Algorithms

Lecture Topic: Linear Programming (Part 3)

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Roadmap of this lecture:

1. Linear Programming (LP)

   1.1 Prove the correctness of the SIMPLEX Algorithm.

   1.2 What if the initial basic solution is infeasible.

SIMPLEX Algorithm

A generic step:

$$z = 27 + \frac{1}{4}x_2 + \frac{1}{2}x_3 - \frac{3}{4}x_6$$

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6$$

$$x_4 = 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6$$

$$x_5 = 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6$$

Basic solution:

$x_1 = 9,\ x_2 = 0,\ x_3 = 0,\ x_4 = 21,\ x_5 = 6,\ x_6 = 0$

Objective value = 27

SIMPLEX Algorithm:

Slack-Form LP 1

SIMPLEX Algorithm

A generic step:

$$z = 27 + \frac{1}{4}x_2 + \frac{1}{2}x_3 - \frac{3}{4}x_6$$

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Basic solution:

$$x_1 = 9,\ x_2 = 0,\ x_3 = 0,\ x_4 = 21,\ x_5 = 6,\ x_6 = 0$$

Objective value = 27

SIMPLEX Algorithm:

Slack-Form LP 1 → Feasible Basic Solution

SIMPLEX Algorithm

A generic step:

$$z = 27 + \frac{1}{4}x_2 + \frac{1}{2}x_3 - \frac{3}{4}x_6$$

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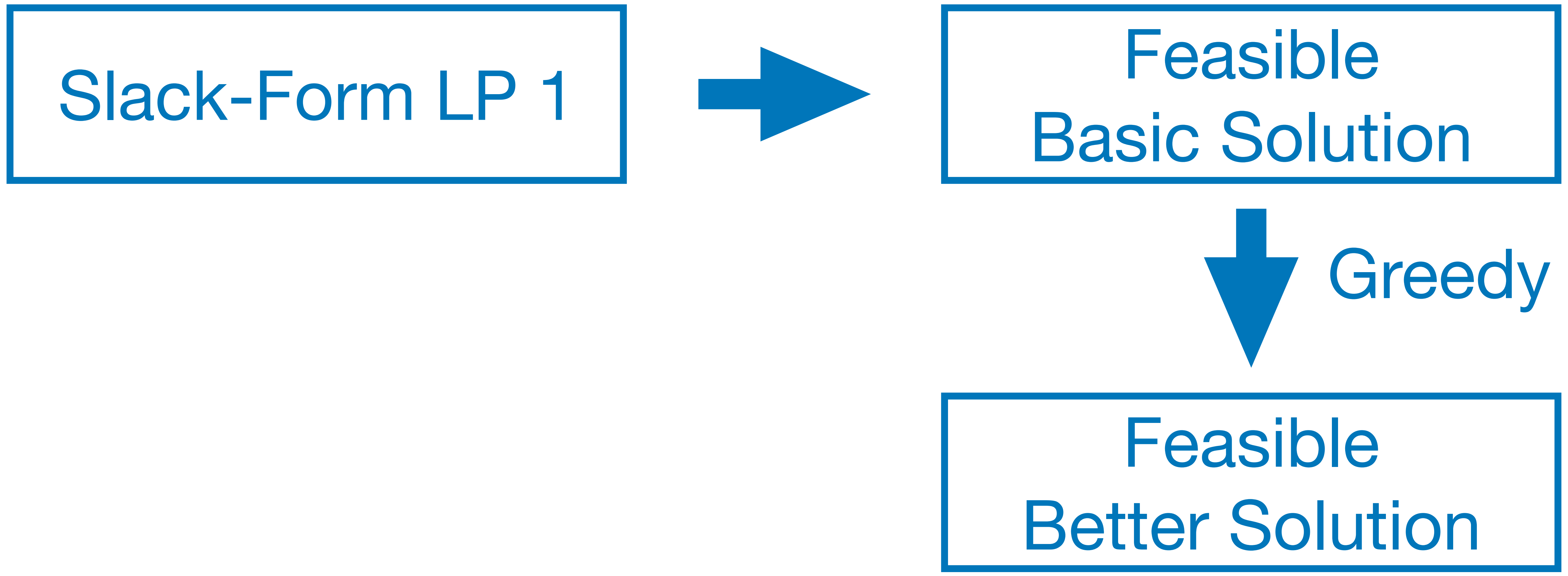
SIMPLEX Algorithm:



SIMPLEX Algorithm

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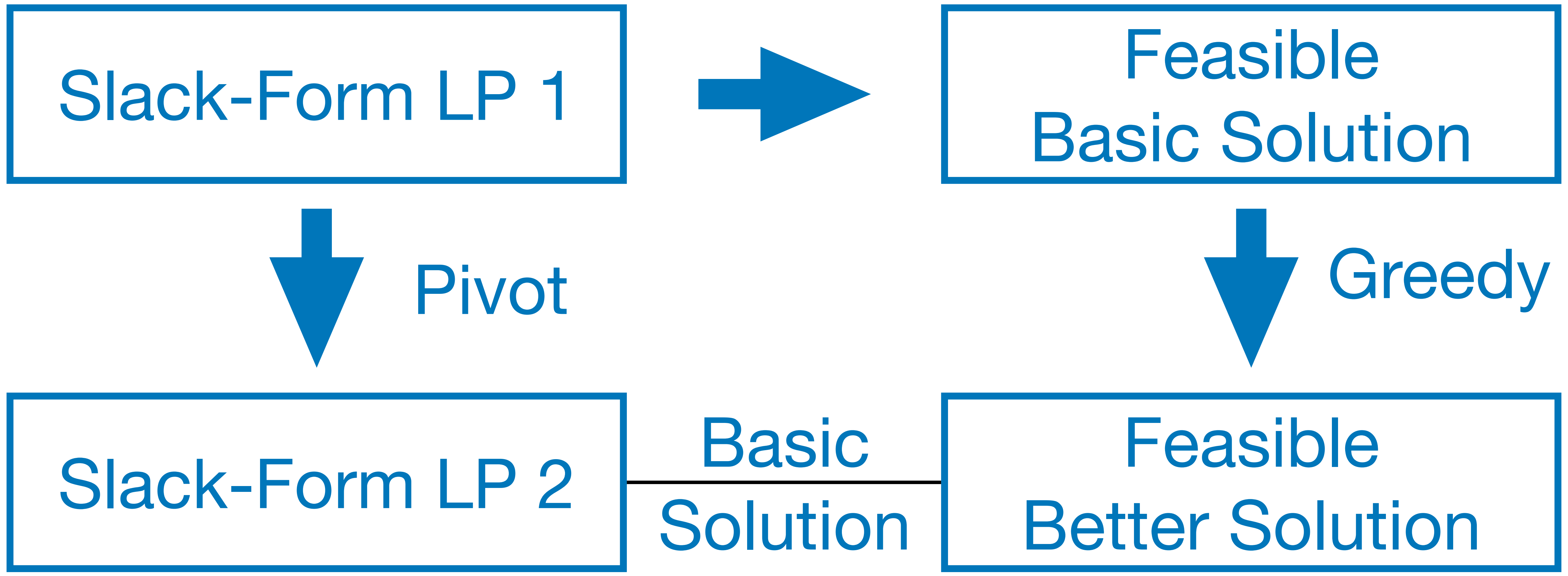

SIMPLEX Algorithm

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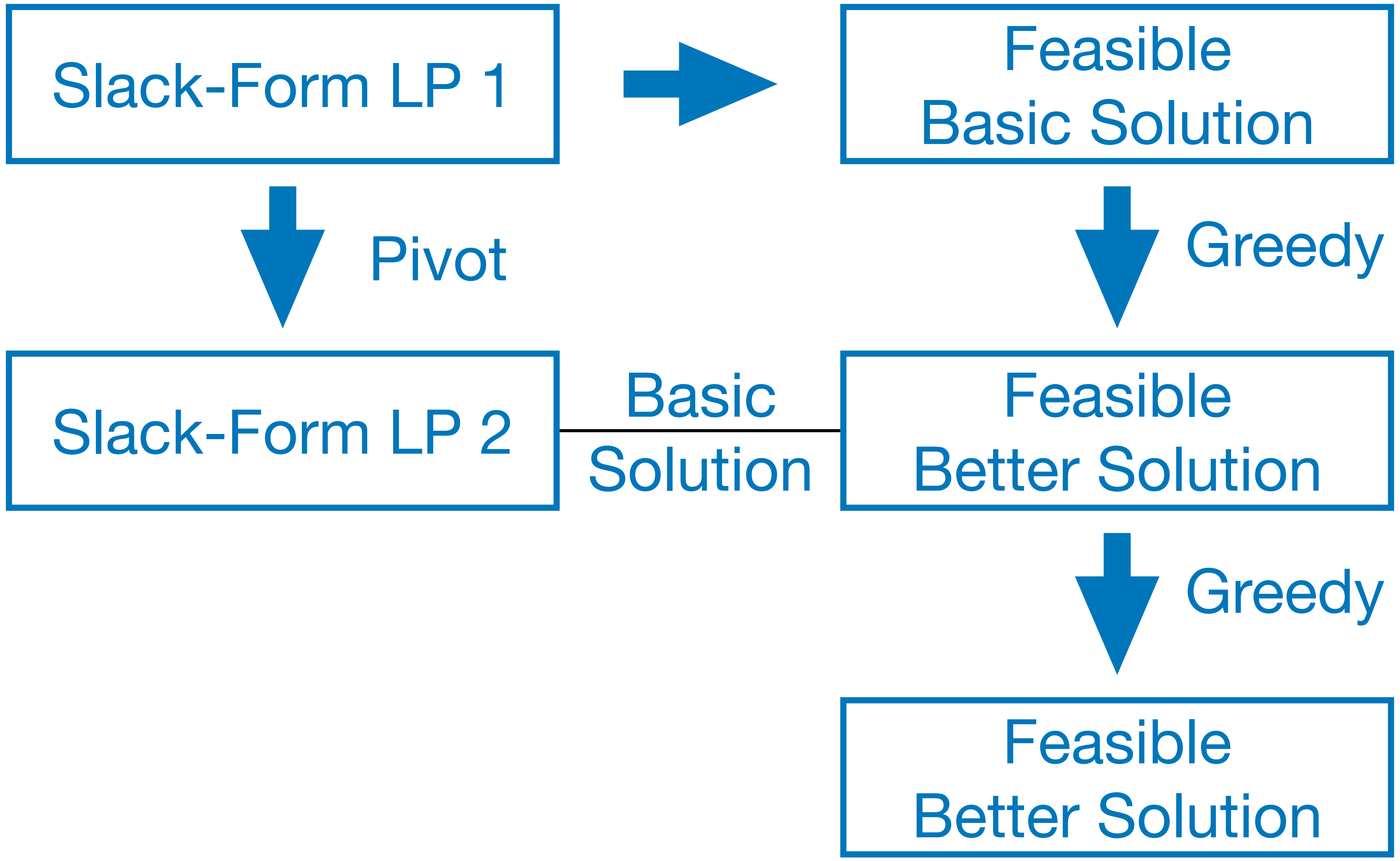

SIMPLEX Algorithm

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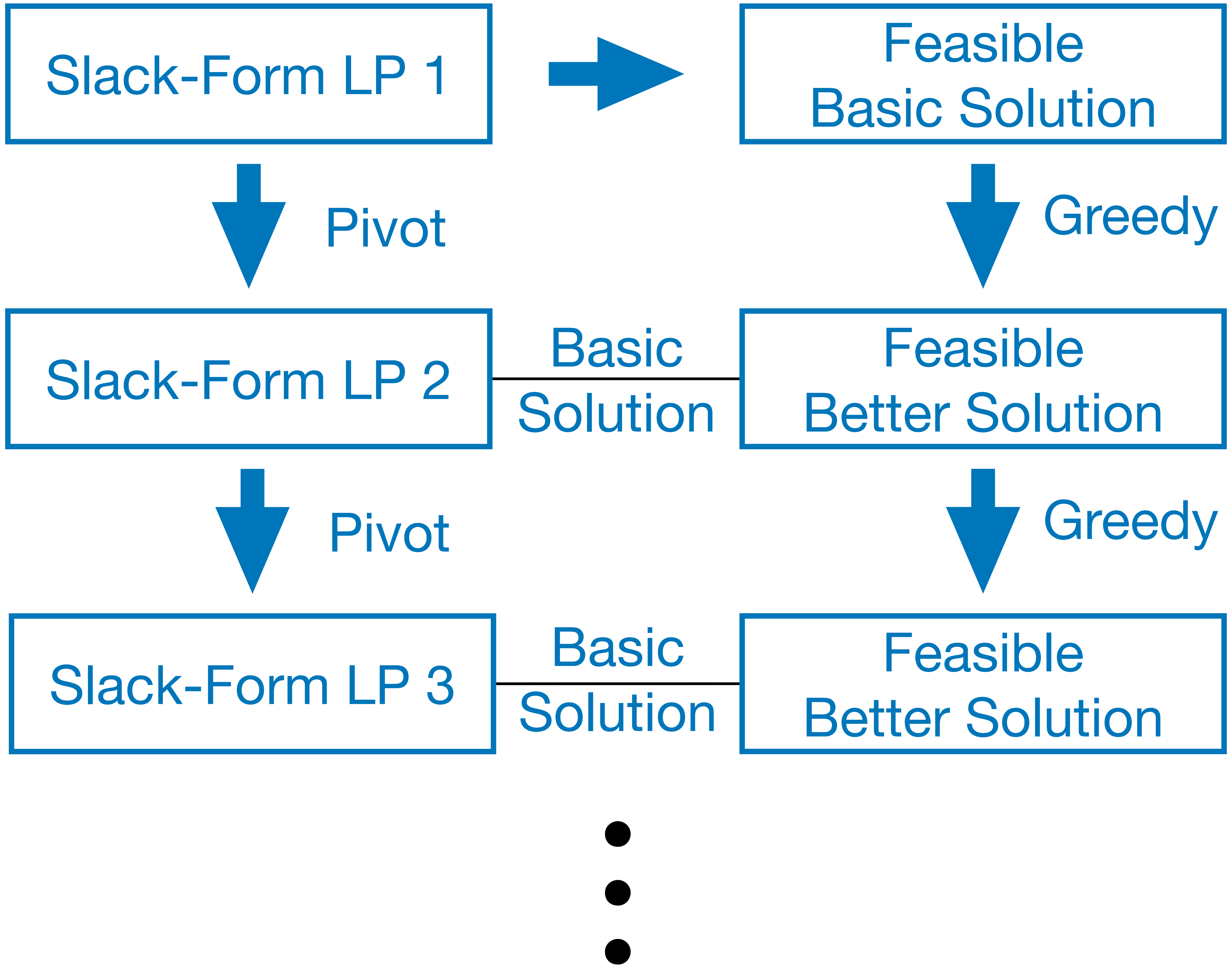
SIMPLEX Algorithm:



Duality

How to prove the SIMPLEX Algorithm returns an optimal solution?

Standard-Form LP (Primal LP)

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \geq 0$$

Duality

How to prove the SIMPLEX Algorithm returns an optimal solution?

Primal LP

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$$x_1,\ x_2,\ \cdots,\ x_n \ge 0$$

Dual LP

minimize $b_1y_1 + b_2y_2 + \cdots + b_my_m$

s.t.

$$a_{1,1}y_1 + a_{2,1}y_2 + \cdots + a_{m,1}y_m \ge c_1$$
$$a_{1,2}y_1 + a_{2,2}y_2 + \cdots + a_{m,2}y_m \ge c_2$$
$$\vdots$$
$$a_{1,n}y_1 + a_{2,n}y_2 + \cdots + a_{m,n}y_m \ge c_n$$
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Duality

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Dual LP

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$$a_{1,n}y_1 + a_{2,n}y_2 + \cdots + a_{m,n}y_m \geq c_n$$
$$y_1,\ y_2,\ \cdots,\ y_m \geq 0$$

Primal LP

maximize $3x_1 + x_2 + 2x_3$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$
$$2x_1 + 2x_2 + 5x_3 \leq 24$$
$$4x_1 + x_2 + 2x_3 \leq 36$$
$$x_1,\ x_2,\ x_3 \geq 0$$

Dual LP

minimize $30y_1 + 24y_2 + 36y_3$

s.t.

$$y_1 + 2y_2 + 4y_3 \geq 3$$
$$y_1 + 2y_2 + y_3 \geq 1$$
$$3y_1 + 5y_2 + 2y_3 \geq 2$$
$$y_1,\ y_2,\ y_3 \geq 0$$

Duality

How to prove the SIMPLEX Algorithm returns an optimal solution?

Primal LP

maximize $\sum_{j=1}^{n} c_j x_j$

s.t.

$\sum_{j=1}^{n} a_{i,j} x_j \leq b_i$, for $i = 1,2,\cdots,m$

$x_j \geq 0$, for $j = 1,2,\cdots,n$

Dual LP

minimize $\sum_{i=1}^{m} b_i y_i$

s.t.

$\sum_{i=1}^{m} a_{i,j} y_i \geq c_j$, for $j = 1,2,\cdots,n$

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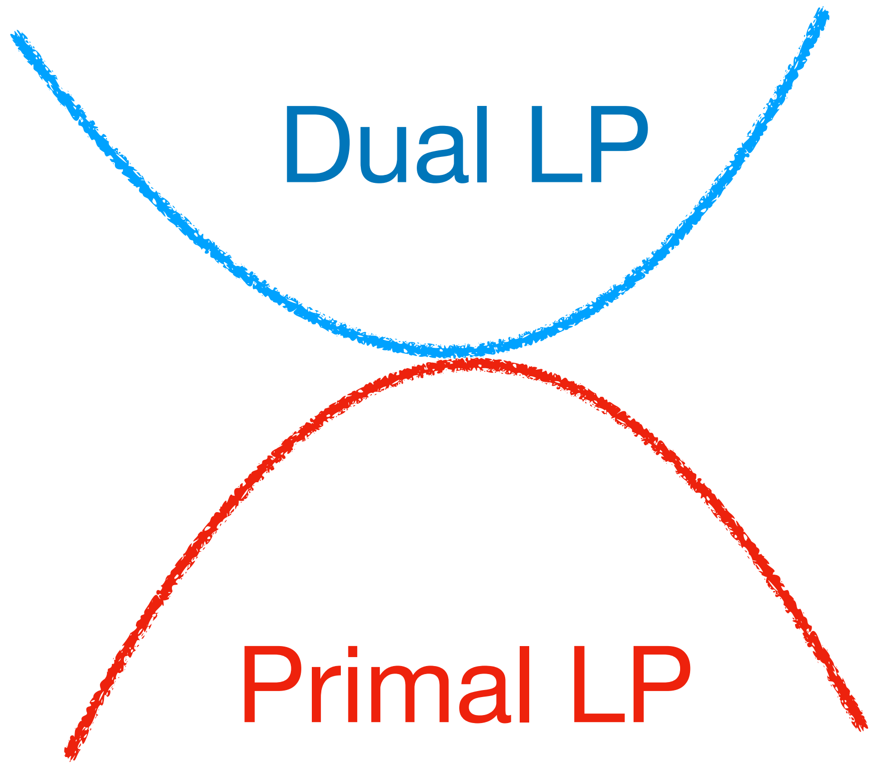
$y_i \ge 0$, for $i = 1,2,\cdots,m$

Theorem: Let $(x_1, x_2, \cdots, x_n)$ be any feasible solution to Primal LP.
Let $(y_1, y_2, \cdots, y_m)$ be any feasible solution to Dual LP.
Then

$$\sum_{j=1}^{n} c_j x_j \le \sum_{i=1}^{m} b_i y_i$$



Duality

How to prove the SIMPLEX Algorithm returns an optimal solution?

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$x_j \geq 0$, for $j = 1,2,\cdots,n$

Dual LP

minimize $\sum_{i=1}^{m} b_i y_i$

s.t.

$\sum_{i=1}^{m} a_{i,j} y_i \geq c_j$, for $j = 1,2,\cdots,n$

$y_i \geq 0$, for $i = 1,2,\cdots,m$

Proof: $\sum_{j=1}^{n} c_j x_j$

$$\leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{i,j} y_i \right) x_j$$

Theorem: Let $(x_1, x_2, \cdots, x_n)$ be any feasible solution to Primal LP.

Let $(y_1, y_2, \cdots, y_m)$ be any feasible solution to Dual LP.

Then

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i$$

Duality

How to prove the SIMPLEX Algorithm returns an optimal solution?

Primal LP

maximize $\sum_{j=1}^{n} c_j x_j$

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$\sum_{j=1}^{n} a_{i,j} x_j \le b_i$, for $i = 1,2,\cdots,m$

$x_j \ge 0$, for $j = 1,2,\cdots,n$

Dual LP

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s.t.

$\sum_{i=1}^{m} a_{i,j} y_i \ge c_j$, for $j = 1,2,\cdots,n$

$y_i \ge 0$, for $i = 1,2,\cdots,m$

Theorem: Let $(x_1, x_2, \cdots, x_n)$ be any feasible solution to Primal LP.
Let $(y_1, y_2, \cdots, y_m)$ be any feasible solution to Dual LP.
Then

$$\sum_{j=1}^{n} c_j x_j \le \sum_{i=1}^{m} b_i y_i$$

Proof: $\sum_{j=1}^{n} c_j x_j$

$$\le \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{i,j} y_i\right) x_j$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{i,j} x_j\right) y_i$$

Duality

How to prove the SIMPLEX Algorithm returns an optimal solution?

Primal LP

maximize $\sum_{j=1}^{n} c_j x_j$

s.t.

$\sum_{j=1}^{n} a_{i,j} x_j \leq b_i$, for $i = 1,2,\cdots,m$

$x_j \geq 0$, for $j = 1,2,\cdots,n$

Dual LP

minimize $\sum_{i=1}^{m} b_i y_i$

s.t.

$\sum_{i=1}^{m} a_{i,j} y_i \geq c_j$, for $j = 1,2,\cdots,n$

$y_i \geq 0$, for $i = 1,2,\cdots,m$

Theorem: Let $(x_1, x_2, \cdots, x_n)$ be any feasible solution to Primal LP.
Let $(y_1, y_2, \cdots, y_m)$ be any feasible solution to Dual LP.
Then

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i$$

Proof:

$$\sum_{j=1}^{n} c_j x_j$$

$$\leq \sum_{j=1}^{n} \Big(\sum_{i=1}^{m} a_{i,j} y_i \Big) x_j$$

$$= \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} a_{i,j} x_j \Big) y_i$$

$$\leq \sum_{i=1}^{m} b_i y_i$$

Weak Duality

Duality

How to prove the SIMPLEX Algorithm returns an optimal solution?

Primal LP

maximize $\sum_{j=1}^{n} c_j x_j$

s.t.

$\sum_{j=1}^{n} a_{i,j} x_j \le b_i$, for $i = 1,2,\cdots,m$

$x_j \ge 0$, for $j = 1,2,\cdots,n$

Dual LP

minimize $\sum_{i=1}^{m} b_i y_i$

s.t.

$\sum_{i=1}^{m} a_{i,j} y_i \ge c_j$, for $j = 1,2,\cdots,n$

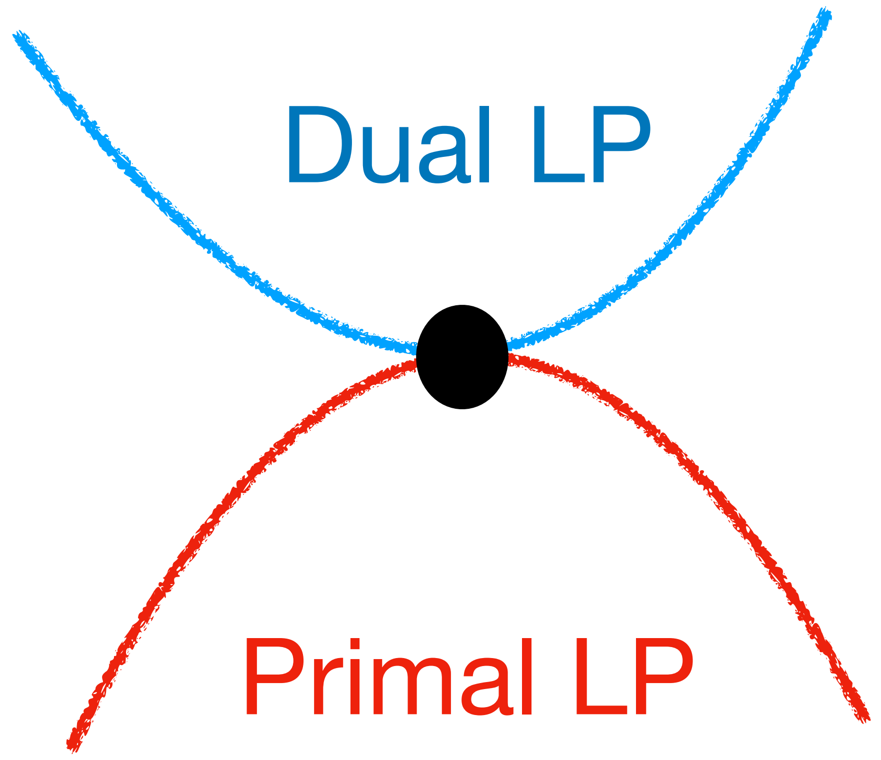
$y_i \ge 0$, for $i = 1,2,\cdots,m$

Theorem: Let $(x_1, x_2, \cdots, x_n)$ be any feasible solution to Primal LP.
Let $(y_1, y_2, \cdots, y_m)$ be any feasible solution to Dual LP.
Then

$$\sum_{j=1}^{n} c_j x_j \le \sum_{i=1}^{m} b_i y_i$$

Proof:

$$\sum_{j=1}^{n} c_j x_j \le \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{i,j} y_i\right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{i,j} x_j\right) y_i \le \sum_{i=1}^{m} b_i y_i$$



Weak Duality

Strong Duality

Duality

How to prove the SIMPLEX Algorithm returns an optimal solution?

Primal LP

maximize $\sum_{j=1}^{n} c_j x_j$

s.t.

$\sum_{j=1}^{n} a_{i,j} x_j \leq b_i$, for $i = 1,2,\cdots,m$

$x_j \geq 0$, for $j = 1,2,\cdots,n$

Dual LP

minimize $\sum_{i=1}^{m} b_i y_i$

s.t.

$\sum_{i=1}^{m} a_{i,j} y_i \geq c_j$, for $j = 1,2,\cdots,n$

$y_i \geq 0$, for $i = 1,2,\cdots,m$

Proof:

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{i,j} y_i\right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{i,j} x_j\right) y_i \leq \sum_{i=1}^{m} b_i y_i$$

Theorem: Let $(x_1, x_2, \cdots, x_n)$ be any feasible solution to Primal LP.
Let $(y_1, y_2, \cdots, y_m)$ be any feasible solution to Dual LP.
Then

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i$$

If $\sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i y_i$, then $(x_1, x_2, \cdots, x_n)$ is an optimal solution to the Primal LP,
and $(y_1, y_2, \cdots, y_m)$ is an optimal solution to the Dual LP.

$\Rightarrow$ Optimality of SIMPLEX Algorithm.

Duality

If $\sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i y_i$, then $(x_1, x_2, \cdots, x_n)$ is an optimal solution to the Primal LP,

and $(y_1, y_2, \cdots, y_m)$ is an optimal solution to the Dual LP.

$\Rightarrow$ Optimality of SIMPLEX Algorithm.

Let's use an example to illustrate how to find such optimal solutions. (For rigorous proof, see textbook.)

Duality

Primal LP

maximize $3x_1 + x_2 + 2x_3$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$
$$2x_1 + 2x_2 + 5x_3 \leq 24$$
$$4x_1 + x_2 + 2x_3 \leq 36$$
$$x_1,\ x_2,\ x_3 \geq 0$$

Dual LP

minimize $30y_1 + 24y_2 + 36y_3$

s.t.

$$y_1 + 2y_2 + 4y_3 \geq 3$$
$$y_1 + 2y_2 + y_3 \geq 1$$
$$3y_1 + 5y_2 + 2y_3 \geq 2$$
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Duality

Primal LP

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$$2x_1 + 2x_2 + 5x_3 \leq 24$$
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$$x_1,\ x_2,\ x_3 \geq 0$$

Final solution
by SIMPLEX Algorithm:

$$x_1 = 8,\ x_2 = 4,\ x_3 = 0$$

Dual LP

minimize $30y_1 + 24y_2 + 36y_3$

s.t.

$$y_1 + 2y_2 + 4y_3 \geq 3$$
$$y_1 + 2y_2 + y_3 \geq 1$$
$$3y_1 + 5y_2 + 2y_3 \geq 2$$
$$y_1,\ y_2,\ y_3 \geq 0$$

Duality

Primal LP

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Dual LP

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s.t.

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Final solution by SIMPLEX Algorithm:

$$x_1 = 8,\ x_2 = 4,\ x_3 = 0$$

Direct derivation

Corresponding solution:

$$y_1 = 0,\ y_2 = \frac{1}{6},\ y_3 = \frac{2}{3}$$

Duality

Primal LP

maximize $3x_1 + x_2 + 2x_3$

s.t.

$$
\begin{aligned}
x_1 + x_2 + 3x_3 &\le 30 \\
2x_1 + 2x_2 + 5x_3 &\le 24 \\
4x_1 + x_2 + 2x_3 &\le 36 \\
x_1,\ x_2,\ x_3 &\ge 0
\end{aligned}
$$

Final solution
by SIMPLEX Algorithm:

$$x_1 = 8,\ x_2 = 4,\ x_3 = 0$$

Direct

derivation

Dual LP

minimize $30y_1 + 24y_2 + 36y_3$

s.t.

$$
\begin{aligned}
y_1 + 2y_2 + 4y_3 &\ge 3 \\
y_1 + 2y_2 + y_3 &\ge 1 \\
3y_1 + 5y_2 + 2y_3 &\ge 2 \\
y_1,\ y_2,\ y_3 &\ge 0
\end{aligned}
$$

All constraints
are satisfied.

Corresponding solution:

$$y_1 = 0,\ y_2 = \frac{1}{6},\ y_3 = \frac{2}{3}$$

Feasible

Duality

Primal LP

maximize $3x_1 + x_2 + 2x_3$

s.t.

$$
\begin{aligned}
x_1 + x_2 + 3x_3 &\le 30\\
2x_1 + 2x_2 + 5x_3 &\le 24\\
4x_1 + x_2 + 2x_3 &\le 36\\
x_1,\ x_2,\ x_3 &\ge 0
\end{aligned}
$$

Dual LP

minimize $30y_1 + 24y_2 + 36y_3$

s.t.

$$
\boxed{
\begin{aligned}
y_1 + 2y_2 + 4y_3 &\ge 3\\
y_1 + 2y_2 + y_3 &\ge 1\\
3y_1 + 5y_2 + 2y_3 &\ge 2\\
y_1,\ y_2,\ y_3 &\ge 0
\end{aligned}
}
$$

All constraints are satisfied.

Final solution by SIMPLEX Algorithm:

$$x_1 = 8,\ x_2 = 4,\ x_3 = 0$$

Direct derivation

Corresponding solution:

$$y_1 = 0,\ y_2 = \frac{1}{6},\ y_3 = \frac{2}{3}$$

Feasible

Objective Value:

$$3 \times 8 + 4 + 2 \times 0 = 28$$

Both Optimal

Objective Value:

$$30 \times 0 + 24 \times \frac{1}{6} + 36 \times \frac{2}{3} = 28$$

Quiz questions:

1. What is a "Primal LP", and what is a "Dual LP"?

2. What is the duality between "Primal LP" and its "Dual LP"?

3. How to prove the SIMPLEX Algorithm outputs an optimal solution?

Roadmap of this lecture:

1. Linear Programming (LP)

   1.1 Prove the correctness of the SIMPLEX Algorithm.

   1.2 What if the initial basic solution is infeasible.

SIMPLEX Algorithm

1) Start with a slack-form LP, whose basic solution is feasible.

2) Repeatedly transform the slack-form LP into new slack-form LPs, such that the basic solution gives higher and higher objective values.

SIMPLEX Algorithm

What if the basic solution is not feasible?

1) Start with a slack-form LP, whose basic solution is feasible.

2) Repeatedly transform the slack-form LP into new slack-form LPs, such that the basic solution gives higher and higher objective values.

Example:

maximize $2x_1 - x_2$

s.t.

$$2x_1 - x_2 \le 2$$

$$x_1 - 5x_2 \le -4$$

$$x_1,\ x_2 \ge 0$$

$\Rightarrow$

Slack-form LP:

$$z = 2x_1 - x_2$$

$$x_3 = 2 - 2x_1 + x_2$$

$$x_4 = -4 - x_1 + 5x_2$$

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Basic solution: $x_1 = 0,\ x_2 = 0,\ x_3 = 2,\ x_4 = -4$

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$$z = 2x_1 - x_2$$
$$x_3 = 2 - 2x_1 + x_2$$
$$x_4 = -4 - x_1 + 5x_2$$

Basic solution: $x_1 = 0,\ x_2 = 0,\ x_3 = 2,\ x_4 = -4$

The basic solution is not feasible.

We will study how to handle this case.