Algorithms

Lecture Topic: Approximation Algorithms (Part 1)

Anxiao (Andrew) Jiang

Roadmap of this lecture:

1. Define "Approximation Algorithm".

2. Understand approximation algorithms by solving the "Vertex Cover Problem".

   2.1 An approximation algorithm for "Vertex Cover Problem".

   2.2 Analyze the approximation ratio of the algorithm.

3. Understand approximation algorithms by solving the "Traveling Salesman Problem (TSP)".

   3.1 An approximation algorithm for TSP with the triangle inequality.

   3.2 Analyze the approximation ratio of the algorithm.

Approximation Algorithms

Why do we need Approximation Algorithms?

Approximation algorithms are “easier” than exact algorithms, since it requires only approximate solutions, not optimal solutions.

How to analyze approximation algorithms?

Approximation Algorithms

Consider an optimization problem.

Let C^* be the cost of an optimal solution.

Let C be the cost of the solution found by our algorithm.

(For simplicity, assume cost > 0.)

Approximation Algorithms

maximization

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If $\dfrac{C^*}{C} \leq \rho$ for all possible instances, then our algorithm is called a ρ-approximation algorithm.

We say our algorithm has an approximation ratio of ρ .

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Quiz questions:

1. What is an "Approximation Algorithm"?

2. What is "approximation ratio"?

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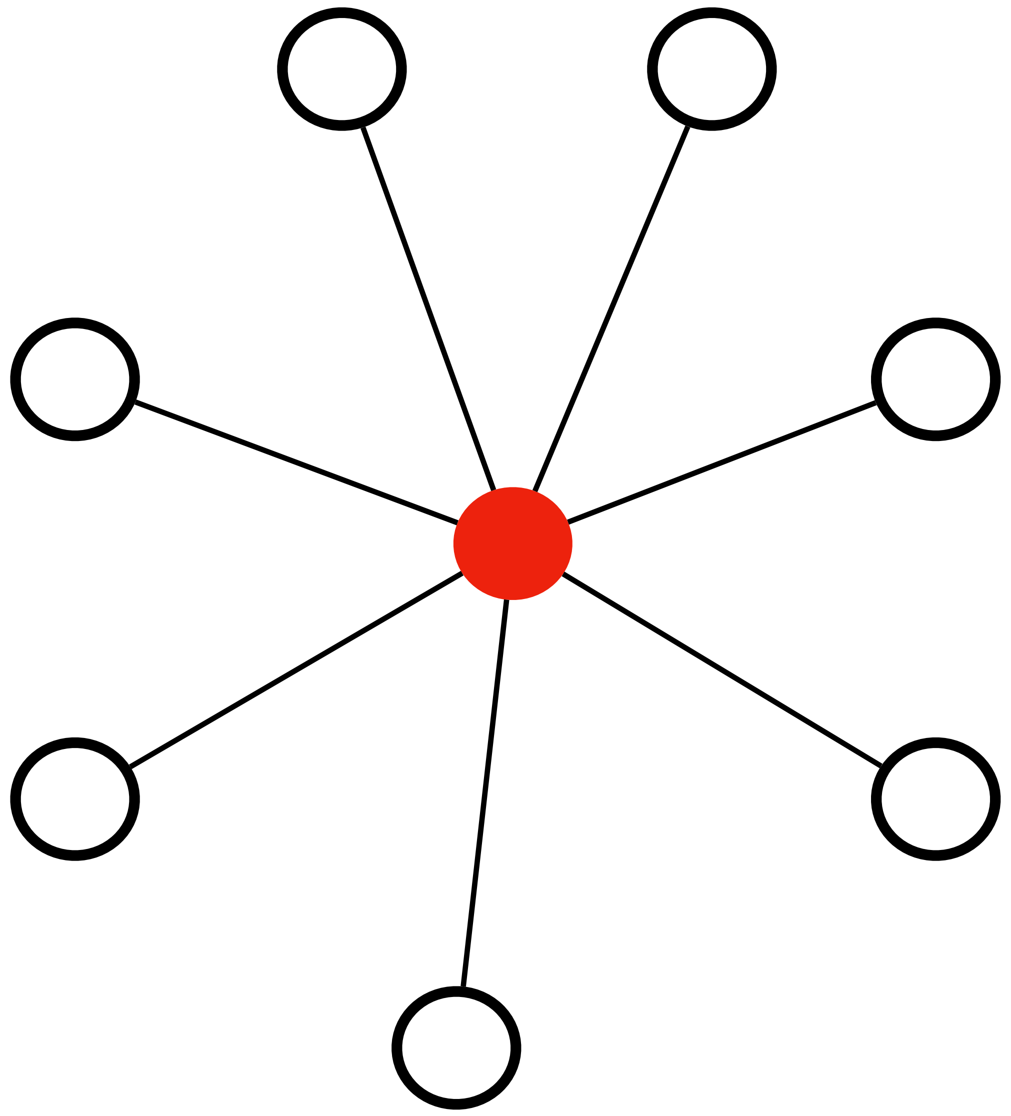
Vertex Cover Problem

Input: An undirected graph G=(V,E).

Output: A vertex cover of G of minimum size.



Vertex Cover Problem

We will show a 2-approximation algorithm.

Input: An undirected graph G=(V,E).

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Algorithm:

1. $S = \emptyset$ — S is the vertex cover.
2. $E' = E$ — E' are the uncovered edges.
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This step seems strange.
Either u or v is enough for covering the edge (u,v).
But we choose both u and v.

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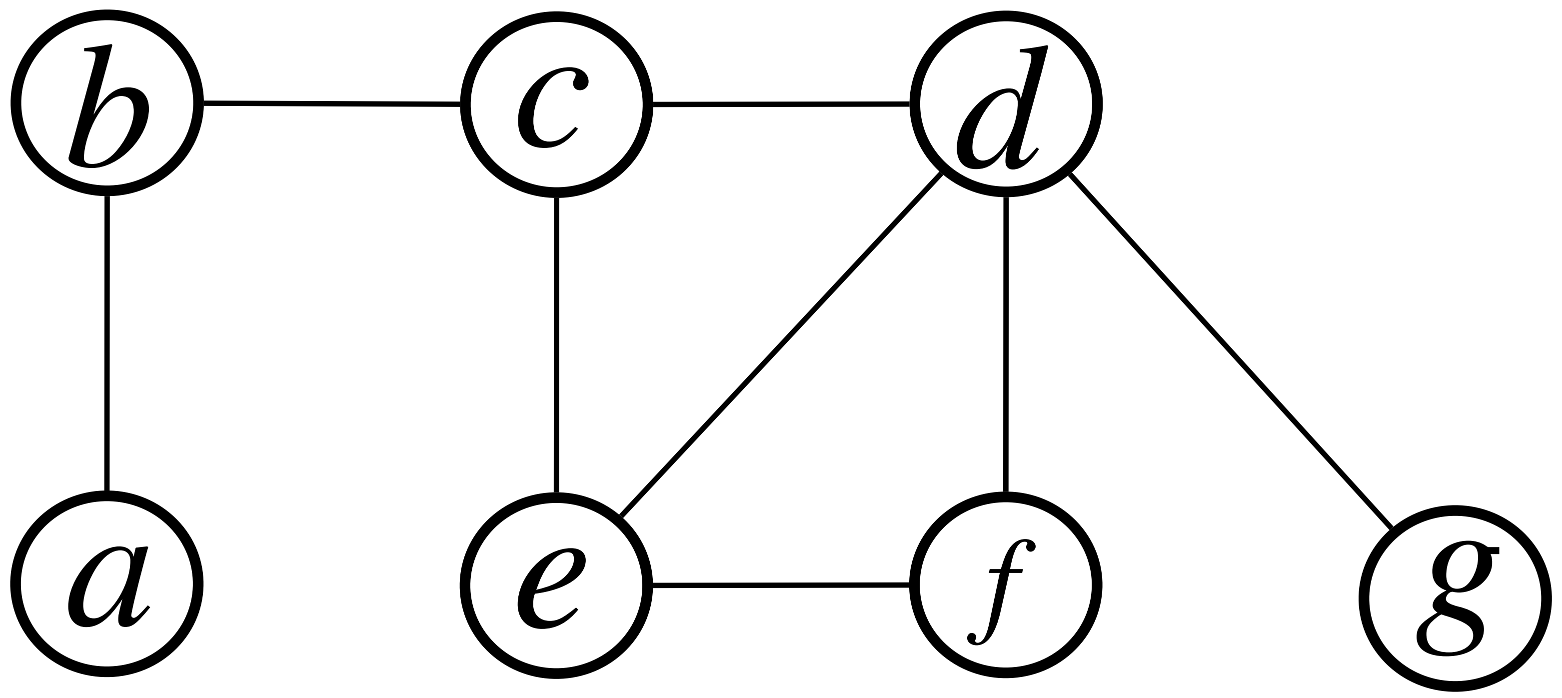

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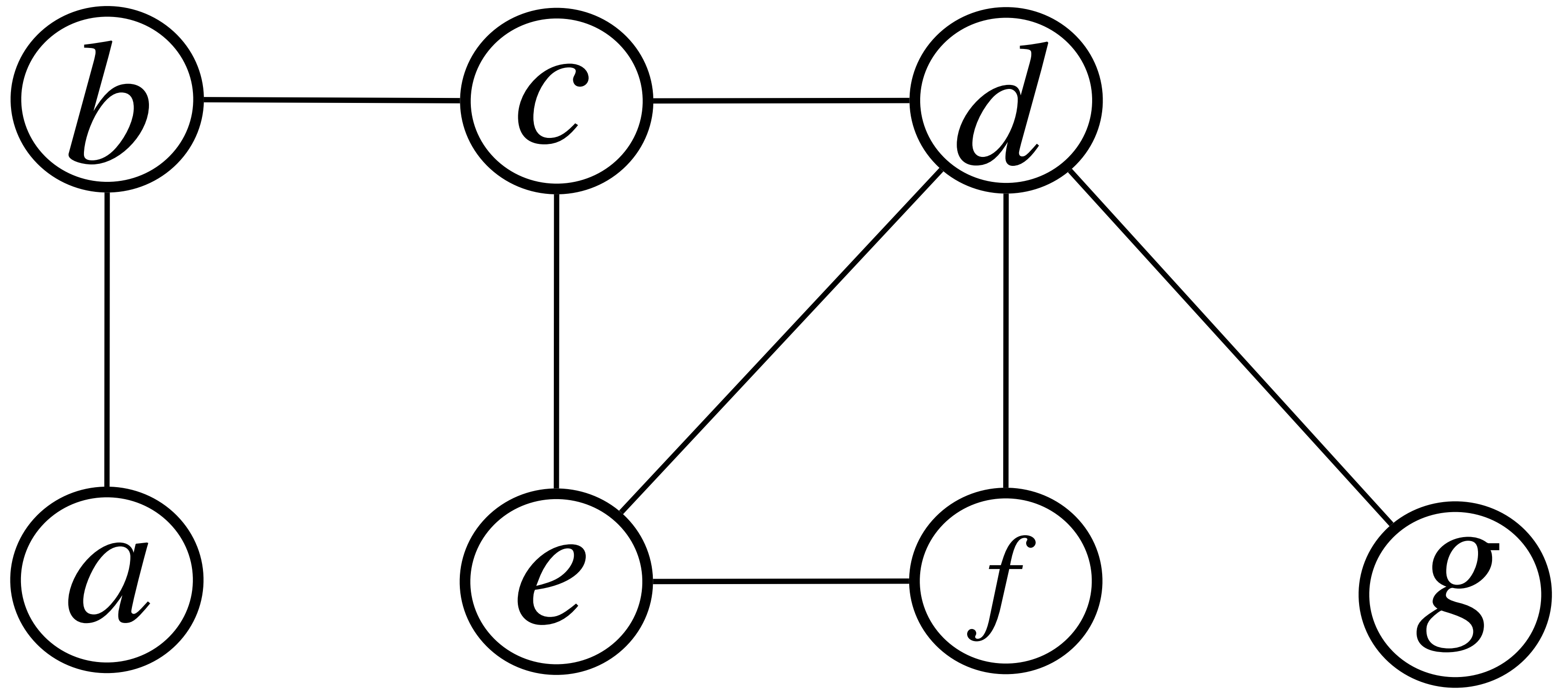

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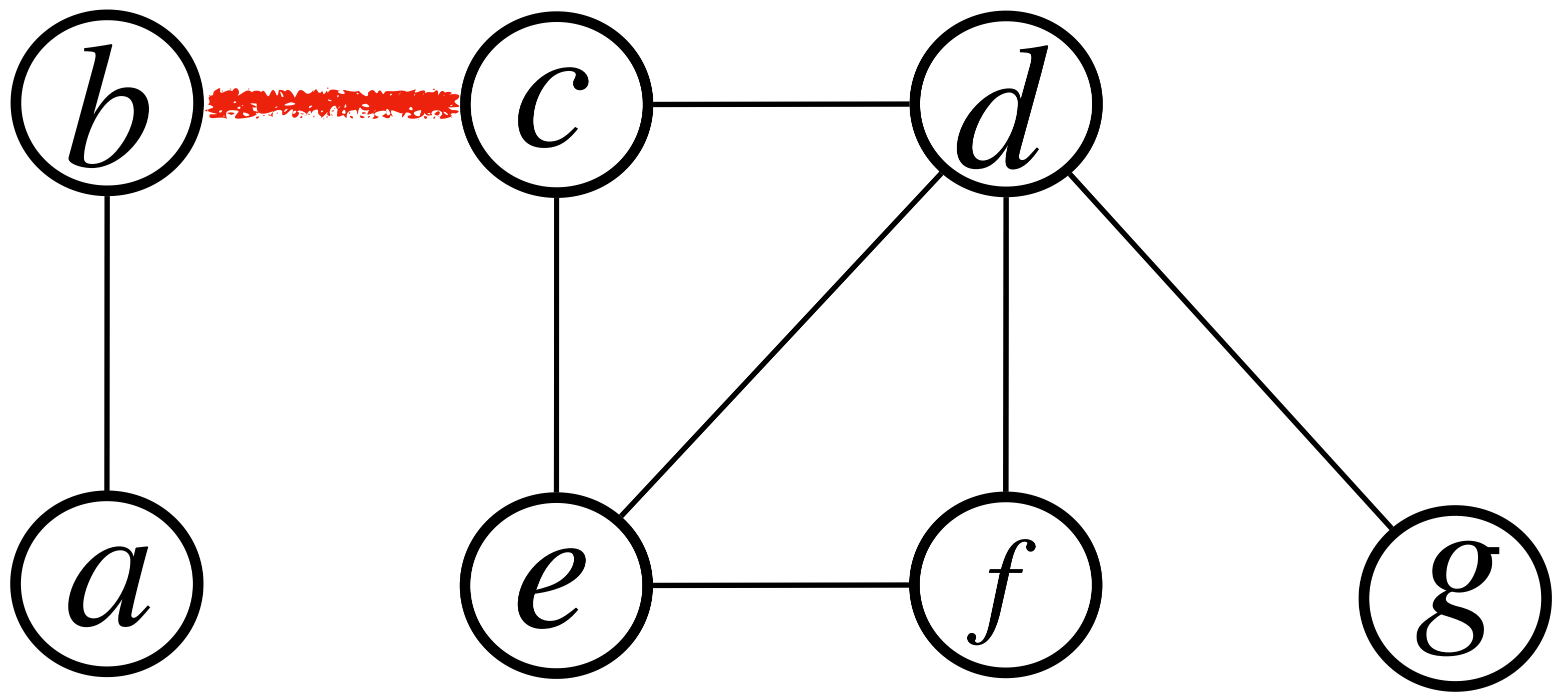

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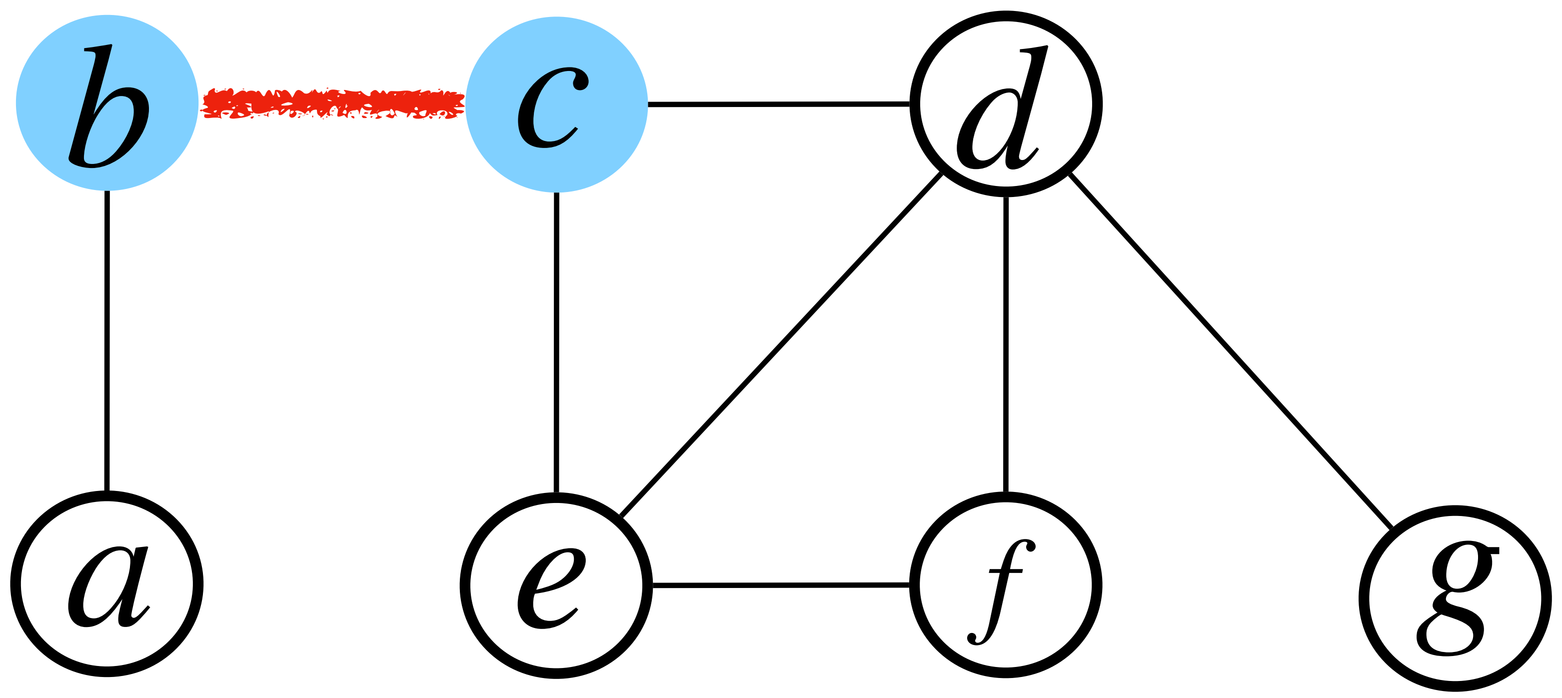

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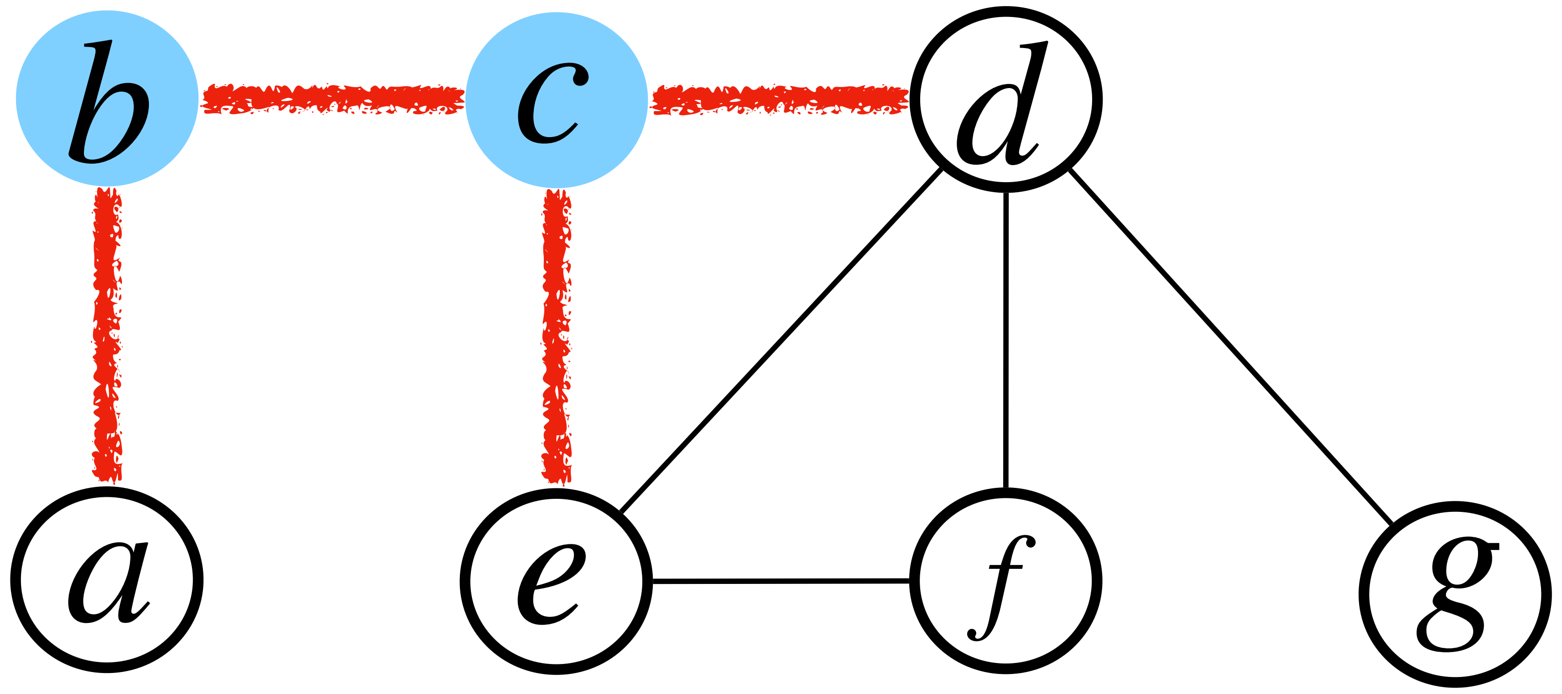

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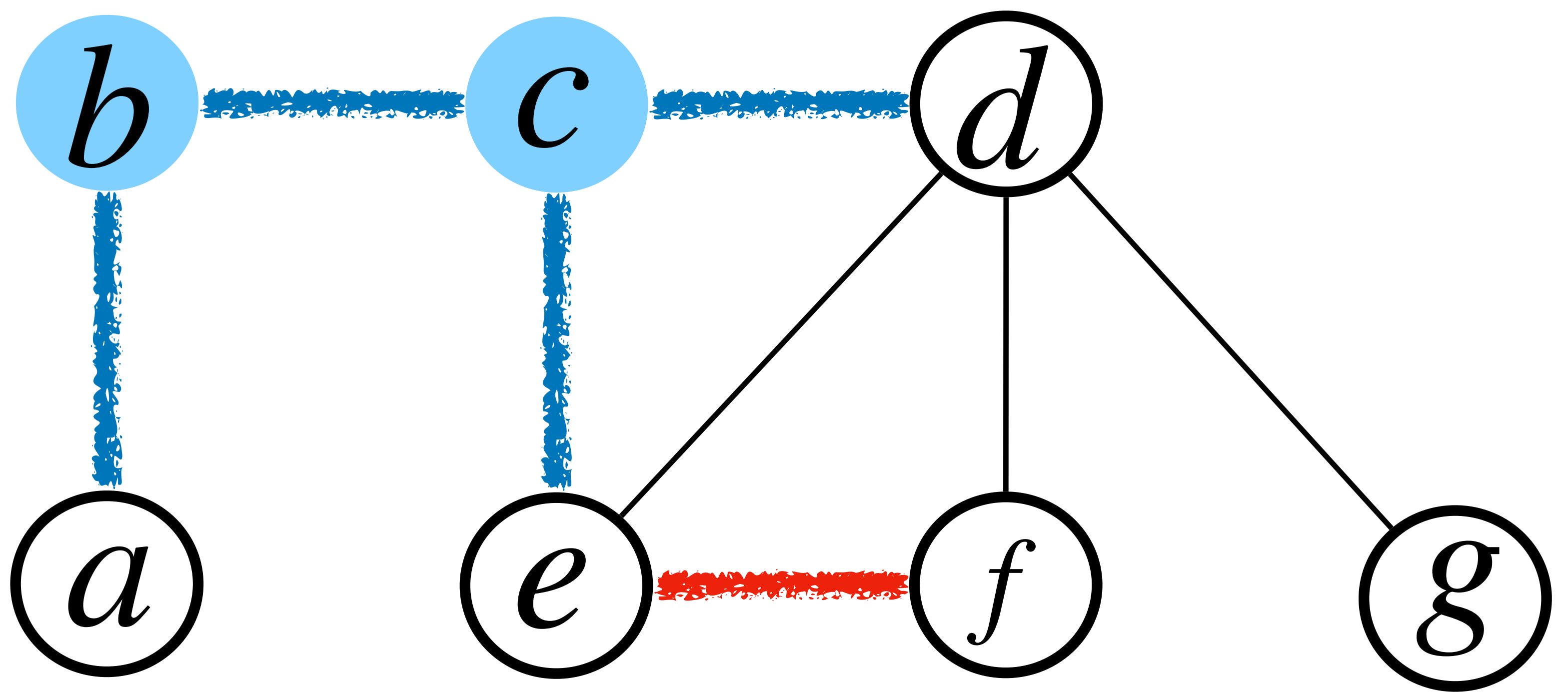

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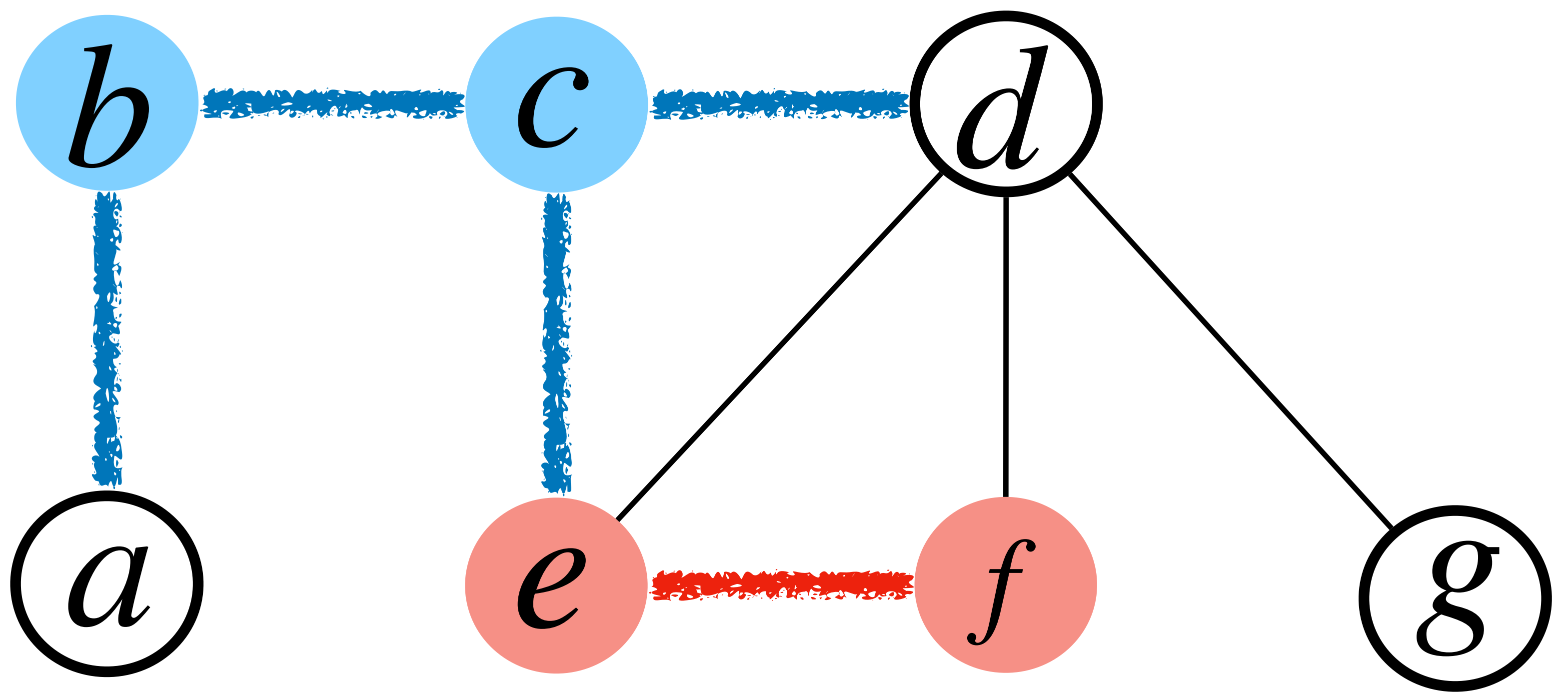

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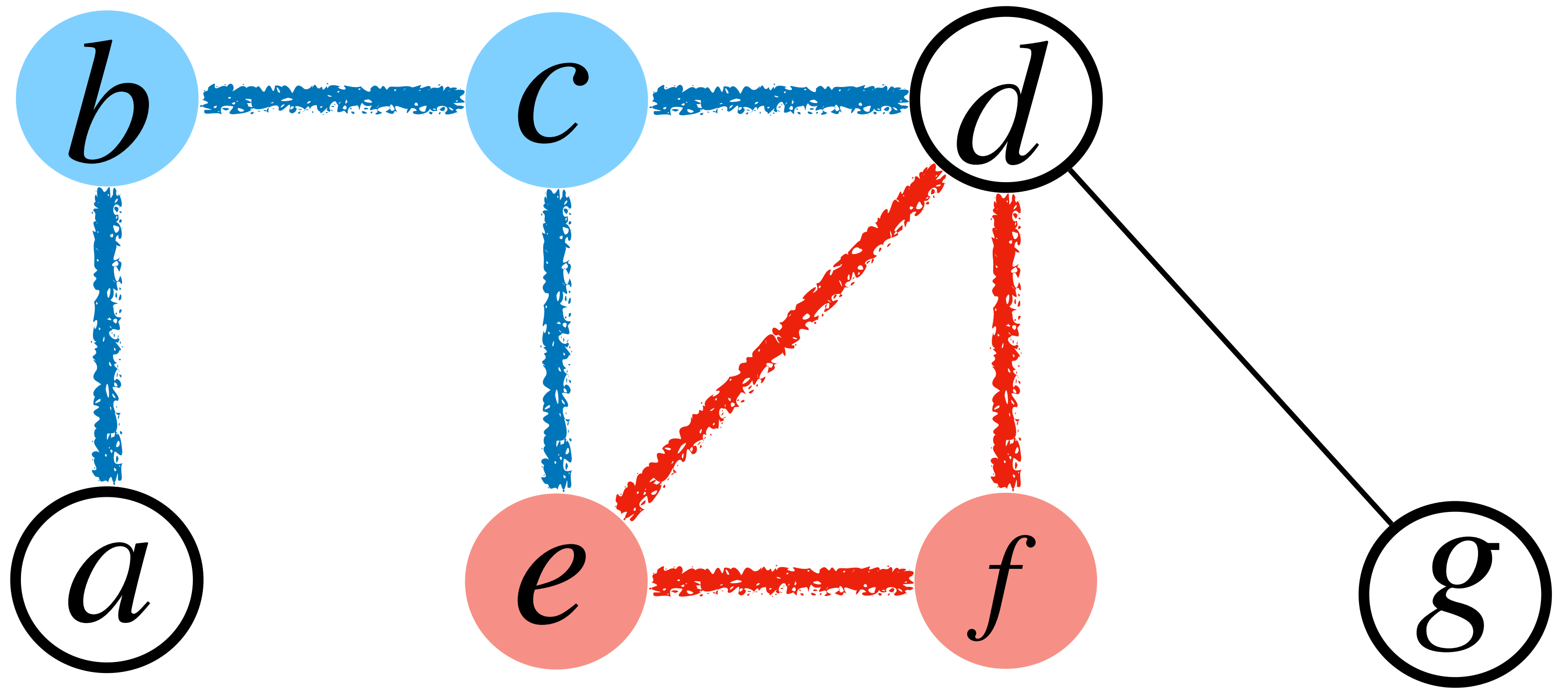

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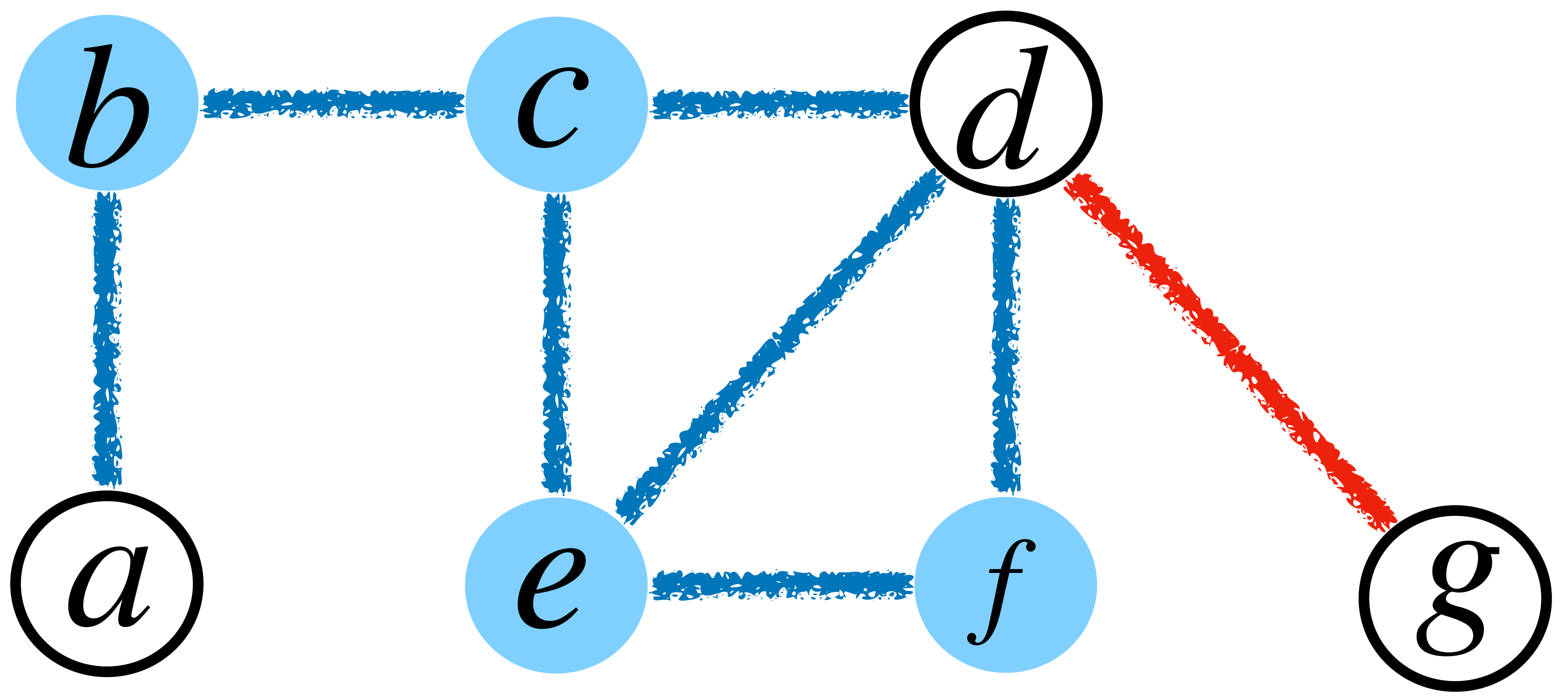

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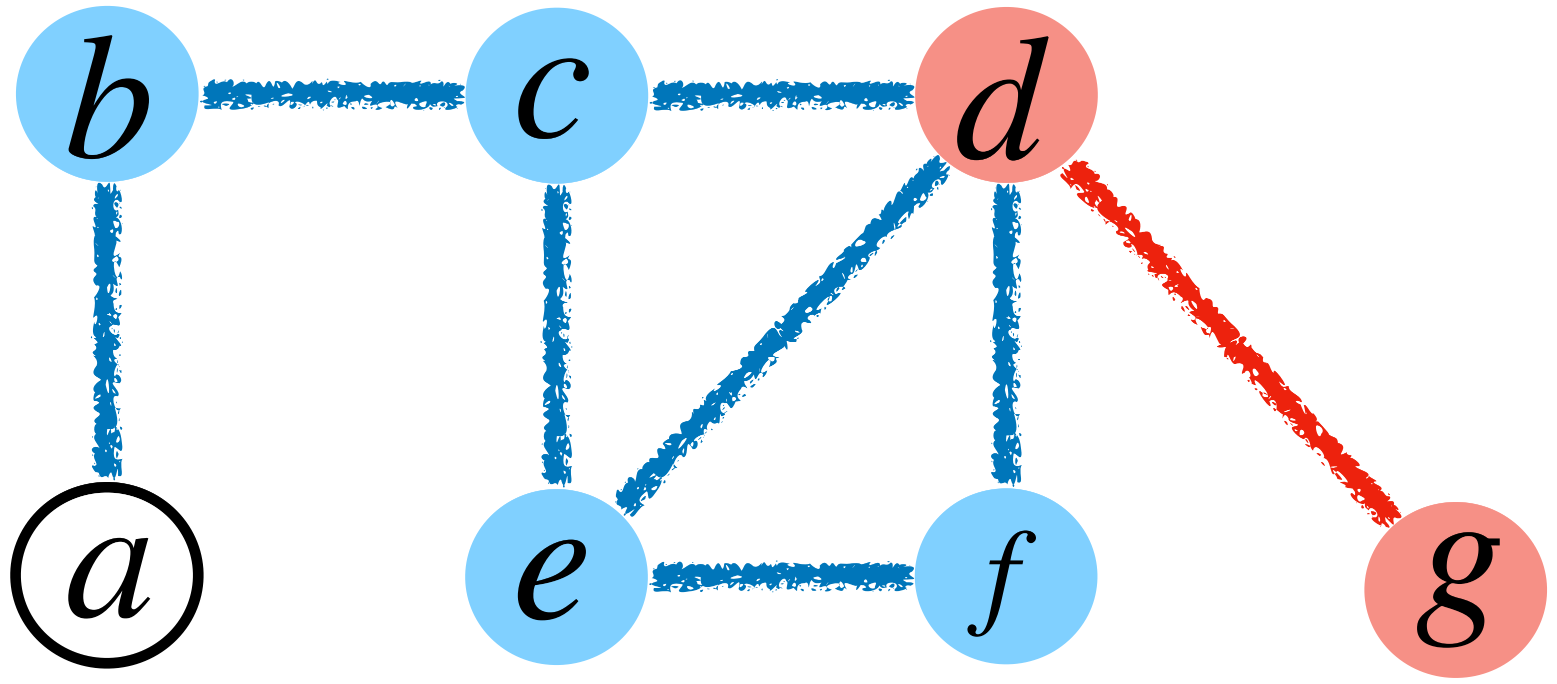

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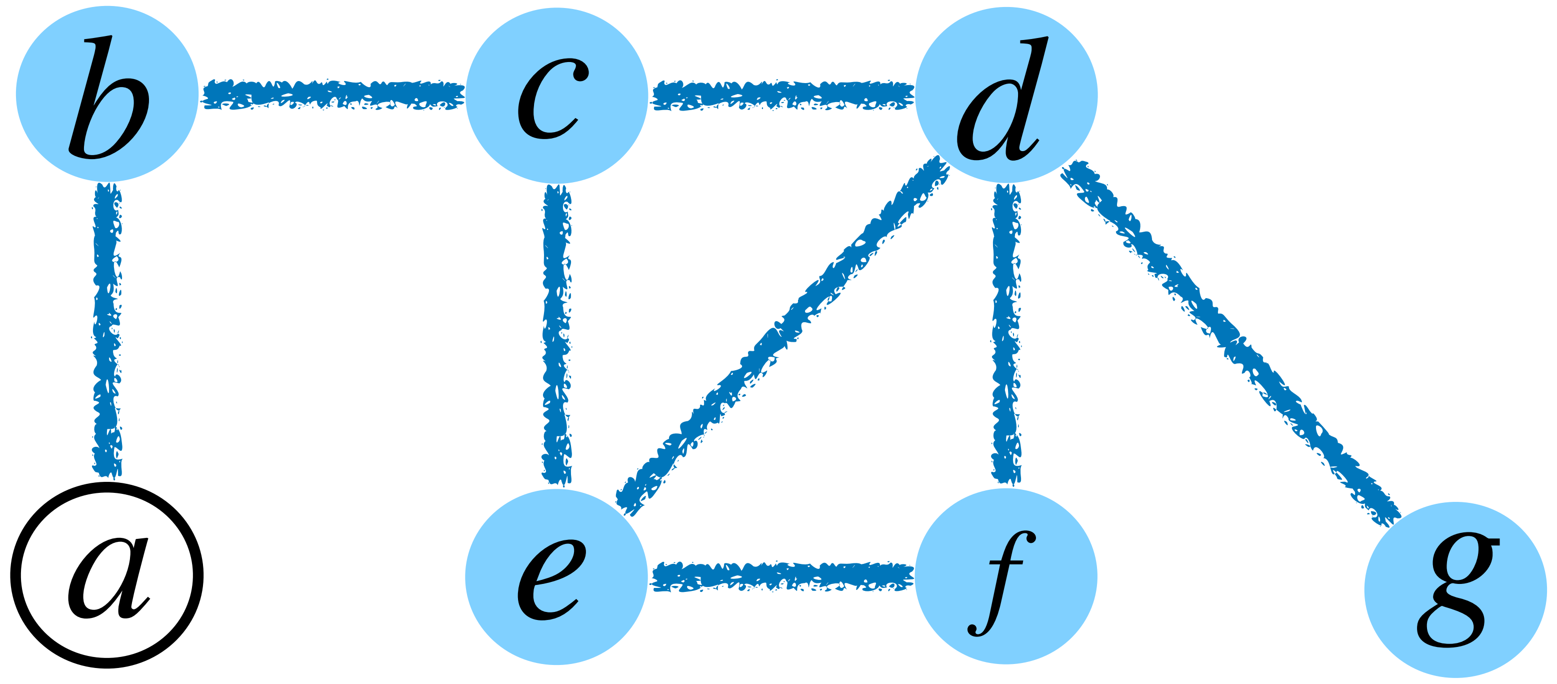
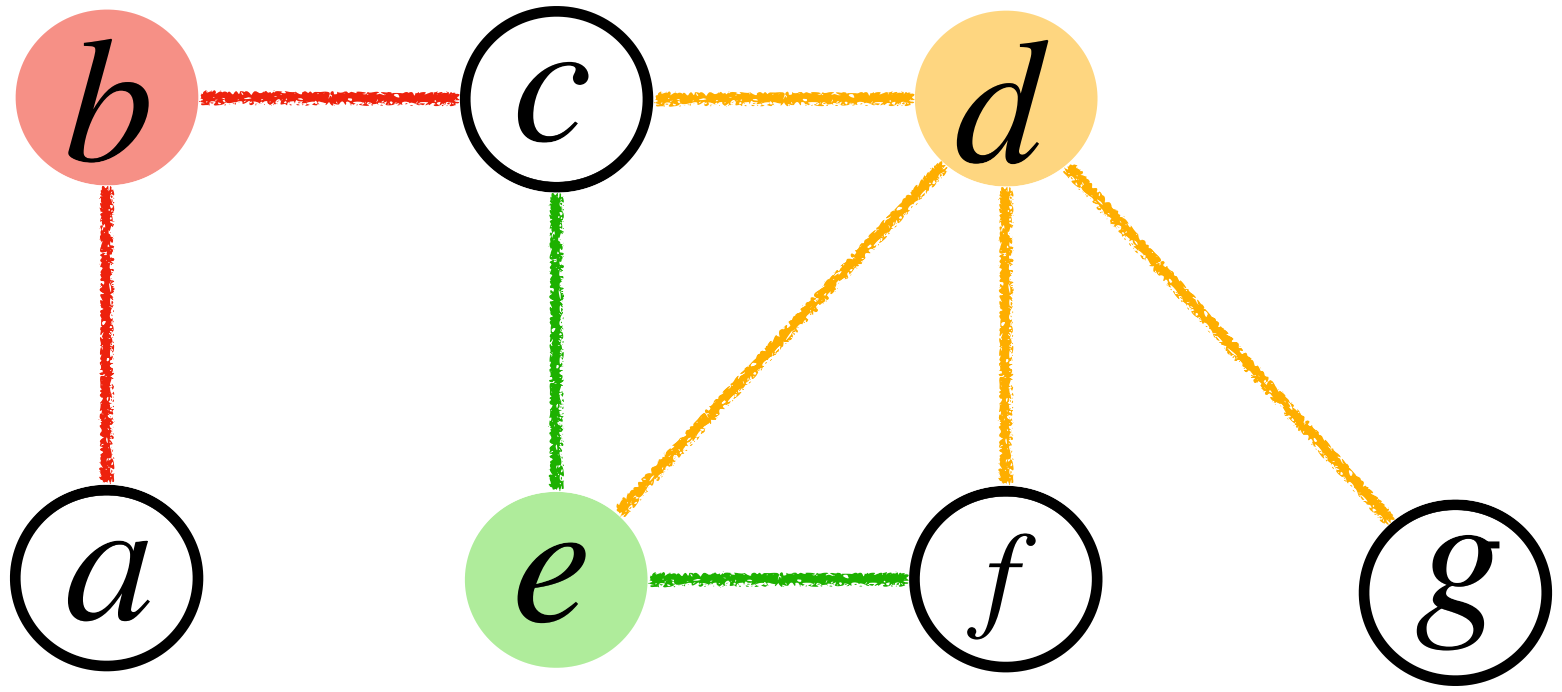


$$S = \{b, c, e, f, d, g\}$$

$$C = |S| = 6$$

$$C^* = 3$$



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We need to show the approximation ratio is at most 2 for all instances.

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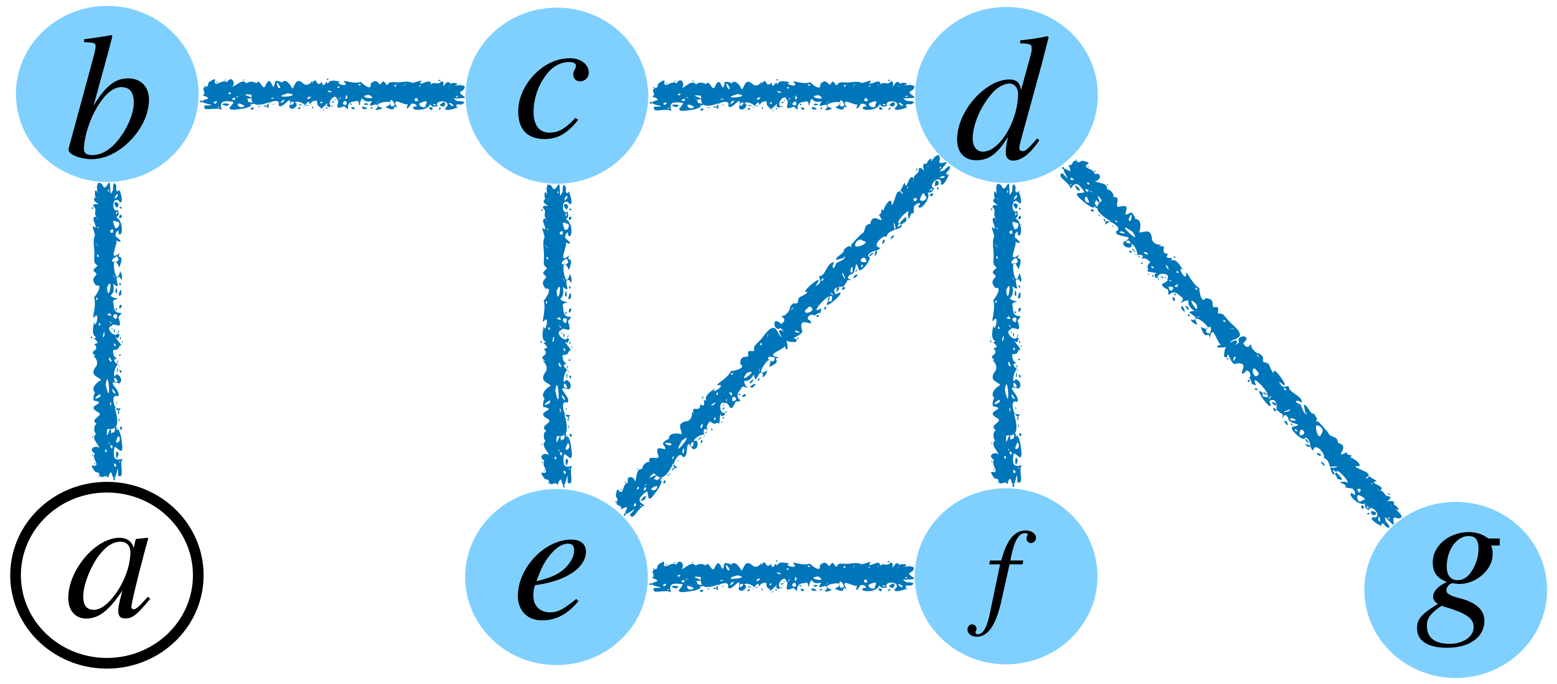

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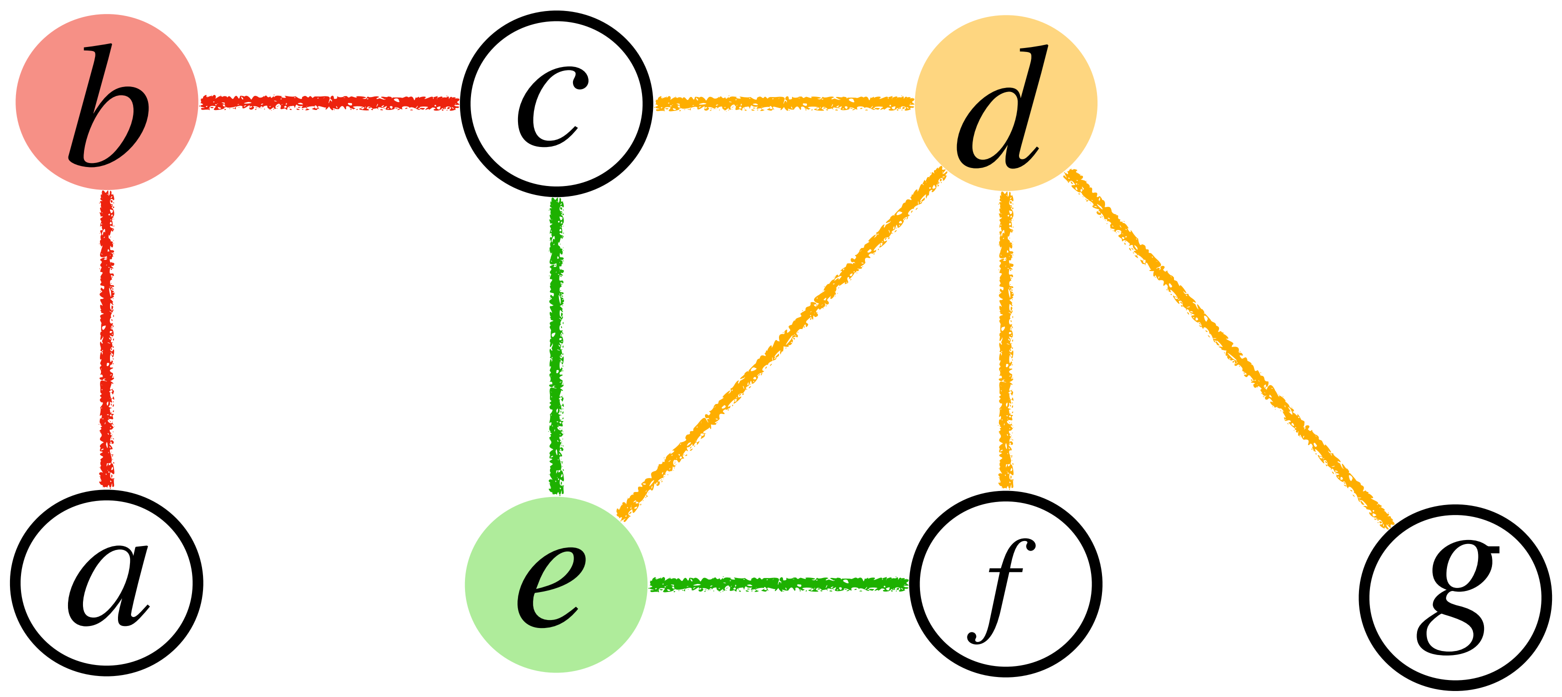

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Quiz questions:

1. What is the main idea of the above approximation algorithm for the "Vertex Cover Problem"?

2. Can you think of an instance for which the above algorithm outputs an optimal solution, and an instance for which it outputs a non-optimal solution?

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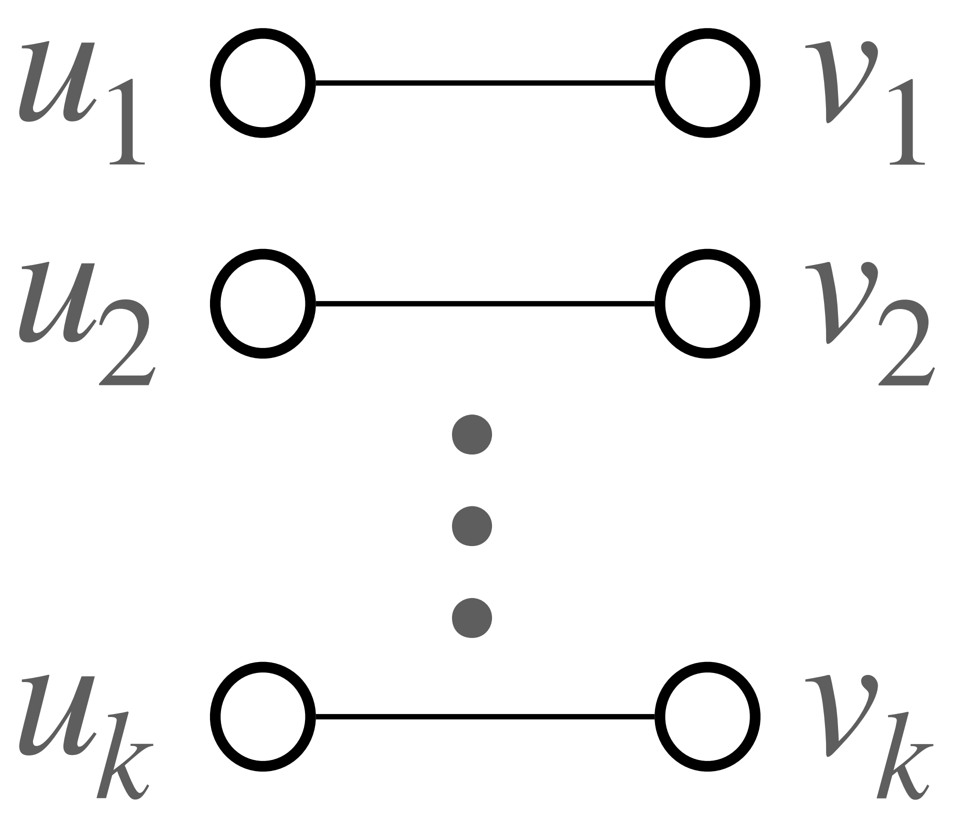

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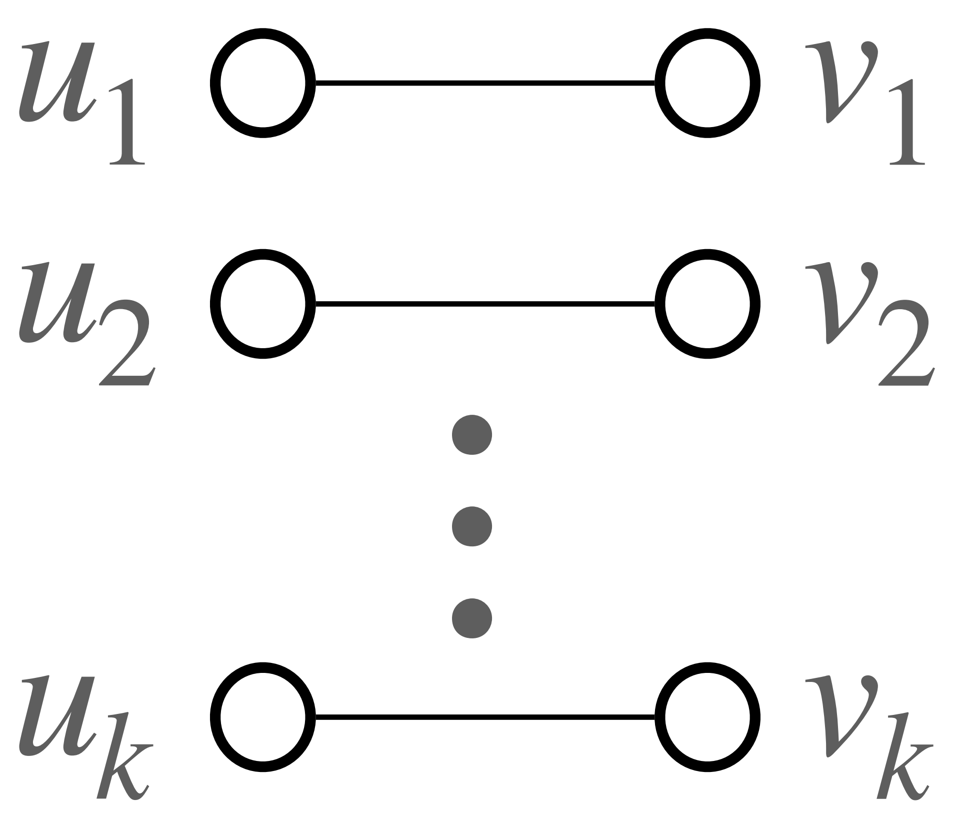
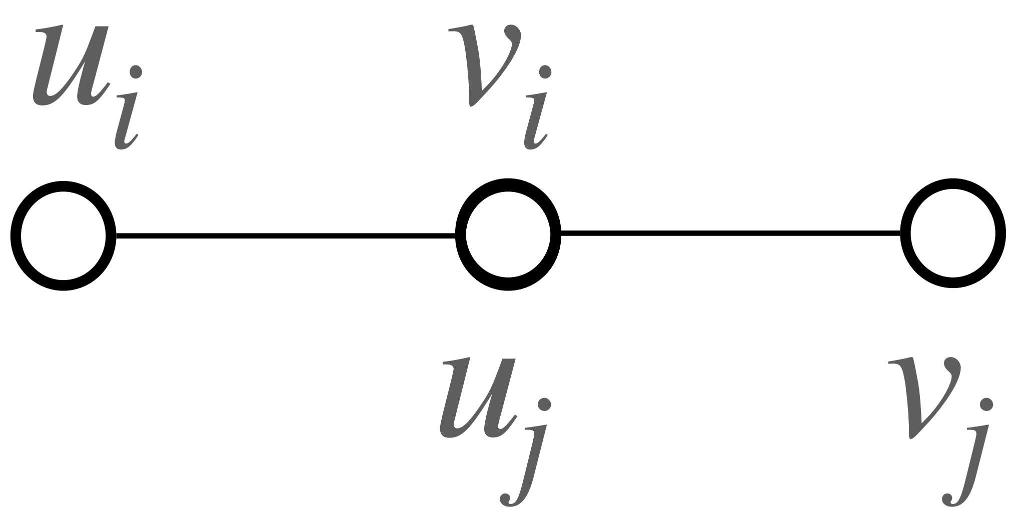


be the k edges chosen by the algorithm.

No two edges above share any common node.

Impossible:



Why ?

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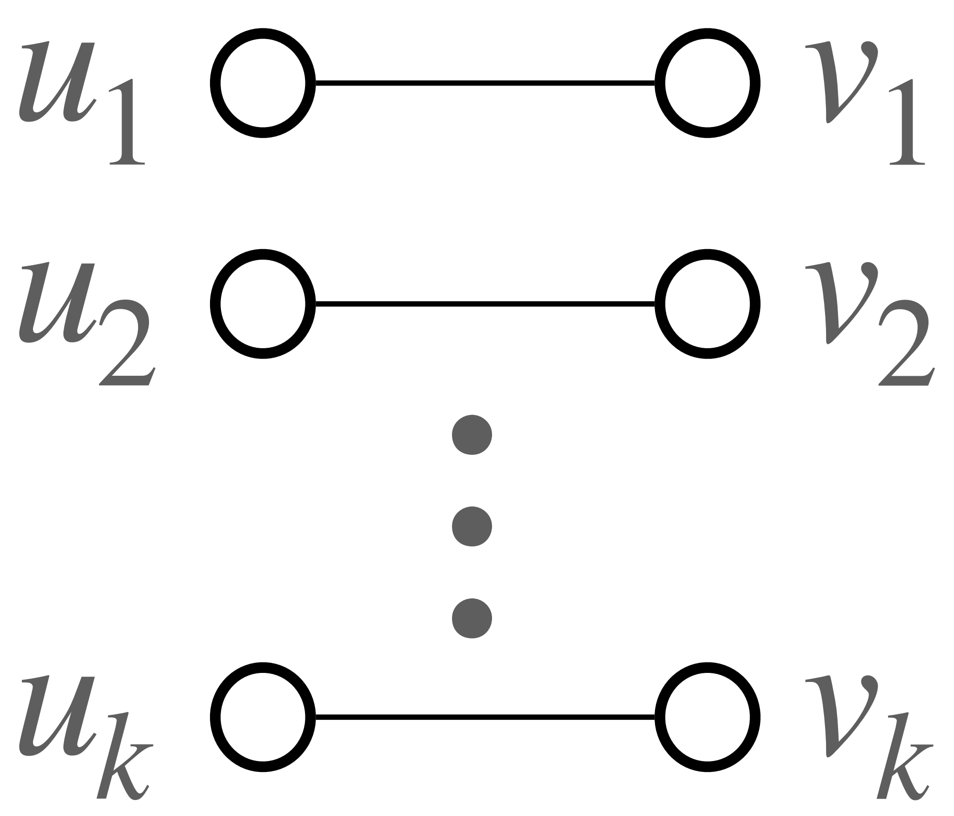

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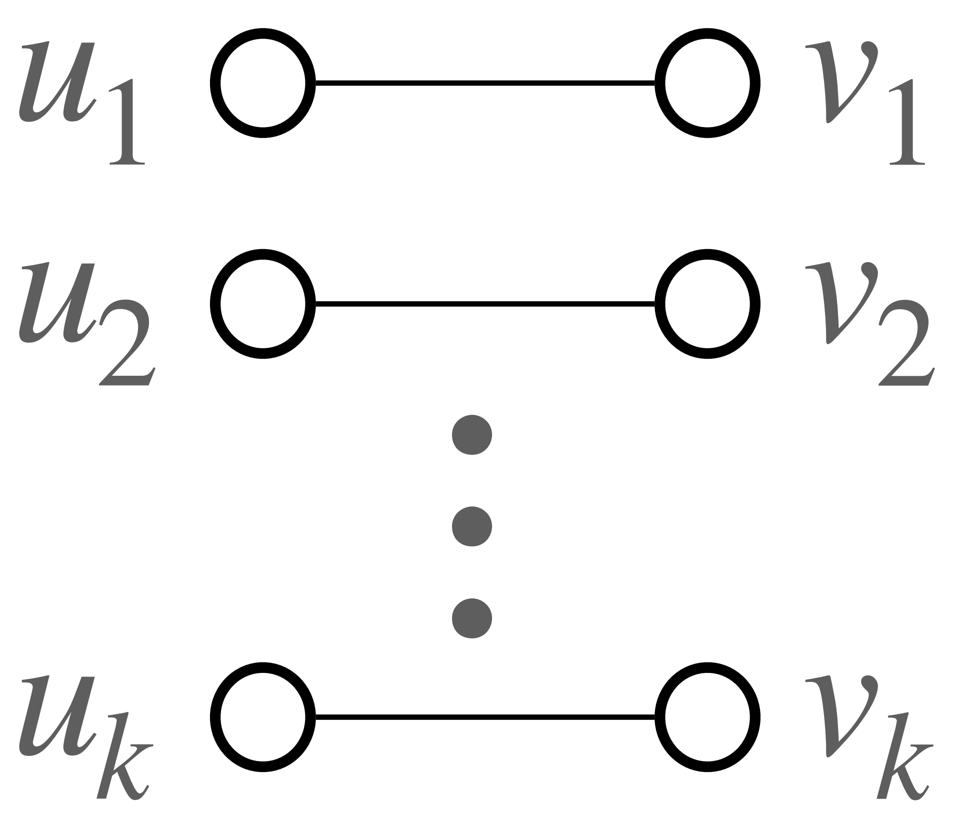

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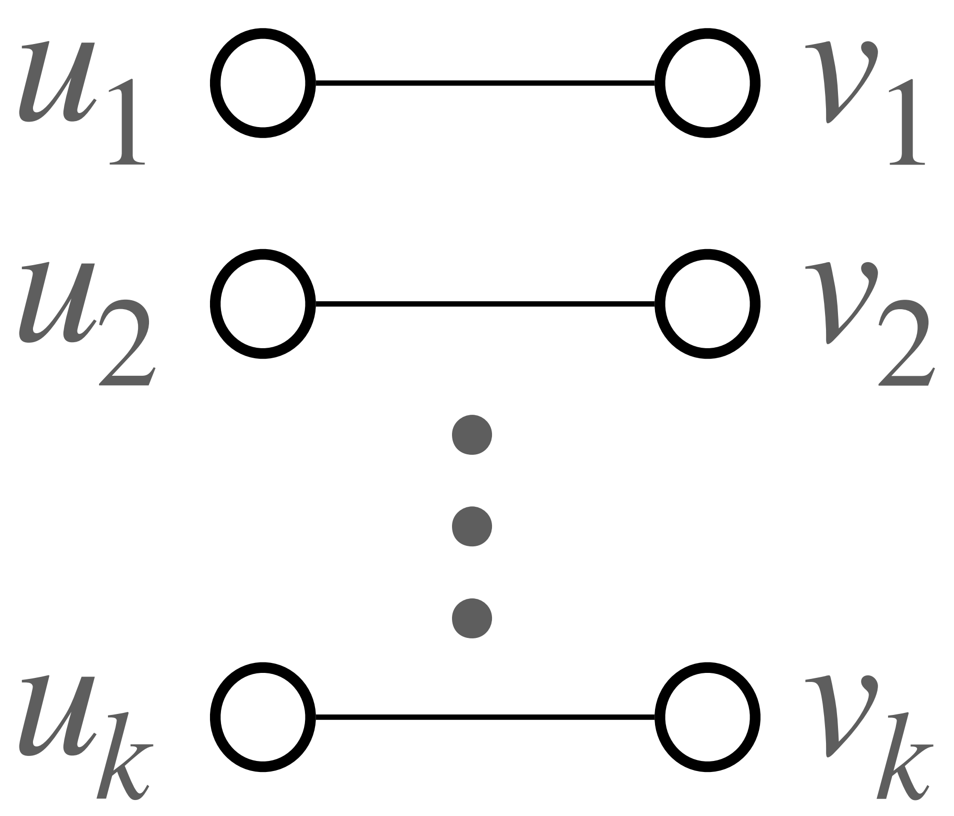

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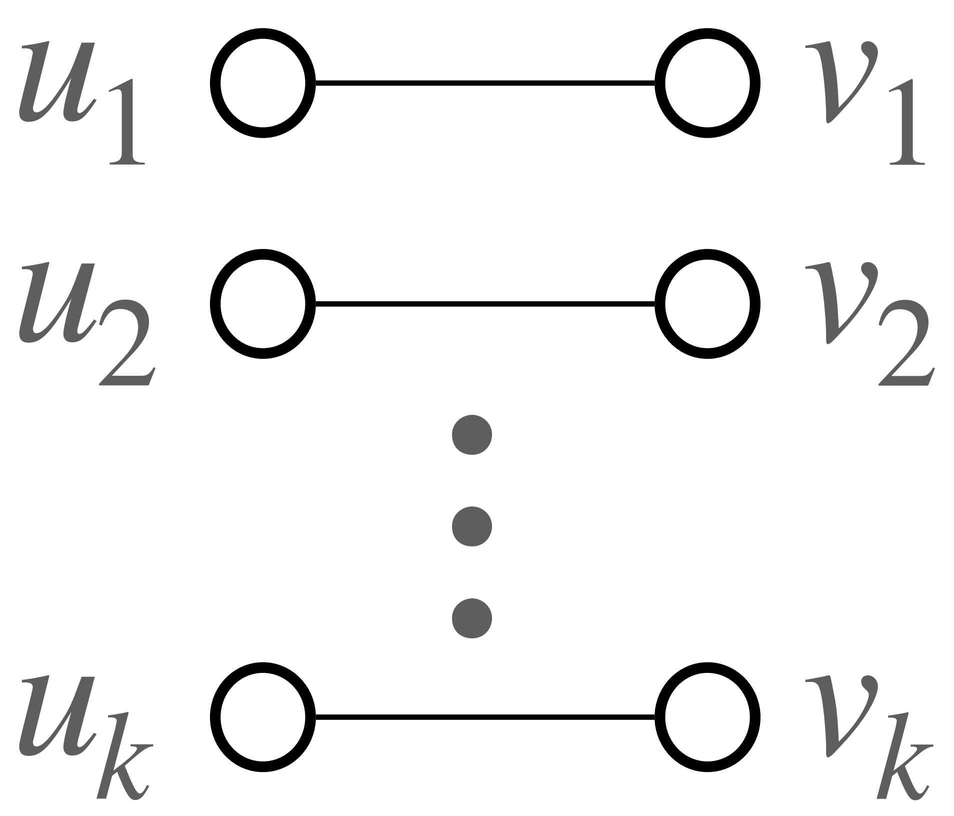

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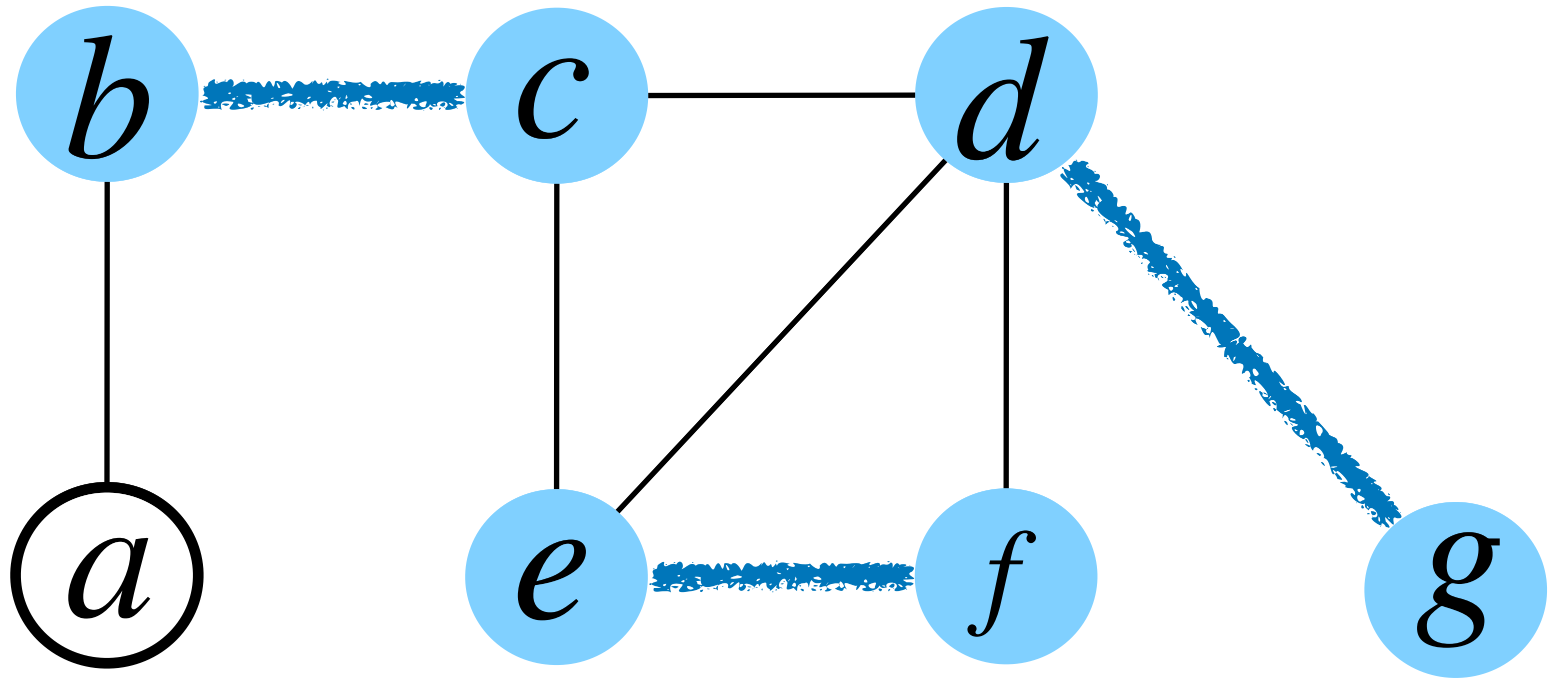
$$\boxed{\text{So} \quad C^* \geq k}$$

As $C = 2k$

$$\boxed{\frac{C}{C^*} \leq \frac{2k}{k} = 2}$$

Let's take a look at our earlier example:



3 disjoint chosen edges

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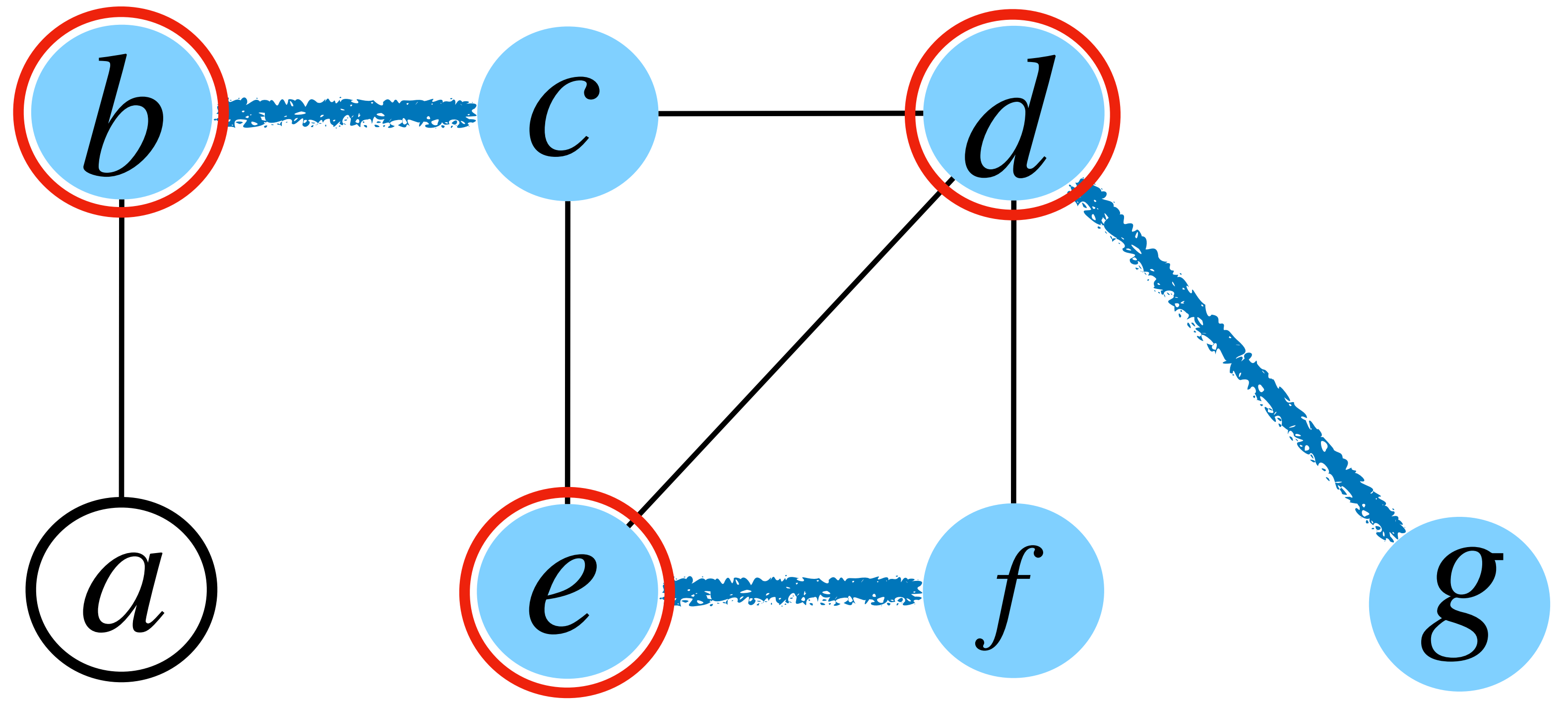

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3 disjoint chosen edges

Every vertex cover will contain at least half of those endpoints.

So the approximation ratio is 2.

How to prove $\frac{C}{C^*} \leq \rho$ without knowing C^* ?

For minimization problem, we just need a lower bound of C^*.

Remember our earlier proof:

$\boxed{\text{So } C^* \geq k}$ Lower bound

As $C = 2k$

$$\boxed{\frac{C}{C^*} \leq \frac{2k}{k} = 2}$$

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maximization **upper bound**

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As $C = 2k$

$$\boxed{\frac{C}{C^*} \leq \frac{2k}{k} = 2}$$

$$\boxed{\frac{C^*}{C} \leq \frac{\text{upper bound of } C^*}{C}}$$

Quiz questions:

1. For the above approximation algorithm for the "Vertex Cover Problem", how did we find the approximation ratio without knowing the optimal cost?

2. The "Vertex Cover Problem" is a minimization problem. For a maximization problem, how can we find the approximation ratio without knowing the optimal cost?

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Traveling Salesman Problem (TSP)

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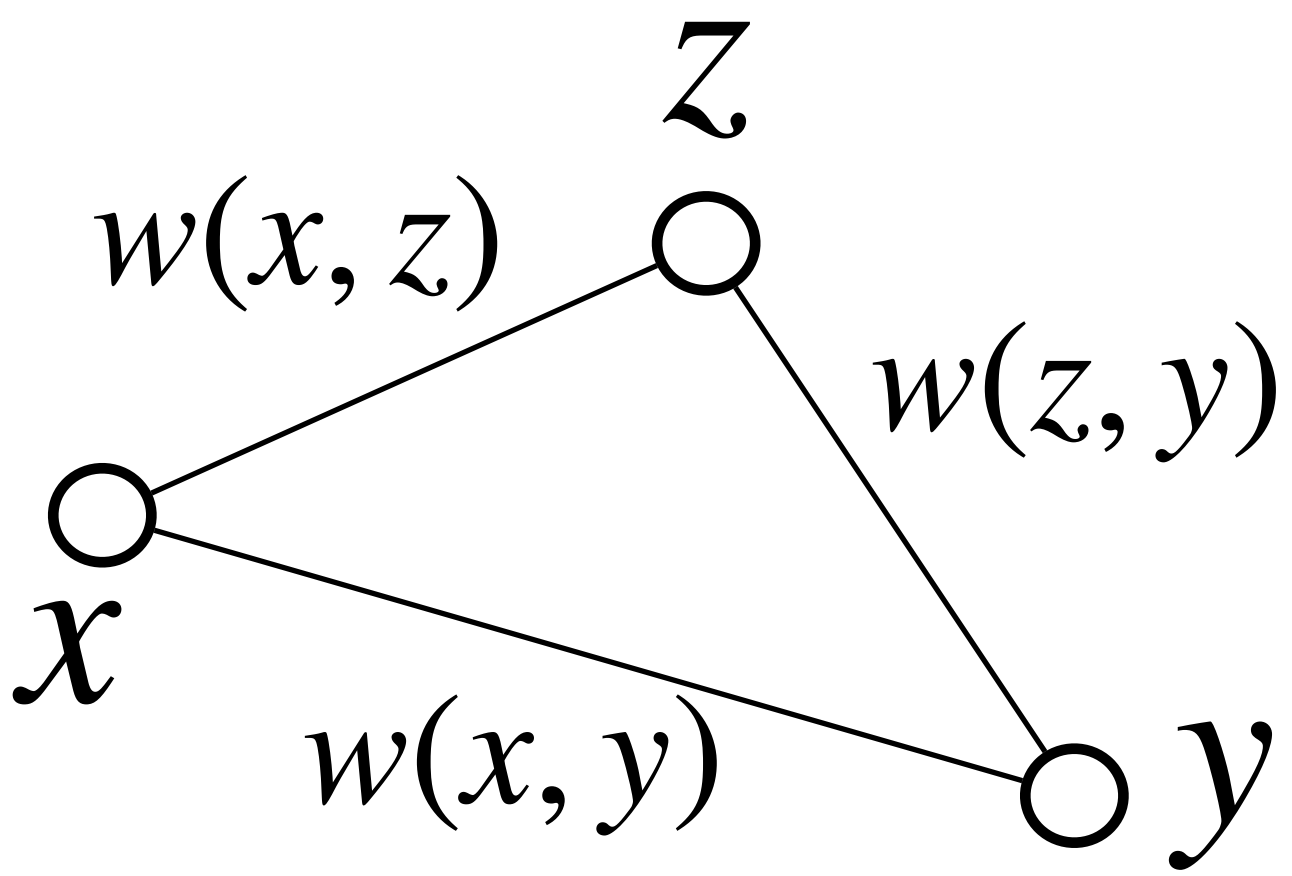

$$w(x, y) \leq w(x, z) + w(z, y)$$

Traveling Salesman Problem (TSP) with the Triangle Inequality

Input: An undirected complete graph G=(V,E), where every edge $(u, v) \in E$ has a non-negative integer weight $w(u, v)$. The edge weights satisfy the triangle inequality.

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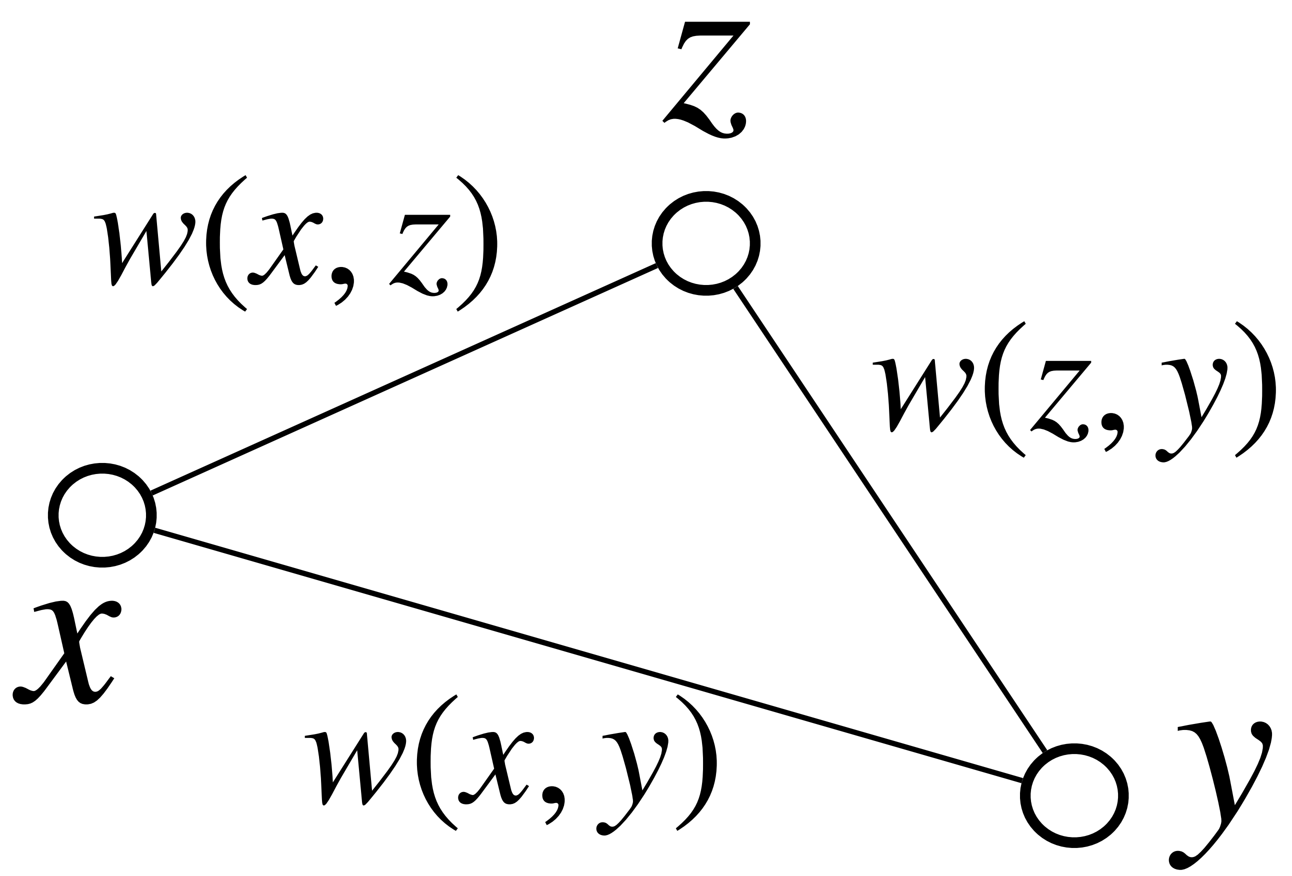

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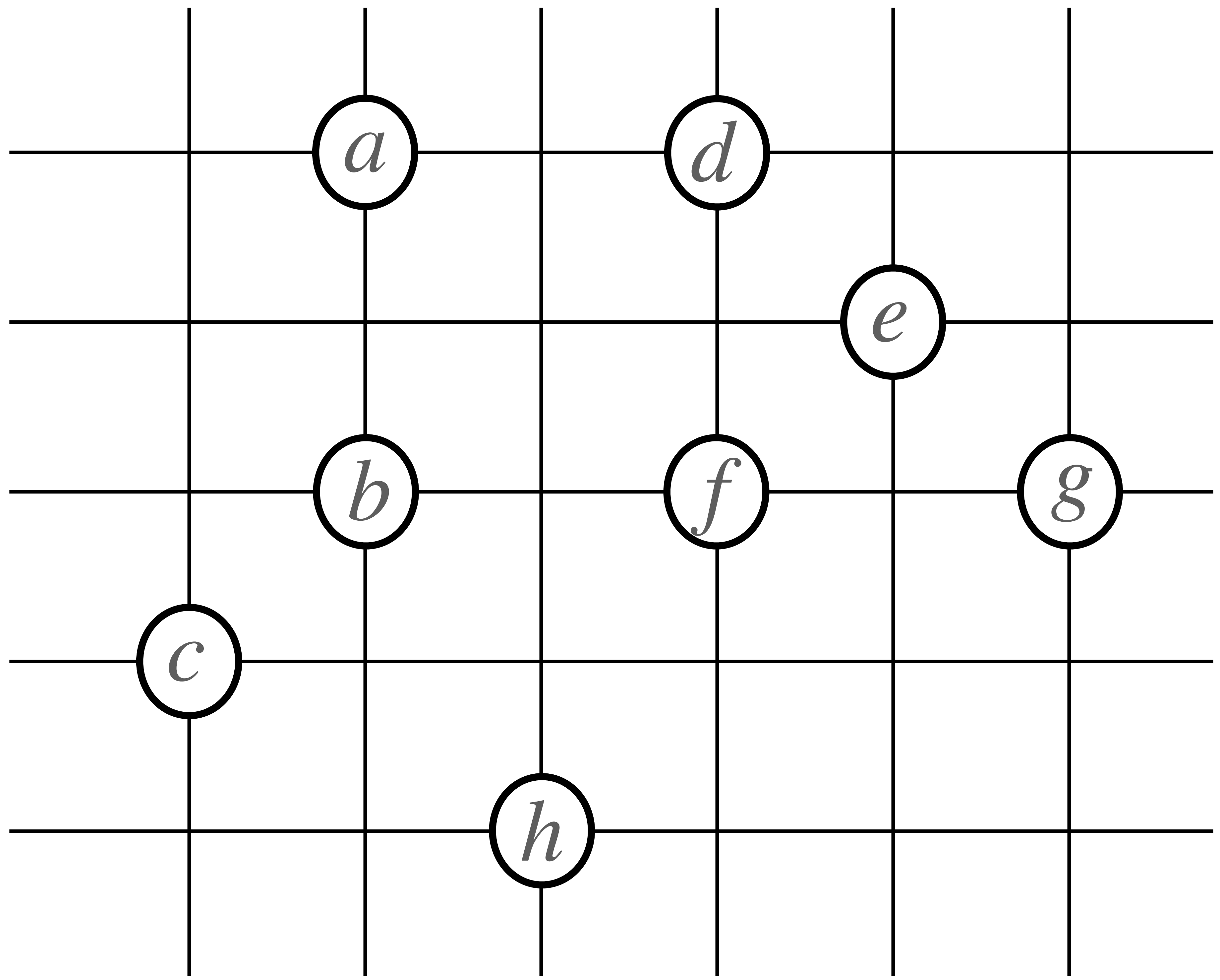
Idea of Algorithm:

1. Build an MST of G.

Edge weight between every two nodes: Their Euclidean distance.



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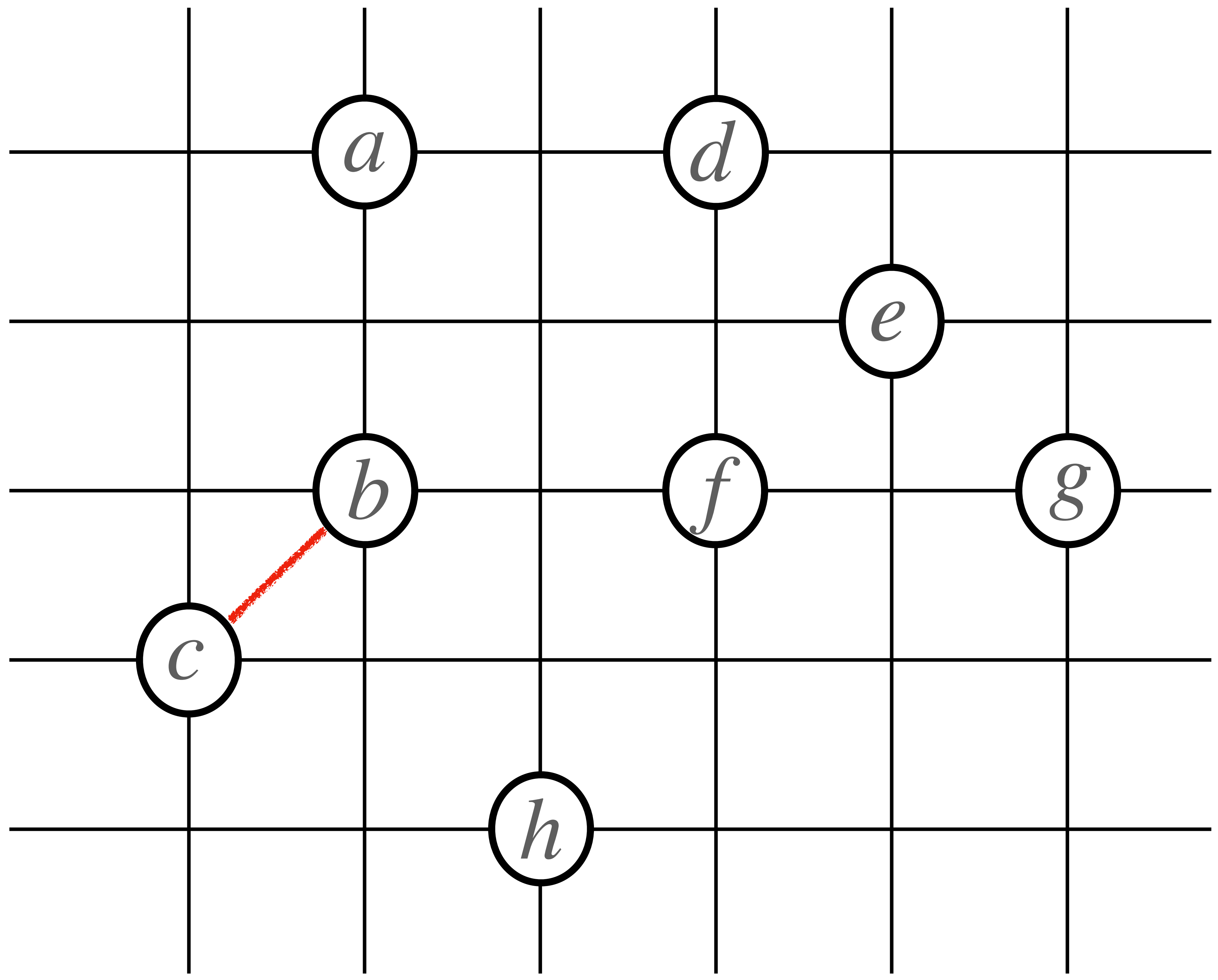

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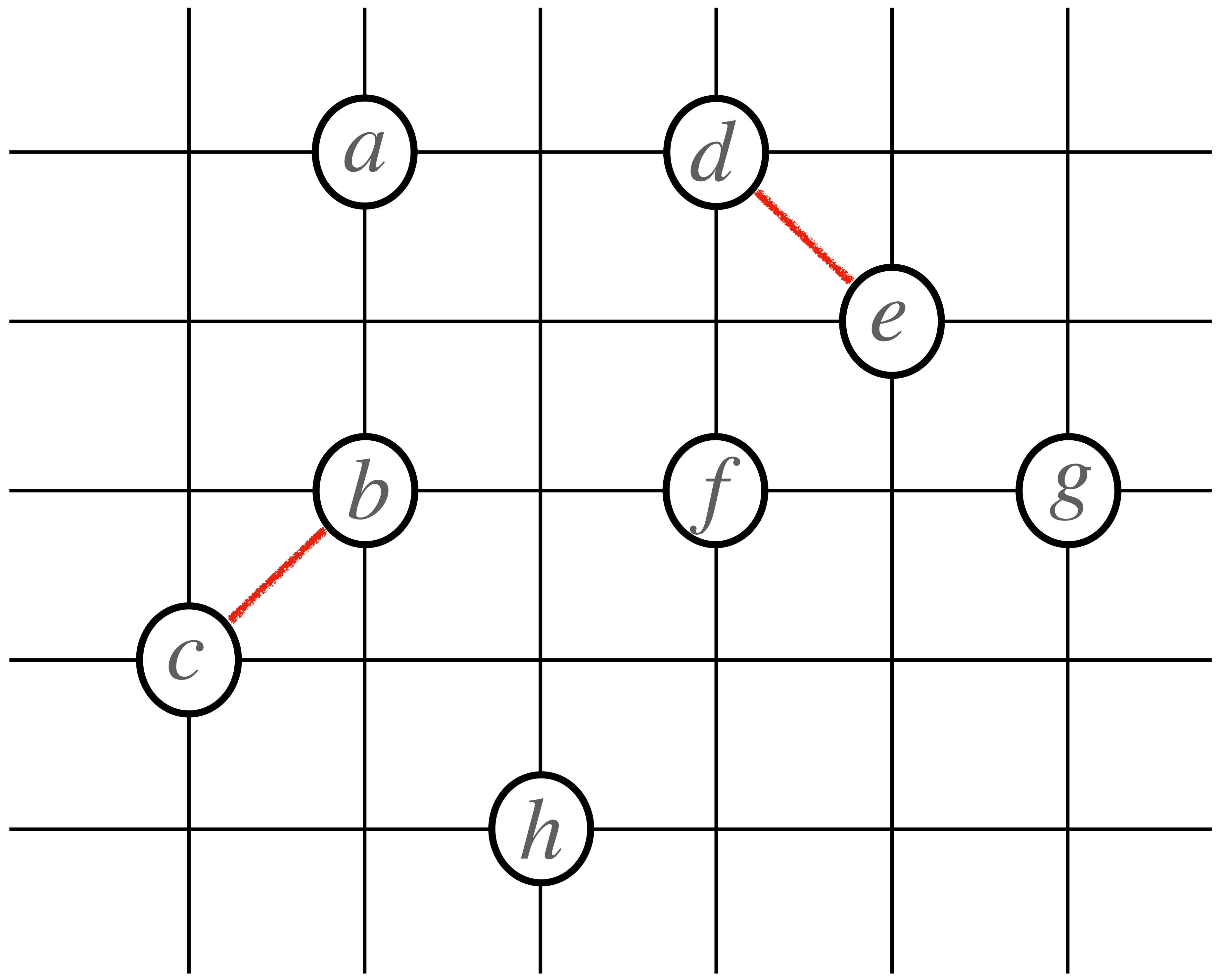

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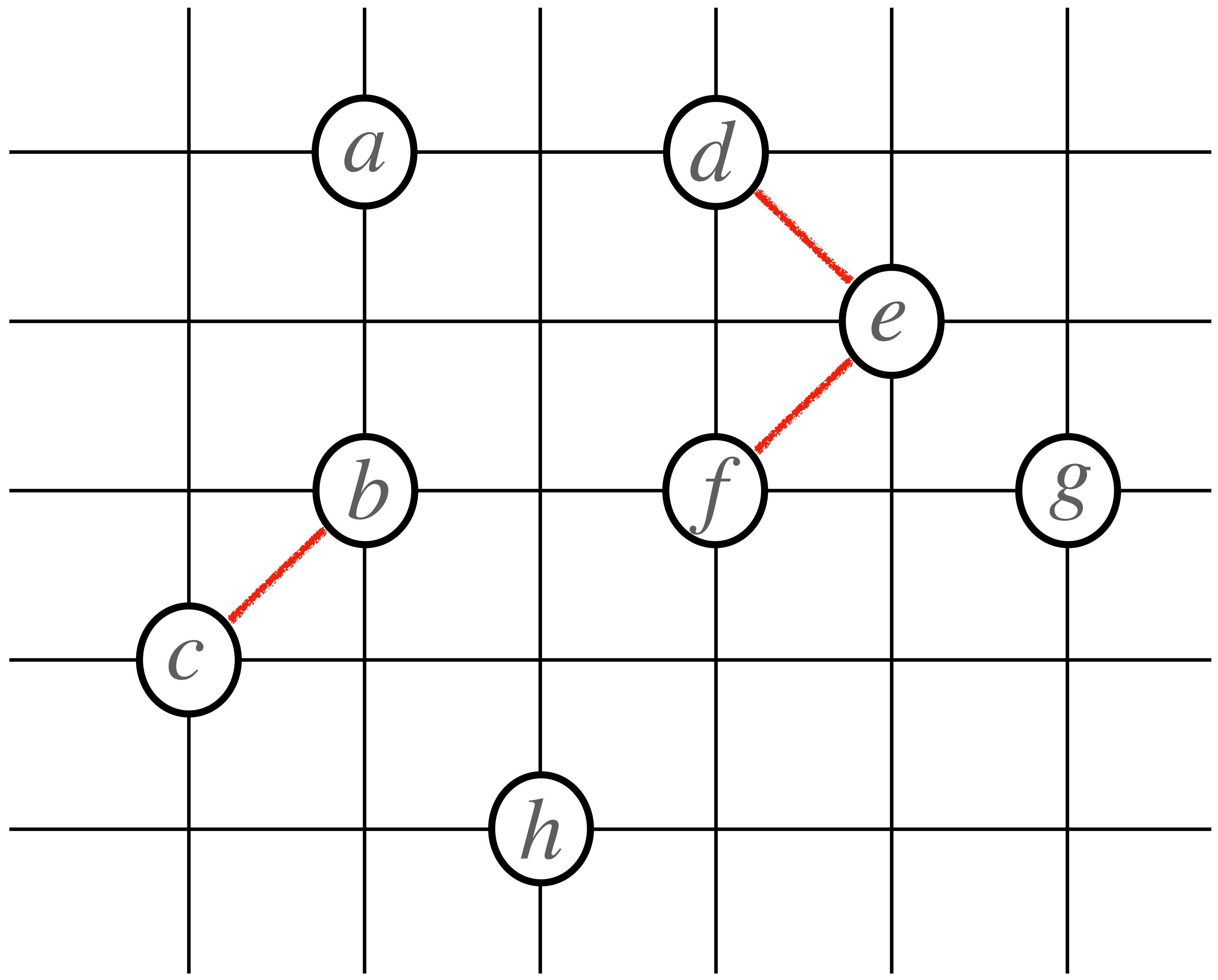

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Output: A Hamiltonian cycle of minimum weight.

Edge weight between every two nodes:
Their Euclidean distance.

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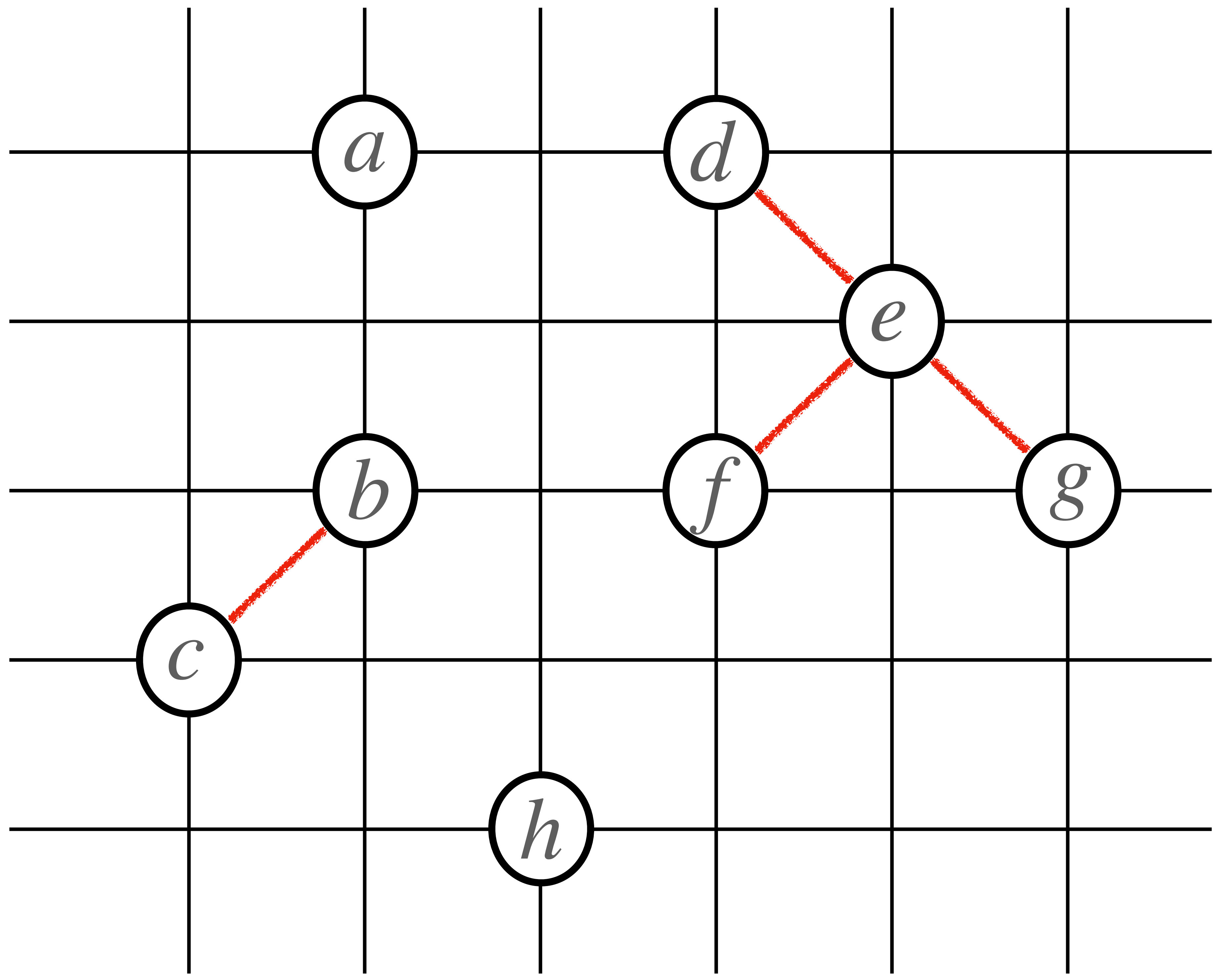

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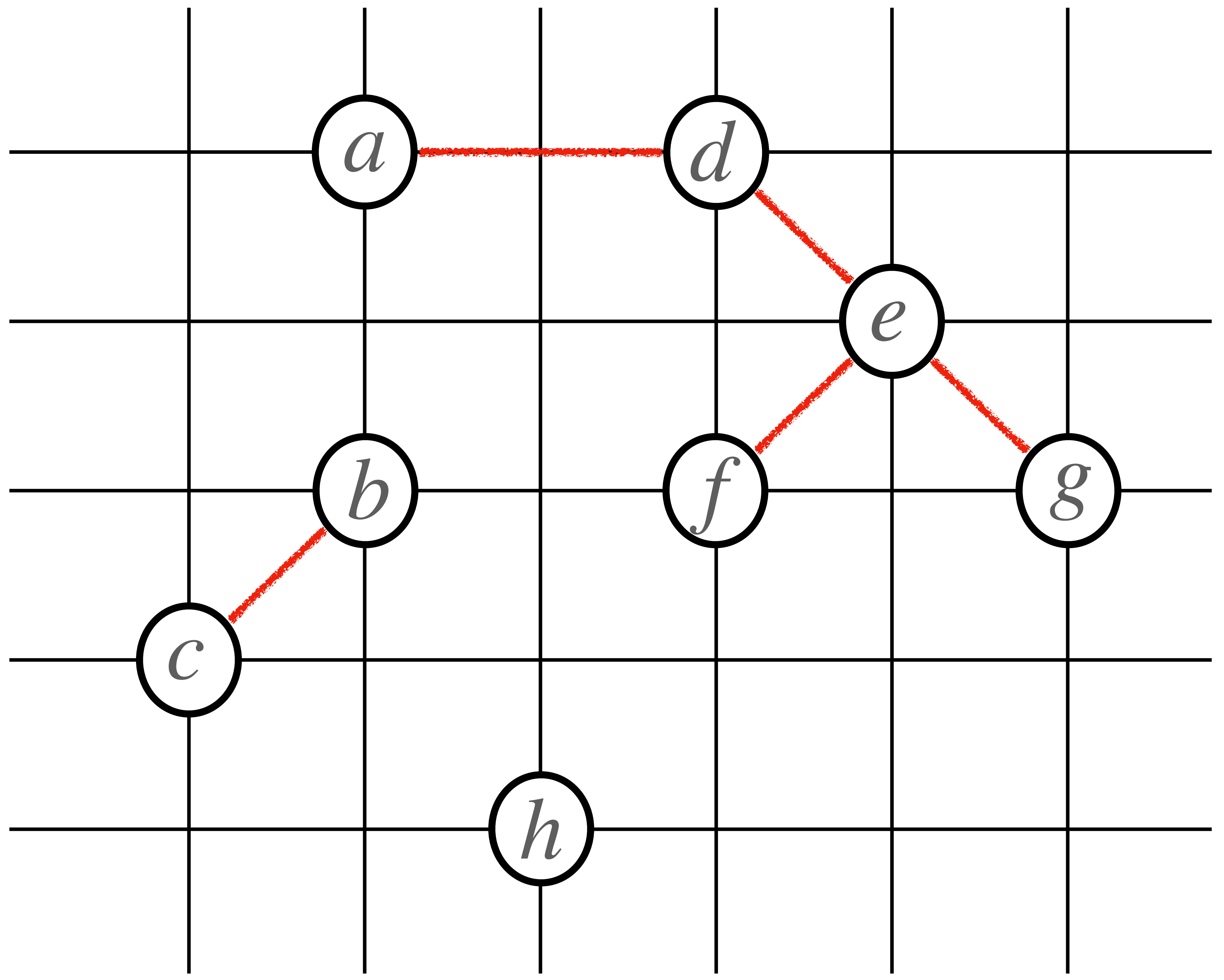

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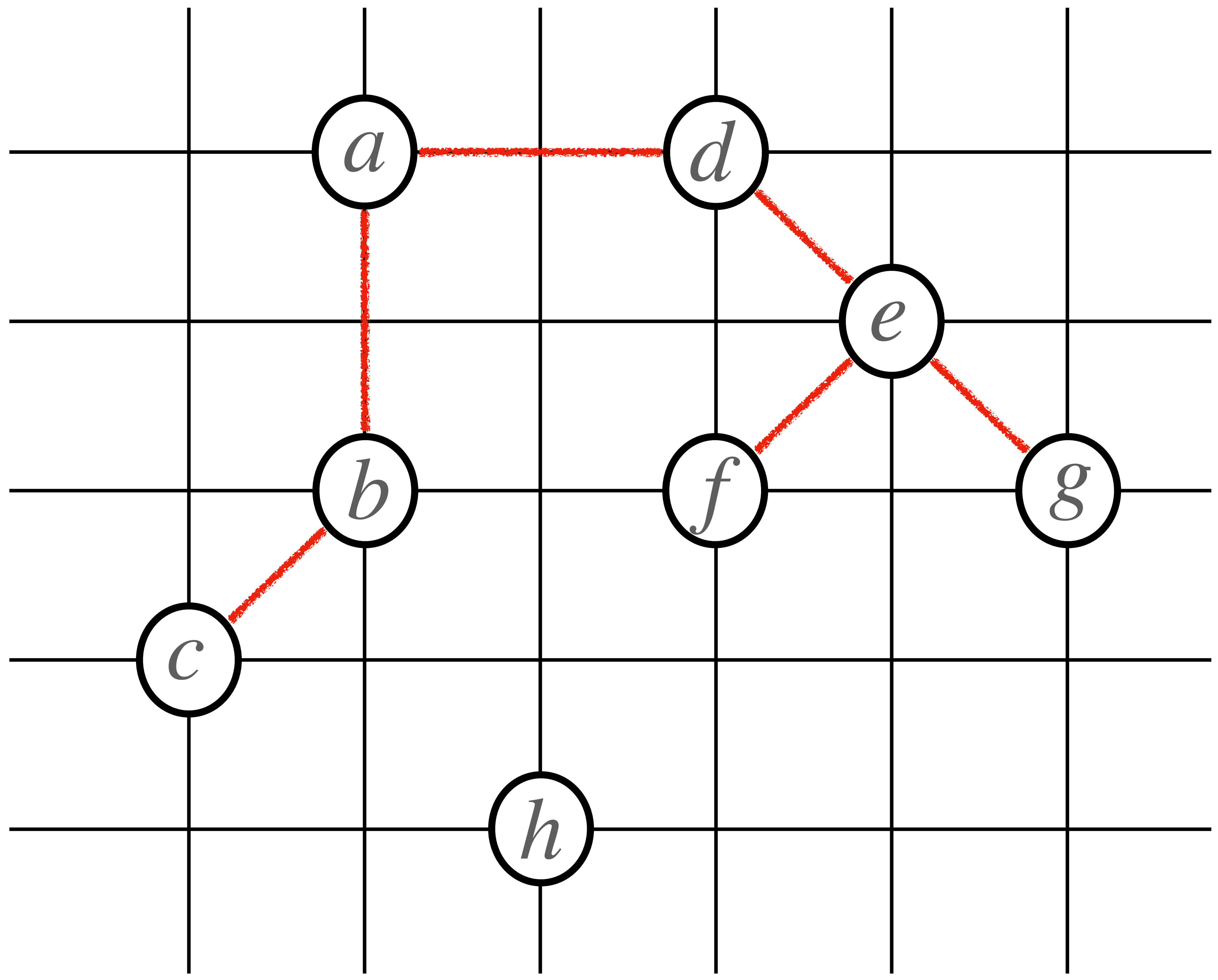

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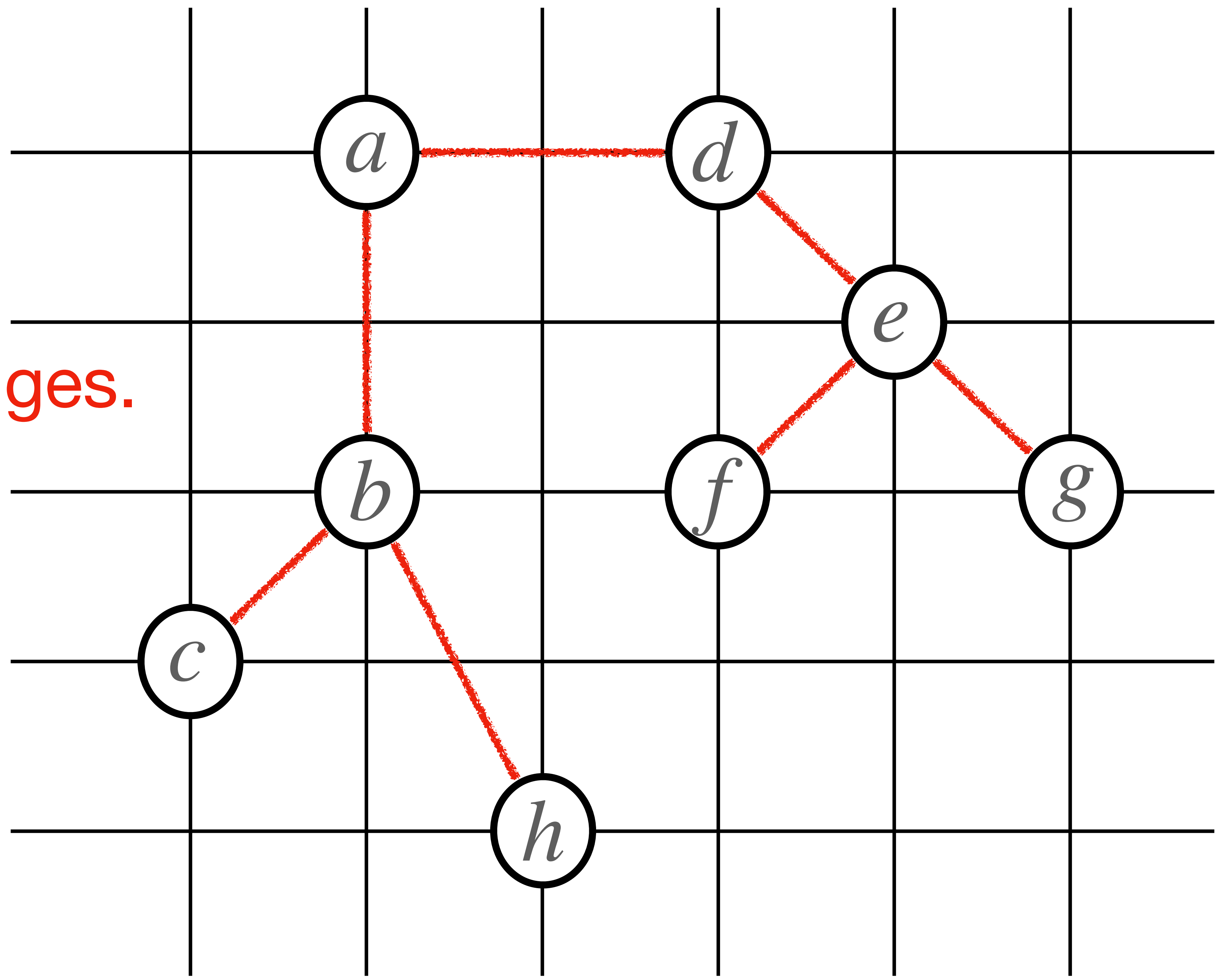

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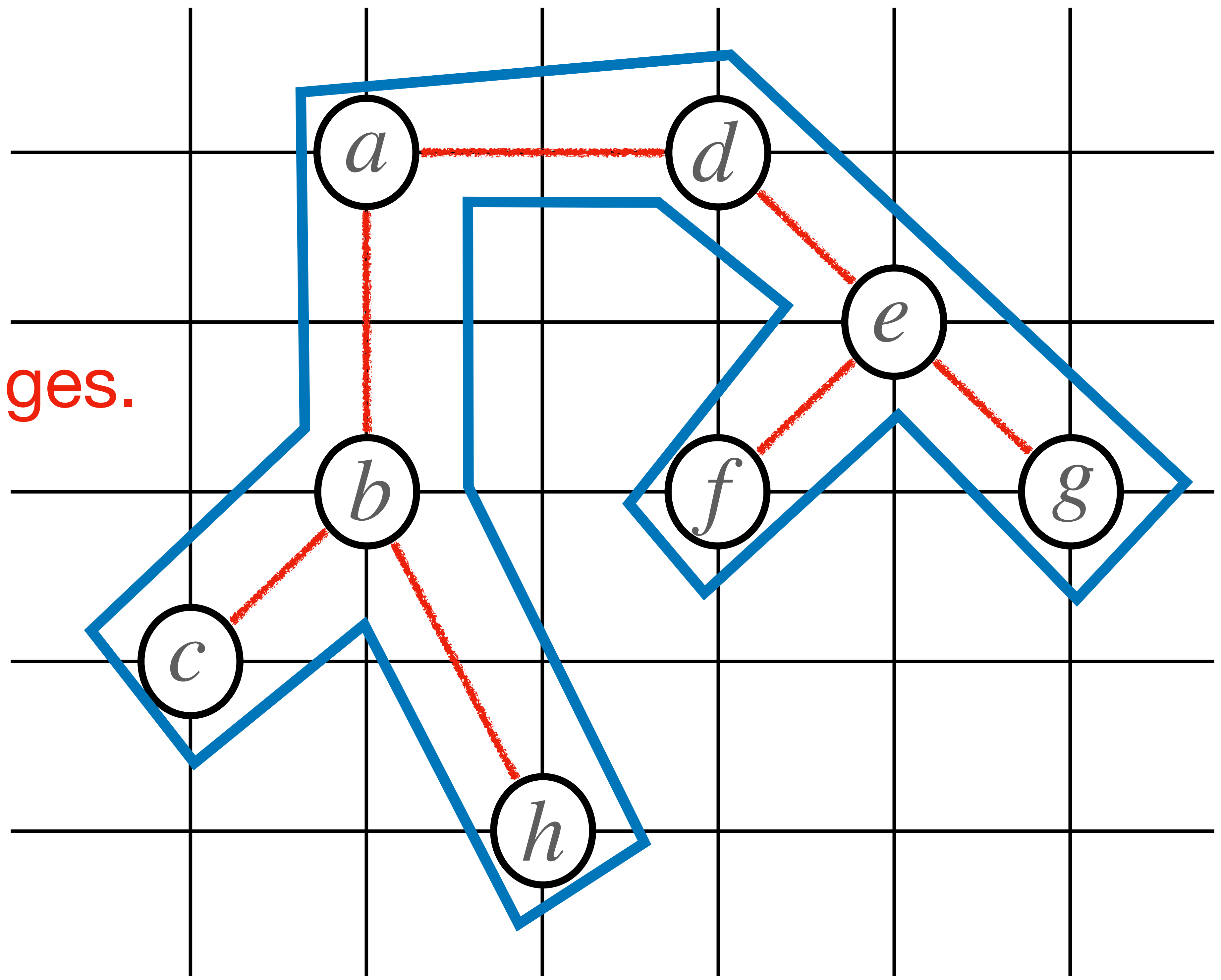

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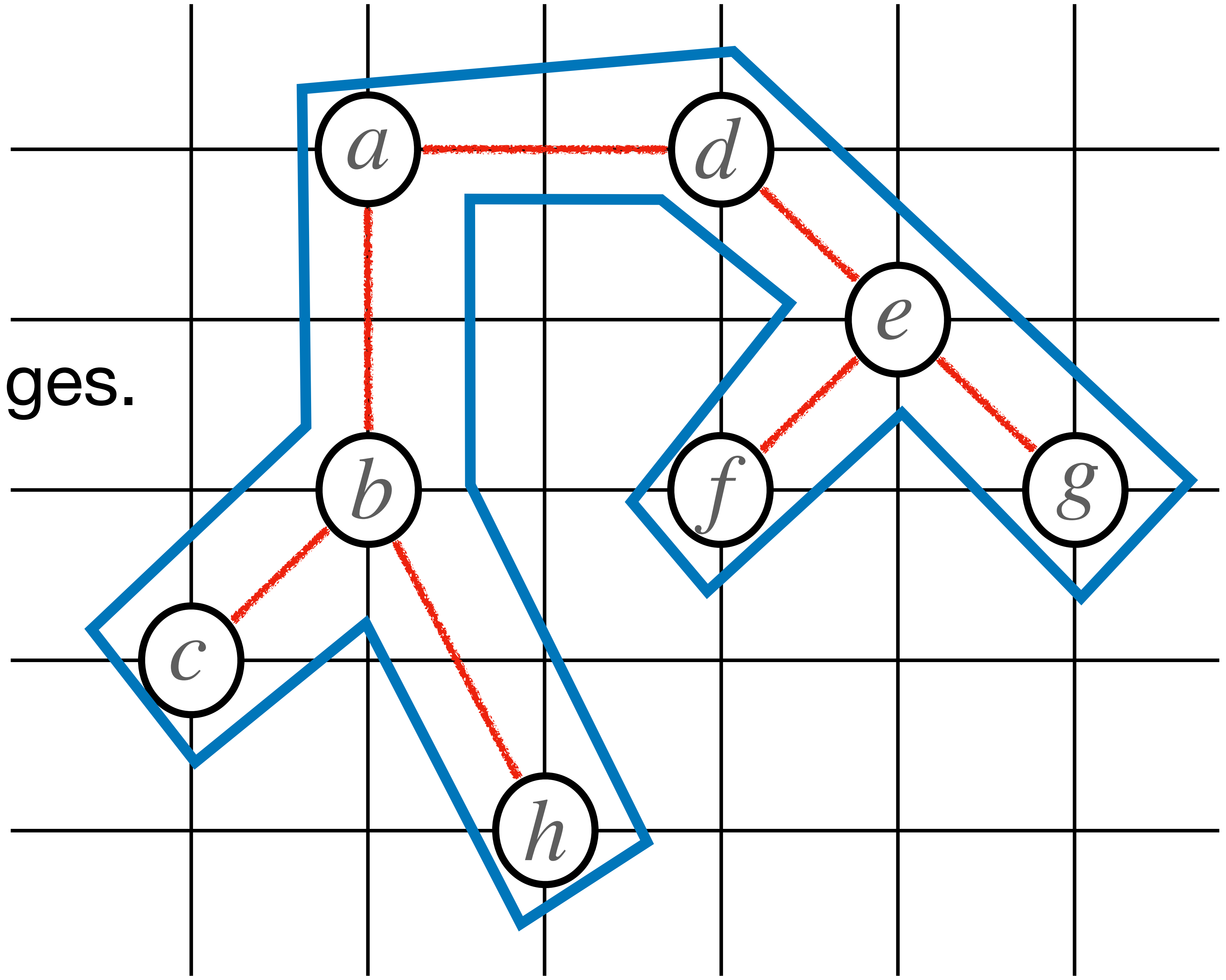

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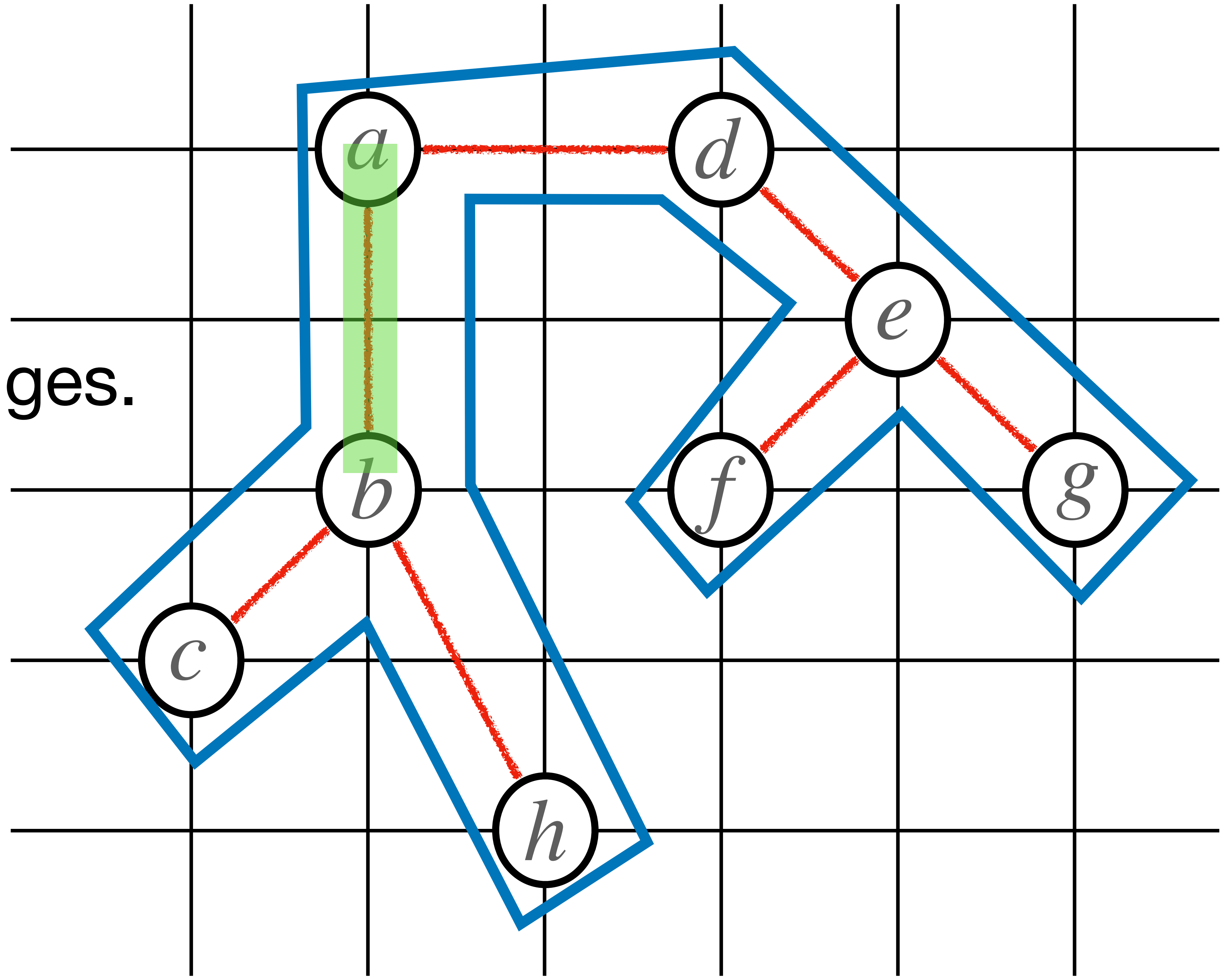

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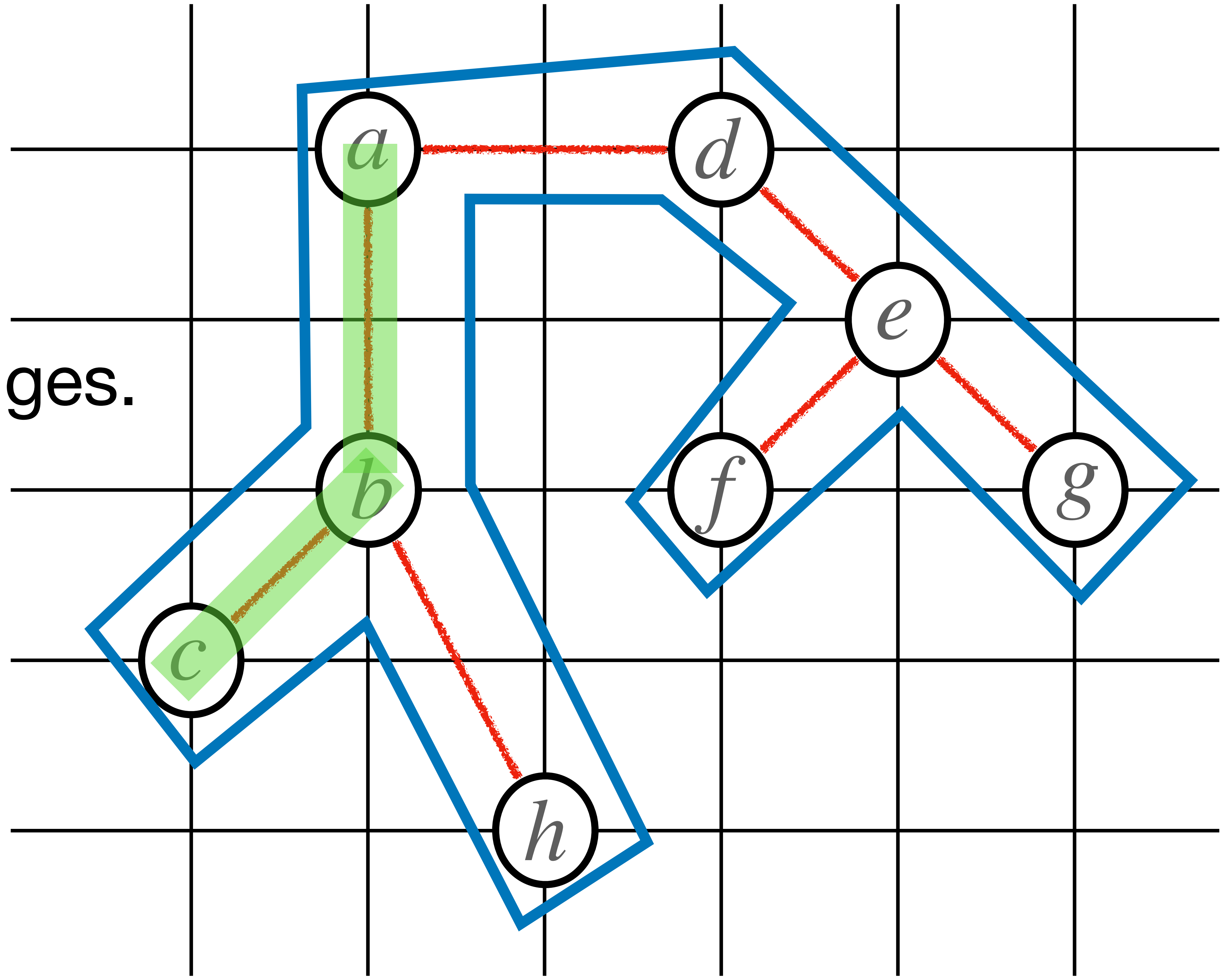

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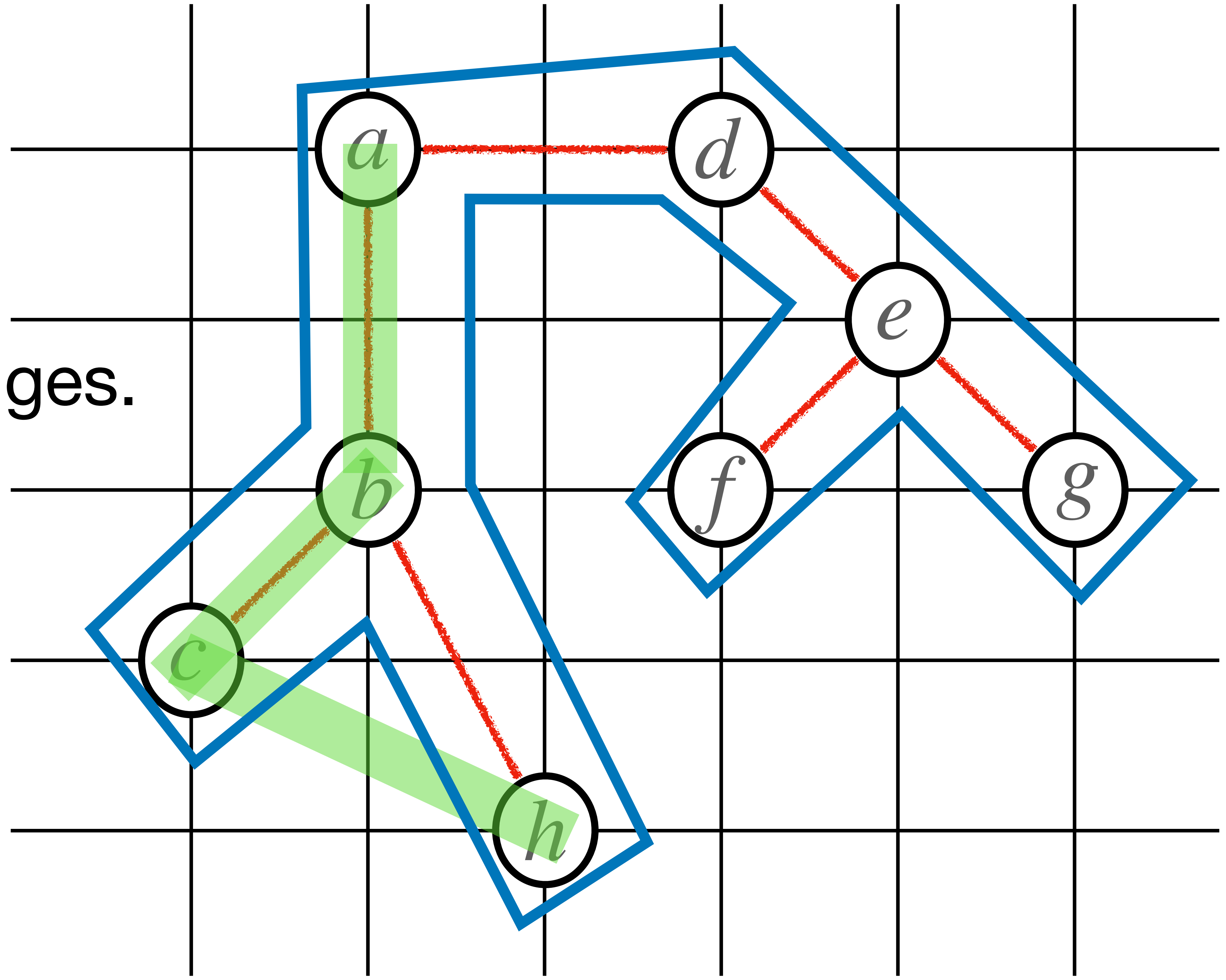

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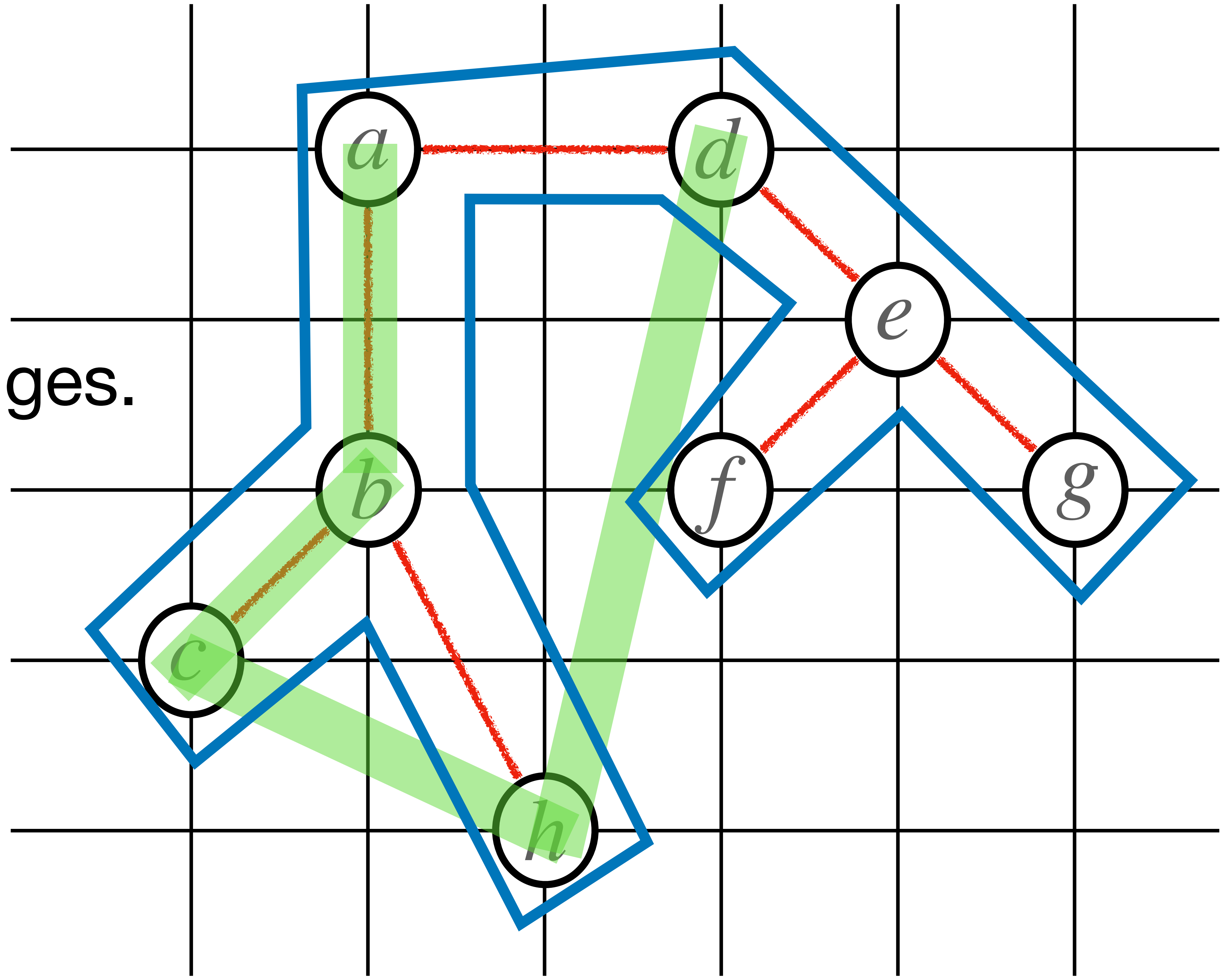

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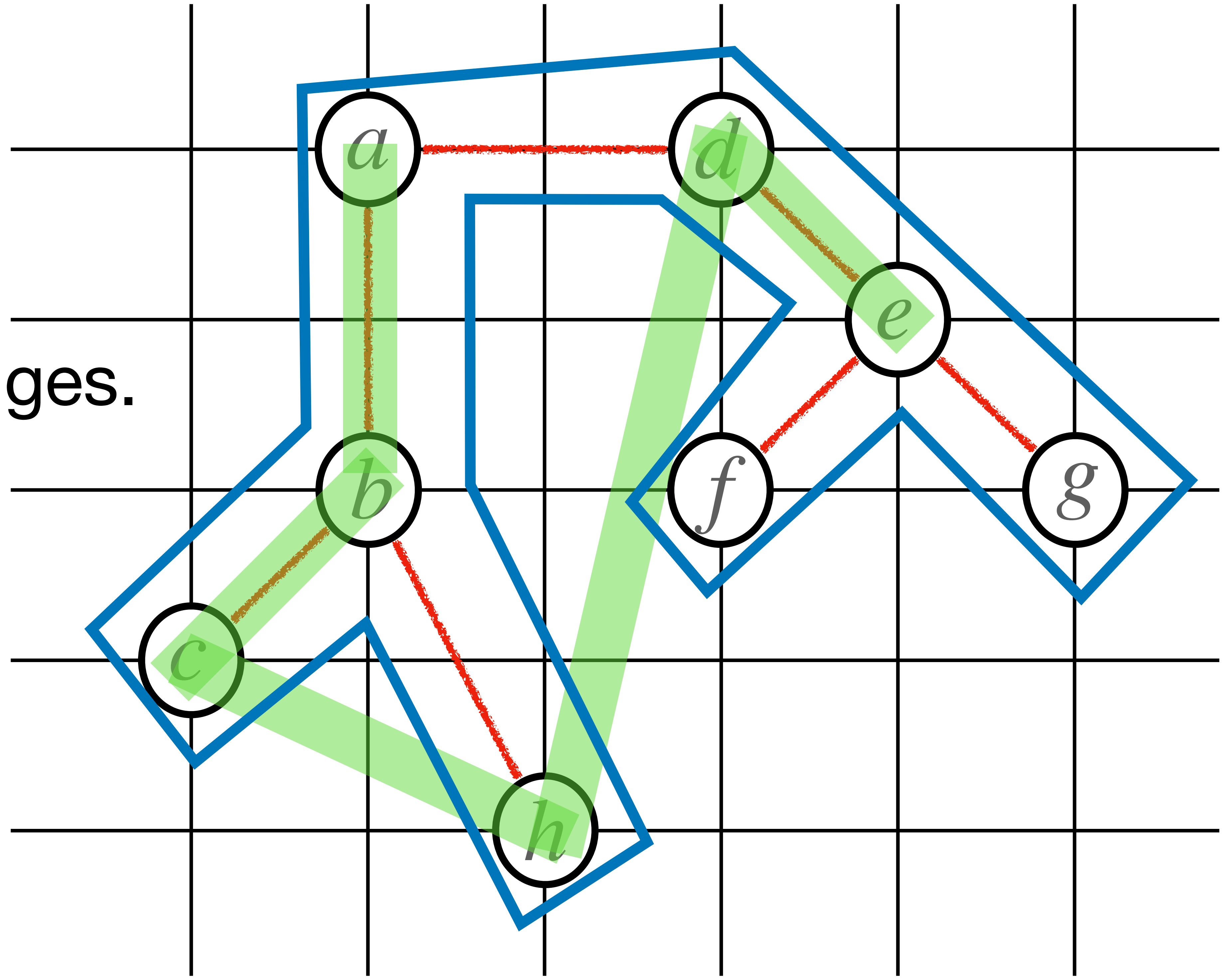

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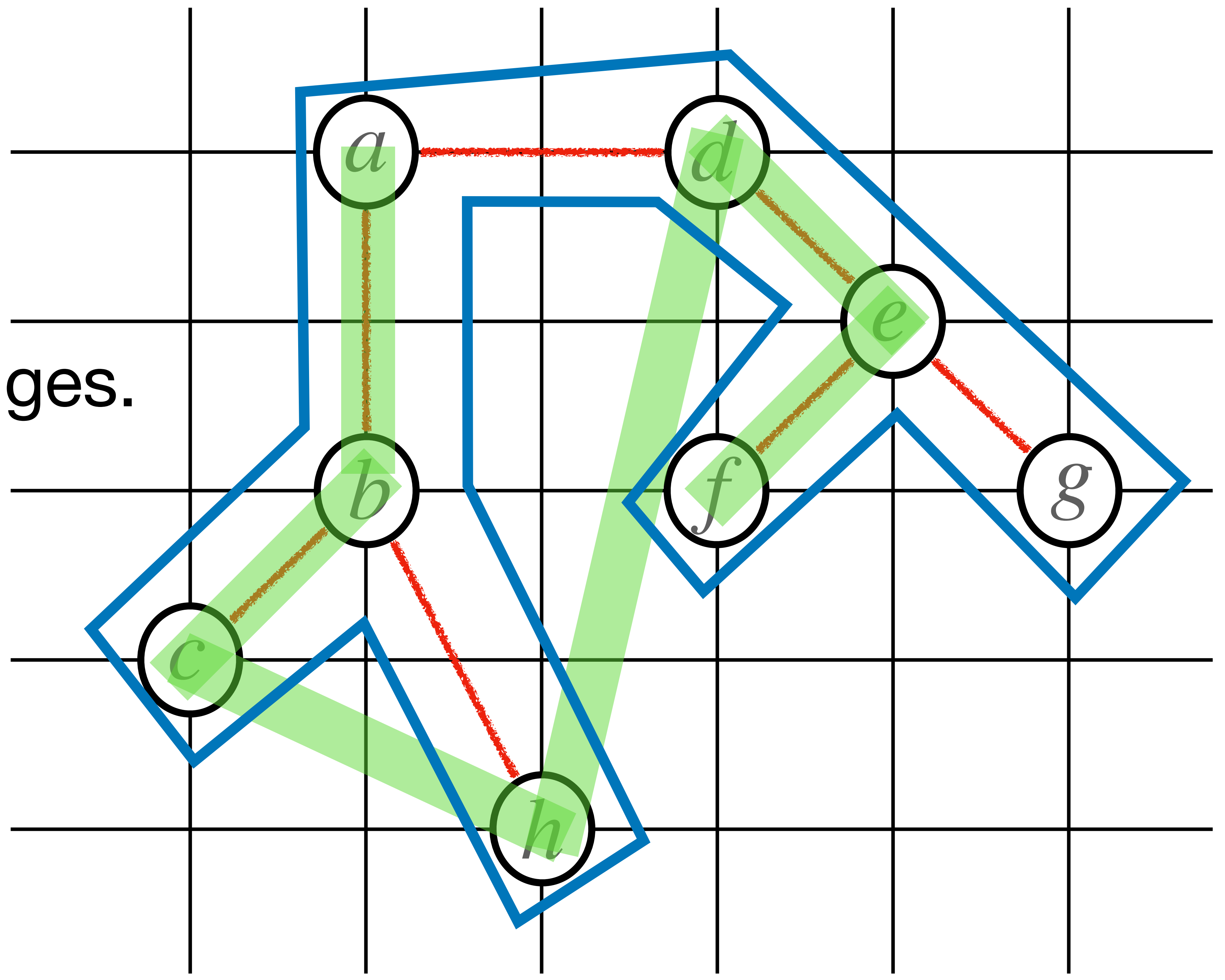

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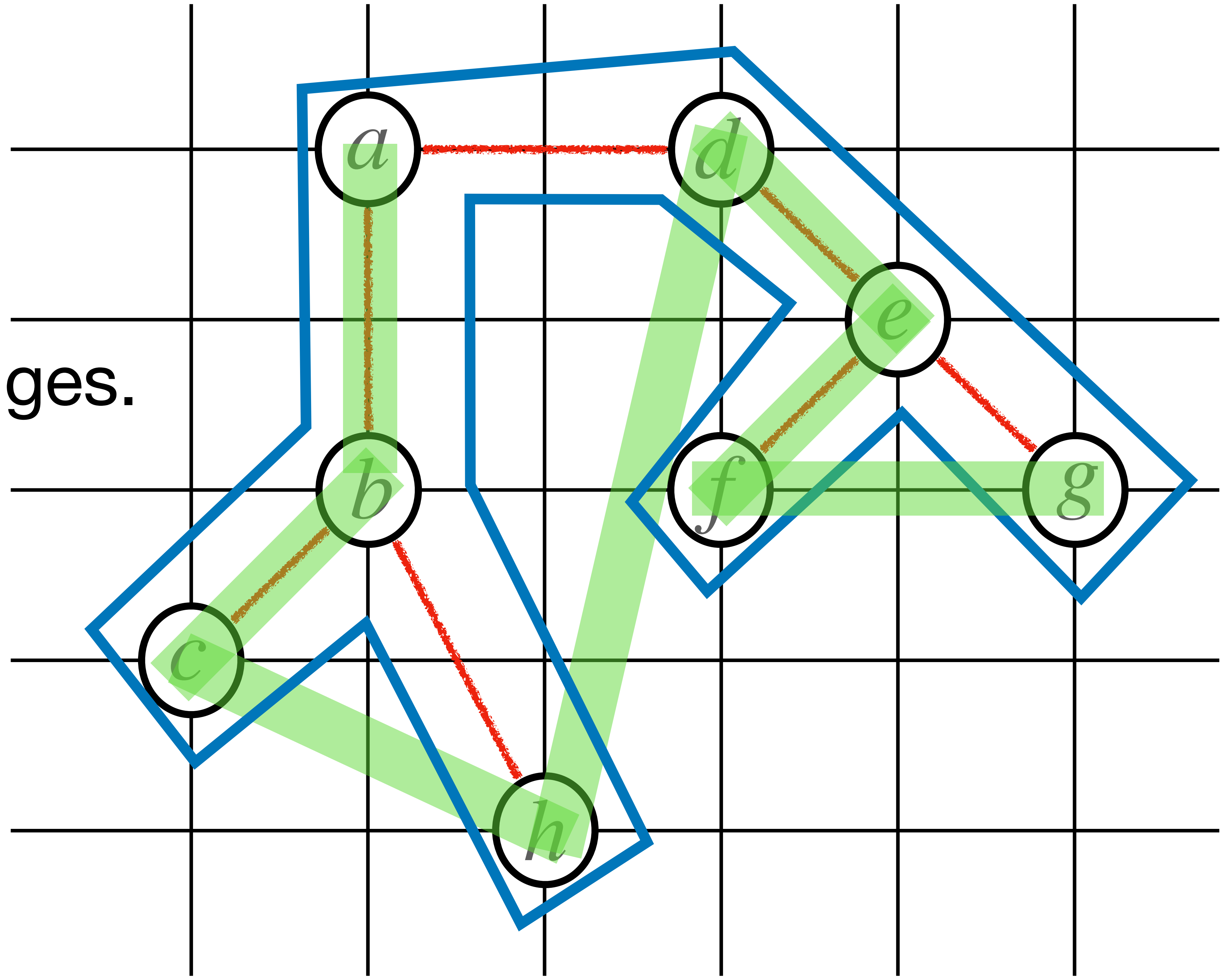

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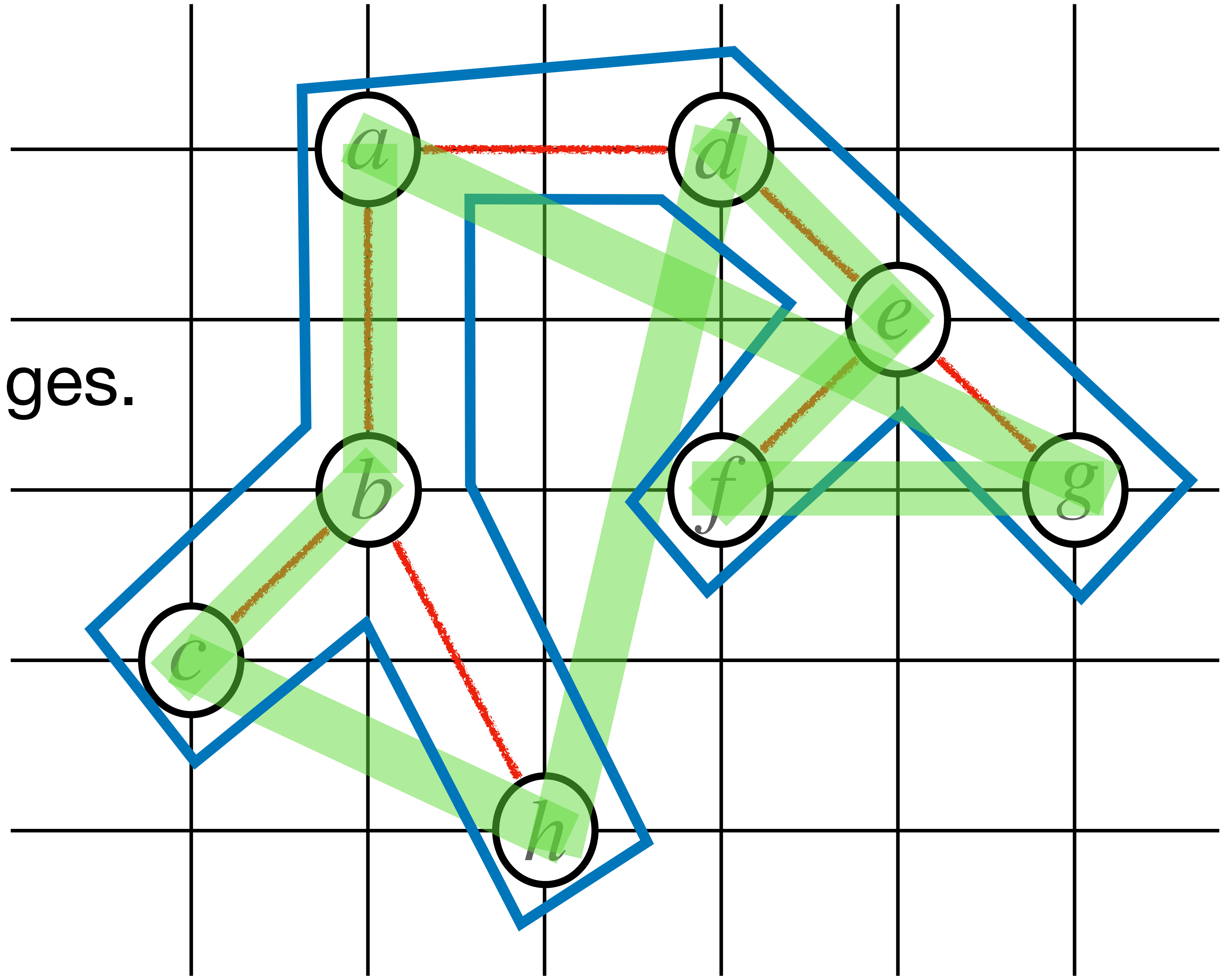

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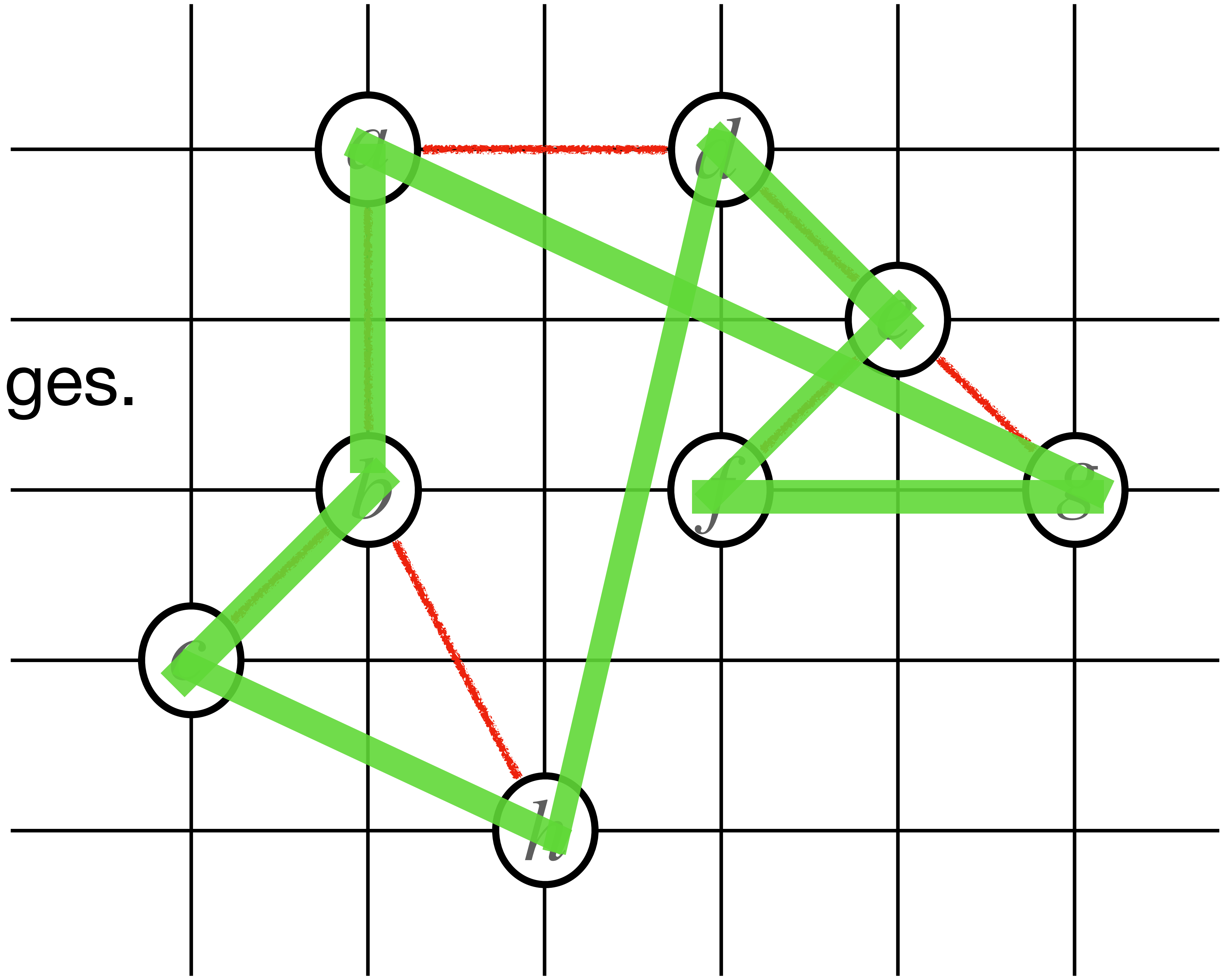
See the Hamiltonian cycle more clearly.



Quiz questions:

1. What is the main idea of the above approximation algorithm for “TSP with triangle inequality”?

2. Can you think of an instance for which the above approximation algorithm will output an optimal solution, and an instance for which it will not?

Roadmap of this lecture:

1. Define "Approximation Algorithm".

2. Understand approximation algorithms by solving the "Vertex Cover Problem".

   2.1 An approximation algorithm for "Vertex Cover Problem".

   2.2 Analyze the approximation ratio of the algorithm.

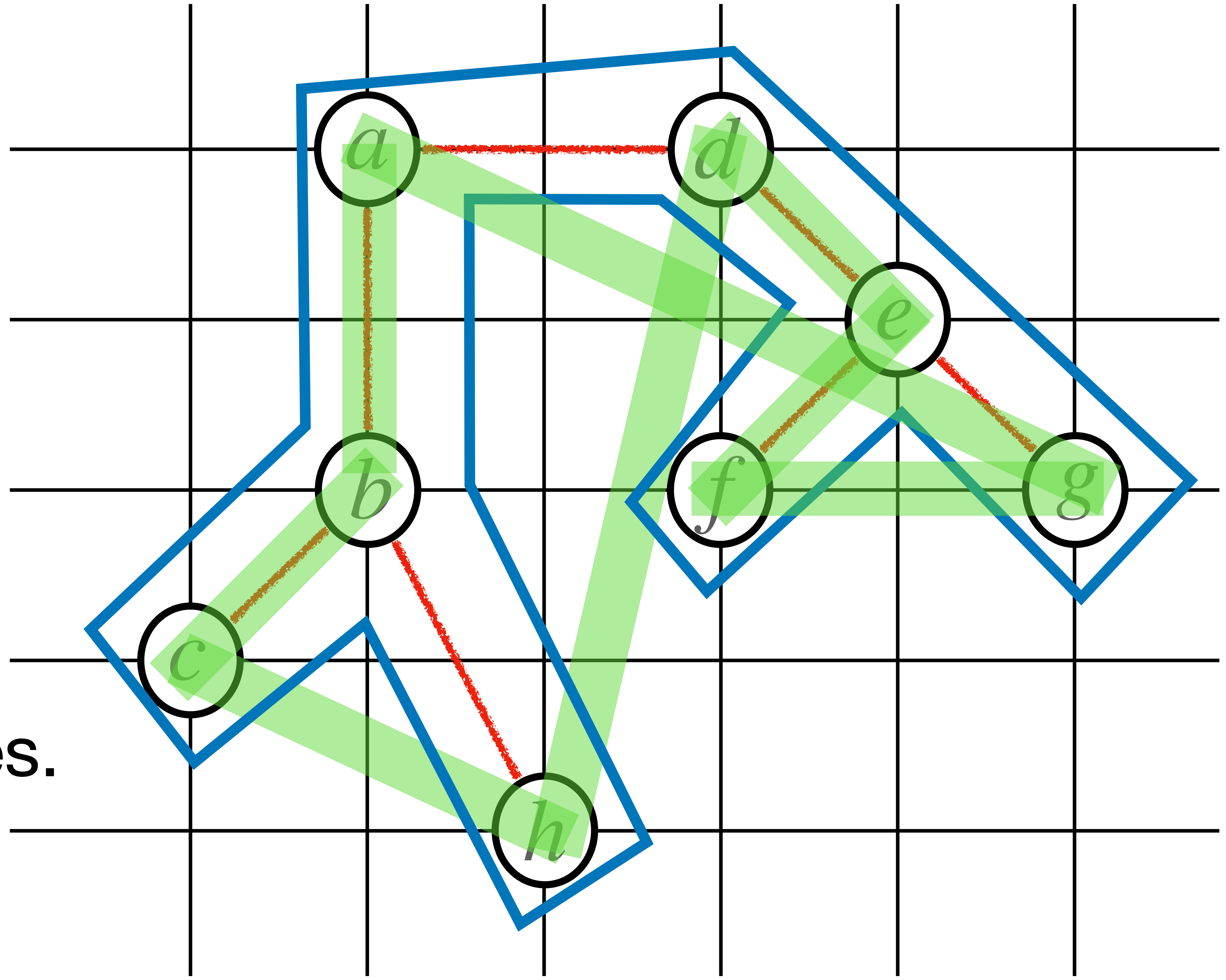
3. Understand approximation algorithms by solving the "Traveling Salesman Problem (TSP)".

   3.1 An approximation algorithm for TSP with the triangle inequality.

   3.2 Analyze the approximation ratio of the algorithm.

Theorem: The above algorithm for TSP with triangle inequality is a polynomial-time 2-approximation algorithm.

Proof:



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C_{DFS} : Weight of the "tour" obtained in Step 2.

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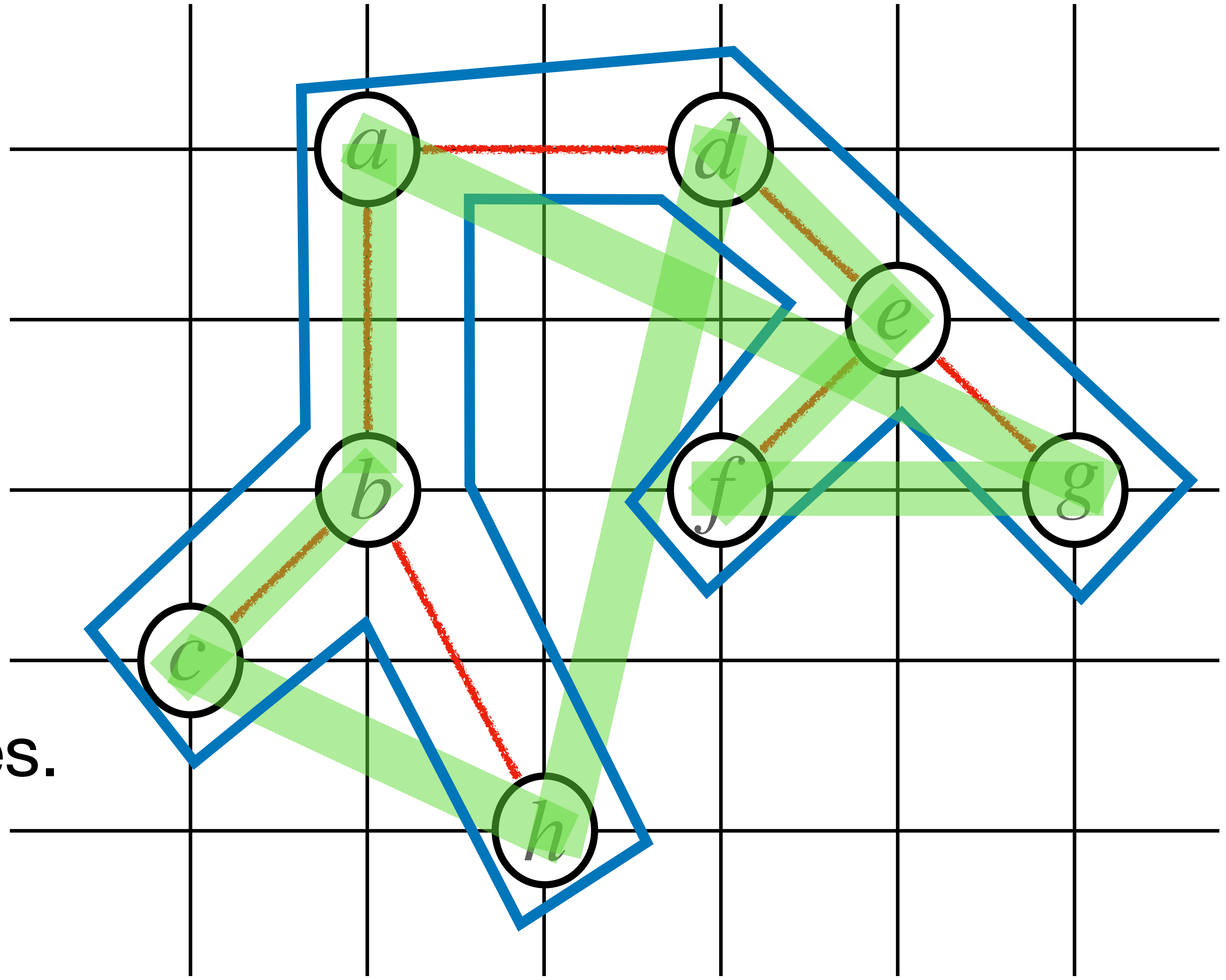
C^* : Minimum weight of an optimal Hamiltonian cycle.



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$\boxed{C_{MST} \leq C^*}$ Why?

Lower bound to C^*

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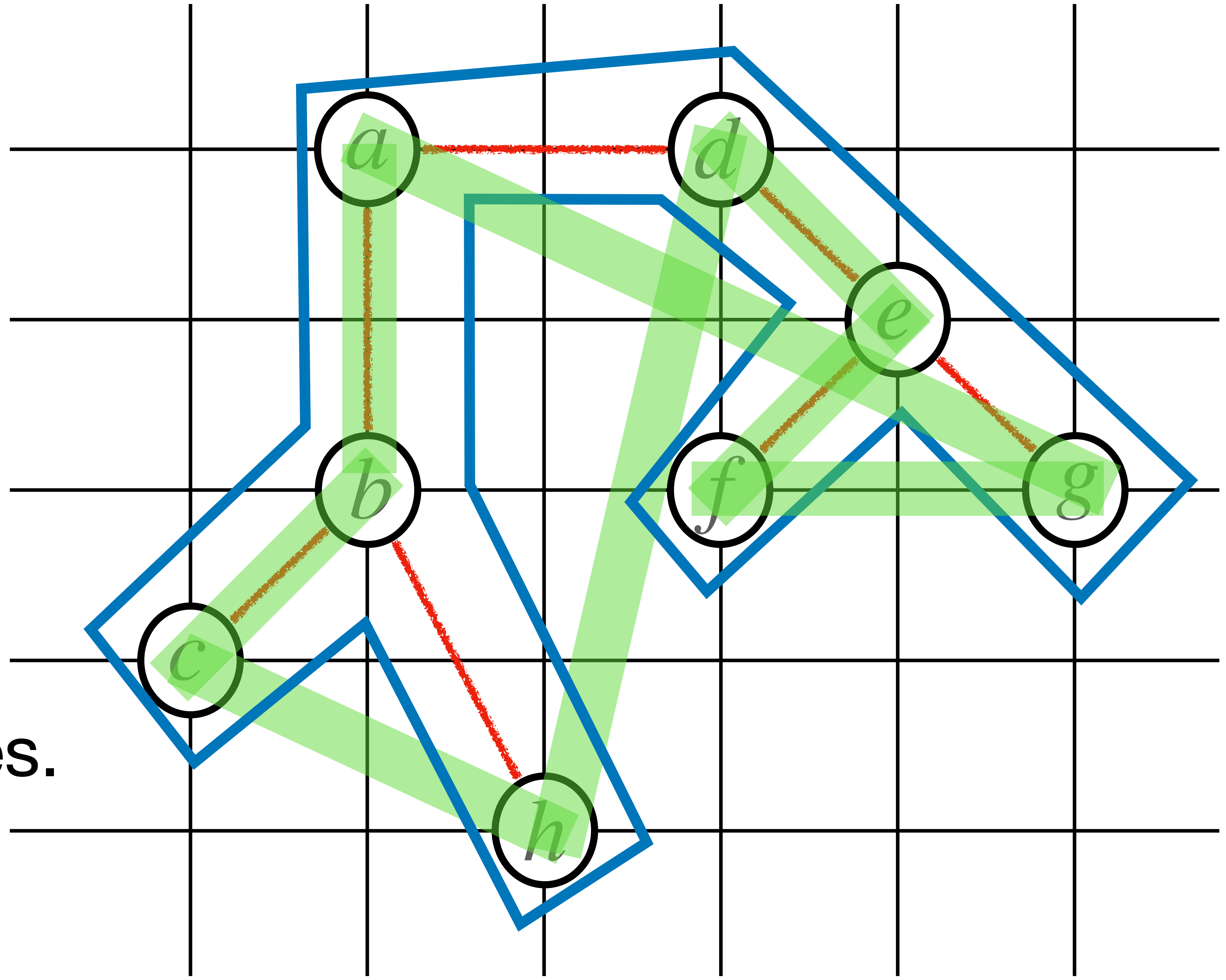

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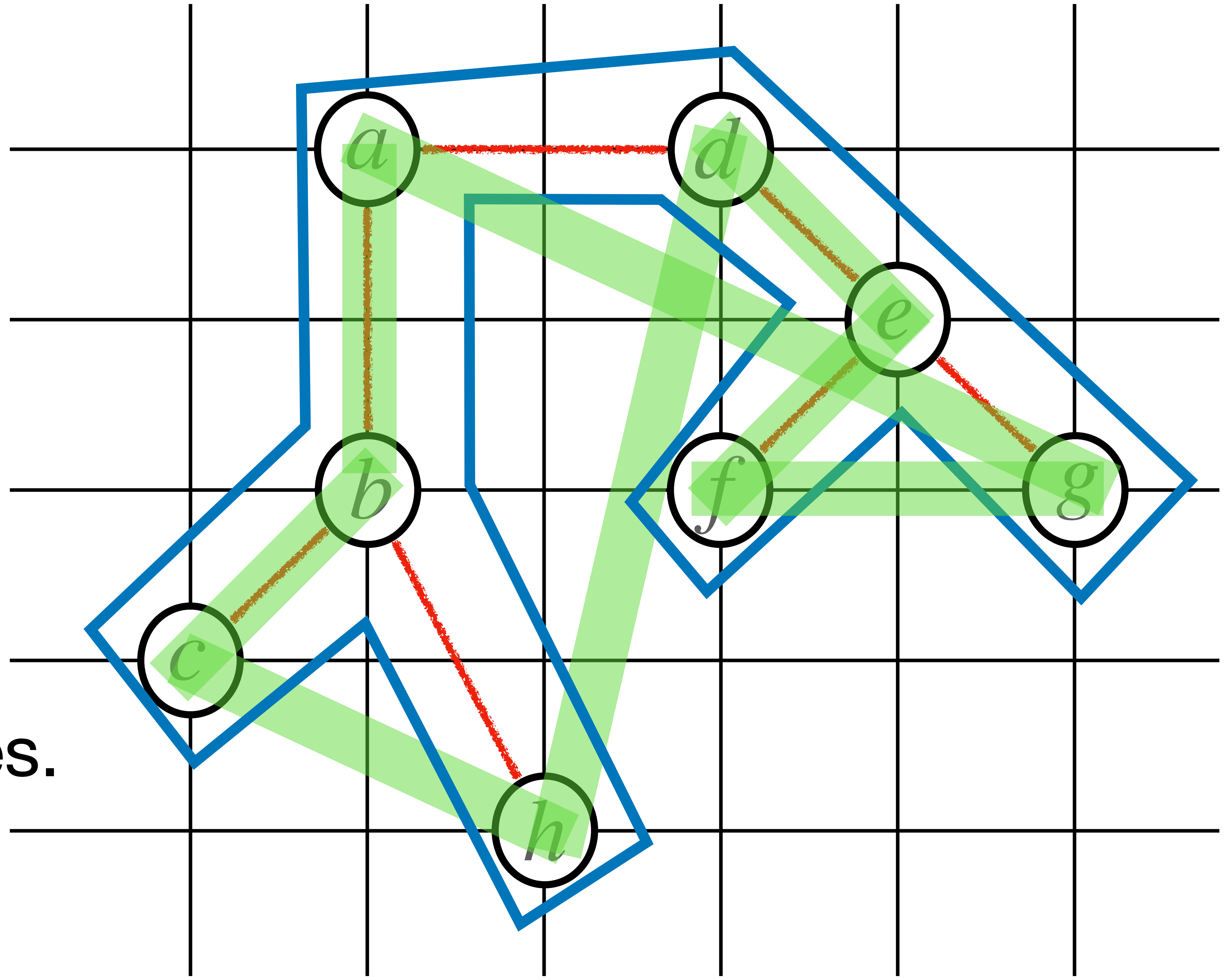

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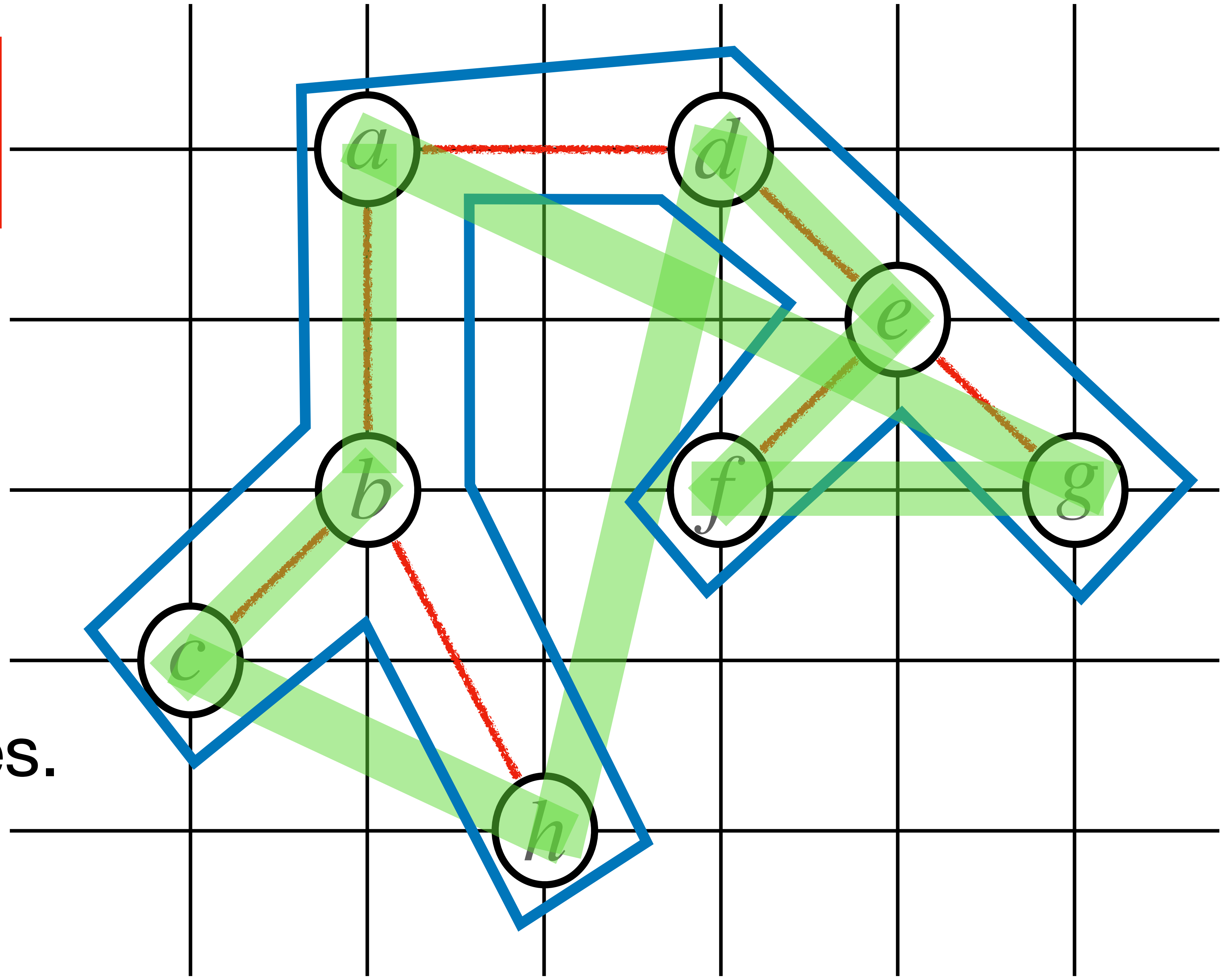

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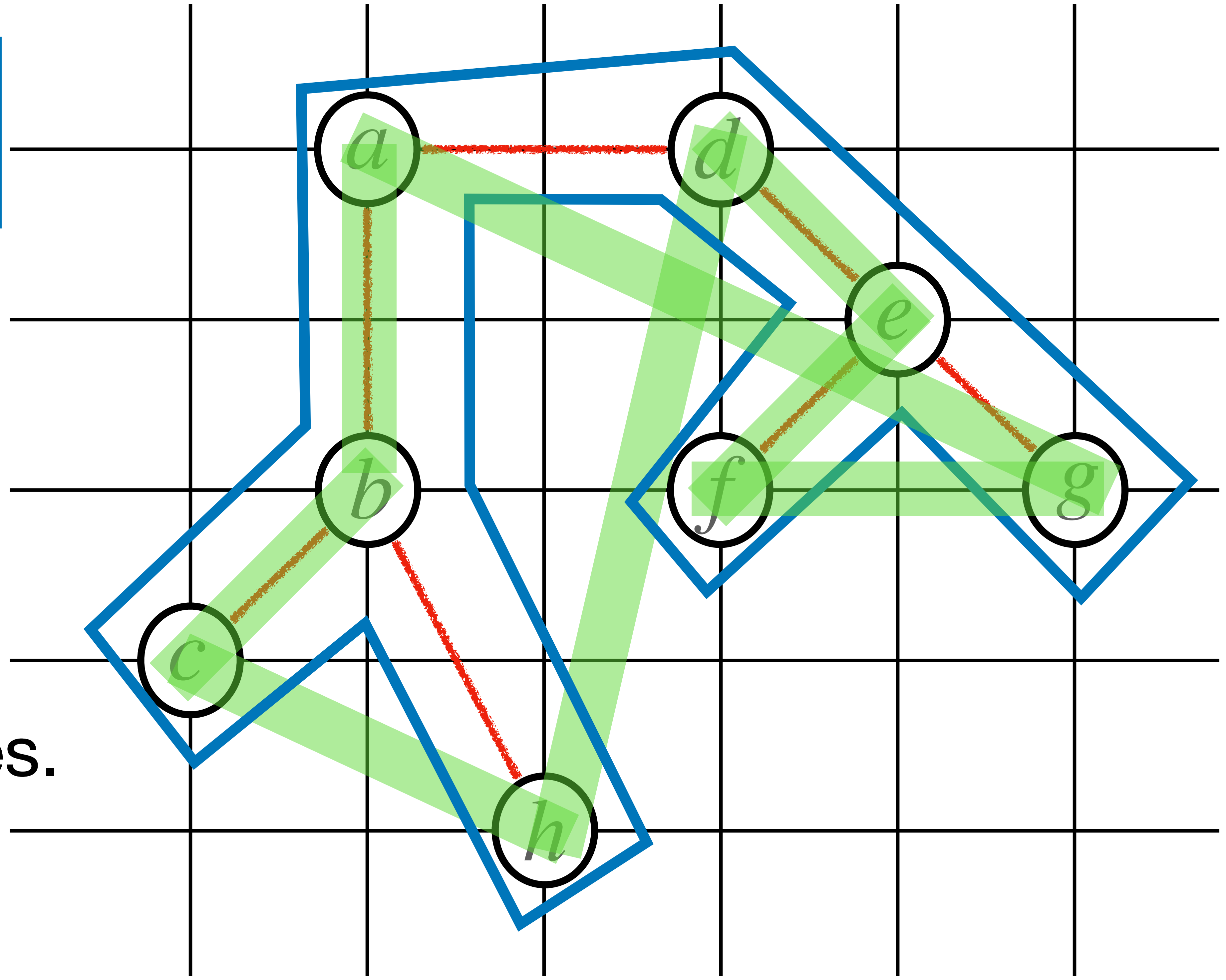

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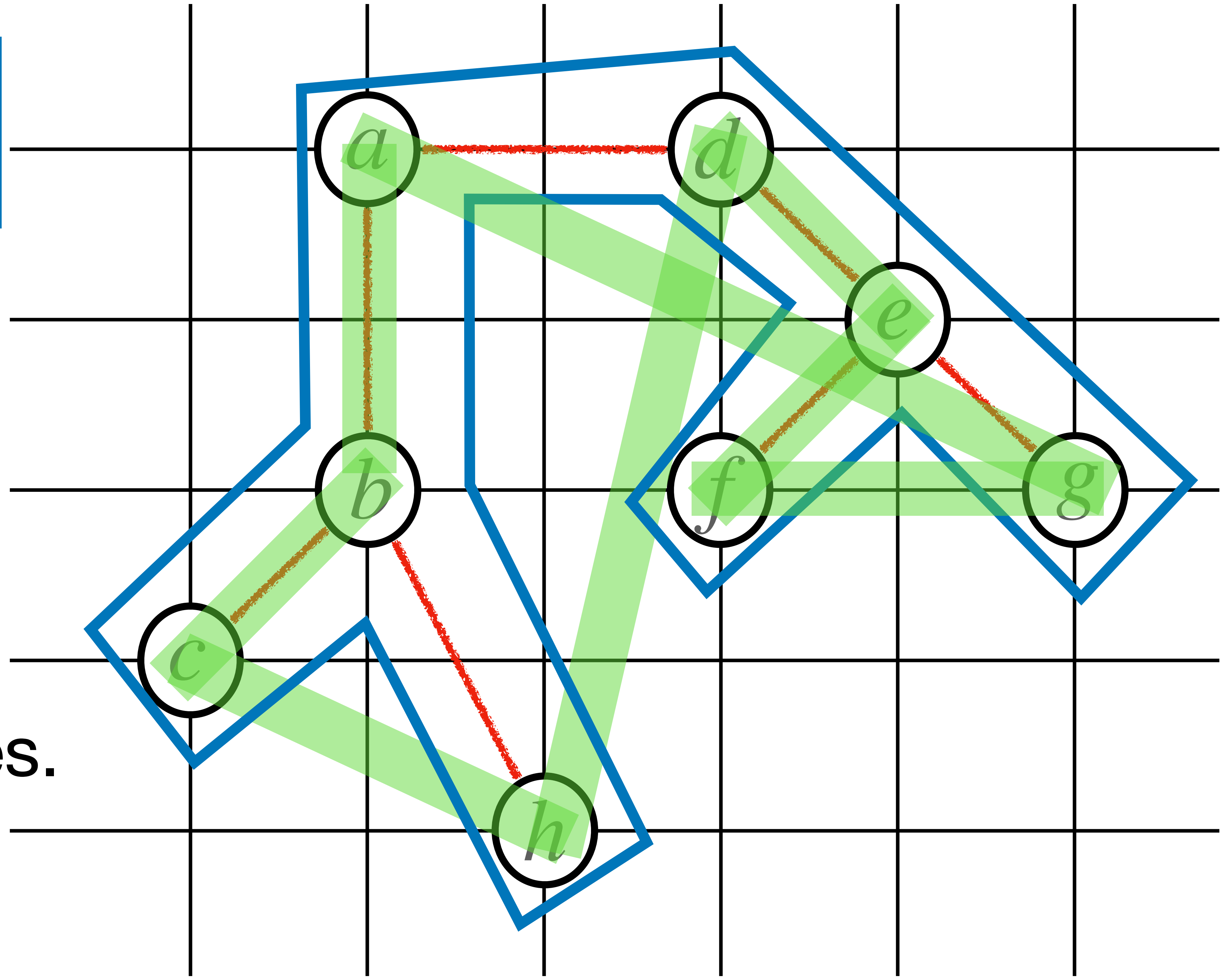
$$\boxed{C \leq C_{DFS} = 2C_{MST} \leq 2C^*} \Rightarrow \frac{C}{C^*} \leq 2$$



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Quiz questions:

1. For the above approximation algorithm for "TSP with triangle inequality", how did we find its approximation ratio without knowing the optimal cost?

2. If we do not have the "triangle inequality condition" for the TSP, will the above proof for the approximation ratio still work?