# Algorithms

## Lecture Topic: Online Algorithms (Part 1)

**Anxiao (Andrew) Jiang**

Roadmap of this lecture:

1. Define "Online Algorithm".

2. Understand "Online Algorithm" by solving the "Elevator-or-Stairs Problem".

   2.1 Define the "Elevator-or-Stairs Problem".

   2.2 Competitive ratios of two intuitive algorithms.

   2.3 A better online algorithm with competitive ratio 2.

**Offline Algorithm:** The whole input is known in advance.

**Online Algorithm:** The algorithm receives the input progressively over time.

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Example: Stock trading

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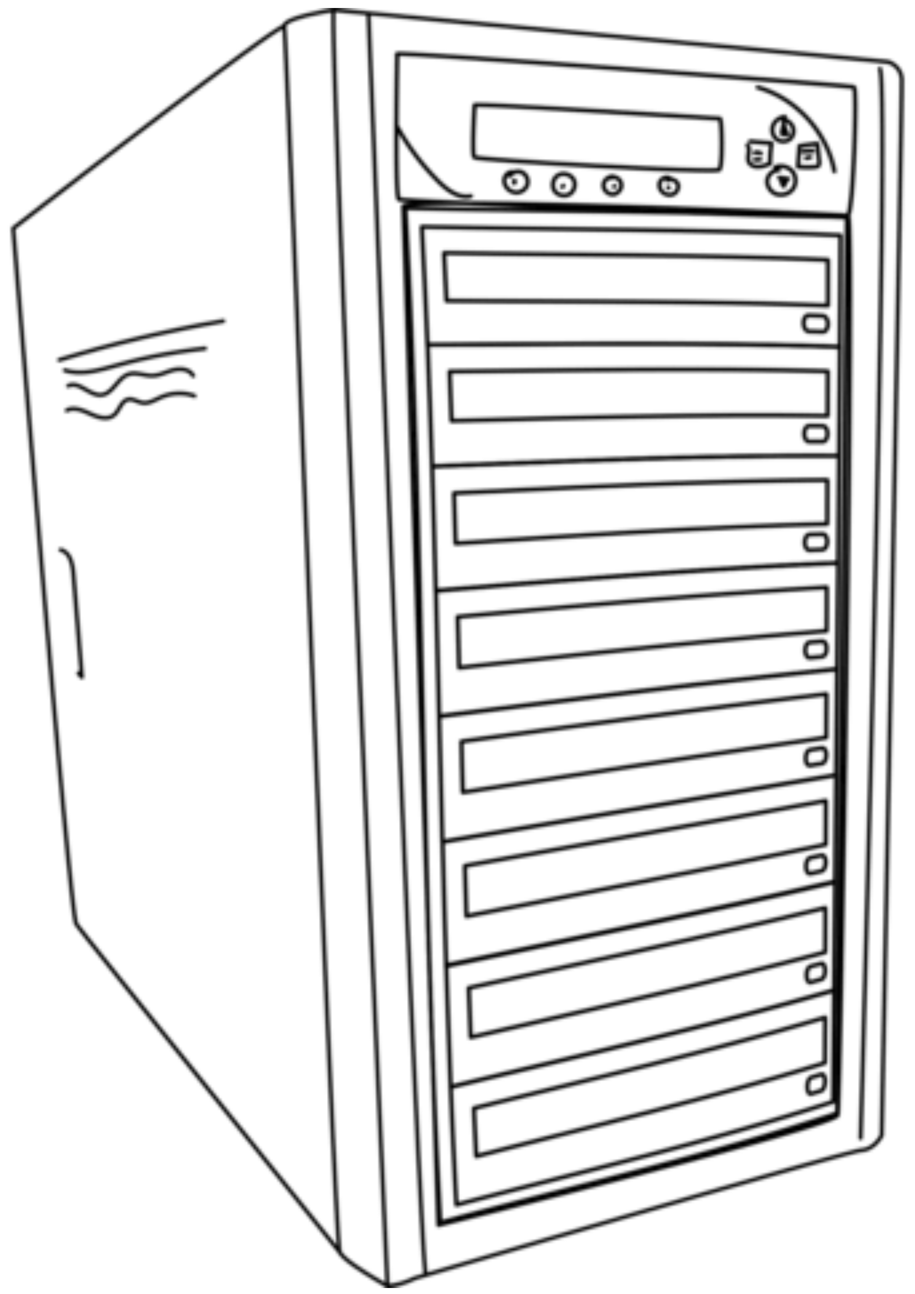
A computer scheduling arriving jobs



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**Example:** Stock trading

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A store deciding when to order more inventory

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**Example:**

- Stock trading
- A computer scheduling arriving jobs
- A store deciding when to order more inventory
- A taxi driver deciding if to pick up a fare

Online Algorithm: The algorithm receives the input progressively over time.

Goal: The online algorithm's performance is always close to the optimal solution's.

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Consider an optimization problem.

Let $C^*$ be the cost of an optimal solution.

Let $C$ be the cost of the solution found by our online algorithm.

(For simplicity, assume cost $> 0$.)

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Consider an optimization problem.

Let $C^*$ be the cost of an optimal solution.

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(For simplicity, assume cost > 0.)

Then $C^* \geq C,$ $\quad \dfrac{C^*}{C} \geq 1.$

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(For simplicity, assume cost > 0.)

Then $C^* \geq C$, $\dfrac{C^*}{C} \geq 1.$

If $\dfrac{C^*}{C} \leq \rho$ for all possible instances, then our algorithm is said to have Competitive Ratio $\rho$.

**Online Algorithm:** The algorithm receives the input progressively over time.

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Consider an optimization problem.

Let $C^*$ be the cost of an optimal solution.

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Then $C \geq C^*$, $\dfrac{C}{C^*} \geq 1.$

If $\dfrac{C}{C^*} \leq \rho$ for all possible instances, then our algorithm is said to have Competitive Ratio $\rho$.

Quiz questions:

1. What is the difference between an “online algorithm” and an “offline algorithm”?

2. How to measure the performance of an “online algorithm”?

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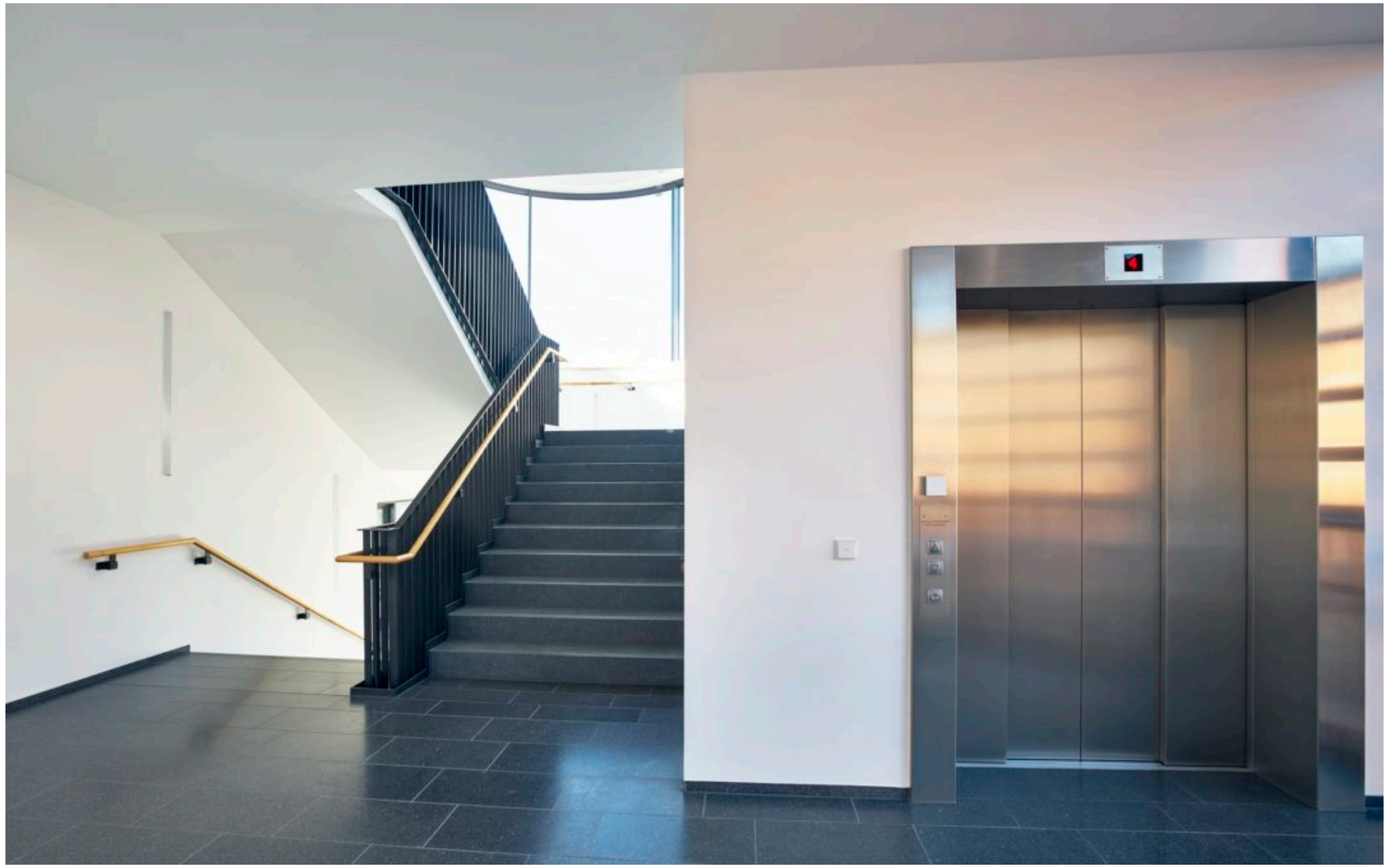
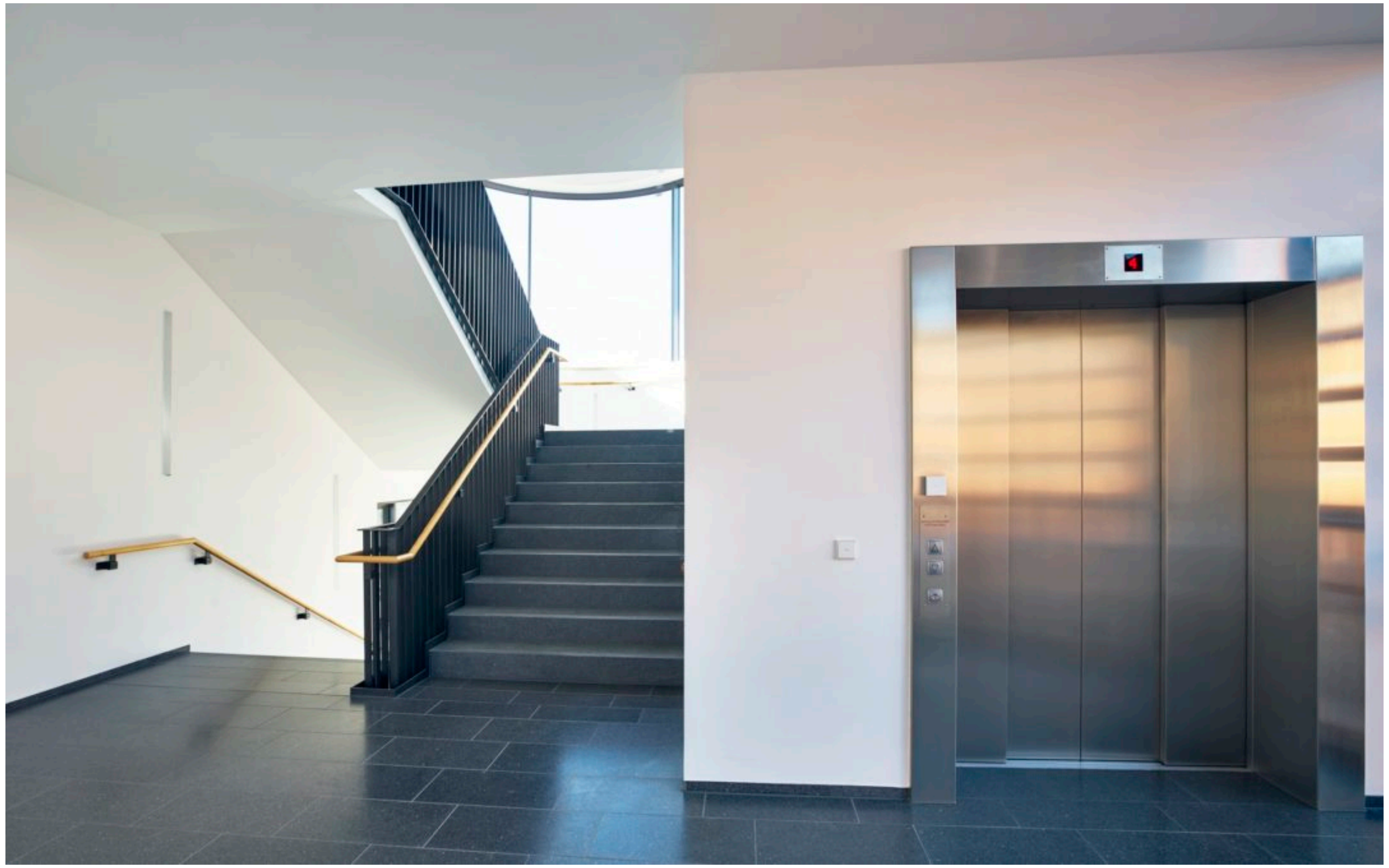
# Elevator-or-Stairs Problem



# Elevator-or-Stairs Problem



Desitnation: k floors up

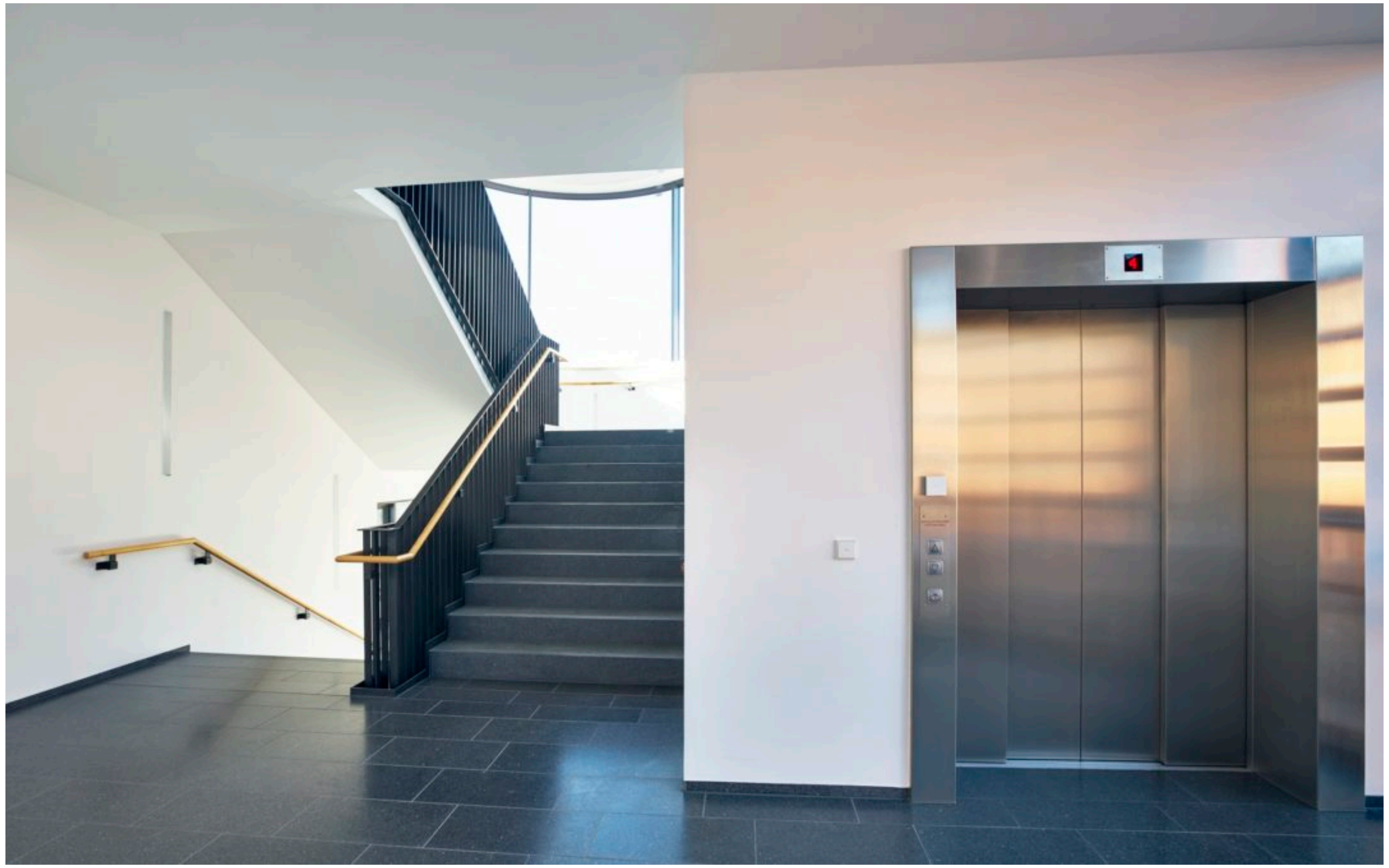
Climb stairs: 1 floor/minute

Elevator: k floors/minute

When elevator will arrive: unknown

Goal: minimize time to reach k-th floor

## Elevator-or-Stairs Problem



Desitnation: k floors up

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When elevator will arrive:

in at most $B - 1$ minutes, with $B \gg k$

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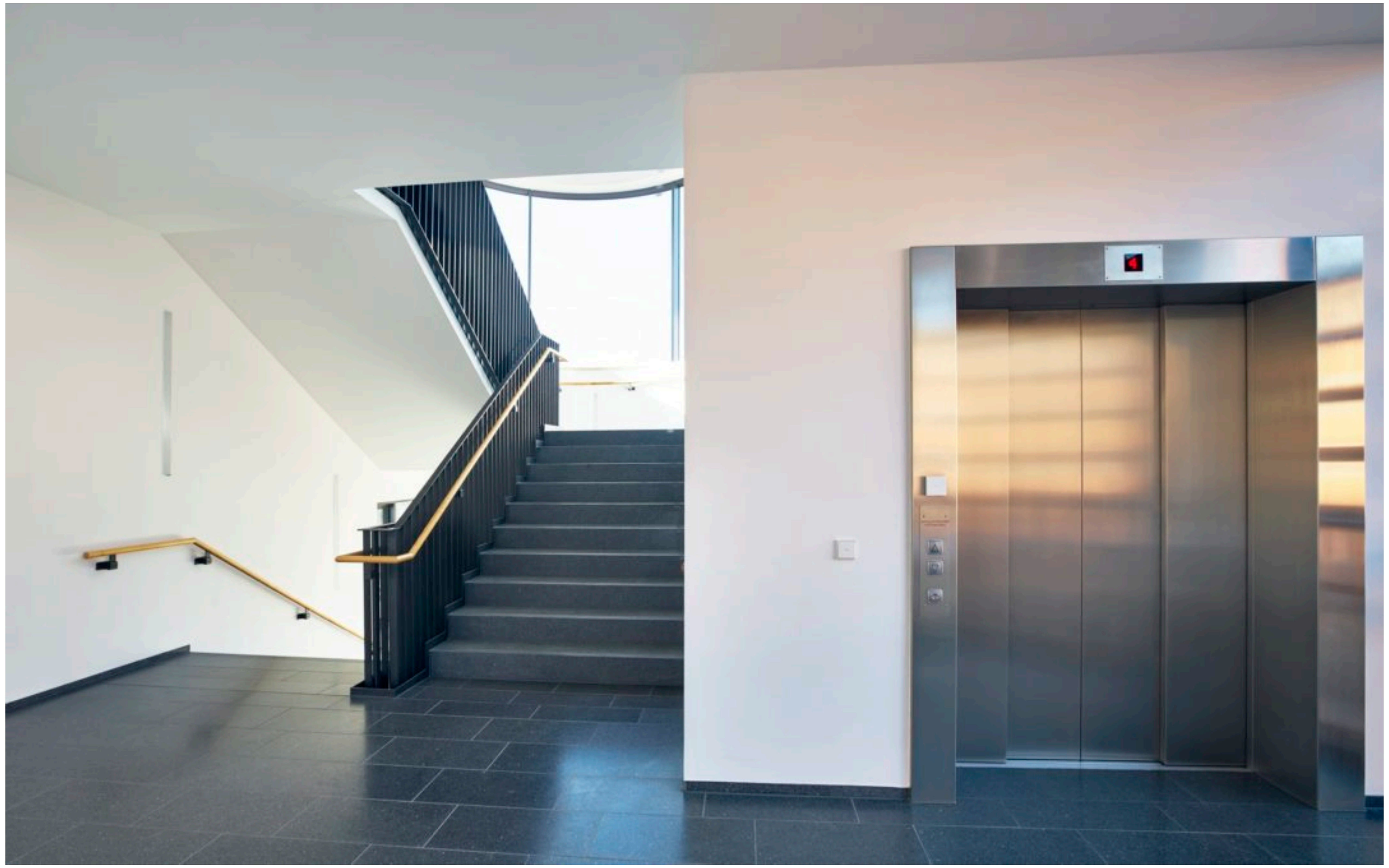

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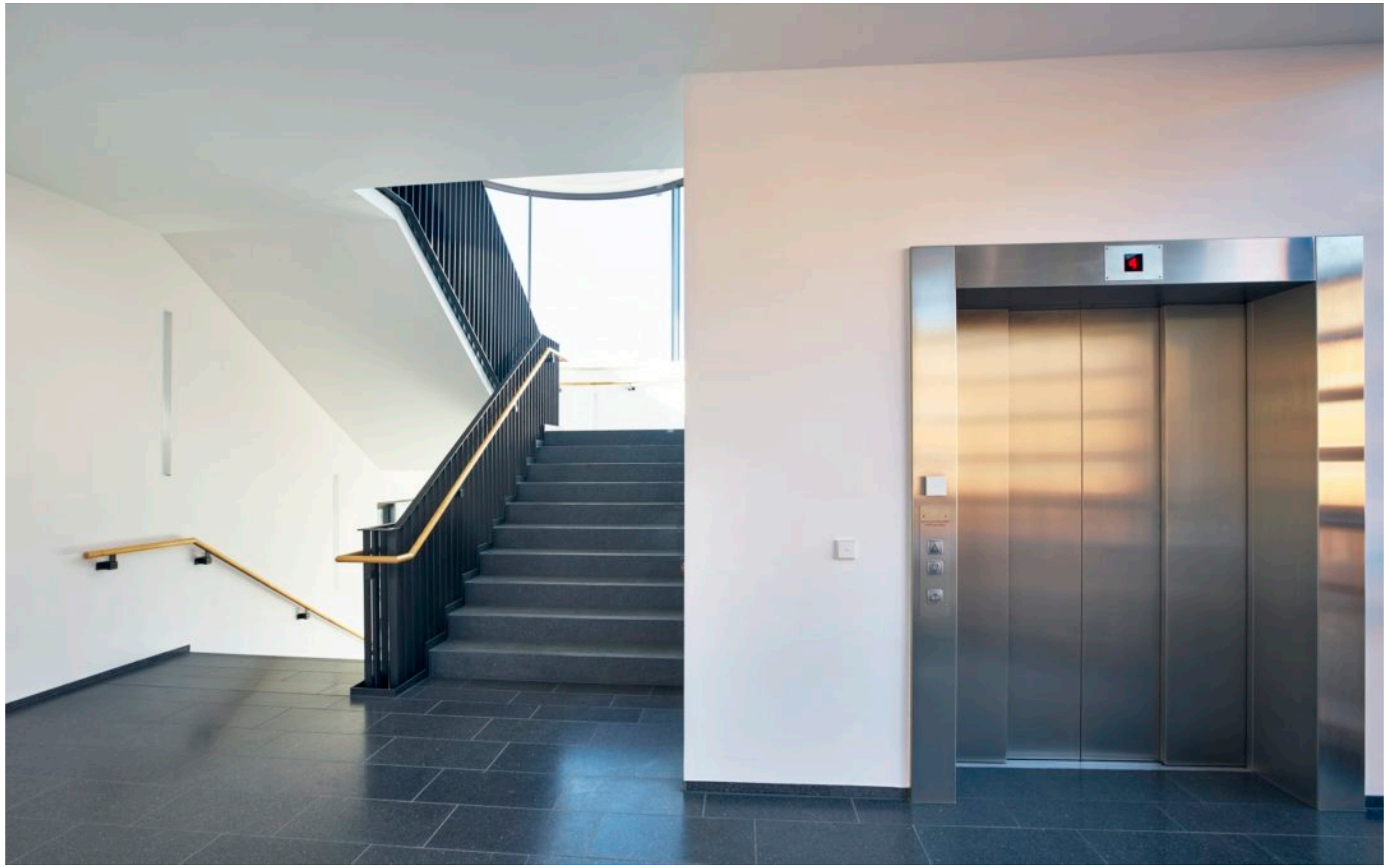

Desitnation: k floors up

Climb stairs: 1 floor/minute

Elevator: k floors/minute

When elevator will arrive:

in at most $B - 1$ minutes, with $B \gg k$

in an integer number of minutes

Take the stairs: $k$ minutes

Take the elevator: between 1 and $B$ minutes

What are their competitive ratios? We need to know the optimal solution (where the whole input is known) first.

Quiz questions:

1. Why do we need an "online algorithm" (instead of an "offline algorithm") for the "Elevator-or-Stairs Problem"?

2. If we know when the elevator will arrive, how would we design an offline algorithm?

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**Input:** Desitnation: k floors up.
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When elevator will arrive:
in at most $B - 1$ minutes, with $B \gg k$.
The elevator will arrive in an integer number of minutes.

**Output:** A method to reach the $k$-th floor as quickly as possible.

Method 1: Take the stairs.

Method 2: Take the elevator.

Seer: a person who can see the whole input, including the future.

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If the elevator arrives soon, take the elevator.

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If the elevator arrives in at most $k-1$ minutes,
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Suppose the elevator arrives in $m$ minutes, where $0 \le m \le B-1$.
Let $t(m)$ be the number of minutes the optimal solution takes.

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$$t(m) = \begin{cases} m+1 & \text{if } m \leq k-1 \\ k & \text{if } m \geq k \end{cases}$$

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**Output:** A method to reach the $k$-th floor as quickly as possible.

The elevator arrives in $m$ minutes.

| $m =$ | $0$ | $1$ | $2$ | $\cdots$ | $k-1$ | $k$ | $k+1$ | $\cdots$ | $B-1$ |
|---|---|---|---|---|---|---|---|---|---|
| Method 1: | $k$ | $k$ | $k$ | $\cdots$ | $k$ | $k$ | $k$ | $\cdots$ | $k$ |
| $t(m)$ : | $1$ | $2$ | $3$ | $\cdots$ | $k$ | $k$ | $k$ | $\cdots$ | $k$ |

Method 1: Take the stairs.
$k$ minutes

Method 2: Take the elevator.

Optimal solution:

$$t(m) = \begin{cases} m+1 & \text{if } m \le k-1 \\ k & \text{if } m \ge k \end{cases}$$

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|---|---|---|---|---|---|---|---|---|---|
| Method 1: | $k$ | $k$ | $k$ | $\cdots$ | $k$ | $k$ | $k$ | $\cdots$ | $k$ |
| $t(m)$ : | $1$ | $2$ | $3$ | $\cdots$ | $k$ | $k$ | $k$ | $\cdots$ | $k$ |
| Ratio: | $\frac{k}{1}$ | $\frac{k}{2}$ | $\frac{k}{3}$ | $\cdots$ | $\frac{k}{k}$ | $\frac{k}{k}$ | $\frac{k}{k}$ | $\cdots$ | $\frac{k}{k}$ |

Method 1: Take the stairs.
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Method 1: Take the stairs.
$k$ minutes

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| Method 1: | $k$ | $k$ | $k$ | $\cdots$ | $k$ | $k$ | $k$ | $\cdots$ | $k$ |
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Competitive Ratio = $k$

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$$m = \quad 0 \quad 1 \quad 2 \quad \cdots \quad k-1 \quad k \quad k+1 \quad \cdots \quad B-1$$

Method 1: Take the stairs.
$k$ minutes

Method 2: Take the elevator.
$m+1$ minutes

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| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $m =$ | $0$ | $1$ | $2$ | $\cdots$ | $k-1$ | $k$ | $k+1$ | $\cdots$ | $B-1$ |
| Method 2: | $1$ | $2$ | $3$ | $\cdots$ | $k$ | $k+1$ | $k+2$ | $\cdots$ | $B$ |

Method 1: Take the stairs.
$k$ minutes

Method 2: Take the elevator.
$m+1$ minutes

Optimal solution:

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| $m =$ | 0 | 1 | 2 | $\cdots$ | $k-1$ | $k$ | $k+1$ | $\cdots$ | $B-1$ |
|---|---|---|---|---|---|---|---|---|---|
| Method 2: | 1 | 2 | 3 | $\cdots$ | $k$ | $k+1$ | $k+2$ | $\cdots$ | $B$ |
| $t(m)$ : | 1 | 2 | 3 | $\cdots$ | $k$ | $k$ | $k$ | $\cdots$ | $k$ |

Method 1: Take the stairs.
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Method 2: Take the elevator.
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Optimal solution:

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| Method 2: | $1$ | $2$ | $3$ | $\cdots$ | $k$ | $k+1$ | $k+2$ | $\cdots$ | $B$ |
| $t(m)$ : | $1$ | $2$ | $3$ | $\cdots$ | $k$ | $k$ | $k$ | $\cdots$ | $k$ |
| Ratio: | $1$ | $1$ | $1$ | $\cdots$ | $1$ | $\frac{k+1}{k}$ | $\frac{k+2}{k}$ | $\cdots$ | $\frac{B}{k}$ |

Method 1: Take the stairs.

$k$ minutes

Method 2: Take the elevator.

$m+1$ minutes

Optimal solution:

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|---|---|---|---|---|---|---|---|---|---|
| Method 2: | $1$ | $2$ | $3$ | $\cdots$ | $k$ | $k+1$ | $k+2$ | $\cdots$ | $B$ |
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| Ratio: | $1$ | $1$ | $1$ | $\cdots$ | $1$ | $\frac{k+1}{k}$ | $\frac{k+2}{k}$ | $\cdots$ | $\frac{B}{k}$ |

Method 1: Take the stairs.
$k$ minutes

Competitive Ratio $= \frac{B}{k}$

Method 2: Take the elevator.

$m+1$ minutes

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Output: A method to reach the $k$-th floor as quickly as possible.

Method 1: Take the stairs.

Competitive ratio = $k$

Method 2: Take the elevator.

Competitive ratio = $B/k$

Optimal solution:

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Quiz questions:

1. How did we find the competitive ratio of the method of "always take the stairs"?

2. How did we find the competitive ratio of the method of "always take the elevator"?

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**Output:** A method to reach the $k$-th floor as quickly as possible.

Method 1: Take the stairs.

Competitive ratio = $k$

Method 2: Take the elevator.

Competitive ratio = $B/k$

Optimal solution:

$$t(m) = \begin{cases} m+1 & \text{if } m \le k-1 \\ k & \text{if } m \ge k \end{cases}$$

Method 3: Wait for the elevator for a while.

If it still does not arrive, take the stairs.

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Wait for the elevator for $k$ minutes.
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Method 3: Wait for the elevator for a while.
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Wait for the elevator for $k$ minutes.
If it still does not arrive, take the stairs.

Time:

$$h(m) = \begin{cases} m+1 & \text{if } m \le k \\ 2k & \text{if } m > k \end{cases}$$

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Method 3: Wait for the elevator for $k$ minutes.
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$$h(m) = \begin{cases} m+1 & \text{if } m \le k \\ 2k & \text{if } m > k \end{cases}$$

The elevator arrives in $m$ minutes.

$$m = \quad 0 \quad 1 \quad 2 \quad \cdots \quad k-1 \quad k \quad k+1 \quad \cdots \quad B-1$$

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| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $m =$ | $0$ | $1$ | $2$ | $\cdots$ | $k-1$ | $k$ | $k+1$ | $\cdots$ | $B-1$ |
| Method 3: | $1$ | $2$ | $3$ | $\cdots$ | $k$ | $k+1$ | $2k$ | $\cdots$ | $2k$ |

## Elevator-or-Stairs Problem

**Input:** Desitnation: k floors up.
Climb stairs: 1 floor/minute.
Elevator: k floors/minute.
When elevator will arrive:
in at most $B-1$ minutes, with $B \gg k$.
The elevator will arrive in an integer number of minutes.

**Output:** A method to reach the $k$-th floor as quickly as possible.

Method 1: Take the stairs.
Competitive ratio = $k$

Method 2: Take the elevator.
Competitive ratio = $B/k$

Optimal solution:

$$t(m) = \begin{cases} m+1 & \text{if } m \le k-1 \\ k & \text{if } m \ge k \end{cases}$$

Method 3: Wait for the elevator for $k$ minutes.
If it still does not arrive, take the stairs.

Time:

$$h(m) = \begin{cases} m+1 & \text{if } m \le k \\ 2k & \text{if } m > k \end{cases}$$

The elevator arrives in $m$ minutes.

| $m =$ | $0$ | $1$ | $2$ | $\cdots$ | $k-1$ | $k$ | $k+1$ | $\cdots$ | $B-1$ |
|---|---|---|---|---|---|---|---|---|---|
| Method 3: | $1$ | $2$ | $3$ | $\cdots$ | $k$ | $k+1$ | $2k$ | $\cdots$ | $2k$ |
| $t(m)$ : | $1$ | $2$ | $3$ | $\cdots$ | $k$ | $k$ | $k$ | $\cdots$ | $k$ |

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|---|---|---|---|---|---|---|---|---|---|
| Method 3: | $1$ | $2$ | $3$ | $\cdots$ | $k$ | $k+1$ | $2k$ | $\cdots$ | $2k$ |
| $t(m)$ : | $1$ | $2$ | $3$ | $\cdots$ | $k$ | $k$ | $k$ | $\cdots$ | $k$ |
| Ratio: | $1$ | $1$ | $1$ | $\cdots$ | $1$ | $\frac{k+1}{k}$ | $2$ | $\cdots$ | $2$ |

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| $m =$ | $0$ | $1$ | $2$ | $\cdots$ | $k-1$ | $k$ | $k+1$ | $\cdots$ | $B-1$ |
|---|---|---|---|---|---|---|---|---|---|
| Method 3: | $1$ | $2$ | $3$ | $\cdots$ | $k$ | $k+1$ | $2k$ | $\cdots$ | $2k$ |
| $t(m)$ : | $1$ | $2$ | $3$ | $\cdots$ | $k$ | $k$ | $k$ | $\cdots$ | $k$ |
| Ratio: | $1$ | $1$ | $1$ | $\cdots$ | $1$ | $\frac{k+1}{k}$ | $2$ | $\cdots$ | $2$ |

Competitive Ratio = 2

**Elevator-or-Stairs Problem**

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Time:
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The elevator arrives in $m$ minutes.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $m =$ | $0$ | $1$ | $2$ | $\cdots$ | $k-1$ | $k$ | $k+1$ | $\cdots$ | $B-1$ |
| Method 3: | $1$ | $2$ | $3$ | $\cdots$ | $k$ | $k+1$ | $2k$ | $\cdots$ | $2k$ |
| $t(m)$ : | $1$ | $2$ | $3$ | $\cdots$ | $k$ | $k$ | $k$ | $\cdots$ | $k$ |
| Ratio: | $1$ | $1$ | $1$ | $\cdots$ | $1$ | $\frac{k+1}{k}$ | $2$ | $\cdots$ | $2$ |

Competitive Ratio = 2

Common philosophy of online algorithms:
guard against any possible worst case.

Quiz questions:

1. How did the above method achieve a constant competitive ratio?

2. What is the value of "guard against all possible worst cases" for online algorithms?