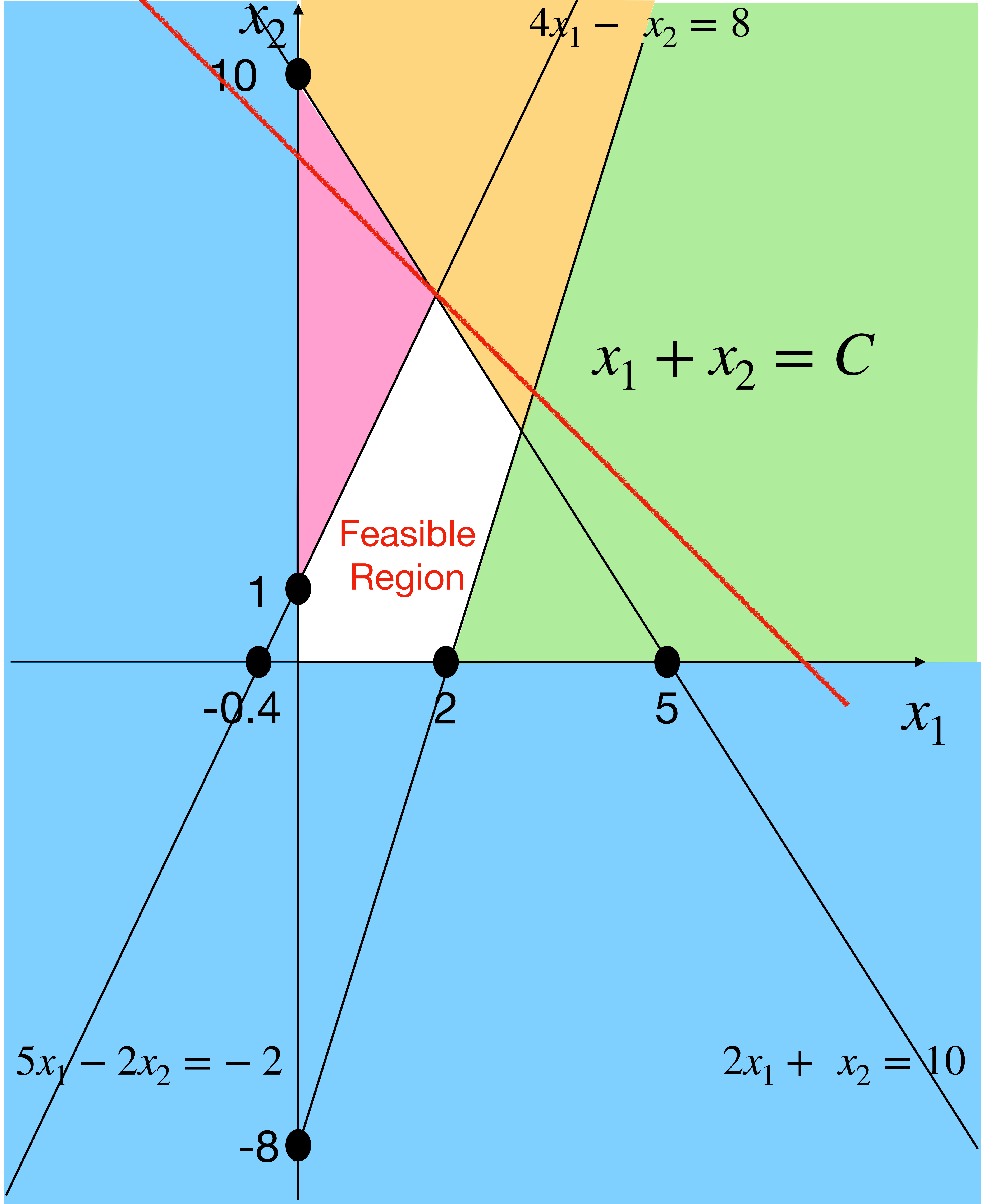
Linear Programming

maximize $x_1 + x_2$

s.t.

$$4x_1 - x_2 \leq 8$$
$$2x_1 + x_2 \leq 10$$
$$5x_1 - 2x_2 \geq -2$$
$$x_1, x_2 \geq 0$$

As the red line just touches the feasible region, we get an optimal solution.

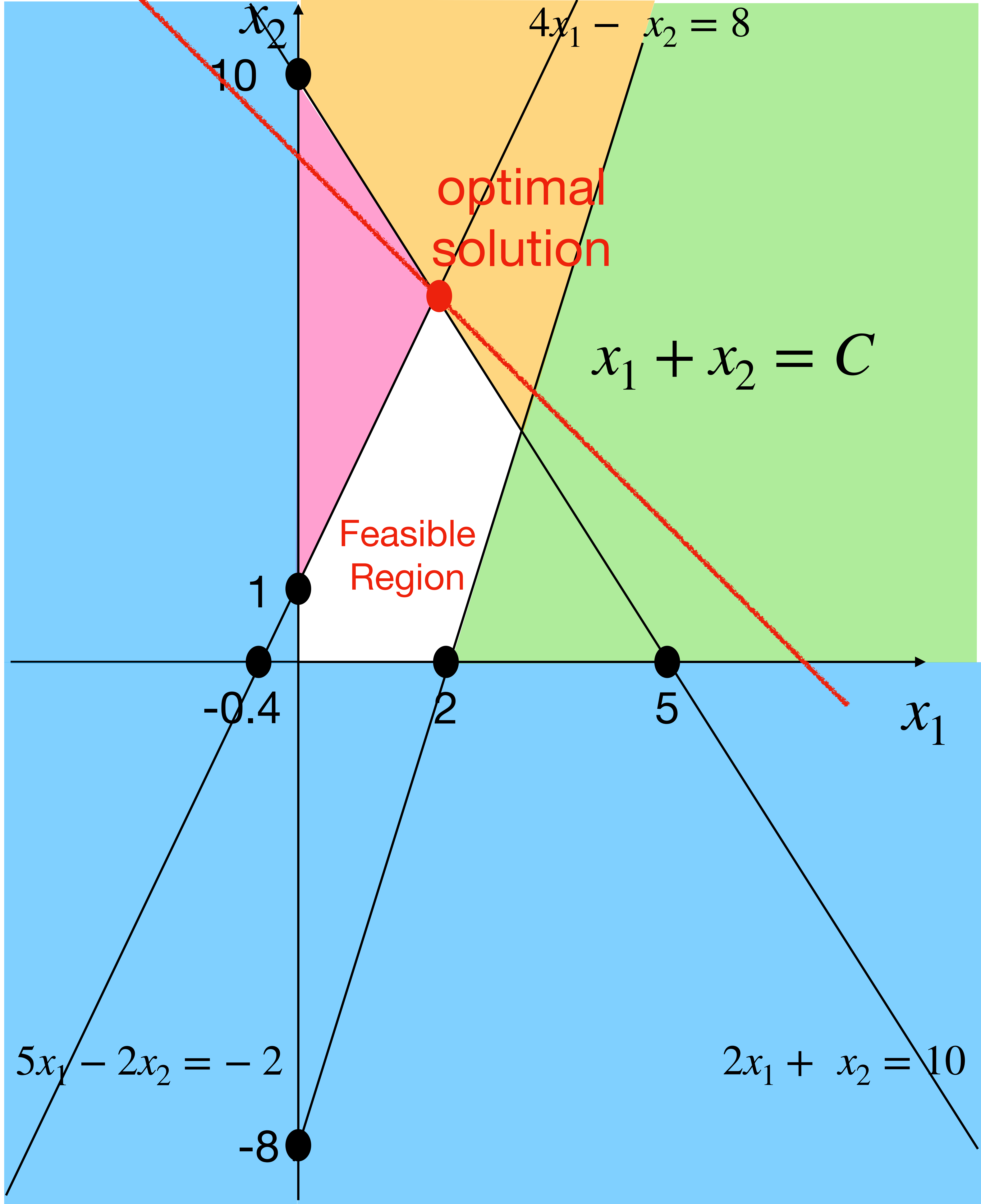


Linear Programming

maximize $x_1 + x_2$

s.t.

$$4x_1 - x_2 \leq 8$$
$$2x_1 + x_2 \leq 10$$
$$5x_1 - 2x_2 \geq -2$$
$$x_1, x_2 \geq 0$$



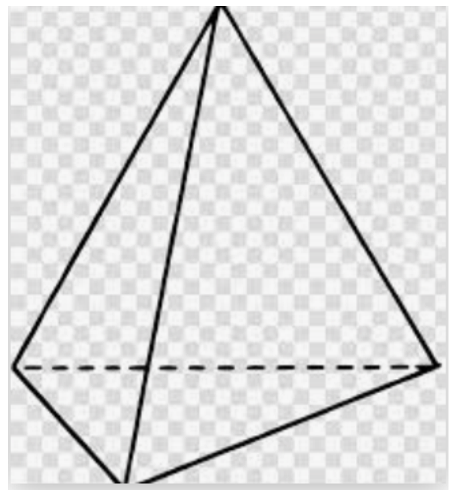
As the red line just touches the feasible region, we get an optimal solution.

The feasible region is a convex polygon. And an optimal solution is a vertex.

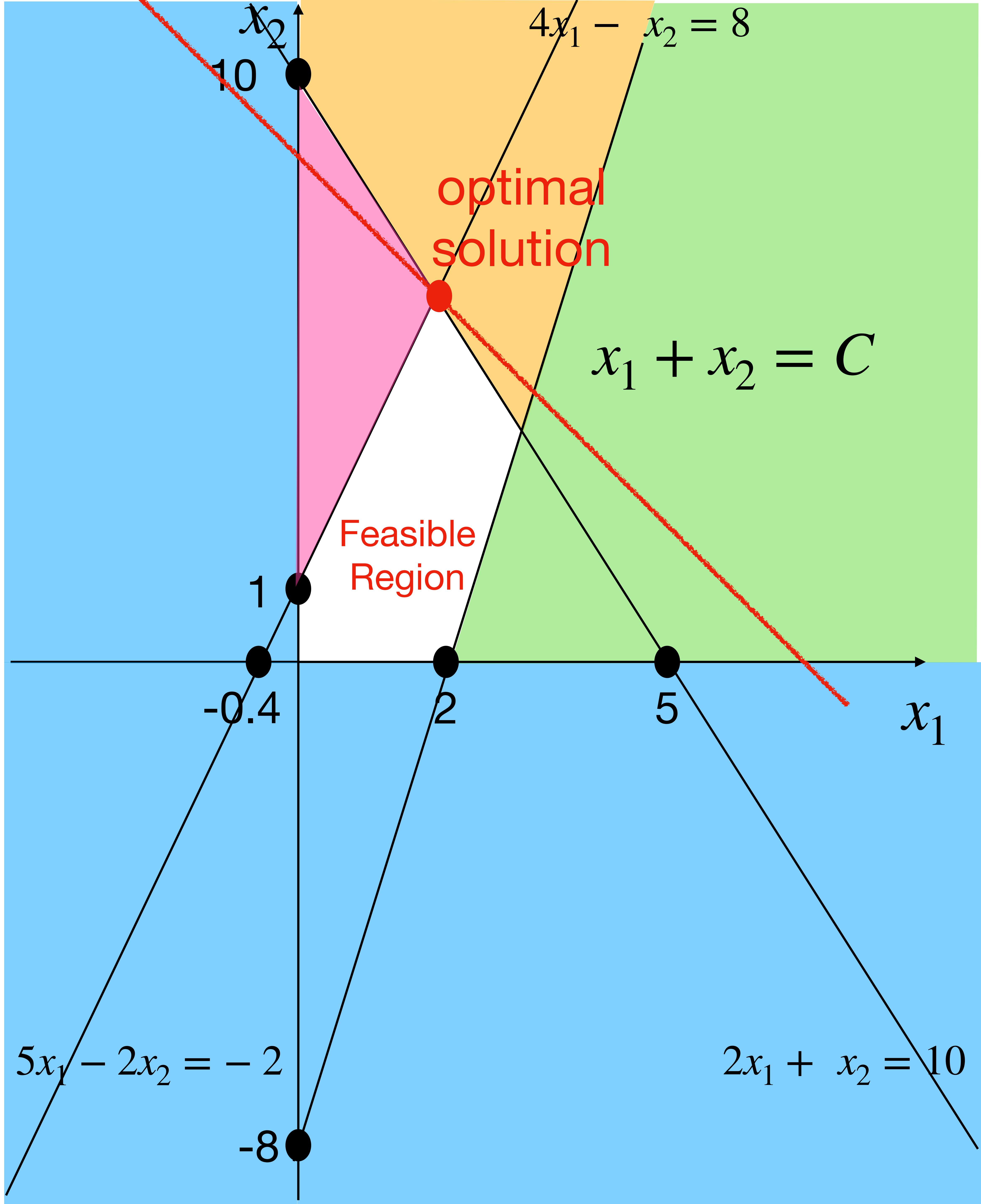
Linear Programming

The above observation can be generalized to n variables.

The feasible region is a convex n-dimensional "polygon" called a SIMPLEX.



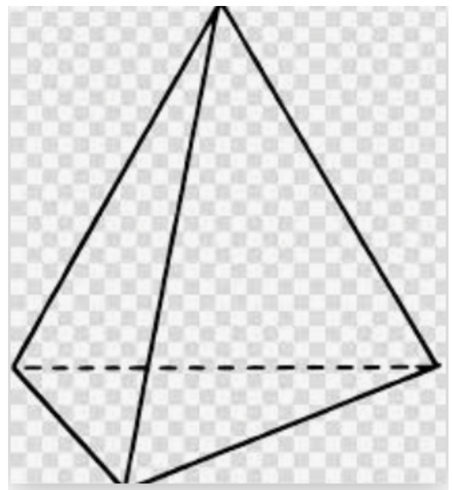
There exists an optimal solution that is a vertex of the SIMPLEX.



Linear Programming

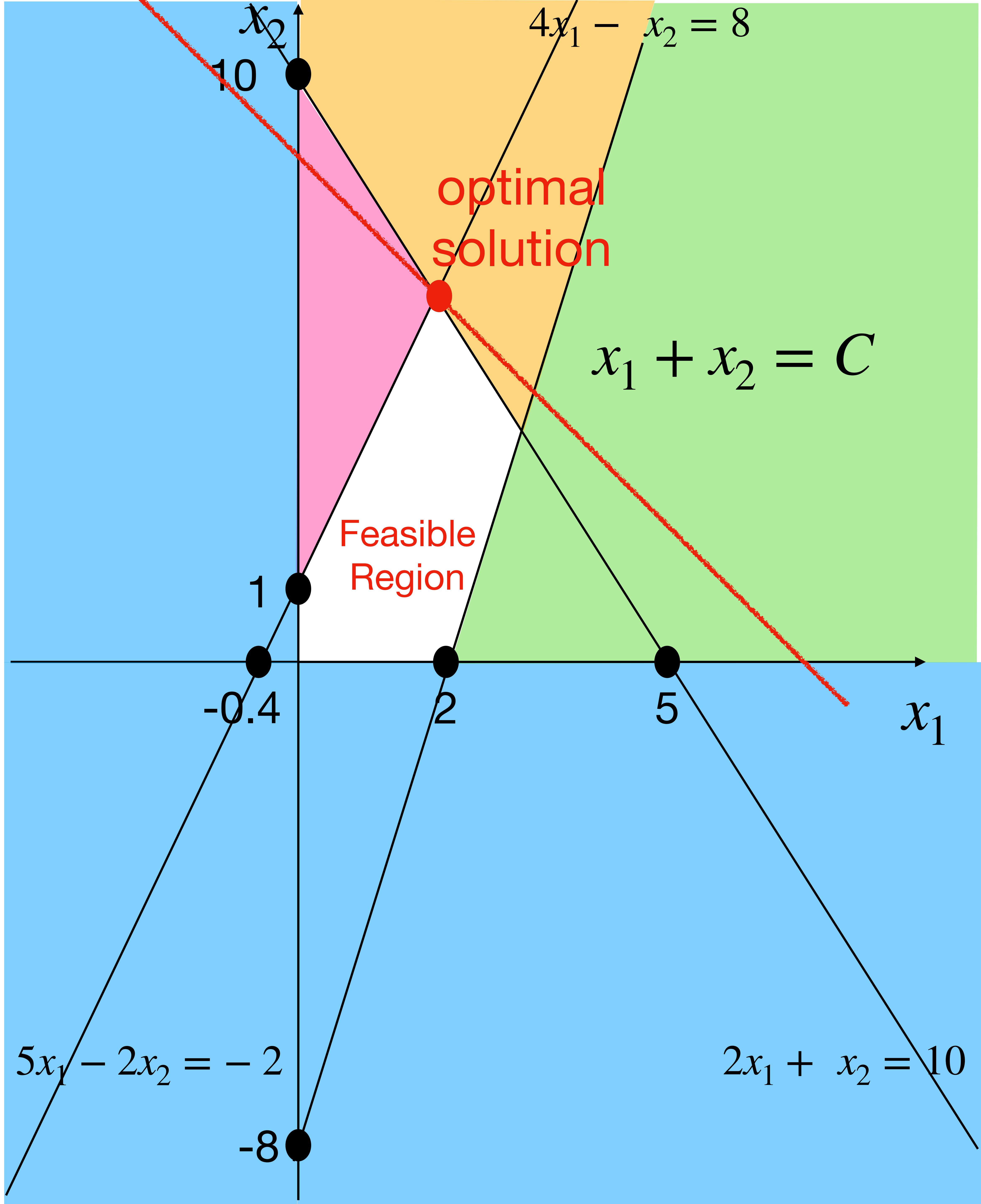
The above observation can be generalized to n variables.

The feasible region is a convex n-dimensional “polygon” called a SIMPLEX.



There exists an optimal solution that is a vertex of the SIMPLEX.

Let’s look for such an optimal solution.



Quiz questions:

1. What is the feasible region of an LP like?

2. How to solve an LP with two variables?

Roadmap of this lecture:

1. Linear Programming (LP)

   1.1 Define "Linear Programming".

   1.2 A basic example of linear programming.

   1.3 Standard-Form LP and Slack-Form LP.

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &\le b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &\le b_2 \\
&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &\le b_m \\
x_1,\ x_2,\ \cdots,\ x_n &\ge 0
\end{aligned}
$$

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \geq 0$$

Can we turn every LP into standard form?

YES.

Linear Programming

How to turn every LP into standard form:

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &\le b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &\le b_2 \\
&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &\le b_m \\
x_1,\ x_2,\ \cdots,\ x_n &\ge 0
\end{aligned}
$$

minimize $-2x_1 + 3x_2$

s.t.

$$
\begin{aligned}
x_1 + x_2 &= 7 \\
x_1 - 2x_2 &\le 4 \\
x_1 &\ge 0
\end{aligned}
$$

Linear Programming

How to turn every LP into standard form:

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \le b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \le b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \le b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \ge 0$$

minimize $-2x_1 + 3x_2$

s.t.

$$x_1 + x_2 = 7$$
$$x_1 - 2x_2 \le 4$$
$$x_1 \ge 0$$

What's wrong?

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \le b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \le b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \le b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \ge 0$$

How to turn every LP into standard form:

minimize $-2x_1 + 3x_2$

s.t.

$$x_1 + x_2 = 7$$
$$x_1 - 2x_2 \le 4$$
$$x_1 \ge 0$$

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \geq 0$$

How to turn every LP into standard form:

minimize $-2x_1 + 3x_2$

s.t.

$$x_1 + x_2 = 7$$
$$x_1 - 2x_2 \leq 4$$
$$x_1 \geq 0$$

$\Rightarrow$

maximize $2x_1 - 3x_2$

s.t.

$$x_1 + x_2 = 7$$
$$x_1 - 2x_2 \leq 4$$
$$x_1 \geq 0$$

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \le b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \le b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \le b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \ge 0$$

How to turn every LP into standard form:

minimize $-2x_1 + 3x_2$

s.t.

$$x_1 + x_2 = 7$$
$$x_1 - 2x_2 \le 4$$
$$x_1 \ge 0$$

$\Rightarrow$

maximize $2x_1 - 3x_2$

s.t.

$$x_1 + x_2 = 7$$
$$x_1 - 2x_2 \le 4$$
$$x_1 \ge 0$$

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \le b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \le b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \le b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \ge 0$$

How to turn every LP into standard form:

minimize $-2x_1 + 3x_2$

s.t.

$$x_1 + x_2 = 7$$
$$x_1 - 2x_2 \le 4$$
$$x_1 \ge 0$$

maximize $2x_1 - 3x_2$

s.t.

$$x_1 + x_2 = 7$$
$$x_1 - 2x_2 \le 4$$
$$x_1 \ge 0$$

Replace x_2 :

$$\boxed{\begin{gathered} x_2 = x_2' - x_2'' \\ x_2' \ge 0,\ x_2'' \ge 0 \end{gathered}}$$

Example:

$$5 = 6 - 1$$
$$0 = 1 - 1$$
$$-5 = 0 - 5$$

Solution space is unchanged.

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \geq 0$$

How to turn every LP into standard form:

minimize $-2x_1 + 3x_2$

s.t.

$$x_1 + x_2 = 7$$
$$x_1 - 2x_2 \leq 4$$
$$x_1 \geq 0$$

$\Rightarrow$

maximize $2x_1 - 3x_2$

s.t.

$$x_1 + x_2 = 7$$
$$x_1 - 2x_2 \leq 4$$
$$x_1 \geq 0$$

$\Downarrow$

maximize $2x_1 - 3(x_2' - x_2'')$

s.t.

$$x_1 + (x_2' - x_2'') = 7$$
$$x_1 - 2(x_2' - x_2'') \leq 4$$
$$x_1, x_2', x_2'' \geq 0$$

$\Leftarrow$

maximize $2x_1 - 3x_2' + 3x_2''$

s.t.

$$x_1 + x_2' - x_2'' = 7$$
$$x_1 - 2x_2' + 2x_2'' \leq 4$$
$$x_1, x_2', x_2'' \geq 0$$

Linear Programming

How to turn every LP into standard form:

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2$$

$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m$$

$$x_1,\ x_2,\ \cdots,\ x_n \geq 0$$

maximize $2x_1 - 3x_2' + 3x_2''$

s.t.

$$x_1 + x_2' - x_2'' = 7$$

$$x_1 - 2x_2' + 2x_2'' \leq 4$$

$$x_1, x_2', x_2'' \geq 0$$

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \le b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \le b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \le b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \ge 0$$

How to turn every LP into standard form:

maximize $2x_1 - 3x_2' + 3x_2''$

s.t.

$$x_1 + x_2' - x_2'' = 7$$
$$x_1 - 2x_2' + 2x_2'' \le 4$$
$$x_1, x_2', x_2'' \ge 0$$

$$\Downarrow$$

maximize $2x_1 - 3x_2' + 3x_2''$

s.t.

$$x_1 + x_2' - x_2'' \ge 7$$
$$x_1 + x_2' - x_2'' \le 7$$
$$x_1 - 2x_2' + 2x_2'' \le 4$$
$$x_1, x_2', x_2'' \ge 0$$

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \le b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \le b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \le b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \ge 0$$

How to turn every LP into standard form:

maximize $2x_1 - 3x_2' + 3x_2''$

s.t.

$$x_1 + x_2' - x_2'' = 7$$
$$x_1 - 2x_2' + 2x_2'' \le 4$$
$$x_1, x_2', x_2'' \ge 0$$

$$\Downarrow$$

maximize $2x_1 - 3x_2' + 3x_2''$

s.t.

$$x_1 + x_2' - x_2'' \ge 7$$
$$x_1 + x_2' - x_2'' \le 7$$
$$x_1 - 2x_2' + 2x_2'' \le 4$$
$$x_1, x_2', x_2'' \ge 0$$

Linear Programming

How to turn every LP into standard form:

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2$$

$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m$$

$$x_1,\ x_2,\ \cdots,\ x_n \geq 0$$

maximize $2x_1 - 3x_2' + 3x_2''$

s.t.

$$x_1 + x_2' - x_2'' = 7$$

$$x_1 - 2x_2' + 2x_2'' \leq 4$$

$$x_1, x_2', x_2'' \geq 0$$

$$\Downarrow$$

maximize $2x_1 - 3x_2' + 3x_2''$

s.t.

$$-x_1 - x_2' + x_2'' \leq -7$$

$$x_1 + x_2' - x_2'' \leq 7$$

$$x_1 - 2x_2' + 2x_2'' \leq 4$$

$$x_1, x_2', x_2'' \geq 0$$

Standard Form:

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \geq 0$$

Slack-Form LP:

We want equations, not inequalities.

Linear Programming

Standard-Form LP:

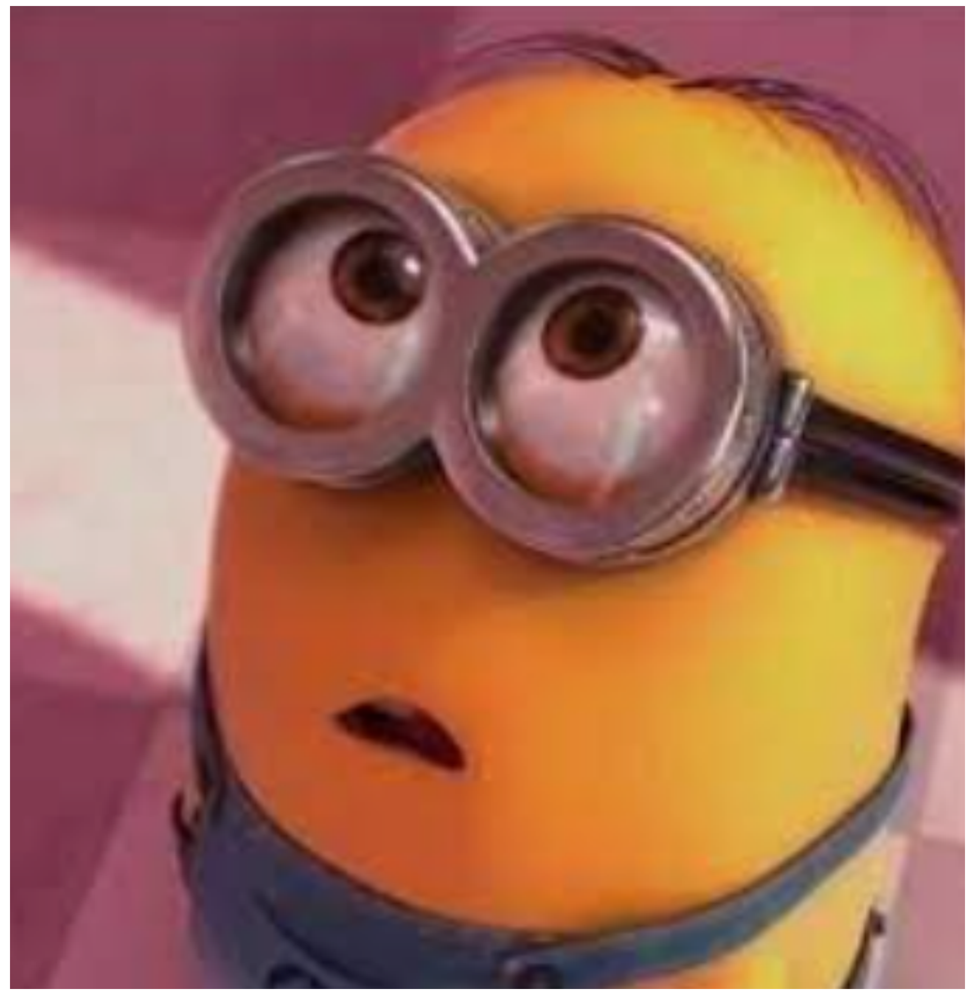
$$\text{maximize} \quad c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \le b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \le b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \le b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \ge 0$$

Slack-Form LP:

How to turn inequality into equality?



In mathematics, it's easy.

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \geq 0$$

$\Rightarrow$

Slack-Form LP:

$$x_{n+1} = b_1 - a_{1,1}x_1 - a_{1,2}x_2 - \cdots - a_{1,n}x_n \geq 0$$

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2$$

$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m$$

$$x_1,\ x_2,\ \cdots,\ x_n \geq 0$$

Slack-Form LP:

$$x_{n+1} = b_1 - a_{1,1}x_1 - a_{1,2}x_2 - \cdots - a_{1,n}x_n \geq 0$$

Auxiliary variable

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \geq 0$$

Slack-Form LP:

Right Hand Side

$$x_{n+1} = b_1 - a_{1,1}x_1 - a_{1,2}x_2 - \cdots - a_{1,n}x_n \geq 0$$

Auxiliary variable

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2$$

$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m$$

$$x_1,\ x_2,\ \cdots,\ x_n \geq 0$$

Slack-Form LP:

RHS Minus the Left Hand Side

$$x_{n+1} = b_1 - a_{1,1}x_1 - a_{1,2}x_2 - \cdots - a_{1,n}x_n \geq 0$$

Auxiliary variable

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \geq 0$$

Slack-Form LP:

RHS

Minus the LHSide

$$x_{n+1} = b_1 - a_{1,1}x_1 - a_{1,2}x_2 - \cdots - a_{1,n}x_n \geq 0$$

Auxiliary variable

RHS >= LHS

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2$$

$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m$$

$$x_1,\ x_2,\ \cdots,\ x_n \geq 0$$

Slack-Form LP:

$$x_{n+1} = b_1 - a_{1,1}x_1 - a_{1,2}x_2 - \cdots - a_{1,n}x_n \geq 0$$

$$x_{n+2} = b_2 - a_{2,1}x_1 - a_{2,2}x_2 - \cdots - a_{2,n}x_n \geq 0$$

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \geq 0$$

Slack-Form LP:

$$x_{n+1} = b_1 - a_{1,1}x_1 - a_{1,2}x_2 - \cdots - a_{1,n}x_n \geq 0$$
$$x_{n+2} = b_2 - a_{2,1}x_1 - a_{2,2}x_2 - \cdots - a_{2,n}x_n \geq 0$$
$$\vdots$$
$$x_{n+m} = b_m - a_{m,1}x_1 - a_{m,2}x_2 - \cdots - a_{m,n}x_n \geq 0$$

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \le b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \le b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \le b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \ge 0$$

Slack-Form LP:

$$x_{n+1} = b_1 - a_{1,1}x_1 - a_{1,2}x_2 - \cdots - a_{1,n}x_n \ge 0$$
$$x_{n+2} = b_2 - a_{2,1}x_1 - a_{2,2}x_2 - \cdots - a_{2,n}x_n \ge 0$$
$$\vdots$$
$$x_{n+m} = b_m - a_{m,1}x_1 - a_{m,2}x_2 - \cdots - a_{m,n}x_n \ge 0$$
$$x_1,\ x_2,\ \cdots,\ x_n, x_{n+1}, x_{n+2}, \cdots, x_{n+m} \ge 0$$

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \geq 0$$

Slack-Form LP:

$$x_{n+1} = b_1 - a_{1,1}x_1 - a_{1,2}x_2 - \cdots - a_{1,n}x_n$$
$$x_{n+2} = b_2 - a_{2,1}x_1 - a_{2,2}x_2 - \cdots - a_{2,n}x_n$$
$$\vdots$$
$$x_{n+m} = b_m - a_{m,1}x_1 - a_{m,2}x_2 - \cdots - a_{m,n}x_n$$
$$x_1,\ x_2,\ \cdots,\ x_n, x_{n+1}, x_{n+2}, \cdots, x_{n+m} \geq 0$$

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \le b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \le b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \le b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \ge 0$$

Slack-Form LP:

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$
$$x_{n+1} = b_1 - a_{1,1}x_1 - a_{1,2}x_2 - \cdots - a_{1,n}x_n$$
$$x_{n+2} = b_2 - a_{2,1}x_1 - a_{2,2}x_2 - \cdots - a_{2,n}x_n$$
$$\vdots$$
$$x_{n+m} = b_m - a_{m,1}x_1 - a_{m,2}x_2 - \cdots - a_{m,n}x_n$$
$$x_1,\ x_2,\ \cdots,\ x_n, x_{n+1}, x_{n+2}, \cdots, x_{n+m} \ge 0$$

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \geq 0$$

Slack-Form LP:

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Objective value

s.t.

$$x_{n+1} = b_1 - a_{1,1}x_1 - a_{1,2}x_2 - \cdots - a_{1,n}x_n$$
$$x_{n+2} = b_2 - a_{2,1}x_1 - a_{2,2}x_2 - \cdots - a_{2,n}x_n$$
$$\vdots$$
$$x_{n+m} = b_m - a_{m,1}x_1 - a_{m,2}x_2 - \cdots - a_{m,n}x_n$$
$$x_1,\ x_2,\ \cdots,\ x_n, x_{n+1}, x_{n+2}, \cdots, x_{n+m} \geq 0$$

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \le b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \le b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \le b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \ge 0$$

Slack-Form LP:

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$x_{n+1} = b_1 - a_{1,1}x_1 - a_{1,2}x_2 - \cdots - a_{1,n}x_n$$
$$x_{n+2} = b_2 - a_{2,1}x_1 - a_{2,2}x_2 - \cdots - a_{2,n}x_n$$
$$\vdots$$
$$x_{n+m} = b_m - a_{m,1}x_1 - a_{m,2}x_2 - \cdots - a_{m,n}x_n$$

[redacted]

By convention, we do not write this, but we know this constraint is there.

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \geq 0$$

Slack-Form LP:

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$x_{n+1} = b_1 - a_{1,1}x_1 - a_{1,2}x_2 - \cdots - a_{1,n}x_n$$
$$x_{n+2} = b_2 - a_{2,1}x_1 - a_{2,2}x_2 - \cdots - a_{2,n}x_n$$
$$\vdots$$
$$x_{n+m} = b_m - a_{m,1}x_1 - a_{m,2}x_2 - \cdots - a_{m,n}x_n$$

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \le b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \le b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \le b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \ge 0$$

Slack-Form LP:

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Equation above: objective function.

Equations below: constraints.

$$x_{n+1} = b_1 - a_{1,1}x_1 - a_{1,2}x_2 - \cdots - a_{1,n}x_n$$
$$x_{n+2} = b_2 - a_{2,1}x_1 - a_{2,2}x_2 - \cdots - a_{2,n}x_n$$
$$\vdots$$
$$x_{n+m} = b_m - a_{m,1}x_1 - a_{m,2}x_2 - \cdots - a_{m,n}x_n$$

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \geq 0$$

Slack-Form LP:

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$
$$x_{n+1} = b_1 - a_{1,1}x_1 - a_{1,2}x_2 - \cdots - a_{1,n}x_n$$
$$x_{n+2} = b_2 - a_{2,1}x_1 - a_{2,2}x_2 - \cdots - a_{2,n}x_n$$
$$\vdots$$
$$x_{n+m} = b_m - \underbrace{a_{m,1}x_1 - a_{m,2}x_2 - \cdots - a_{m,n}x_n}$$

LHS variables:

Basic Variables

RHS variables:

Non-Basic Variables

Linear Programming

Standard-Form LP:

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \le b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \le b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \le b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \ge 0$$

Slack-Form LP:

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$
$$x_{n+1} = b_1 - a_{1,1}x_1 - a_{1,2}x_2 - \cdots - a_{1,n}x_n$$
$$x_{n+2} = b_2 - a_{2,1}x_1 - a_{2,2}x_2 - \cdots - a_{2,n}x_n$$
$$\vdots$$
$$x_{n+m} = b_m - \underbrace{a_{m,1}x_1 - a_{m,2}x_2 - \cdots - a_{m,n}x_n}_{\text{RHS variables: Non-Basic Variables}}$$

LHS variables:

Basic Variables

RHS variables:

Non-Basic Variables

Each variable is either on the LHS or RHS,
but never on both sides.

So each variable is either a Basic Variable, or a Non-Basic Variable, but never both.

When we move variables later, a basic variable can change to a non-basic variable, or vice versa.

Quiz questions:

1. What is the difference between a Standard-Form LP and a Slack-Form LP?
2. How to turn a general LP into a Standard-Form LP?
3. How to turn a general LP into a Slack-Form LP?