Algorithms

Lecture Topic: Topological Sort

Anxiao (Andrew) Jiang

Roadmap of this lecture:

1. Application of DFS: Topological Sort.

    1.1 Define "Topological Sort".

    1.2 Algorithm for topological sort.

Topological Sort

Input: A directed acyclic graph G=(V,E).

Output: Sort vertices of G linearly such that all the edges are from left to right.

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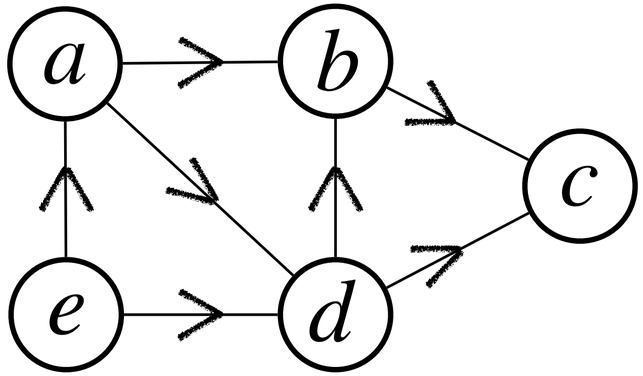

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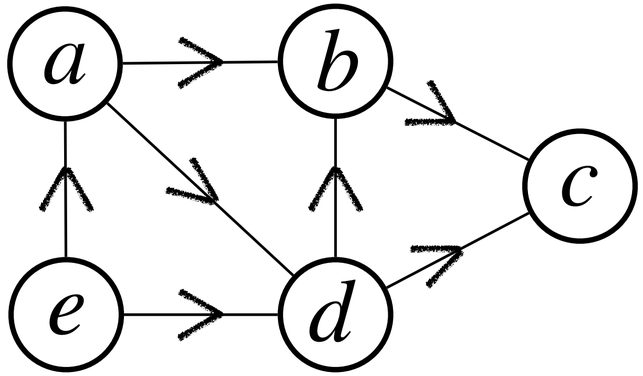

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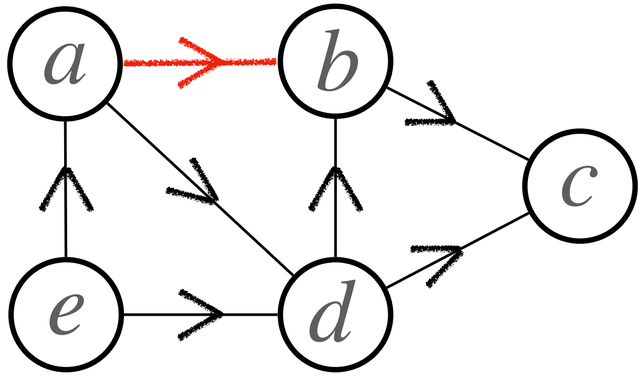

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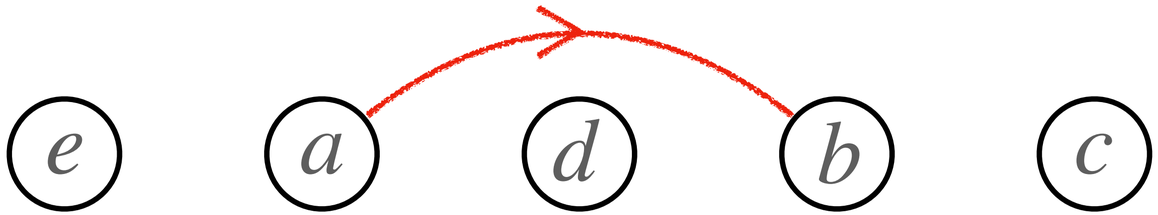

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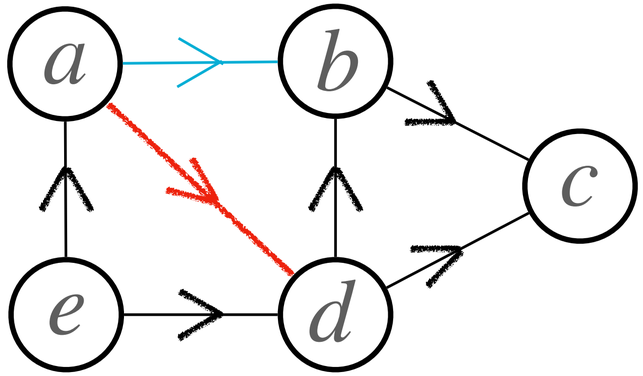

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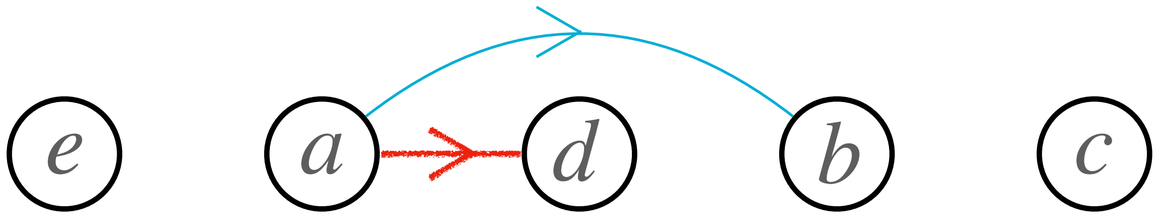

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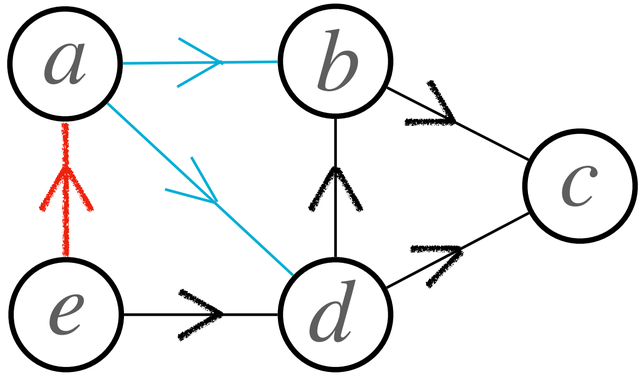

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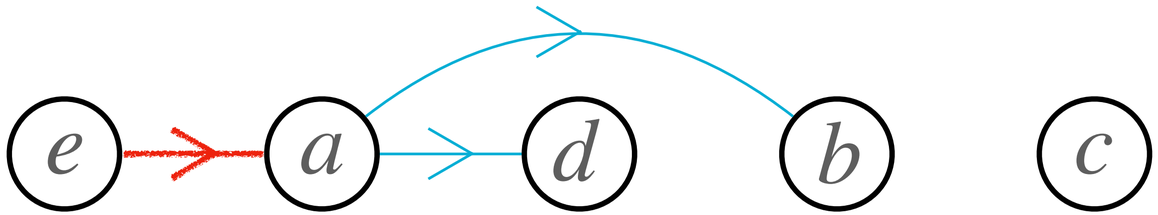

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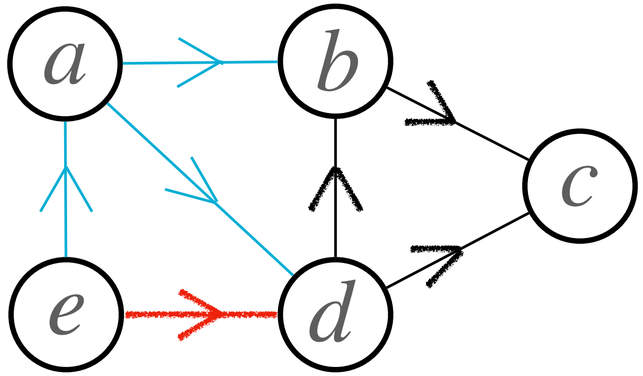

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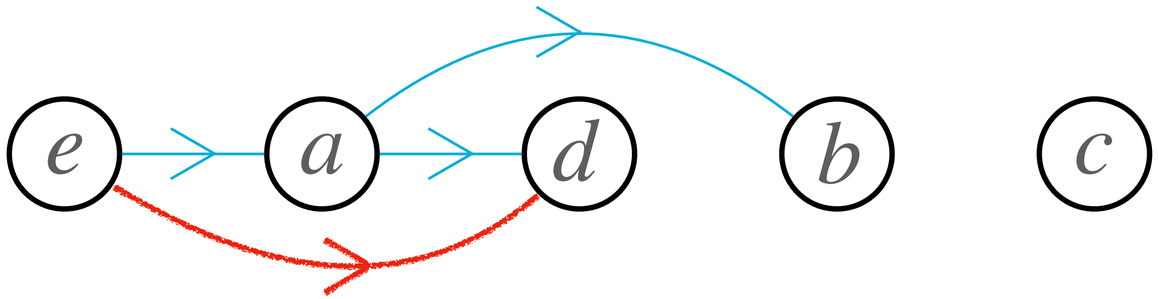

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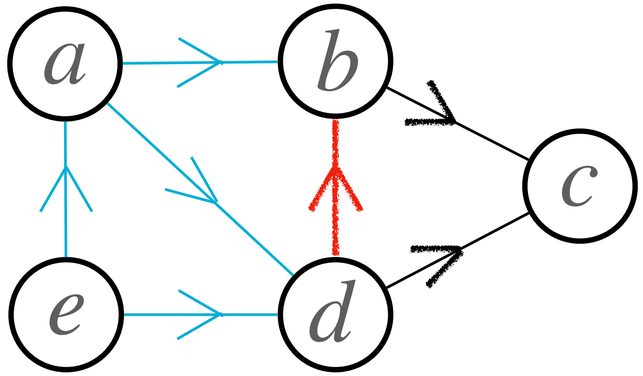

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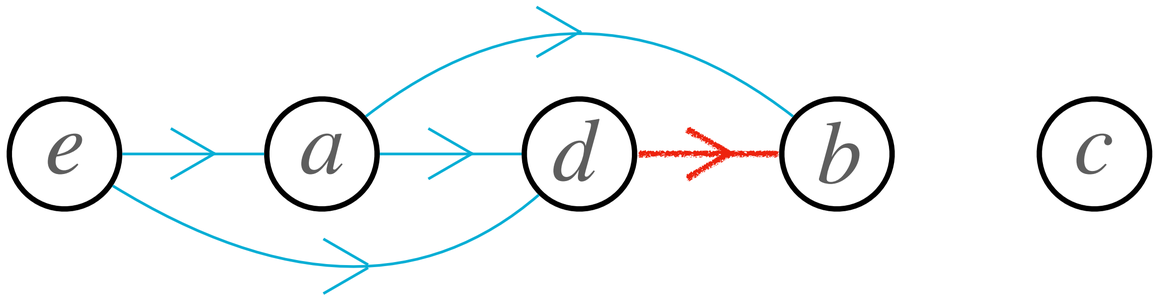

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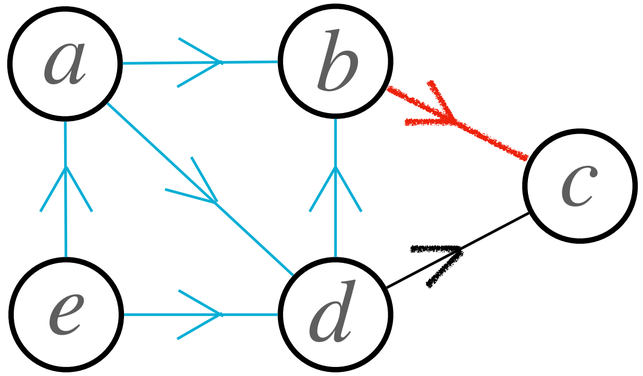

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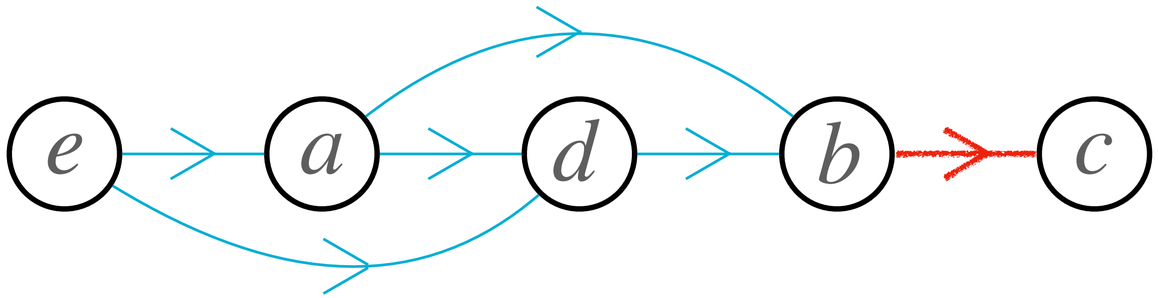

Topological Sort

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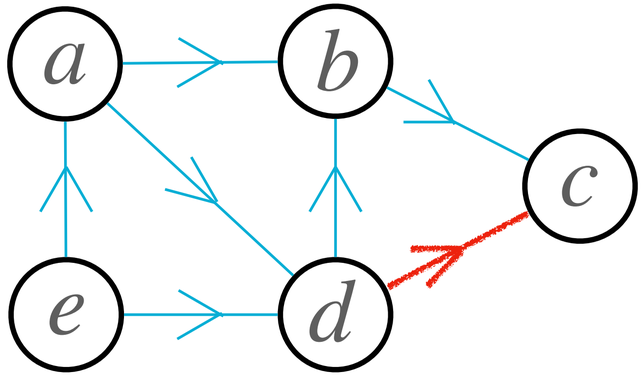

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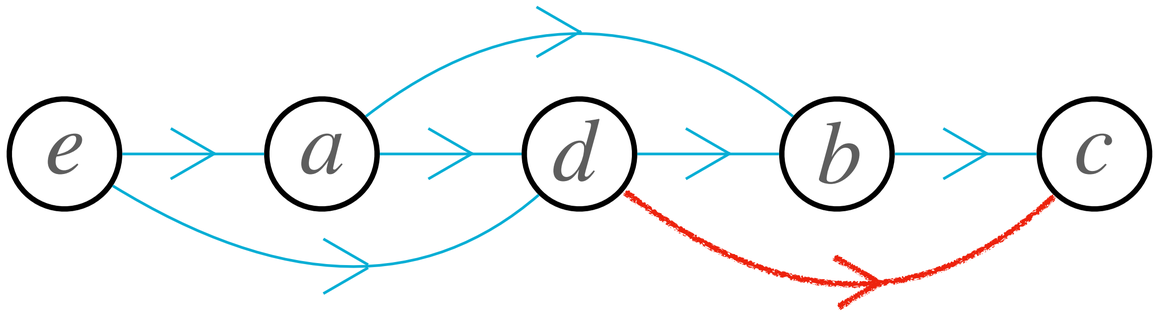

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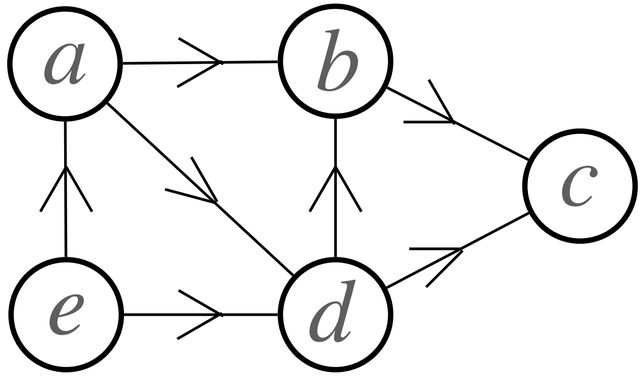

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Why topological sort must exist for DAG?

Sink: a node without outgoing edges.

A DAG must have a sink.

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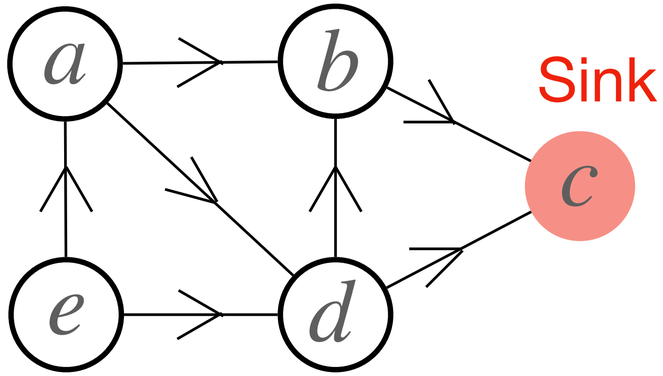

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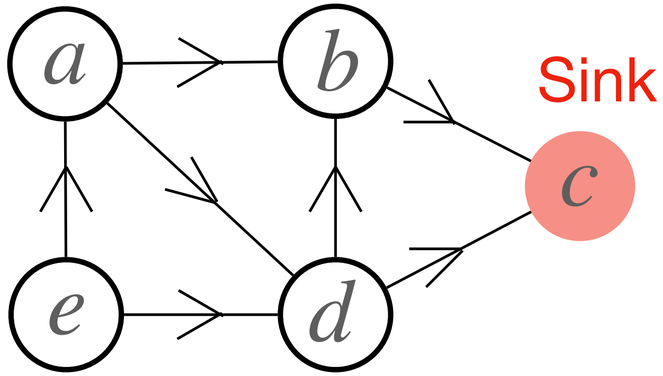

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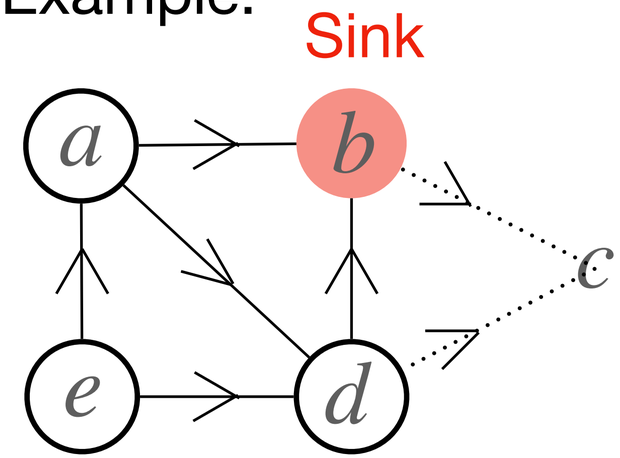

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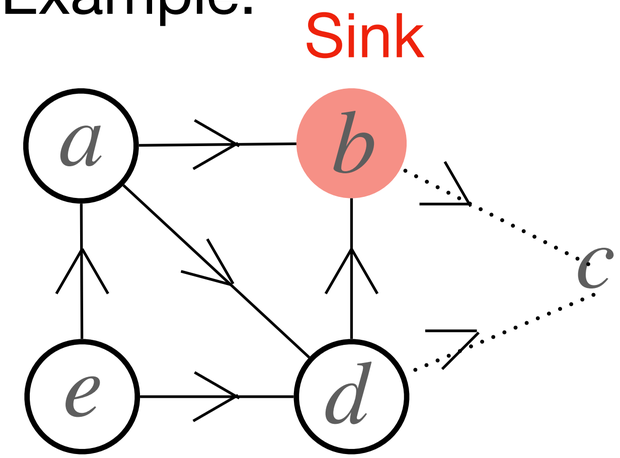

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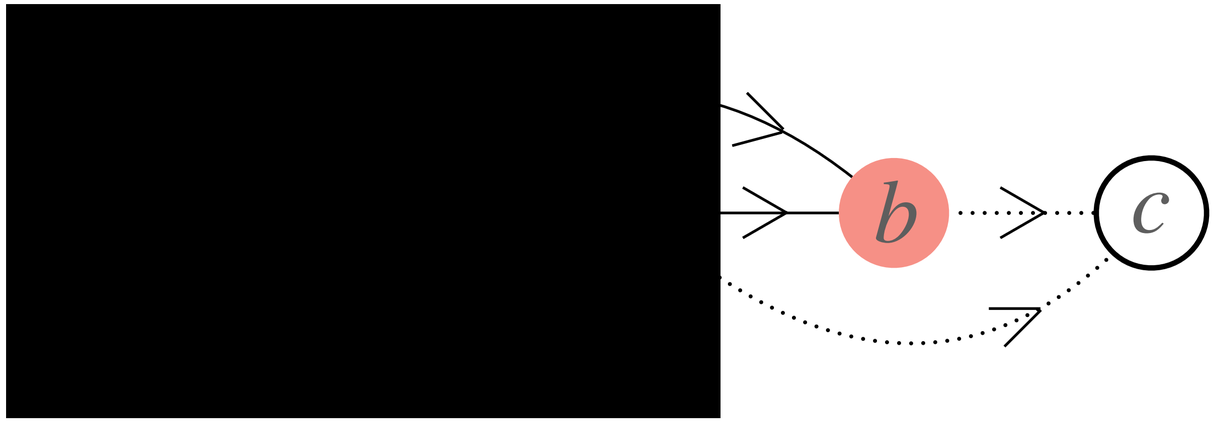

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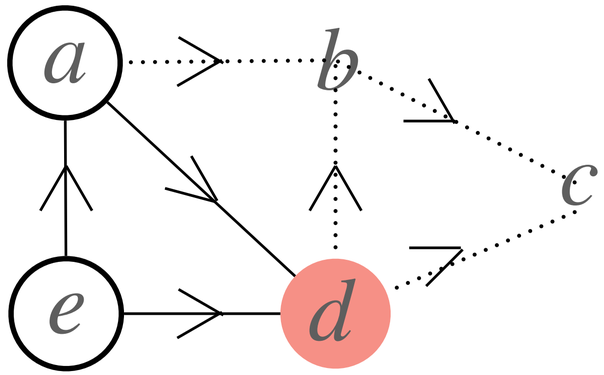

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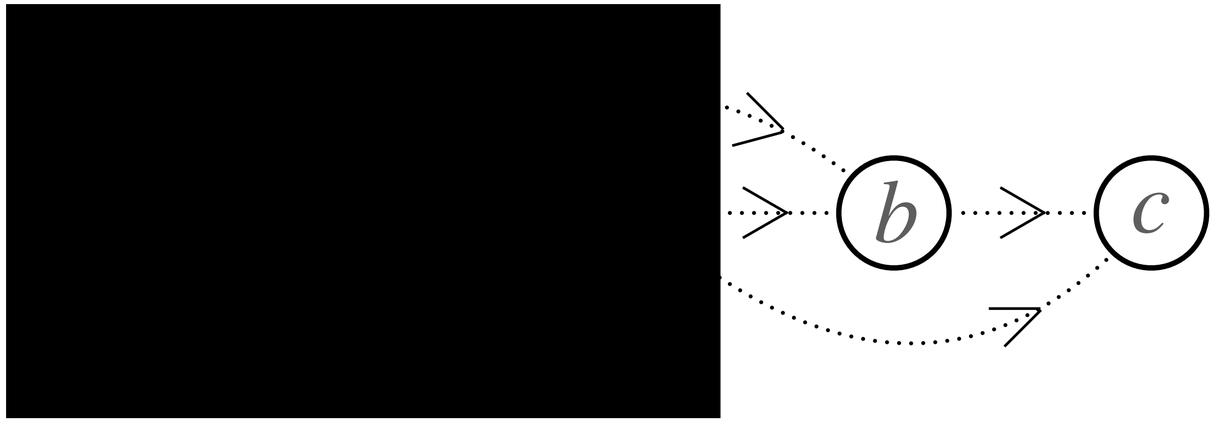

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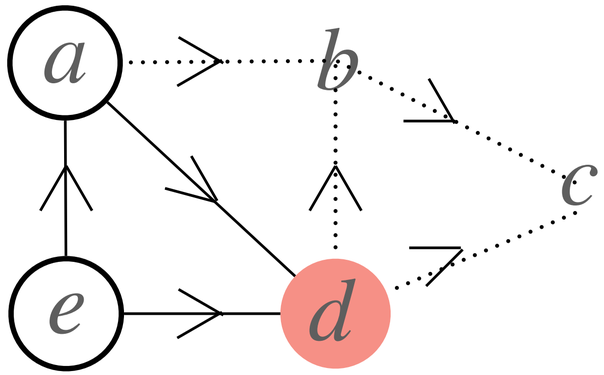

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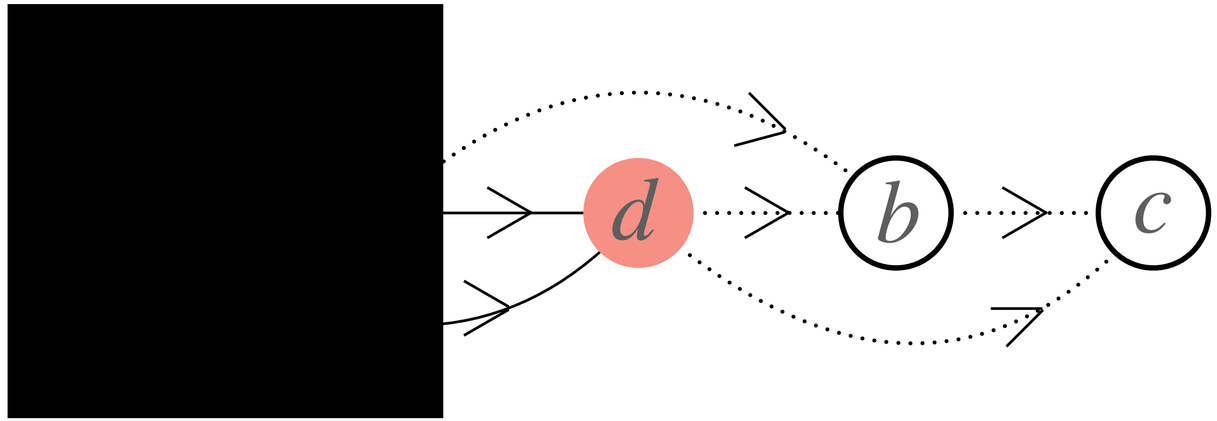

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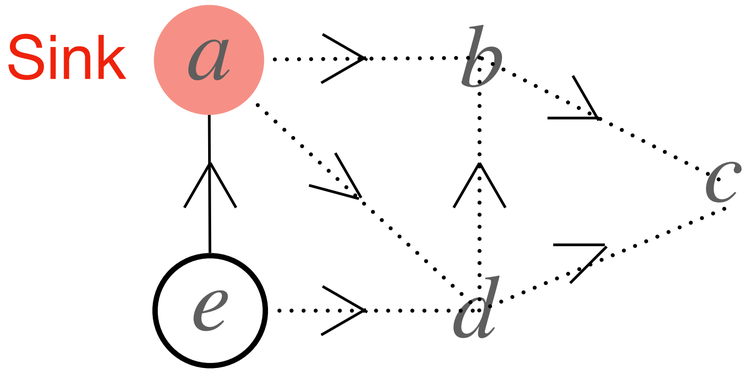

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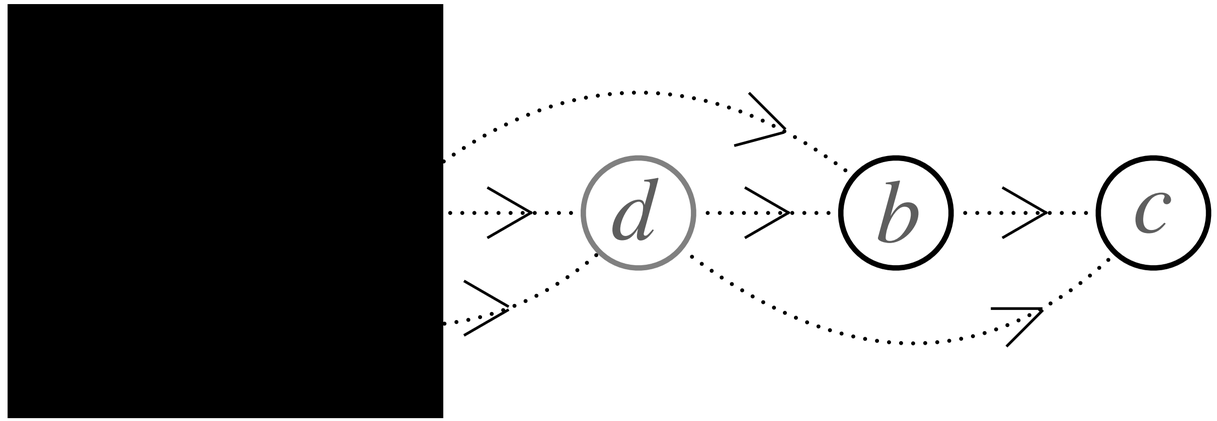

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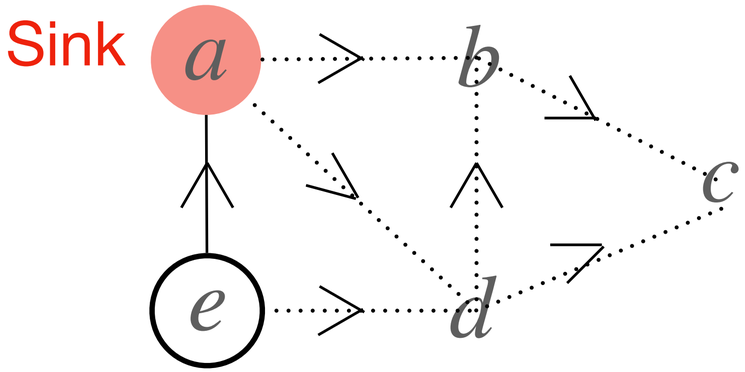

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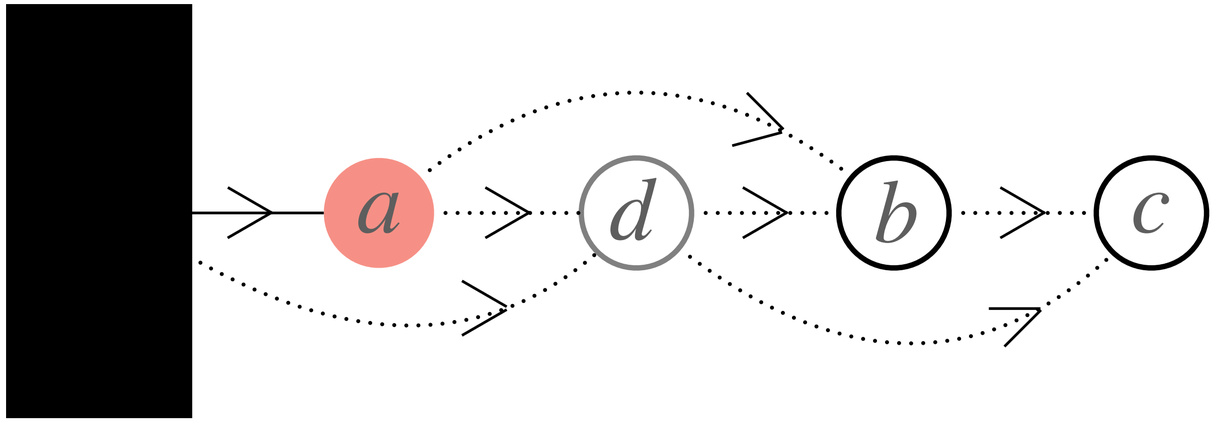

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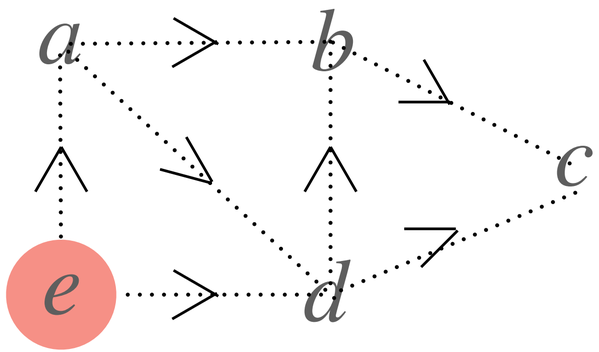

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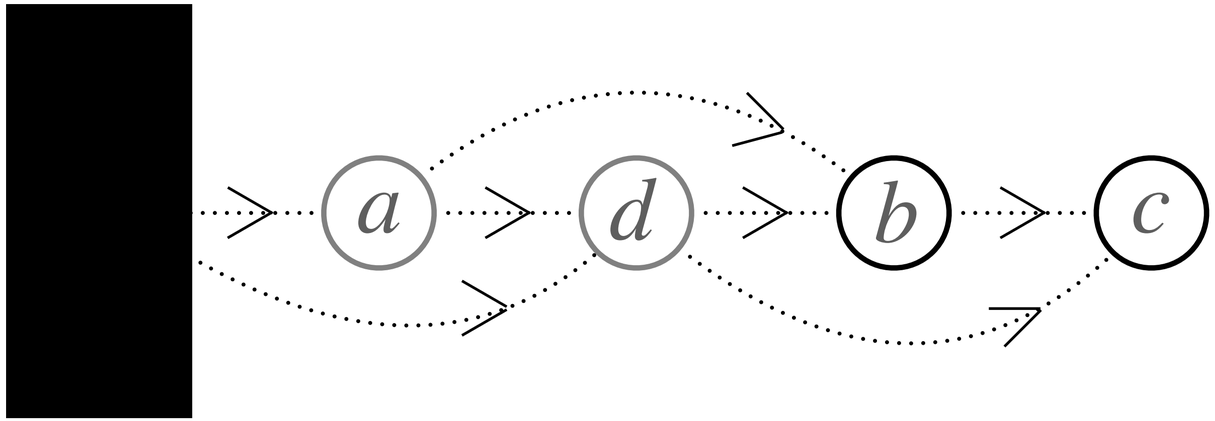

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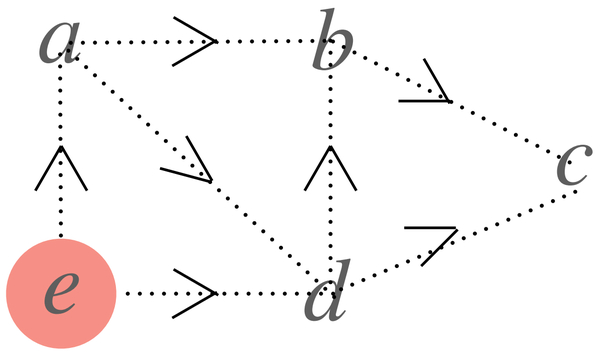

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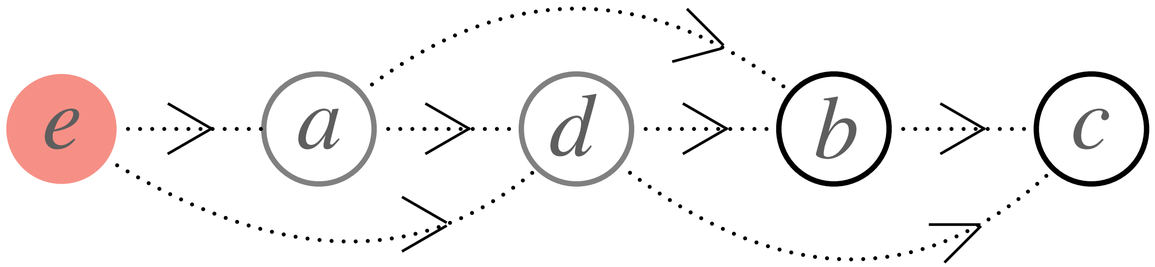

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We actually have an algorithm for “Topological Sort” now!

But is it efficient?

How to get an efficient algorithm of time complexity $O(V + E)$?

Quiz questions:

1. Why does topological sort exist only for acyclic graphs?

2. What are the applications of topological sort?

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1. Application of DFS: Topological Sort.

    1.1 Define "Topological Sort".

    1.2 Algorithm for topological sort.

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Theorem: Let G=(V,E) be a directed acyclic graph. When we run DFS on G, for any edge $(u, v) \in E$, we must have $f(u) > f(v)$.

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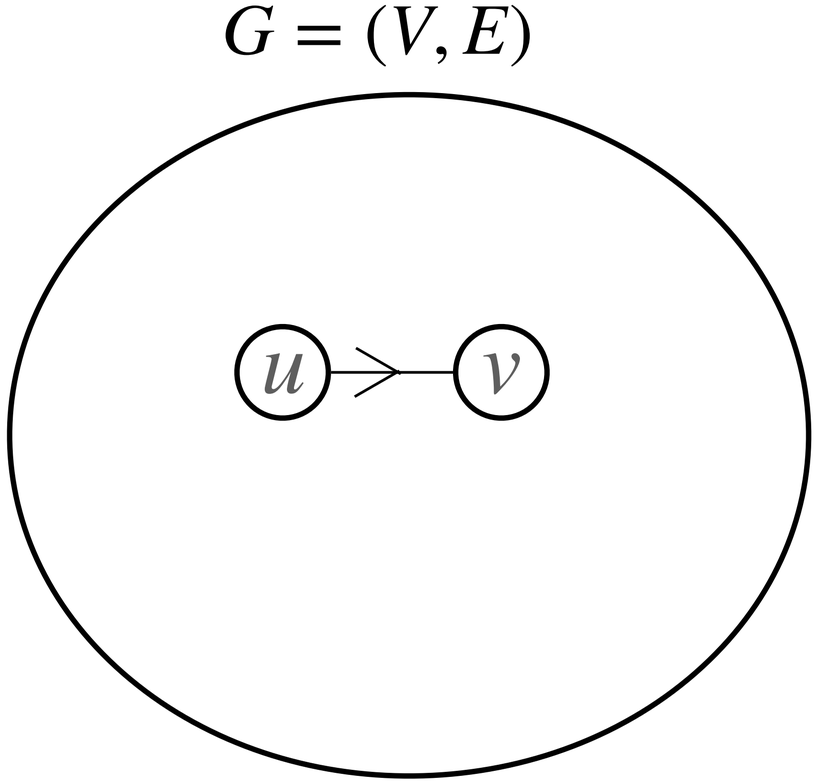

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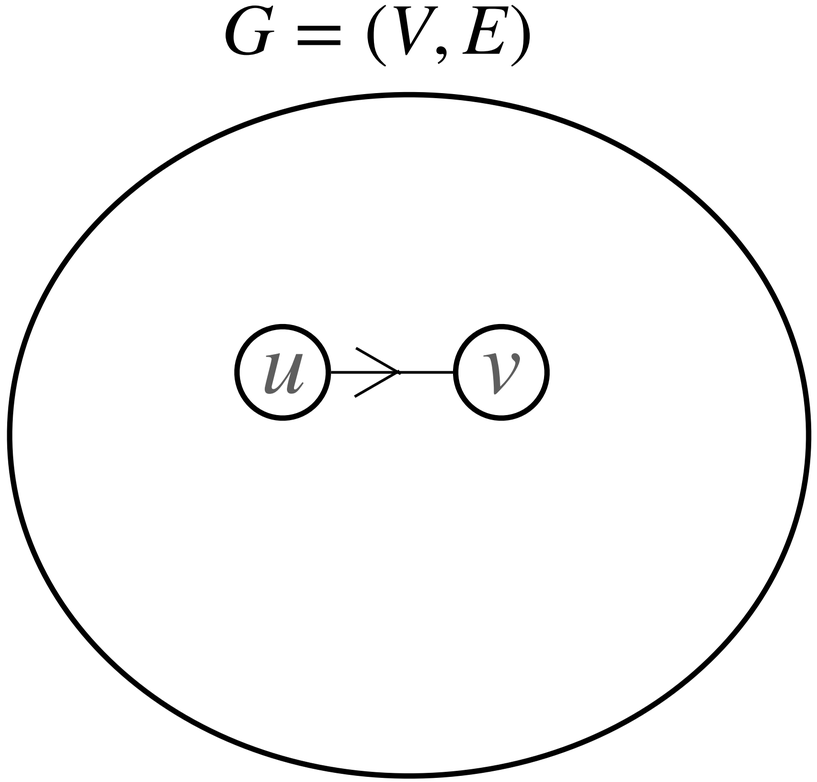

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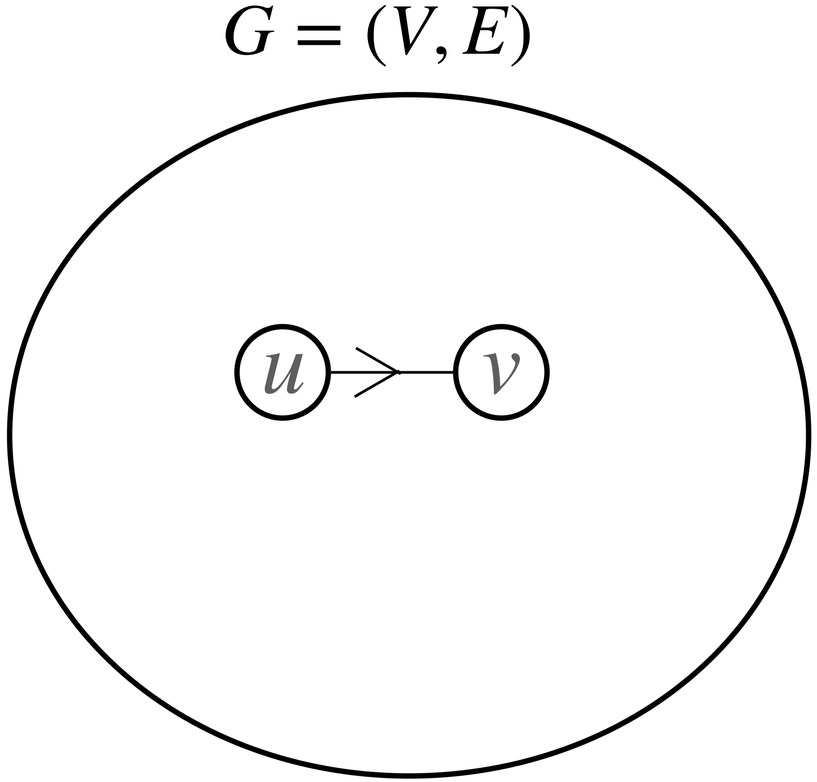

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G is acyclic → v cannot reach u.

v will finish before DFS discovers u.

$f(v) < d(u) < f(u)$.

Topological Sort

We can sort edges by their finish time (which is a linear ordering)!

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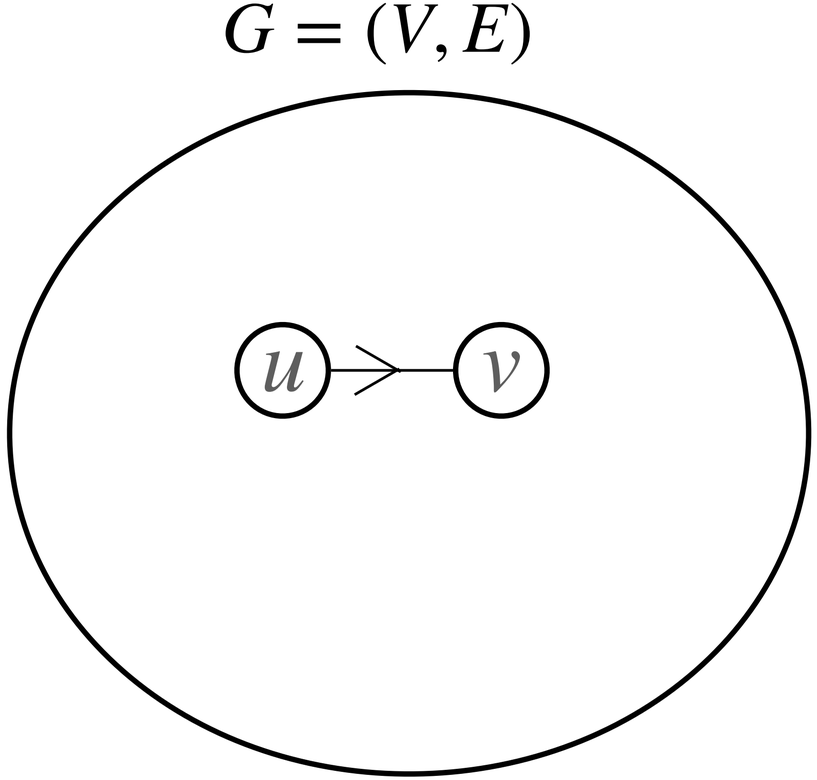

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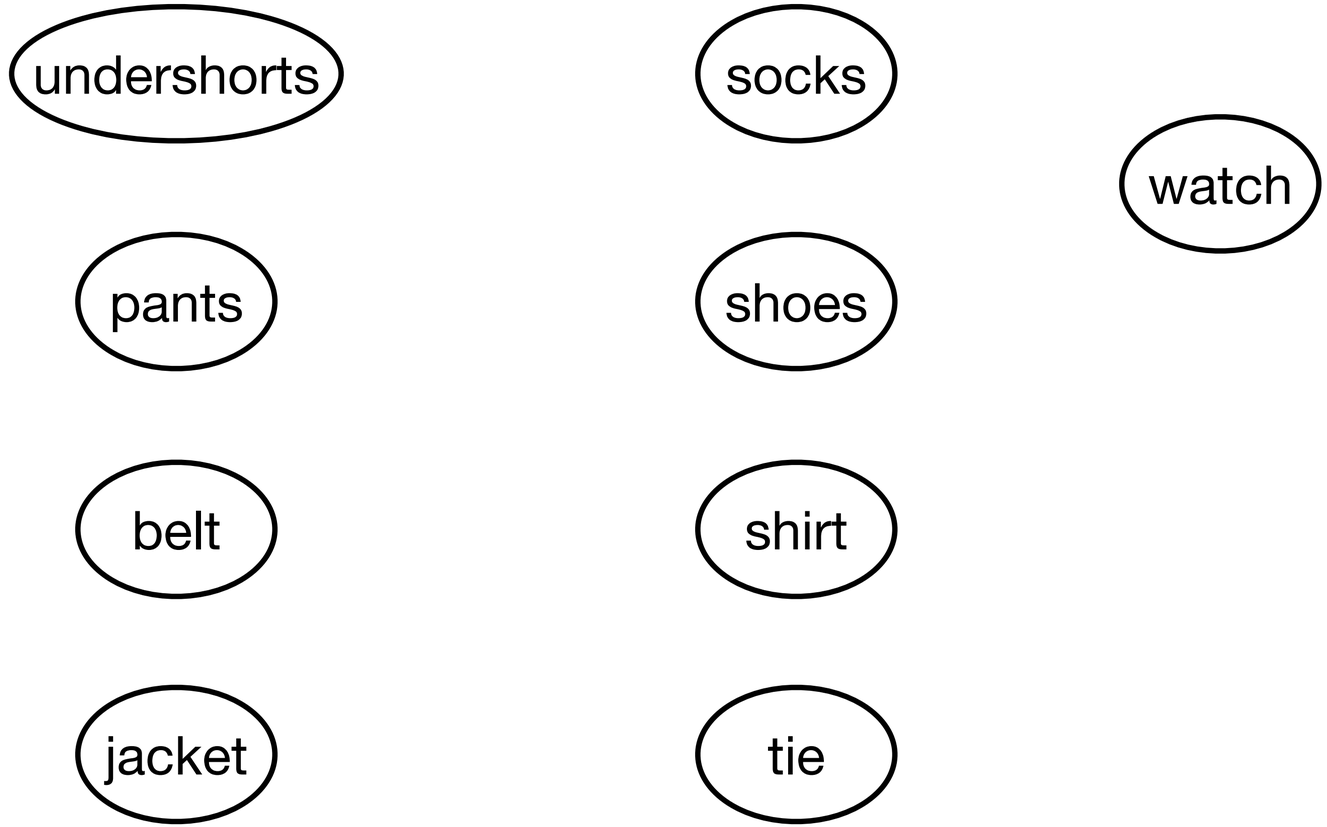

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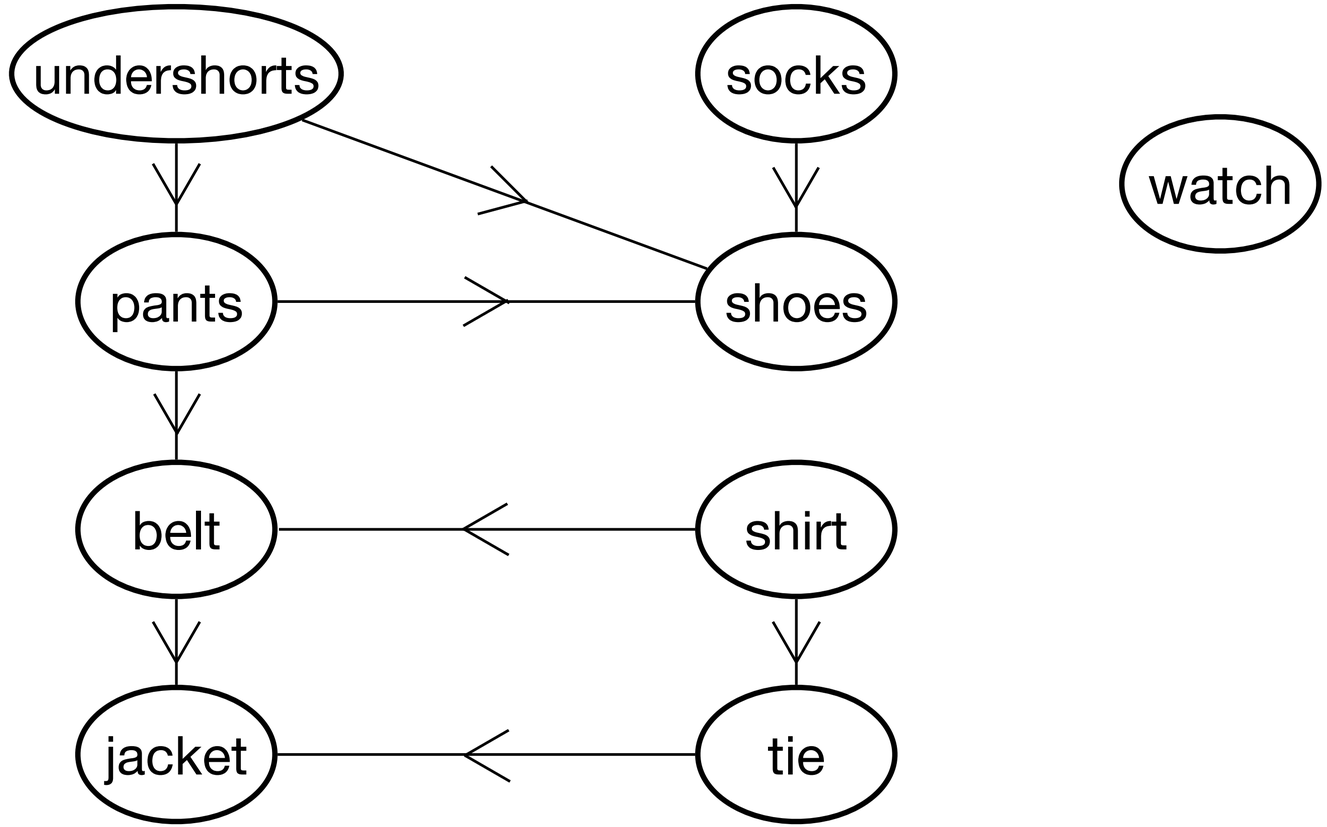

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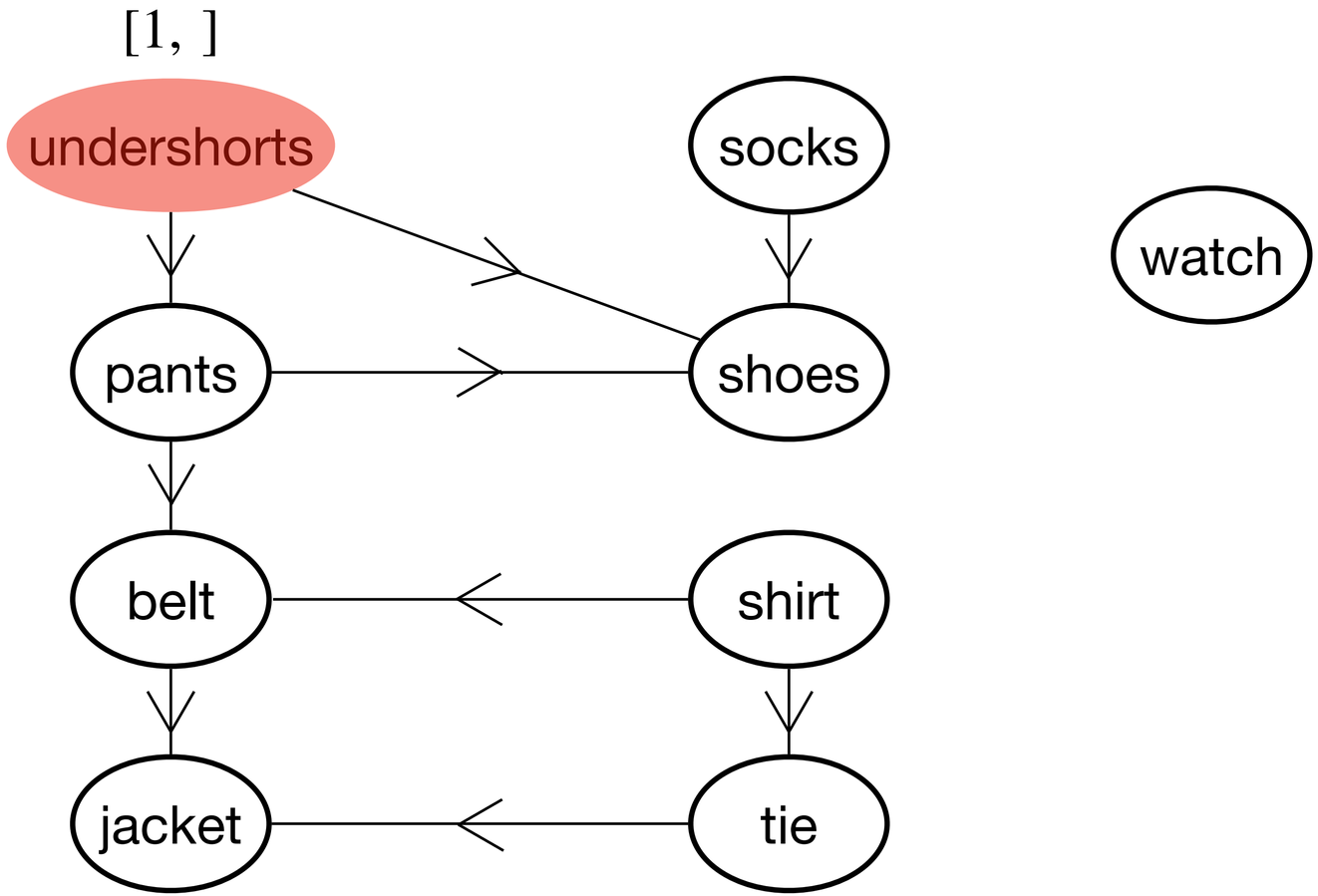

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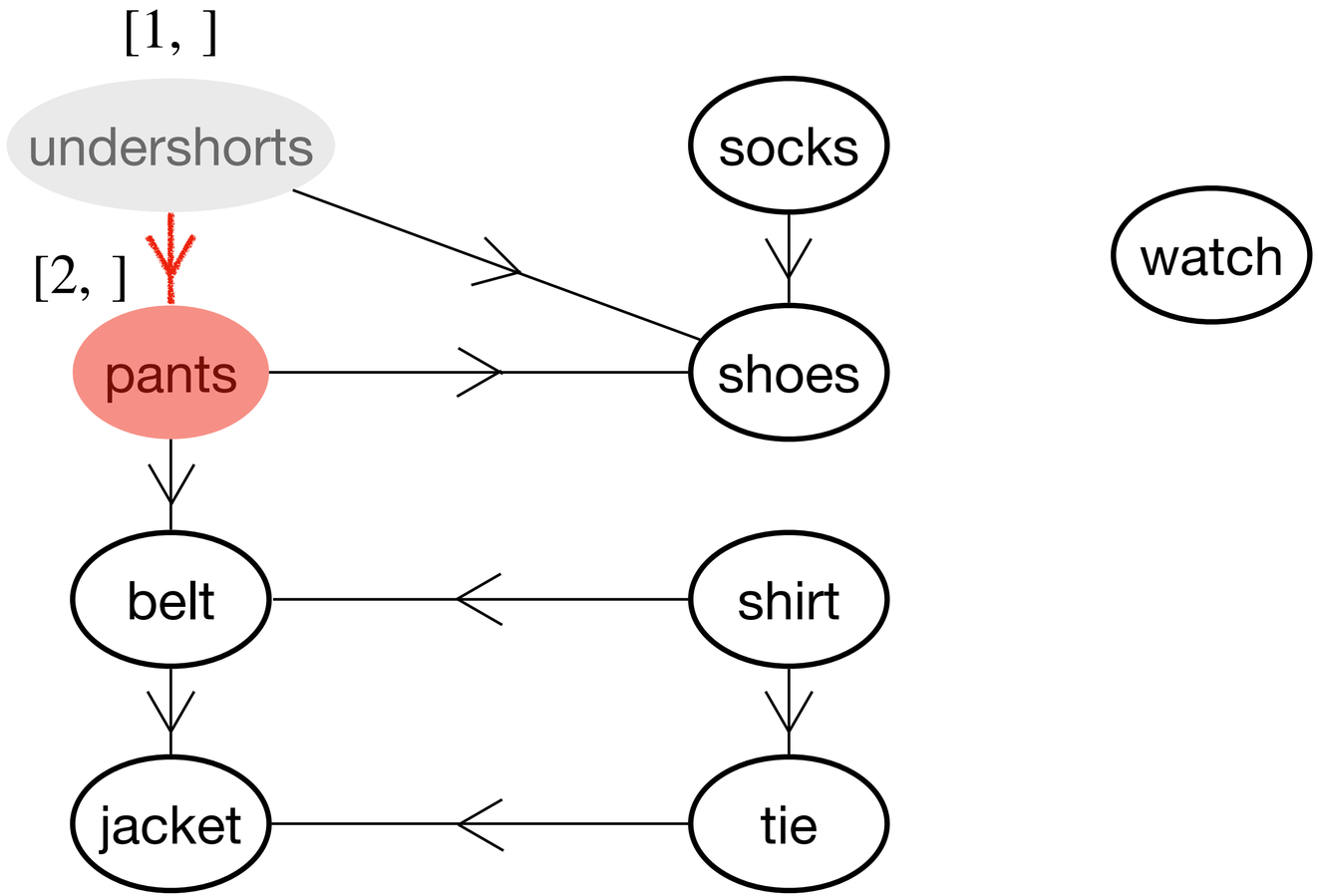

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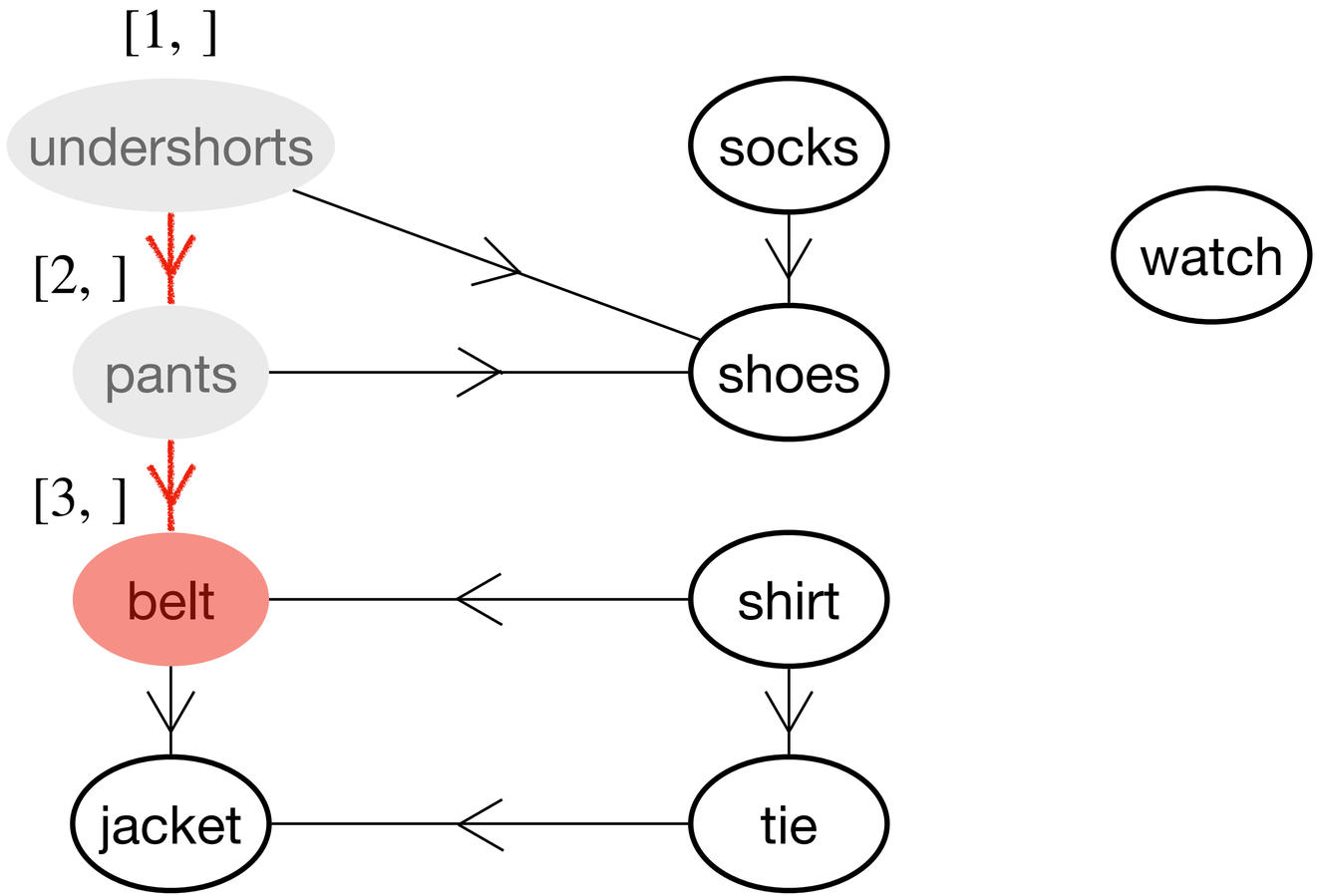

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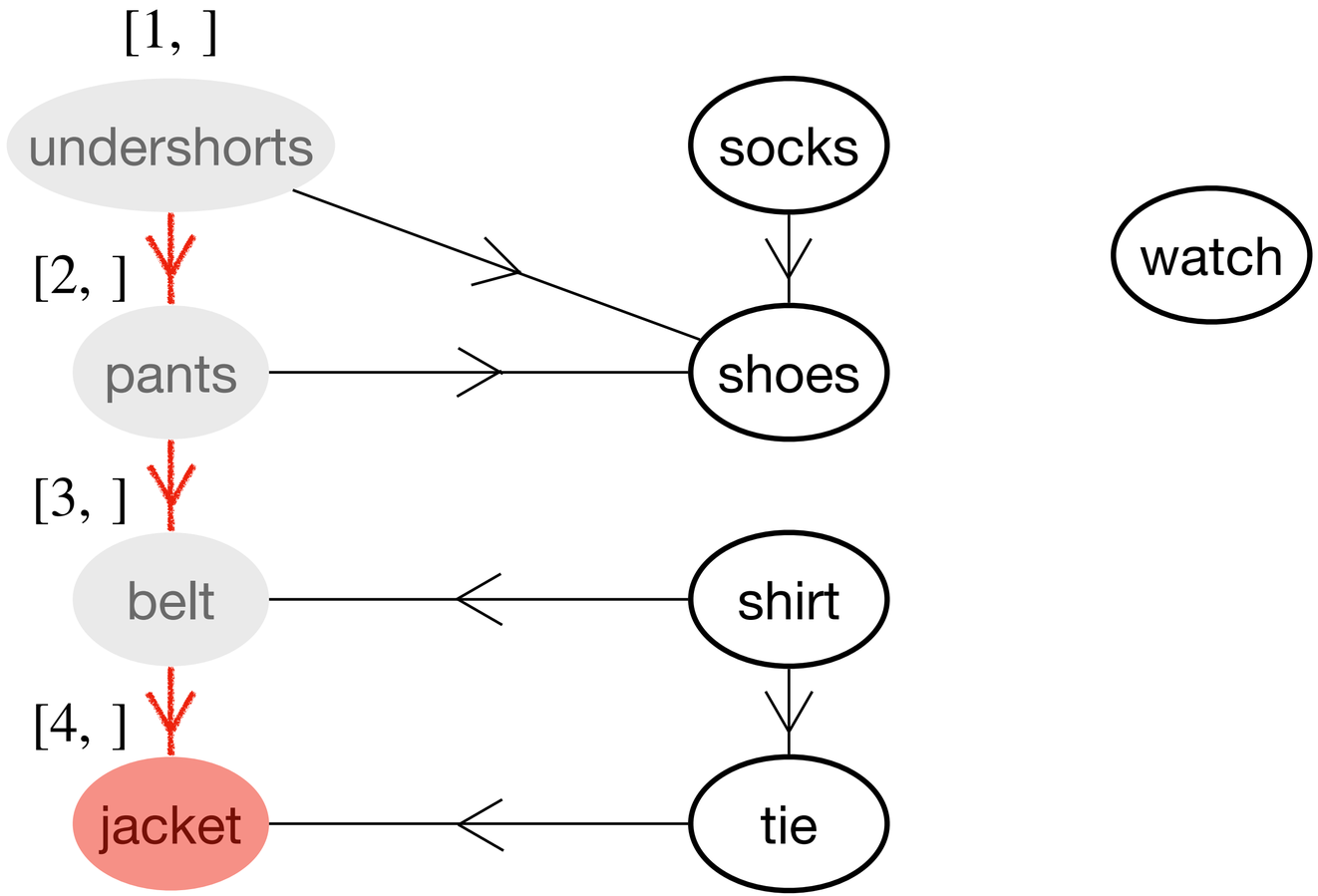

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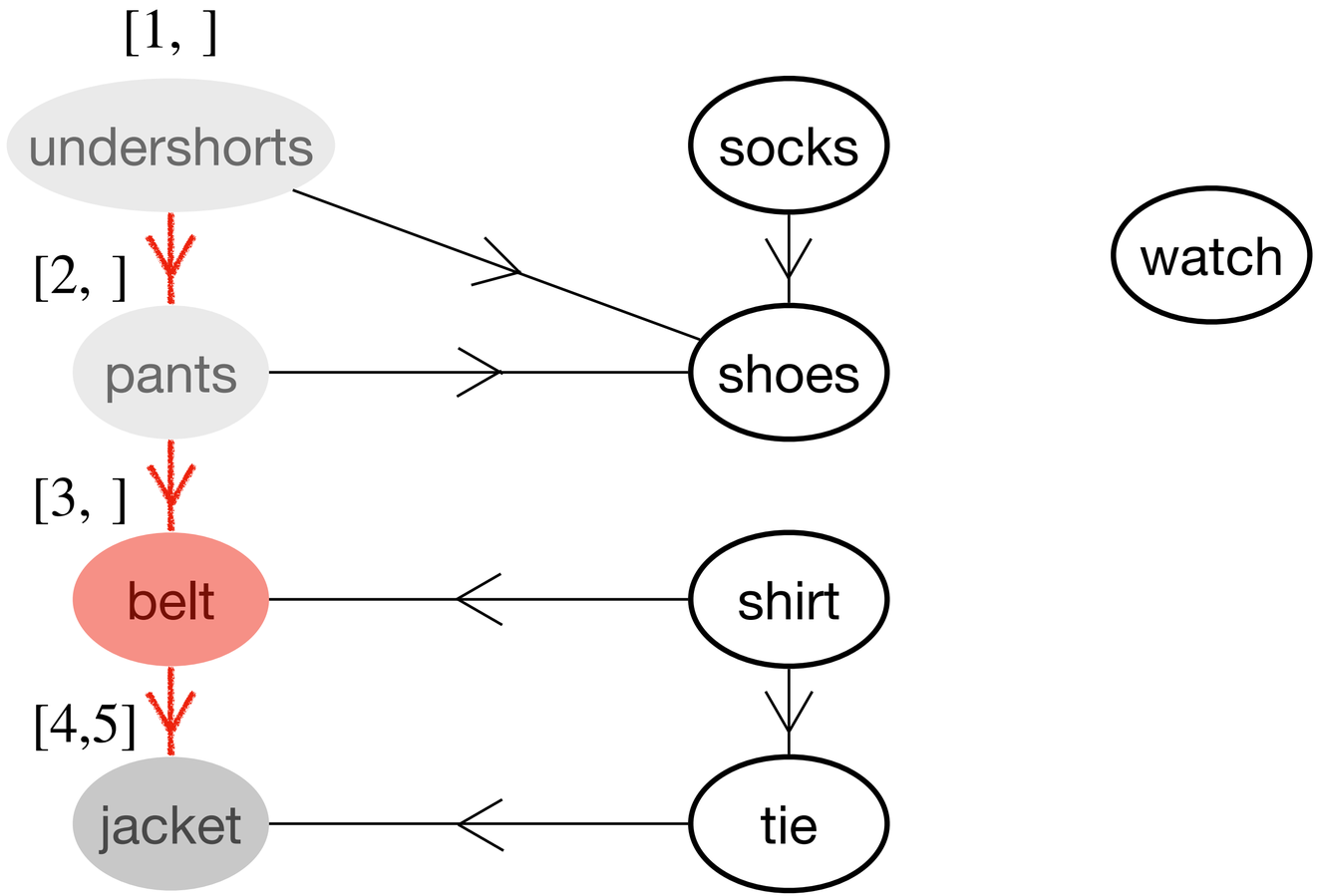

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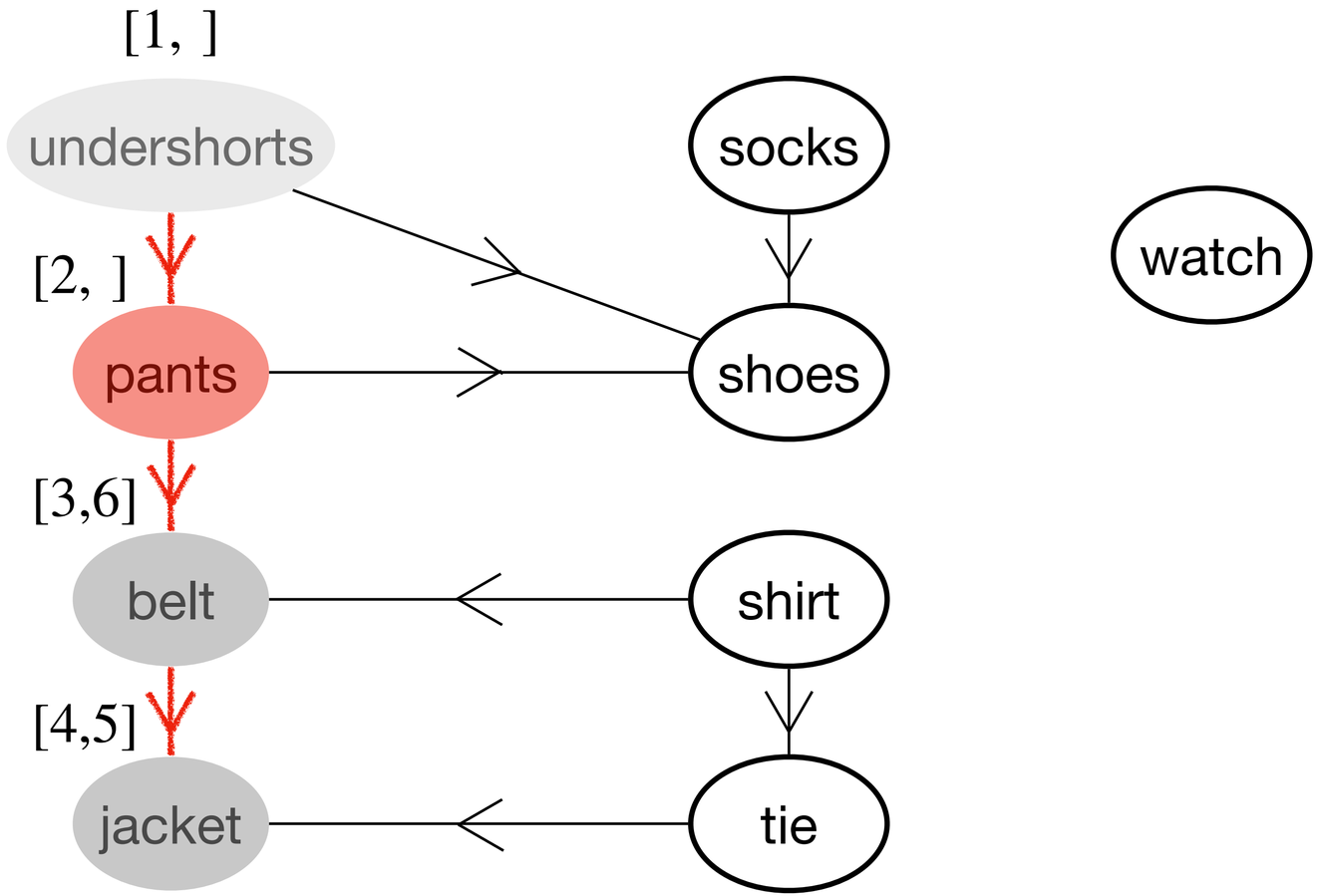

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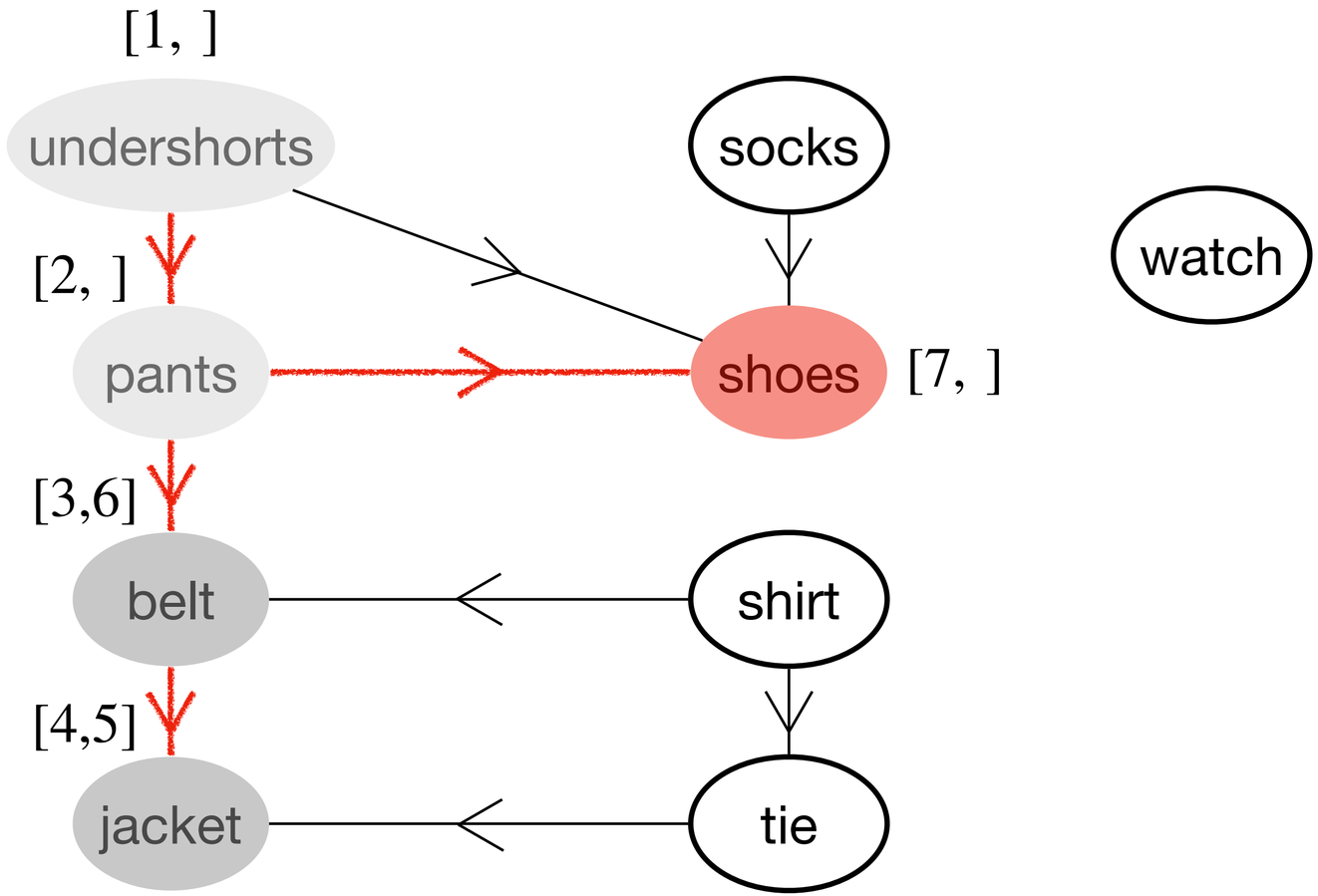

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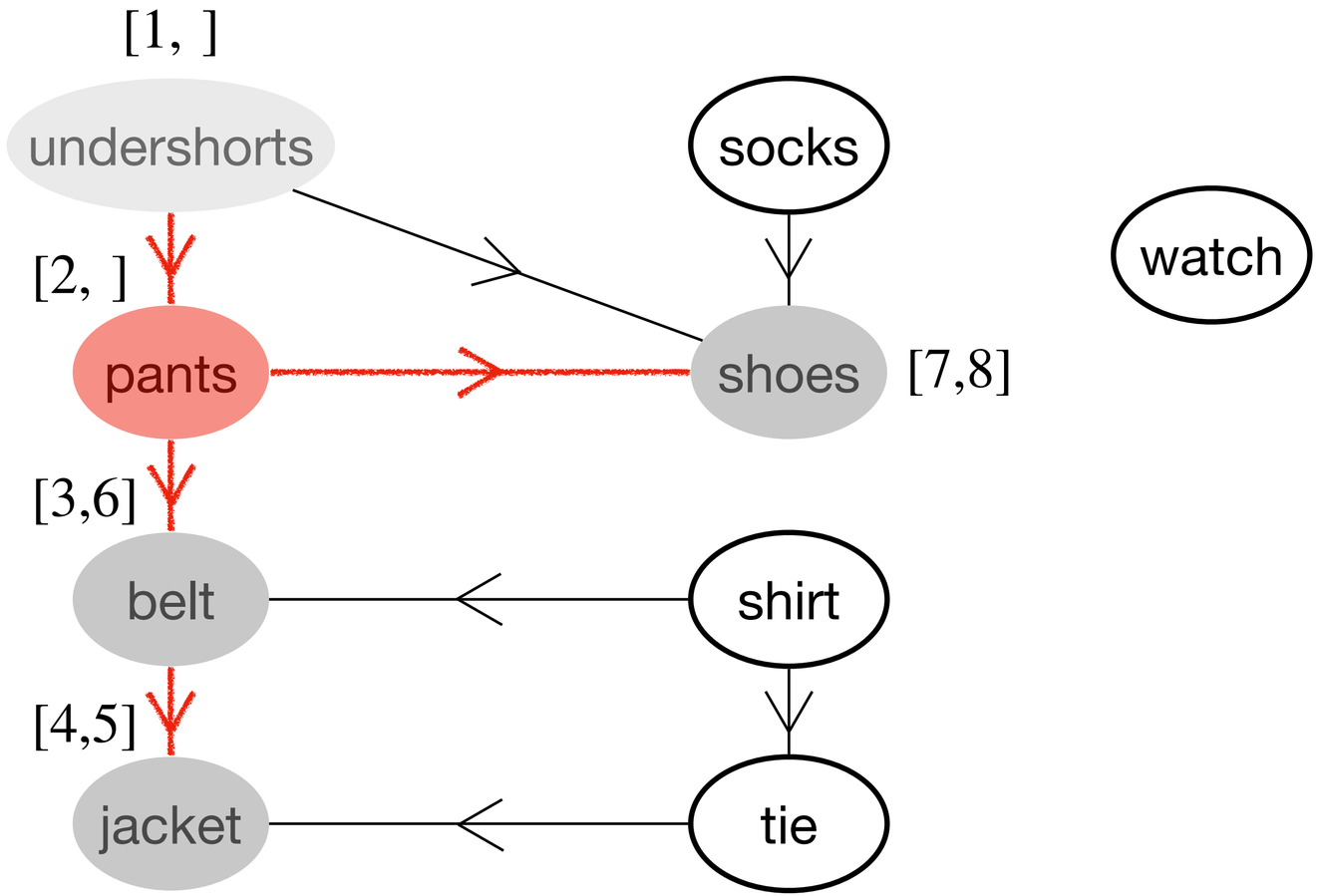

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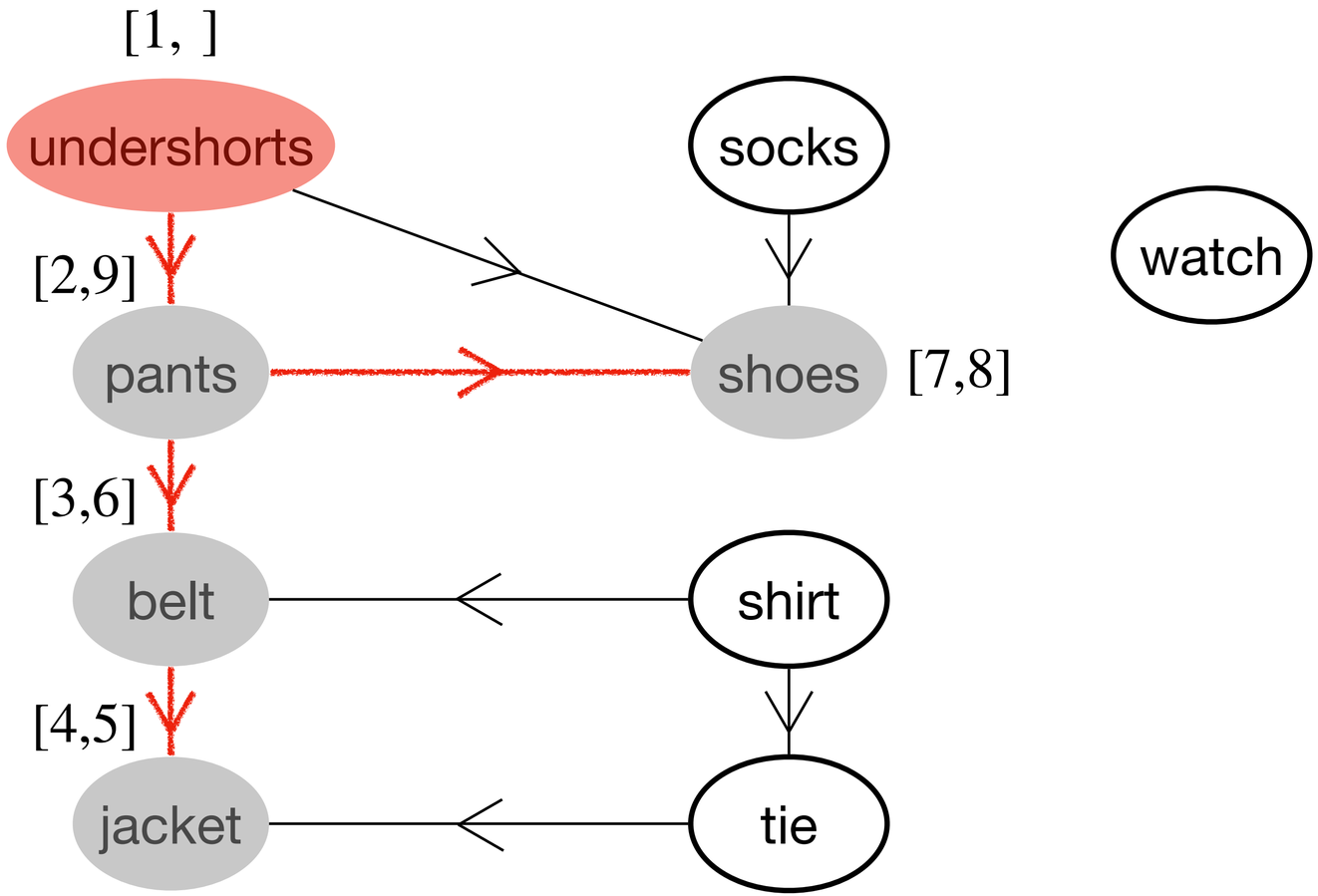

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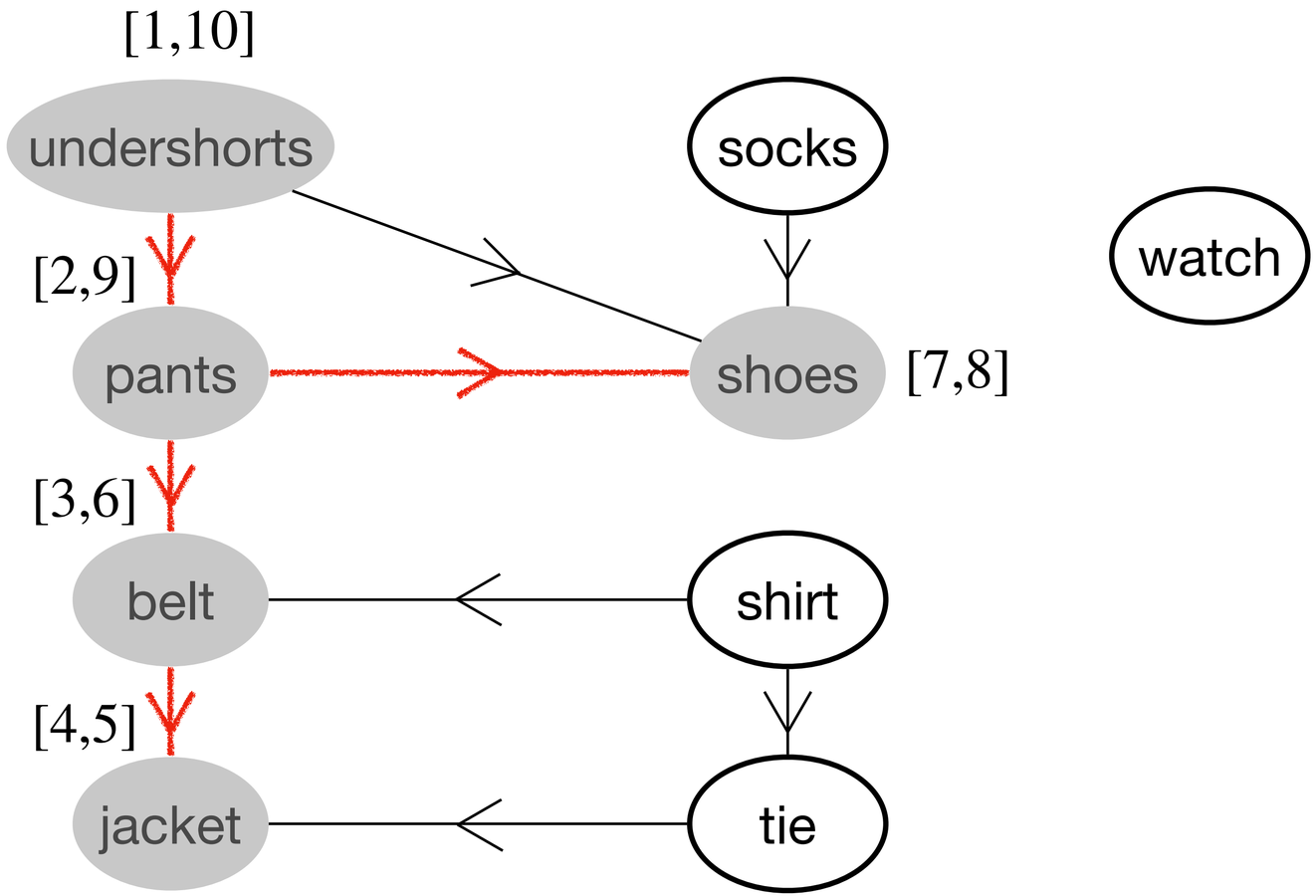

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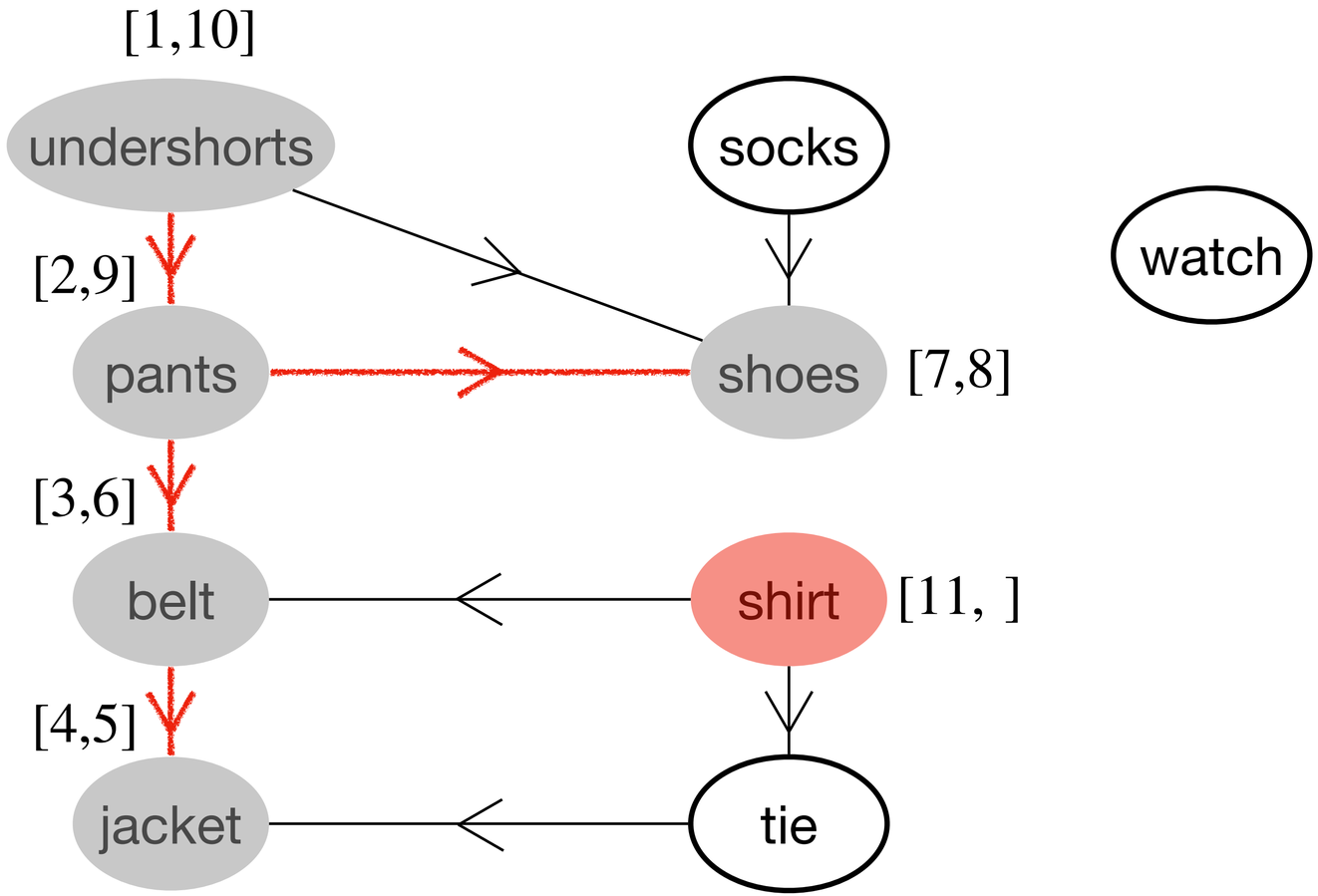

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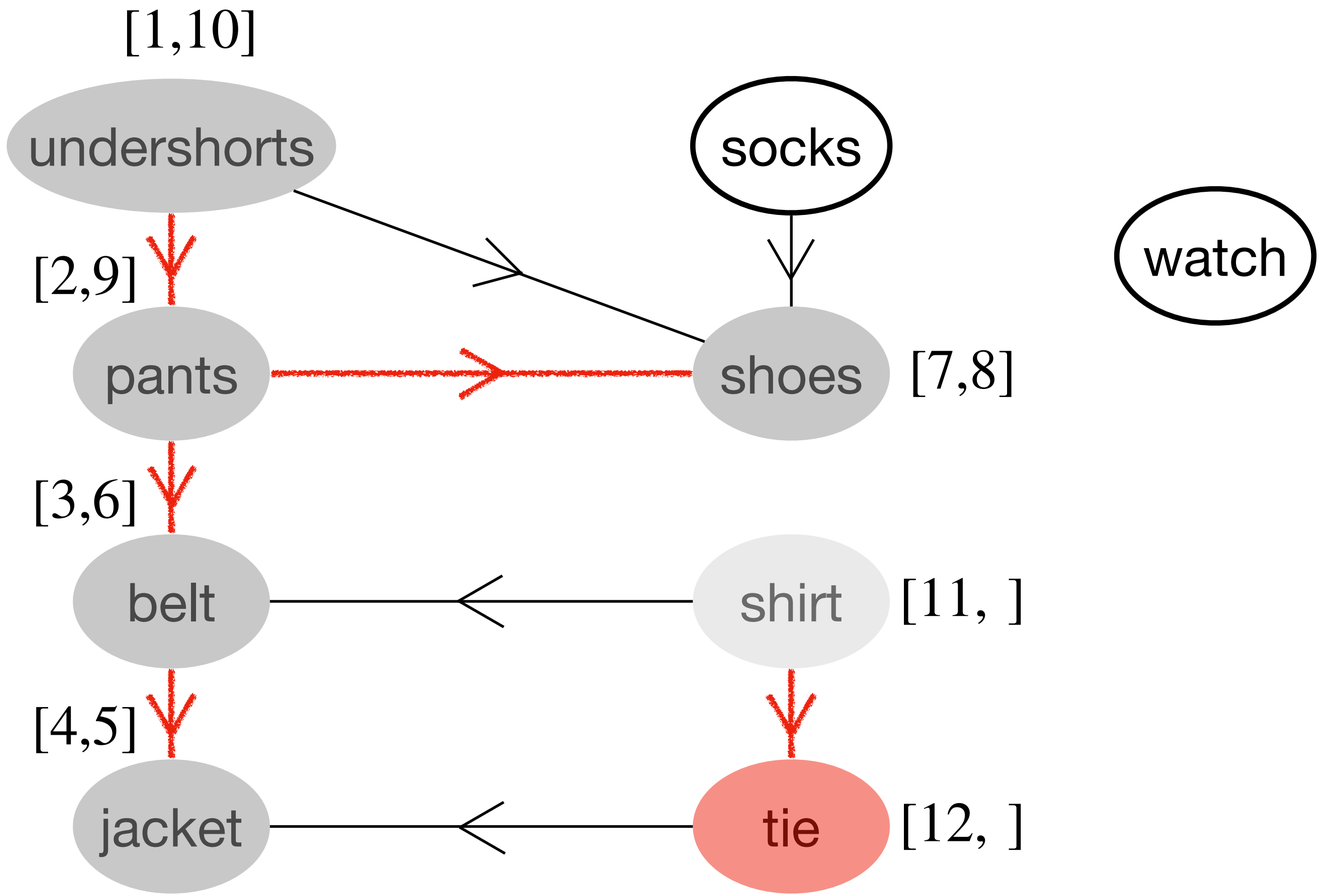

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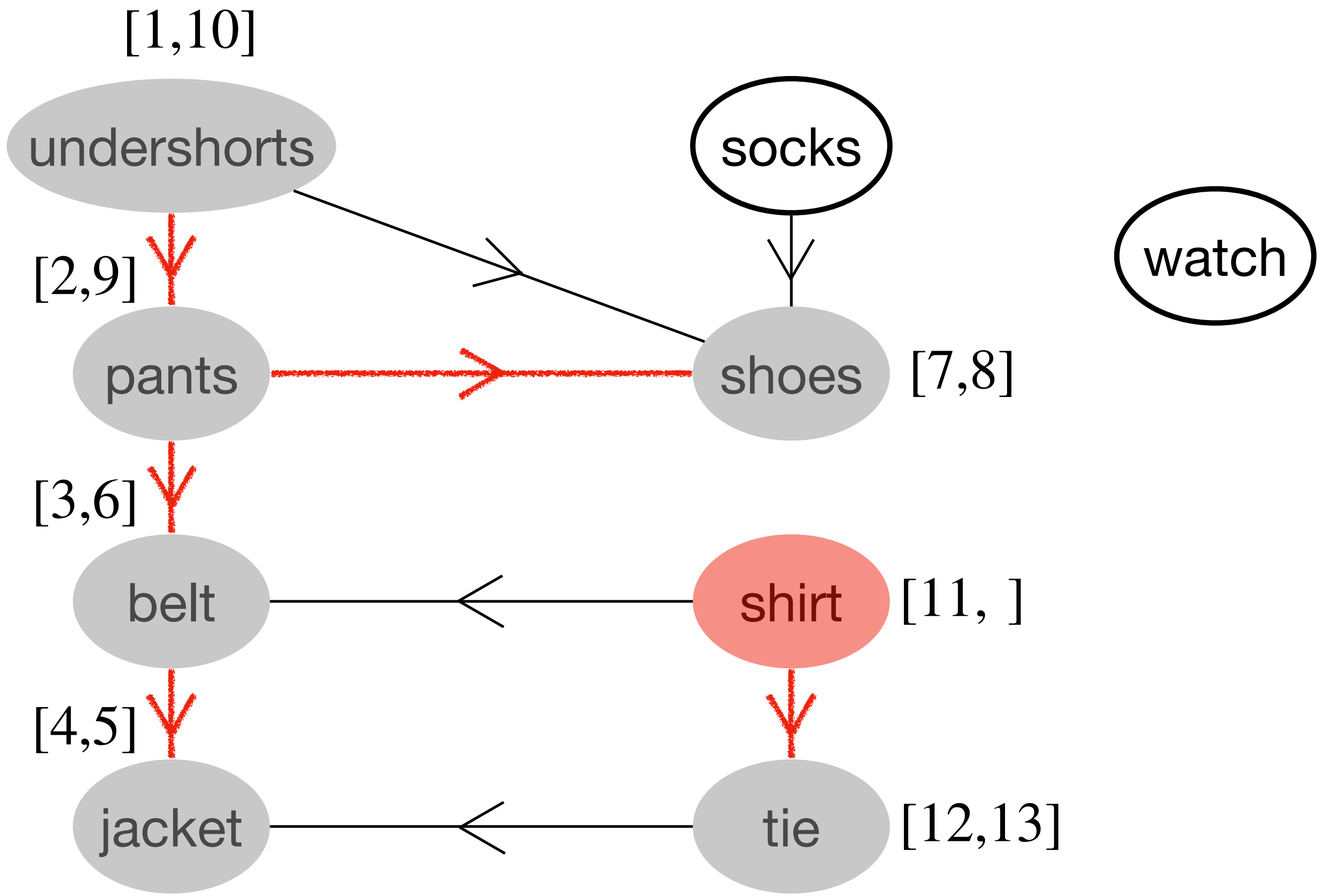

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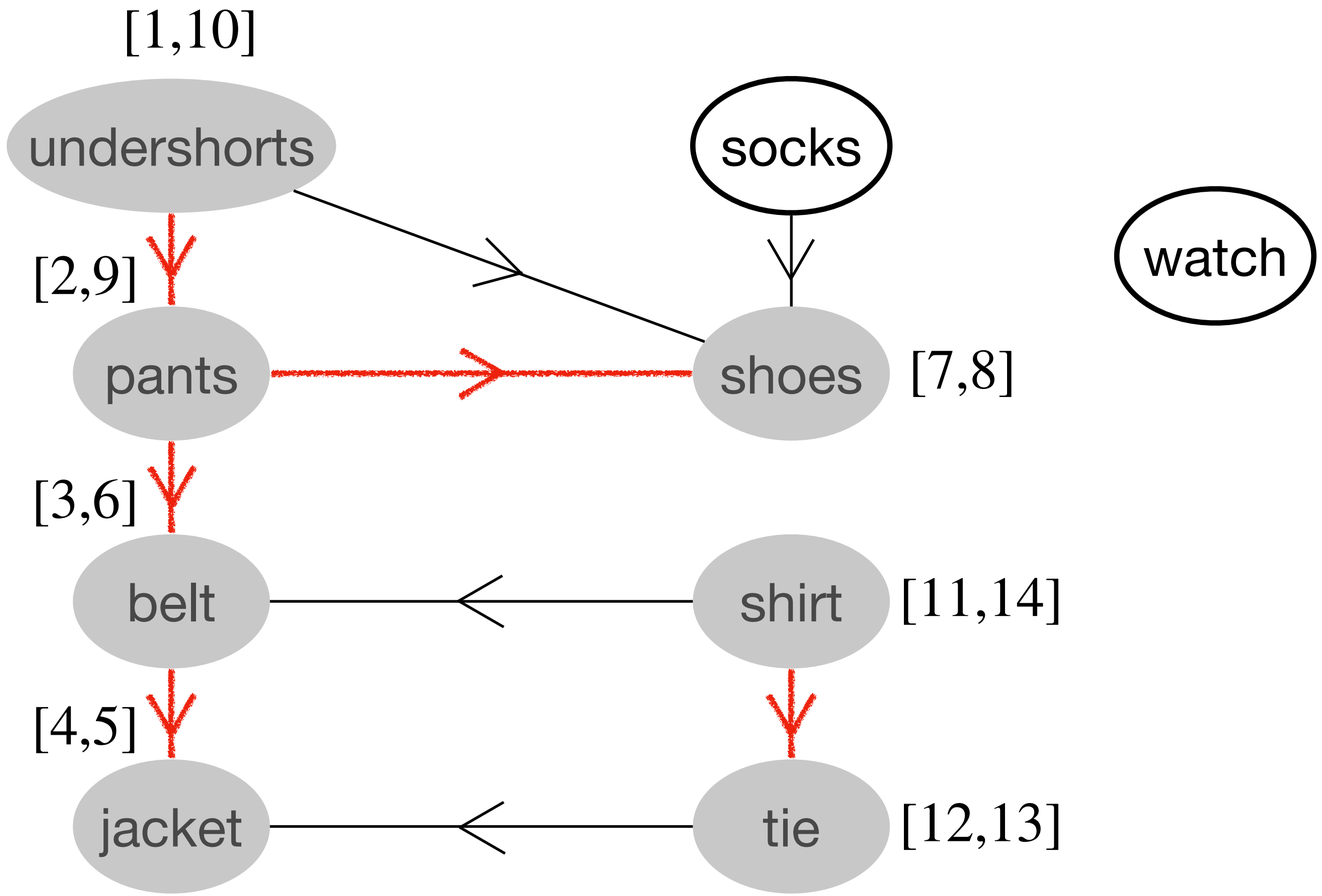

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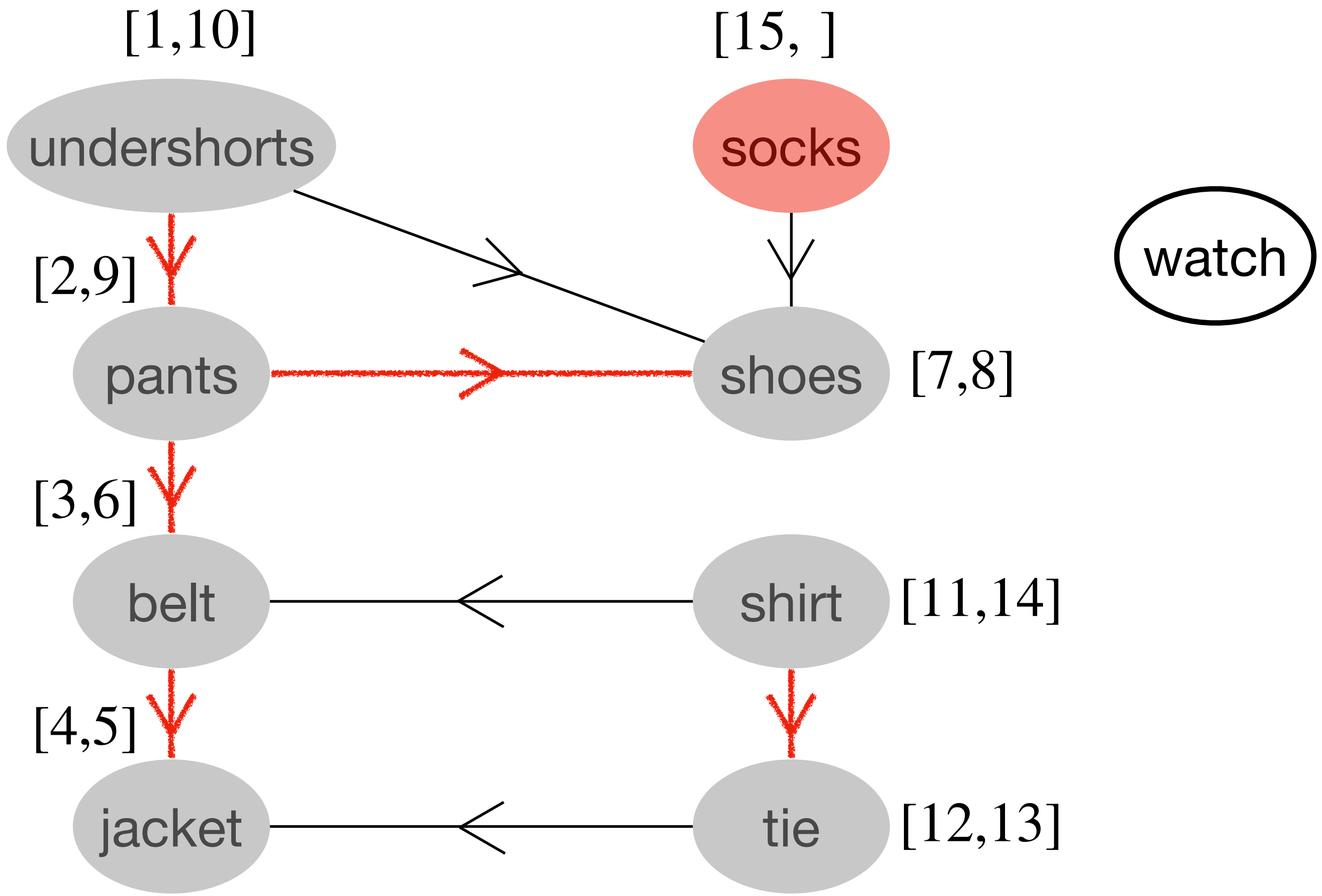

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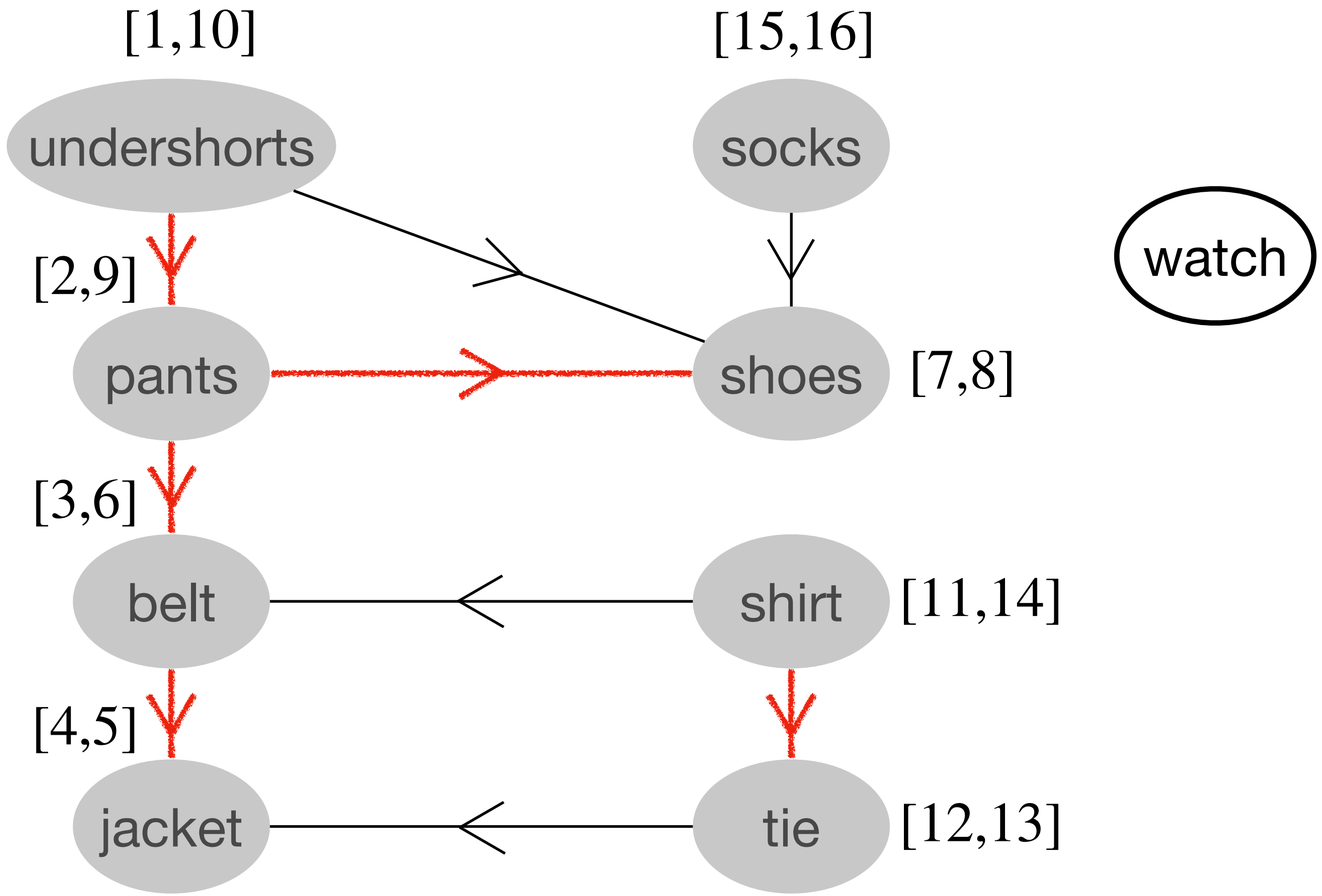

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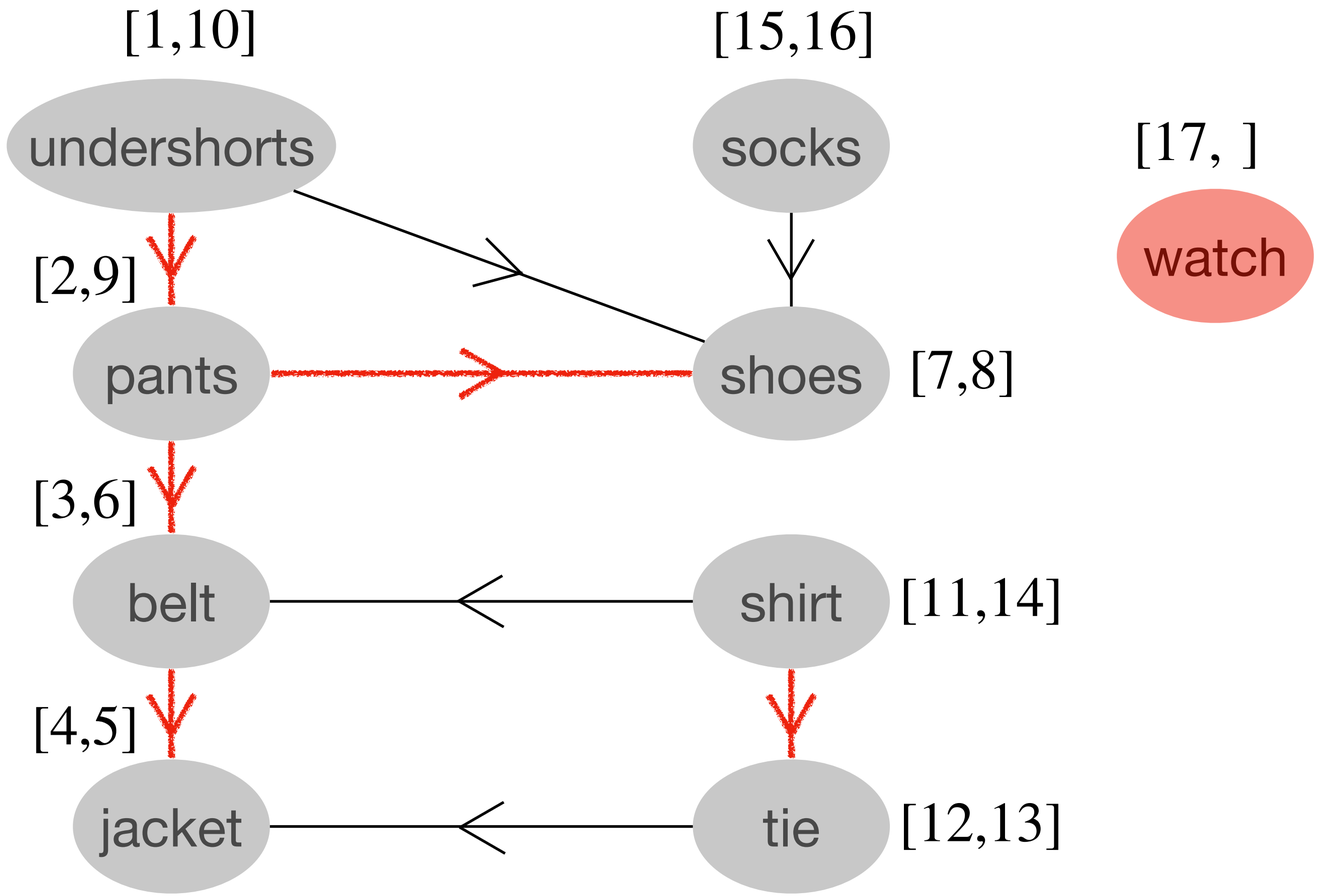

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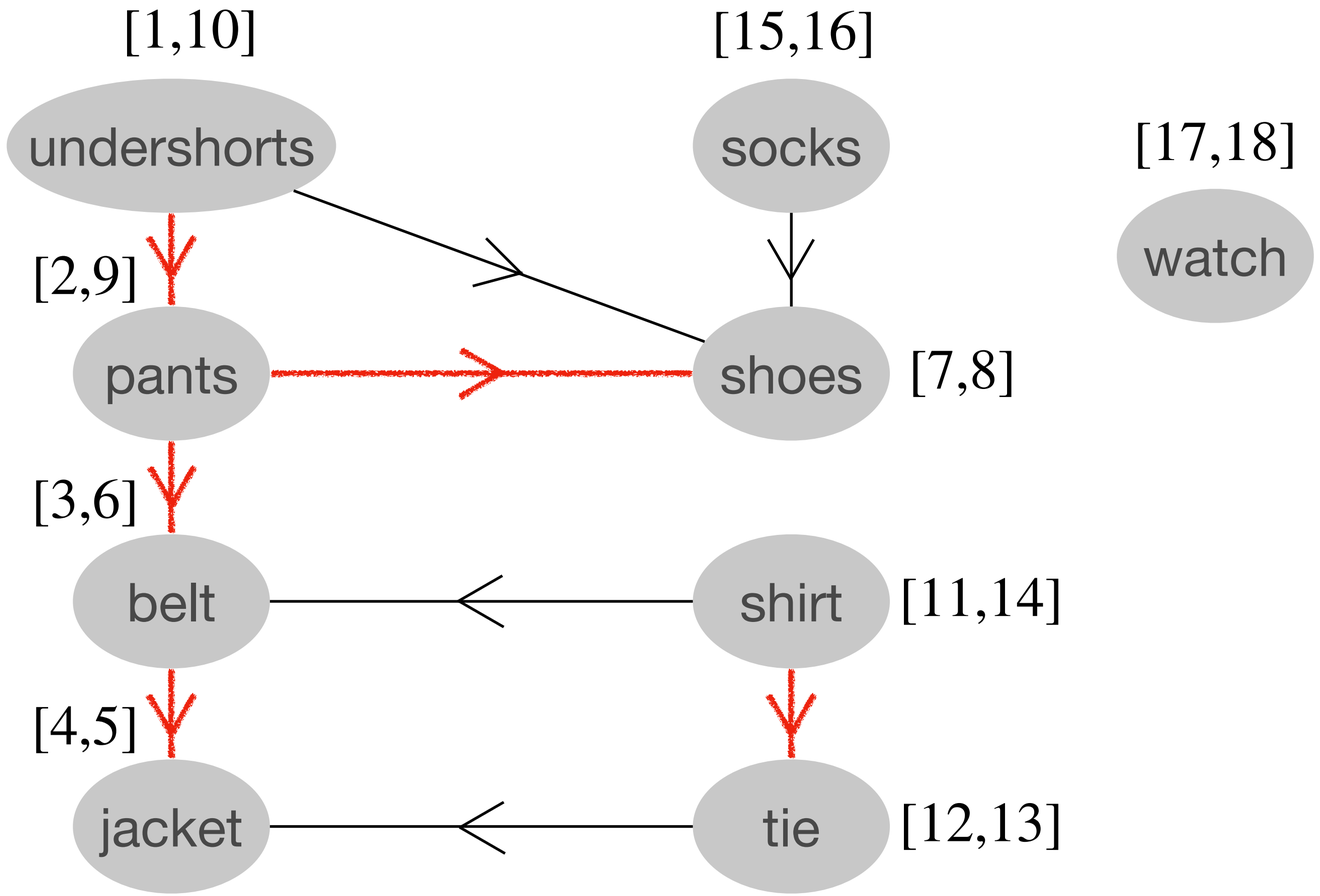

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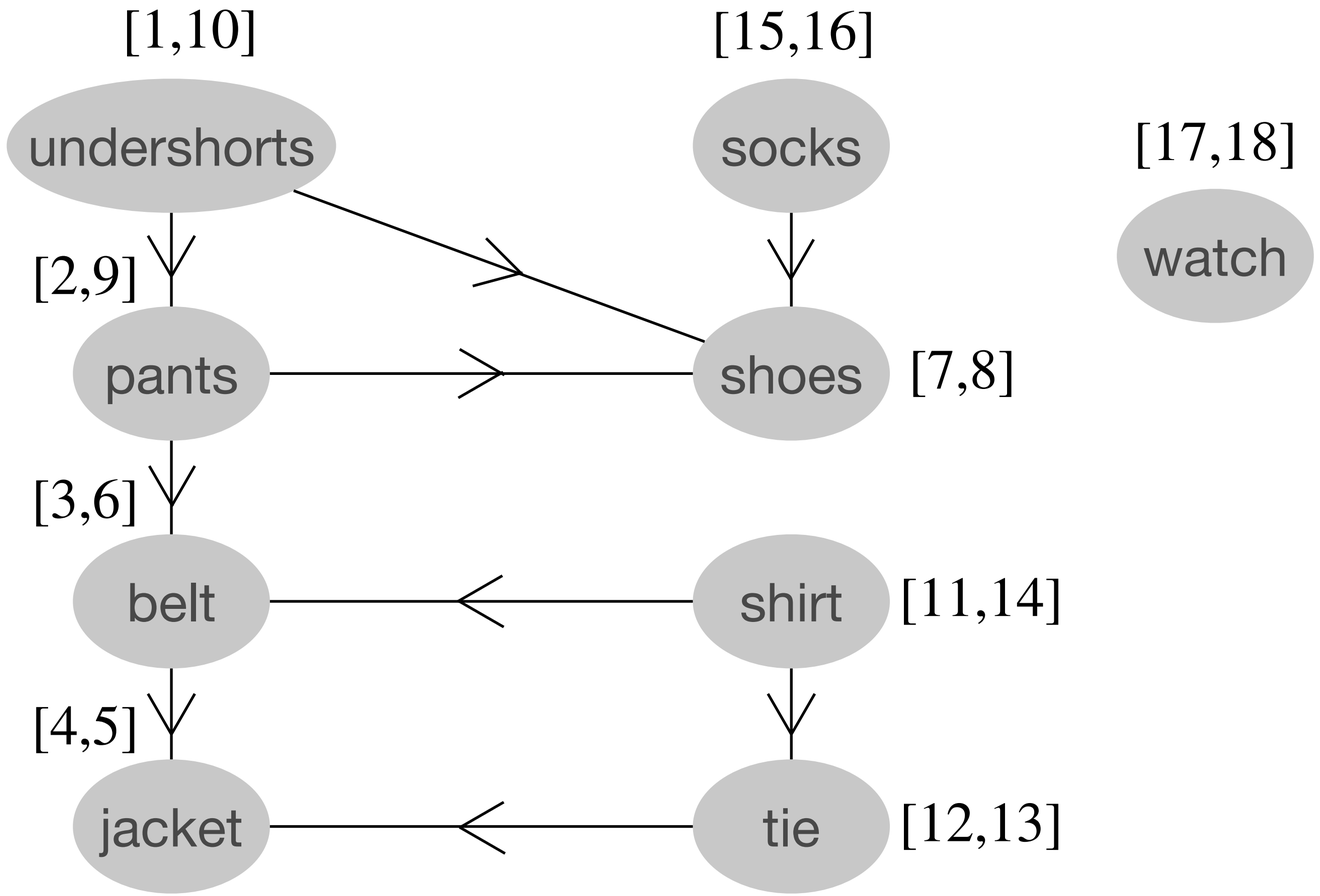

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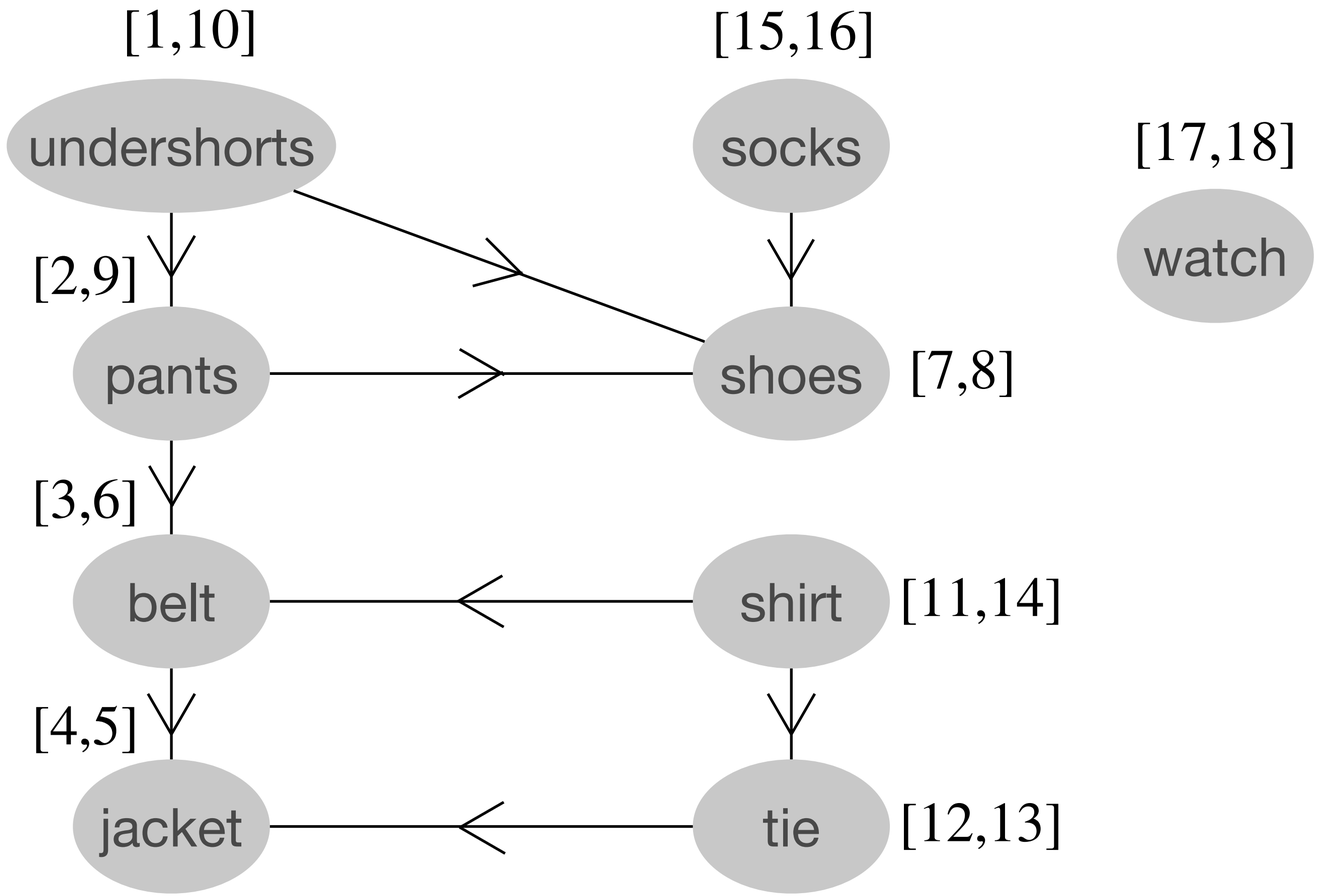

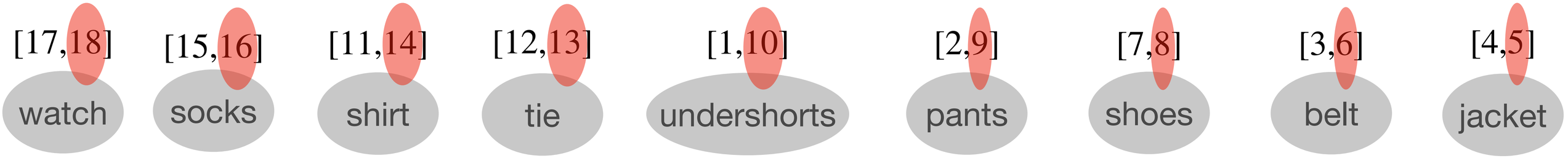

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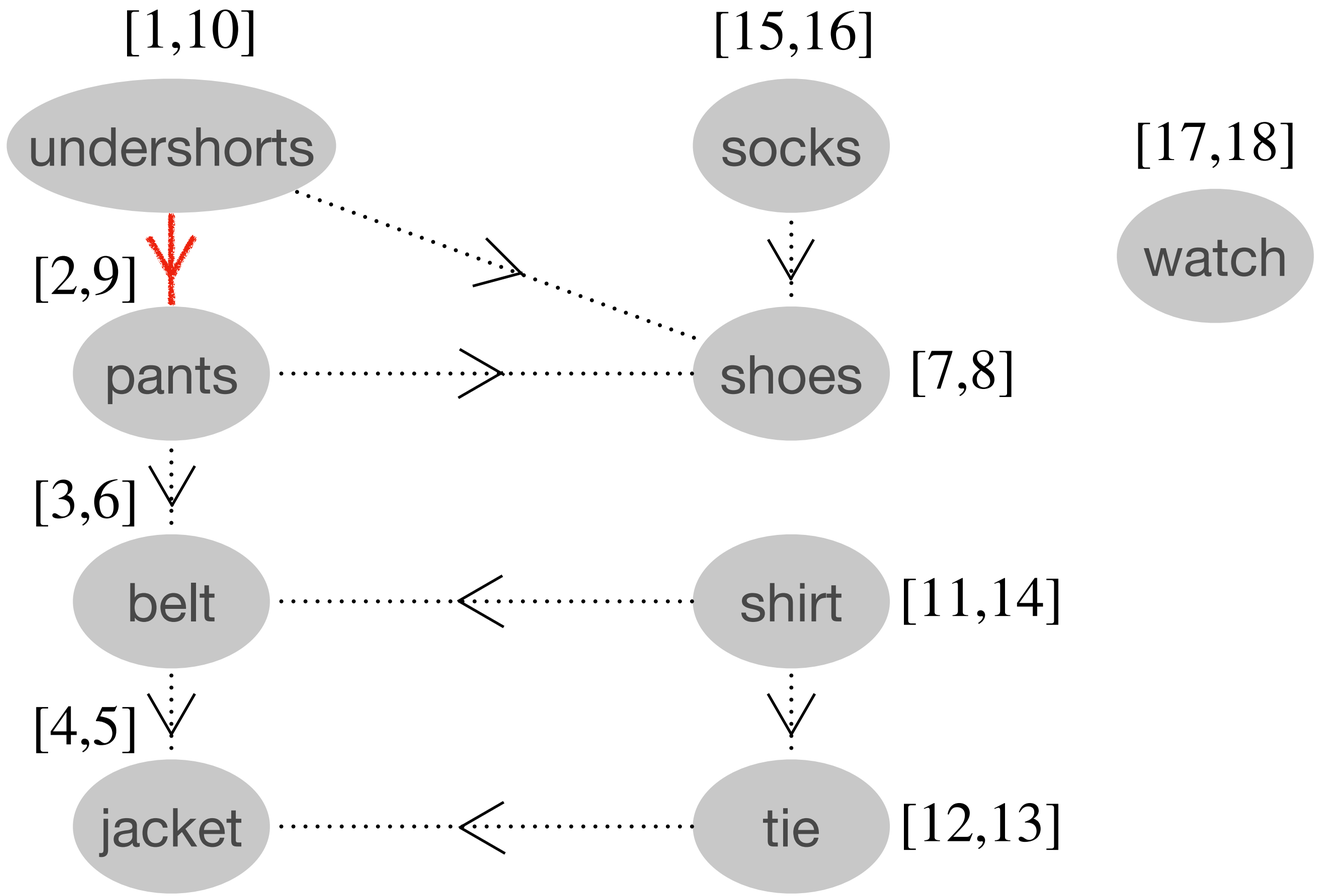

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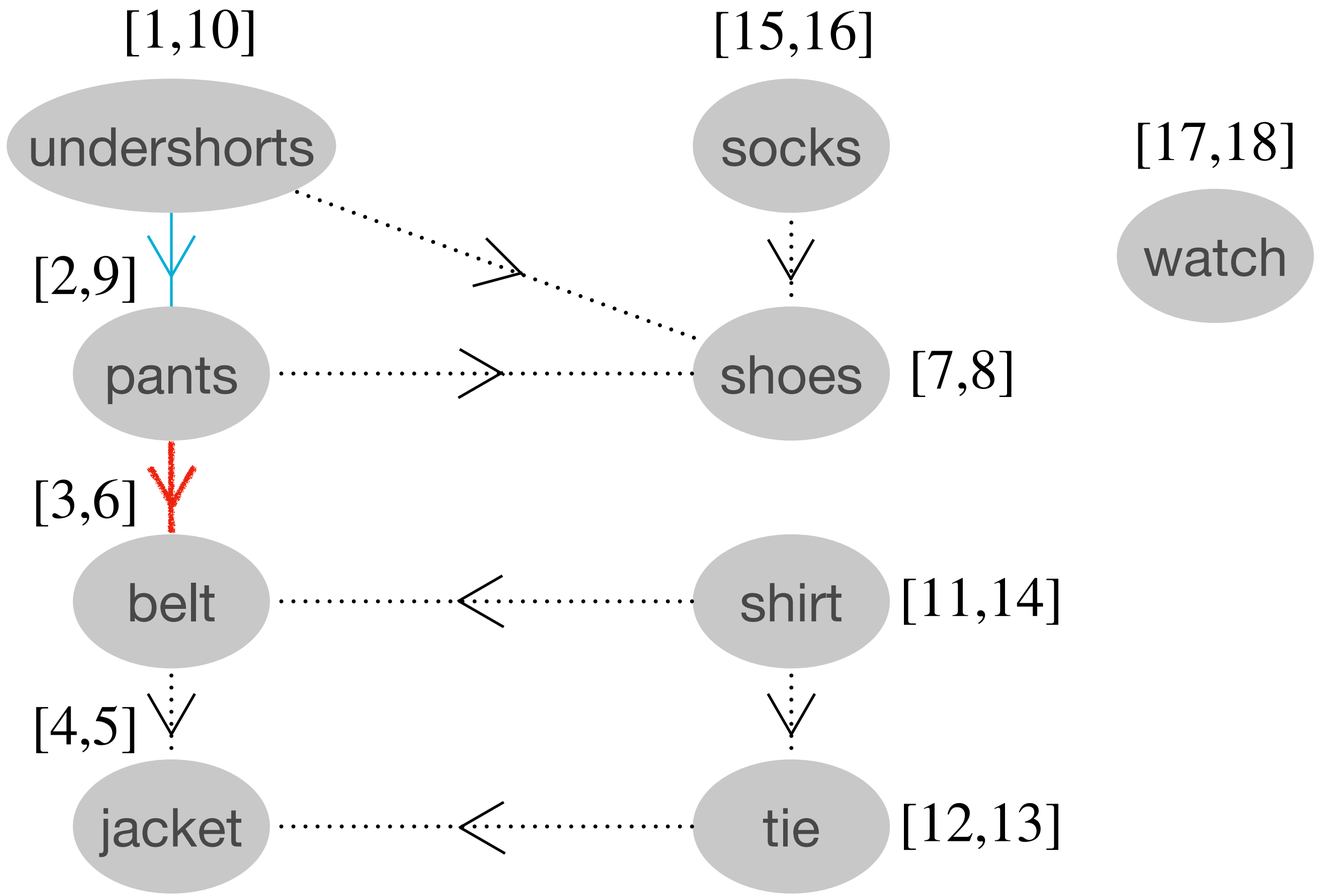

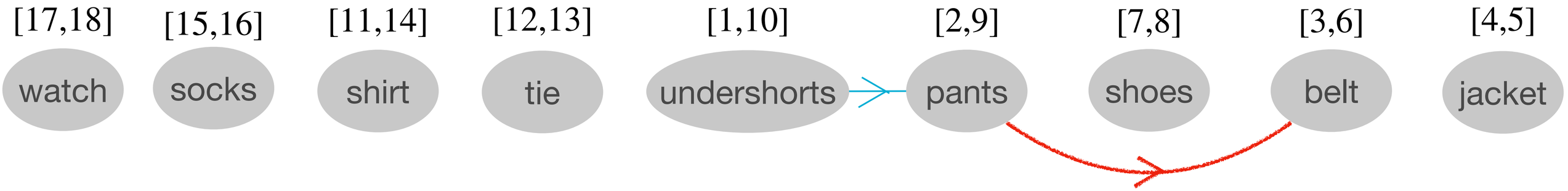

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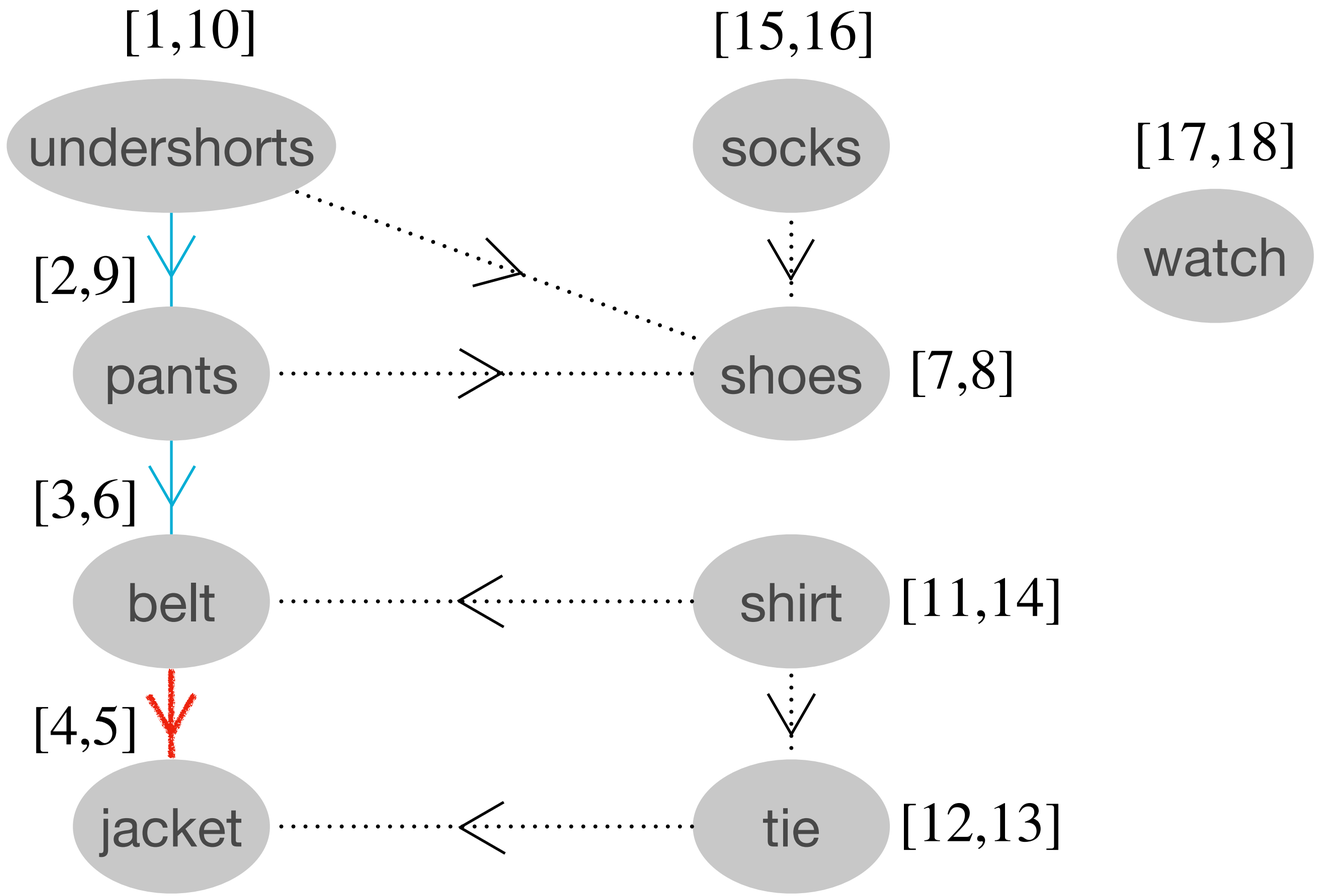

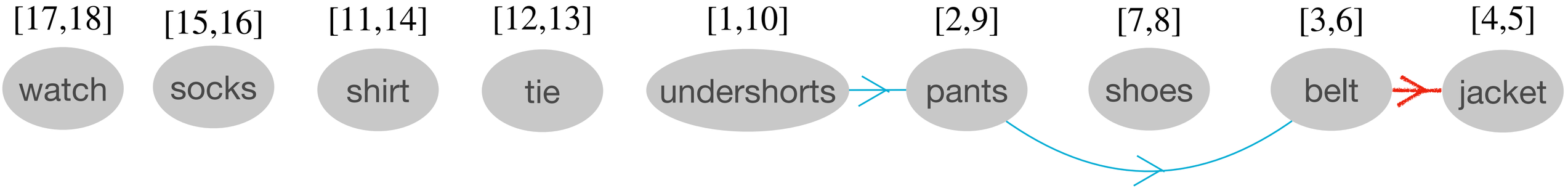

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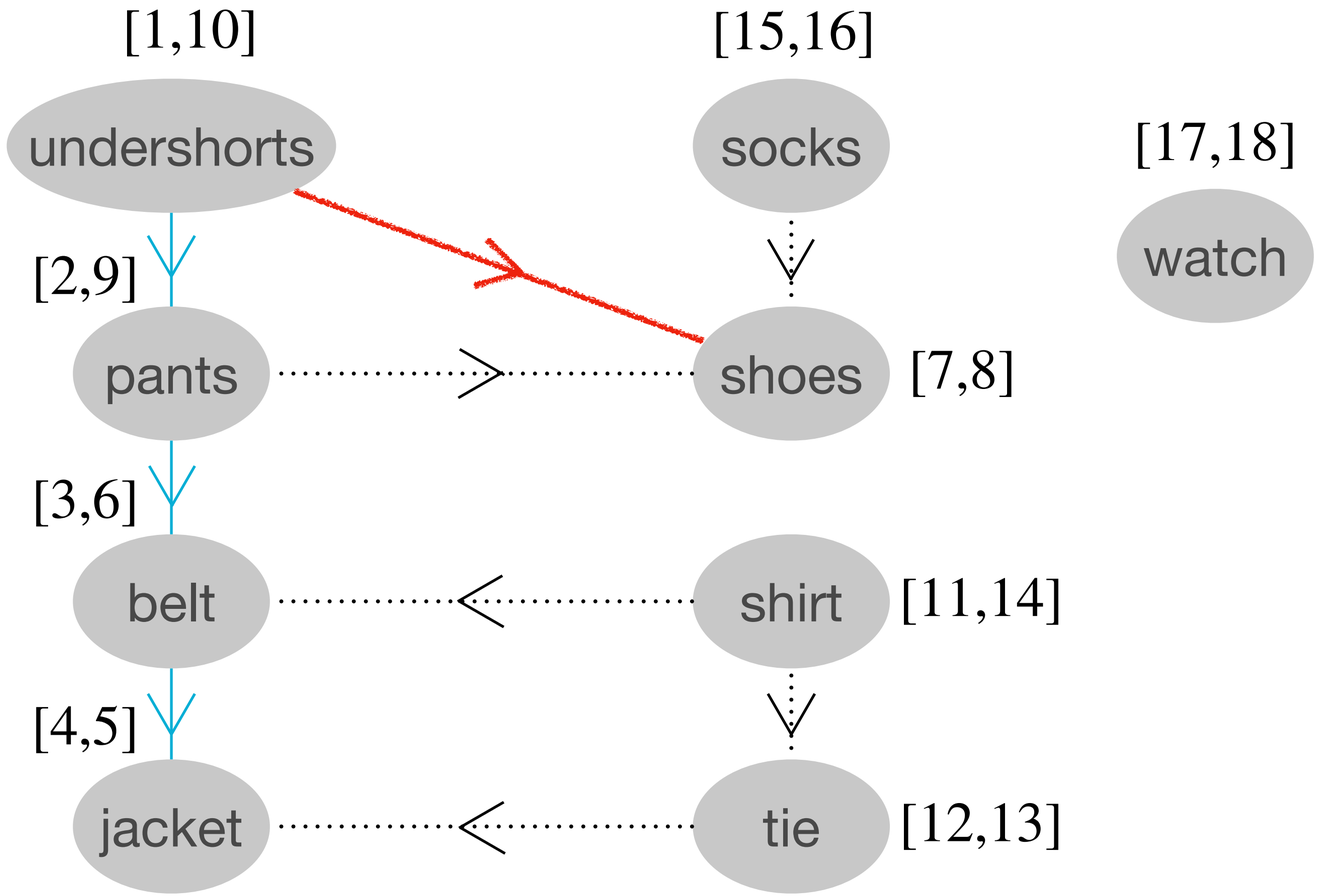

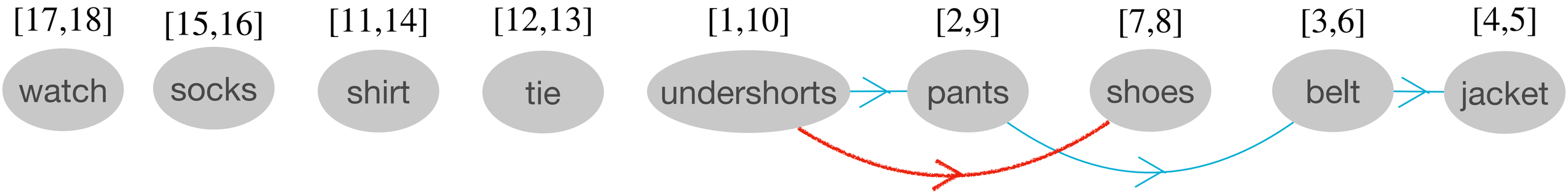

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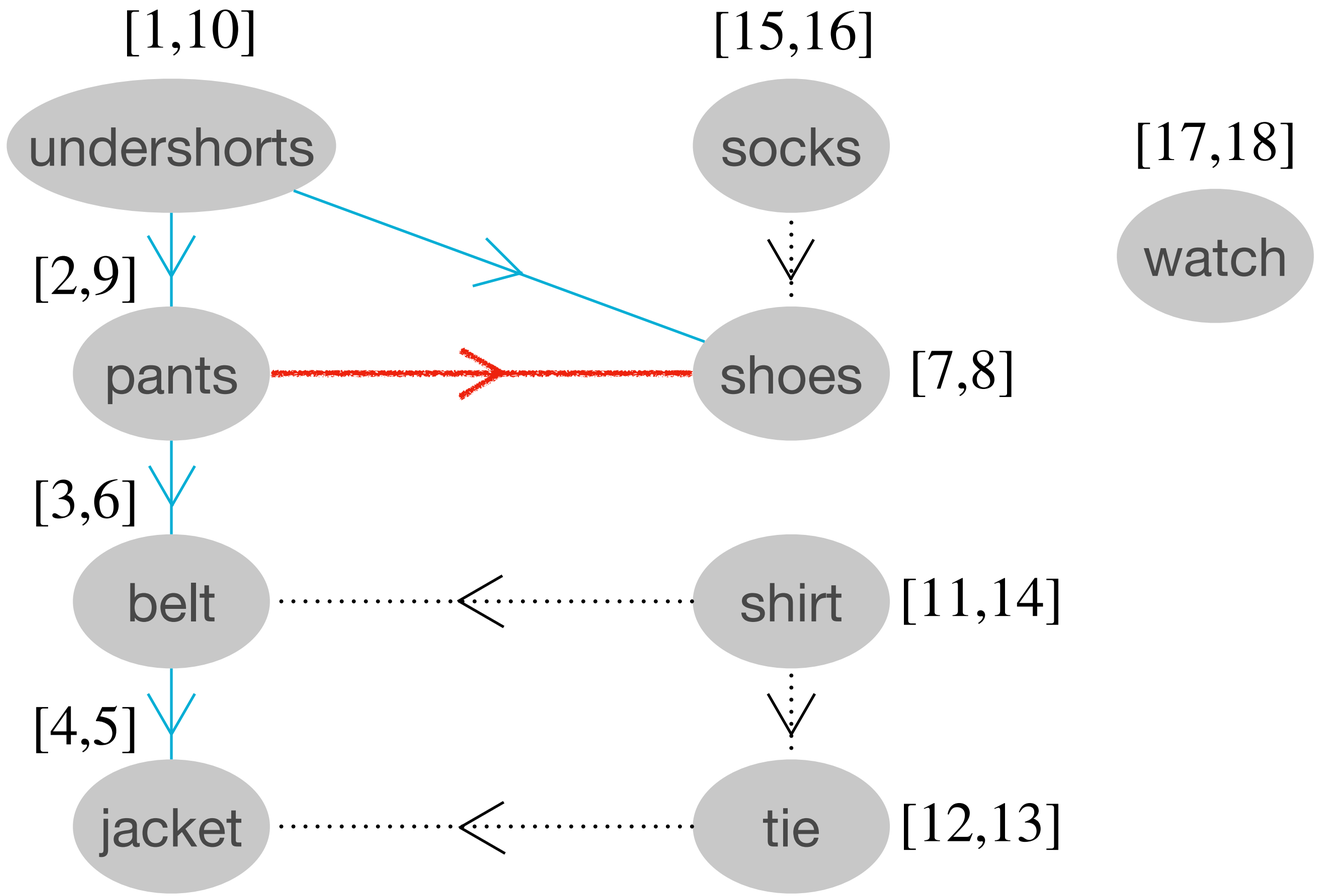

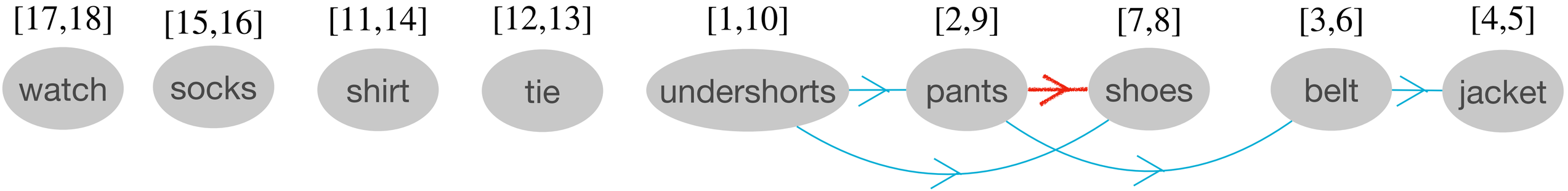

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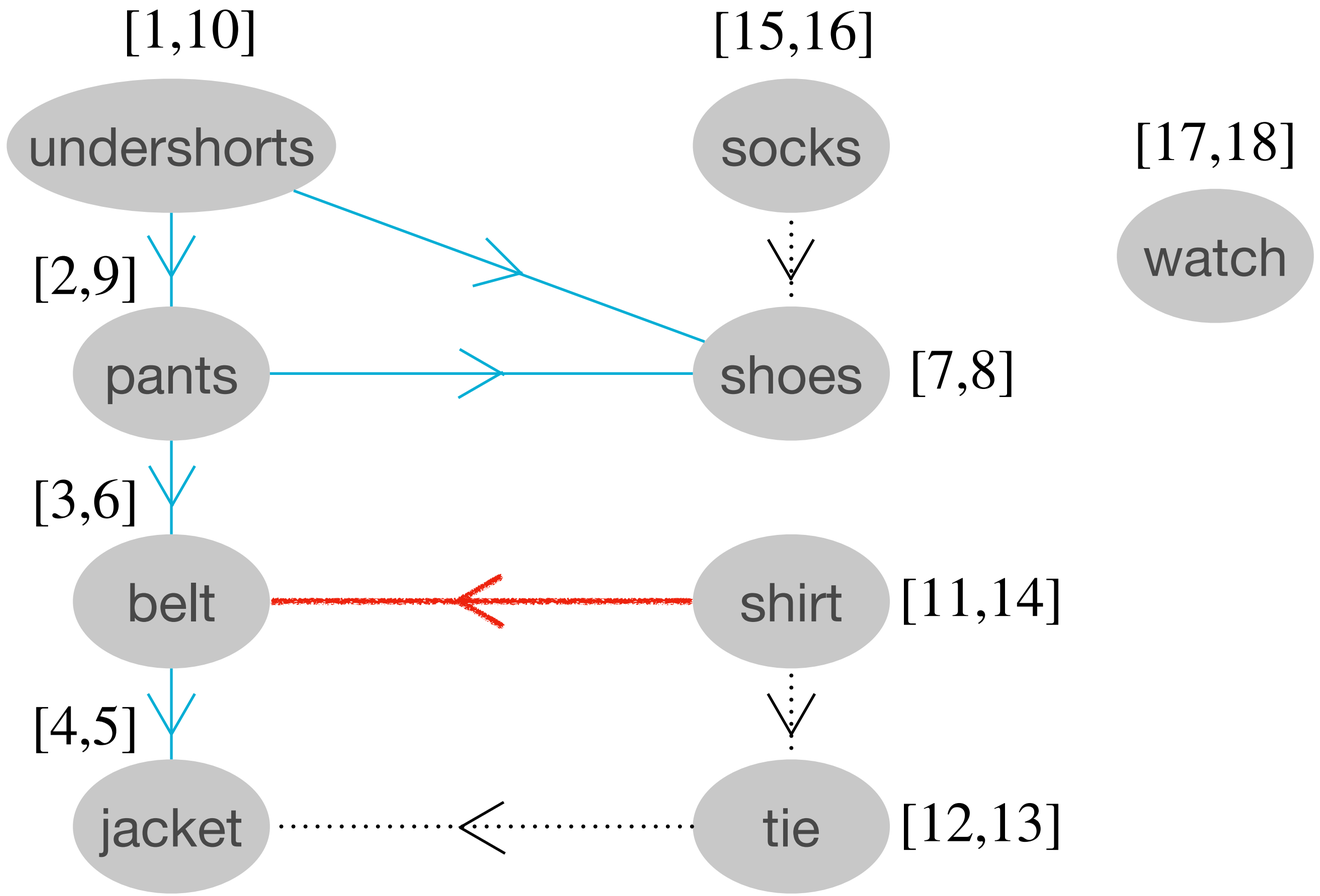

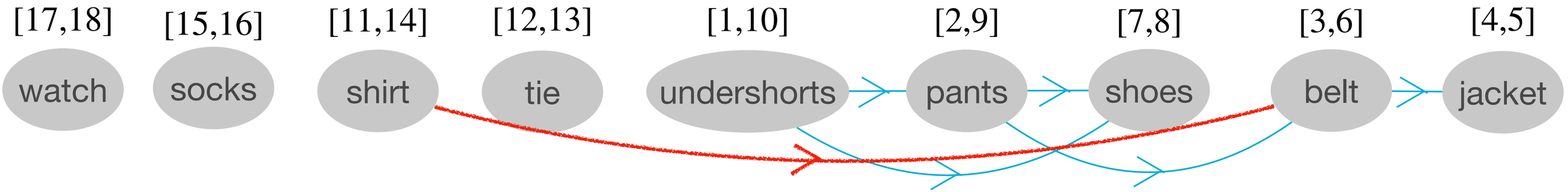

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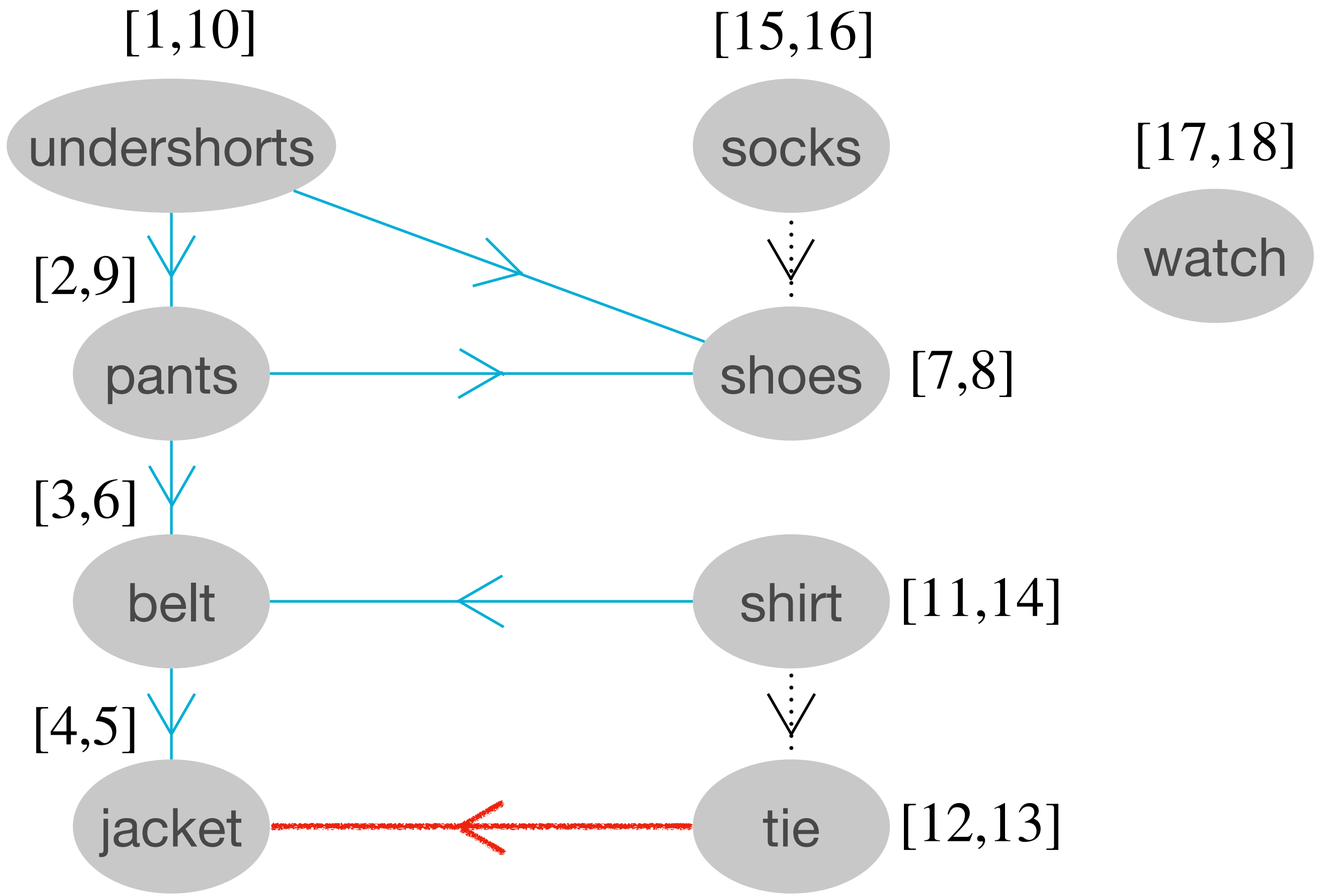

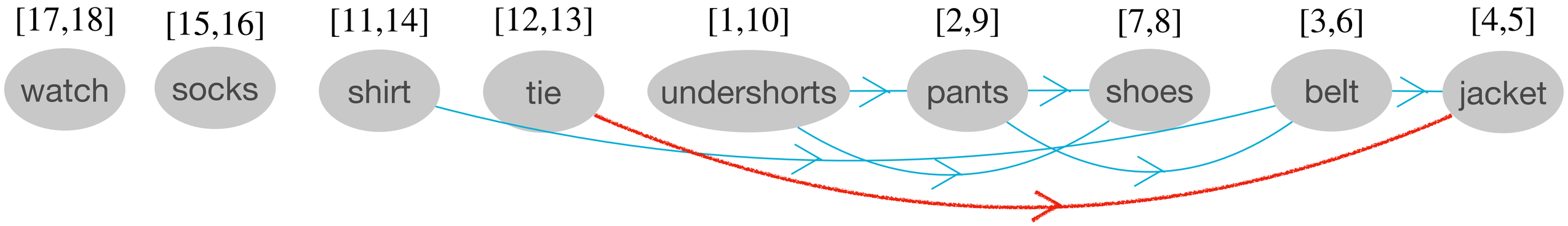

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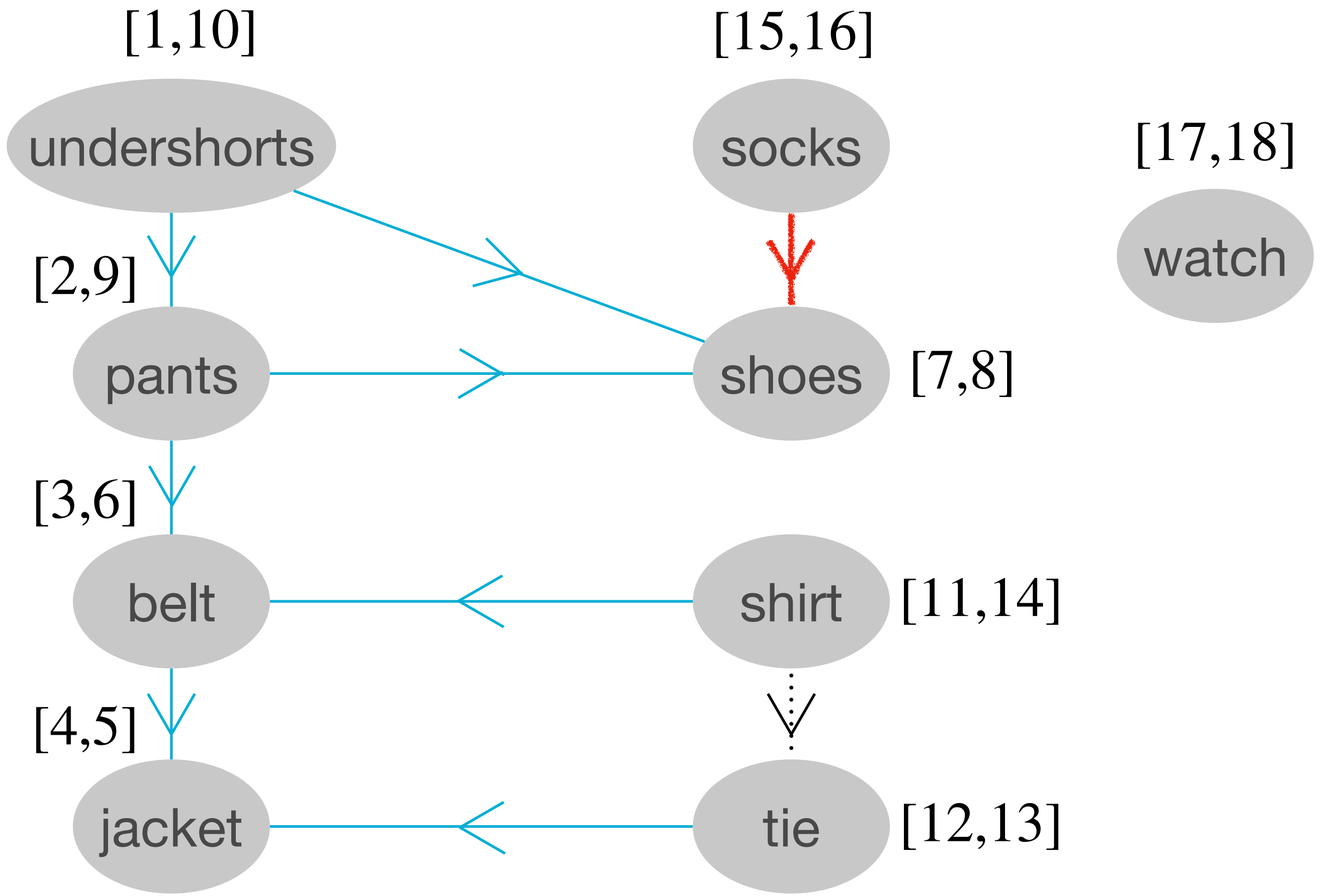

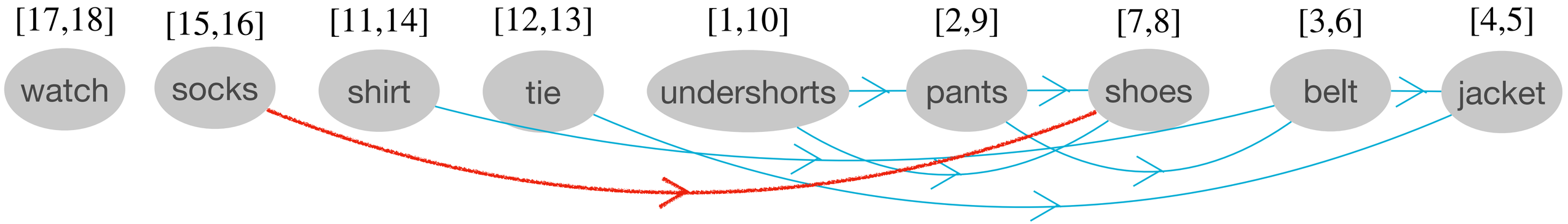

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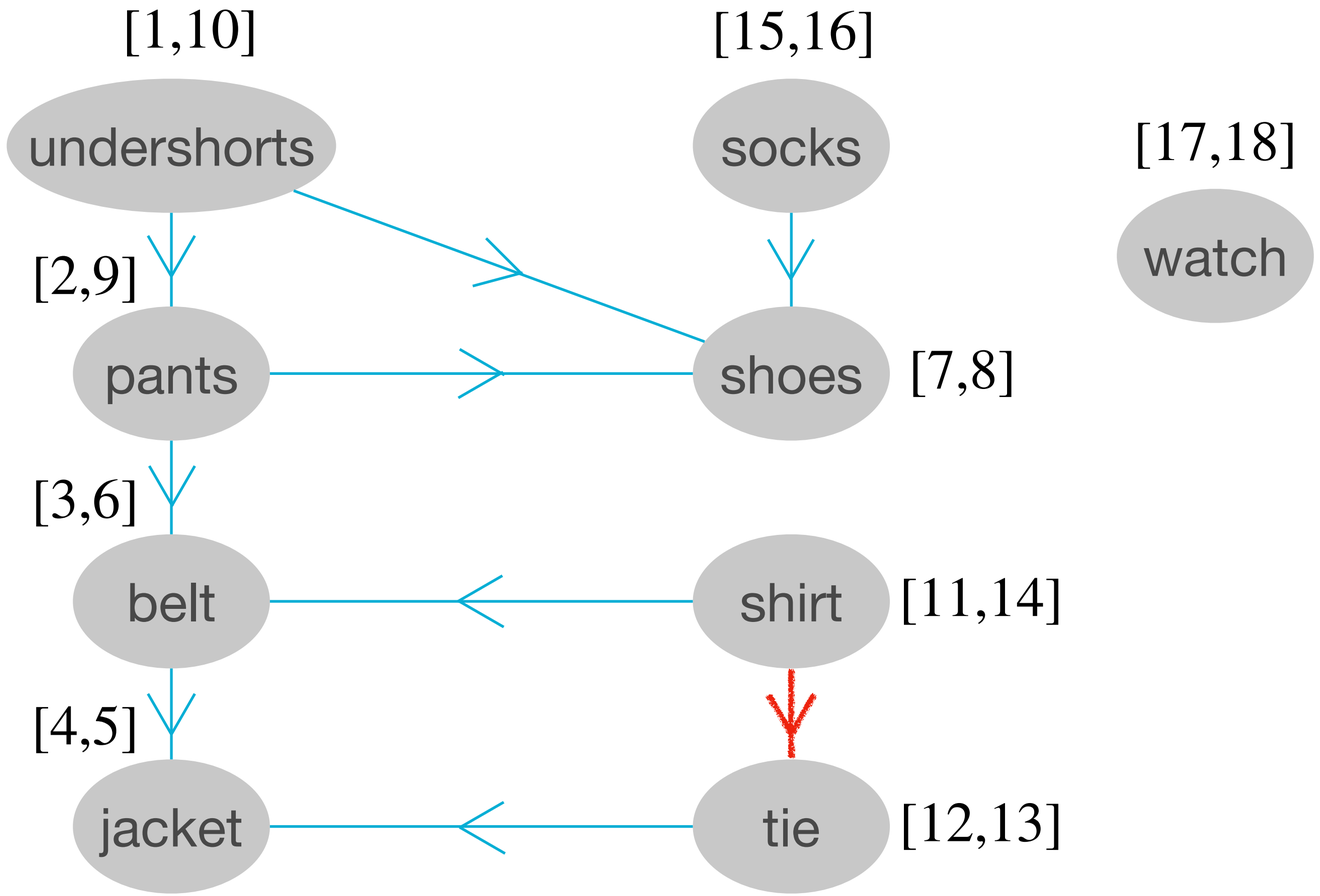

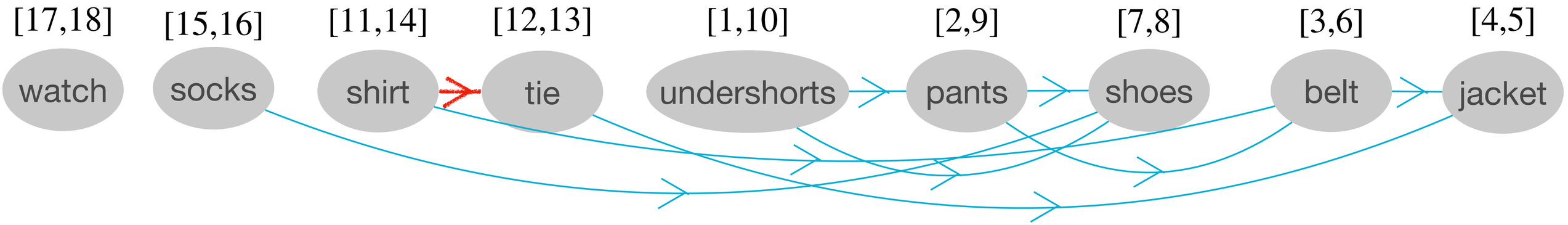

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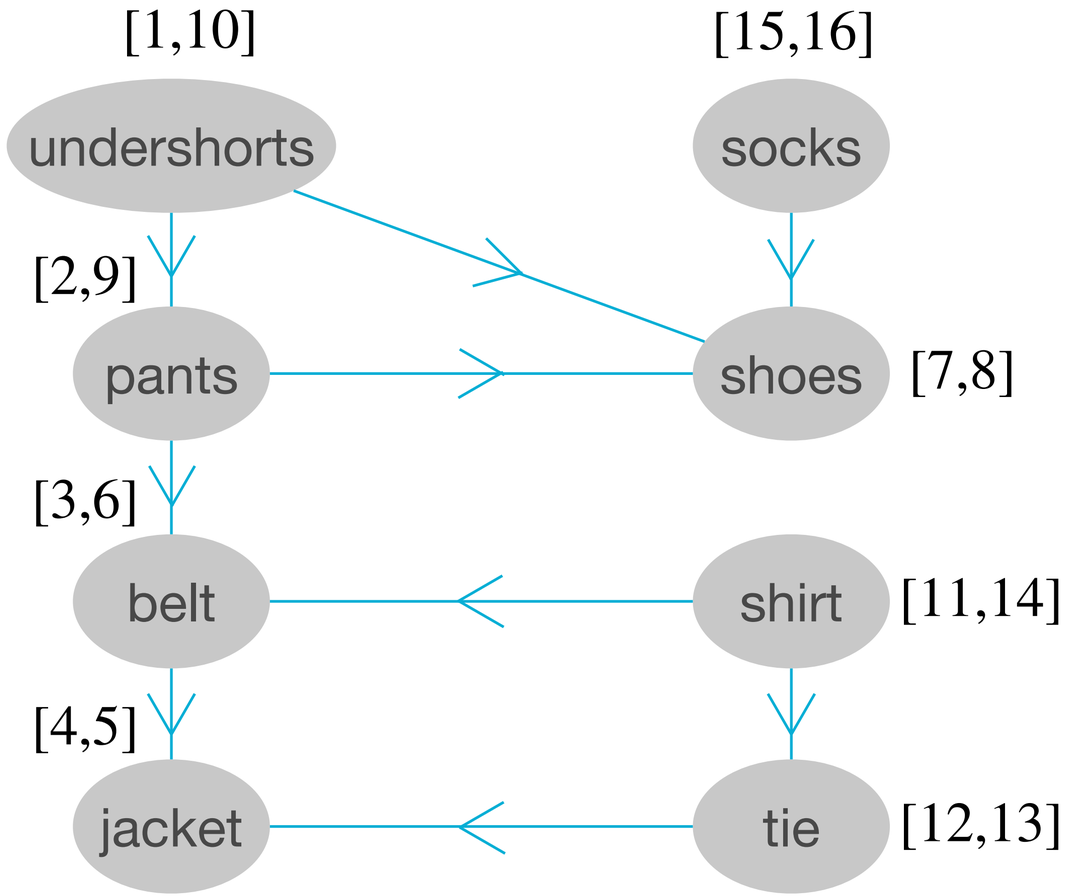
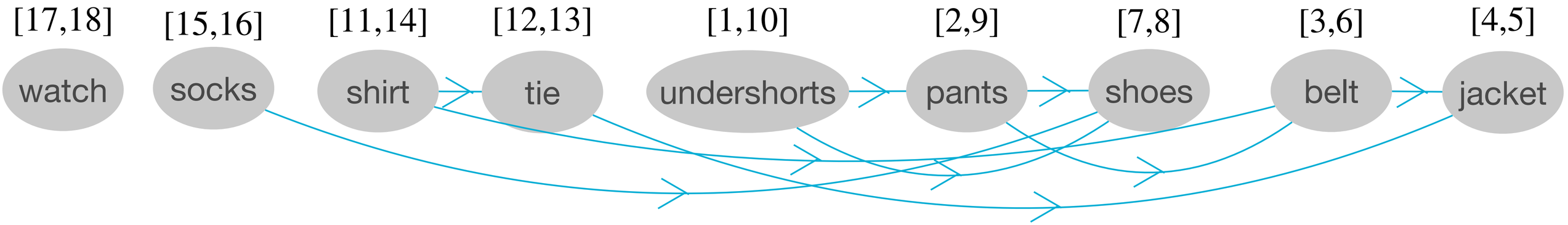


Time Complexity:

$$O(V+E)$$



Quiz questions:

1. When we run DFS on a DAG, what is the relationship between the finish times of the two endpoints of an edge?

2. How is the above property used by the topological sort algorithm?