Jordan Normal Forms and Jordan Blocks using Eigen Values and Eigen Vectors

G10_EV&V_A2

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***Abstract*— In this paper, we have tried to design methods that can help to find whether a matrix is diagonalizable or not and also its equivalent Jordan Normal form. We also describe real world applications of Jordan Normal form and also, Intuitation behind the concepts of Jordan Blocks and Jordan Boxes. This paper is more towards Mathematical Understanding because to solve some real problem we need to understand the core concepts behind the Jordan blocks and Eigen values.**

***Keywords*— Algebraic Multiplicity, Geometric Multiplicity, Jordan Normal Forms, Jordan blocks, Jordan boxes, Kernel Matrix, Dimension of Matrix, Diagonalizability.**

I. Introduction

In linear algebra, Eigen vectors of a linear transformations ,is the vector which is scaled when the corresponding linear transformation applied on it. The factor which is responsible for Eigen vectors is called as Eigen values. Eigen values and Eigen vectors are of great importance in determining the diagonalizable property of a matrix (Mainly square matrices). For a matrix to be diagonalizable, there is some conditions which need to be followed, First one all its Eigen values should be distinct as well as corresponding Eigen vector of each Eigen value should be linearly independent. Concept of algebraic multiplicity and geometric multiplicity also comes into light in determining diagonalizable behaviour of matrix. Algebraic multiplicity, Geometric multiplicity index of an Eigen values is used to determine its power in the characteristic polynomial. Whereas geometric multiplicity is equal to nullity of matrix, where nullity is the cardinality of rows with all the elements as zero Algebraic multiplicity should be equal to geometric multiplicity, for a matrix to be diagonalizable. The number of distinct Eigen values gives count of Jordan blocks and geometric multiplicity gives the number of smaller Jordan boxes present in big Jordan block.[1]

II. Background

A. Description

In, this project we discussed the core concept of Jordan Normal Forms and Jordan Blocks and "How this concept is useful to many real world applications" our discussion is going to be purely mathematical and in last we implemented the basic code to show the Jordan matrix and we use this to perform further tasks in different areas. There are many applications of Jordan Normal form in field of Applied Mathematics, Bio-Physics, Markov processes and many more.
The Concept of Jordan Normal form is used only when the preciseness is needed because the Jordan Block is high sensitive towards the preciseness of floating point values (i.e. very negligible manipulation of value in input Matrix can make a very huge difference in the values of Jordan Matrix.)[2]

B. Basic Idea

Consider a square matrix A, then there exists an Invertible Matrix X of same size such that this matrix forms the matrix A into a diagonal matrix.
More concretely this means,

$$(X^{-1}\,.A\,.\,X) = D(Daigonal\ Matrix)$$

In another way, we can say that we are decomposing the matrix A into 3 matrices, i.e.

$$A = X\ .D.\ X^{-1}(Matix\ Decomposition)$$

The below method only work when given matrix A is Daigonalizable,
By using this method we can calculate the powers of matrix A, We can use the decomposition and need the power of diagonal matix D, (Calculating power of diagonal matrix is very easy we just need the entries of the the diagonal elements and calculate the power of these elements).
Now, we are generalizing the above method for calculating the power for all square matrix.

C. *Generalization of Technique*

Now we are generalizing the above method for calculating the power for all the square matrices. In general case, there always exists Jordan Normal form.

$A \in C^{m*m}$: *There always exists Jordan Matrix J*

A: Set of all Square matrices
C: Set of all matrices with dimension m*m and which contains all complex numbers (also include real numbers)
J: J is a Jordan matix equivalent to A.

Fact:
Sometimes, matrix A contains only Real numbers but the Jordan Matrix J doesn't always contain only Real numbers, Complex number may be possible in Jordan Matrix J.[3]

D. *Algorithm to Find Jordan Block*

The **algorithm** to find the Jordan is given below[8],

1. ***Calculate all the Eigen values of A : (*** $\boldsymbol{\lambda_1, \lambda_2, \lambda_3, \ldots .\lambda_K}$***)***

2. ***For all Eigen values from 1 to k***

   I. Determine the Algebraic multiplicity of $\boldsymbol{\lambda_i}$ ***and then calculate the Geometric multiplicity of*** $\boldsymbol{\lambda_i}$.

   Geometric multiplicity = dim($\boldsymbol{Ker(A - \lambda_i . I)}$***)***

   II. If needed we can calculate further until dimension doesn't change

$$\boldsymbol{(A - \lambda_i . I)^2}$$

   $\boldsymbol{dim(Ker(A - \lambda_i . I)^2),\ dim(Ker(A - \lambda_i . I)^3) \ldots\ldots\ldots}$

3. **End**

E. *Working of algorithm in example*

Now we are taking the example to explaining the algorithm in 4*4 matrix.

$$A = \begin{matrix} 3 & 1 & 0 & 1 \\ -1 & 5 & 4 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{matrix}$$

Eigen values: det(A–**λ.**I) =

$$= \det\begin{pmatrix} \begin{pmatrix} 3-\boldsymbol{\lambda} & 1 \\ -1 & 5-\boldsymbol{\lambda} \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 4 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 2-\boldsymbol{\lambda} & 0 \\ 0 & 4-\boldsymbol{\lambda} \end{pmatrix} \end{pmatrix}$$

$$= \det\begin{pmatrix} 3-\boldsymbol{\lambda} & 1 \\ -1 & 5-\boldsymbol{\lambda} \end{pmatrix} \cdot \det\begin{pmatrix} 2-\boldsymbol{\lambda} & 0 \\ 0 & 4-\boldsymbol{\lambda} \end{pmatrix}$$

$$= \big((3-\boldsymbol{\lambda})(5-\boldsymbol{\lambda})+1\big).(2-\boldsymbol{\lambda})(4-\boldsymbol{\lambda})$$

$$= (16 - 8\boldsymbol{\lambda} + \boldsymbol{\lambda^2})(2-\boldsymbol{\lambda})(4-\boldsymbol{\lambda})$$

$$= (4-\boldsymbol{\lambda})^2\ (2-\boldsymbol{\lambda})(4-\boldsymbol{\lambda}) = \ (2-\boldsymbol{\lambda})^1.(4-\boldsymbol{\lambda})^3$$

$= \boldsymbol{\lambda_1} = 2$ with $\propto(\boldsymbol{\lambda_1}) = 1$ {Jordan Block of 1x1}

$= \boldsymbol{\lambda_2} = 4$ with $\propto(\boldsymbol{\lambda_2}) = 3$ {Jordan Block of 3x3}

$$J = \begin{array}{|c|cccc} \hline 2 & \multicolumn{1}{|c}{} & & & \\ \hline & \multicolumn{1}{|c}{4} & & & \multicolumn{1}{c|}{} \\ & \multicolumn{1}{|c}{} & 4 & & \multicolumn{1}{c|}{} \\ & \multicolumn{1}{|c}{} & & 4 & \multicolumn{1}{c|}{} \\ \cline{2-5} \end{array}$$

For placing 4 there are 3 possibilities which can be denoted by

Geometric multiplicity,

1[st] possibility: $\begin{matrix} \boxed{4} & & \\ & \boxed{4} & \\ & & \boxed{4} \end{matrix}$ $Geometric\ Multiplicity(\boldsymbol{\lambda_2})$ =3

2[nd] possibility: $\begin{matrix} \boxed{\begin{matrix} 4 & 1 \\ & 4 \end{matrix}} & \\ & \boxed{4} \end{matrix}$ $Geometric\ Multiplicity(\boldsymbol{\lambda_2})$ =2

3[rd] possibility: $\boxed{\begin{matrix} 4 & & \\ & 4 & \\ & & 4 \end{matrix}}$ $Geometric\ Multiplicity(\boldsymbol{\lambda_2})$ =1

Eigen space for$(\boldsymbol{\lambda}_2)$ = 4

$$Ker(\boldsymbol{A}-\boldsymbol{\lambda}_2.I) = \ker\begin{pmatrix} 3-4 & 1 & 0 & 1 \\ -1 & 5-4 & 4 & 1 \\ 0 & 0 & 2-4 & 0 \\ 0 & 0 & 0 & 4-4 \end{pmatrix}$$

Now we are doing Gaussian elimination,

$$= ker\begin{pmatrix} -1 & 1 & 0 & 1 \\ -1 & 1 & 4 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now, 2nd row = (2nd row – 1st row),

$$= ker\begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now, 3rd row = (3rd row $-\frac{1}{2} * 2^{nd}$ row),

$$\begin{array}{c} = ker\begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{cccc} & \uparrow & & \uparrow \\ & x_2 & & x_4 \end{array} \end{array}$$

Free variables: $\{ x_2, x_4\}$

$\dim(Ker(\boldsymbol{A}-\boldsymbol{\lambda}_2.I)) = 2$

Now, we select the possibility of Geometric multiplicity is 2. So, the Jordan Block for giving example is given below,

$$J = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

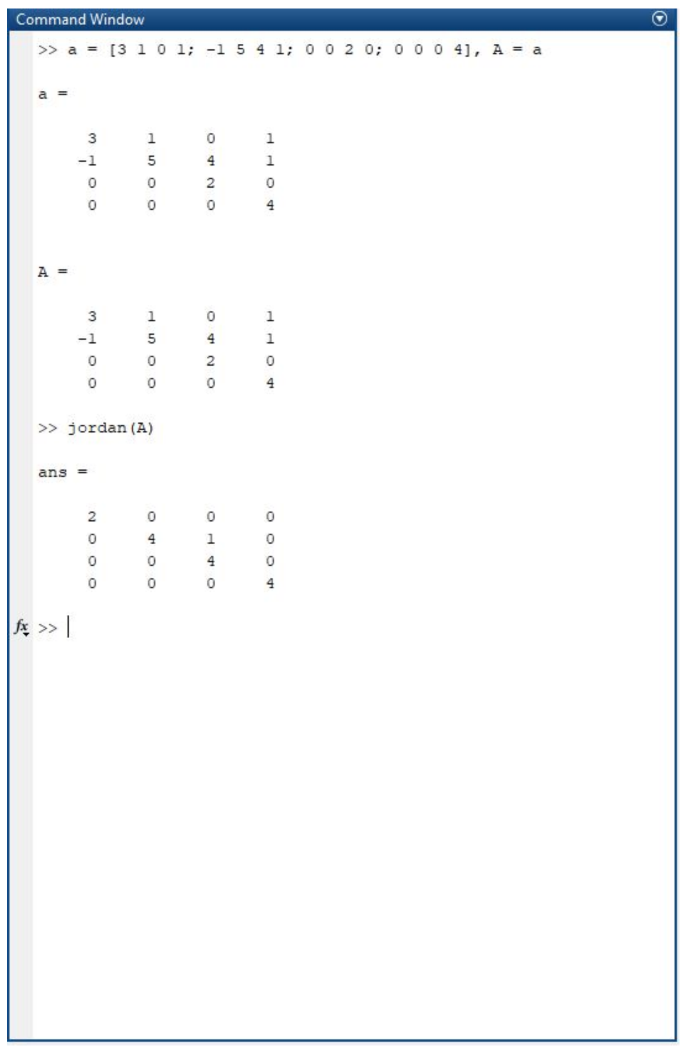
III. Application of Jordan Normal Forms and Jordan Blocks

The concept of Jordan Block which helps to solve the Tough problems like,

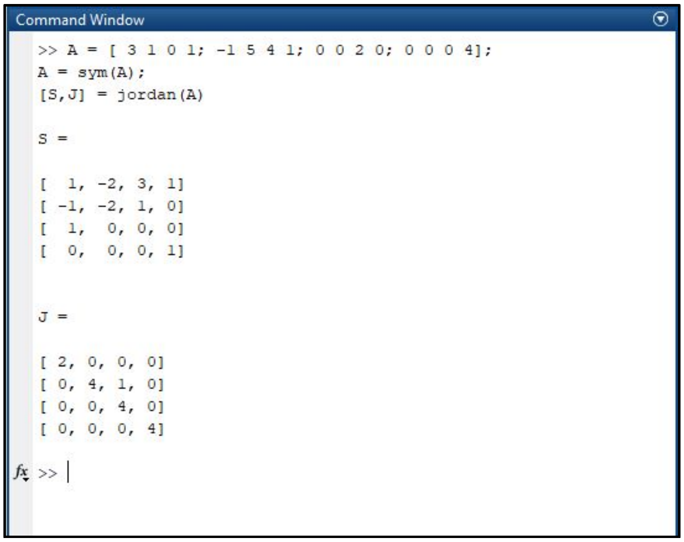
1. Linear Quadratic Mechanism problem
2. Validating configuration of Mean Vector
3. Reducing subspaces by using Block Daigonalization technique.
4. Similarity Transformations.

IV. Experiment and Analysis

For, experiments we extensively used the MATLAB's Symbolic Math Toolbox™ for performing the analysis and experiments in our project.

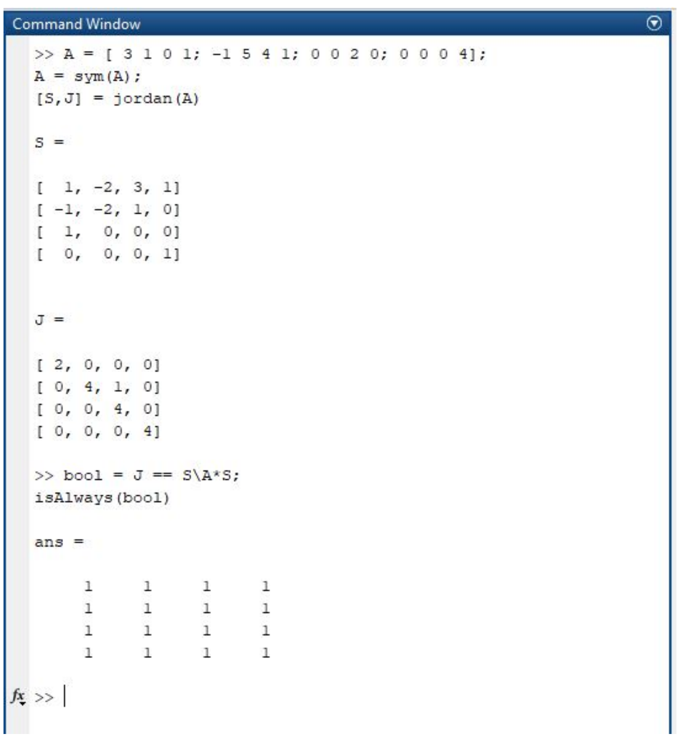

```
Command Window
>> a = [3 1 0 1; -1 5 4 1; 0 0 2 0; 0 0 0 4], A = a

a =

     3     1     0     1
    -1     5     4     1
     0     0     2     0
     0     0     0     4


A =

     3     1     0     1
    -1     5     4     1
     0     0     2     0
     0     0     0     4

>> jordan(A)

ans =

     2     0     0     0
     0     4     1     0
     0     0     4     0
     0     0     0     4

>>
```

In above given code we validated our calculation using traditional method and we find out that our calculated Jodan Block is same as Matlab output.
Now we are going to calculate Jordan Matrix and similarity transform Matrix



```
Command Window
>> A = [ 3 1 0 1; -1 5 4 1; 0 0 2 0; 0 0 0 4];
A = sym(A);
[S,J] = jordan(A)

S =

[  1, -2, 3, 1]
[ -1, -2, 1, 0]
[  1,  0, 0, 0]
[  0,  0, 0, 1]


J =

[ 2, 0, 0, 0]
[ 0, 4, 1, 0]
[ 0, 0, 4, 0]
[ 0, 0, 0, 4]

>>
```

Now we are verifying the condition S\A*S =J or not,



```
Command Window
>> A = [ 3 1 0 1; -1 5 4 1; 0 0 2 0; 0 0 0 4];
A = sym(A);
[S,J] = jordan(A)

S =

[  1, -2, 3, 1]
[ -1, -2, 1, 0]
[  1,  0, 0, 0]
[  0,  0, 0, 1]


J =

[ 2, 0, 0, 0]
[ 0, 4, 1, 0]
[ 0, 0, 4, 0]
[ 0, 0, 0, 4]

>> bool = J == S\A*S;
isAlways(bool)

ans =

     1     1     1     1
     1     1     1     1
     1     1     1     1
     1     1     1     1

>>
```

V. Conclusion

This report draws attention to the mathematical techniques required to find the equivalent Jordan Normal Form of a square matrix. We have also tried to give glimpse of certain topics like algebraic multiplicity and geometric multiplicity which are crucial in mathematical calculation of Jordan blocks and Jordan boxes. The concept of diagonalization of matrices and Jordan normal form can be generalized to solve many practical problems in the long run.

VI. Appendix

MATLAB Code:

A = [3 1 0 1; -1 5 4 1; 0 0 2 0; 0 0 0 4];

A = sym(A);

[S,J] = jordan(A)

*bool = J == S\A*S;*

isAlways(bool)

VII. Acknowledgment

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