Technical University Braunschweig **Summer Term 2023**
Institute for Partial Differential Equations

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Formulas for the exams in
Numerical Methods for ordinary and partial differential equations
Computational Sciences in Engineering

Numerical Basics

Lagrange basis polynomials

$$\text{grad } L_n^N = N, \quad L_n^N(x_i) = \delta_{in}$$

B-Spline recurssion grad $B_{i,k} = k-1$,

$$B_{i,k}(x) = \frac{x - x_i}{x_{i+k} - x_i} B_{i,k-1}(x) + \frac{x_{i+k+1} - x}{x_{i+k+1} - x_{i+1}} B_{i+1,k-1}(x)$$

Newton polynomial

$$w(x) = \prod_{i=0}^{N} (x - x_i), \quad \text{grad } N+1$$

Interpolation error

$$r(x) = -\frac{w(x)}{(N+1)!} f^{(N+1)}(\xi(x))$$

Quadrature formula

$$I(f) \approx I(p) = \sum_{n=0}^{N} f(x_n) \int_a^b L_n^N(x)\mathrm{d}x$$

Newton Cotes formulas

$$a = x_0,\, b = x_N,\;\; x_{i+1} - x_i = h$$

midpoint $N = 0$, trapezoid $N = 1$,
Kepler $N = 2$

Gaussian quadrature

$$I(f) = I(p), \forall p:\ \text{grad } p < 2N+2$$

Quadrature error

$$|I(f) - I(p)| \le \int_a^b |r(x)|\mathrm{d}x$$

Kepler $\quad |R(f)| \le \frac{(b-a)^5}{2880} M_4$

Trapezoid $\quad |R(f)| \le \frac{(b-a)^3}{12} M_2$

Romberg's extrapolation

$$Q_{new} = \frac{2^q Q_{2J}^{old} - Q_J^{old}}{2^q - 1}$$

Difference quotient

$$\frac{f(x+h) - f(x)}{h} \approx f'(x)$$

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \approx f''(x)$$

Banach fixed-point $\vartheta < 1$

$$\|x_k - x^*\| \le \frac{\vartheta^k}{1-\vartheta} \|x_1 - x_0\|$$

$$\|x_k - x^*\| \le \frac{\vartheta}{1-\vartheta} \|x_k - x_{k-1}\|$$

Newton-Method etc.

$$x_{k+1} = x_k - \lambda D^{-1} f(x_k)$$

$$\text{e.g. } \lambda = 1,\ D = \nabla f(x_k)$$

Ordinary differential equations

Euler-Heun method

$$y_{i+1} = y_i + \frac{h_i}{2}\left[f(t_i, y_i) + f(t_{i+1}, y_i + h_i f(t_i, y_i))\right]$$

Improved Euler method

$$y_{i+1} = y_i + h_i f\left(t_i + \frac{h_i}{2}, y_i + \frac{h_i}{2} f(t_i, y_i)\right)$$

Crank-Nicolson method

$$y_{i+1} = y_i + \frac{h_i}{2}[f(t_i, y_i) + f(t_{i+1}, y_{i+1})]$$

classical Runge-Kutta

in general with increment function

$$y_{i+1} = y_i + h_i \Phi(t_i, y_i, h_i)$$

Truncation error

$$\tau = \frac{y(t+h) - y(t)}{h} - \Phi(t, y(t), h)$$

Consistency order q $\tau = \mathcal{O}(h^q)$ for $h \to 0$

Runge-Kutta, s-stages

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array}$$

Model problem

$$y' = -ay, \quad \mathrm{Re}\, a \geq 0$$

Stability region

$$S = \{\mu \in \mathbb{C} : \, |p(\mu)| \leq 1\}$$

A-stability

$$\{z : \, \mathrm{Re}\, z \leq 0\} \subset S$$

Stiff differential equations: step size for explicit methods "unnecessarily" small

Adams-Bashforth method

$m = 2$:

$$y_{i+1} = y_i + h\left[\frac{3}{2} f(t_i, y_i) - \frac{1}{2} f(t_{i-1}, y_{i-1})\right]$$

$m = 3$:

$$y_{i+1} = y_i + h\left[\frac{23}{12} f(t_i, y_i) - \frac{16}{12} f(t_{i-1}, y_{i-1}) + \frac{5}{12} f(t_{i-2}, y_{i-2})\right]$$

Adams-Moulton method

$m = 2$:

$$y_{i+1} = y_i + h\left[\frac{5}{12} f(t_{i+1}, y_{i+1}) + \frac{8}{12} f(t_i, y_i) - \frac{1}{12} f(t_{i-1}, y_{i-1})\right]$$

BDF

$$m = 2: \quad \frac{3}{2} y_{i+1} - 2y_i + \frac{1}{2} y_{i-1} = h f(t_{i+1}, y_{i+1})$$

Partial differential equations

Dirichlet BC

$$u = g \quad \text{on } \partial\Omega$$

Neumann boundary condition

$$\mathbf{I} \cdot \mathbf{n} = p \quad \text{on } \partial\Omega$$

Method of lines $u(t, x_i) \approx u_j(t)$

$$\mathrm{spec}\begin{pmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & 1 \\ & \ddots & 1 & -2 \end{pmatrix} = \left\{-2 + 2\cos\frac{k\pi}{n}, \, k = 1, \dots, n-1\right\}$$

CFL-condition

$$u_{,t} = a u_{,xx} \quad a\Delta t < \frac{1}{2}\Delta x^2$$

Discretization of Δu in $\mathbb{R}^2$