Algorithms

Lecture Topic: Maximum Flow (Part 2)

Anxiao (Andrew) Jiang

Roadmap of this lecture:

1. Maximum Flow.

   1.1 Analyze the correctness and time complexity of the Ford-Fulkerson Method.

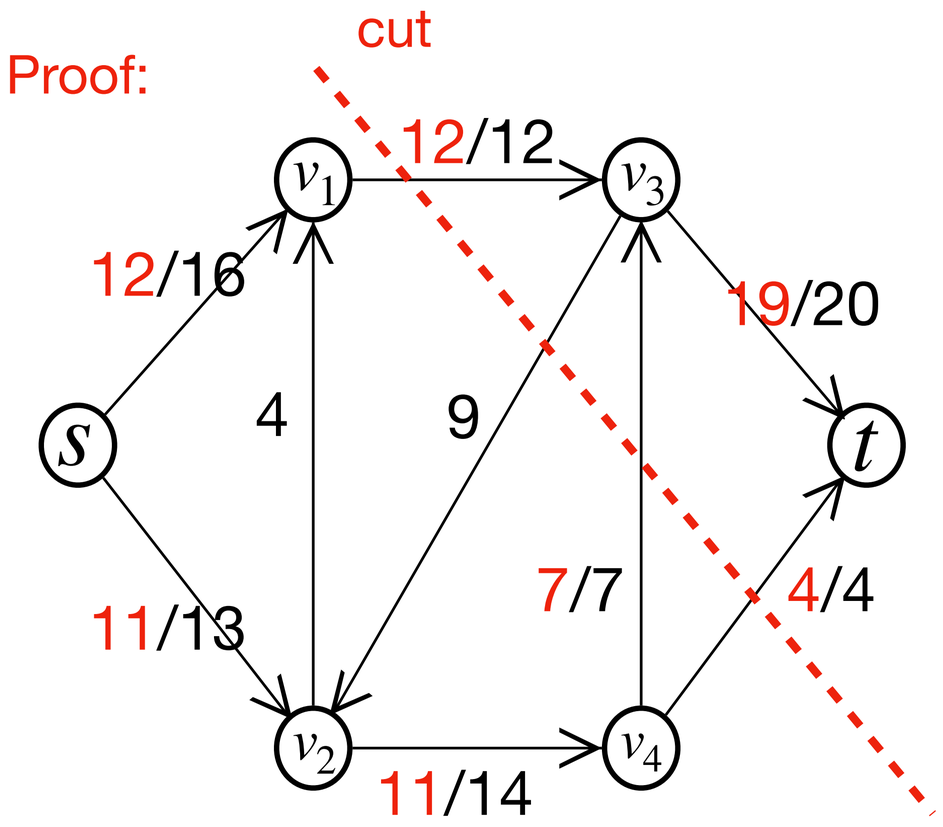
   1.2 Edmonds-Karp Algorithm for maximum flow.

   1.3 Use maximum flow to solve the "Maximum Bipartite Matching Problem".

Maximum Flow

Ford-Fulkerson Method

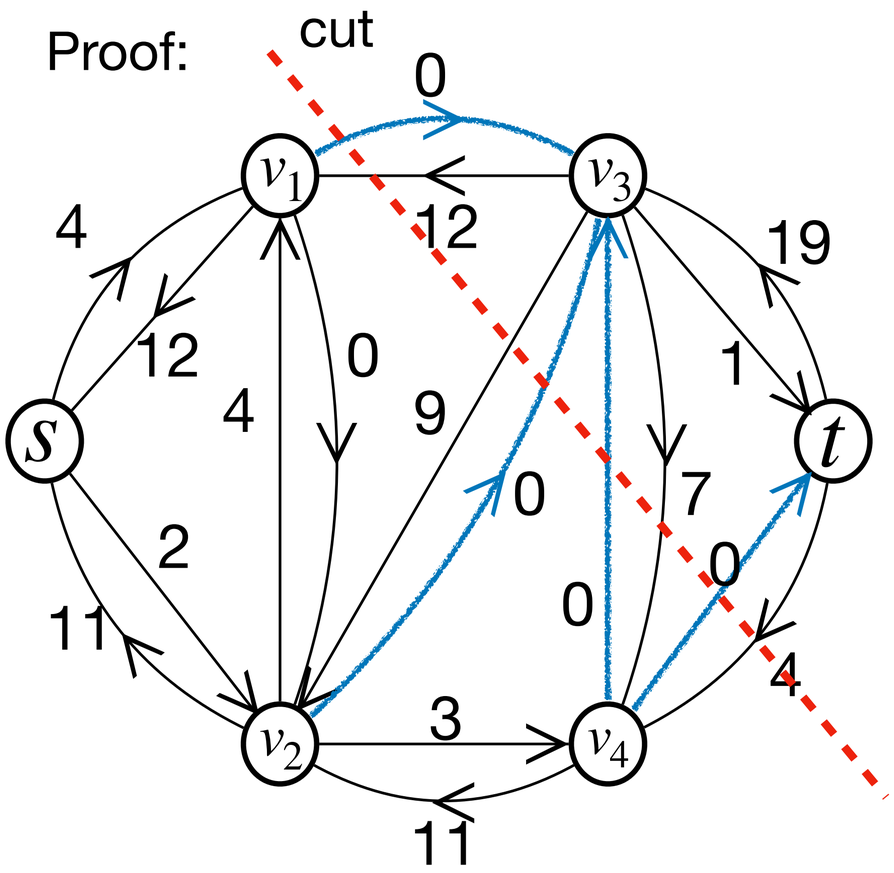
Original network



Size of flow: 12+11=19+4=23

Is it optimal? YES.

Residual network



Is there still a path from s to t of residual capacity > 0?

NO

Theorem: The Ford-Fulkerson Method finds a maximum flow.

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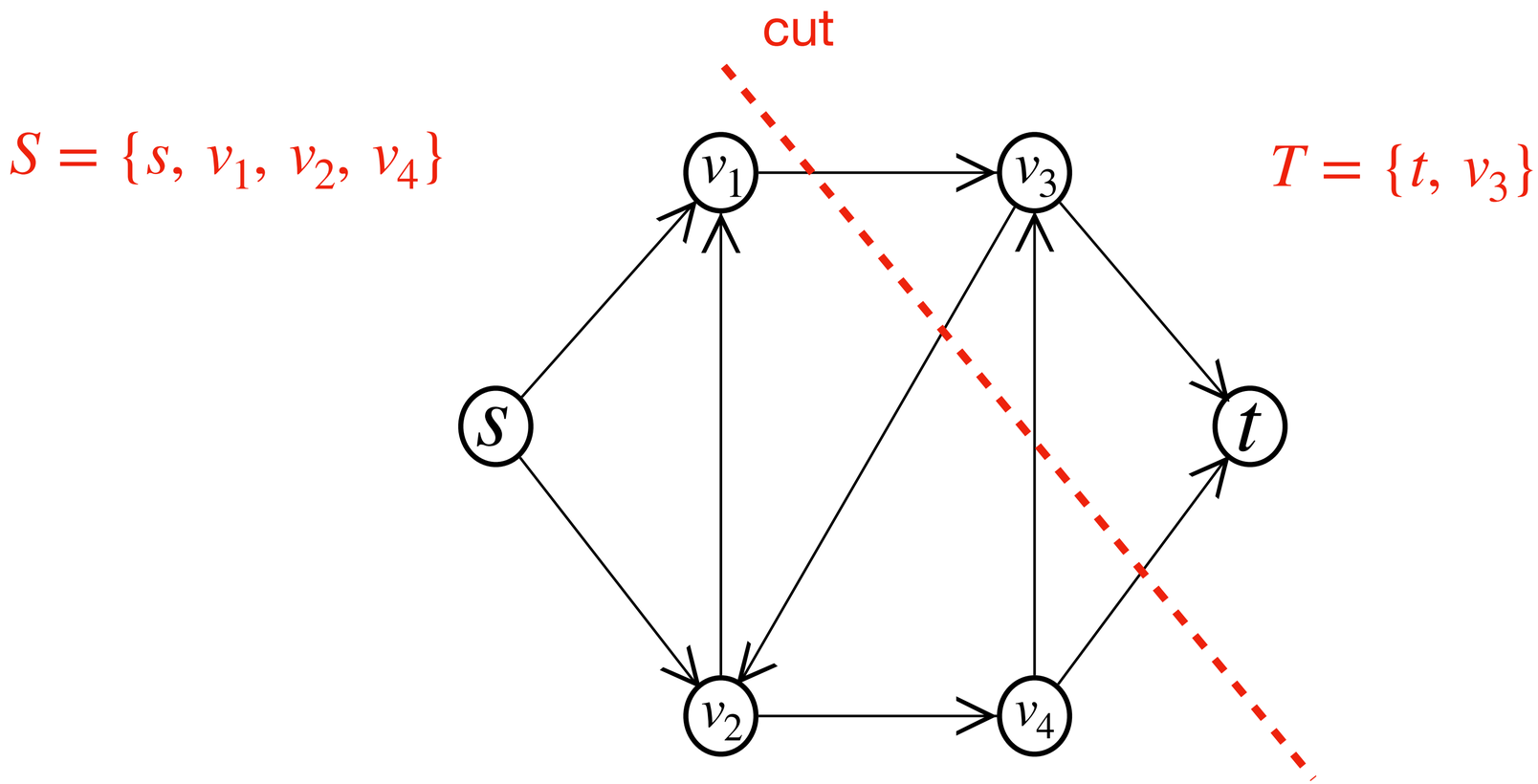
Cut: A partition of the nodes in the network G=(V,E) into two sets S and T, such that $s \in S$ and $t \in T$.



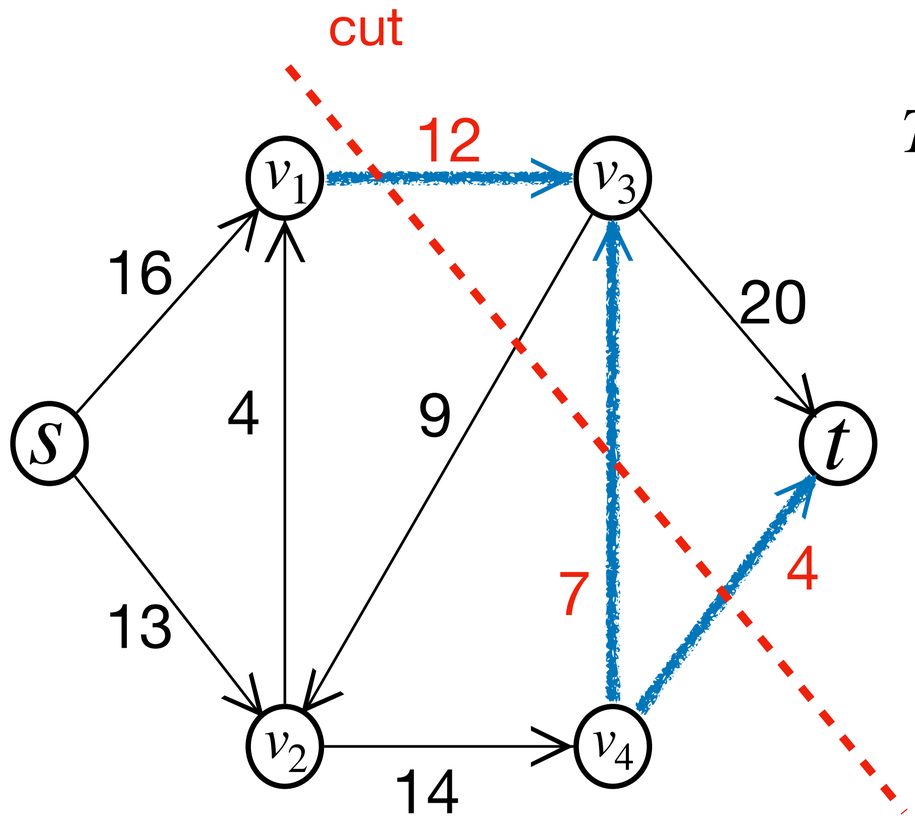
Theorem: The Ford-Fulkerson Method finds a maximum flow.

Cut: A partition of the nodes in the network G=(V,E) into two sets S and T, such that $s \in S$ and $t \in T$.

Capacity of Cut (S,T): the total capacity of the edges from S to T.

$S = \{s,\ v_1,\ v_2,\ v_4\}$

$T = \{t,\ v_3\}$

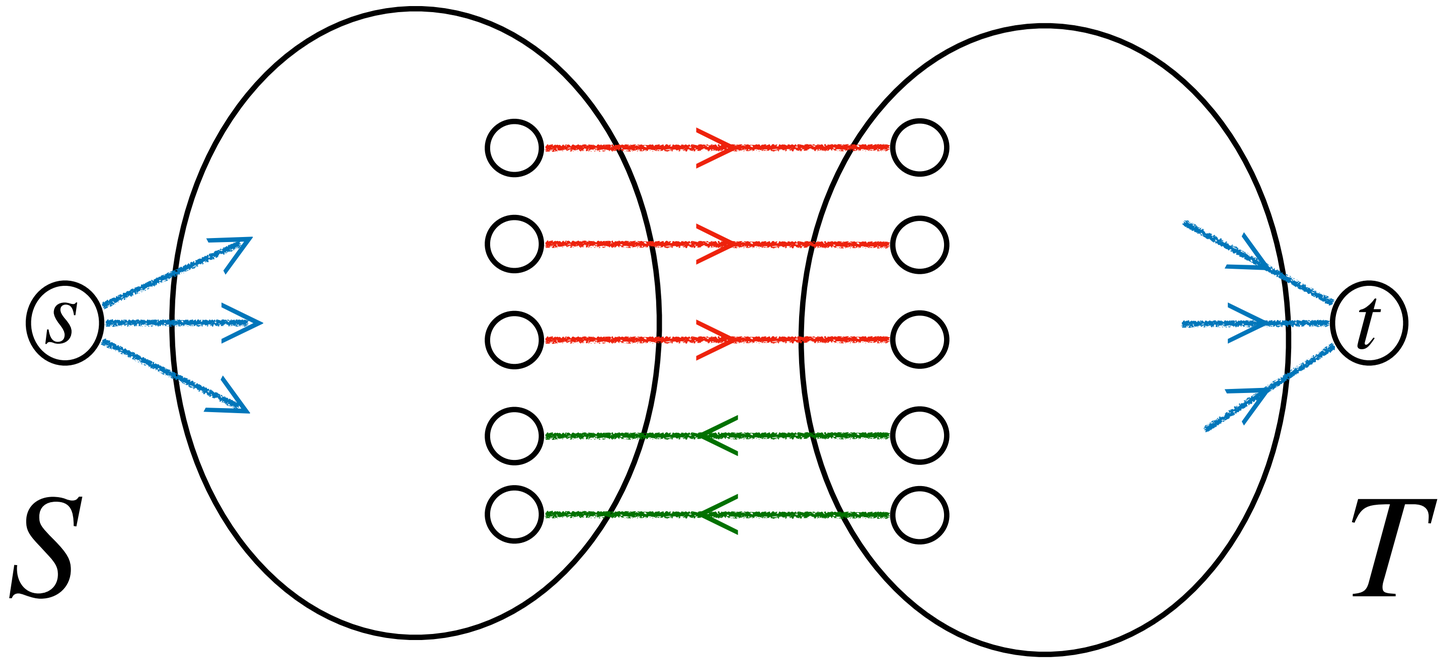


Capacity of cut (S, T)

$= 12 + 7 + 4$

$= 23$

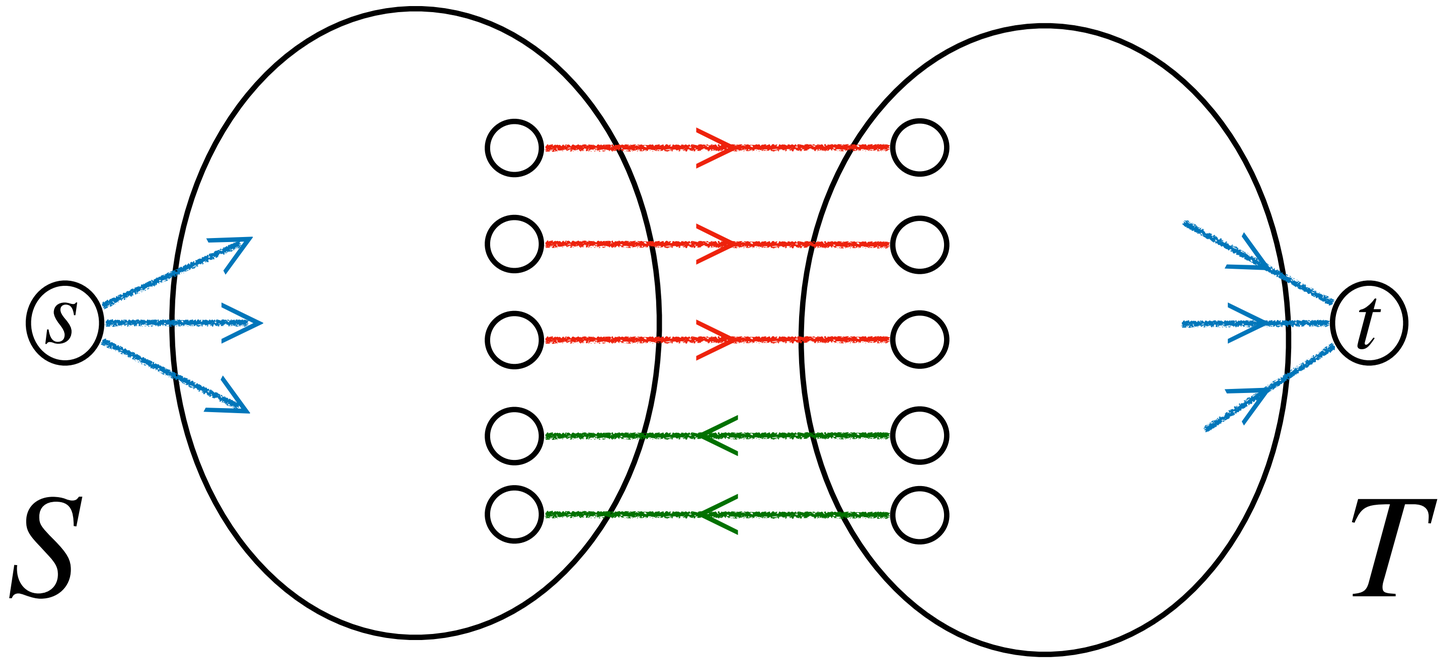
Lemma: The size of any flow is less than or equal to the capacity of any cut.



Lemma: The size of any flow is less than or equal to the capacity of any cut.

Proof: Size of flow (leaving source s or entering sink t)

= flow from S to T - flow from T to S



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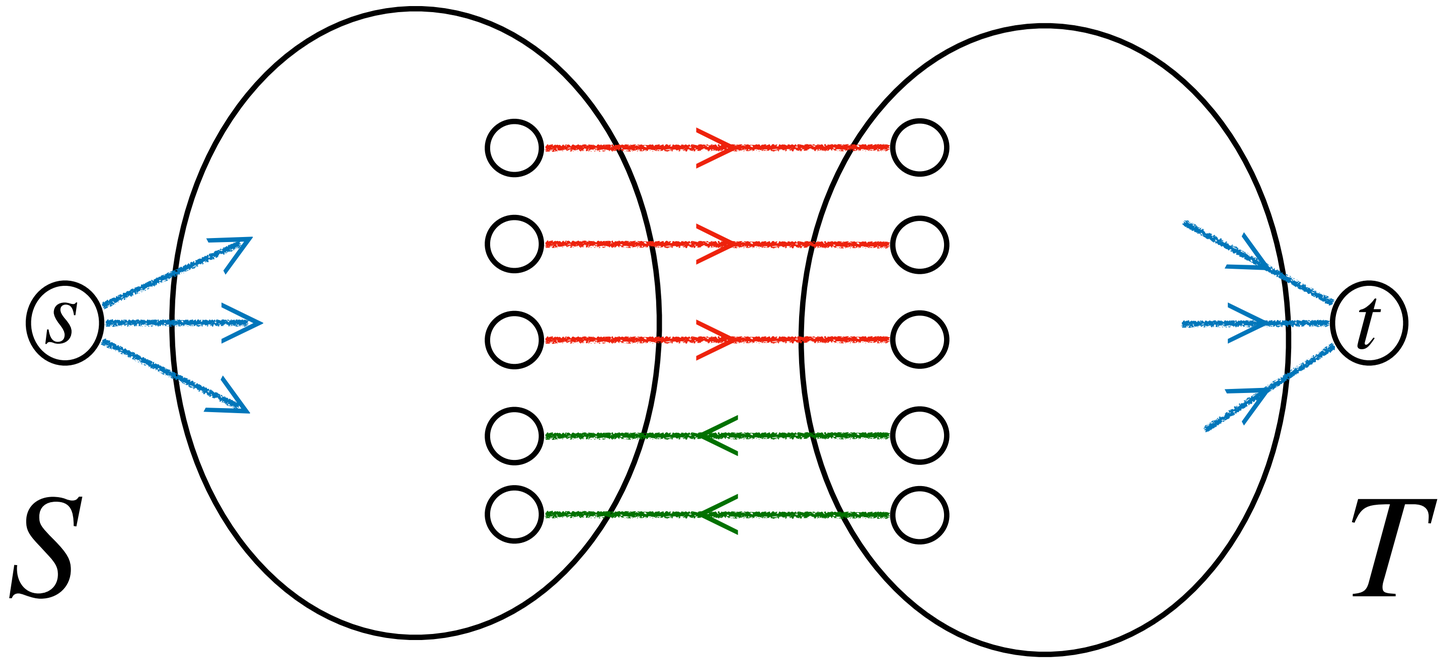
$\leq$ flow from S to T



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Proof: Size of flow (leaving source s or entering sink t)

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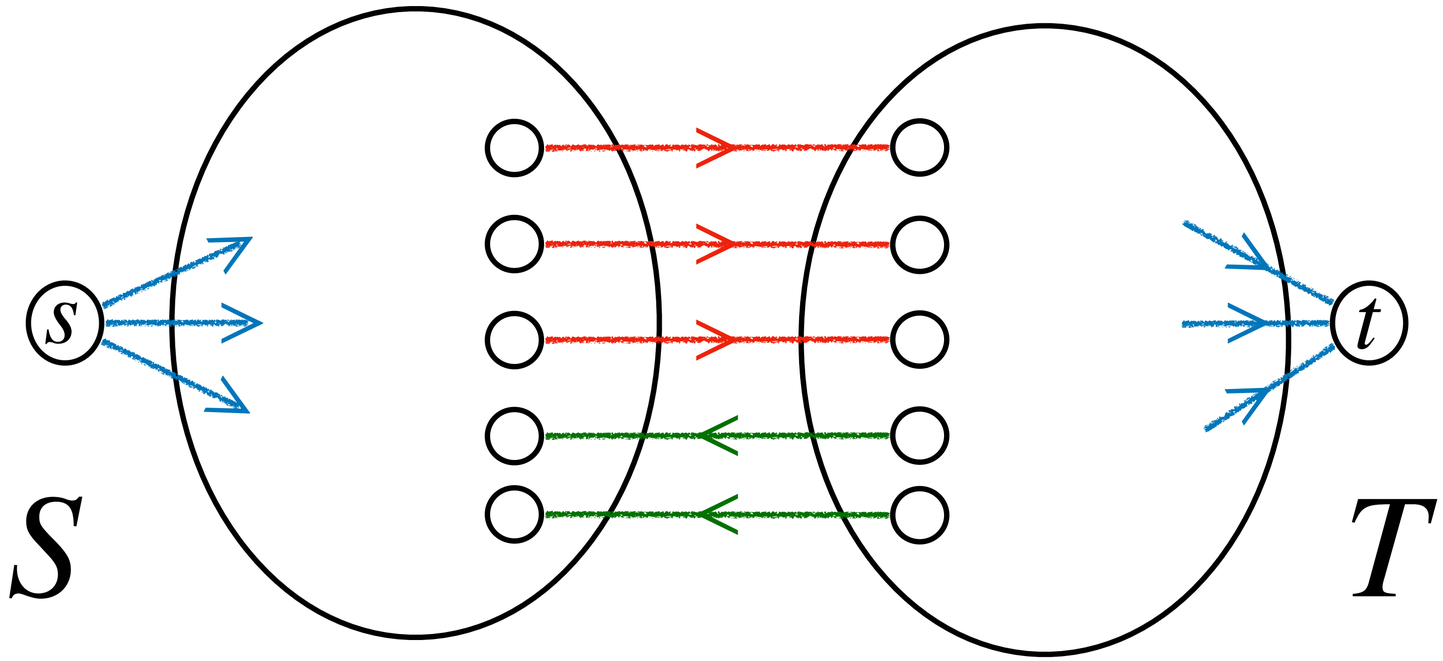
$\leq$ flow from S to T

$\leq$ total capacity of the edges from S to T



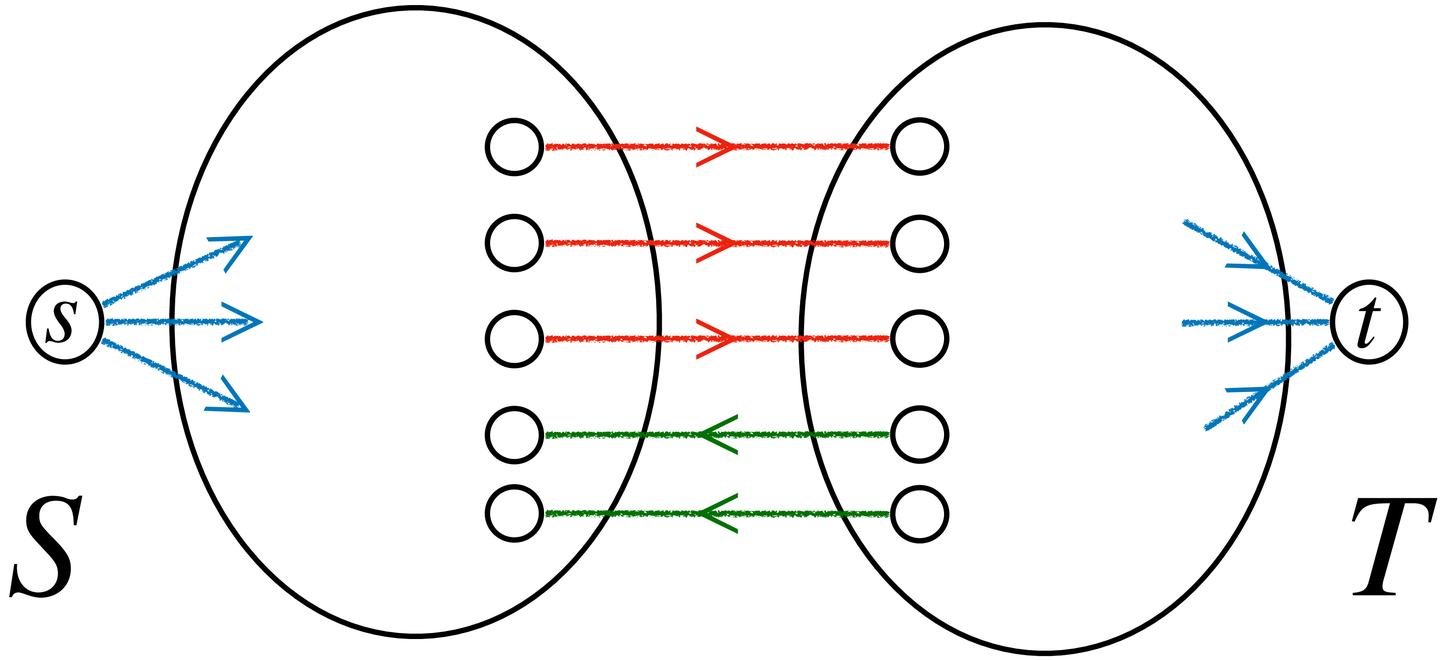
Lemma: The size of any flow is less than or equal to the capacity of any cut.

Proof: Size of flow (leaving source s or entering sink t)

= flow from S to T - flow from T to S

$\leq$ flow from S to T

$\leq$ total capacity of the edges from S to T = capacity of the cut (S,T)



Lemma: The size of any flow is less than or equal to the capacity of any cut.

Maximum flow: a flow from s to t of maximum size.

Minimum cut: a cut (S,T) of minimum capacity.

$$\text{size of maximum flow} \leq \text{capacity of minimum cut}$$

Question: is it true that

$$\text{size of maximum flow} = \text{capacity of minimum cut}$$

Answer: Yes. We will prove that.

Theorem: Let f be a flow in a network $G = (V, E)$ with source s and sink t.

Then these three conditions are equivalent:

1. f is a maximum flow in G.
2. The residual network G_f has no path from s to t with positive residual capacity.
3. The size of the flow f equals the capacity of some cut (S, T).

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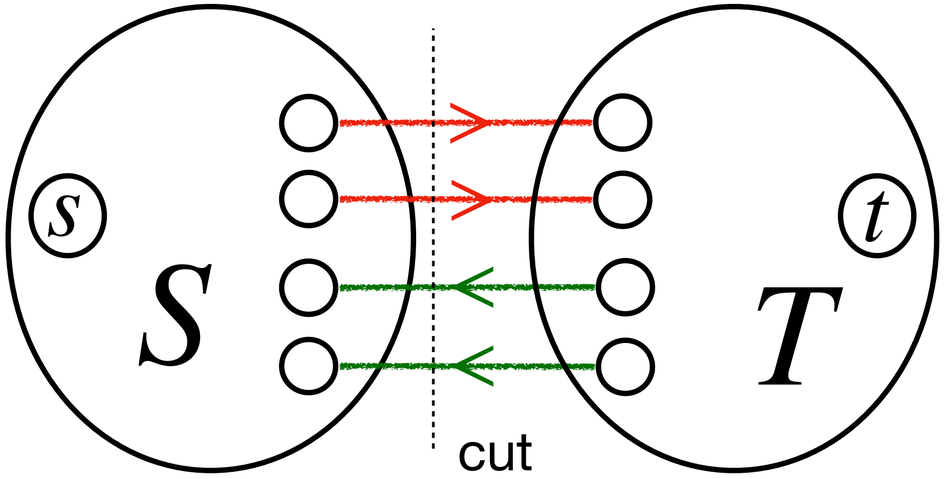
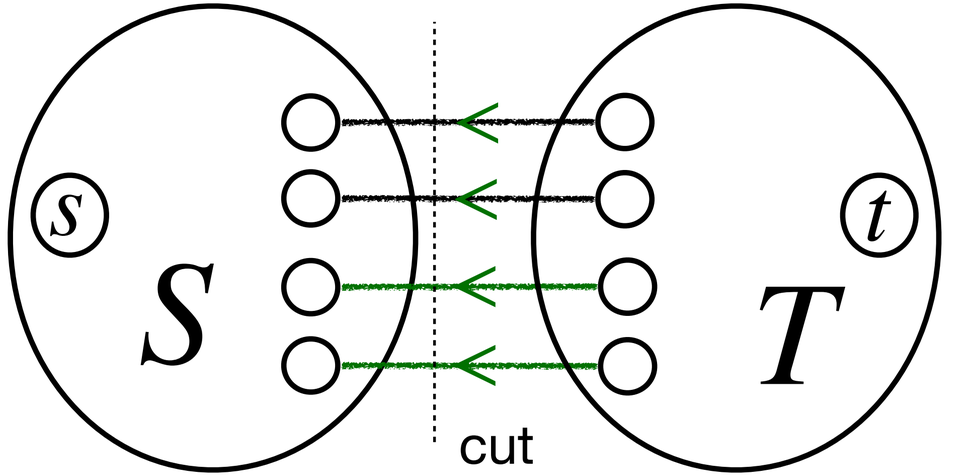
$2 \rightarrow 3$: S : the set of nodes that the source s can reach in the residual network G_f

T : the remaining nodes

Original network G



Residual network G_f



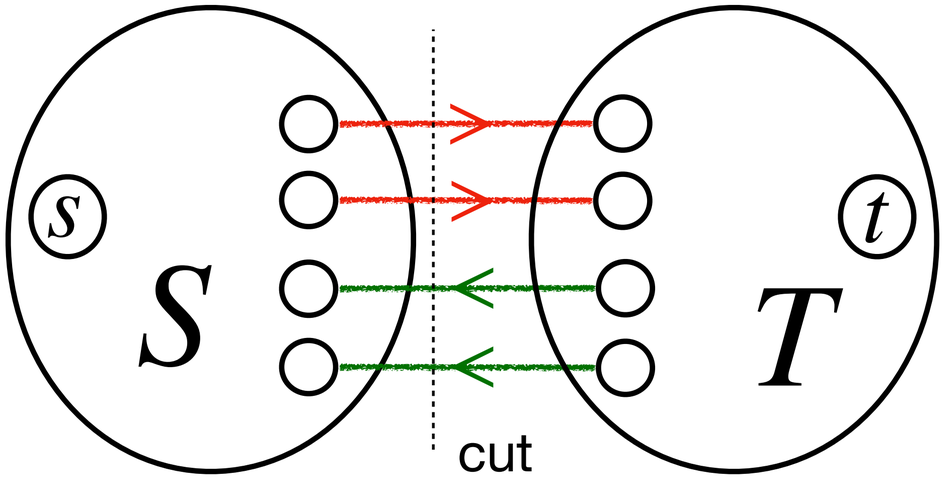
In the original network G:

"Forward" edges from S to T are saturated by flow

"Backward" edges from T to S have no flow at all

Therefore, the size of the flow f equals the capacity of this cut (S, T) .

Original network G



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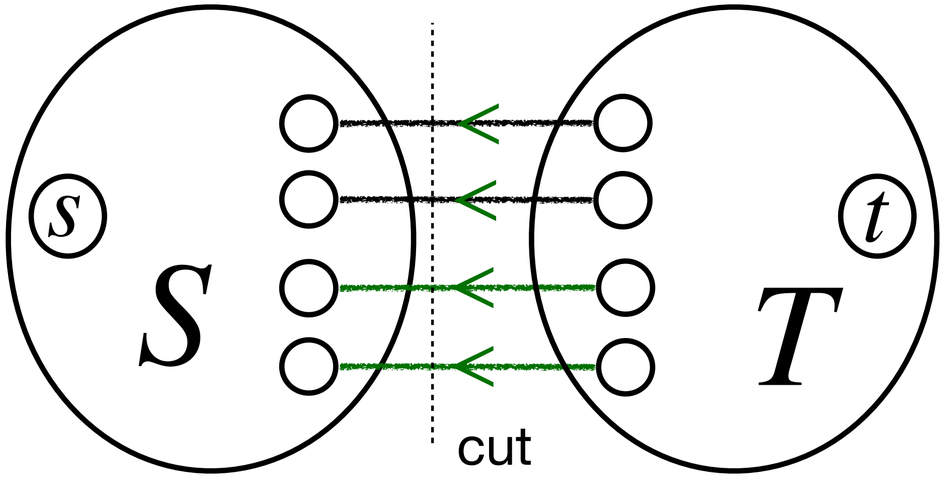

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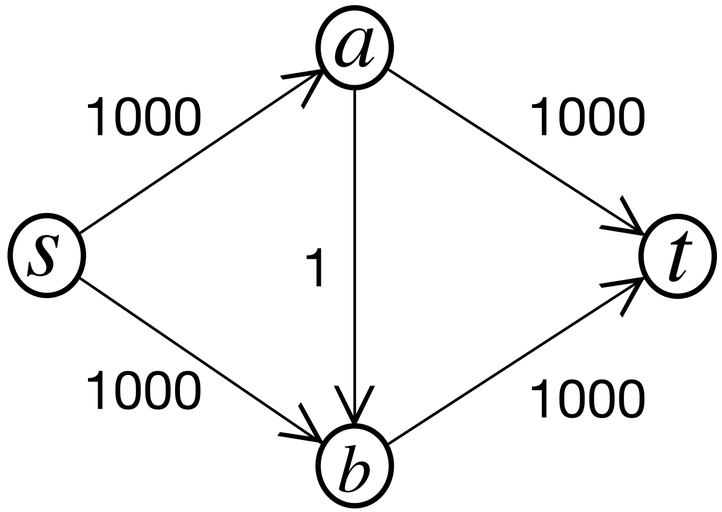
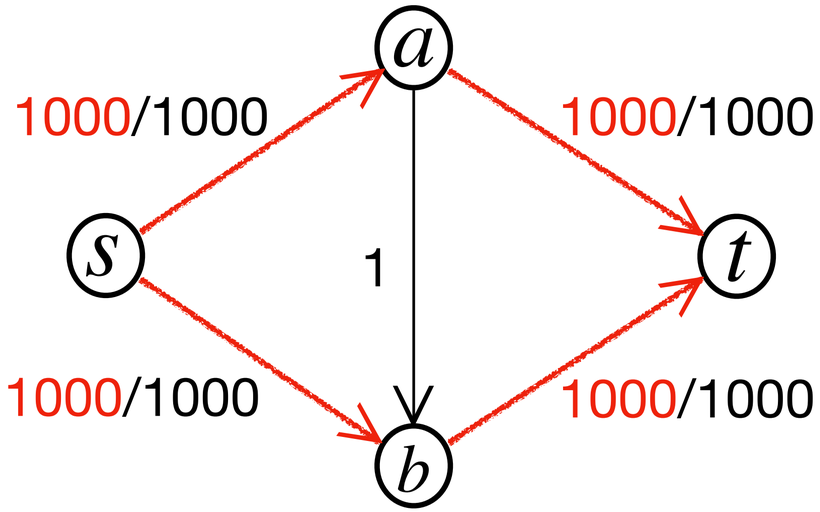
Theorem: The Ford-Fulkerson Method finds a maximum flow.

Time complexity of the Ford-Fulkerson Method



Time complexity of the Ford-Fulkerson Method

Maximum Flow: 2000



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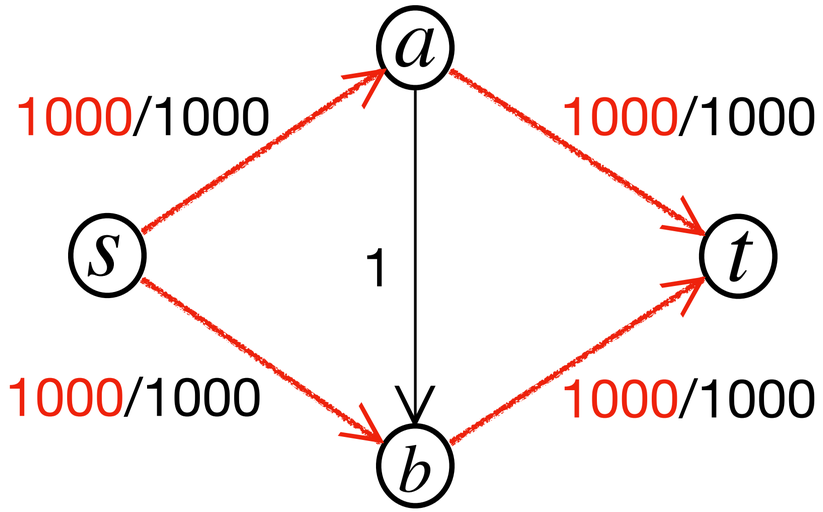
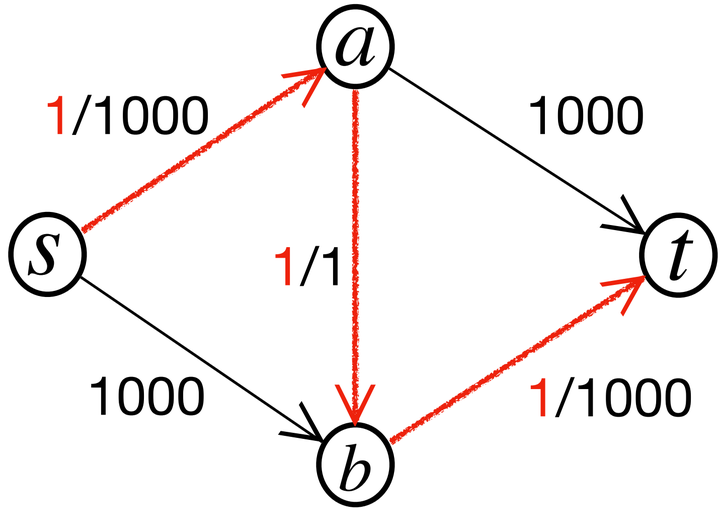


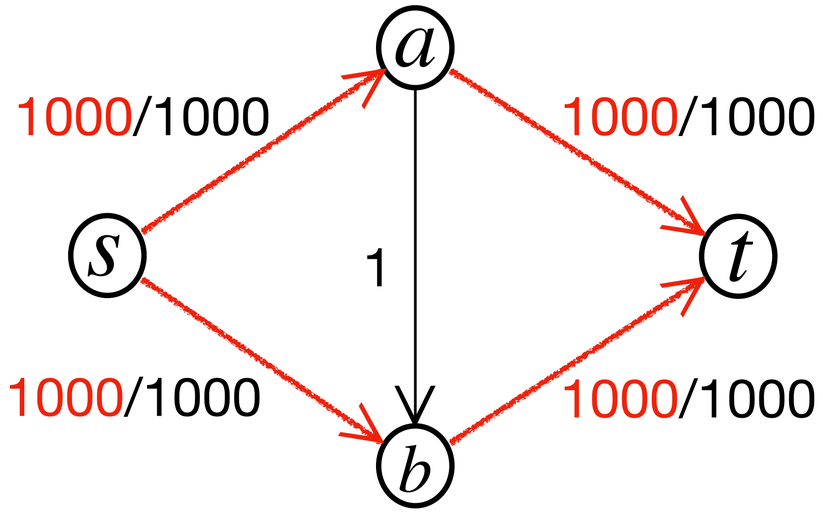
Naive way to run Ford-Fulkerson Method:



Current flow size = 1

Time complexity of the Ford-Fulkerson Method

Maximum Flow: 2000



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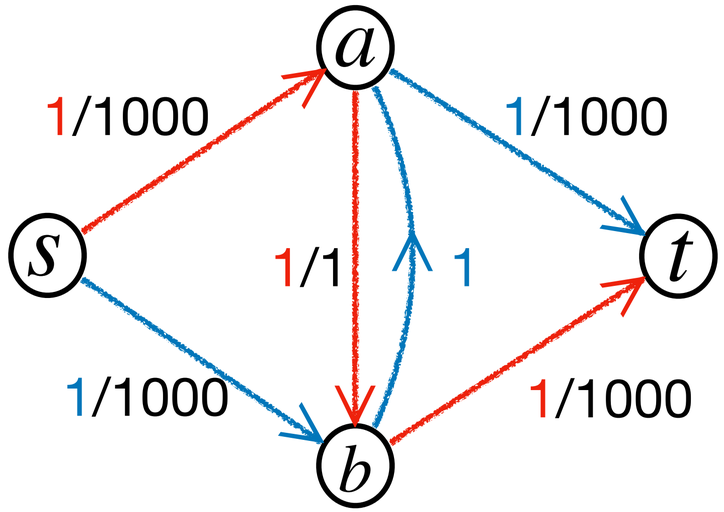
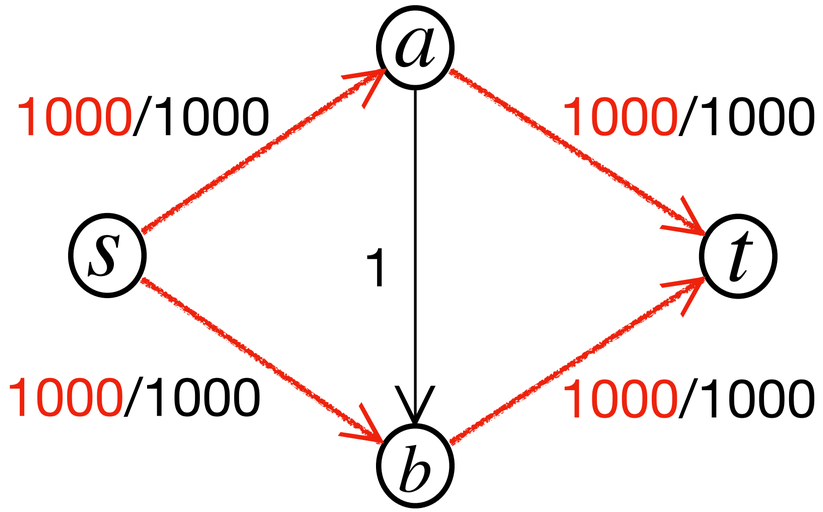


Current flow size = 2

Time complexity of the Ford-Fulkerson Method

Maximum Flow: 2000



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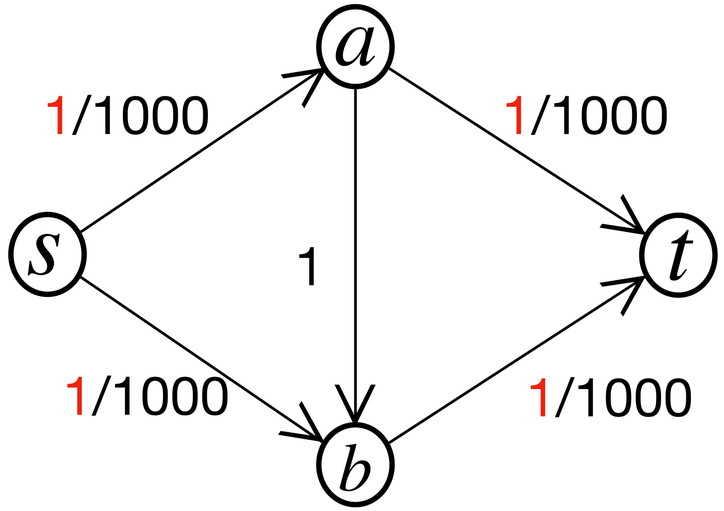
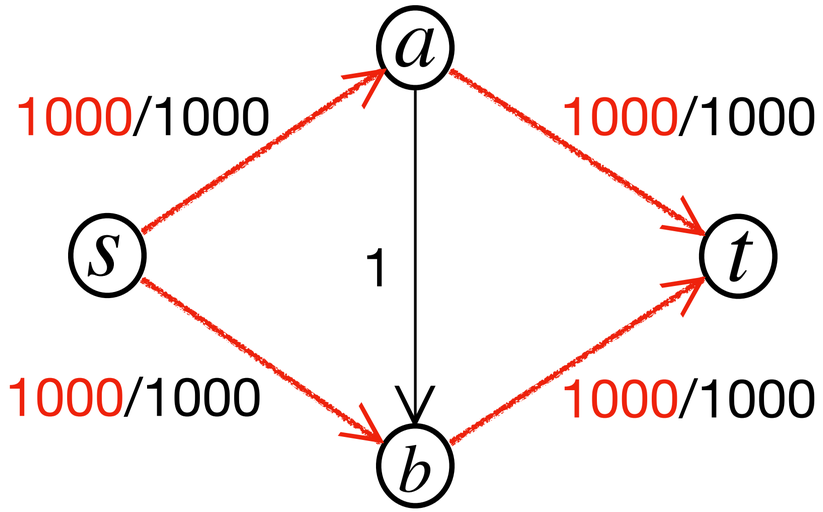


Current flow size = 2

Time complexity of the Ford-Fulkerson Method

Maximum Flow: 2000



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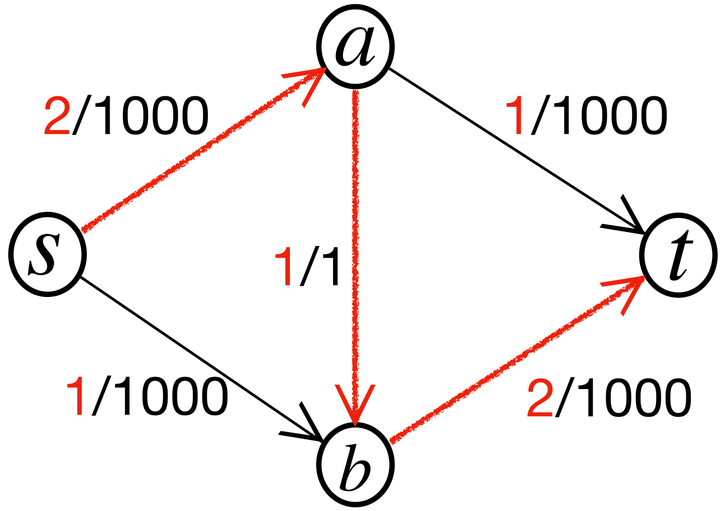
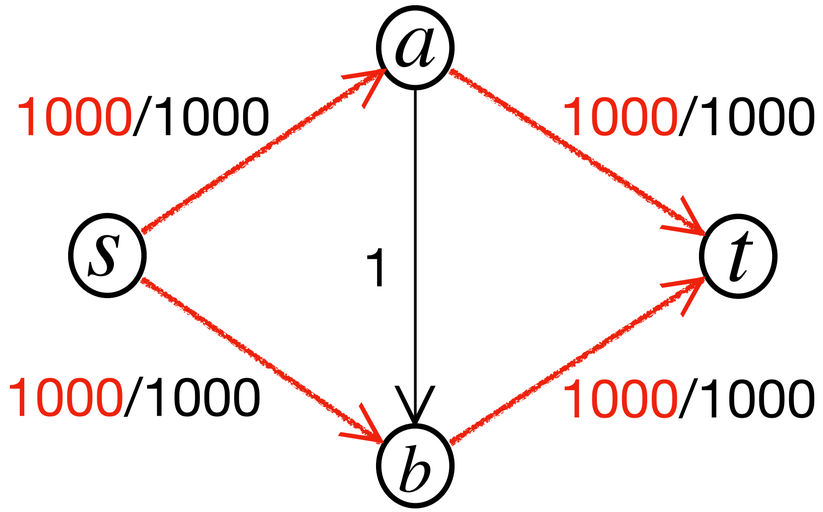


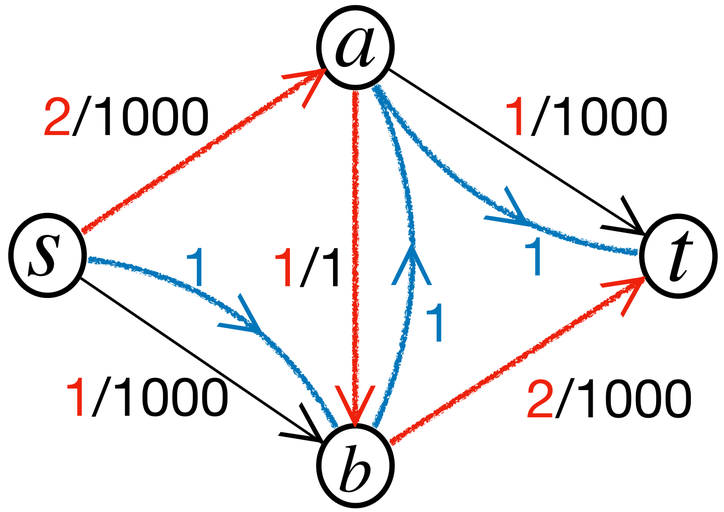
Current flow size = 3

Time complexity of the Ford-Fulkerson Method

Maximum Flow: 2000



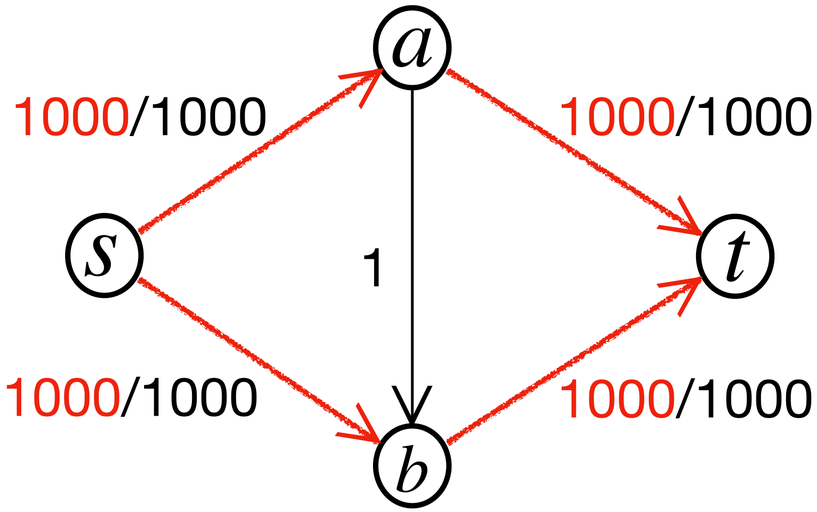
Naive way to run Ford-Fulkerson Method:



Current flow size = 4

Time complexity of the Ford-Fulkerson Method

Maximum Flow: 2000



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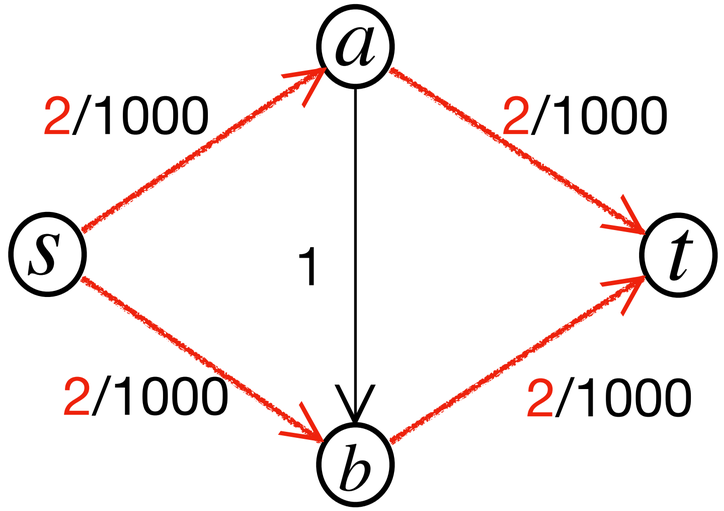


Current flow size = 4

Time complexity: too high

Quiz questions:

1. What does the “Max-Flow Min-Cut” Theorem tell us?

2. Why does the Ford-Fulkerson Method have high time complexity?

Roadmap of this lecture:

1. Maximum Flow.

    1.1 Analyze the correctness and time complexity of the Ford-Fulkerson Method.

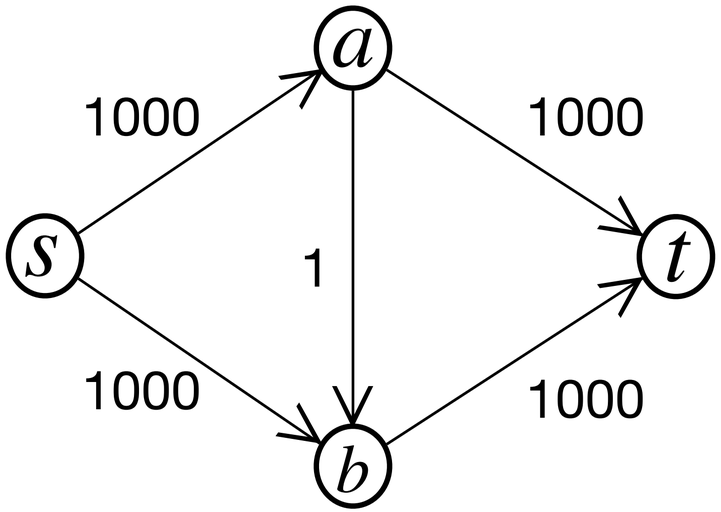
    1.2 Edmonds-Karp Algorithm for maximum flow.

    1.3 Use maximum flow to solve the "Maximum Bipartite Matching Problem".

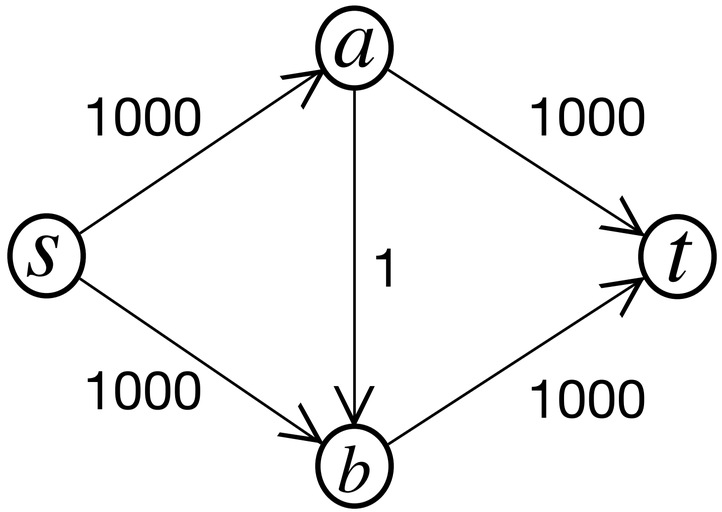
Edmonds-Karp Algorithm (an efficient way to implement the Ford-Fulkerson Method):

Each time, pick the shortest path from s to t in the residual network,
and augment the flow along that path.

Original network



Residual network



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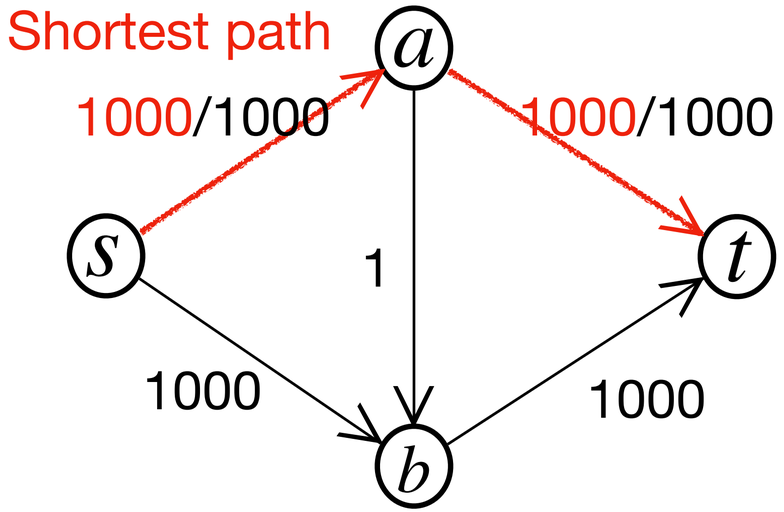

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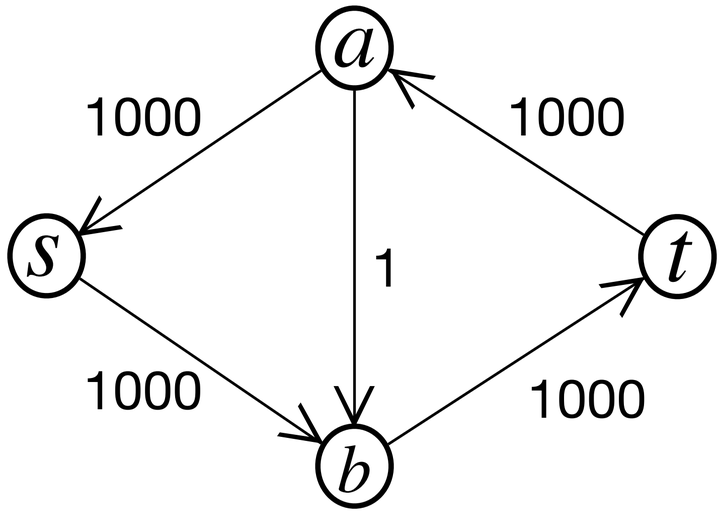

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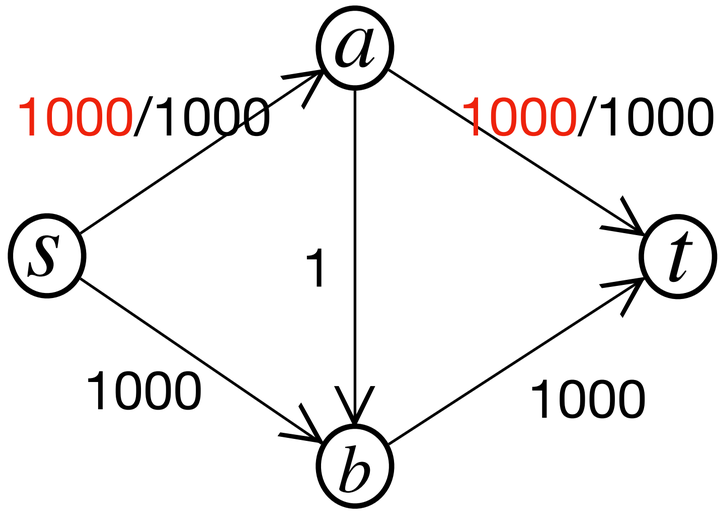

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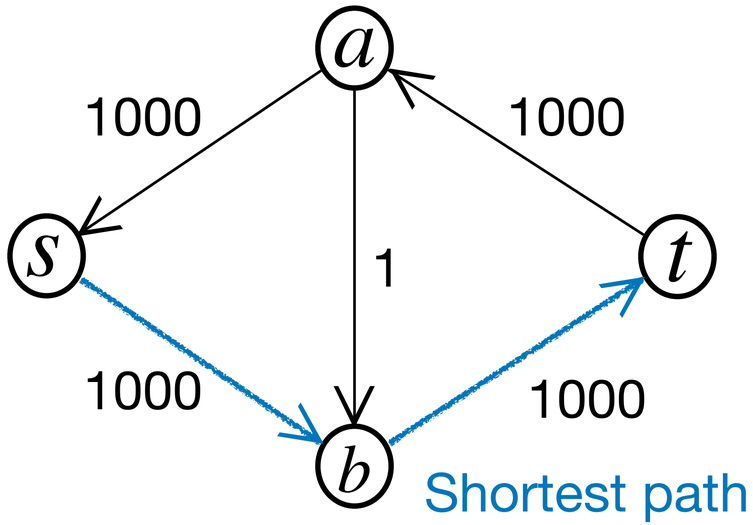

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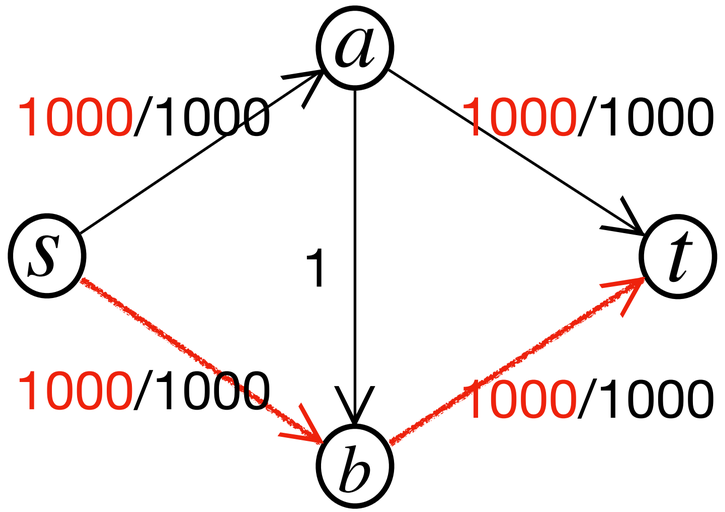

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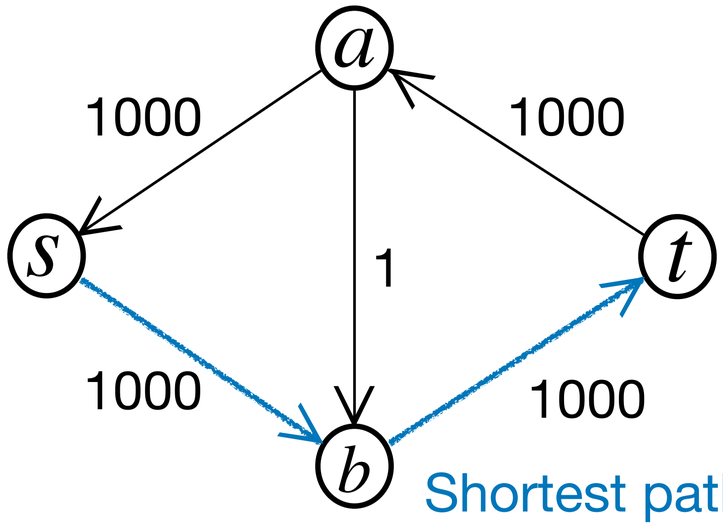


It works!

Edmonds-Karp Algorithm (an efficient way to implement the Ford-Fulkerson Method):

Each time, pick the shortest path from s to t in the residual network,
and augment the flow along that path.

Time Complexity: $O(VE^2)$

Quiz question:

1. What is the difference between the “Edmonds-Karp Algorithm” and the “Ford-Fulkerson Method”?

Roadmap of this lecture:

1. Maximum Flow.

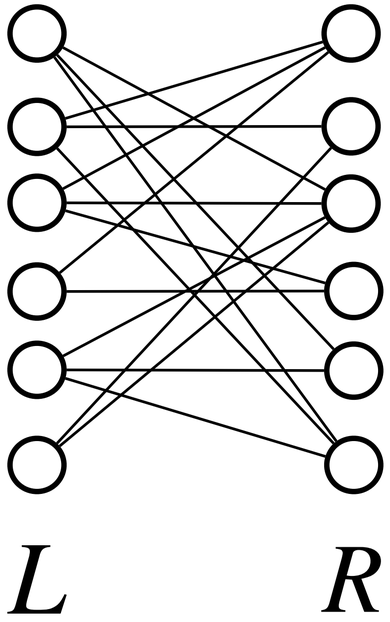
    1.1 Analyze the correctness and time complexity of the Ford-Fulkerson Method.

    1.2 Edmonds-Karp Algorithm for maximum flow.

    1.3 Use maximum flow to solve the "Maximum Bipartite Matching Problem".

Application: Maximum Bipartite Matching

Bipartite Graph: A graph $G = (V, E)$ is bipartite if we can partition the nodes into L and R, such that every edge is between a node in L and a node in R.



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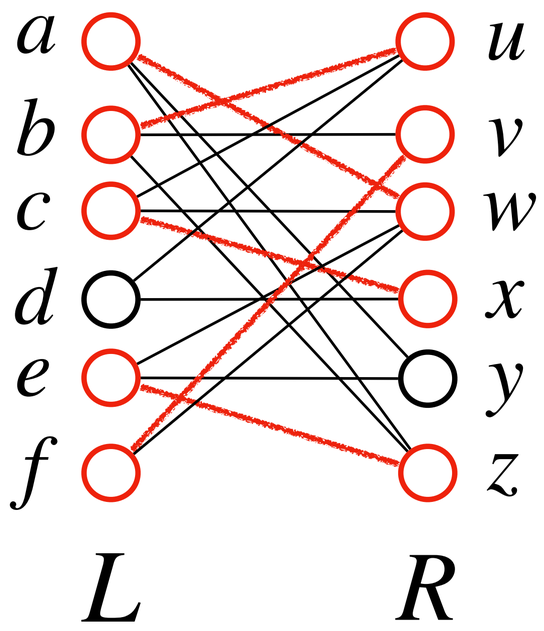
Matching: Given an undirected graph $G = (V, E)$, a matching is a subset of edges $S \subseteq E$ such that every node $v \in V$ is an endpoint of at most one edge in S.



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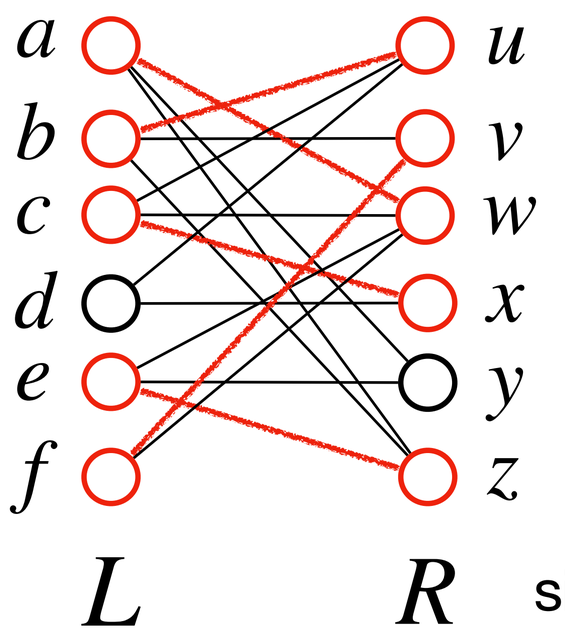

size of matching $= 5$

Bipartite Matching Problem:

Input: A bipartite graph $G = (L \cup R, E)$.

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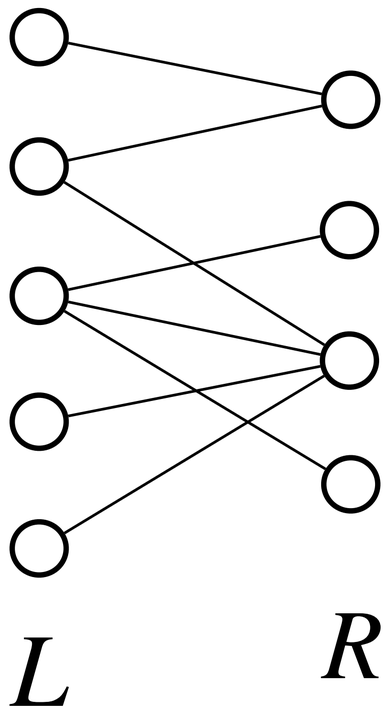

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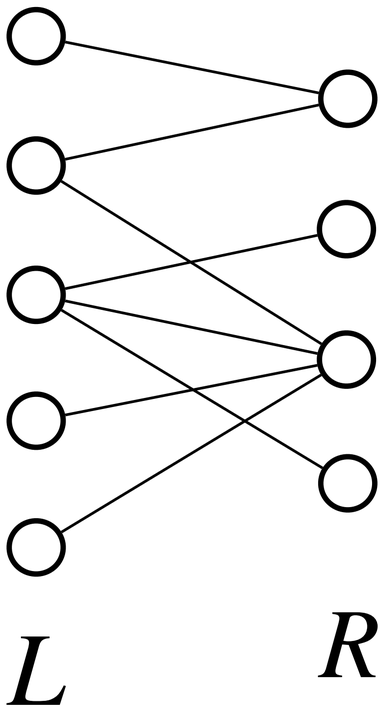

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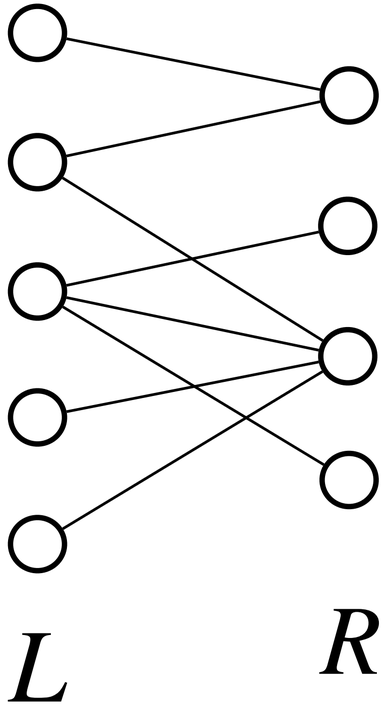

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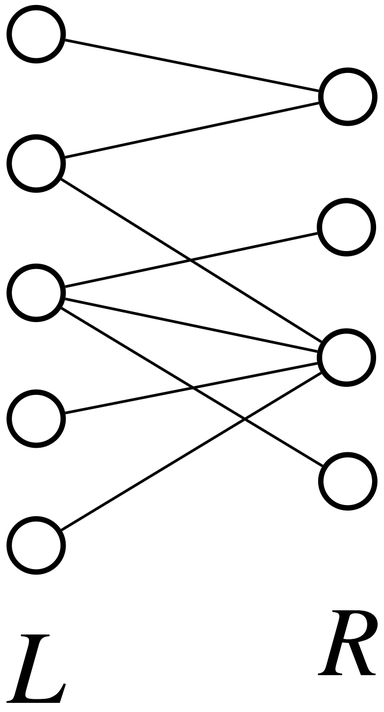

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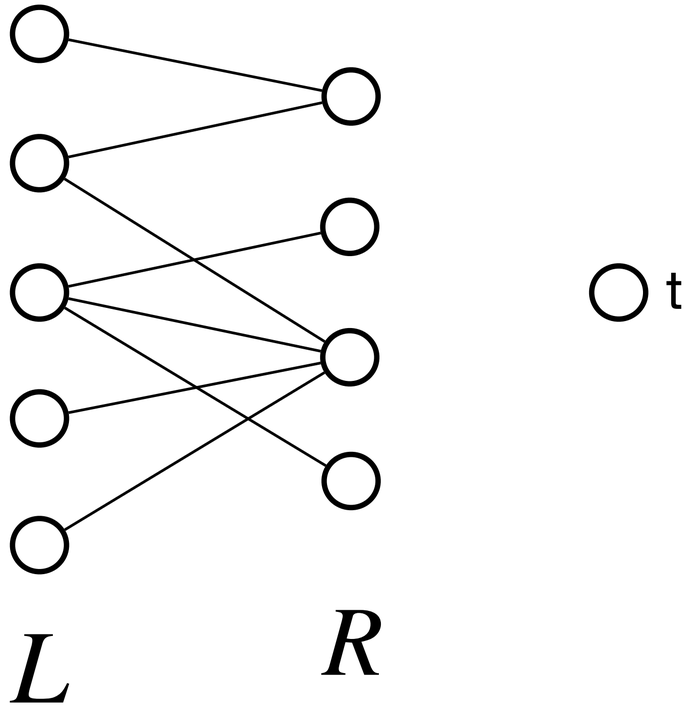

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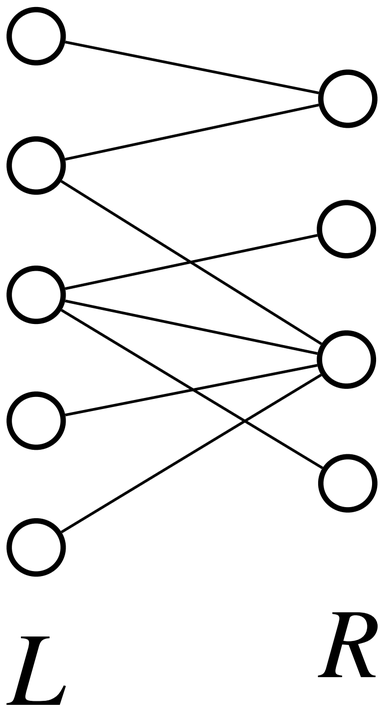

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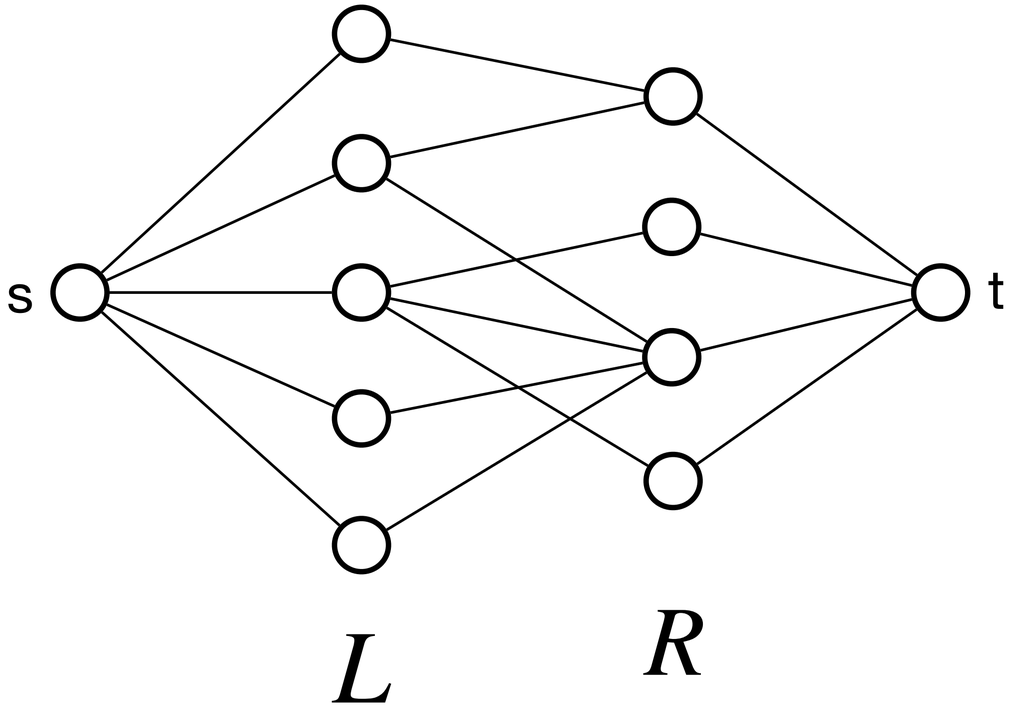

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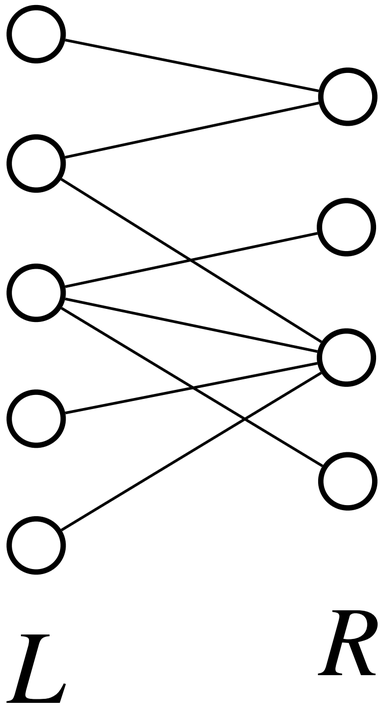
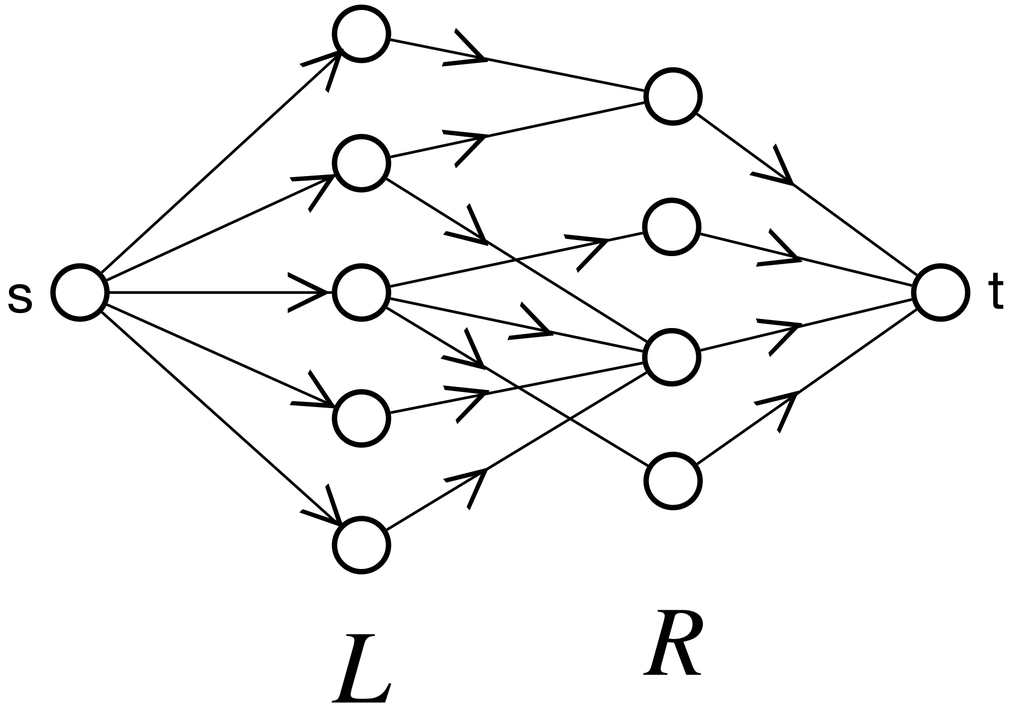


Maximum Flow Problem:

All edges have capacity 1



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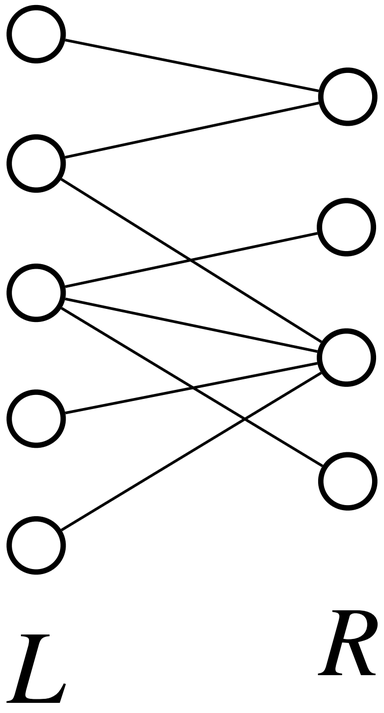

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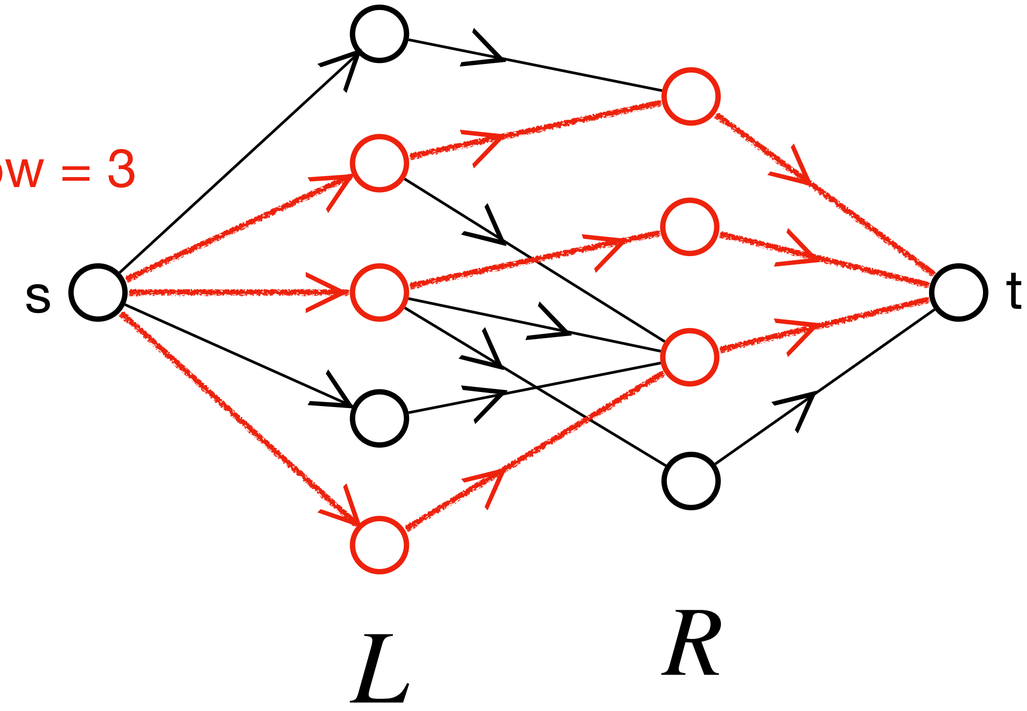
Max flow = 3



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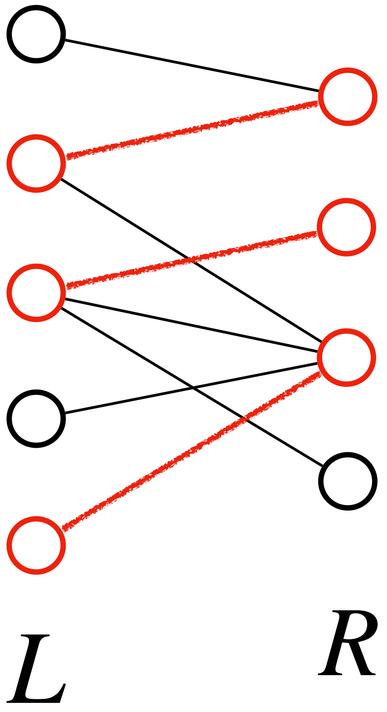
Max matching = 3



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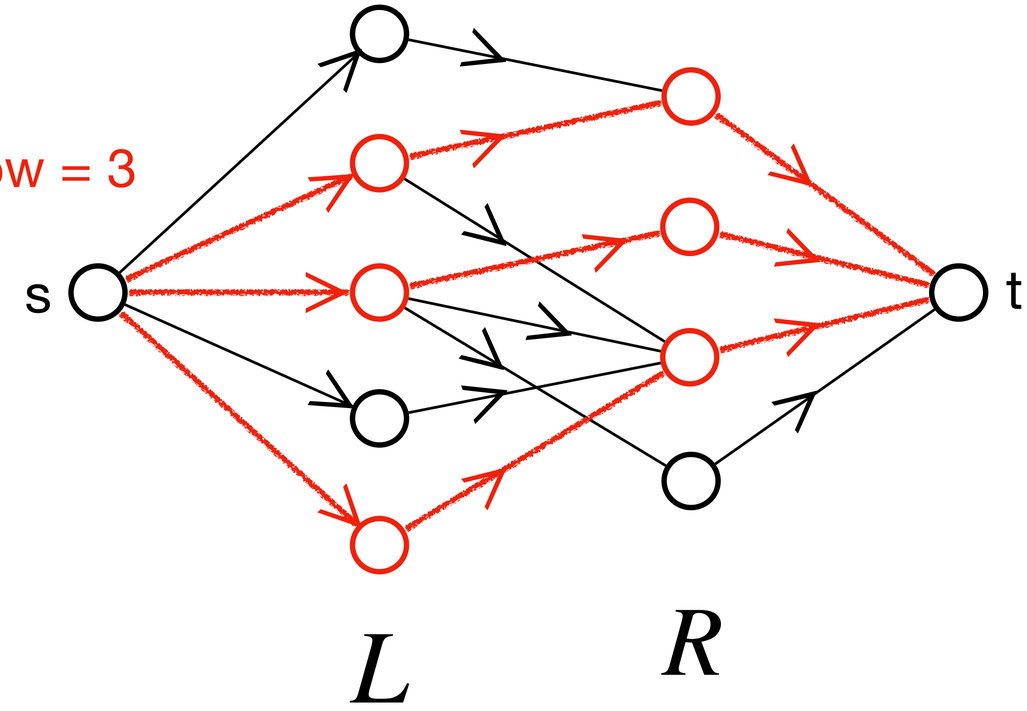

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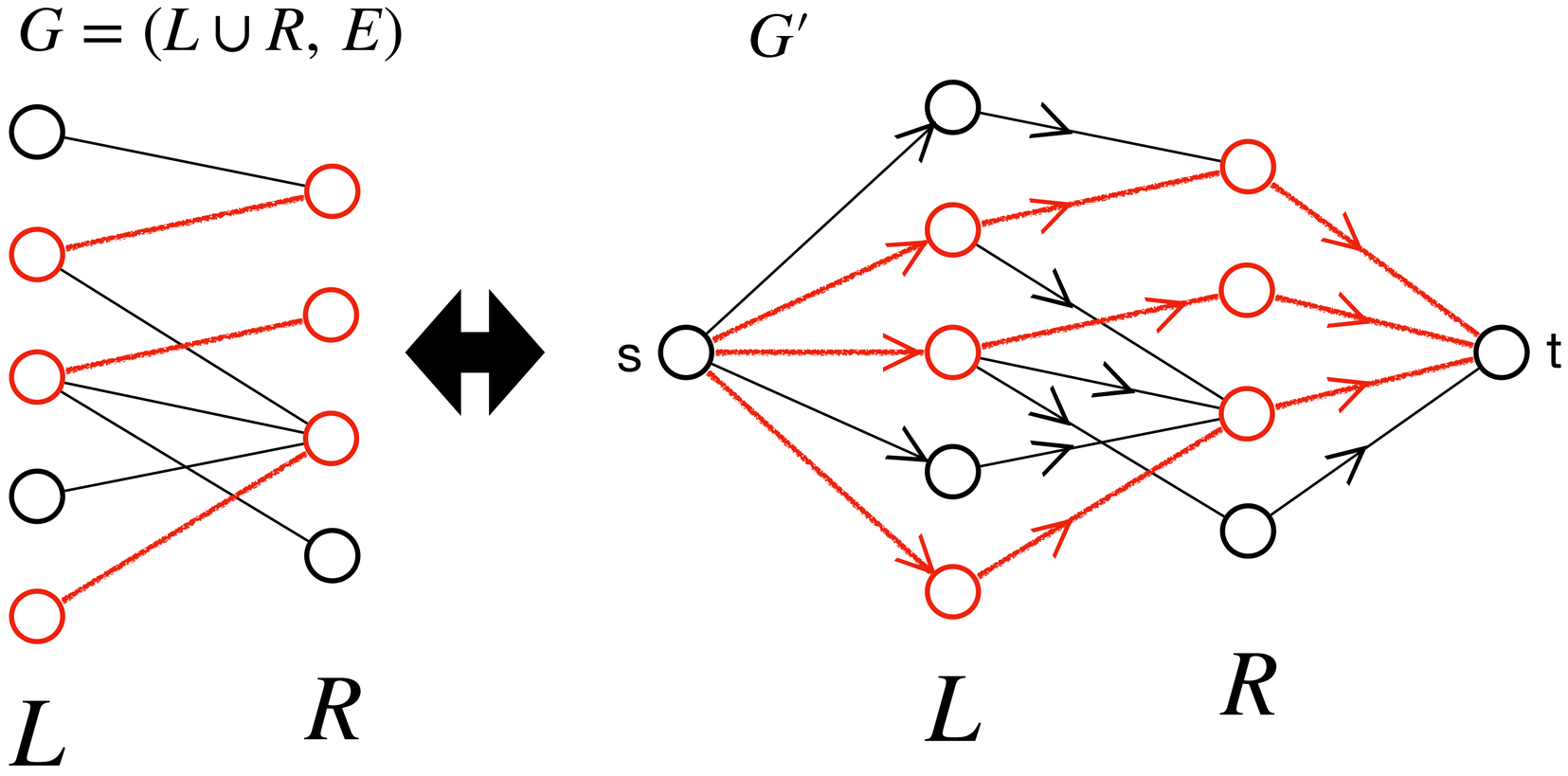
Theorem: G has a matching of size k if and only if G' has a flow of size k.



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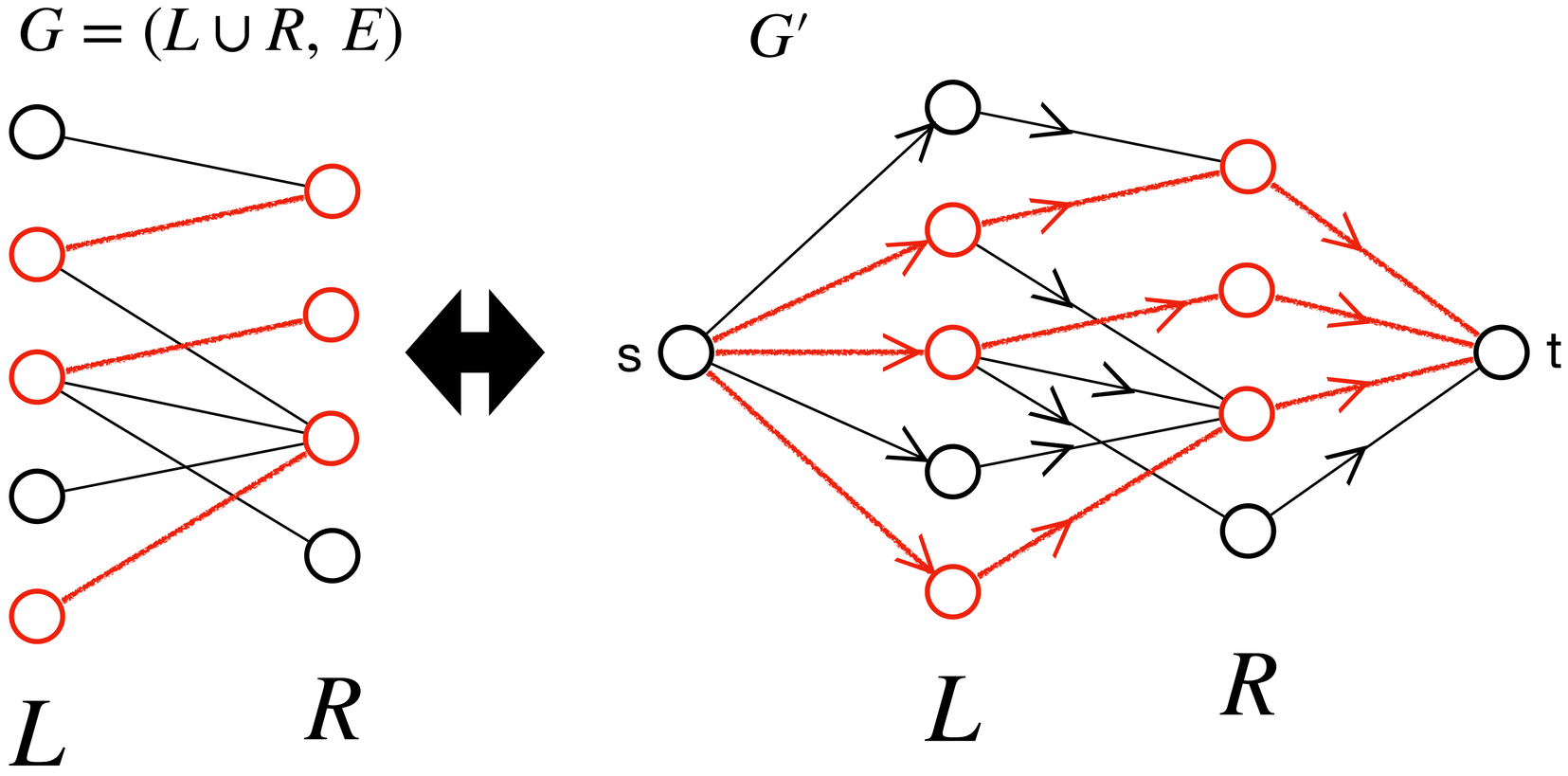

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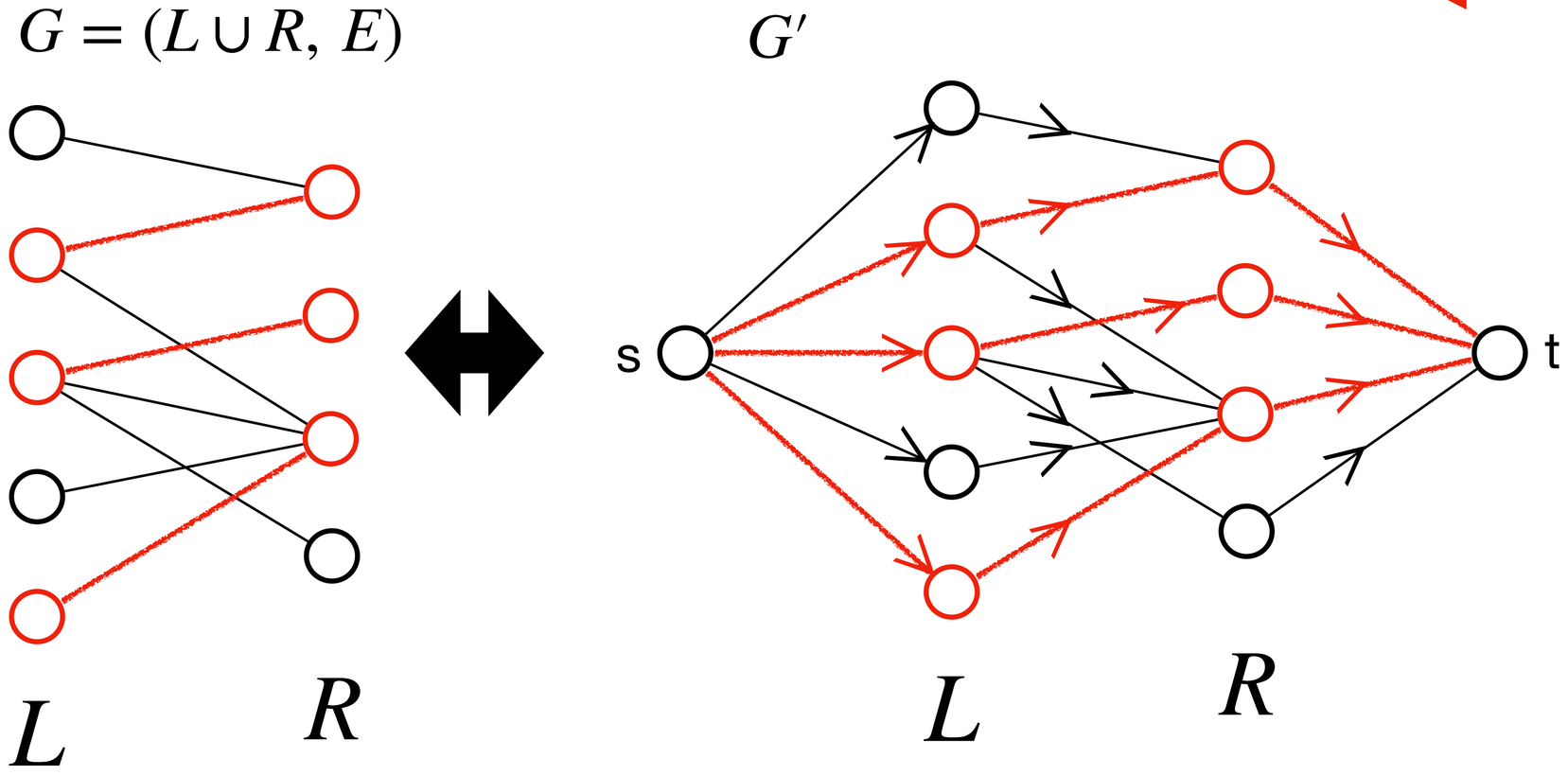

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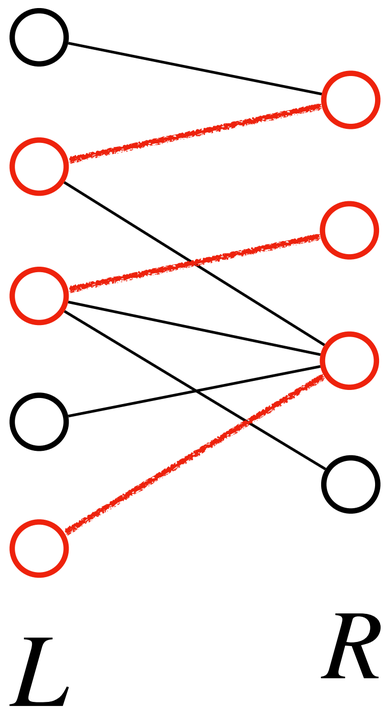
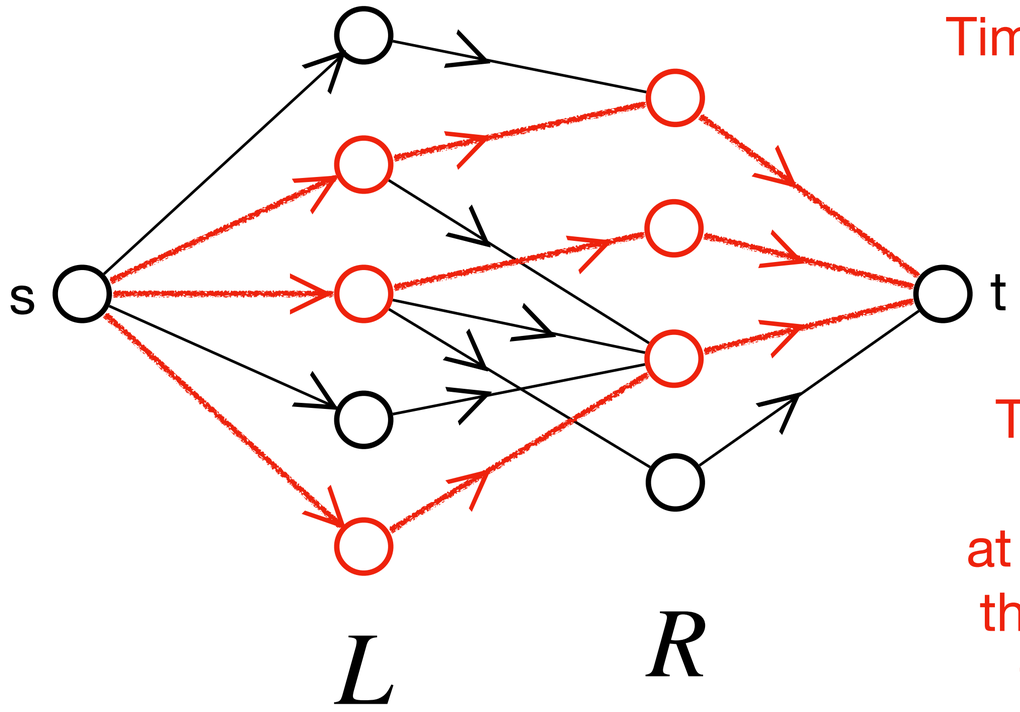
$G = (L \cup R, E)$



G'



Time complexity: $O(VE)$

$|f| \leq |L| \leq |V|$

The Edmonds-Karp algorithm needs to augment the flow at most |V| times, and each time the BFS algorithm uses O(V+E) time to find a shortest path.

Formulate Maximum Flow Problem as a Linear Program

Input: A directed graph $G = (V, E)$, where every edge $(u, v) \in E$ has a positive capacity $c(u, v)$.

Let $s \in V$ be a source node, and let $t \in V$ be a sink node.

Output: A maximum flow from s to t.

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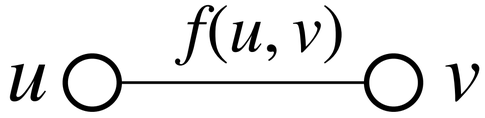

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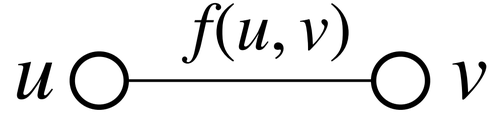

Objective function and constraints:

$$\text{maximize} \quad \sum_{(s,v)\in E} f(s, v)$$

$$\text{subject to:} \quad 0 \le f(u, v) \le c(u, v), \qquad \forall (u, v) \in E$$

$$\sum_{(u,v)\in E} f(u, v) = \sum_{(v,u)\in E} f(v, u), \qquad \forall v \in V - \{s, t\}$$

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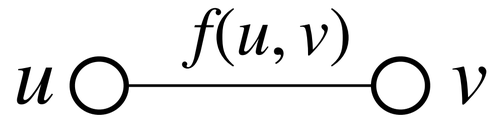

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Why do we still need Edmonds-Karp Algorithm? Lower time complexity.

Quiz questions:

1. How do we map the “Maximum Bipartite Matching Problem” to a “Maximum Flow Problem”?

2. What is the benefit of designing a specialized algorithm for a more special problem than using a more general algorithm for a more general problem?