Algorithms

Lecture Topic: NP Completeness (Part 2)

Anxiao (Andrew) Jiang

Roadmap of this lecture:

1. NP Completeness

    1.1 Define P and NP.

    1.2 Define "NP complete (NPC)".

    1.3 How to prove a problem is NPC.

NP Completeness

We focus on “Decision Problems”.

P: the set of decision problems with this property: for every instance of the problem,
there exists an algorithm that can determine if the answer is YES or NO in polynomial time.

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1) The process of finding a solution.
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Sub-case 2: answer is “NO”

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~~1) The process of finding a solution.~~

2) Verify the correctness of the solution (so it knows when to end).

The constraint for NP is weaker:

} Sub-case 1: answer is “YES”

~~Sub-case 2: answer is “NO”~~

$$P \subseteq NP$$

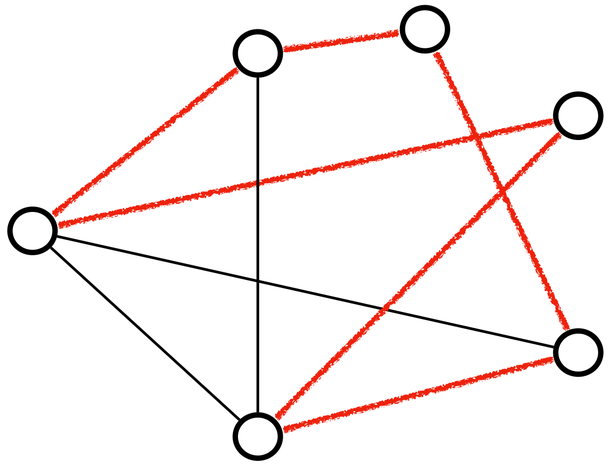
NP Completeness

$P \subseteq NP$

Hamiltonian cycle Problem:

Input: A graph G=(V,E).

Question: Does G have a Hamiltonian cycle?



Finding a Hamiltonian cycle might be hard ……

But given a Hamiltonian cycle, it is easy to verify it is indeed A Hamiltonian cycle.

Hamiltonian cycle Problem $\in NP$

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Co-NP: the set of decision problems with this property: for every instance of the problem,

If the answer is “NO”, there exists some data with which an algorithm can verify that the answer is indeed “NO” in polynomial time.

NP Completeness

$$P \subseteq NP$$

$$P \subseteq \text{Co-NP}$$

NP Completeness

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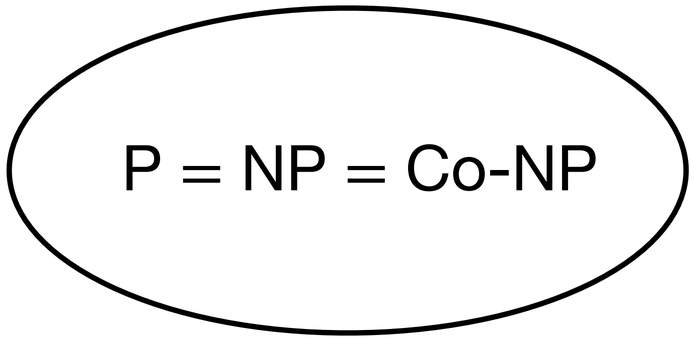
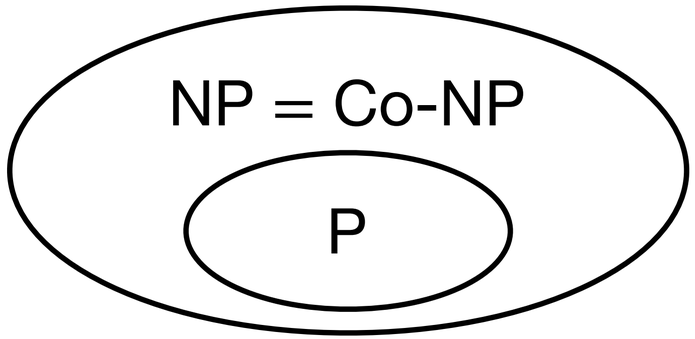
Four possible cases:

(a)



(b)



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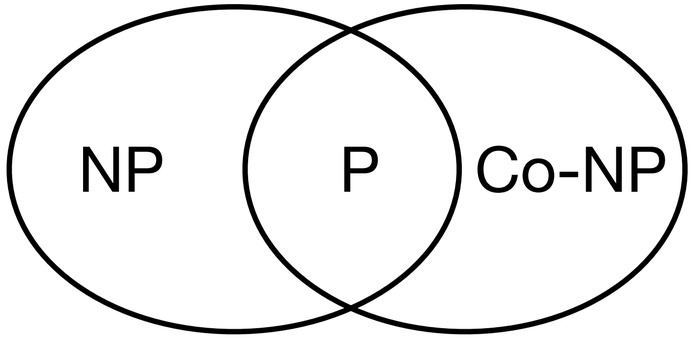

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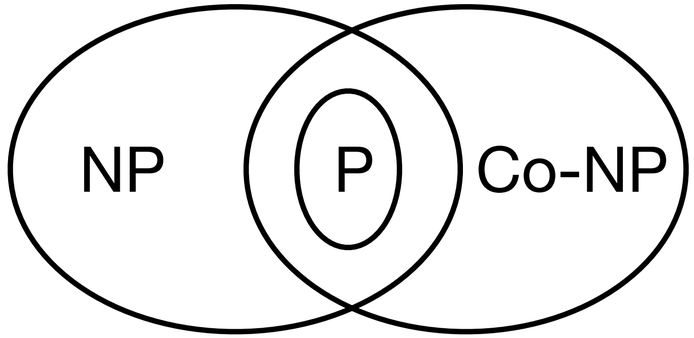

Quiz questions:

1. What does “certificate” mean for NP?

2. What is the difference between P and NP?

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    1.3 How to prove a problem is NPC.

NP Completeness

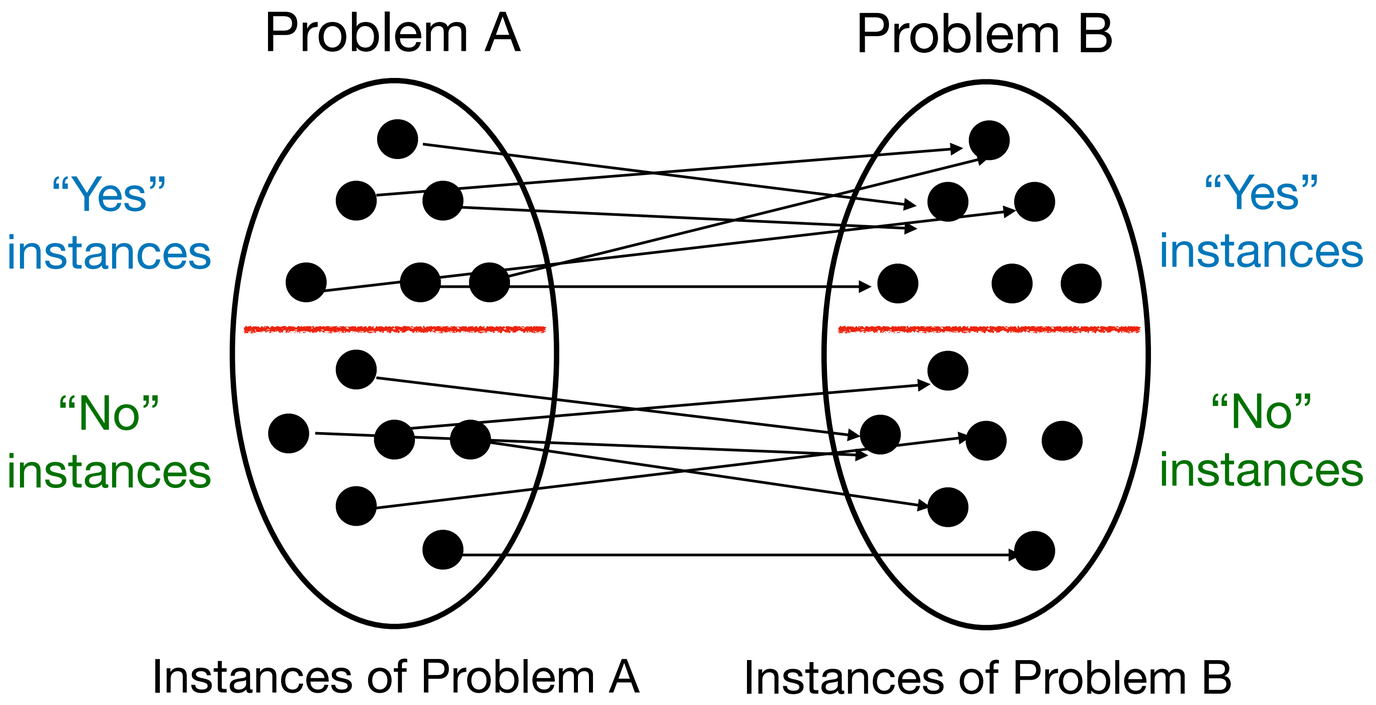
Polynomial-time reduction
from Problem A to Problem B:

$$A \leq_p B$$

Easier Harder

If Problem A is known to be “hard”,
then we can use the above relation
to show that Problem B is also “hard”.

Polynomial-time reduction:

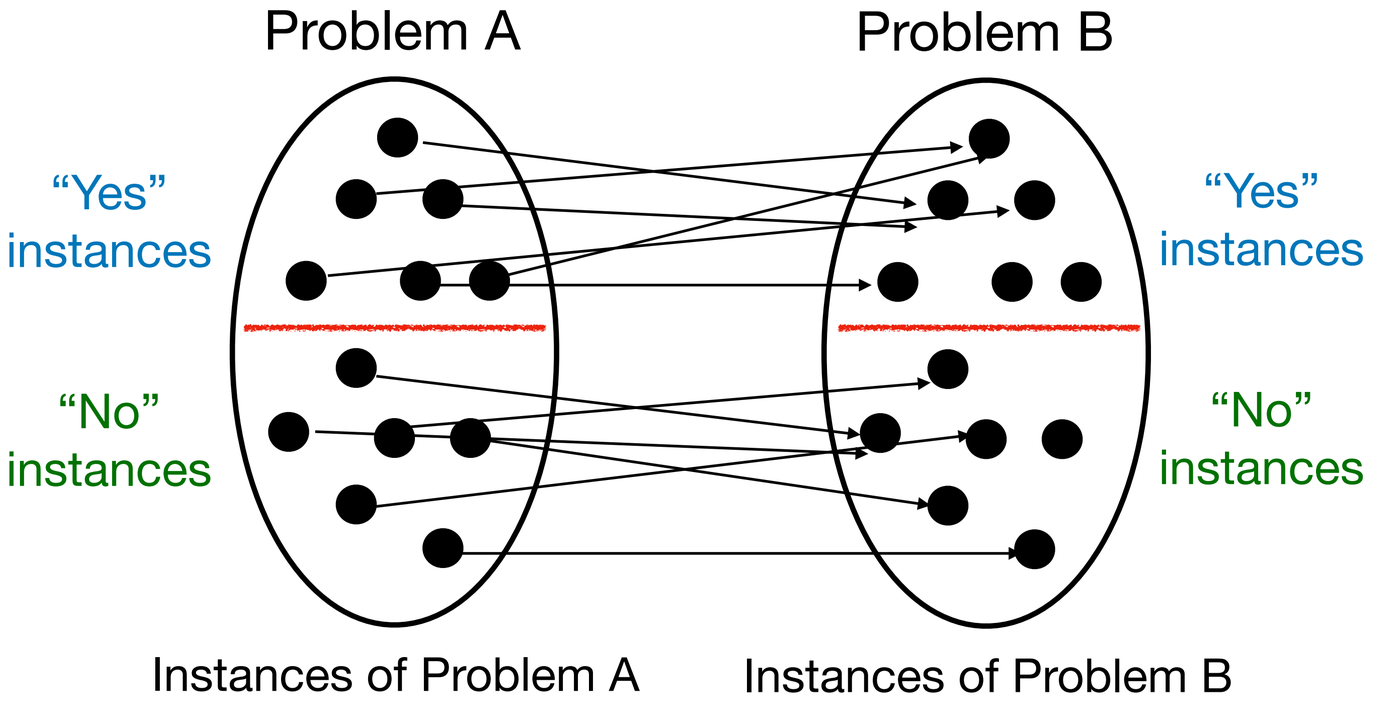


NP Completeness

Lemma: If $A \leq_p B$, then $B \in P$ implies $A \in P$.

Proof: We can use the reduction,
to solve A through solving B,
all in polynomial time.

Polynomial-time reduction:



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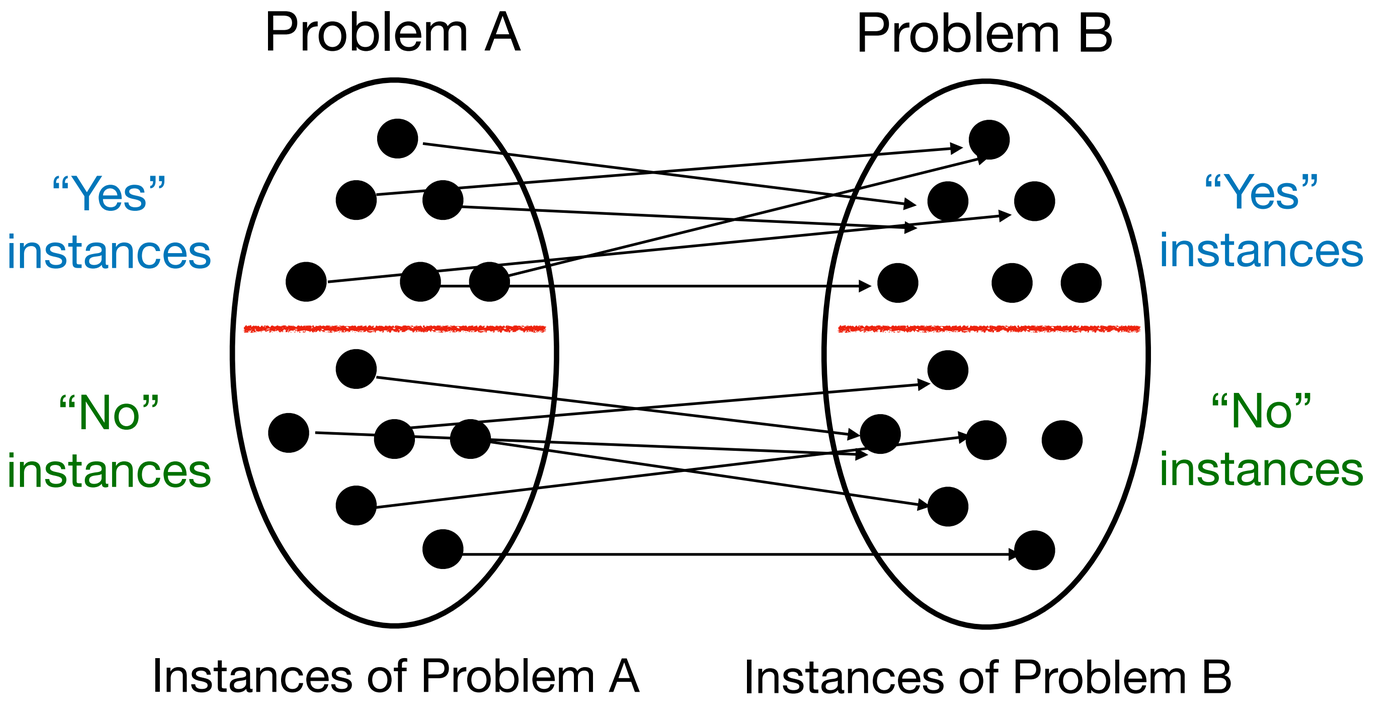

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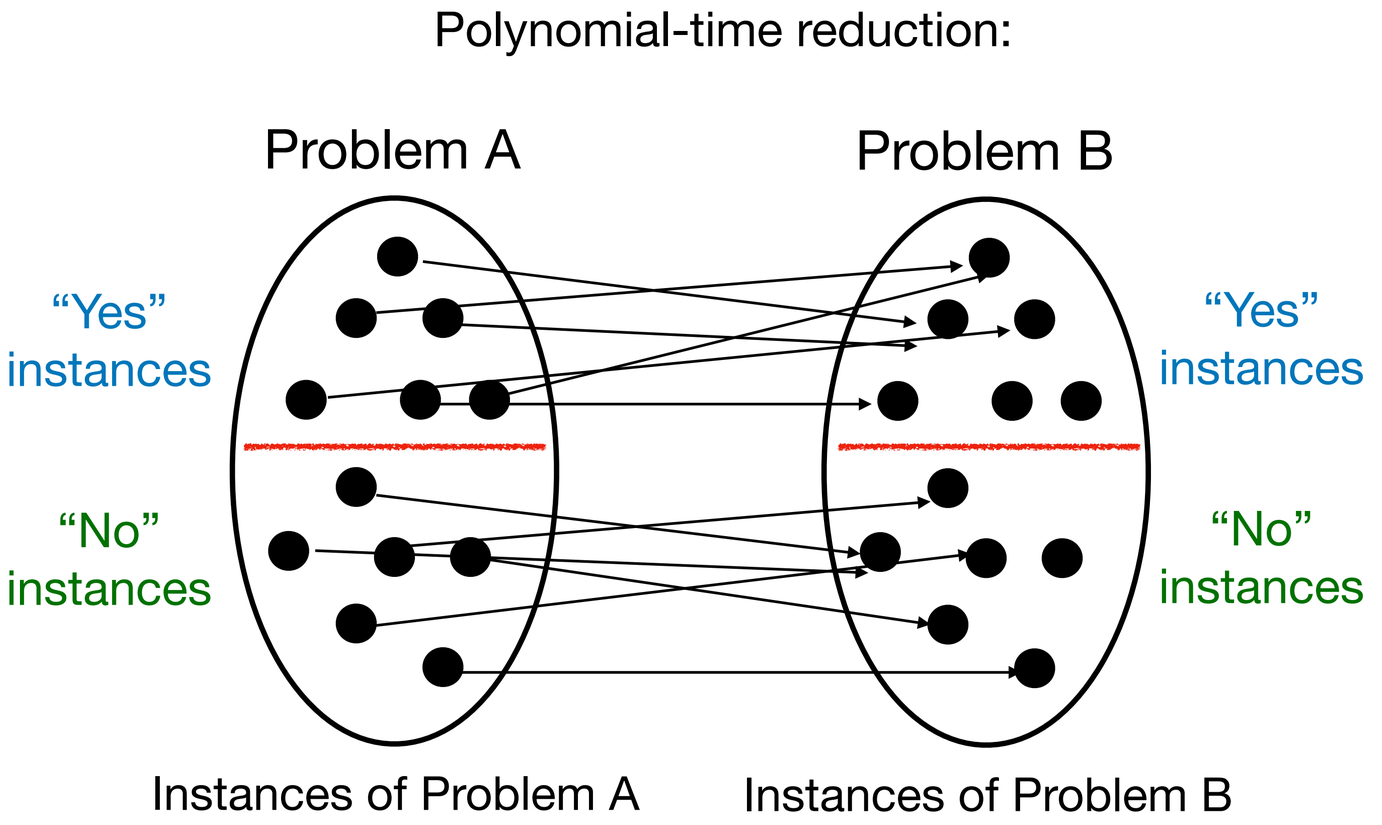

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Theorem: If any NP-complete (NPC) problem can be solved in polynomial time, then P=NP.

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Theorem: If any NP-complete (NPC) problem can be solved in polynomial time, then P=NP.

If $P \neq NP$, then no NPC problem can be solved in polynomial time.

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If only condition 2 is satisfied,
L is called “NP-hard”.

Quiz questions:

1. What is an NP-complete problem?

2. Are there known NP-complete problems?

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Lemma: For any problem $A \in NPC$,

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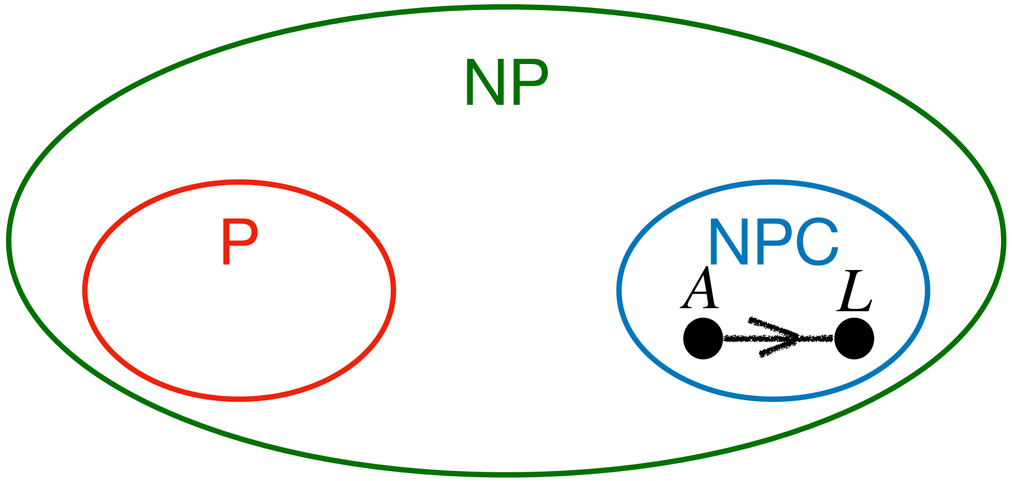

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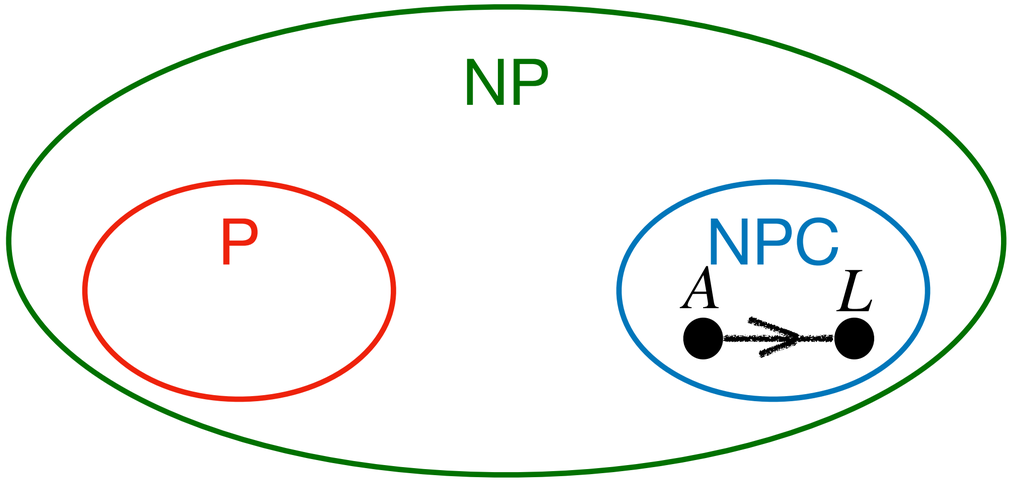

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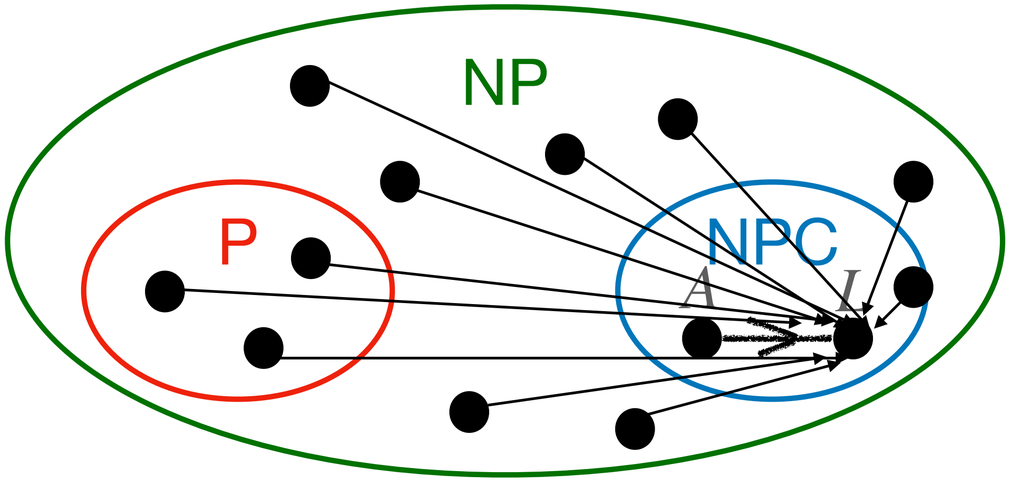

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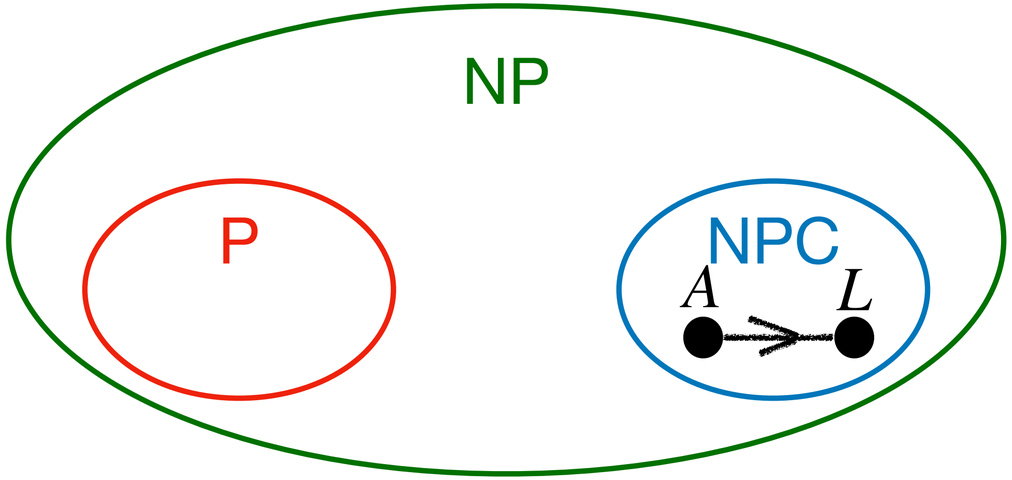

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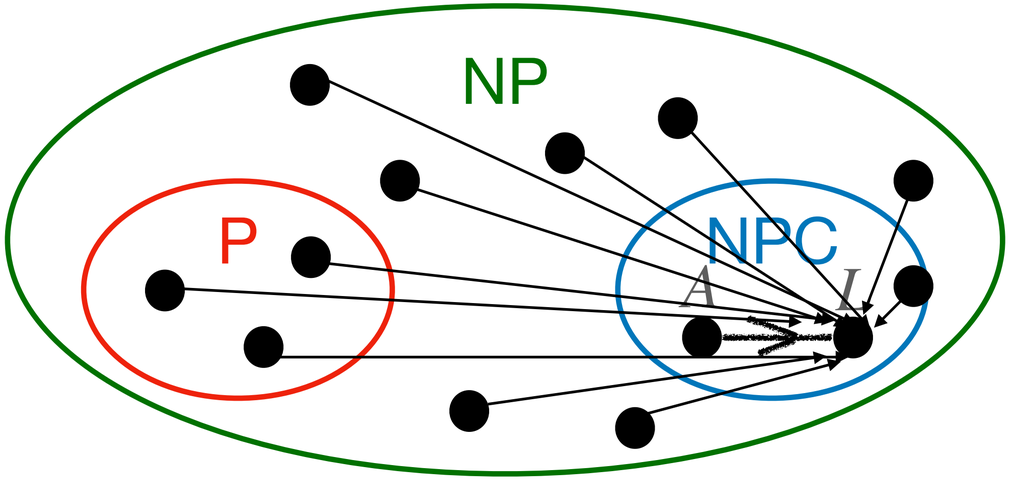

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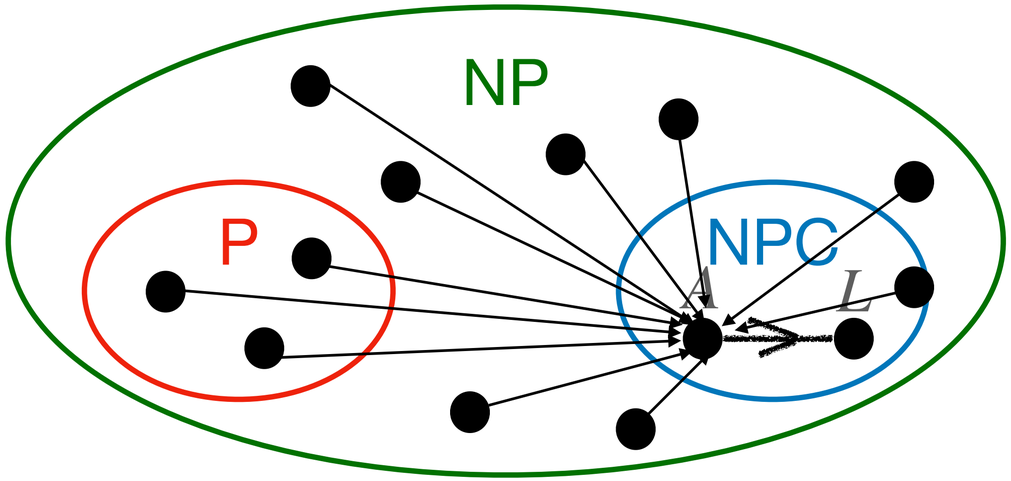

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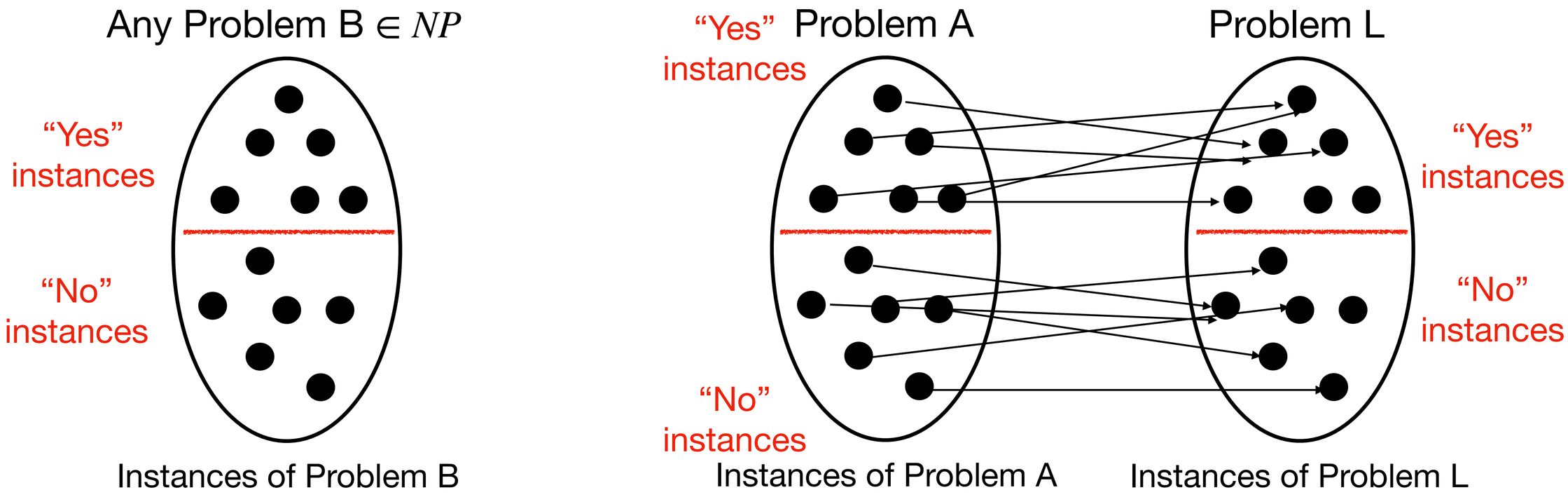
Proof:

Polynomial-time reduction
From A to L:



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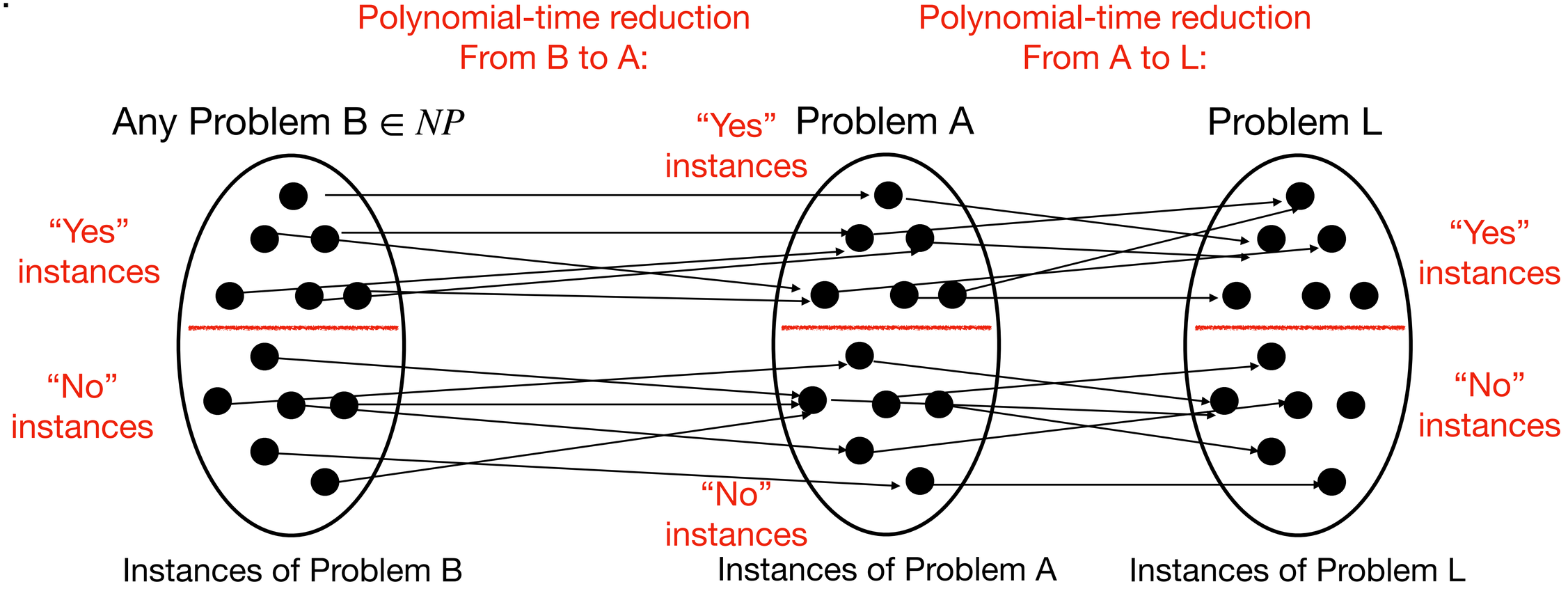

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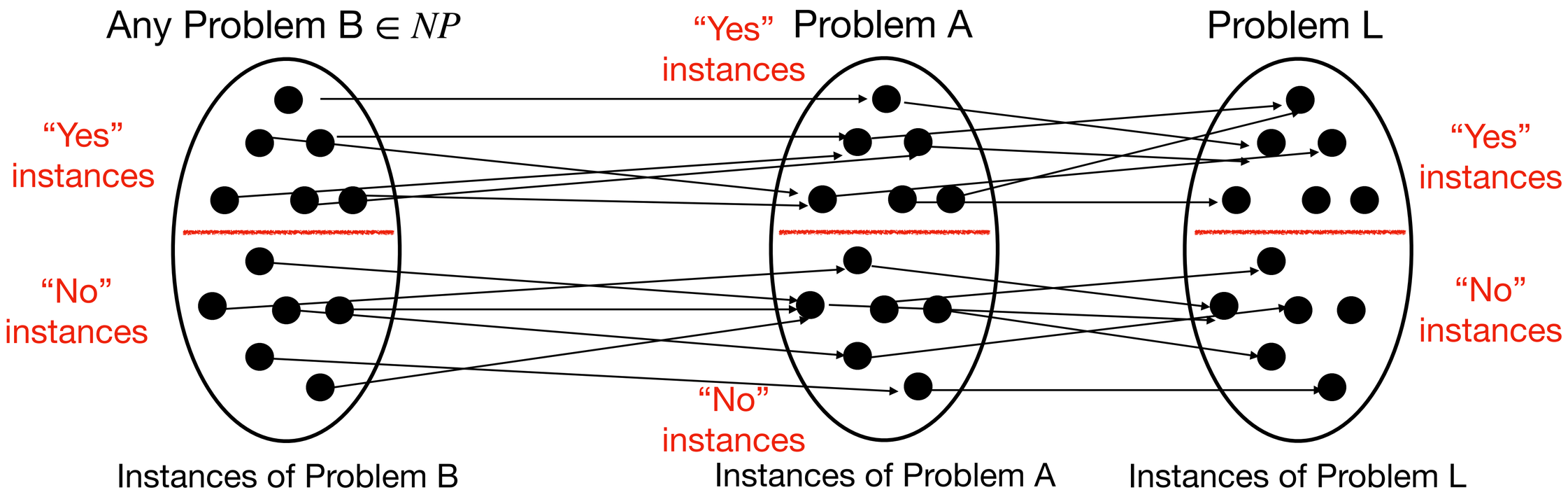
Polynomial-time reduction from B to L:



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Standard technique for proving NP-completeness today

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Quiz question:

1. Why can we prove a problem to be NPC by using the reduction from only one NPC problem, instead of using reductions from all NP problems?