SLEPIAN WOLF ENCODING

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DISTRIBUTED SOURCE CODING

- An important problem in information theory and communication.
- DSC problems regard the compression of multiple correlated information sources that do not communicate with each other.
- By modeling the correlation between multiple sources at the decoder side together with channel codes, DSC is able to shift computation complexity from encoder side to the decoder side.
- Provides appropriate frameworks for video/multimedia compression and sensor networks.

SLEPIAN-WOLF BOUND

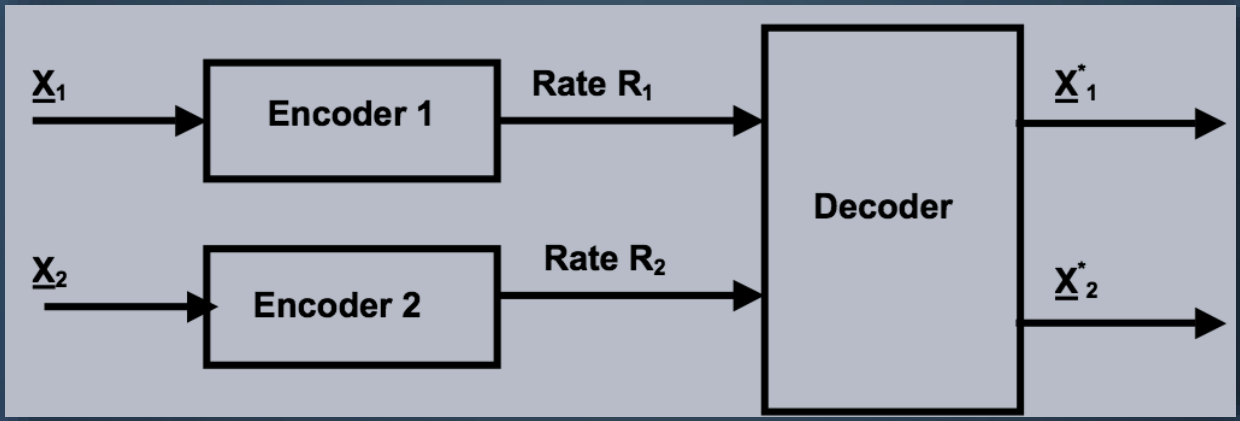
- Suppose Alice has X, and Bob has Y.
- Alice wants to compress data using entropy H(X)
- Bob wants to compress data using entropy H(Y)
- They use one decoder and the rate of compression is $R(X)+R(Y) = H(X)+H(Y)$.
- But, Slepian Wolf encoding states that even if there is no link between Alice and Bob, you can do with H(X,Y) compression rate, knowing the correlation between X and Y.

AN EXAMPLE

- Suppose, weather in 2 neighboring towns-Janestown and Thomasville, is correlated.
- Each town has a weather "good" or "bad" with equal probabilities, and the probability that the 2 towns have different weather on a given day is *p*.
- Jane, a resident in Janestown wants to send last year's weather to Tom in Thomasville.
  - If Jane knows the weather in Thomasville, she can just send the difference string, which will have $365p$ bits as "1" and $365(1-p)$ as a "0". The difference string can be compressed to $365h_2(p)$ length h_2 being the entropy of difference string.
  - Even if Jane is unaware of the weather in Thomasville, she can send the same length signal.

    This follows from **Slepian Wolf encoding.**

FORMULATION



Let X and Y be two discrete sources, with probabilities p(x) and p(y), which are formed from N-independent drawings from a joint distribution p(x,y).

An encoding function f_1 is a map from X^n (n length drawing from X) to 2^{nR1} of a codeword,

$$f_1: X^n \rightarrow \{1,2,\dots.2^{nR_1}\}$$

Similarly for Y,

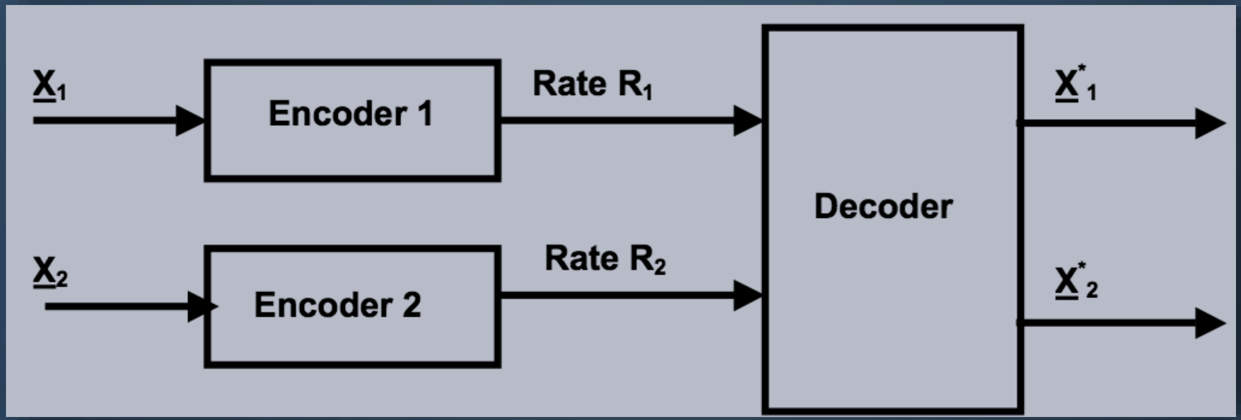
$$f_2: Y^n \rightarrow \{1,2,\dots.2^{nR_2}\}$$

Then there is a single decoding function g, which is:

$$g: \{1,2,\dots.2^{nR_1}\}x\{1,2,\dots.2^{nR_2}\} \rightarrow X^n \times Y^n$$

This gives us back the sequences. In this process the error probability $Pr(g(f_1(x^n),f_2(x^n)) \neq (x^n,y^n))) = P_e^{(n)}$. (R_1,R_2) are achievable if ∃ encoding and decoding functions, such that $P_e^{(n)} \rightarrow 0$, and $R_1 \geq H(X)$ and $R_2 \geq H(Y)$.

ADMISSIBLE RATE PAIRS



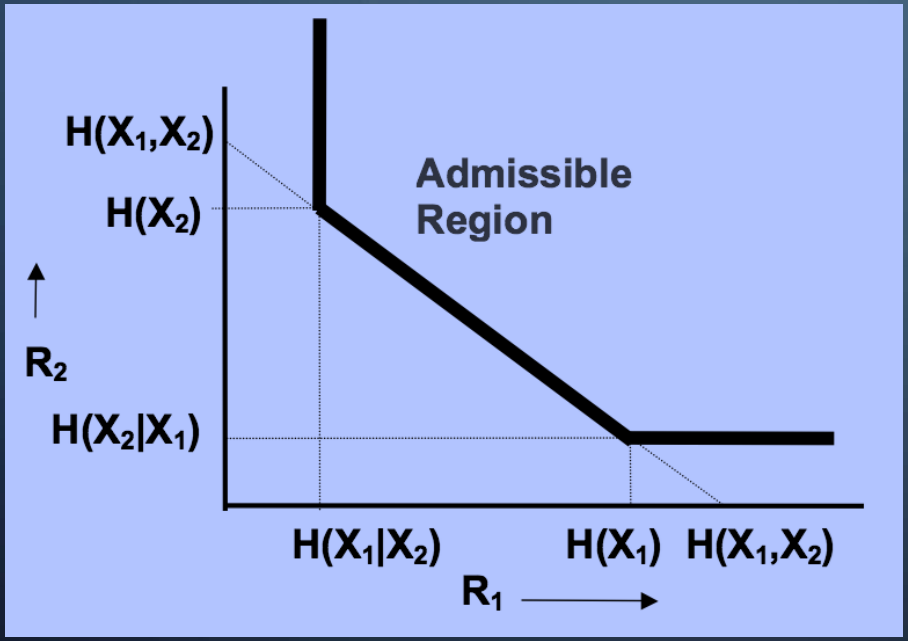
The systems of interest are those for which the probability that X_1^* doesn't equal X_1 and X_2^* doesn't equal X_2 can be made as small as possible by choosing sufficiently large n.

- Such systems are said to be *ADMISSIBLE SYSTEMS.*
- The rate pair (R1,R2) for an admissible system is called an *ADMISSIBLE RATE PAIR.*
- The closure of all such admissible pairs is called the *ADMISSIBLE RATE REGION.*

SLEPIAN WOLF THEOREM

states that:

The admissible rate region for the pair of rates (R1, R2) is the set of points that satisfy the three inequalities:



$$R_1 \geq H(X_1 \mid X_2)$$

$$R_2 \geq H(X_2 \mid X_1)$$

$$R_1 + R_2 \geq H(X_1, X_2)$$

SIGNIFICANCE OF THE THEOREM

- The significance of the theorem is seen by comparing it to the entropy bound for the compression rates of single sources.
- Separate encoders which ignore the source correlation can achieve rates only upto:

$$R_1+R_2 \geq H(X_1)+H(X_2).$$

- However for Slepian-Wolf coding the encoders exploit the knowledge of correlation to achieve the same rates as an optimal joint encoder, namely

$$R_1+R_2 \geq H\ (X_1,X_2).$$

Note that: $H(X_1)+H(X_2) \geq H(X1,X2)$

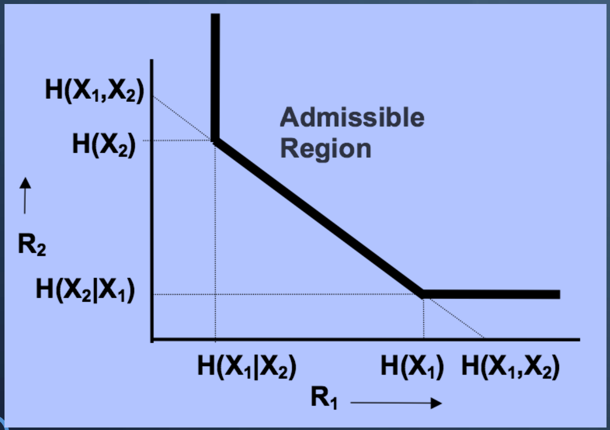
PROOF

- Necessity:
  - The necessity of the 3 inequalities follows by considering a modified system in which the pair of source sequences, X_1 and X_2 are input to a single encoder.
    - The output of this single encoder must have a rate at least $H(X_1, X_2)$. Thus, $R_1 + R_2 \geq H(X_1, X_2)$
    - Furthermore if the decoder somehow knows X_2, the single encoder would require a code whose rate is at least $H(X_1|X_2)$. Thus, $R_1 \geq H(X_1|X_2)$
    - By symmetry, $R_2 \geq H(X_2|X_1)$

PROOF (CONTD.)

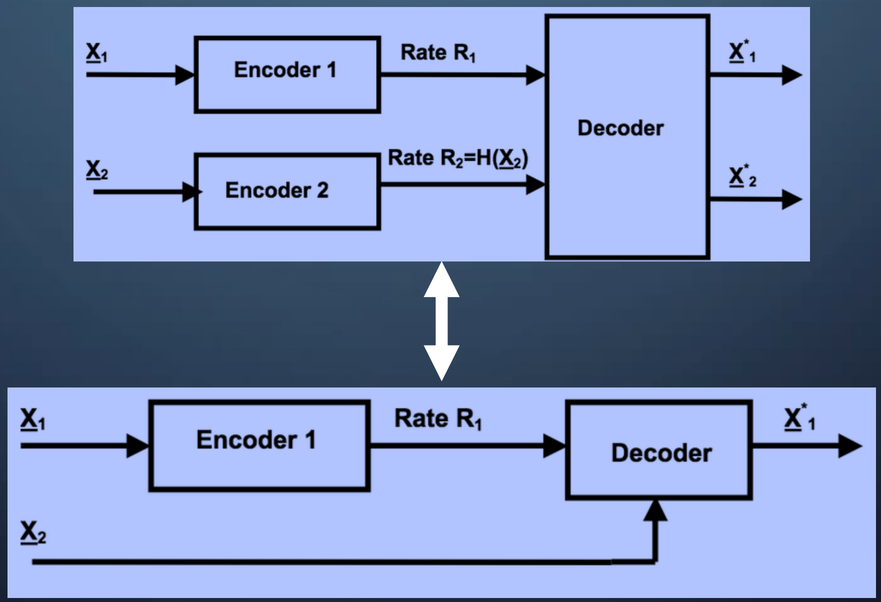
SUFFICIENCY:

- Claim: Proving the admissibility at $R_1=H(X_1|X_2)$, $R_2=H(X_2)$ is established, then we can say that the claimed region is admissible.
  - Proof:
    i. the admissibility of the rate pair $R_1=H(X_1)$, $R_2=H(X_2|X_1)$ follows from symmetry;
    ii. the admissibility of all rate pairs on the straight line connecting the rate pair $R_1=H(X_1|X_2)$, $R_2=H(X_2)$ with the rate pair $R_1=H(X_1)$, $R_2=H(X_2|X_1)$ follows from time-sharing; and
    iii. the admissibility of all rates within the admissibility region but not on the boundaries follows from wasting bits.



PROOF

- Sufficieny:
  - Consider the particular rate pair R_1=$H(X_1 | X_2)$ and R_2=$H(X_2)$.
  - Note that if R_1=$H(X_1 | X_2)$, R_2=$H(X_2)$, then the output of Encoder-2 suffices to reconstruct X_2. So, the system becomes equivalent to the below system.



PROOF (CONTD.)

- The original construction of an admissible system at the rate point $R_1=H(X_1|X_2)$, $R_2=H(X_2)$ was found for a particular model of the statistics of the correlated source pair called the **twin binary symmetric source.**
  - A twin binary symmetric source is a memoryless source whose outputs X_1 and X_2 are binary random variables (taking on the values 0 and 1) described by
    - $P(X_1=0)=P(X_1=1)$
    - $P(X_2=0|X_1=1)=P(X_2=1|X_1=0)=p$
    - $P(X_2=0|X_1=0)=P(X_2=1|X_1=1)=1\text{-}p,$

    $p,(1-p)$ where p is a parameter satisfying $0\leq p\leq 1$. Note that $P(X_2=0)=P(X_2=1)=1/2$.
  - Define $h_2(p)=-[p\log_2(p)+(1-p)\log_2(1-p)]$. For the twin binary symmetric source:
    - $H(X_1)=1$
    - $H(X_2)=1$
    - $H(X_2|X_1)=H(X_1|X_2)=h_2(p)$
    - $H(X_1,X_2)=1+h_2(p)$.

    For the twin binary symmetric source, the rate point of interest has $R1=H(X_1|X_2)=h_2(p)$ and $R_2=H(X_2)=1$.

PROOF (CONTD.)

- Compressing X_1:
  - The problem of compressing X_1 can be thought of as the problem where we transmit X_1 and X_2 is received at the other end of a Binary Symmetric channel. What we need is a parity check code!
  - It is known that a parity check code exists for a BSC with approx. $2^{n(1-h2(p))}$ codewords such that when X_2 is seen on the channel, we can reliably tell which codeword was actually sent (i.e. X_1).
  - But there are 2^n binary strings that can be X_1.
  - So, we do a construction called co-set decomposition.
  - Each co-set has a length $2^{n(1-h2(p))}$, therefore the number of cosets are: $2^n / 2^{n(1-h2(p))} = 2^{nh2(p)}$
  - Now the encoder of X_1 can send the information about X_1 in $nh_2(p)$ bits.
  - Decoder uses this information and the already present information X_2 to construct X_1 reliably.

PROOF (CONCLUSION)

- Applying this construction to the problem at hand, X_1 must be in one of the cosets of the group code.
- If the source encoder transmits to the decoder the identity of the coset containing X_1 , the decoder can determine X_1 from this knowledge and its knowledge of X_2 by using a decoder for the coset code that operated on the "received" word X_2 .
- Since there are $2^{nh2(p)}$ cosets, the encoder must transmit $nh_2(p)$ binary digits.

The rate of transmission is $h_2(p)=H(X_1|X_2)$. This establishes the admissibility of the rate point $R_1=H(X_1|X_2)=h_2(p)$ and $R_2=H(X_2)=1$.

The entire admissible rate region then follows from symmetry, time-sharing and wasting bits.

REMARKS

- The Slepian-Wolf theorem applies to a much wider class of problems than that described here.
- The statistical description of the source sequences can be of a much more general nature,
  - there can be more than two correlated source sequences, and
  - the configuration of the encoders and decoder can be of many different types.