Algorithms

Lecture Topic: Matchings in Bipartite Graphs

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Roadmap of this lecture:

1. Matching in Bipartite Graphs

   1.1 Define "Maximum Bipartite Matching Problem".

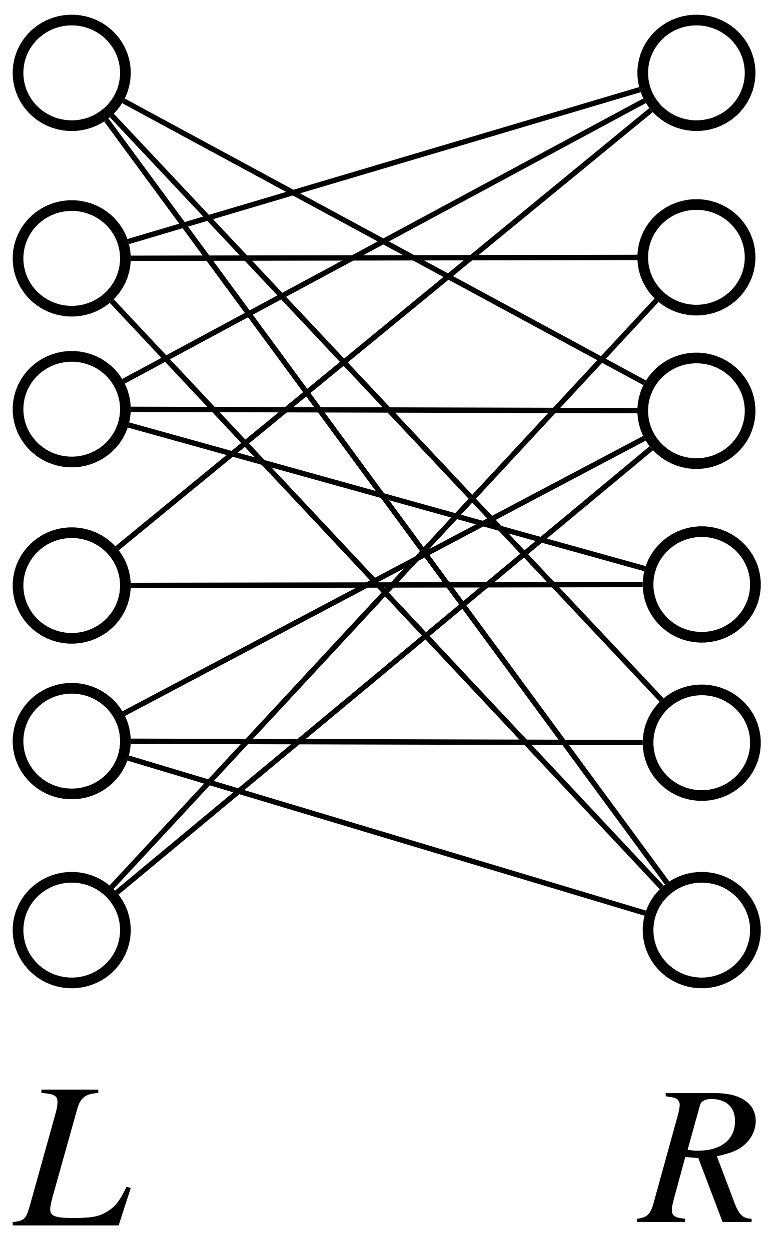
   1.2 Concepts useful for augmenting matching.

   1.3 Hopcroft-Karp Algorithm for "Maximum Bipartite Matching Problem".

   1.4 Time complexity of Hopcroft-Karp Algorithm.

Maximum Bipartite Matching (revisited)

Bipartite Graph: A graph $G = (V, E)$ is bipartite if we can partition the nodes into L and R, such that every edge is between a node in L and a node in R.



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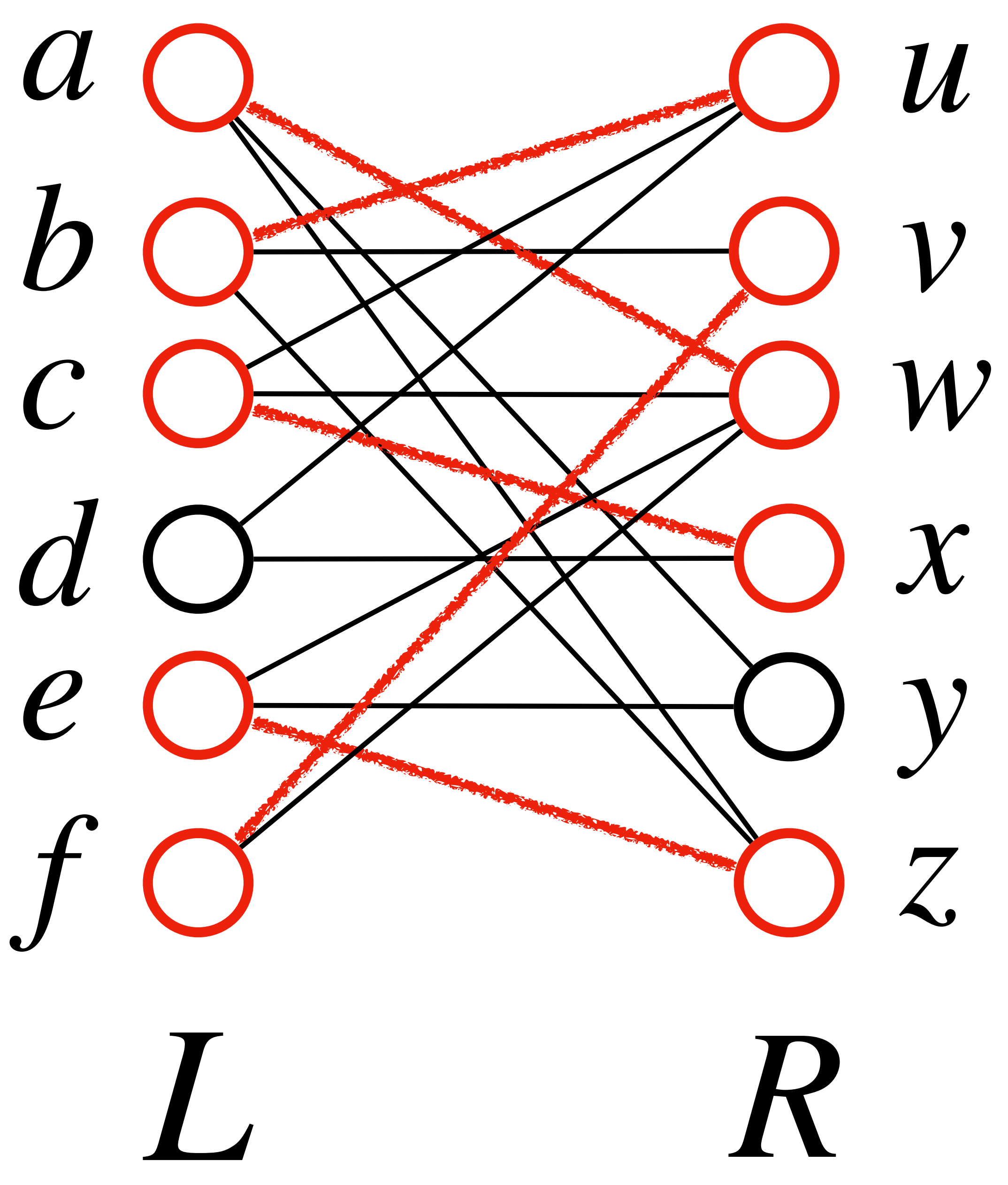

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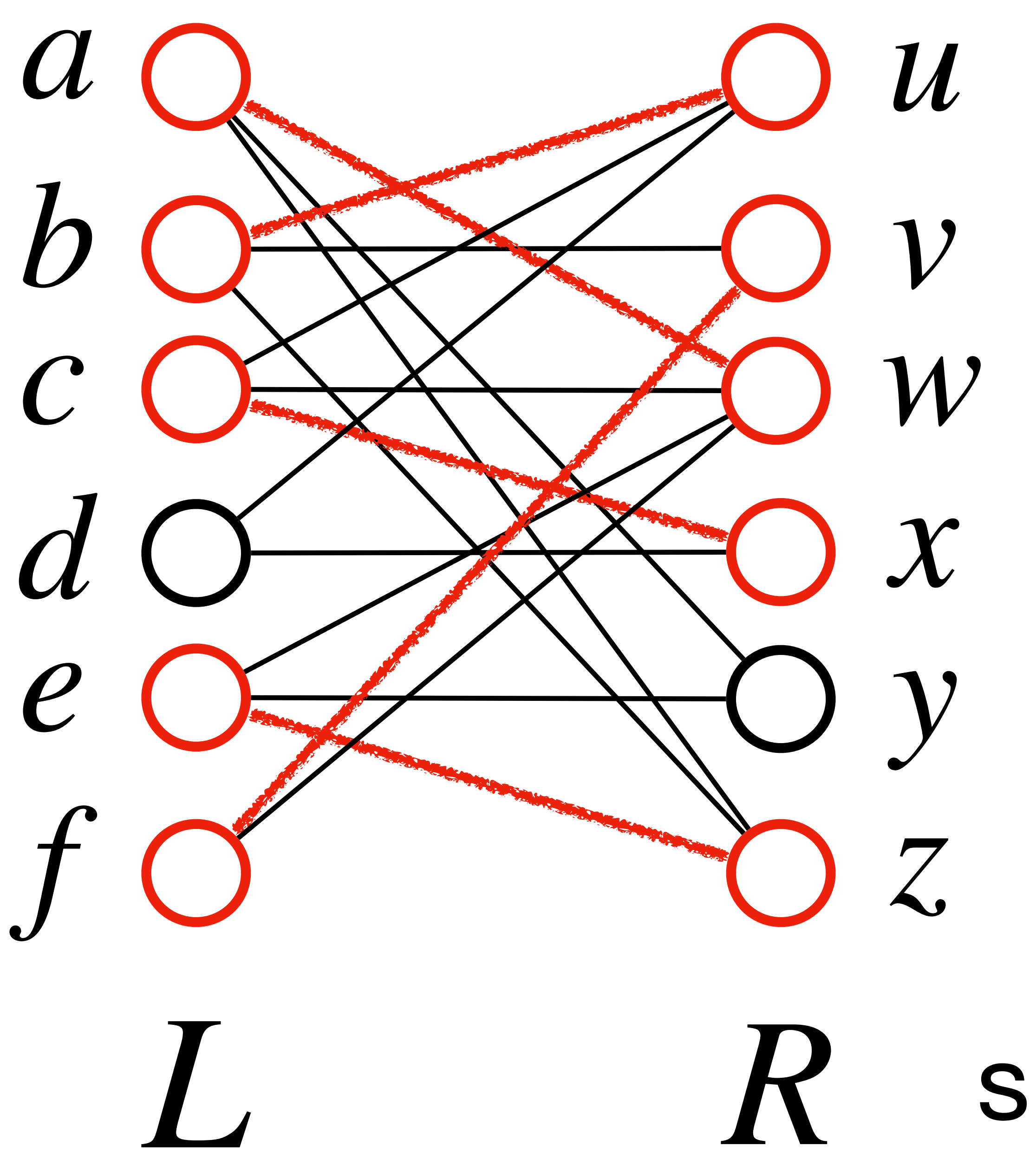

Bipartite Matching Problem:

Input: A bipartite graph $G = (L \cup R, E)$.

Output: A matching of maximum size.

size of matching $= 5$

Quiz question:

1. What are the applications of the "Maximum Bipartite Matching Problem"?

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Maximum Bipartite Matching

Algorithm based on maximum flow: $O(VE)$

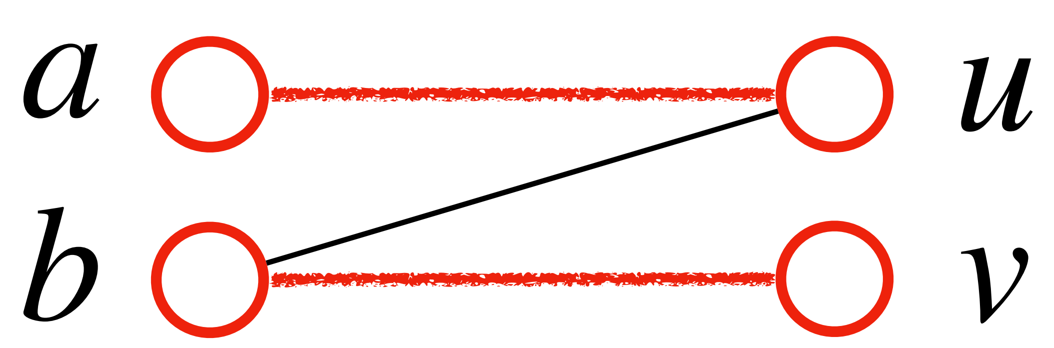
Hopcroft-Karp Algorithm: $O(\sqrt{V}\ E)$

Maximum Matching vs. Maximal Matching

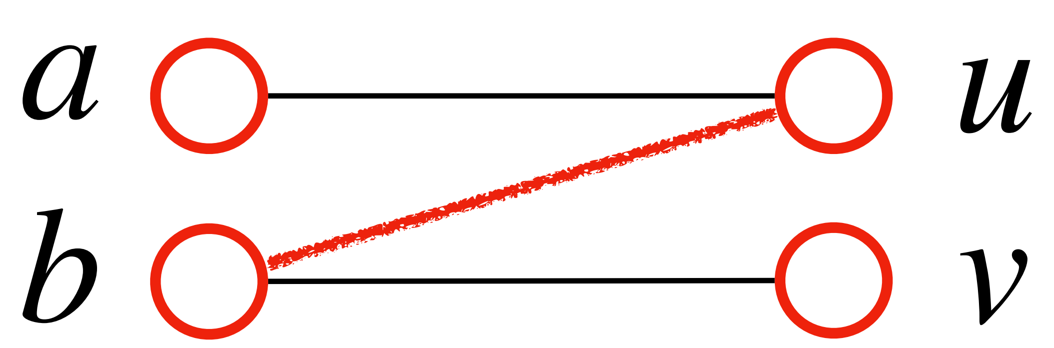
Maximum Matching: a matching of maximum size (number of edges).

Maximal Matching: A maximal matching is a matching $M \subseteq E$ in the graph $G = (V, E)$ to which no other edges can be added, that is, for every edge $e \in E - M$, the edge set $M \cup \{e\}$ fails to be a matching.

Maximum Matching



Maximal Matching

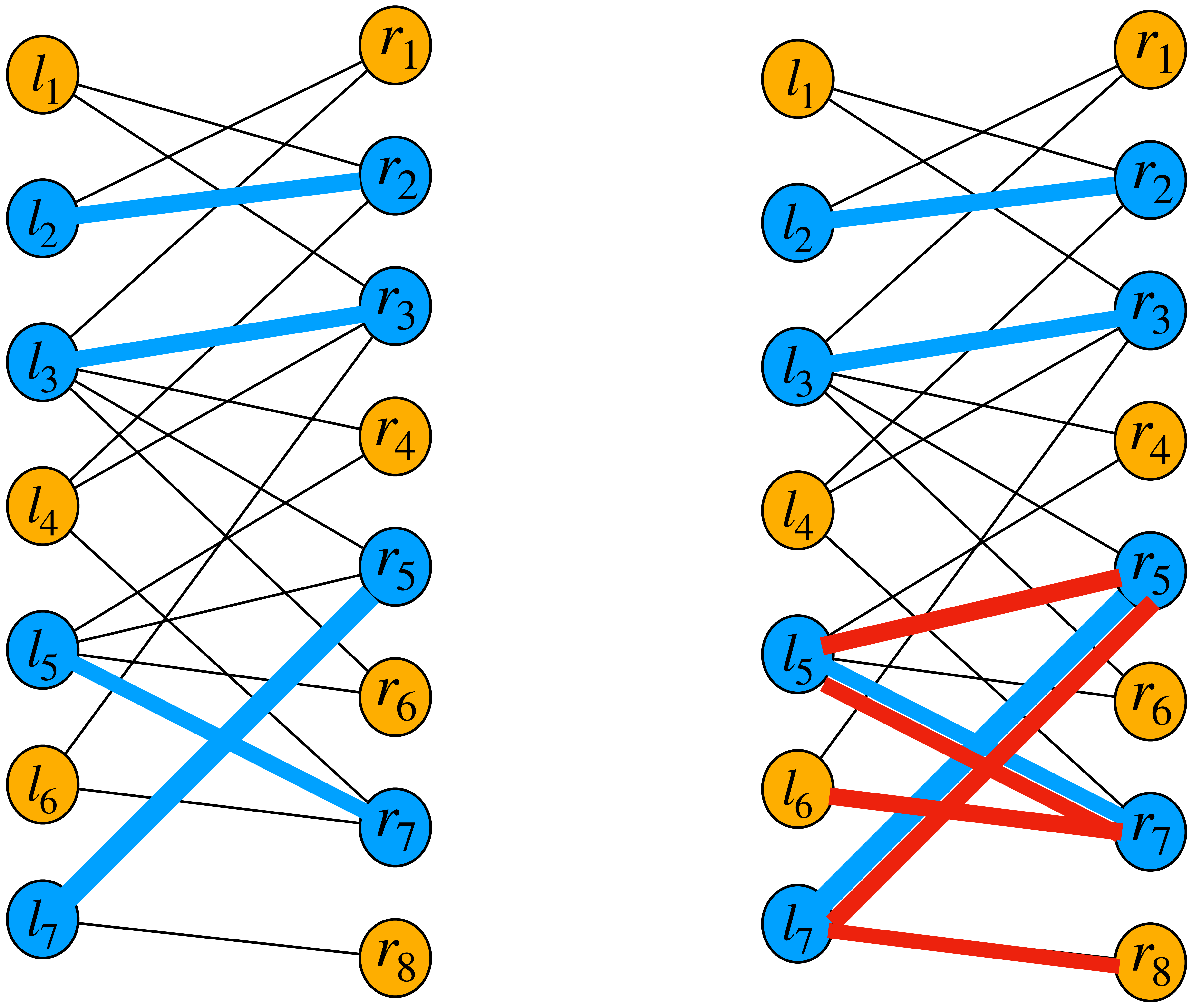


A Maximum Matching is also a Maximal Matching.
A Maximal Matching is not always a Maximum Matching.

Maximum Matching: globally optimal.

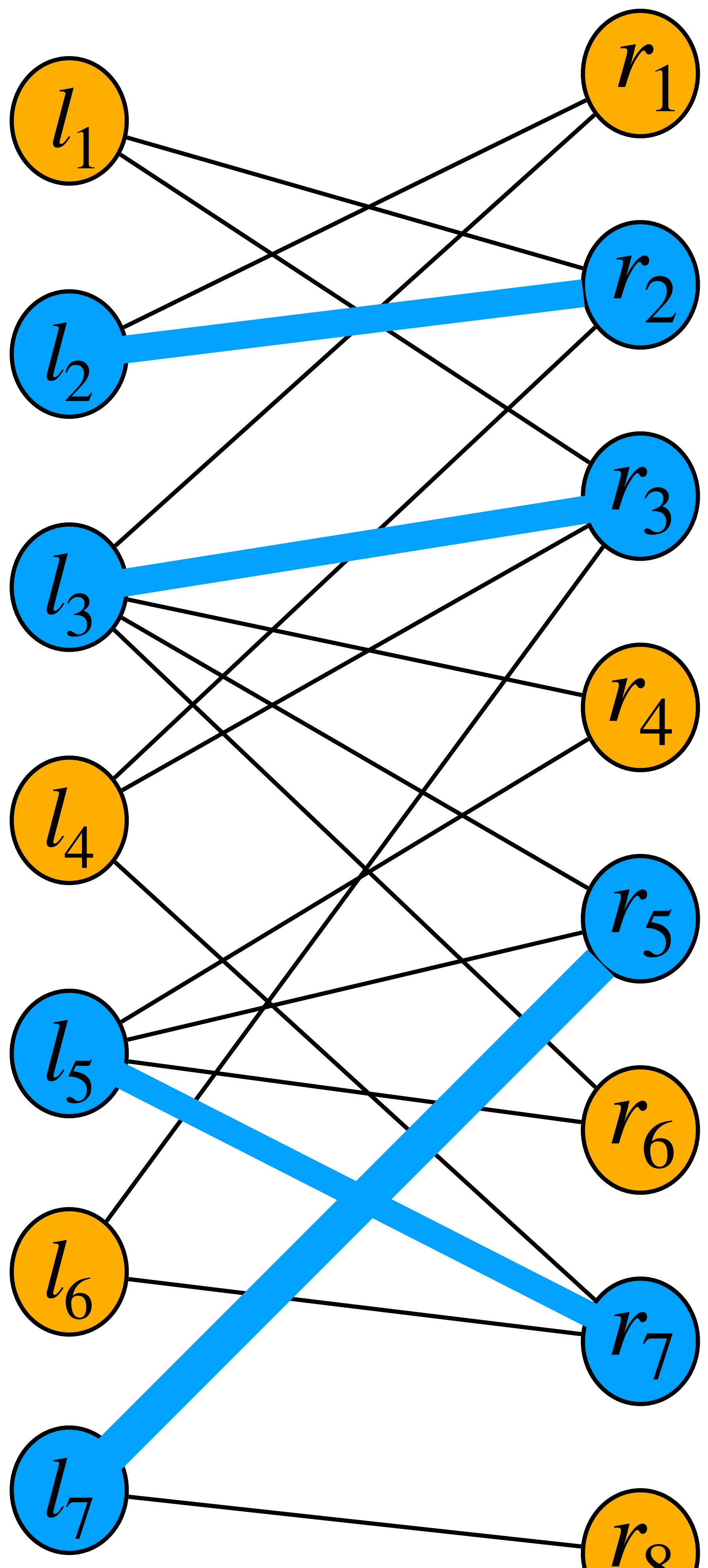
Maximal Matching: locally optimal.

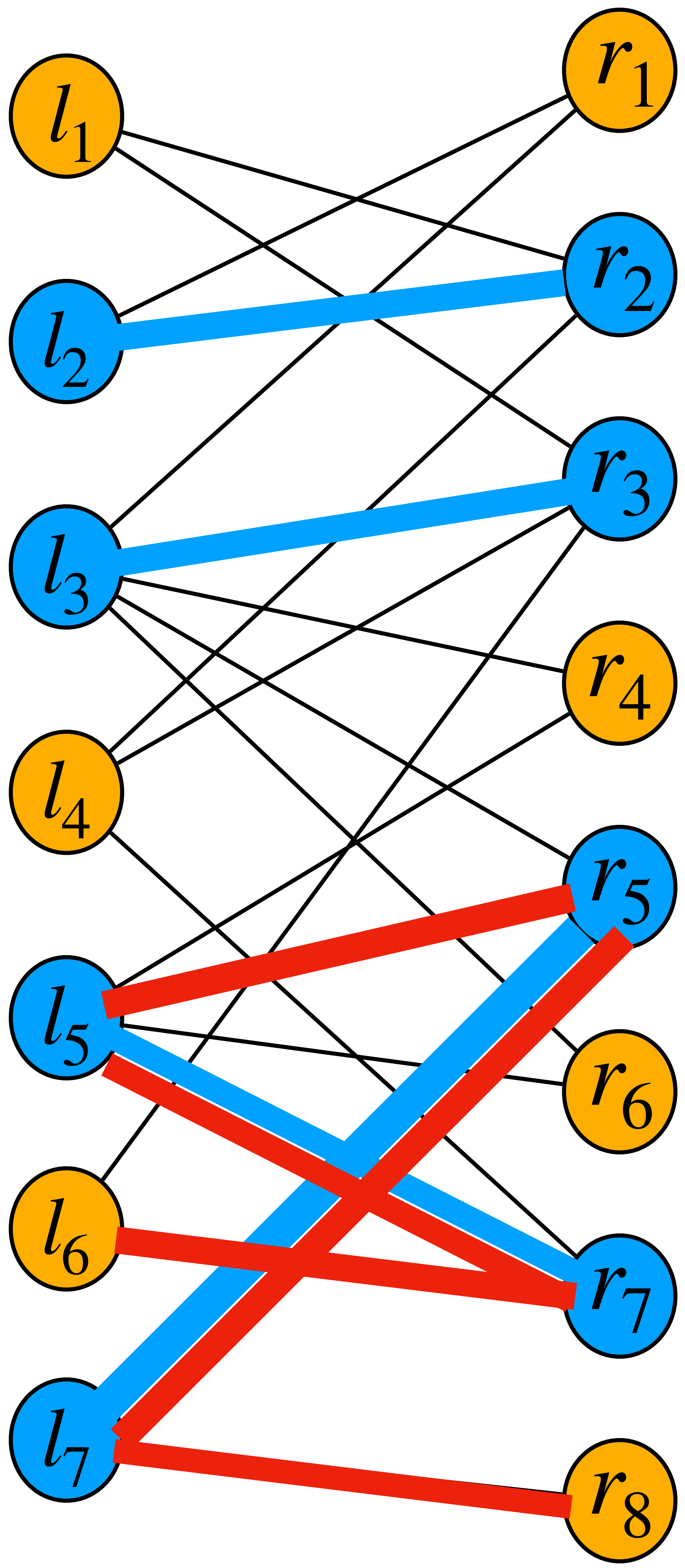
M-alternating Path: Given a matching M in an undirected graph $G = (V, E)$, an M-alternating path is a simple path whose edges alternate between being in M and being in $E - M$.



M-alternating Path: Given a matching M in an undirected graph $G = (V, E)$, an M-alternating path is a simple path whose edges alternate between being in M and being in $E - M$.

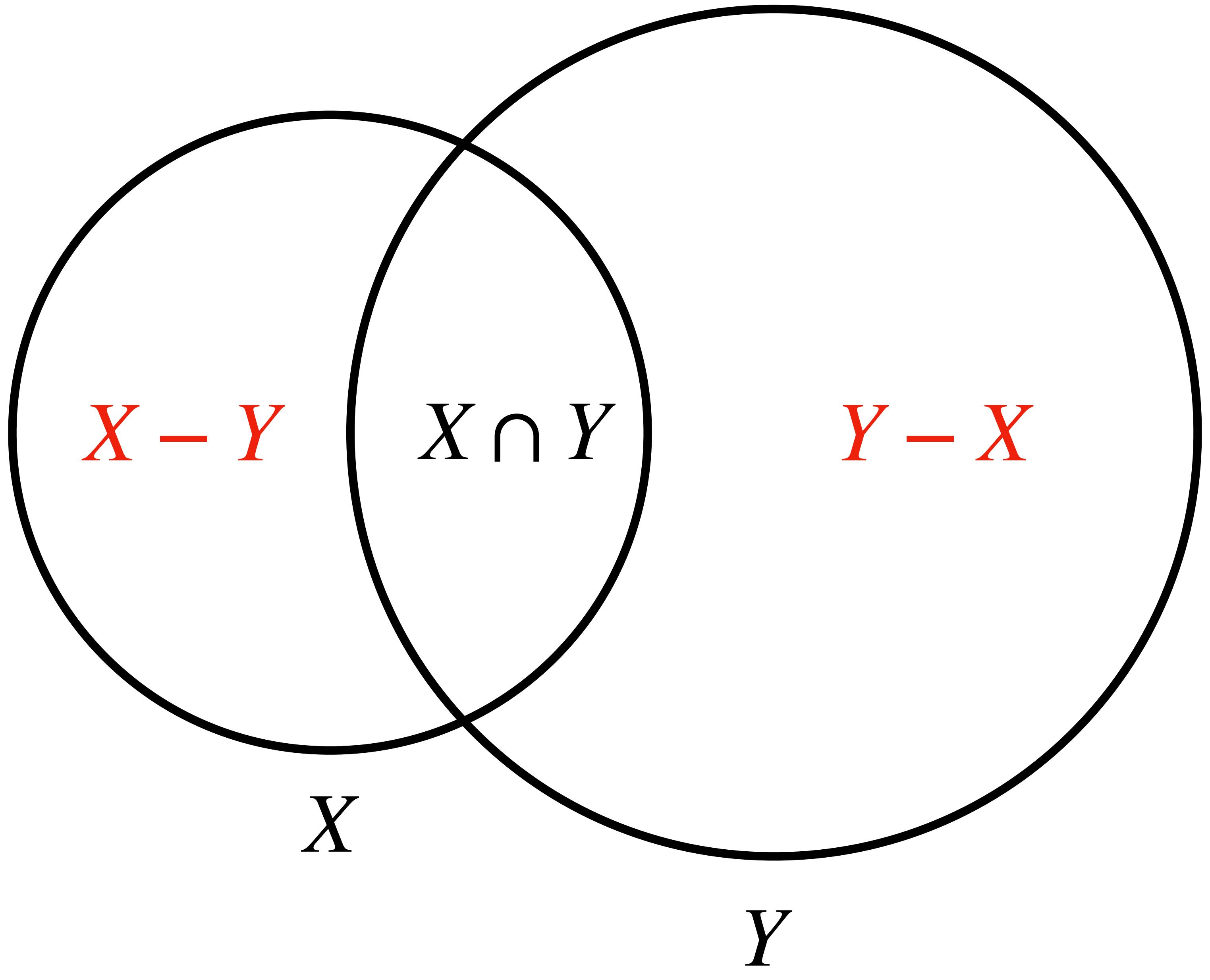
M-augmenting Path: an M-alternating path whose first and last vertices are not matched by M.





Symmetric Difference: given two sets X and Y, their symmetric difference

$$X \oplus Y = (X - Y) \cup (Y - X)$$

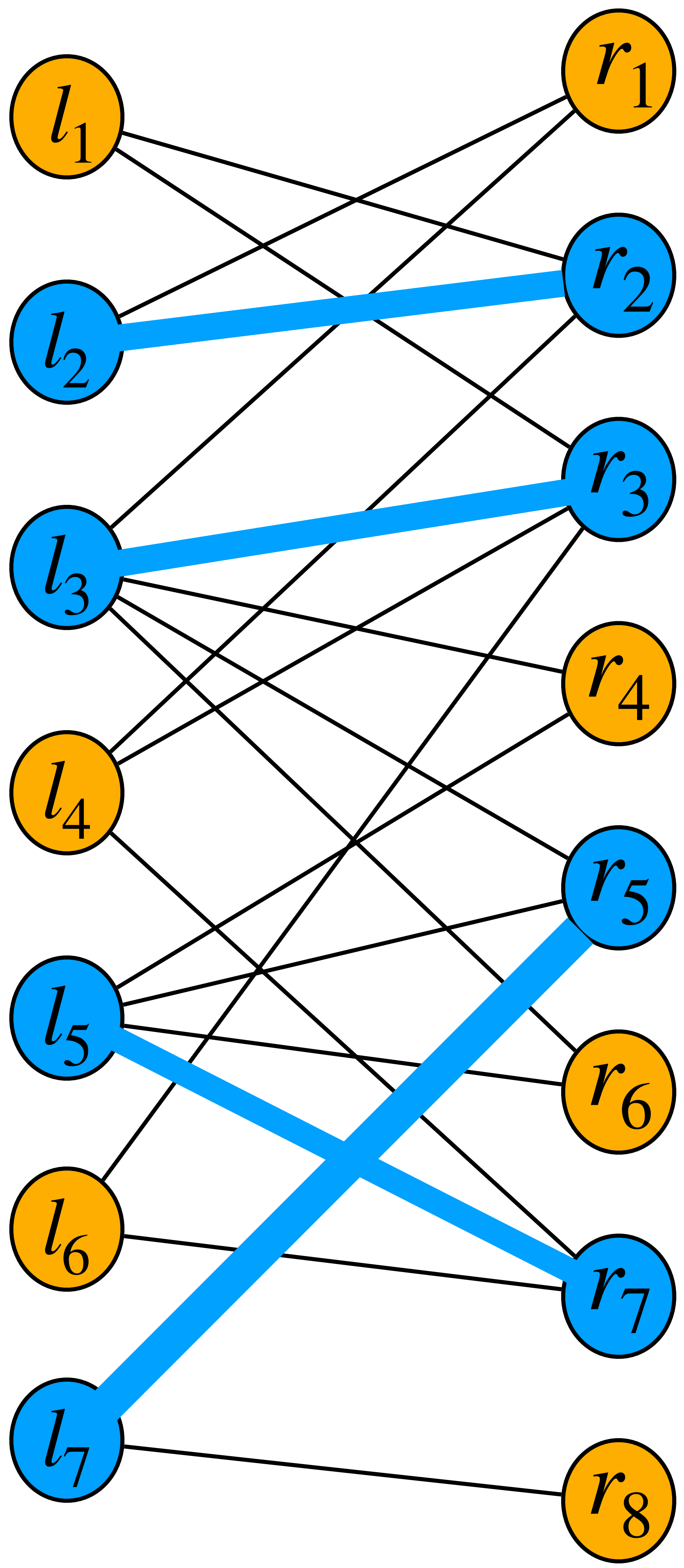


$$X \oplus Y = (X \cup Y) - (X \cap Y)$$

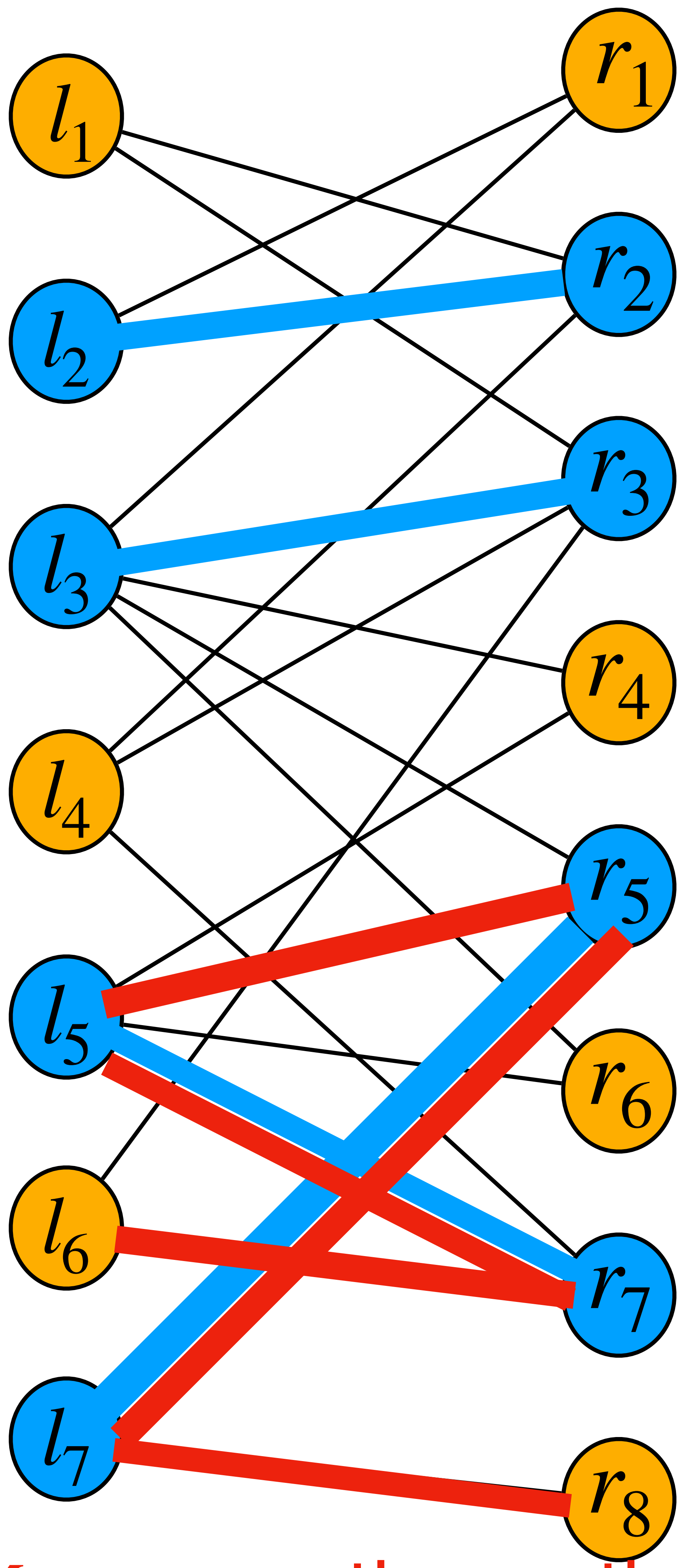
The operator $\oplus$ is commutative and associative: $X \oplus Y = Y \oplus X$, $(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z)$

Furthermore, $X \oplus X = \varnothing$, $X \oplus \varnothing = \varnothing \oplus X = X$

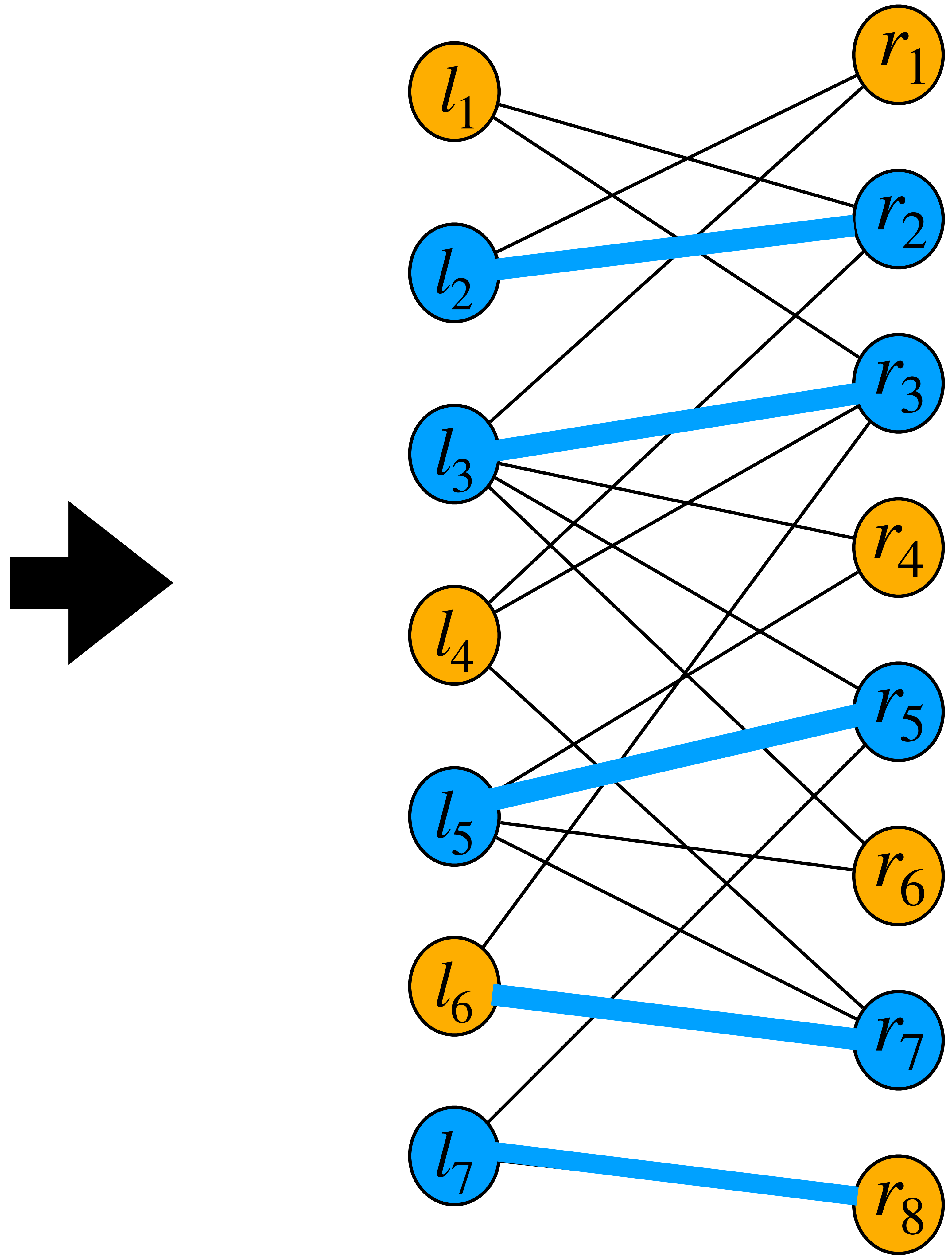
Lemma: Let M be a matching in any undirected graph $G = (V, E)$, and let P be an M-augmenting path. Then, the set of edges $M' = M \oplus P$ is also a matching in G with $|M'| = |M| + 1$.



Matching M



M-augmenting path P



Larger Matching M'

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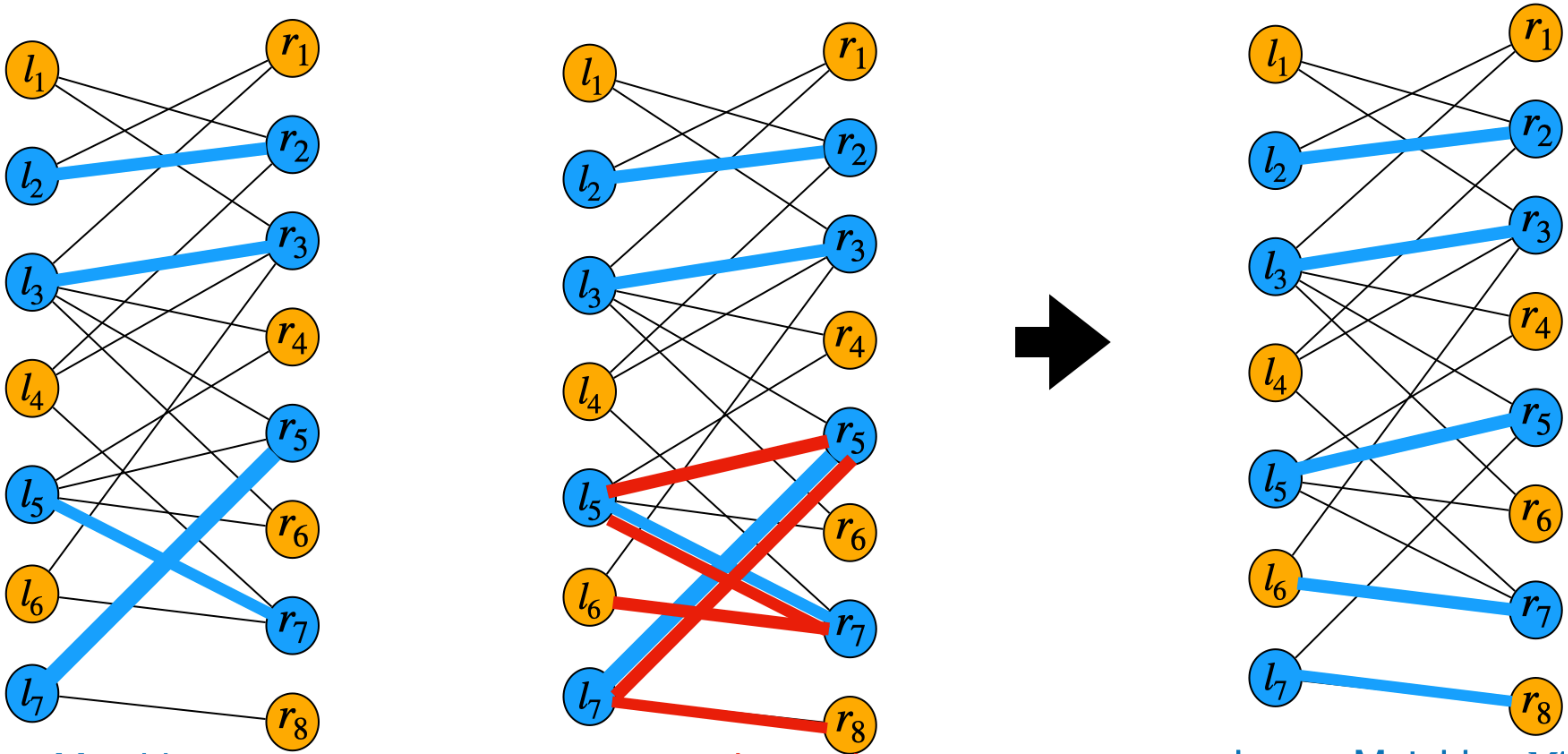
Proof: P has an odd number of edges. #edges not in M = # edges in M + 1



Matching M — M-augmenting path P — Larger Matching M'

Corollary: Let M be a matching in any undirected graph $G = (V, E)$ and $P_1, P_2, \cdots, P_k$ be vertex-disjoint M-augmenting paths. Then the set of edges $M' = M \oplus (P_1 \cup P_2 \cup \cdots \cup P_k)$ is a matching in G with $|M'| = |M| + k$.

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Proof: Since $P_1,\ P_2,\ \cdots,\ P_k$ are vertex-disjoint, $P_1 \cup P_2 \cup \cdots \cup P_k = P_1 \oplus P_2 \oplus \cdots \oplus P_k$.

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Since $\oplus$ is associative,

$$M \oplus (P_1 \cup P_2 \cup \cdots \cup P_k) = M \oplus (P_1 \oplus P_2 \oplus \cdots \oplus P_k)$$

$$= (\cdots((M \oplus P_1) \oplus P_2) \oplus \cdots \oplus P_{k-1}) \oplus P_k$$

matching of size $|M| + 1$

matching of size $|M| + 2$

matching of size $|M| + 3$

matching of size $|M| + k - 1$

matching of size $|M| + k$

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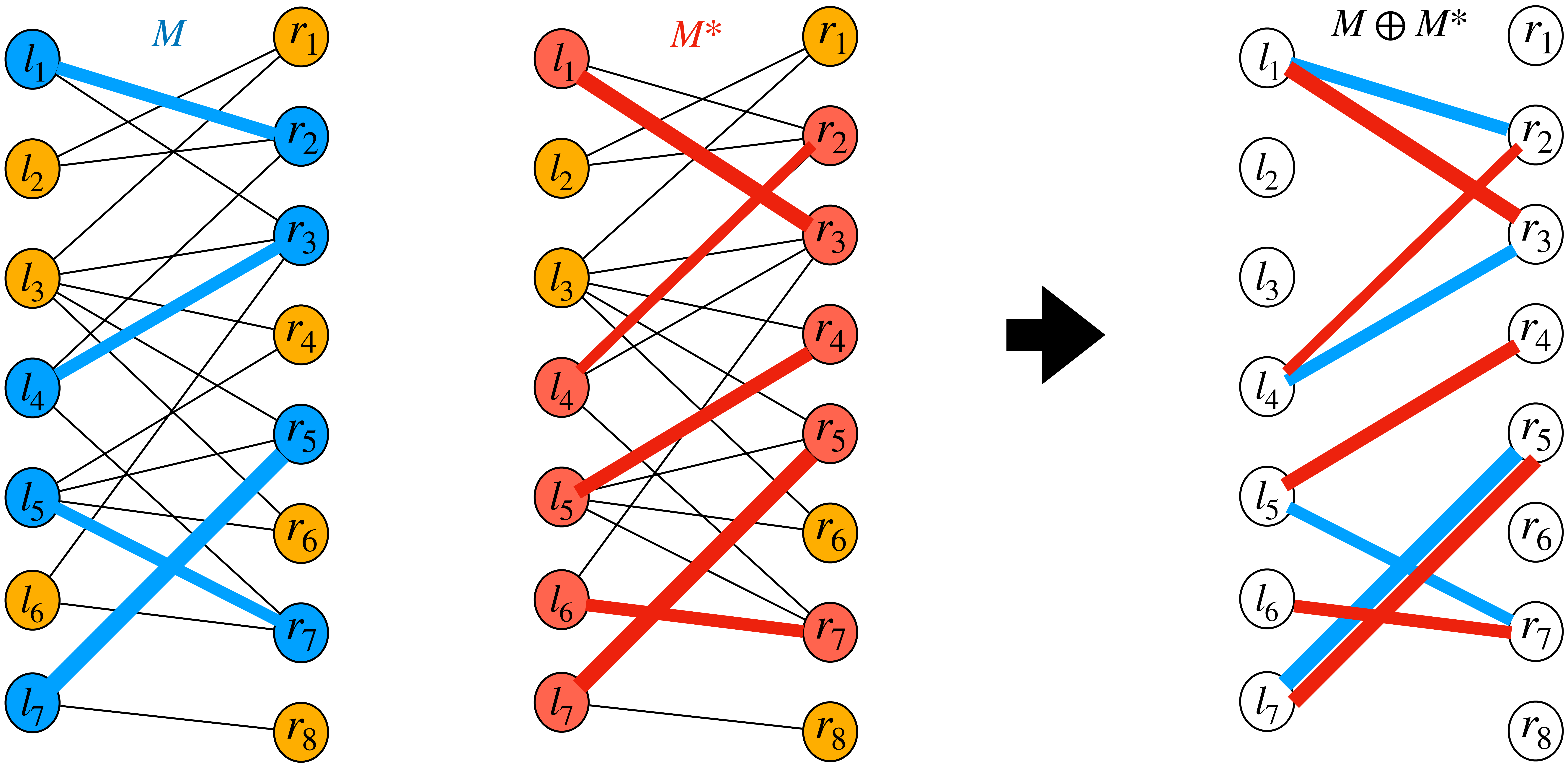

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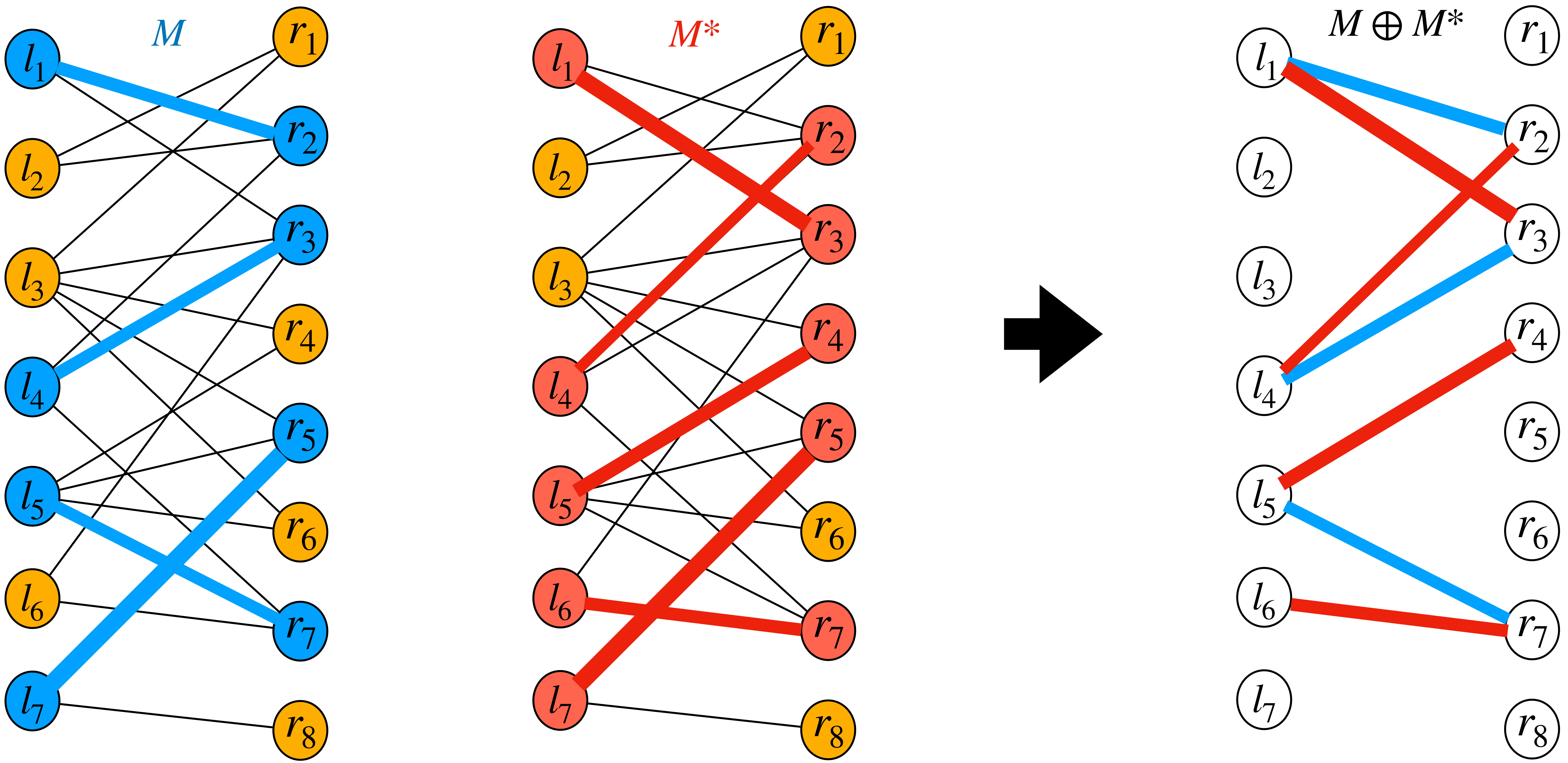

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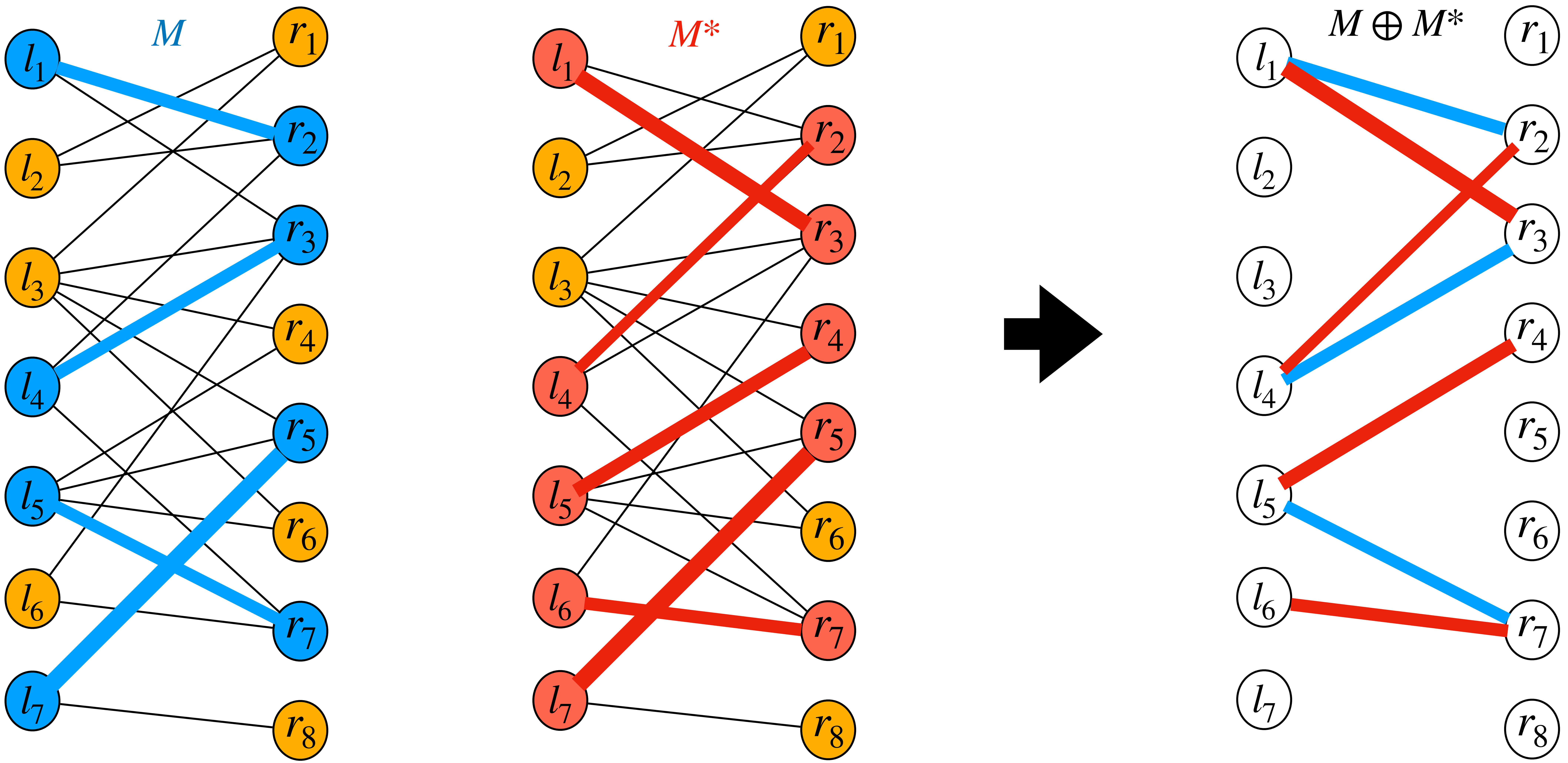

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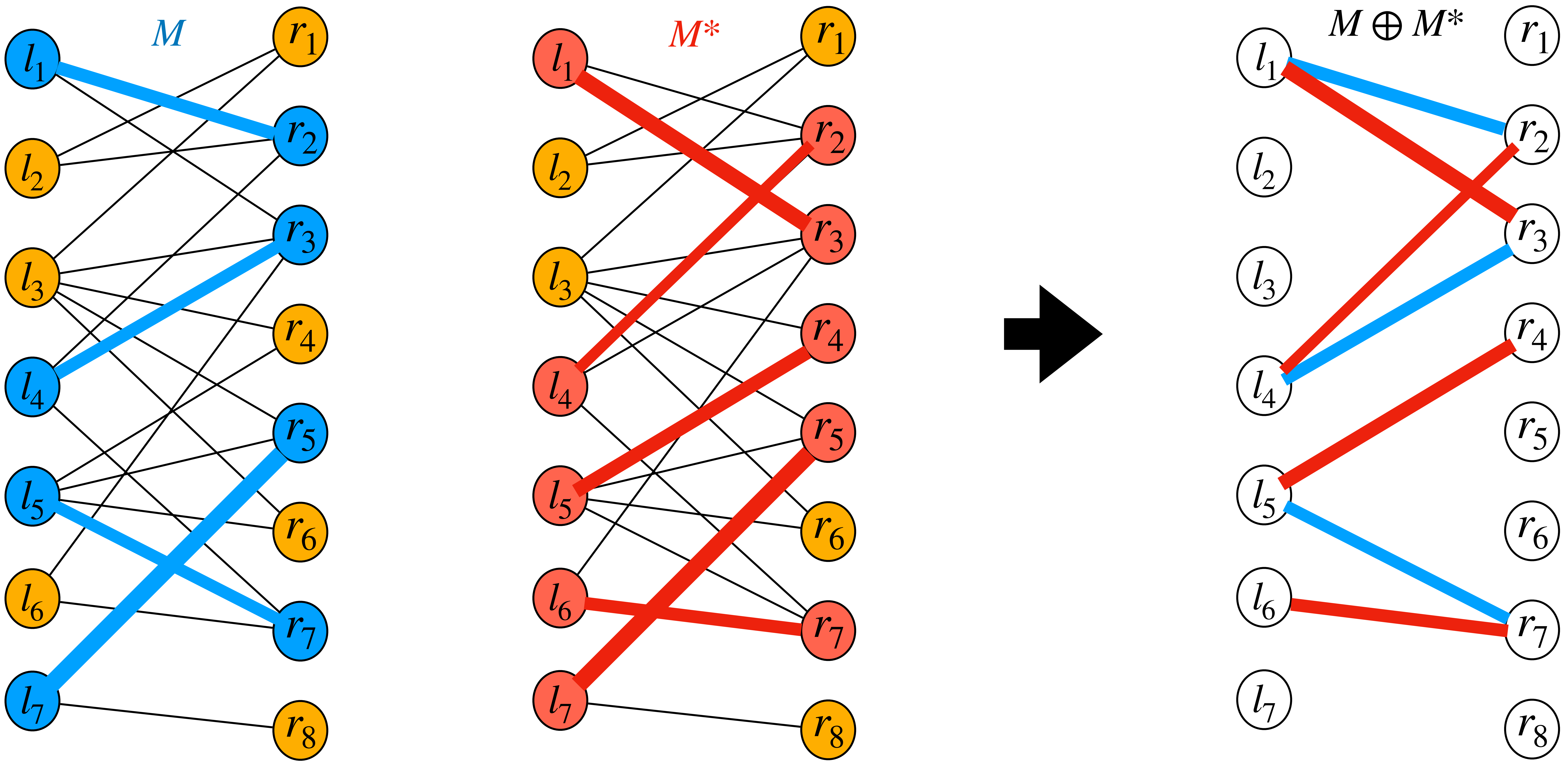

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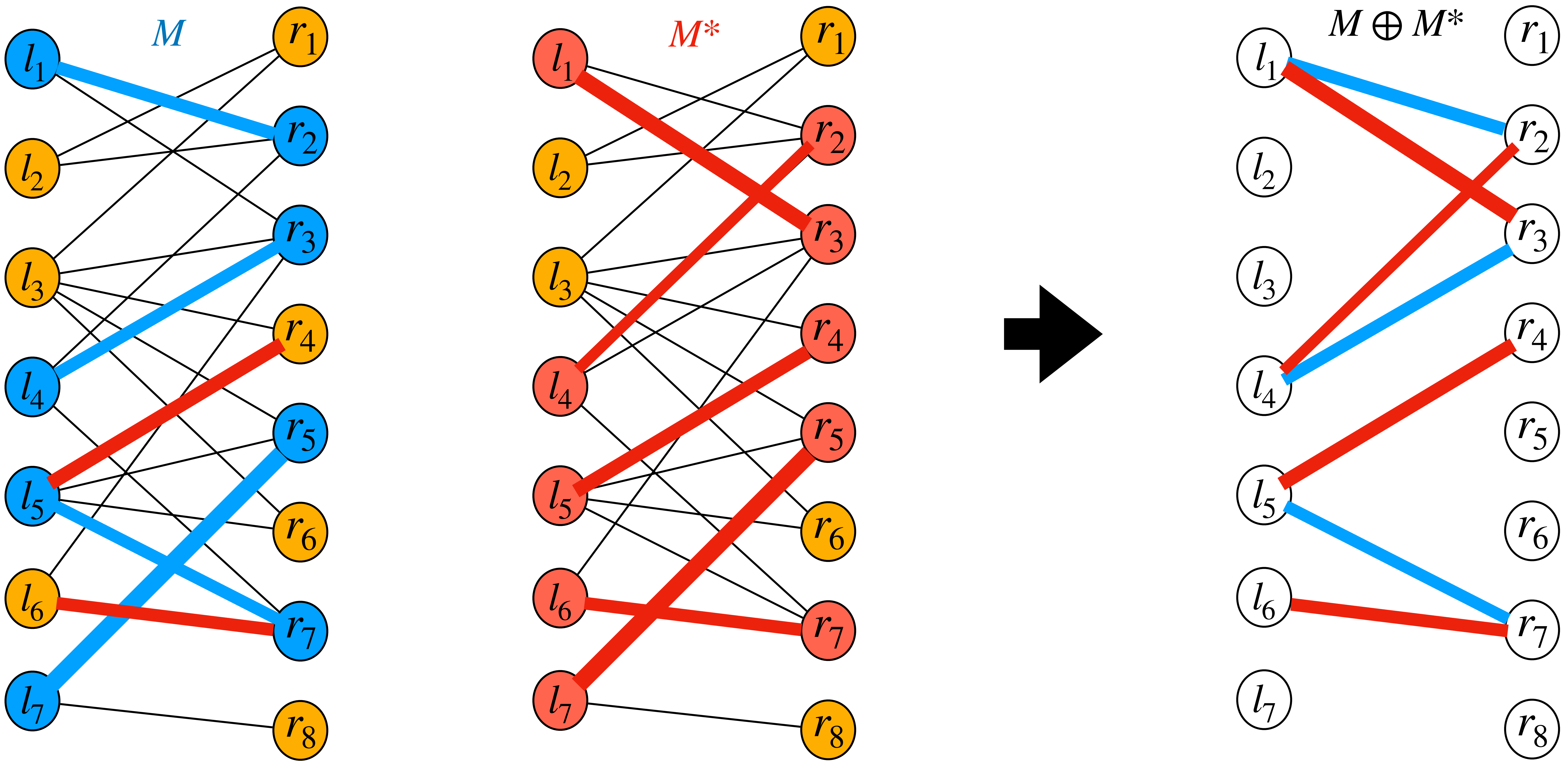

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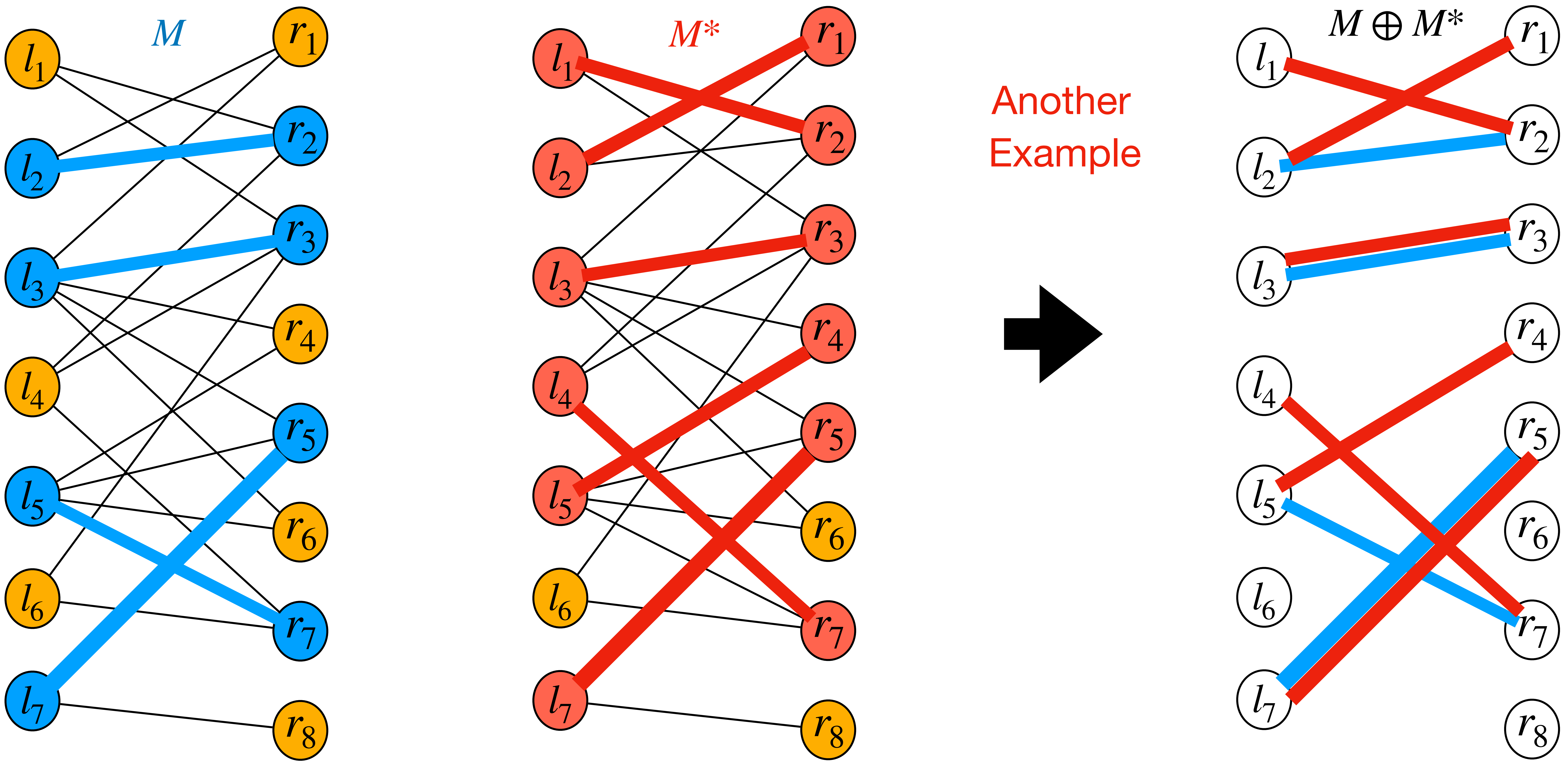

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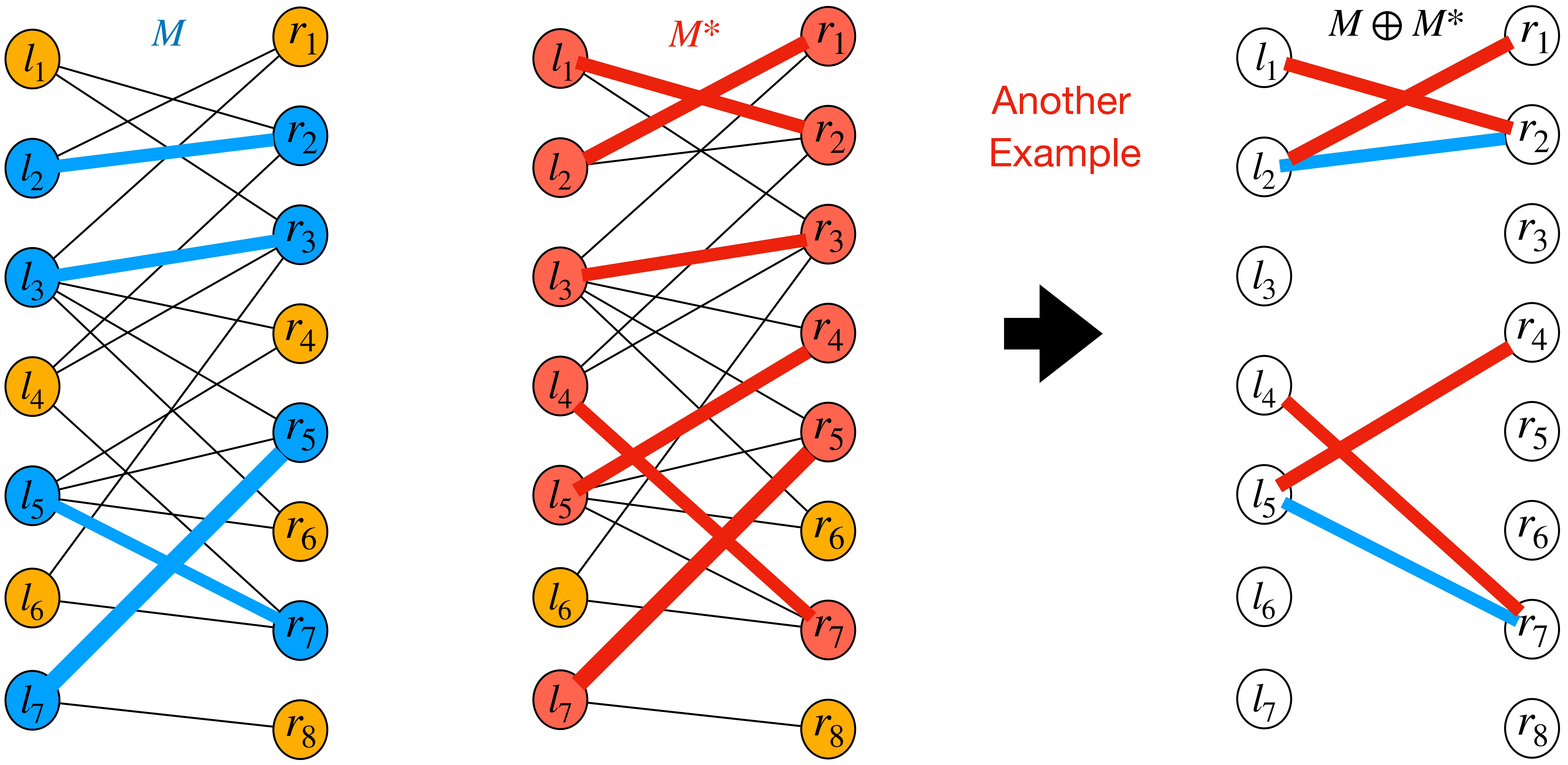

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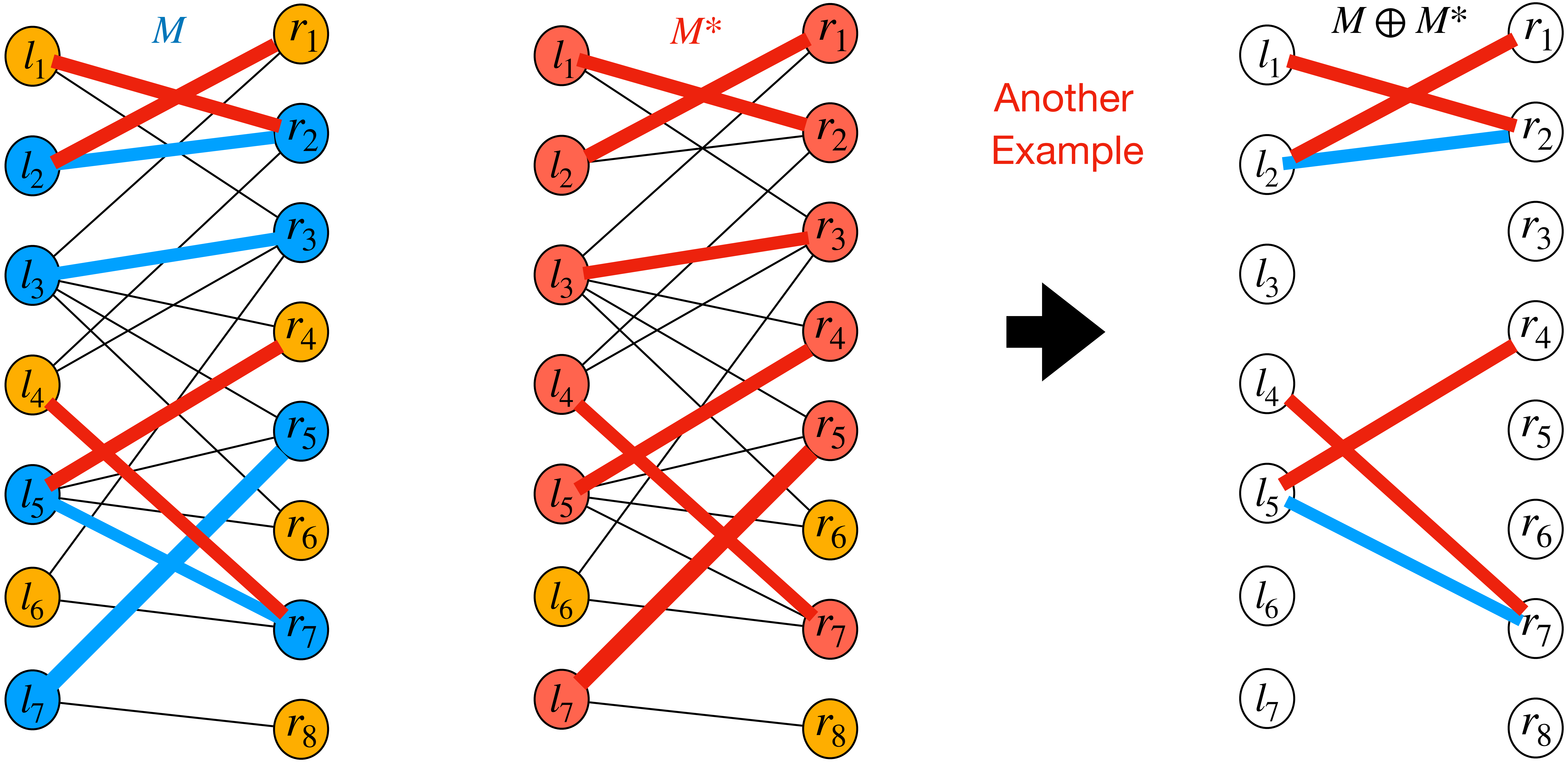
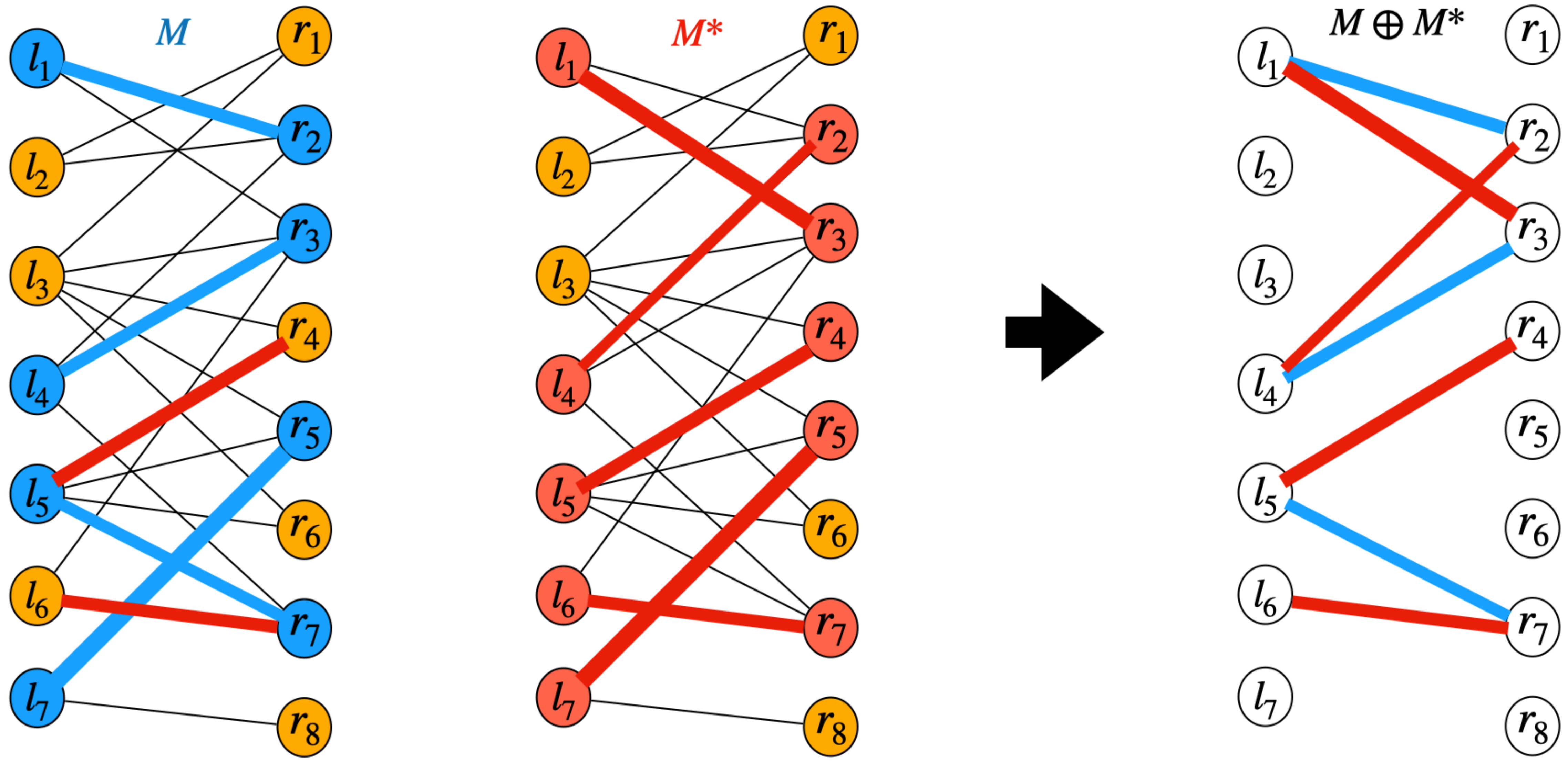


Another Example

Lemma: Let M and M^* be matchings in graph $G=(V,E)$ and consider the graph $G'=(V,E')$, where $E'=M \oplus M^*$. Then, G' is a disjoint union of simple paths, simple cycles, and/or isolated vertices. The edges in each such simple path or simple cycle alternate between M and M^*. If $|M^*|>|M|$, then G' contains at least $|M^*|-|M|$ vertex-disjoint M-augmenting paths.

Proof: Each vertex in G' has degree 0, 1 or 2, since at most two edges of E' can be incident on a vertex: at most one edge from M and at most one edge from M^*.



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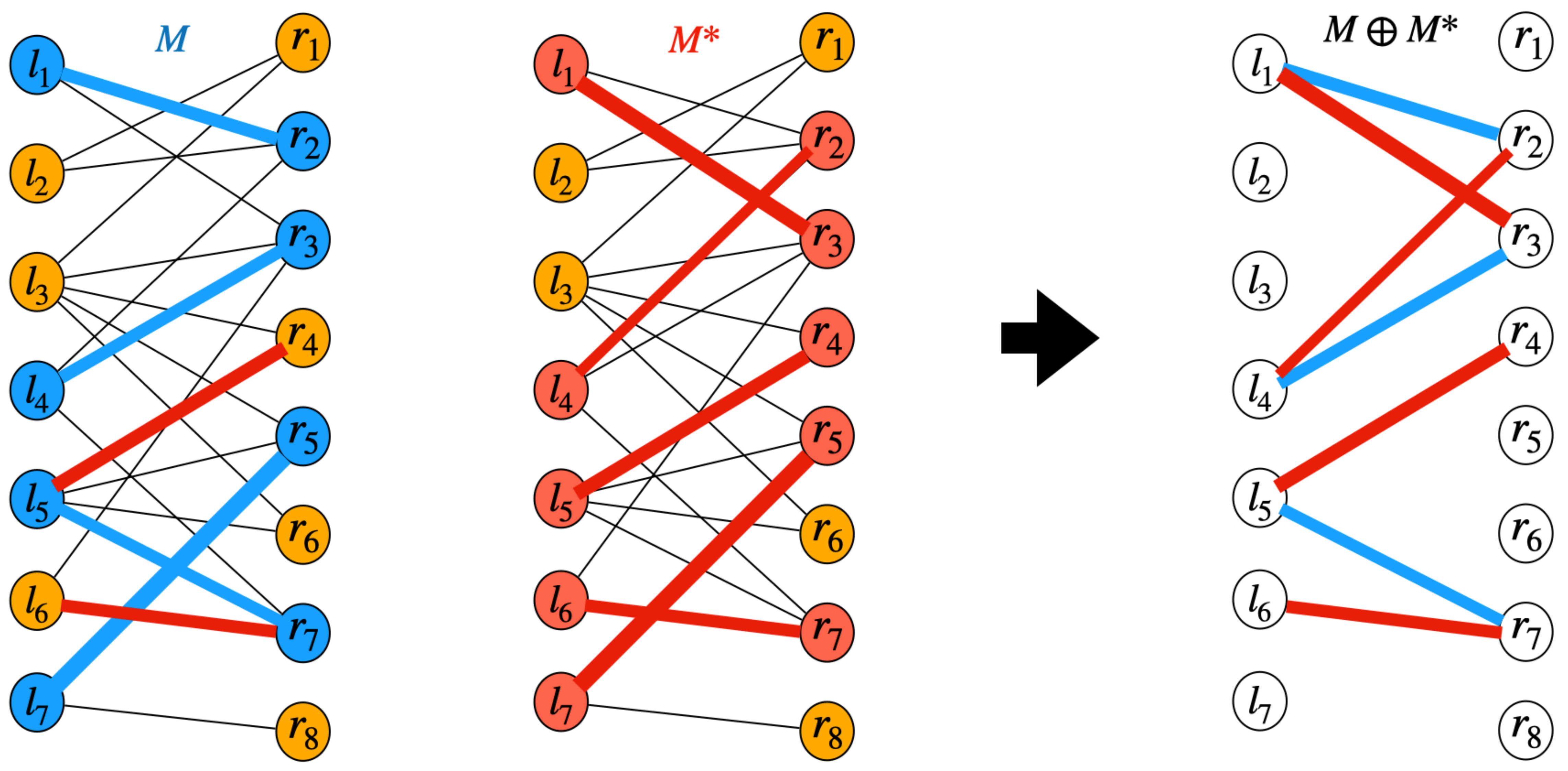

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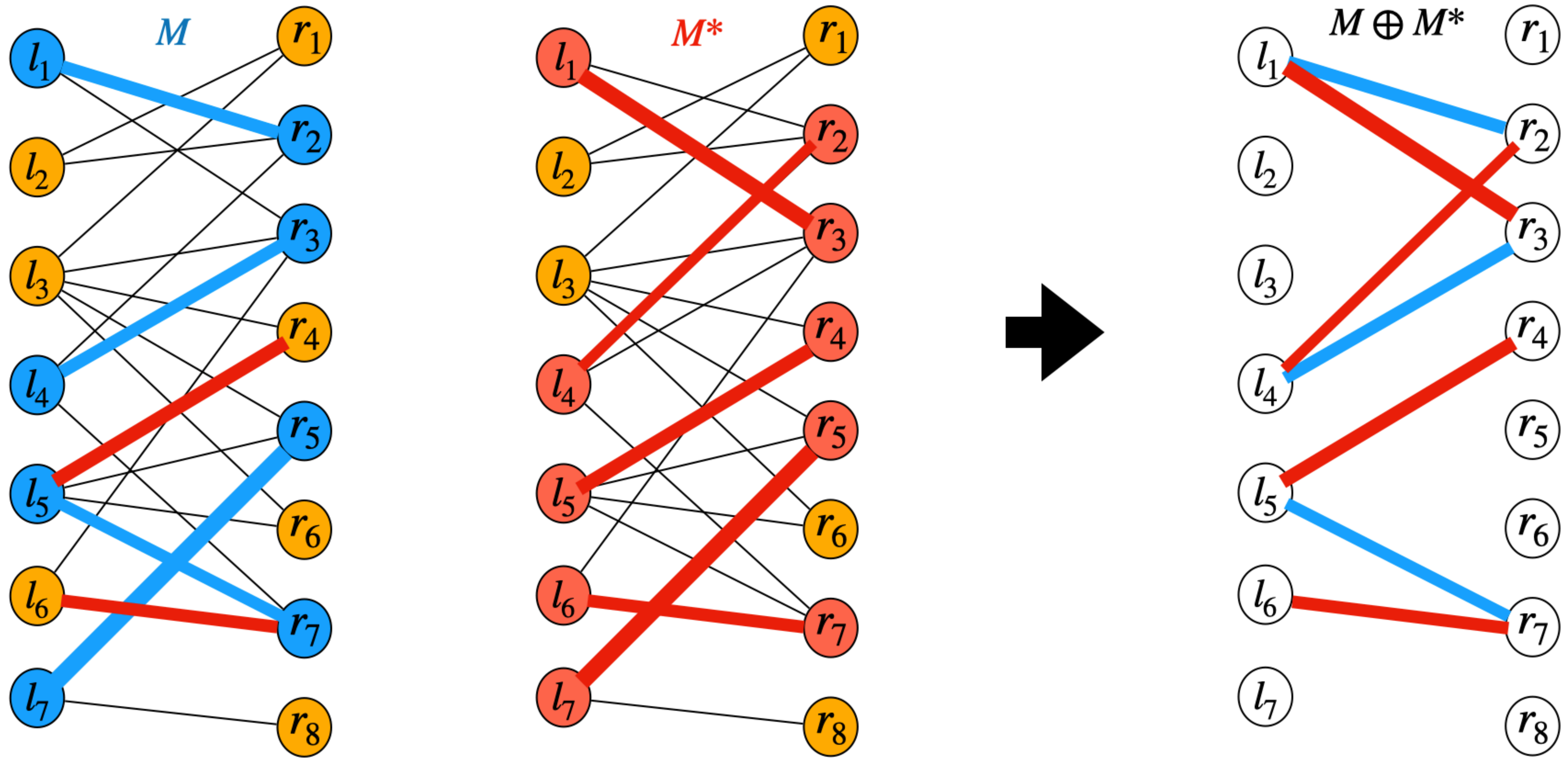

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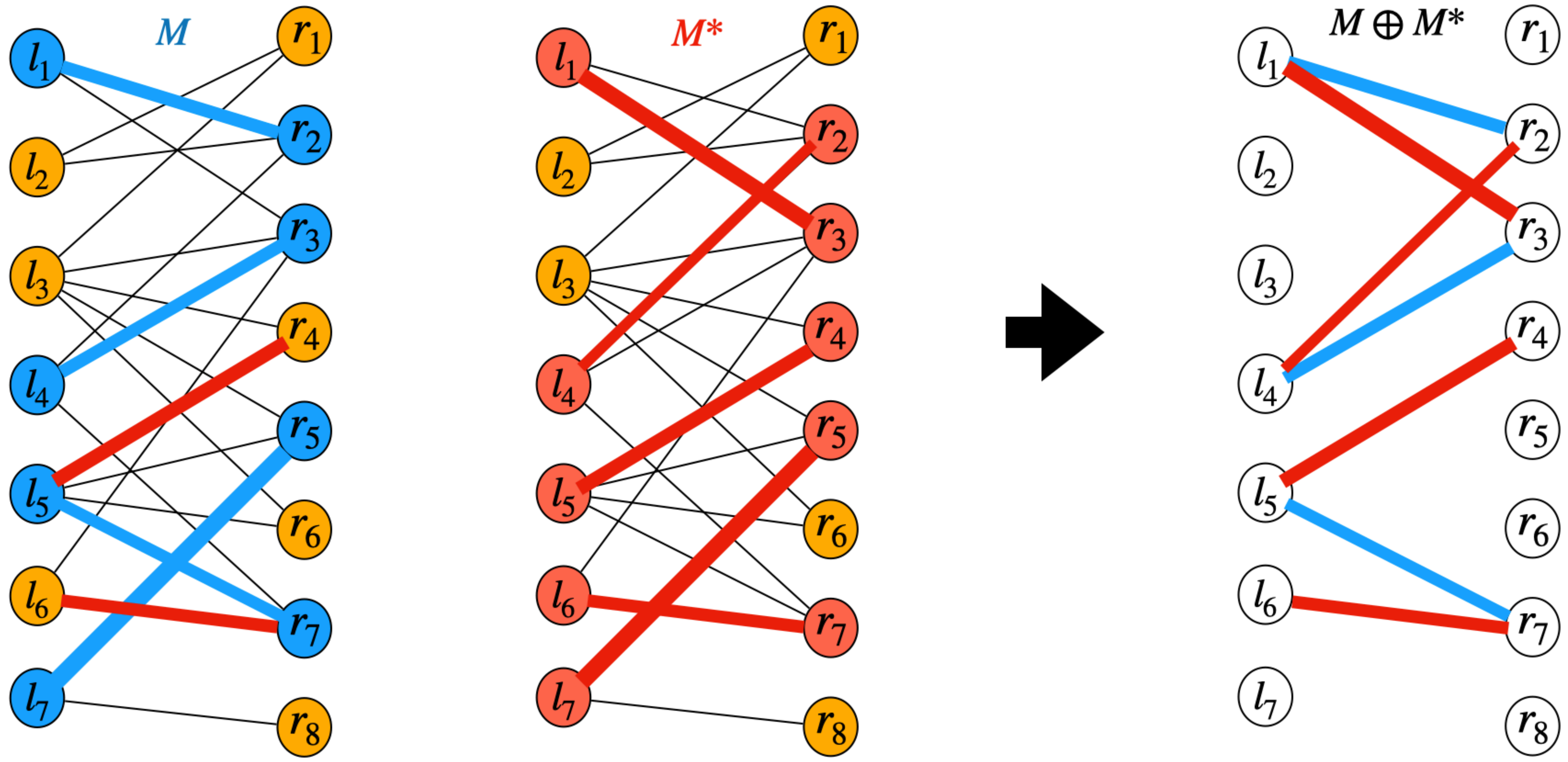

Corollary: Let M be a matching in any undirected graph $G = (V, E)$ and $P_1, P_2, \cdots, P_k$ be vertex-disjoint M-augmenting paths. Then the set of edges $M' = M \oplus (P_1 \cup P_2 \cup \cdots \cup P_k)$ is a matching in G with $|M'| = |M| + k$.

For a non-optimal matching, if there exist augmenting paths, we can use them to augment the matching to a larger size.

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Corollary: Matching M in graph $G = (V, E)$ is a maximum matching if and only if G contains no M-augmenting path.

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Corollary: Matching M in graph $G = (V, E)$ is a maximum matching if and only if G contains no M-augmenting path.

To find an optimal matching, keep looking for augmenting paths, until no such path exists.

Quiz questions:

1. What is an M-augmenting path?

2. How can M-augmenting paths be used to augment a matching?

3. How can we tell if a matching is of maximum size or not?

Roadmap of this lecture:

1. Matching in Bipartite Graphs

   1.1 Define "Maximum Bipartite Matching Problem".

   1.2 Concepts useful for augmenting matching.

   1.3 Hopcroft-Karp Algorithm for "Maximum Bipartite Matching Problem".

   1.4 Time complexity of Hopcroft-Karp Algorithm.

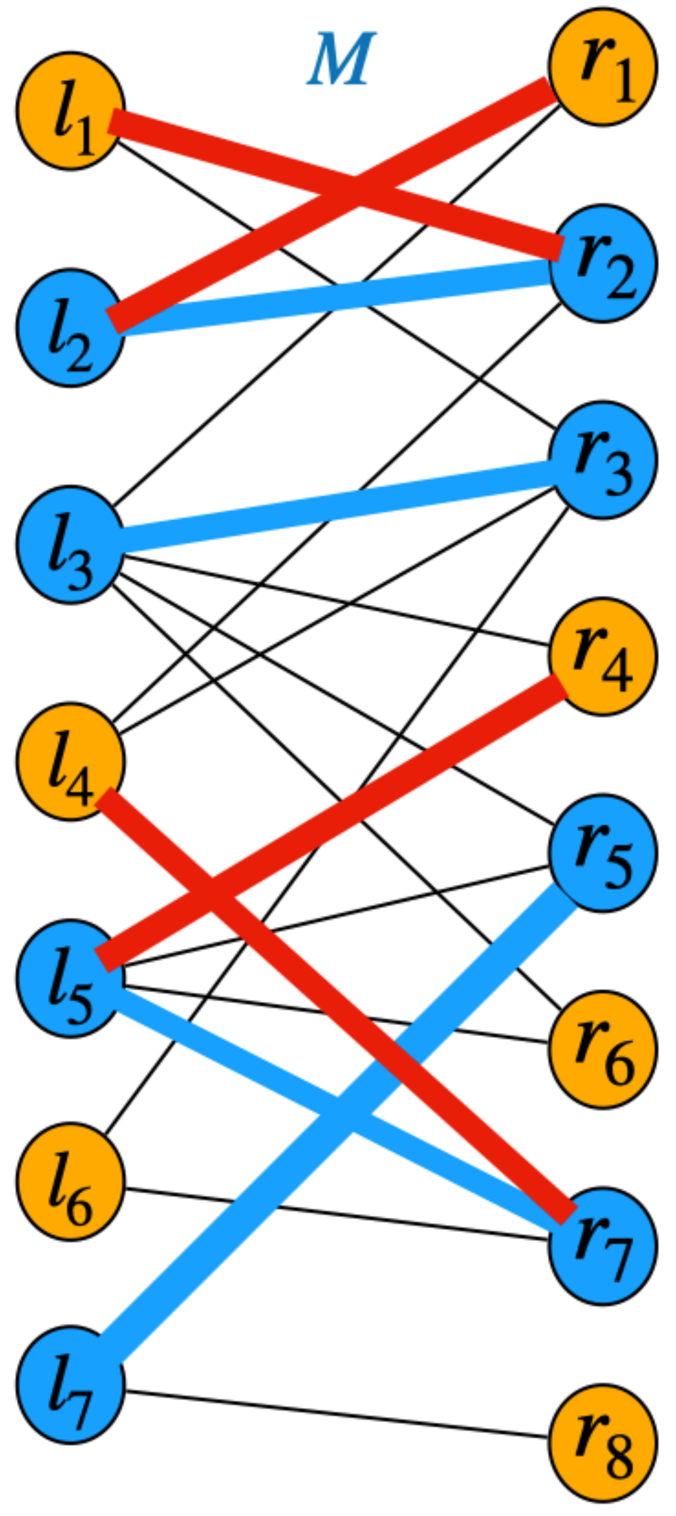
Idea for a Maximum-Matching Algorithm of time complexity $O(VE)$:

1. Start with an empty matching M.

2. Repeatedly run a variant of either BFS or DFS from an unmatched vertex that takes alternating path until we find anotehr unmatched vertex.

   Use the resulting M-augmenting path to increase the size of M by 1.

   (End the algorithm when no more augmenting path exists.)



Hopcroft-Karp Algorithm: Use the above idea and improve the complexity to $O(\sqrt{V}\ E)$.

Hopcroft-Karp Algorithm:

Hopcroft-Karp(G)

1. $M = \varnothing$
2. repeat
3. let $\mathscr{P} = \{P_1, P_2, \cdots, P_k\}$ be a maximal set of vertex-disjoint shortest M-augmenting paths
4. $M = M \oplus (P_1 \cup P_2 \cup \cdots \cup P_k)$
5. until $\mathscr{P}$ is empty
6. return M

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The algorithm finds a maximum matching: based on our previous analysis

Hopcroft-Karp Algorithm:

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To prove that the time complexity is $O(\sqrt{V}\,E)$, we need to show:

1. the repeat loop iterates $O(\sqrt{V})$ times
2. how to implement step 3 of the algorithm to make it run in $O(E)$ time

Let's first see how to find a maximal set of vertex-disjoint shortest M-augmenting paths in $O(E)$ time.

There are 3 phases:

1. Form a directed version G_M of the undirected bipartite graph G.
2. Create a directed acyclic graph H from G_M via a variant of BFS.
3. Find a maximal set of vertex-disjoint shortest M-augmenting paths by running a variant of DFS on the transpose H^T of H. (Recall that the transpose of a directed graph reverses the direction of each edge. Since H is acyclic, so is H^T.)

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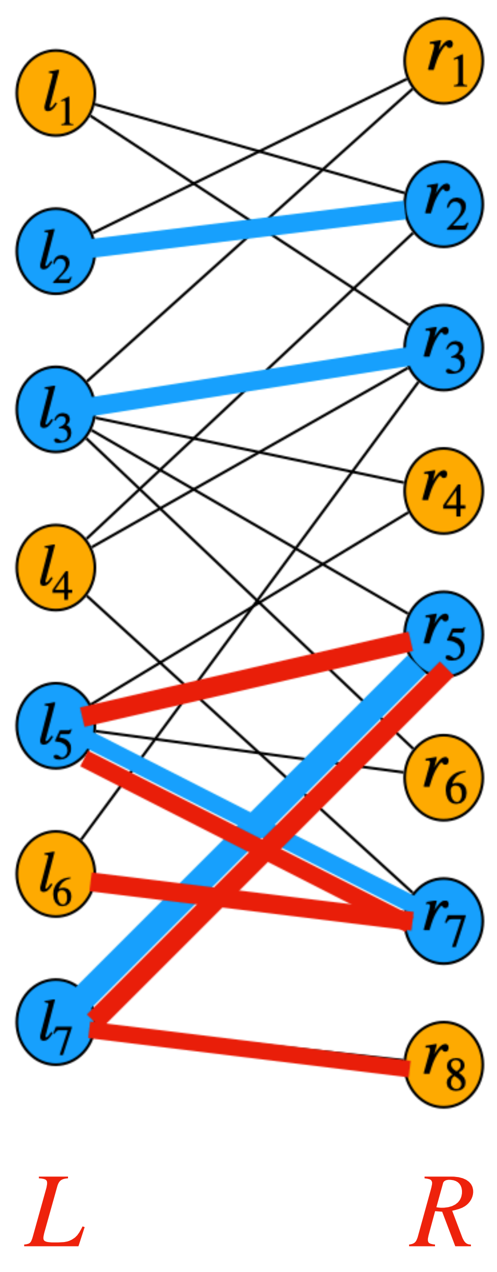
What an M-augmenting path looks like:

1. It starts at an unmatched vertex in L.
2. it traverses an odd number of edges.
3. it ends at an unmatched vertex in R.

4. The edges it traverses from L to R must belong to $E - M$.
5. The edges it traverses from R to L must belong to M.



Let's first see how to find a maximal set of vertex-disjoint shortest M-augmenting paths in $O(E)$ time.

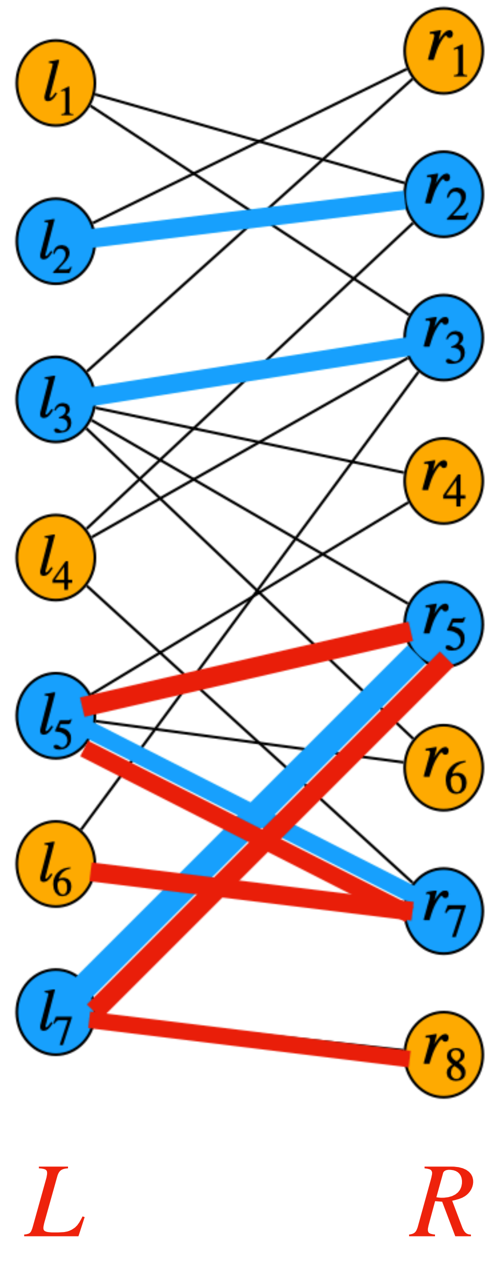
There are 3 phases:

1. Form a directed version G_M of the undirected bipartite graph G.
2. Create a directed acyclic graph H from G_M via a variant of BFS.
3. Find a maximal set of vertex-disjoint shortest M-augmenting paths by running a variant of DFS on the transpose H^T of H. (Recall that the transpose of a directed graph reverses the direction of each edge. Since H is acyclic, so is H^T.)

What an M-augmenting path looks like:

1. It starts at an unmatched vertex in L.
2. it traverses an odd number of edges.
3. it ends at an unmatched vertex in R.

4. The edges it traverses from L to R must belong to $E - M$.
5. The edges it traverses from R to L must belong to M.



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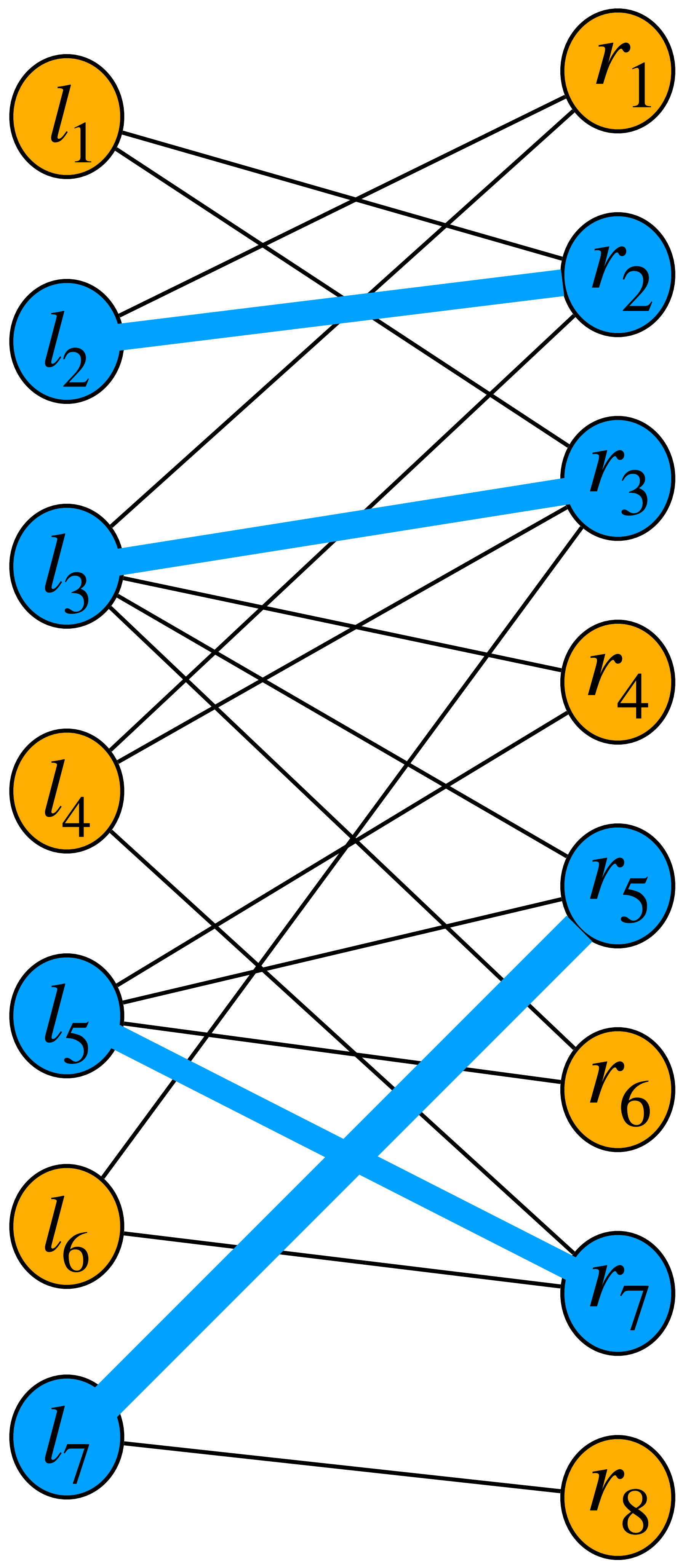

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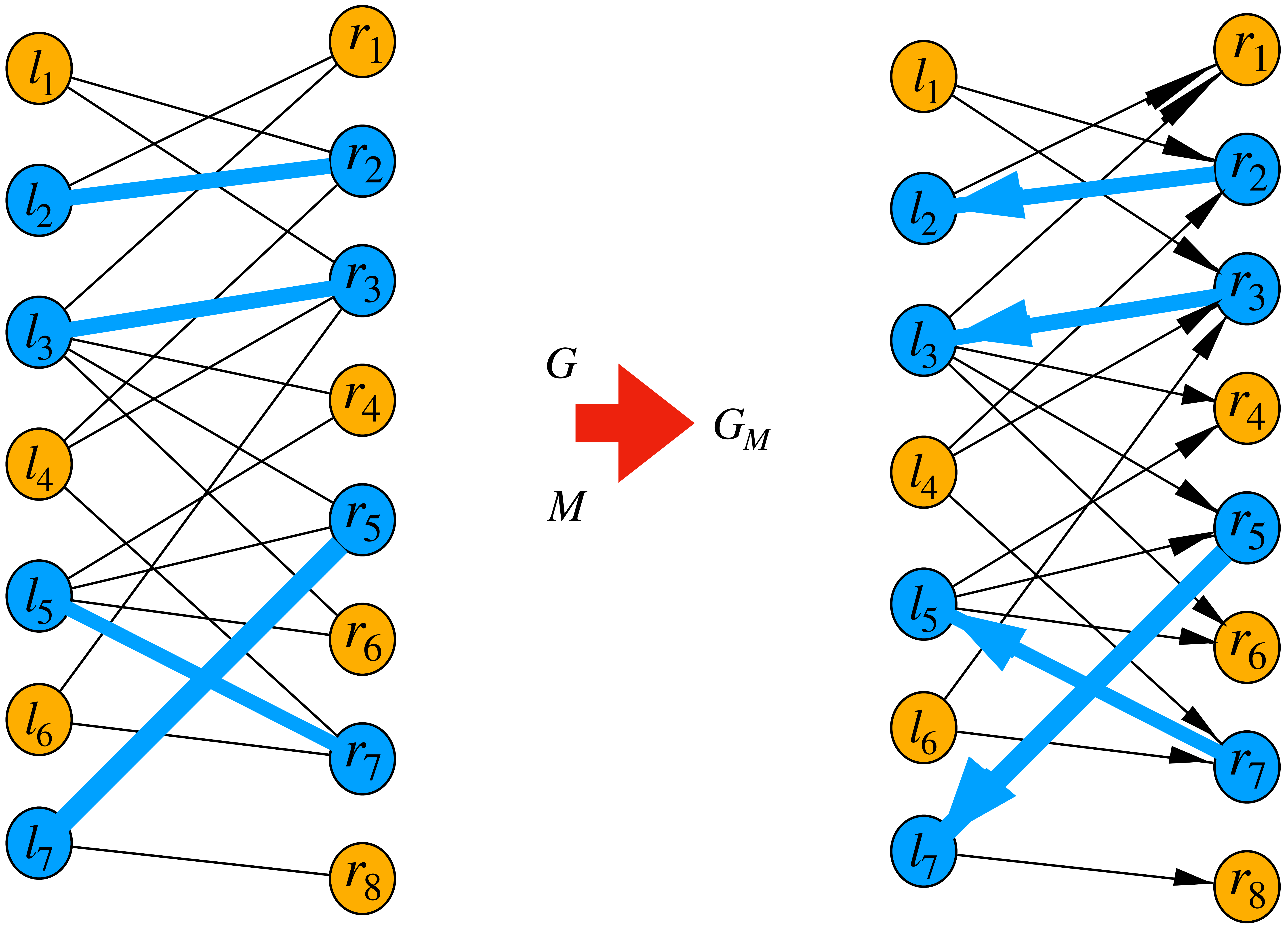

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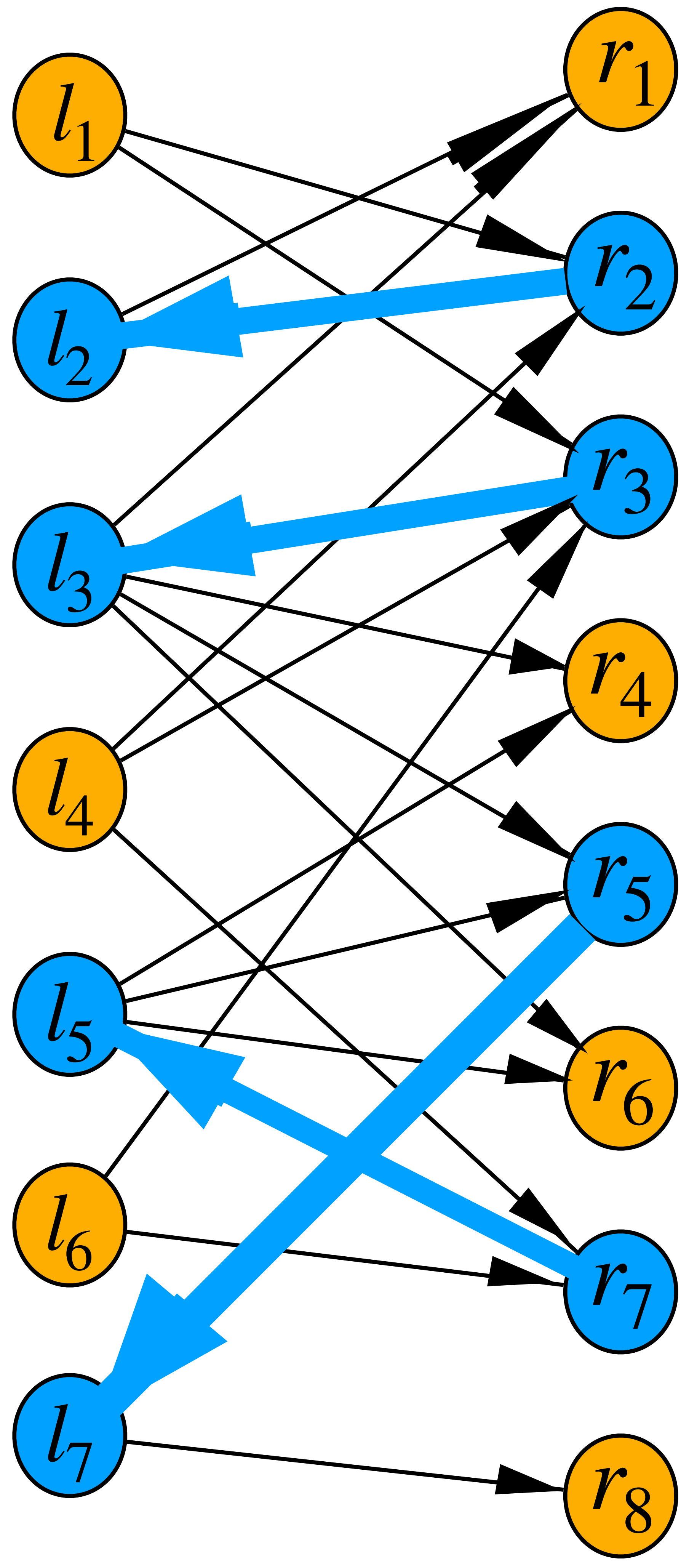

$G_M \Rightarrow H = (V_H, E_H)$

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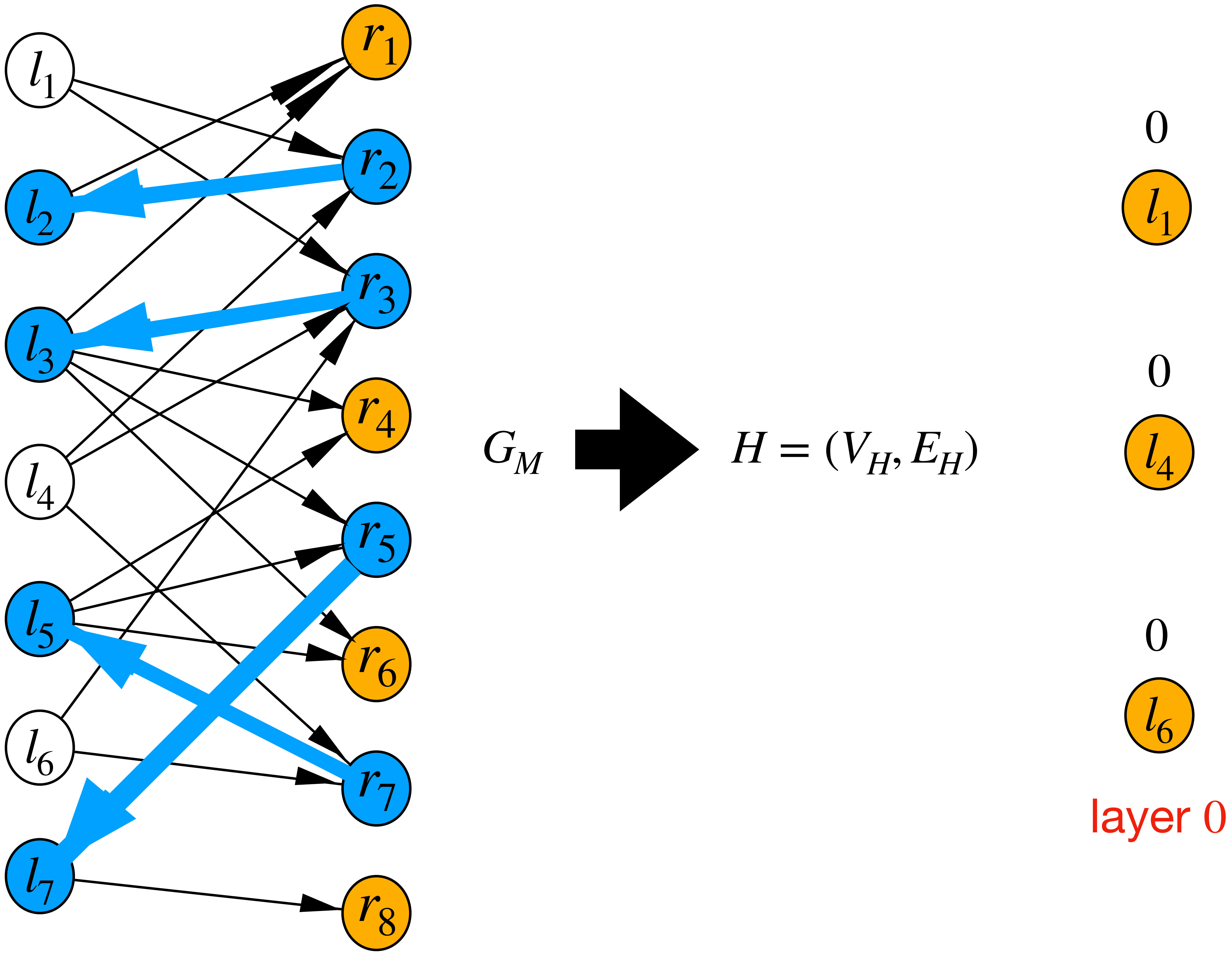

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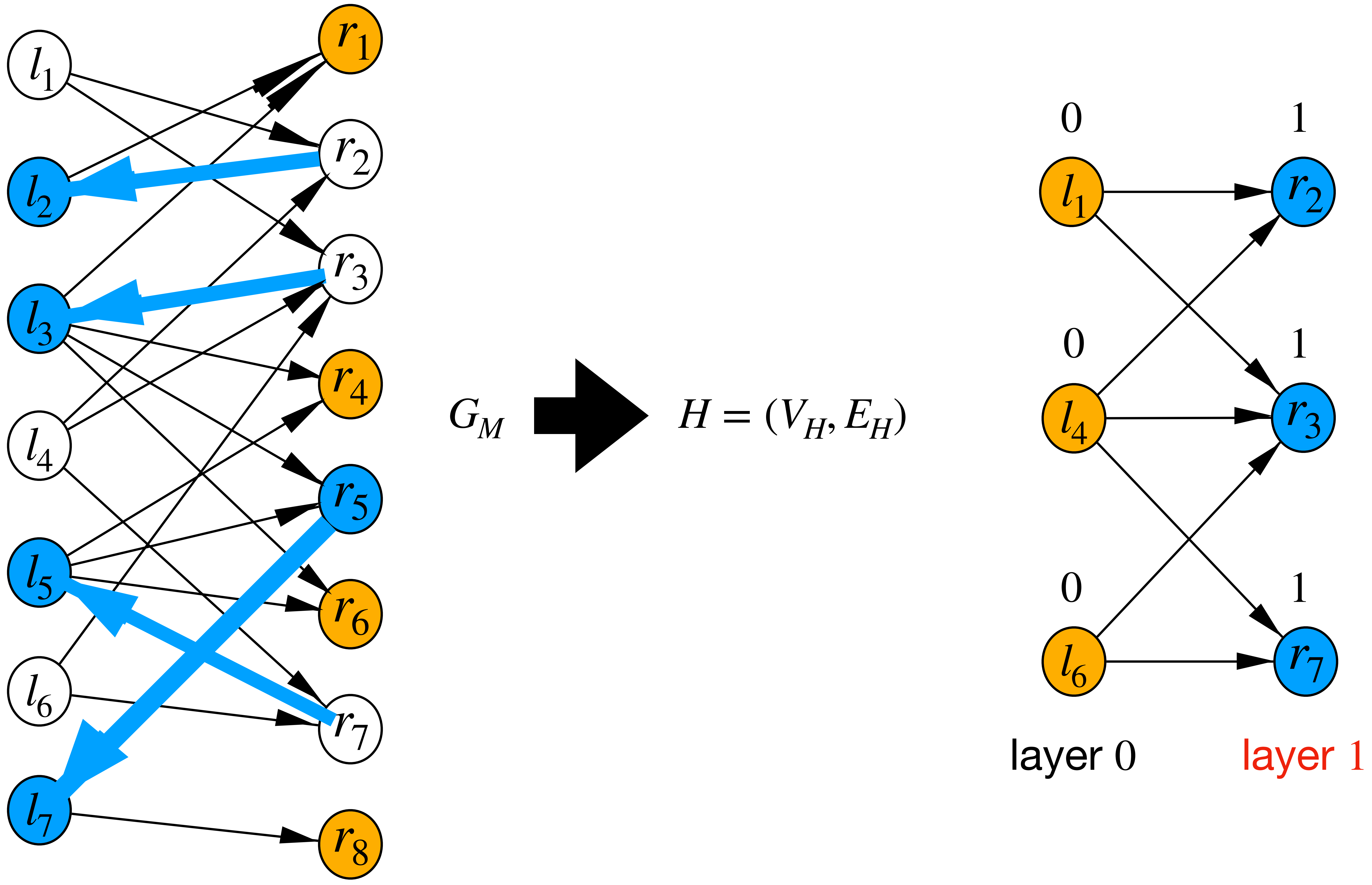

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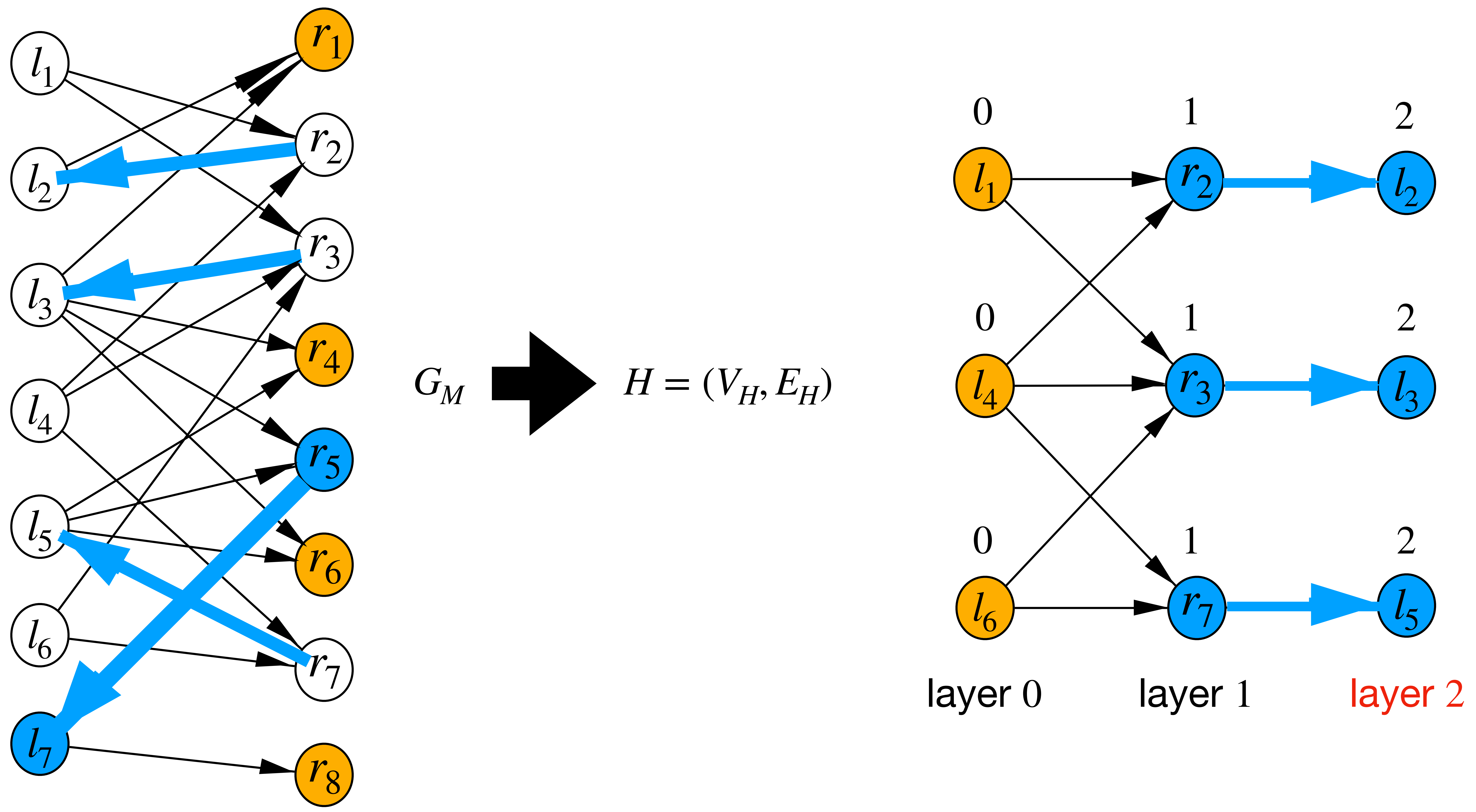

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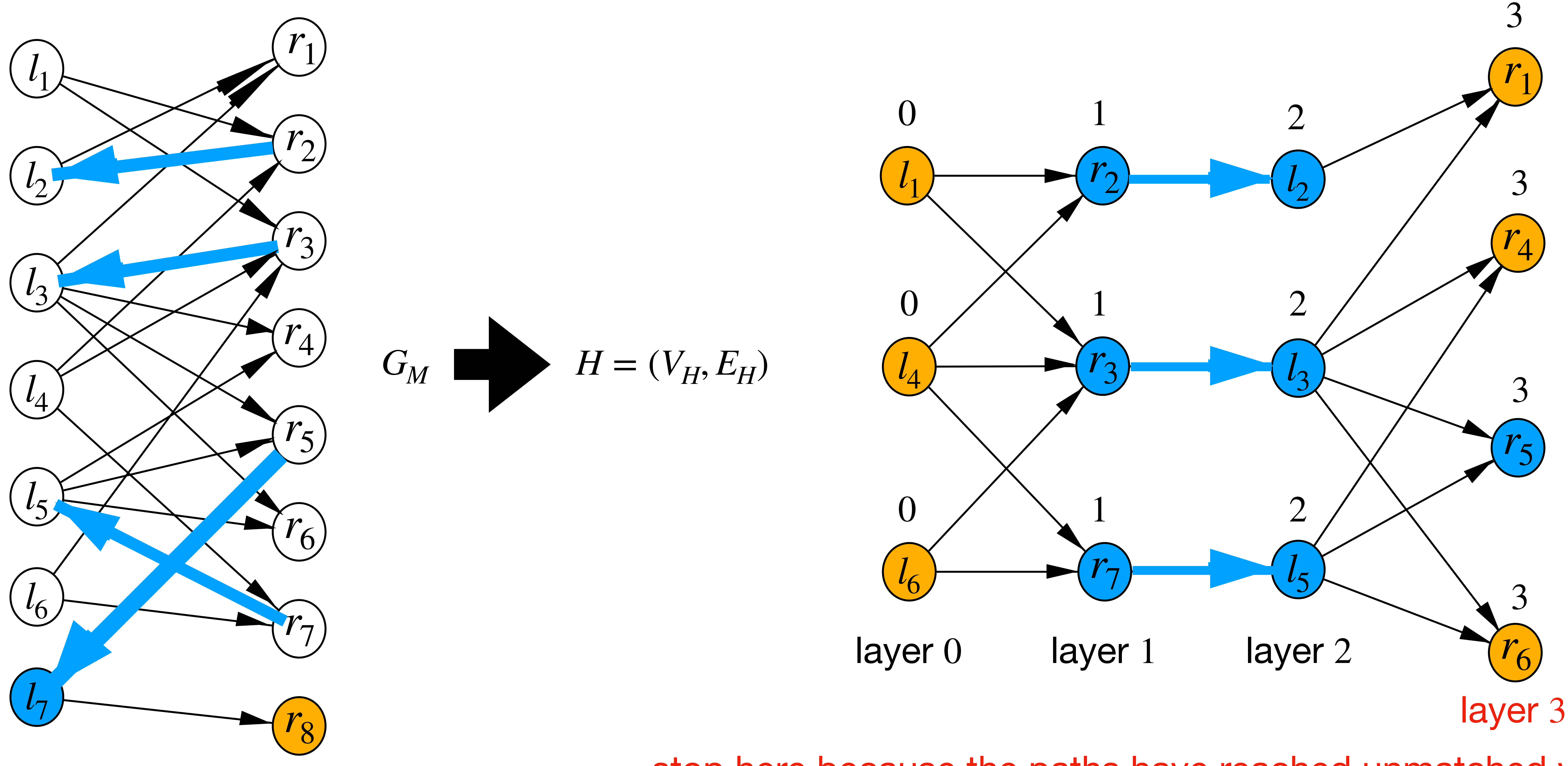
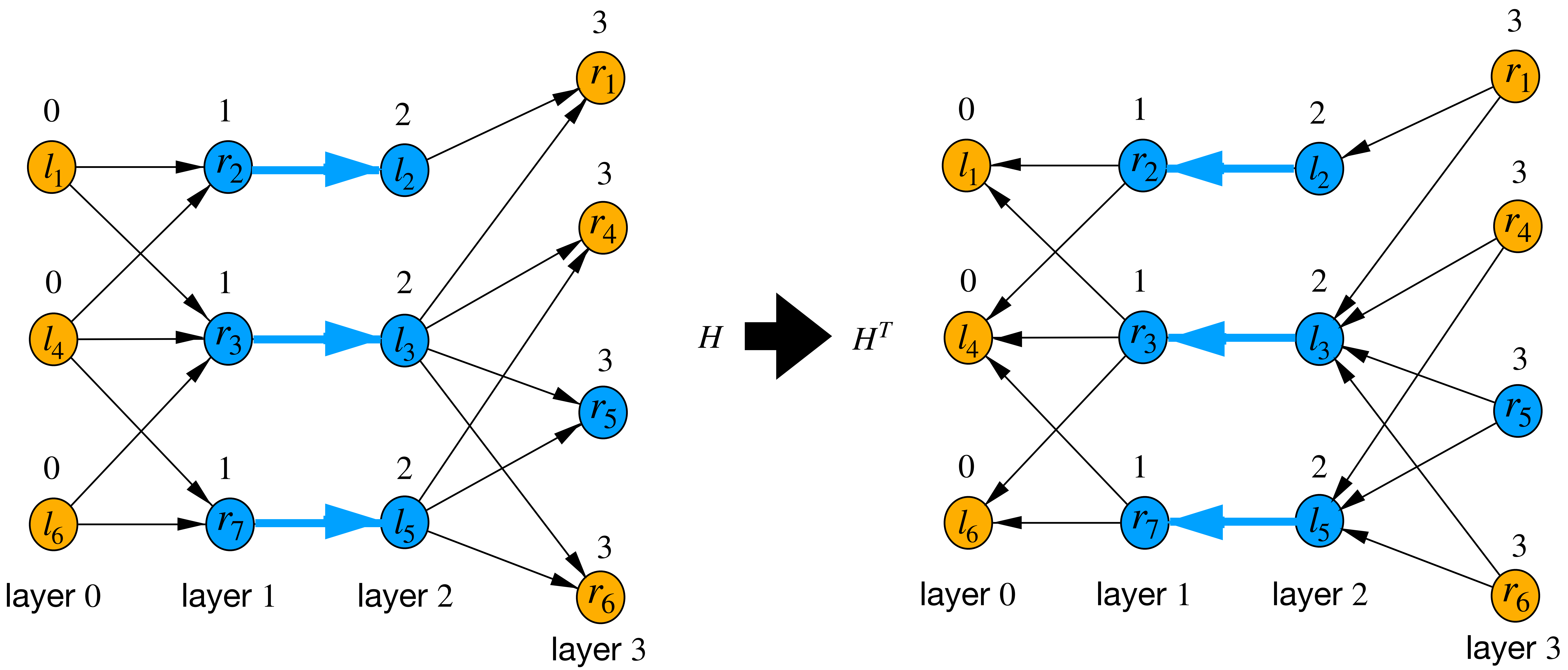


stop here because the paths have reached unmatched vertices
(so we have found shortest M-augmenting paths)

3. Find a maximal set of vertex-disjoint shortest M-augmenting paths by running a variant of DFS on the transpose H^T of H.

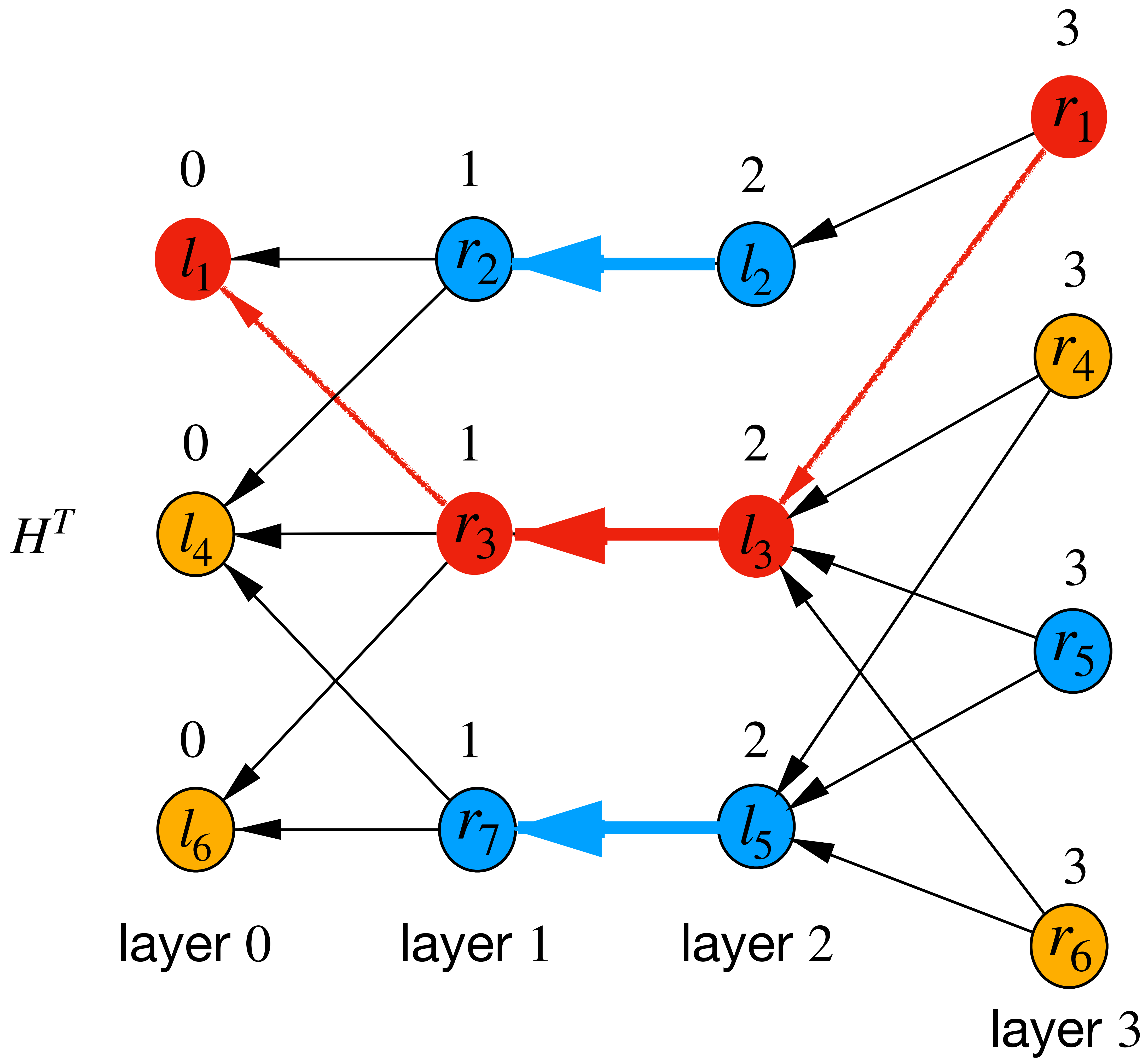


H contains every shortest M-augmenting path in G.

Why transpose H? Make sure each DFS path not only starts with an unmatched vertex, but also ends with an unmatched vertex.

3. Find a maximal set of vertex-disjoint shortest M-augmenting paths by running a variant of DFS on the transpose H^T of H.

Found 1st DFS path: $r_1 \to l_3 \to r_3 \to l_1$



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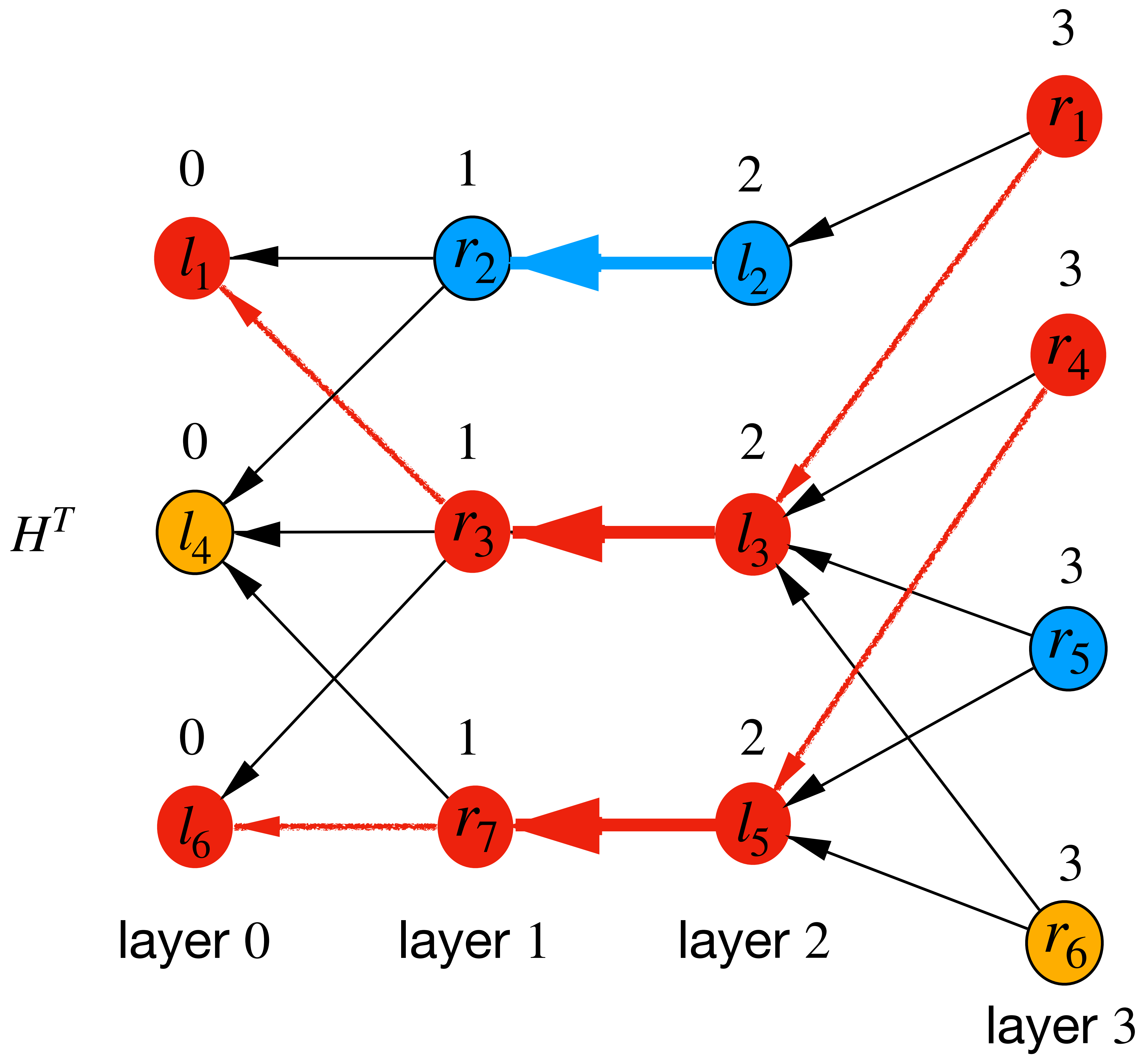

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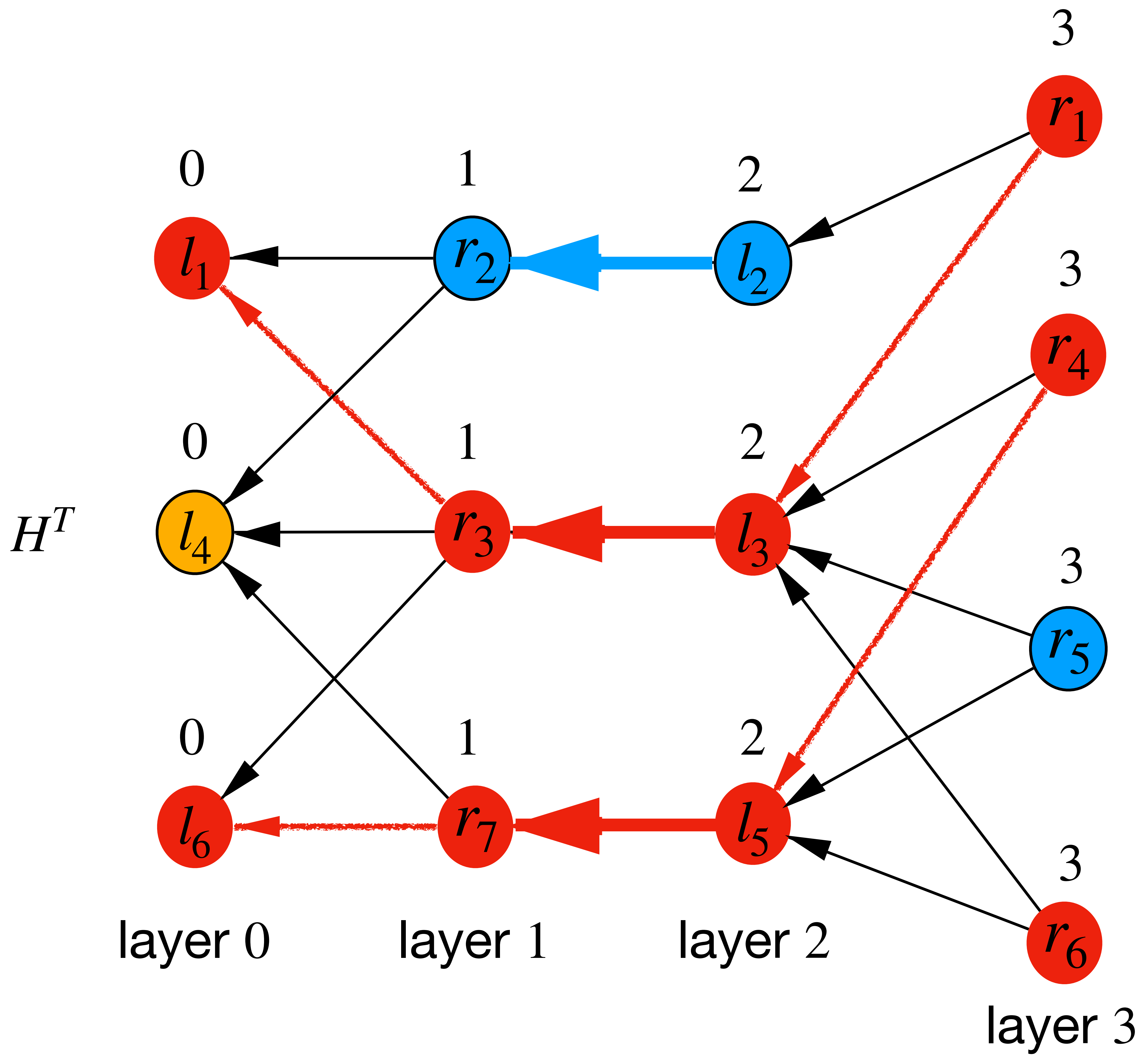

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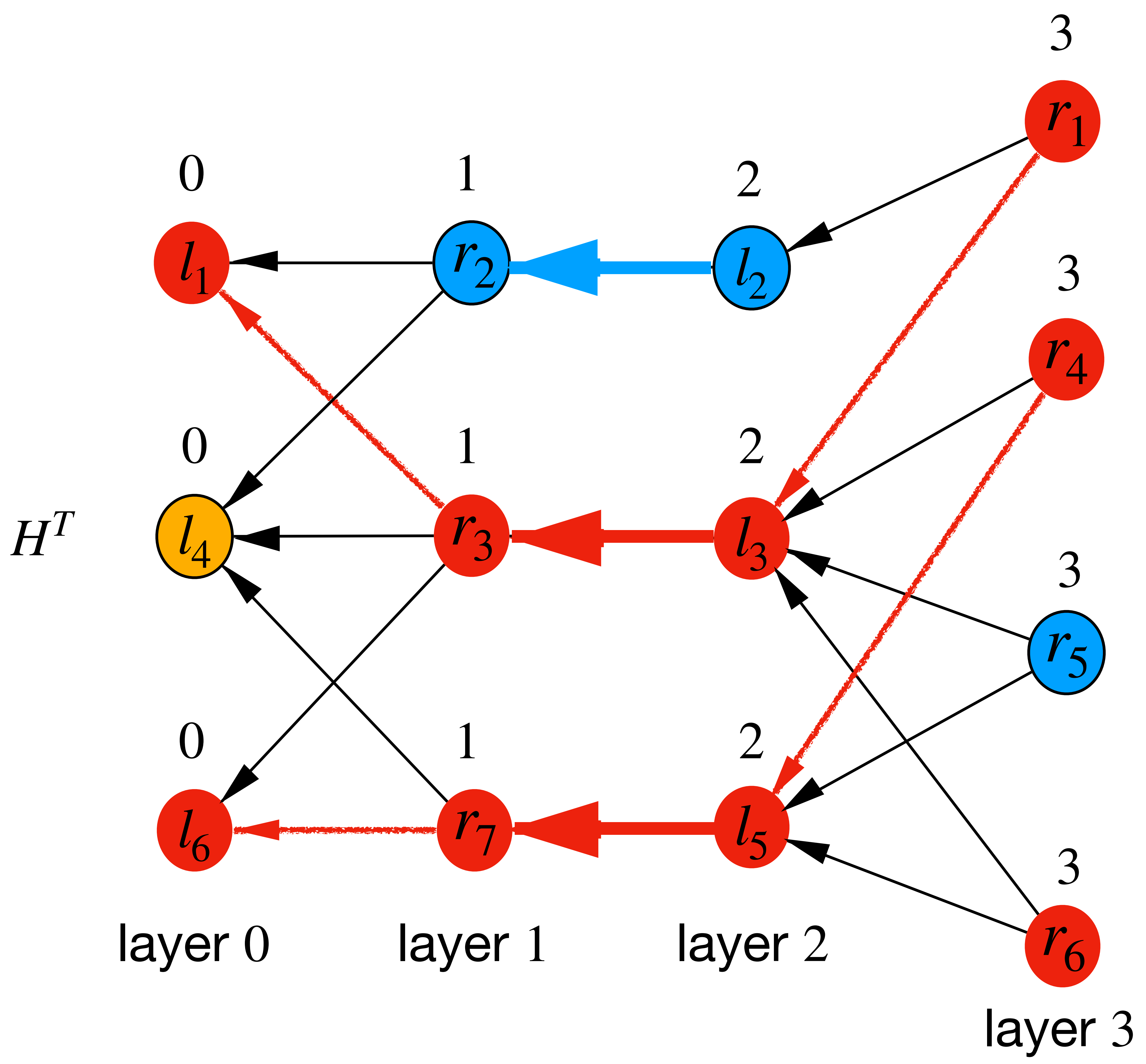
The step ends when we have run DFS starting with every unmatched vertex in the last layer.



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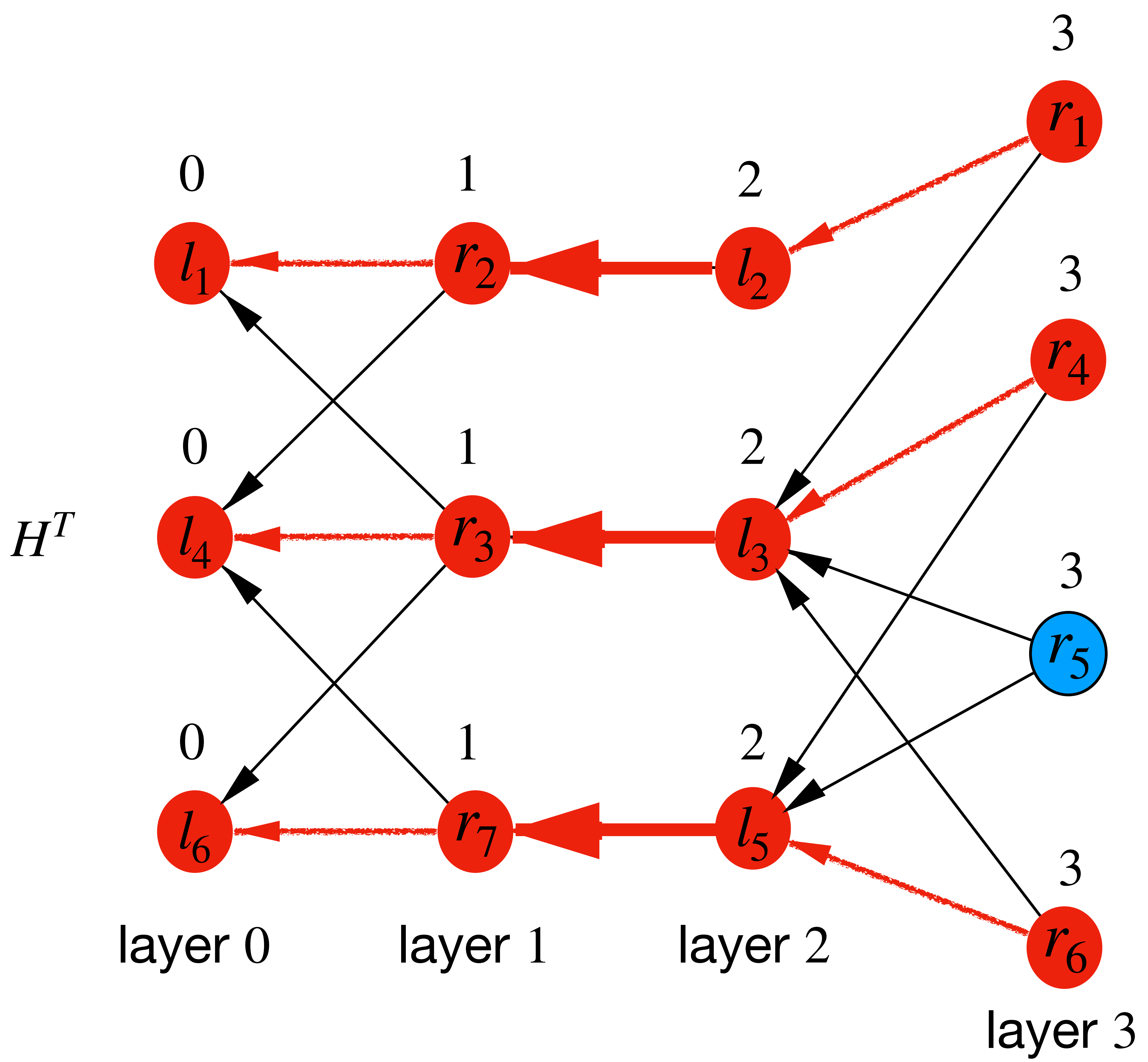
~~(It failed to reach layer 0. Discard it.)~~

The step ends when we have run DFS starting with every unmatched vertex in the last layer.

We have found 2 M-augmenting paths, which is a maximal set, not a maximum set.

But that is all right.



a maximum set of M-augmenting paths

Quiz questions:

1. What is the main idea of the Hopcroft-Karp Algorithm?

2. How does the Hopcroft-Karp Algorithm find augmenting paths?

Roadmap of this lecture:

1. Matching in Bipartite Graphs

   1.1 Define "Maximum Bipartite Matching Problem".

   1.2 Concepts useful for augmenting matching.

   1.3 Hopcroft-Karp Algorithm for "Maximum Bipartite Matching Problem".

   1.4 Time complexity of Hopcroft-Karp Algorithm.

Let's first see how to find a maximal set of vertex-disjoint shortest M-augmenting paths in $O(E)$ time.

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Hopcroft-Karp Algorithm:

Hopcroft-Karp(G)

1. $M = \varnothing$
2. repeat
3. let $\mathscr{P} = \{P_1, P_2, \cdots, P_k\}$ be a maximal set of vertex-disjoint shortest M-augmenting paths
4. $M = M \oplus (P_1 \cup P_2 \cup \cdots \cup P_k)$
5. until $\mathscr{P}$ is empty
6. return M

To prove that the time complexity is $O(\sqrt{V}\, E)$, we need to show:

1. the repeat loop iterates $O(\sqrt{V})$ times
2. how to implement step 3 of the algorithm to make it run in $O(E)$ time

Lemma: Let $G = (V, E)$ be an undirected bipartite graph with matching M, and let q be the length of a shortest M-augmenting path. Let $\mathscr{P} = \{P_1, P_2 \cdots, P_k\}$ be a maximal set of vertex-disjoint M-augmenting paths of length q. Let $M' = M \oplus (P_1 \cup P_2 \cup \cdots \cup P_k)$, and suppose that P is a shortest M'-augmenting path. Then P has more than q edges.

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matching M $\Rightarrow$ matching M'

length of shortest M-augmenting path $= q$ $\Rightarrow$ length of shortest M'-augmenting path $> q$

The lengths of shortest augmenting paths keep increasing from iteration to interation.

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Proof: Case 1: P is vertex-disjoint from the augmenting paths in $\mathscr{P}$.

Case 2: P is not vertex-disjoint from the augmenting paths in $\mathscr{P}$.

Lemma: Let $G = (V, E)$ be an undirected bipartite graph with matching M, and let q be the length of a shortest M-augmenting path. Let $\mathscr{P} = \{P_1, P_2 \cdots, P_k\}$ be a maximal set of vertex-disjoint M-augmenting paths of length q. Let $M' = M \oplus (P_1 \cup P_2 \cup \cdots \cup P_k)$, and suppose that P is a shortest M'-augmenting path. Then P has more than q edges.

Proof: Case 1: P is vertex-disjoint from the augmenting paths in $\mathscr{P}$.

Since P contains edges that are in M but are not in any of $P_1, P_2, \cdots, P_k$, P is also an M-augmenting path.

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Since P is disjoint from $P_1, P_2, \cdots, P_k$ but is also an M-augmenting path, and since $\mathscr{P}$ is a maximal set of shortest M-augmenting paths, P must be longer than any of th augmenting paths in $\mathscr{P}$, each of which has length q. Therefore, P has more than q edges.

Lemma: Let $G = (V, E)$ be an undirected bipartite graph with matching M, and let q be the length of a shortest M-augmenting path. Let $\mathscr{P} = \{P_1, P_2 \cdots, P_k\}$ be a maximal set of vertex-disjoint M-augmenting paths of length q. Let $M' = M \oplus (P_1 \cup P_2 \cup \cdots \cup P_k)$, and suppose that P is a shortest M'-augmenting path. Then P has more than q edges.

Proof: Case 2: P is not vertex-disjoint from the augmenting paths in $\mathscr{P}$.

P visits at least one vertex from the M-augmenting paths in $\mathscr{P}$.

M' is a matching in G with $|M'| = |M| + k$.

Corollary: Let M be a matching in any undirected graph $G = (V, E)$ and $P_1, P_2, \cdots, P_k$ be vertex-disjoint M-augmenting paths. Then the set of edges $M' = M \oplus (P_1 \cup P_2 \cup \cdots \cup P_k)$ is a matching in G with $|M'| = |M| + k$.

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A contains at least $|M' \oplus P| - |M| = k + 1$ vertex-disjoint M-augmenting paths.

Lemma: Let M and M^* be matchings in graph $G = (V, E)$ and consider the graph $G' = (V, E')$, where $E' = M \oplus M^*$. Then, G' is a disjoint union of simple paths, simple cycles, and/or isolated vertices. The edges in each such simple path or simple cycle alternate between M and M^*. If $|M^*| > |M|$, then G' contains at least $|M^*| - |M|$ vertex-disjoint M-augmenting paths.

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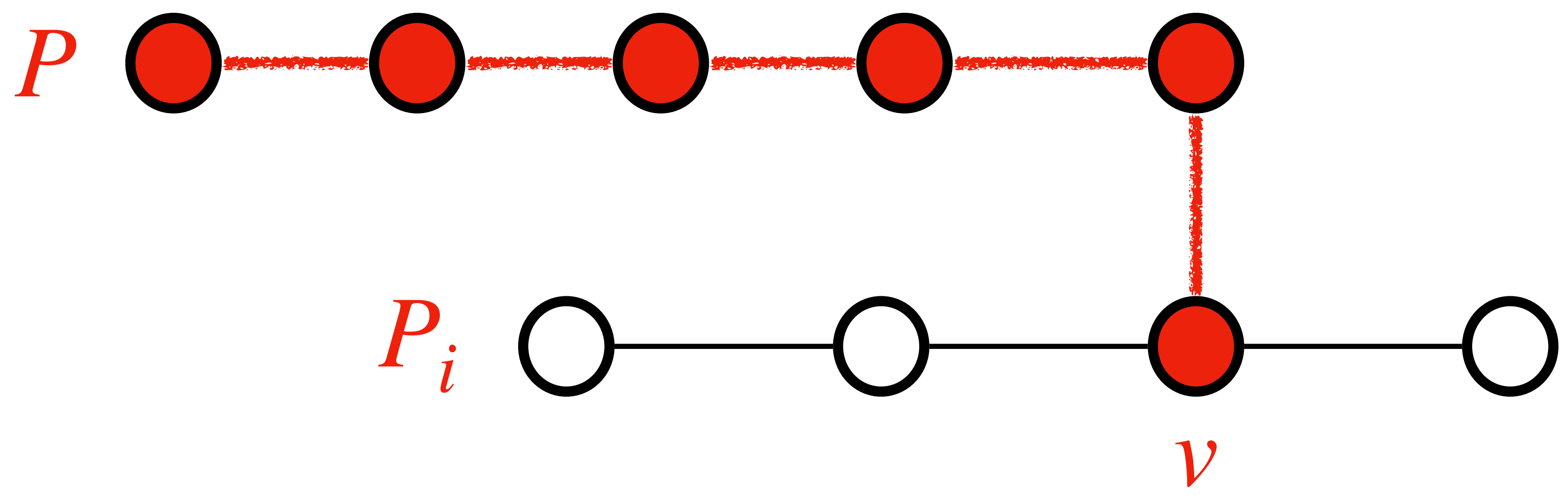

v cannot be an endpoint of P,
because the endpoints of P are unmatched under M'.

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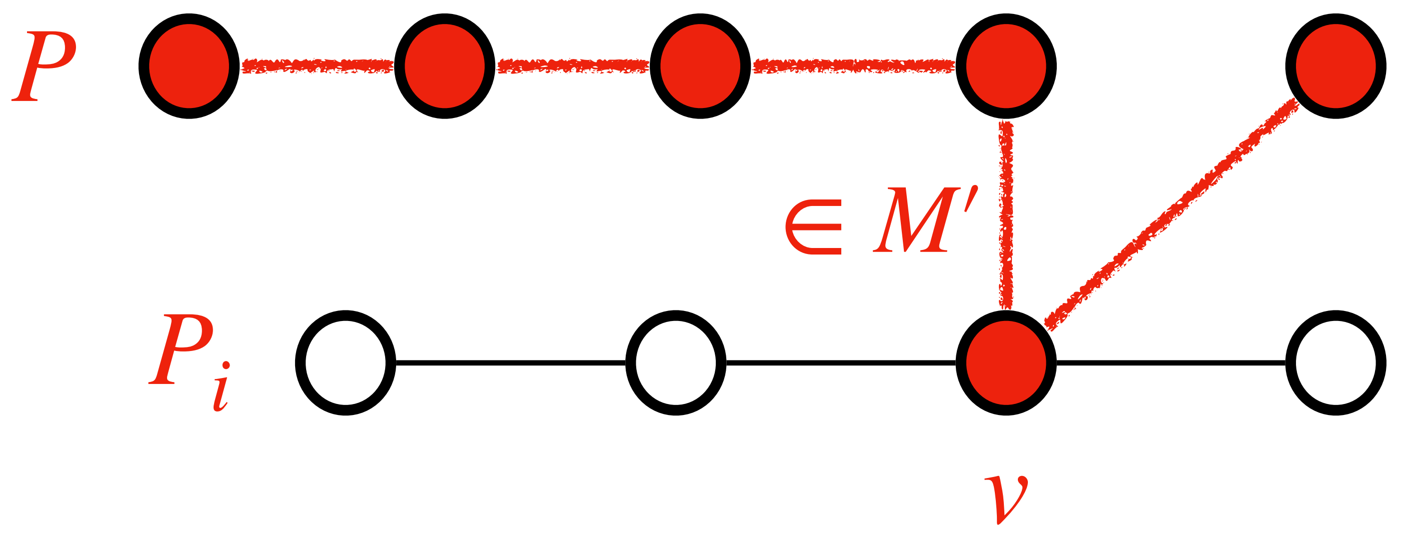

v has an incident edge in P that belongs to M'.

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Proof: Case 2: P is not vertex-disjoint from the augmenting paths in $\mathscr{P}$.

M' is a matching in G with $|M'|=|M|+k$.

$M'\oplus P$ is a matching with $|M'\oplus P|=|M'|+1=|M|+k+1$.

Let $A=M\oplus M'\oplus P=(P_1\cup P_2\cup\cdots\cup P_k)\oplus P$.

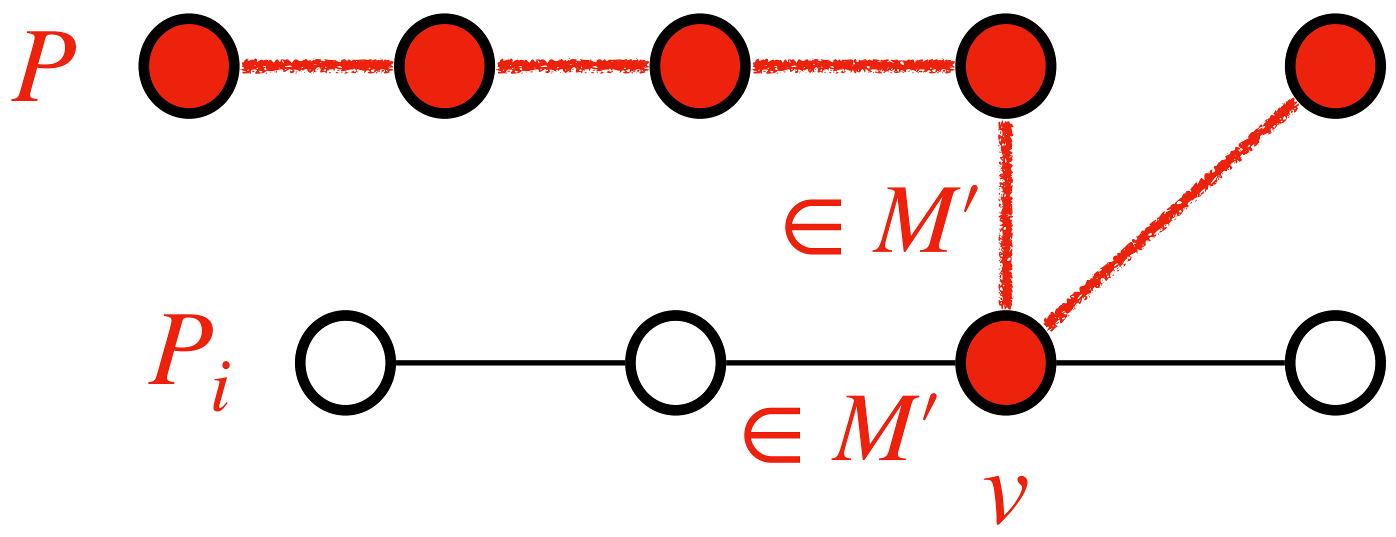
A contains at least $|M'\oplus P|-|M|=k+1$ vertex-disjoint M-augmenting paths.

Since each such M-augmenting path has at least q edges, we have $|A|\geq(k+1)q=kq+q$.

Claim: P shares at least one edge with some M-augmenting path in $\mathscr{P}$.

Under the matching M', every vertex in each M-augmenting path in $\mathscr{P}$ is matched.

Suppose that P shares a vertex v with some path $P_i\in\mathscr{P}$.



v has an incident edge in P that belongs to M'.

Since any vertex has at most one incident edge in a matching, this edge must also belong to P_i.

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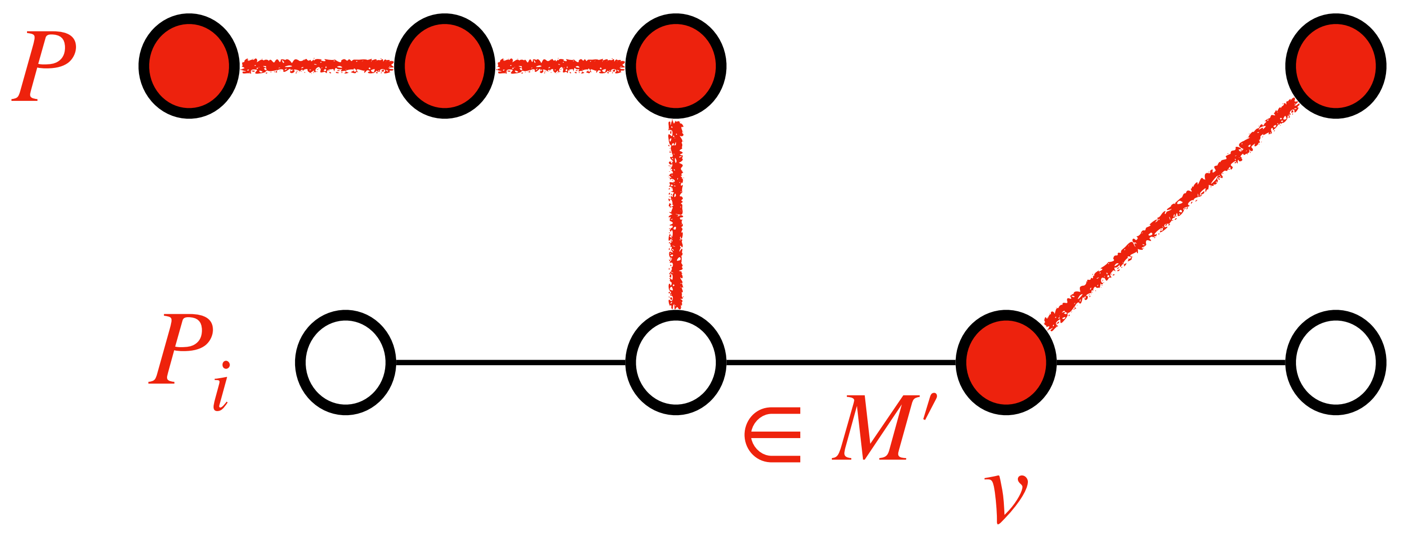

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The claim is true.

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$|P| \;>\; q$

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The above lemma bounds the size of a maximum matching, based on the length of a shortest augmenting path.

Lemma: Let M be a matching in graph $G = (V, E)$, and let a shortest M-augmenting path in G contain q edges. Then the size of a maximum matching in G is at most $|M| + |V|/(q+1)$.

Proof: Let M^* be a maximum matching in G.

G contains at least $|M^*| - |M|$ vertex-disjoint M-augmenting paths.

Lemma: Let M and M^* be matchings in graph $G = (V, E)$ and consider the graph $G' = (V, E')$, where $E' = M \oplus M^*$. Then, G' is a disjoint union of simple paths, simple cycles, and/or isolated vertices. The edges in each such simple path or simple cycle alternate between M and M^*. If $|M^*| > |M|$, then G' contains at least $|M^*| - |M|$ vertex-disjoint M-augmenting paths.

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Each of these paths contains at least q edges, and hence at least $q+1$ vertices.

Since these paths are vertex-disjoint, we have $(|M^*| - |M|)(q+1) \leq |V|$, so that

$|M^*| \leq |M| + |V|/(q+1)$.

Lemma: When the Hopcroft-Karp algorithm runs on an undirected bipartite graph $G = (V, E)$, the repeat loop of lines 2-5 iterates $O(\sqrt{V})$ times.

Hopcroft-Karp Algorithm:

Hopcroft-Karp(G)

1. $M = \varnothing$
2. repeat
3. let $\mathscr{P} = \{P_1, P_2, \cdots, P_k\}$ be a maximal set of vertex-disjoint
4. $M = M \oplus (P_1 \cup P_2 \cup \cdots \cup P_k)$
5. until $\mathscr{P}$ is empty
6. return M

Lemma: When the Hopcroft-Karp algorithm runs on an undirected bipartite graph $G = (V, E)$, the repeat loop of lines 2-5 iterates $O(\sqrt{V})$ times.

Proof: The length q of the shortest M-augmenting paths found in line 3 increases from iteration to iteration.

Lemma: Let $G = (V, E)$ be an undirected bipartite graph with matching M, and let q be the length of a shortest M-augmenting path. Let $\mathscr{P} = \{P_1, P_2 \cdots, P_k\}$ be a maximal set of vertex-disjoint M-augmenting paths of length q. Let $M' = M \oplus (P_1 \cup P_2 \cup \cdots \cup P_k)$, and suppose that P is a shortest M'-augmenting path. Then P has more than q edges.

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Consider the situation after the first time line 4 executes with M-augmenting paths whose length is at least $\lceil \sqrt{|V|} \rceil$.

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Since the size of a matching increases by at least one edge per iteration, the number of additional iterations before achieving a maximum matching is at most

$$\frac{|V|}{\lceil \sqrt{|V|} \rceil + 1} < \frac{|V|}{\sqrt{|V|}} = \sqrt{|V|}.$$

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Hence, the total number of loop iterations is less than $2\sqrt{|V|}$.

Time complexity of the Hopcroft-Karp algorithm: $O(\sqrt{V}\ E)$.

Quiz question:

1. What is the main idea used to prove the time complexity of the Hopcroft-Karp Algorithm?