Algorithms

Lecture Topic: Linear Programming (Part 4)

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Roadmap of this lecture:

1. Linear Programming (LP)

   1.1 Find initial basic feasible solution for LP.

Initial Basic Feasible Solution

How to solve an LP:

1) Decide if LP is feasible
2) If LP is feasible, turn it into a slack form whose basic solution is feasible.
3) Run SIMPLEX algorithm.

Initial Basic Feasible Solution

How to solve an LP:

1) Decide if LP is feasible
2) If LP is feasible, turn it into a slack form whose basic solution is feasible.
3) Run SIMPLEX algorithm.

maximize $2x_1 - x_2$

s.t.

$$\begin{aligned} 2x_1 - x_2 &\leq 2 \\ x_1 - 5x_2 &\leq -4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Is this LP feasible?

That is, does this LP have a solution to x_1 and x_2 that satisfies all the constraints?

Note that feasibility has nothing to do with the objective function.

Initial Basic Feasible Solution

How to solve an LP:

1) Decide if LP is feasible
2) If LP is feasible, turn it into a slack form whose basic solution is feasible.
3) Run SIMPLEX algorithm.

$$
\begin{aligned}
\text{maximize} \quad & 2x_1 - x_2 \\
\text{s.t.} \quad & \\
& 2x_1 - x_2 \le 2 \\
& x_1 - 5x_2 \le -4 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

Is this LP feasible?

That is, does this LP have a solution to x_1 and x_2 that satisfies all the constraints?

Note that feasibility has nothing to do with the objective function.

$\Downarrow$

Slack-form LP:

$$z = 2x_1 - x_2$$

$$x_3 = 2 - 2x_1 + x_2$$

$$x_4 = -4 - x_1 + 5x_2$$

The slack-form LP does not have a feasible basic solution.

Does it mean the LP is infeasible? Not at all.

Basic solution: $x_1 = 0,\ x_2 = 0,\ x_3 = 2,\ x_4 = -4$

Initial Basic Feasible Solution

How to solve an LP:

1) Decide if LP is feasible
2) If LP is feasible, turn it into a slack form whose basic solution is feasible.
3) Run SIMPLEX algorithm.

$$
\begin{aligned}
\text{maximize} \quad & 2x_1 - x_2 \\
\text{s.t.} \quad & \\
& 2x_1 - x_2 \le 2 \\
& x_1 - 5x_2 \le -4 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

Is this LP feasible?

Whether this LP is feasible or not,

we can always add a "helper" to make it feasible.

Initial Basic Feasible Solution

How to solve an LP:

1) Decide if LP is feasible
2) If LP is feasible, turn it into a slack form whose basic solution is feasible.
3) Run SIMPLEX algorithm.

Original LP:

maximize $2x_1 - x_2$

s.t.

$$\begin{aligned} 2x_1 - x_2 &\leq 2 \\ x_1 - 5x_2 &\leq -4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Is this LP feasible?

Whether this LP is feasible or not,

we can always add a “helper” to make it feasible.

Auxiliary LP:

maximize [redacted]

s.t.

$$\begin{aligned} 2x_1 - x_2 - x_0 &\leq 2 \\ x_1 - 5x_2 - x_0 &\leq -4 \\ x_0, x_1, x_2 &\geq 0 \end{aligned}$$

x_0 is our “helper”.

This auxiliary LP is ALWAYS feasible. Why?

Initial Basic Feasible Solution

How to solve an LP:

1) Decide if LP is feasible
2) If LP is feasible, turn it into a slack form whose basic solution is feasible.
3) Run SIMPLEX algorithm.

Original LP:

maximize $2x_1 - x_2$

s.t.

$$\begin{aligned} 2x_1 - x_2 &\le 2 \\ x_1 - 5x_2 &\le -4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Is this LP feasible?

Whether this LP is feasible or not,

we can always add a "helper" to make it feasible.

Auxiliary LP:

maximize [redacted]

s.t.

$$\begin{aligned} 2x_1 - x_2 - x_0 &\le 2 \\ x_1 - 5x_2 - x_0 &\le -4 \\ x_0, x_1, x_2 &\ge 0 \end{aligned}$$

x_0 is our "helper".

This auxiliary LP is ALWAYS feasible. Why?

Because we can always make x_0 sufficiently large to satisfy all constraints.

For example, here we can make $x_0 = 4,$ and a feasible solution is

$$x_0 = 4,\ x_1 = 0,\ x_2 = 0$$

Initial Basic Feasible Solution

How to solve an LP:

1) Decide if LP is feasible
2) If LP is feasible, turn it into a slack form whose basic solution is feasible.
3) Run SIMPLEX algorithm.

Original LP:

$$
\begin{array}{ll}
\text{maximize} & 2x_1 - x_2 \\
\text{s.t.} & \\
& 2x_1 - x_2 \leq 2 \\
& x_1 - 5x_2 \leq -4 \\
& x_1, x_2 \geq 0
\end{array}
$$

Is this LP feasible?

Whether this LP is feasible or not,

we can always add a “helper” to make it feasible.

Auxiliary LP:

maximize [redacted]

$$
\begin{array}{ll}
\text{s.t.} & \\
& 2x_1 - x_2 - x_0 \leq 2 \\
& x_1 - 5x_2 - x_0 \leq -4 \\
& x_0, x_1, x_2 \geq 0
\end{array}
$$

x_0 is our “helper”. This auxiliary LP is ALWAYS feasible.

But how large does x_0 have to be to make this auxiliary LP feasible?

If x_0 can be 0, then no “help” is needed, which means the original LP is feasible.

If x_0 has to be greater than 0, then the original LP is not feasible.

Which case is it? Let’s find out.

Initial Basic Feasible Solution

How to solve an LP:

1) Decide if LP is feasible
2) If LP is feasible, turn it into a slack form whose basic solution is feasible.
3) Run SIMPLEX algorithm.

Original LP:

$$\begin{aligned} \text{maximize} \quad & 2x_1 - x_2 \\ \text{s.t.} \quad & \\ & 2x_1 - x_2 \leq 2 \\ & x_1 - 5x_2 \leq -4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Is this LP feasible?

$$\Downarrow$$

Auxiliary LP:

$$\begin{aligned} \text{maximize} \quad & -x_0 \\ \text{s.t.} \quad & \\ & 2x_1 - x_2 - x_0 \leq 2 \\ & x_1 - 5x_2 - x_0 \leq -4 \\ & x_0, x_1, x_2 \geq 0 \end{aligned}$$

How small can x_0 be?

In particular, can x_0 be as small as 0?

Initial Basic Feasible Solution

Theorem: Let L be an LP

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &\le b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &\le b_2 \\
&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &\le b_m \\
x_1,\ x_2,\ \cdots,\ x_n &\ge 0
\end{aligned}
$$

Let L_{aux} be its auxiliary LP

maximize $-x_0$

s.t.

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n - x_0 &\le b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n - x_0 &\le b_2 \\
&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n - x_0 &\le b_m \\
x_0,\ x_1,\ x_2,\ \cdots,\ x_n &\ge 0
\end{aligned}
$$

Then L is feasible if and only if the optimal objective value for L_{aux} is 0.

Initial Basic Feasible Solution

Theorem: Let L be an LP. Let L_{aux} be its auxiliary LP

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &\le b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &\le b_2 \\
&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &\le b_m \\
x_1,\ x_2,\ \cdots,\ x_n &\ge 0
\end{aligned}
$$

maximize $-x_0$

s.t.

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n - x_0 &\le b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n - x_0 &\le b_2 \\
&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n - x_0 &\le b_m \\
x_0,\ x_1,\ x_2,\ \cdots,\ x_n &\ge 0
\end{aligned}
$$

Then L is feasible if and only if the optimal objective value for L_{aux} is 0.

Proof: $\Rightarrow$ If L has a feasible solution $(x_1,\ x_2,\ \cdots,\ x_n)$,

then L_{aux} has an optimal solution $(x_0 = 0,\ x_1,\ x_2,\ \cdots,\ x_n)$.

Initial Basic Feasible Solution

Theorem: Let L be an LP

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &\le b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &\le b_2 \\
&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &\le b_m \\
x_1,\ x_2,\ \cdots,\ x_n &\ge 0
\end{aligned}
$$

Let L_{aux} be its auxiliary LP

maximize $-x_0$

s.t.

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n - x_0 &\le b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n - x_0 &\le b_2 \\
&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n - x_0 &\le b_m \\
x_0,\ x_1,\ x_2,\ \cdots,\ x_n &\ge 0
\end{aligned}
$$

Then L is feasible if and only if the optimal objective value for L_{aux} is 0.

Proof: $\Leftarrow$

If L_{aux} has an optimal solution $(x_0 = 0,\ x_1,\ x_2,\ \cdots,\ x_n)$,

then L has a feasible solution $(x_1,\ x_2,\ \cdots,\ x_n)$.

Initial Basic Feasible Solution

Original LP:

$$\begin{aligned} &\text{maximize} \quad 2x_1 - x_2 \\ &\text{s.t.} \\ &\qquad 2x_1 - x_2 \leq 2 \\ &\qquad x_1 - 5x_2 \leq -4 \\ &\qquad x_1, x_2 \geq 0 \end{aligned}$$

Is this LP feasible?

Let's solve the Auxiliary LP to find out.

Auxiliary LP:

$$\begin{aligned} &\text{maximize} \quad \boxed{-x_0} \\ &\text{s.t.} \\ &\qquad 2x_1 - x_2 - x_0 \leq 2 \\ &\qquad x_1 - 5x_2 - x_0 \leq -4 \\ &\qquad x_0, x_1, x_2 \geq 0 \end{aligned}$$

How small can x_0 be?

In particular, can x_0 be as small as 0?

Auxiliary LP:

maximize $-x_0$

s.t.

$$2x_1 - x_2 - x_0 \leq 2$$
$$x_1 - 5x_2 - x_0 \leq -4$$
$$x_0, x_1, x_2 \geq 0$$

$$z = -x_0$$
$$x_3 = 2 + x_0 - 2x_1 + x_2$$
$$x_4 = -4 + x_0 - x_1 + 5x_2$$

Auxiliary LP:

maximize $-x_0$

s.t.

$$2x_1 - x_2 - x_0 \leq 2$$
$$x_1 - 5x_2 - x_0 \leq -4$$
$$x_0, x_1, x_2 \geq 0$$

$$z = -x_0$$
$$x_3 = 2 + x_0 - 2x_1 + x_2$$
$$x_4 = -4 + x_0 - x_1 + 5x_2$$

Do a pivot:

1) Move x_0 to left.

2) Pick a basic variable whose constant term is the most negative, and move it to right.

Auxiliary LP:

maximize $-x_0$

s.t.

$$2x_1 - x_2 - x_0 \leq 2$$
$$x_1 - 5x_2 - x_0 \leq -4$$
$$x_0, x_1, x_2 \geq 0$$

$$z = -x_0$$
$$x_3 = 2 + x_0 - 2x_1 + x_2$$
$$x_4 = -4 + x_0 - x_1 + 5x_2$$

Most negative

Do a pivot:

1) Move x_0 to left.

2) Pick a basic variable whose constant term is the most negative, and move it to right.

Auxiliary LP:

maximize $\quad -x_0$

s.t.

$$2x_1 - x_2 - x_0 \leq 2$$
$$x_1 - 5x_2 - x_0 \leq -4$$
$$x_0, x_1, x_2 \geq 0$$

$$z = \quad - \boxed{x_0}$$
$$x_3 = 2 + x_0 - 2x_1 + x_2$$
$$\boxed{x_4} = -4 + x_0 - x_1 + 5x_2$$

Do a pivot:

1) Move x_0 to left.

2) Pick a basic variable whose constant term is the most negative, and move it to right.

Auxiliary LP:

maximize $-x_0$

s.t.

$$2x_1 - x_2 - x_0 \le 2$$
$$x_1 - 5x_2 - x_0 \le -4$$
$$x_0, x_1, x_2 \ge 0$$

$$z = -x_0$$
$$x_3 = 2 + x_0 - 2x_1 + x_2$$
$$x_4 = -4 + x_0 - x_1 + 5x_2$$

$$z = -4 - x_1 + 5x_2 - x_4$$
$$x_3 = 6 - x_1 - 4x_2 + x_4$$
$$x_0 = 4 + x_1 - 5x_2 + x_4$$

Do a pivot:

1) Move x_0 to left.

2) Pick a basic variable whose constant term is the most negative, and move it to right.

Auxiliary LP:

$$\begin{aligned} &\text{maximize} \quad -x_0 \\ &\text{s.t.} \\ &\quad 2x_1 - x_2 - x_0 \leq 2 \\ &\quad x_1 - 5x_2 - x_0 \leq -4 \\ &\quad x_0, x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} z &= -x_0 \\ x_3 &= 2 + x_0 - 2x_1 + x_2 \\ x_4 &= -4 + x_0 - x_1 + 5x_2 \end{aligned}$$

$$\begin{aligned} z &= -4 - x_1 + 5x_2 - x_4 \\ x_3 &= 6 - x_1 - 4x_2 + x_4 \\ x_0 &= 4 + x_1 - 5x_2 + x_4 \end{aligned}$$

Positive

Do a pivot:

1) Move x_0 to left.

2) Pick a basic variable whose constant term is the most negative, and move it to right.

Auxiliary LP:

$$\begin{aligned} &\text{maximize} \quad -x_0 \\ &\text{s.t.} \\ &\quad 2x_1 - x_2 - x_0 \leq 2 \\ &\quad x_1 - 5x_2 - x_0 \leq -4 \\ &\quad x_0, x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} z &= -x_0 \\ x_3 &= 2 + x_0 - 2x_1 + x_2 \\ x_4 &= -4 + x_0 - x_1 + 5x_2 \end{aligned}$$

$$\begin{aligned} z &= -4 - x_1 + 5x_2 - x_4 \\ x_3 &= 6 - x_1 - 4x_2 + x_4 \\ x_0 &= 4 + x_1 - 5x_2 + x_4 \end{aligned}$$

Where does this 6 come from?

2 + 4 = 6

Do a pivot:

1) Move x_0 to left.

2) Pick a basic variable whose constant term is the most negative, and move it to right.

Auxiliary LP:

$$\begin{aligned} &\text{maximize} \quad -x_0 \\ &\text{s.t.} \\ &\quad 2x_1 - x_2 - x_0 \le 2 \\ &\quad x_1 - 5x_2 - x_0 \le -4 \\ &\quad x_0, x_1, x_2 \ge 0 \end{aligned}$$

What if

$$\begin{aligned} z &= -\boxed{x_0} \\ x_3 &= -2 + x_0 - 2x_1 + x_2 \\ \boxed{x_4} &= -4 + x_0 - x_1 + 5x_2 \end{aligned}$$

$$\begin{aligned} z &= -4 - x_1 + 5x_2 - x_4 \\ x_3 &= 6 - x_1 - 4x_2 + x_4 \\ x_0 &= 4 + x_1 - 5x_2 + x_4 \end{aligned}$$

-2 + 4 = 2

Still non-negative!

Do a pivot:

1) Move x_0 to left.

2) Pick a basic variable whose constant term is the most negative, and move it to right.

Auxiliary LP:

maximize $\quad -x_0$

s.t.

$$2x_1 - x_2 - x_0 \leq 2$$
$$x_1 - 5x_2 - x_0 \leq -4$$
$$x_0, x_1, x_2 \geq 0$$

$$z = \quad -\boxed{x_0}$$
$$x_3 = 2 + x_0 - 2x_1 + x_2$$
$$\boxed{x_4} = -4 + x_0 - x_1 + 5x_2$$

$$z = \quad -4 - x_1 + 5x_2 - x_4$$
$$x_3 = 6 - x_1 - 4x_2 + x_4$$
$$x_0 = 4 + x_1 - 5x_2 + x_4$$

All constant terms are non-negative.

Its basic solution is feasible,
So we can run the SIMPLEX Algorithm now!

Do a pivot:

1) Move x_0 to left.

2) Pick a basic variable whose constant term is the most negative, and move it to right.

Auxiliary LP:

maximize $-x_0$

s.t.

$$
\begin{aligned}
2x_1 - x_2 - x_0 &\le 2 \\
x_1 - 5x_2 - x_0 &\le -4 \\
x_0, x_1, x_2 &\ge 0
\end{aligned}
$$

$$
\begin{aligned}
z &= -x_0 \\
x_3 &= 2 + x_0 - 2x_1 + x_2 \\
x_4 &= -4 + x_0 - x_1 + 5x_2
\end{aligned}
$$

$$
\begin{aligned}
z &= -4 - x_1 + 5x_2 - x_4 \\
x_3 &= 6 - x_1 - 4x_2 + x_4 \\
x_0 &= 4 + x_1 - 5x_2 + x_4
\end{aligned}
$$

Basic solution:

$x_0 = 4,\ x_1 = 0,\ x_2 = 0,\ x_3 = 6,\ x_4 = 0$

Auxiliary LP:

maximize $-x_0$

s.t.

$$2x_1 - x_2 - x_0 \leq 2$$
$$x_1 - 5x_2 - x_0 \leq -4$$
$$x_0, x_1, x_2 \geq 0$$

$$z = -x_0$$
$$x_3 = 2 + x_0 - 2x_1 + x_2$$
$$x_4 = -4 + x_0 - x_1 + 5x_2$$

$$z = -4 - x_1 + 5x_2 - x_4$$
$$x_3 = 6 - x_1 - 4x_2 + x_4$$
$$x_0 = 4 + x_1 - 5x_2 + x_4$$

Run SIMPLEX Algorithm

$$z = -x_0$$
$$x_2 = \frac{4}{5} - \frac{1}{5}x_0 + \frac{1}{5}x_1 + \frac{1}{5}x_4$$
$$x_3 = \frac{14}{5} + \frac{4}{5}x_0 - \frac{9}{5}x_1 + \frac{1}{5}x_4$$

Auxiliary LP:

$$\begin{aligned} &\text{maximize} \quad -x_0 \\ &\text{s.t.} \\ &\quad 2x_1 - x_2 - x_0 \leq 2 \\ &\quad x_1 - 5x_2 - x_0 \leq -4 \\ &\quad x_0, x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} z &= -x_0 \\ x_3 &= 2 + x_0 - 2x_1 + x_2 \\ x_4 &= -4 + x_0 - x_1 + 5x_2 \end{aligned}$$

$$\begin{aligned} z &= -4 - x_1 + 5\boxed{x_2} - x_4 \\ x_3 &= 6 - x_1 - 4x_2 + x_4 \\ \boxed{x_0} &= 4 + x_1 - 5x_2 + x_4 \end{aligned}$$

Run SIMPLEX Algorithm

$$\begin{aligned} z &= -x_0 \\ x_2 &= \frac{4}{5} - \frac{1}{5}x_0 + \frac{1}{5}x_1 + \frac{1}{5}x_4 \\ x_3 &= \frac{14}{5} + \frac{4}{5}x_0 - \frac{9}{5}x_1 + \frac{1}{5}x_4 \end{aligned}$$

Optimal Objective Value = 0

Original LP:

$$\begin{aligned} &\text{maximize} \quad 2x_1 - x_2 \\ &\text{s.t.} \\ &\quad 2x_1 - x_2 \leq 2 \\ &\quad x_1 - 5x_2 \leq -4 \\ &\quad x_1, x_2 \geq 0 \end{aligned}$$

It is feasible!

Auxiliary LP:

maximize $-x_0$

s.t.

$$\begin{aligned} 2x_1 - x_2 - x_0 &\leq 2 \\ x_1 - 5x_2 - x_0 &\leq -4 \\ x_0, x_1, x_2 &\geq 0 \end{aligned}$$

$$\begin{aligned} z &= -x_0 \\ x_3 &= 2 + x_0 - 2x_1 + x_2 \\ x_4 &= -4 + x_0 - x_1 + 5x_2 \end{aligned}$$

$$\begin{aligned} z &= -4 - x_1 + 5\boxed{x_2} - x_4 \\ x_3 &= 6 - x_1 - 4x_2 + x_4 \\ \boxed{x_0} &= 4 + x_1 - 5x_2 + x_4 \end{aligned}$$

Run SIMPLEX Algorithm

Now we need to solve the Original LP. But how?

$$\begin{aligned} z &= -x_0 \\ x_2 &= \frac{4}{5} - \frac{1}{5}x_0 + \frac{1}{5}x_1 + \frac{1}{5}x_4 \\ x_3 &= \frac{14}{5} + \frac{4}{5}x_0 - \frac{9}{5}x_1 + \frac{1}{5}x_4 \end{aligned}$$

Optimal Objective Value = 0

Original LP:

maximize $2x_1 - x_2$

s.t.

$$\begin{aligned} 2x_1 - x_2 &\leq 2 \\ x_1 - 5x_2 &\leq -4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

It is feasible!

Auxiliary LP:

maximize $\quad -x_0$

s.t.

$$
\begin{aligned}
2x_1 - x_2 - x_0 &\le 2 \\
x_1 - 5x_2 - x_0 &\le -4 \\
x_0, x_1, x_2 &\ge 0
\end{aligned}
$$

$$
\begin{aligned}
z &= -x_0 \\
x_3 &= 2 + x_0 - 2x_1 + x_2 \\
x_4 &= -4 + x_0 - x_1 + 5x_2
\end{aligned}
$$

[redacted]

$$
\begin{aligned}
z &= -x_0 \\
x_2 &= \frac{4}{5} - \frac{1}{5}x_0 + \frac{1}{5}x_1 + \frac{1}{5}x_4 \\
x_3 &= \frac{14}{5} + \frac{4}{5}x_0 - \frac{9}{5}x_1 + \frac{1}{5}x_4
\end{aligned}
$$

Now we need to solve the Original LP. But how?

Original LP:

maximize $\quad 2x_1 - x_2$

s.t.

$$
\begin{aligned}
2x_1 - x_2 &\le 2 \\
x_1 - 5x_2 &\le -4 \\
x_1, x_2 &\ge 0
\end{aligned}
$$

Auxiliary LP:

maximize $-x_0$

s.t.

$$\begin{aligned} 2x_1 - x_2 - x_0 &\le 2 \\ x_1 - 5x_2 - x_0 &\le -4 \\ x_0, x_1, x_2 &\ge 0 \end{aligned}$$

$$z = -x_0$$

$$\begin{aligned} x_3 &= 2 + x_0 - 2x_1 + x_2 \\ x_4 &= -4 + x_0 - x_1 + 5x_2 \end{aligned}$$

[redacted]

Now we need to solve the Original LP. But how?

Original LP:

maximize $2x_1 - x_2$

s.t.

$$\begin{aligned} 2x_1 - x_2 &\le 2 \\ x_1 - 5x_2 &\le -4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

$$z = -x_0$$

$$\begin{aligned} x_2 &= \frac{4}{5} - \frac{1}{5}x_0 + \frac{1}{5}x_1 + \frac{1}{5}x_4 \\ x_3 &= \frac{14}{5} + \frac{4}{5}x_0 - \frac{9}{5}x_1 + \frac{1}{5}x_4 \end{aligned}$$

Auxiliary LP:

maximize $-x_0$

s.t.

$$\begin{aligned} 2x_1 - x_2 - x_0 &\le 2 \\ x_1 - 5x_2 - x_0 &\le -4 \\ x_0, x_1, x_2 &\ge 0 \end{aligned}$$

$$z = -x_0$$

$$\begin{aligned} x_3 &= 2 + x_0 - 2x_1 + x_2 \\ x_4 &= -4 + x_0 - x_1 + 5x_2 \end{aligned}$$

$$z = -x_0$$

$$\begin{aligned} x_2 &= \frac{4}{5} - \frac{1}{5}x_0 + \frac{1}{5}x_1 + \frac{1}{5}x_4 \\ x_3 &= \frac{14}{5} + \frac{4}{5}x_0 - \frac{9}{5}x_1 + \frac{1}{5}x_4 \end{aligned}$$

Original LP:

maximize $2x_1 - x_2$

s.t.

$$\begin{aligned} 2x_1 - x_2 &\le 2 \\ x_1 - 5x_2 &\le -4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Auxiliary LP:

$$\begin{array}{ll} \text{maximize} & -x_0 \\ \text{s.t.} & \\ & 2x_1 - x_2 - x_0 \le 2 \\ & x_1 - 5x_2 - x_0 \le -4 \\ & x_0, x_1, x_2 \ge 0 \end{array}$$

$\Rightarrow$

$$z = -x_0$$

$$\boxed{\begin{aligned} x_3 &= 2 + x_0 - 2x_1 + x_2 \\ x_4 &= -4 + x_0 - x_1 + 5x_2 \end{aligned}}$$

$\Rightarrow$

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$$\boxed{\begin{aligned} x_2 &= \frac{4}{5} - \frac{1}{5}x_0 + \frac{1}{5}x_1 + \frac{1}{5}x_4 \\ x_3 &= \frac{14}{5} + \frac{4}{5}x_0 - \frac{9}{5}x_1 + \frac{1}{5}x_4 \end{aligned}}$$

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$\Rightarrow$

$$z = 2x_1 - x_2$$

$$\boxed{\begin{aligned} x_3 &= 2 - 2x_1 + x_2 \\ x_4 &= -4 - x_1 + 5x_2 \end{aligned}}$$

Auxiliary LP:

maximize $-x_0$

s.t.

$$\begin{aligned} 2x_1 - x_2 - x_0 &\leq 2 \\ x_1 - 5x_2 - x_0 &\leq -4 \\ x_0, x_1, x_2 &\geq 0 \end{aligned}$$

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The constraints are equivalent when $x_0 = 0$

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maximize $2x_1 - x_2$

s.t.

$$\begin{aligned} 2x_1 - x_2 &\leq 2 \\ x_1 - 5x_2 &\leq -4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

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Auxiliary LP:

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$$\begin{array}{ll} \text{maximize} & 2x_1 - x_2 \\ \text{s.t.} & \\ & 2x_1 - x_2 \leq 2 \\ & x_1 - 5x_2 \leq -4 \\ & x_1,\ x_2 \geq 0 \end{array}$$

$$z = 2x_1 - x_2$$

$$\boxed{\begin{aligned} x_3 &= 2 - 2x_1 + x_2 \\ x_4 &= -4 - x_1 + 5x_2 \end{aligned}}$$

$$\begin{aligned} z &= 2x_1 - x_2 \\ &= 2x_1 - \left(\frac{4}{5} + \frac{1}{5}x_1 + \frac{1}{5}x_4\right) \\ &= -\frac{4}{5} + \frac{9}{5}x_1 - \frac{1}{5}x_4 \end{aligned}$$

$$\boxed{\begin{aligned} x_2 &= \frac{4}{5} + \frac{1}{5}x_1 + \frac{1}{5}x_4 \\ x_3 &= \frac{14}{5} - \frac{9}{5}x_1 + \frac{1}{5}x_4 \end{aligned}}$$

This slack-form LP
has a feasible basic solution.

Now we can solve it using
the SIMPLEX Algorithm.

Original LP:

$$\begin{aligned} &\text{maximize} \quad 2x_1 - x_2 \\ &\text{s.t.} \\ &\quad 2x_1 - x_2 \leq 2 \\ &\quad x_1 - 5x_2 \leq -4 \\ &\quad x_1,\ x_2 \geq 0 \end{aligned}$$

$$z = 2x_1 - x_2$$

$$\boxed{\begin{aligned} x_3 &= 2 - 2x_1 + x_2 \\ x_4 &= -4 - x_1 + 5x_2 \end{aligned}}$$

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Auxiliary LP:

$$\begin{array}{ll} \text{maximize} & -x_0 \\ \text{s.t.} & \\ & 2x_1 - x_2 - x_0 \leq 2 \\ & x_1 - 5x_2 - x_0 \leq -4 \\ & x_0, x_1, x_2 \geq 0 \end{array}$$

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$$\boxed{\begin{aligned} x_2 &= \frac{4}{5} - \frac{1}{5}x_0 + \frac{1}{5}x_1 + \frac{1}{5}x_4 \\ x_3 &= \frac{14}{5} + \frac{4}{5}x_0 - \frac{9}{5}x_1 + \frac{1}{5}x_4 \end{aligned}}$$

Last question: What if x_0 are a basic variable in these constraint equations?

Original LP:

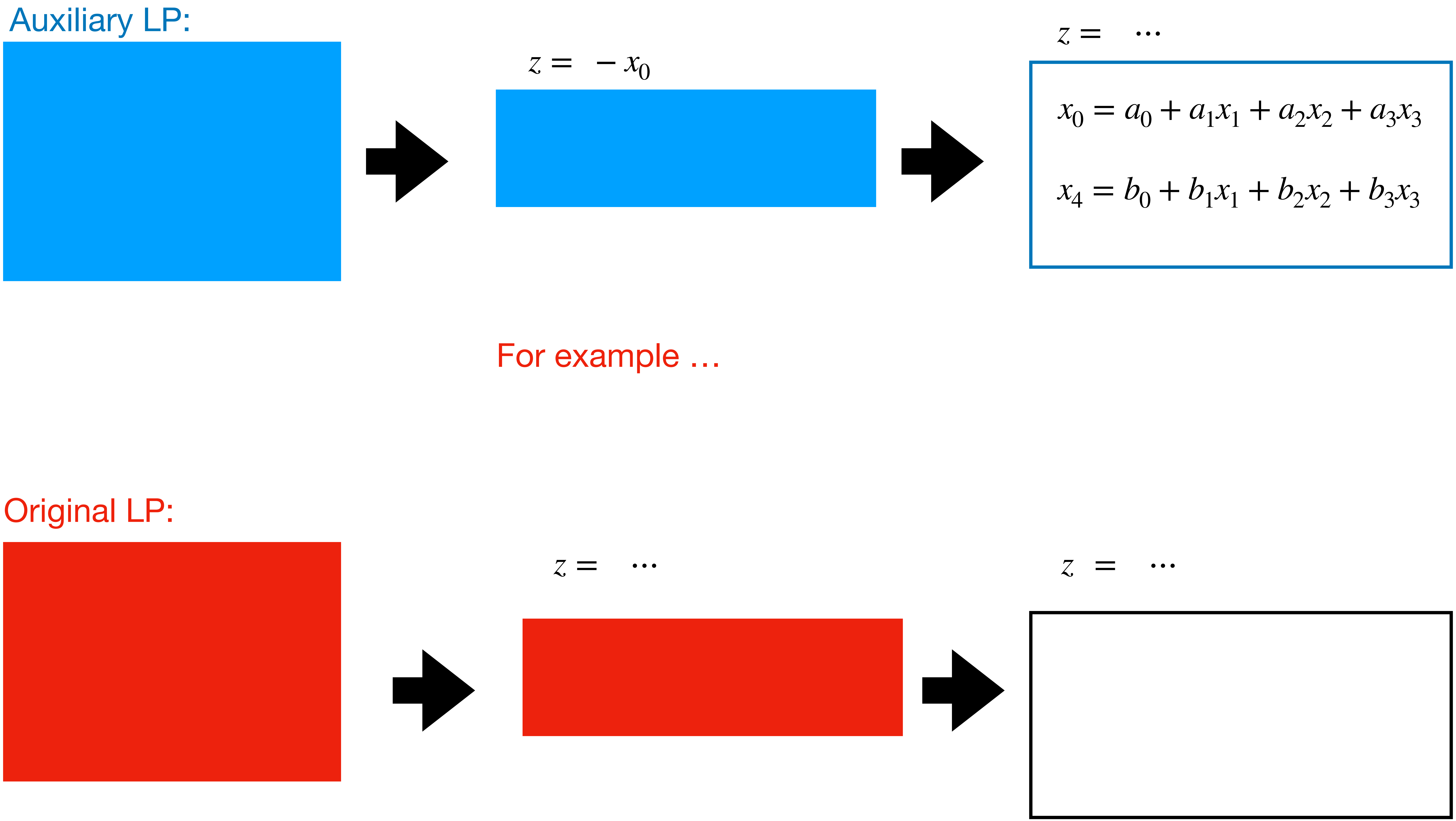
$$\begin{array}{ll} \text{maximize} & 2x_1 - x_2 \\ \text{s.t.} & \\ & 2x_1 - x_2 \leq 2 \\ & x_1 - 5x_2 \leq -4 \\ & x_1, x_2 \geq 0 \end{array}$$

$$z = 2x_1 - x_2$$

$$\boxed{\begin{aligned} x_3 &= 2 - 2x_1 + x_2 \\ x_4 &= -4 - x_1 + 5x_2 \end{aligned}}$$

$$z = -\frac{4}{5} + \frac{9}{5}x_1 - \frac{1}{5}x_4$$

$$\boxed{\begin{aligned} x_2 &= \frac{4}{5} + \frac{1}{5}x_1 + \frac{1}{5}x_4 \\ x_3 &= \frac{14}{5} - \frac{9}{5}x_1 + \frac{1}{5}x_4 \end{aligned}}$$



Auxiliary LP:
$z = -x_0$
$z = \cdots$
$x_0 = a_0 + a_1x_1 + a_2x_2 + a_3x_3$
$x_4 = b_0 + b_1x_1 + b_2x_2 + b_3x_3$
For example …
Original LP:
$z = \cdots$
$z = \cdots$

Auxiliary LP:

$z = -x_0$

$z = \cdots$

$$x_0 = a_0 + a_1x_1 + a_2x_2 + a_3x_3$$

$$x_4 = b_0 + b_1x_1 + b_2x_2 + b_3x_3$$

For example …

Optimal solution:

$$x_0 = a_0,\ x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = b_0$$

Optimal objective value:

$$z = -x_0 = 0$$

Original LP:

$z = \cdots$

$z = \cdots$

Auxiliary LP:

$z = -x_0$

$z = \cdots$

$x_0 = a_0 + a_1x_1 + a_2x_2 + a_3x_3$

$x_4 = b_0 + b_1x_1 + b_2x_2 + b_3x_3$

$a_0 = 0$

Optimal solution:

$x_0 = a_0,\ x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = b_0$

Optimal objective value:

$z = -x_0 = 0$

Original LP:

$z = \cdots$

$z = \cdots$

Auxiliary LP:

$$z = -x_0$$

$$z = \cdots$$

$$x_0 = \quad a_1x_1 + a_2x_2 + a_3x_3$$

$$x_4 = b_0 + b_1x_1 + b_2x_2 + b_3x_3$$

$a_0 = 0$

Optimal solution:

$$x_0 = a_0,\ x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = b_0$$

Optimal objective value:

$$z = -x_0 = 0$$

Original LP:

$$z = \cdots$$

$$z = \cdots$$

Auxiliary LP:

$z = \; -x_0$

$z = \; \cdots$

$x_0 = \; a_1 x_1 + a_2 x_2 + a_3 x_3$

$x_4 = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3$

$a_0 = 0$

Optimal solution:

$x_0 = a_0,\; x_1 = 0,\; x_2 = 0,\; x_3 = 0,\; x_4 = b_0$

Optimal objective value:

$z = \; -x_0 = 0$

Original LP:

$z = \; \cdots$

$z = \; \cdots$

Auxiliary LP:

$$z = -x_0$$

$$z = \cdots$$

$$x_1 = \frac{1}{a_1}x_0 - \frac{a_2}{a_1}x_2 - \frac{a_3}{a_1}x_3$$

$$x_4 = b_0 + b_1x_1 + b_2x_2 + b_3x_3$$

$a_0 = 0$

Optimal solution:

$$x_0 = a_0,\ x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = b_0$$

Optimal objective value:

$$z = -x_0 = 0$$

Original LP:

$$z = \cdots$$

$$z = \cdots$$

Auxiliary LP:

$z = -x_0$

$z = \cdots$

$$x_1 = \frac{1}{a_1}x_0 - \frac{a_2}{a_1}x_2 - \frac{a_3}{a_1}x_3$$

$$x_4 = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3$$

$a_0 = 0$

Optimal solution:

$x_0 = a_0,\ x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = b_0$

Optimal objective value:

$z = -x_0 = 0$

Original LP:

$z = \cdots$

$z = \cdots$

Auxiliary LP:

$z = -x_0$

The constants here are still 0 and $b_0 \geq 0$

$z = \cdots$

$$x_1 = \frac{1}{a_1}x_0 - \frac{a_2}{a_1}x_2 - \frac{a_3}{a_1}x_3$$

$$x_4 = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3$$

$a_0 = 0$

Optimal solution:

$x_0 = a_0,\ x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = b_0$

Optimal objective value:

$z = -x_0 = 0$

Original LP:

$z = \cdots$

$z = \cdots$

Auxiliary LP:

$z = \; -x_0$

$z = \; \cdots$

$$x_1 = \frac{1}{a_1}x_0 - \frac{a_2}{a_1}x_2 - \frac{a_3}{a_1}x_3$$

$$x_4 = b_0 + b_1\boxed{x_1} + b_2x_2 + b_3x_3$$

The constants here are still 0 and $b_0 \geq 0$

We now have a slack-form LP to start running SIMPLEX Algorithm with.

Original LP:

$z = \; \cdots$

$z \; = \; \cdots$

Let $x_0 = 0$

$$x_1 = \frac{1}{a_1}x_0 - \frac{a_2}{a_1}x_2 - \frac{a_3}{a_1}x_3$$

$$x_4 = b_0 + b_1\boxed{x_1} + b_2x_2 + b_3x_3$$

Is the SIMPLEX Algorithm a polynomial-time algorithm? No

But in practice it is very efficient. Why?

Can LP be solved in polynomial time? Yes

Quiz questions:

1. How to tell if an LP is feasible or not?

2. If an LP is feasible, how to find an initial basic feasible solution?