Homework 4 : Analysis of Algorithm

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Q1.> In "concentration solitaire", a game for one person, you have n pairs of matching cards. Give an algorithm to play concentration solitaire that has a competitive ratio of 2.

Main Idea:

The goal is to find matching pairs of cards from a set of cards that are initially face-down. This is a online algorithm problem as we do not know future data.
~~the~~ Lets divide the cards into two sets of equal size and randomly permute the cards in each set. We will then play the game as per the rules. If the cards match remove them from the board and if they do not match we will flip them back and remember their location.
The best case for this will be when for every cards picked from first set matches the card picked from second set. This will complete the game with cost of ~~2n~~ n.

The worst case for the above game can be achieved if ~~f~~ no card picked from first set matches the card picked from second ~~card~~ set untill every pair of card is flipped once. After this we will know the exact location of all cards and will take another n flip to complete game. The total cost in this worst case will therefore be 2n.
From the best and worst case we will be able to conclude that our proposed algorithm will have competitive ratio of 2.

Cards : $\underbrace{C_1, C_2 \cdots\cdots C_n}_{\text{Sequence in which cards are drawn in Set 1}} \quad \underbrace{C_{n+1}\ C_{n+2} \cdots\cdots C_{2n}}_{\text{Sequence in which cards are drawn in Set 2}}$

- Best (Optimal) Case: pick one card from each set such that.

$$\left.\begin{aligned} C_1 &= C_{n+1} \\ C_2 &= C_{n+2} \\ &\vdots \\ C_i &= C_{n+i} \end{aligned}\right\} \text{It will take n trials to complete game.}$$

$$\text{Optimal cost} = n \qquad -(i)$$

- Worst Case : cards are picked such that.

$$\left.\begin{aligned} C_1 &\neq C_{n+1} \\ C_2 &\neq C_{n+2} \\ &\vdots \\ C_i &\neq C_{n+i} \end{aligned}\right\} \text{It will take n trials to know all positions and another n to complete game.}$$

$$\text{Worst case cost} = n + n = 2n. \qquad -(ii)$$

from eqn (i) and (ii)

Competitive ratio $= \dfrac{\text{Worst case cost}}{\text{Best case cost}}$

$= \dfrac{2n}{n} = 2$

Hence Proved.

Q.2.) If for moving x anywhere earlier in the list has cost 0. Show that MOVE-TO-FRONT is 2-competitive in this cost model.
Use potential function: $\phi_i = I(L_i^M, L_i^F)$

Ans. Let L_i^M be the list maintained by MOVE-TO-FRONT immediately after the i^{th} search.
Let L_i^F be FORESEE's list immediately after the i^{th} search.

Let c_i^M be the cost incurred by MOVE-TO-FRONT on i^{th} calls
Let c_i^F be the cost incurred by FORESEE in the i^{th} calls.

Let no. of swaps FORESEE performs in its i^{th} call be t_i.

Therefore if i^{th} operation is a search for element x then,

$$c_i^M = r_{L_{i-1}^M}(x) \quad \text{—(i)}$$

$$c_i^F = r_{L_{i-1}^F}(x). \quad \text{—(ii)}$$

, where $r_L(x)$ is the cost to search element x in list L.

In order to compare these costs more carefully, let's break down the elements into subsets depending on their positions in the two lists before the i^{th} search, relative to the element x being searched for in the i^{th} search.

Let $BB = \{$elements before x in both L^M_{i-1} & $L^F_{i-1}\}$
$BA = \{$elements before x in L^M_{i-1} but after x in $L^F_{i-1}\}$
$AB = \{$elements after x in L^M_{i-1} but before x in $L^F_{i-1}\}$

Equating the above sets to the position of element x in both L^F_{i-1} and L^M_{i-1}, we get,

$$r_{L^M_{i-1}}(x) = |BB| + |BA| + 1 \qquad \text{-(iii)}$$

$$r_{L^F_{i-1}}(x) = |BB| + |AB| + 1 \qquad \text{-(iv)}$$

Consider a swap with an element $y \in BB$. Before the swap, y precedes x in both L^M_{i-1} and L^F_{i-1}. After the swap, x precedes y in L^M_i and L^F_{i-1} does not change. Therefore, the inversion count increases by 1 for each element in BB. Now consider a swap with an element $z \in BA$. Before the swap, z precedes x in L^M_{i-1} but x precedes z in L^F_{i-1}. After the swap, x precedes z in both lists. Therefore, the inversion count decreases by 1 for each element in BA.

$\therefore$ altogether, the inversion count increases by.

$$I(L^M_i, L^F_{i-1}) - I(L^M_{i-1}, L^F_{i-1}) = |BB| - |BA| \qquad \text{-(v)}$$

The value Φ_i of potential fun after the i^{th} calls of MOVE-TO-FRONT and FORESEE depends on the inversion count. Since cost of swapping is considered as 0, therefore Φ_i can be written as $\Phi_i = I(L_i^M, L_i^F)$.

$$-(vi)$$

Using eqn^x (v) and (vi), we can ~~say~~ that after ~~Φ_i~~ i^{th} call of MOVE-to-front but before the i^{th} call of FORESEE, the potential increases by $|BB| - |BA|$.

Assuming that MOVE-to front and foresee start with the same list, the initial potential ϕ_0 is 0 so $\Phi_i \geq \Phi_0$ for all i.

Amortized cost $\hat{c}_i^M$ of i^{th} MOVE-TO-FRONT can be written as,

$$\hat{c}_i^M = \underbrace{c_i^M}_{\text{actual cost of } i^{th} \text{ MOVE-TO-FRONT.}} + \phi_i - \phi_{i-1} \qquad \text{— (vii)}$$

$$c_i^M = r_{L_{i-1}^M}(x) \qquad \text{— (viii)}$$

Using eqn (vii) (viii) and (v), we get.

$$\begin{aligned}
\hat{c}_i^M &= c_i^M + \phi_i - \phi_{i-1} \\
&\leq r_{L_{i-1}^M}(x) + |BB| - |BA| \\
&= \cancel{r_{L_{i-1}^M}(x)} + |BB| + |BB| - \cancel{r_{L_{i-1}^M}(x)} + 1 \\
&= 2|BB| + 1 \\
&\leq 2|BB| + 2|AB| + 2. \\
&= 2(|BB| + |AB| + 1) \qquad \text{(using iv)} \\
&= 2 r_{L_{i-1}^F} \\
&< 2 c_i^F. \qquad \text{--- (ix)}
\end{aligned}$$

$$\sum_{i=1}^{m} \hat{c}_i^M = \sum_{i=1}^{m} c_i^M + \Phi_m - \Phi_0$$

$$\geq \sum_{i=1}^{m} c_i^M \qquad (\because \Phi_m \geq \Phi_0)$$

$\therefore$ Therefore, $\sum_{i=1}^{m} c_i^M \leq \sum_{i=1}^{m} \hat{c}_i^M$

$$\leq \sum_{i=1}^{m} 2 c_i^F \qquad \text{--using (ix)}$$

$$= 2 \sum_{i=1}^{m} c_i^F .$$

Thus the total cost of the m MOVE-TO-FRONT operations is at most 2 times the total cost of the m FORESEE operations, ~~so~~ MOVE-TO-FRONT is 2-competitive.

Hence Proved.