221116 Linear ODE - Variation of constants

We have linear ODE

$$\boxed{y'(t) + a(t)\,y(t) = \underbrace{p(t)}_{\text{External excitation.}}} \quad -(1)$$

If $p(t) = 0$, ODE is Homogeneous

If $p(t) \neq 0$, ODE is non-Homogeneous

* We solve the Homogeneous equation first, ie put $p(t) = 0$

What's $y_f = ?$

By separation of variables, we get

$$\boxed{y_h(t) = c \cdot y_f(t)} \quad (2)$$

Where, $y_h(t) \rightarrow$ Homogeneous solution.

$$y_f(t) = \exp\left(-\int_{t_0}^{t} a(\tau)\,d\tau\right)$$

* Now, We solve the non-Homogeneous equation.

We use the "Variation of Constant Method"

We take

$$\boxed{y_p(t) = c(t) \cdot y_f(t)} \quad -(3)$$

, where $y_p \rightarrow$ Particular solution

$$\Rightarrow y_p'(t) = c'(t)\,y_f(t) + c(t)\,y_f'(t)$$

substitute values of y_p' & y_h into the Linear ODE ($y'(t) + a(t)y(t) = p(t)$)

$$\Rightarrow \underbrace{c'(t)\,y_f(t) + c(t)\,y_f'(t)}_{y_p'} + a(t)\,\underbrace{c\,y_f(t)}_{y_h} = p(t)$$

$$c'(t)\,y_f(t) + c(t)\left\{\underbrace{y_f'(t) + a(t)\,y_f(t)}_{\text{because } y_f = 0\ ?}\right\} = p(t)$$

Particular = One out of the bundle

$$\Rightarrow c'(t)\,y_f(t) = p(t)$$

$$\boxed{c(t) = \int_{t_0}^{t} \frac{p(\tau)}{y_f(\tau)}\,d\tau + \tilde{c}} \quad -(4)$$

$\tilde{c}$ arbitrary because only particular solution is needed.

substitute (4) in (3)

$$\Rightarrow \boxed{y_p(t) = y_f(t) \int_{t_0}^{t} \frac{p(\tau)}{y_f(\tau)}\,d\tau + \tilde{c}\,y_f(\tau)}$$

, where $\tilde{c} \in \mathbb{R} \sim 0$

Since Particular