(P1) $\ldots$

(P2) $\ldots$

$\vdots$

(C) $\ldots$

(P1) Zeno is a philosopher.

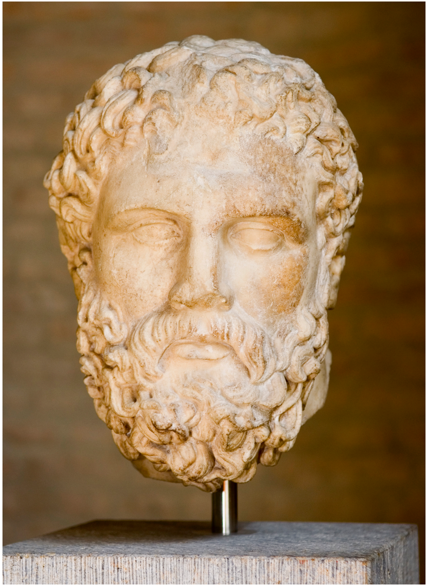
(P2) If Zeno is a philosopher, then he puts forward arguments.

(C) Zeno puts forward arguments.

(P1) A.

(P2) If A, then B.

(C) B.



Paradox:

a logically valid argument in which each premise is plausible if taken by itself, but where the conclusion of the argument is absurd.

$$\left.\begin{array}{ccl} \text{(P1)} & \ldots & \text{[plausible]} \\ \text{(P2)} & \ldots & \text{[plausible]} \\ \vdots & & \\ \hline \\ \text{(C)} & \ldots & \text{[absurd]} \end{array}\right\}\ \text{logically valid}$$

(P1) At the beginning of the race, Achilles is at point x_0, and the tortoise is at point x_1, where x_0 is to the left of x_1.

(P1) At the beginning of the race, Achilles is at point x_0, and the tortoise is at point x_1, where x_0 is to the left of x_1.

(P2) When Achilles is at point x_1, the tortoise is at point x_2, where x_1 is to the left of x_2, and Achilles does not overtake the tortoise at any point between x_0 and x_1.

(P1) At the beginning of the race, Achilles is at point x_0, and the tortoise is at point x_1, where x_0 is to the left of x_1.

(P2) When Achilles is at point x_1, the tortoise is at point x_2, where x_1 is to the left of x_2, and Achilles does not overtake the tortoise at any point between x_0 and x_1.
When Achilles is at point x_2, the tortoise is at point x_3, where x_2 is to the left of x_3, and Achilles does not overtake the tortoise at any point between x_1 and x_2.

(P1) At the beginning of the race, Achilles is at point x_0, and the tortoise is at point x_1, where x_0 is to the left of x_1.

(P2) When Achilles is at point x_1, the tortoise is at point x_2, where x_1 is to the left of x_2, and Achilles does not overtake the tortoise at any point between x_0 and x_1.
When Achilles is at point x_2, the tortoise is at point x_3, where x_2 is to the left of x_3, and Achilles does not overtake the tortoise at any point between x_1 and x_2.

$\vdots$

(P1) At the beginning of the race, Achilles is at point x_0, and the tortoise is at point x_1, where x_0 is to the left of x_1.

(P2) For all $n \geq 1$, when Achilles is at x_n, the tortoise is at x_{n+1}, where x_n is to the left of x_{n+1}, and Achilles does not overtake the tortoise at any point between x_{n-1} and x_n.

(P1) At the beginning of the race, Achilles is at point x_0, and the tortoise is at point x_1, where x_0 is to the left of x_1.

(P2) For all $n \geq 1$, when Achilles is at x_n, the tortoise is at x_{n+1}, where x_n is to the left of x_{n+1}, and Achilles does not overtake the tortoise at any point between x_{n-1} and x_n.

(P3) If P1 and P2 are the case, then
Achilles does not overtake the tortoise at any of x_1, x_2,... nor at any point in between any two of them.

(P4) It takes Achilles a positive amount T_0 of time to get from x_0 to x_1.
It takes Achilles a positive amount T_1 of time to get from x_1 to x_2.

$\vdots$

(P4) For all n, it takes Achilles a positive amount T_n of time to get from x_n to x_{n+1}.

(P4) For all n, it takes Achilles a positive amount T_n of time to get from x_n to x_{n+1}.

(P5) If P4 is the case,
then for all n, it takes Achilles $T_0 + \ldots + T_n$ of time to get from x_0 to x_{n+1}, where each of $T_0, \ldots, T_n$ is a positive amount of time.

(P4) For all n, it takes Achilles a positive amount T_n of time to get from x_n to x_{n+1}.

(P5) If P4 is the case,
then for all n, it takes Achilles $T_0 + \ldots + T_n$ of time to get from x_0 to x_{n+1}, where each of $T_0, \ldots, T_n$ is a positive amount of time.

(P6) If for all n, it takes Achilles $T_0 + \ldots + T_n$ of time to get from x_0 to x_{n+1}, where each of $T_0, \ldots, T_n$ is a positive amount of time,
then the amount of time $T_0 + T_1 + T_2 + \ldots$ in which Achilles is to pass each of $x_0, x_1, x_2, \ldots$ is not bounded by any finite amount of time.

(P4) For all n, it takes Achilles a positive amount T_n of time to get from x_n to x_{n+1}.

(P5) If P4 is the case,
then for all n, it takes Achilles $T_0 + \ldots + T_n$ of time to get from x_0 to x_{n+1}, where each of $T_0, \ldots, T_n$ is a positive amount of time.

(P6) If for all n, it takes Achilles $T_0 + \ldots + T_n$ of time to get from x_0 to x_{n+1}, where each of $T_0, \ldots, T_n$ is a positive amount of time,
then the amount of time $T_0 + T_1 + T_2 + \ldots$ in which Achilles is to pass each of $x_0, x_1, x_2, \ldots$ is not bounded by any finite amount of time.

(P7) If the amount of time $T_0 + T_1 + T_2 + \ldots$ in which Achilles is to pass each of $x_0, x_1, x_2, \ldots$ is not bounded by any finite amount of time, then Achilles never actually passes all of $x_0, x_1, x_2, \ldots$

(P4) For all n, it takes Achilles a positive amount T_n of time to get from x_n to x_{n+1}.

(P5) If P4 is the case,
then for all n, it takes Achilles $T_0 + \ldots + T_n$ of time to get from x_0 to x_{n+1}, where each of $T_0, \ldots, T_n$ is a positive amount of time.

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(P7) If the amount of time $T_0 + T_1 + T_2 + \ldots$ in which Achilles is to pass each of $x_0, x_1, x_2, \ldots$ is not bounded by any finite amount of time, then Achilles never actually passes all of $x_0, x_1, x_2, \ldots$

(P8) If Achilles never actually passes all of $x_0, x_1, x_2, \ldots$, then he is always somewhere to the left of at least one of $x_1, x_2, \ldots$

(P9) If Achilles does not overtake the tortoise at any of $x_1, x_2, \ldots$ nor at any point in between either of them, and if he is always somewhere to the left of at least one of $x_1, x_2, \ldots$,
then Achilles never overtakes the tortoise.

(P1) At the beginning of the race, Achilles is at point x_0, and the tortoise is at point x_1, where x_0 is to the left of x_1.

$\vdots$

(P9) If Achilles does not overtake the tortoise at any of $x_1, x_2, \ldots$ nor at any point in between either of them, and if he is always somewhere to the left of at least one of $x_1, x_2, \ldots$,
then Achilles never overtakes the tortoise.

(C) Achilles never overtakes the tortoise.

(P1) At the beginning of the race, Achilles is at point x_0, and the tortoise is at point x_1, where x_0 is to the left of x_1.

(P2) For all $n \geq 1$, when Achilles is at x_n, the tortoise is at x_{n+1}, where x_n is to the left of x_{n+1}, and Achilles does not overtake the tortoise at any point between x_{n-1} and x_n.

(P3) If P1 and P2 are the case, then
Achilles does not overtake the tortoise at any of x_1, x_2,... nor at any point in between any two of them.

(P4) For all n, it takes Achilles a positive amount T_n of time to get from x_n to x_{n+1}.

(P5) If P4 is the case,
then for all n, it takes Achilles $T_0 + \ldots + T_n$ of time to get from x_0 to x_{n+1}, where each of $T_0, \ldots, T_n$ is a positive amount of time.

(P6) If for all n, it takes Achilles $T_0 + \ldots + T_n$ of time to get from x_0 to x_{n+1}, where each of $T_0, \ldots, T_n$ is a positive amount of time,
then the amount of time $T_0 + T_1 + T_2 + \ldots$ in which Achilles is to pass each of $x_0, x_1, x_2, \ldots$ is not bounded by any finite amount of time.

(P7) If the amount of time $T_0 + T_1 + T_2 + \ldots$ in which Achilles is to pass each of $x_0, x_1, x_2, \ldots$ is not bounded by any finite amount of time, then Achilles never actually passes all of $x_0, x_1, x_2, \ldots$

(P8) If Achilles never actually passes all of $x_0, x_1, x_2, \ldots$, then he is always somewhere to the left of at least one of $x_1, x_2, \ldots$

(P9) If Achilles does not overtake the tortoise at any of $x_1, x_2, \ldots$ nor at any point in between either of them, and also he is always somewhere to the left of at least one of $x_1, x_2, \ldots$,
then Achilles never overtakes the tortoise.

Therefore:

(C) Achilles never overtakes the tortoise.

(P6) If for all n, it takes Achilles $T_0 + \ldots + T_n$ of time to get from x_0 to x_{n+1}, where each of $T_0, \ldots, T_n$ is a positive amount of time,
then the amount of time $T_0 + T_1 + T_2 + \ldots$ in which Achilles is to pass each of $x_0, x_1, x_2, \ldots$ is not bounded by any finite amount of time.

$$T_0 = \frac{1}{2}$$

$$T_1 = \frac{1}{4}$$

$$T_2 = \frac{1}{8}$$

$$\vdots$$

$$T_0 = \frac{1}{2}$$

$$T_1 = \frac{1}{4}$$

$$T_2 = \frac{1}{8}$$

$$\vdots$$

$$T_n = \frac{1}{2^{n+1}}$$

$$T_0 + T_1 + \ldots + T_n = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{n+1}}$$

$$T_0 + T_1 + \ldots + T_n = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{n+1}} < 1$$

$$T_0 + T_1 + T_2 + \ldots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1$$

Natural numbers: $0, 1, 2, 3, \ldots$

$$\{2, 1, 3\} = \{1, 2, 3\}$$

$$\{1, 2, 2, 3\} = \{1, 2, 3\}$$

$\{$Hannes Leitgeb$\}$

$$\{\text{Hannes Leitgeb}\} = \{\text{the philosopher who is giving this lecture}\}$$

{}

Principle of Extensionality:

For all sets X, Y: $X = Y$ if and only if
for all z it holds: z is a member of X if and only if z is a member of Y.

("Two sets are identical if and only if they have the same members.")

$\{1\}$ is a subset of $\{1, 2\}$.

$\{1, 2\}$ is a subset of $\{1, 2, 3\}$.

Subsets:

> For all sets X, Y: X is a subset of Y if and only if
> for all z it holds: if z is a member of X, then z is a member of Y.

("One set is a subset of another if and only if every member of the former set is also a member of the latter set.")

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$\{1, 2\}$ is a subset of $\{1, 2, 3\}$.

$\{1, 2, 3\}$ is not a subset of $\{1, 2\}$.

Subsets:

> For all sets X, Y: X is a subset of Y if and only if
> for all z it holds: if z is a member of X, then z is a member of Y.

("One set is a subset of another if and only if every member of the former set is also a member of the latter set.")

$\{1, 2\}$ is a subset of $\{1, 2, 3\}$.

$\{1, 2, 3\}$ is not a subset of $\{1, 2\}$.

$\{1, 2\}$ is a proper subset of $\{1, 2, 3\}$.

Proper Subsets:

> For all sets X, Y: X is a proper subset of Y if and only if X is a subset of Y, but Y is not a subset of X.

("One set is a proper subset of another if and only if the former is a subset of the latter, but not the other way around.")

(P1) If X is a proper subset of Y, then X does not have equally many members as Y.

(P1) If X is a proper subset of Y, then X does not have equally many members as Y.

$\{1, 2\}$ is a proper subset of $\{1, 2, 3\}$.

Indeed: $\{1, 2\}$ does not have equally many members as $\{1, 2, 3\}$.

(P1) If X is a proper subset of Y, then X does not have equally many members as Y.

(P2) If there is a pairing off between the members of X and the members of Y, then X has equally many members as Y.

(P1) If X is a proper subset of Y, then X does not have equally many members as Y.

(P2) If there is a pairing off between the members of X and the members of Y, then X has equally many members as Y.

(P3) The set of even natural numbers is a proper subset of the set of positive natural numbers.

$\{2, 4, 6, \ldots\}$ is a proper subset of $\{1, 2, 3, 4, 5, 6, \ldots\}$.

(P1) If X is a proper subset of Y, then X does not have equally many members as Y.

(P2) If there is a pairing off between the members of X and the members of Y, then X has equally many members as Y.

(P3) The set of even natural numbers is a proper subset of the set of positive natural numbers.

(P4) There is a pairing off between the even natural numbers and the positive natural numbers.

(P1) If X is a proper subset of Y, then X does not have equally many members as Y.

(P2) If there is a pairing off between the members of X and the members of Y, then X has equally many members as Y.

(P3) The set of even natural numbers is a proper subset of the set of positive natural numbers.

(P4) There is a pairing off between the even natural numbers and the positive natural numbers.

(C) The set of even natural numbers does not have equally many members as the set of positive natural numbers, and the set of even natural numbers does have equally many members as the set of positive natural numbers.

(P1*) If X is a proper subset of Y, then X does not have equally$_1$ many members as Y.

(P2*) If there is a pairing off between the members of X and the members of Y, then X has equally$_2$ many members as Y.

(P3) The set of even natural numbers is a proper subset of the set of positive natural numbers.

(P4) There is a pairing off between the even natural numbers and the positive natural numbers.

(C*) The set of even natural numbers does not have equally$_1$ many members as the set of positive natural numbers, and the set of even natural numbers does have equally$_2$ many members as the set of positive natural numbers.

Equal Size (Equinumerosity, Equipollence):

> For all sets X, Y: X has equally$_2$ many members as Y if and only if there is a pairing off between the members of X and those of Y.

Smaller Size:

For all sets X, Y: X has less$_2$ members than Y if and only if
X has equally$_2$ many members as a proper subset of Y,
but X does not have equally$_2$ many members as Y.

For all sets X and Y:

Either X has less$_2$ many members than Y,
or X has equally$_2$ many members as Y,
or Y has less$_2$ many members than X.

Infinity and Finiteness:

For all sets X: X is infinite$_2$ if and only if X has equally$_2$ many members as a proper subset of X.

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For all sets X: X is infinite$_2$ if and only if
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X is finite$_2$ if and only if X is not infinite$_2$.

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> For all sets X: X is infinite$_2$ if and only if
> X has equally$_2$ many members as a proper subset of X.

X is finite$_2$ if and only if X is not infinite$_2$.

$\{1, 2\}$ is finite$_2$.

Infinity and Finiteness:

For all sets X: X is infinite$_2$ if and only if
X has equally$_2$ many members as a proper subset of X.

X is finite$_2$ if and only if X is not infinite$_2$.

$\{1, 2\}$ is finite$_2$.

$\{1, 2, 3, 4, 5, 6, \ldots\}$ is infinite$_2$.

Theorem:

The set of natural numbers has less members than the set of real numbers.

Theorem:

The set of natural numbers has less members than the set of real numbers.

The real numbers: $0, 1, \frac{1}{2}, -\frac{1}{3}, -[18233 + 16/17], \sqrt{2}, \pi, \ldots$

$$\begin{array}{ccrl}
0 & = & & 0.000000000000000\ldots \\
1 & = & & 1.000000000000000\ldots \\
\frac{1}{2} & = & & 0.500000000000000\ldots \\
-\frac{1}{3} & = & - & 0.333333333333333\ldots \\
-[18233 + 16/17] & = & - & 18233.94117647058\ldots \\
\sqrt{2} & = & & 1.414213562373095\ldots \\
\pi & = & & 3.141592653589793\ldots
\end{array}$$

1. The set of natural numbers has equally many members as a proper subset of the set of real numbers,
2. but the set of natural numbers does not have equally many members as the set of real numbers itself.

1. The set of natural numbers has equally many members as a proper subset of the set of real numbers,
2. but the set of natural numbers does not have equally many members as the set of real numbers itself.

$$\begin{array}{ccc} 0 & \leftrightarrow & 0.000000000000000\ldots \\ 1 & \leftrightarrow & 1.000000000000000\ldots \\ 2 & \leftrightarrow & 2.000000000000000\ldots \\ \vdots & & \vdots \end{array}$$

$$\begin{array}{ccc} 0 & \leftrightarrow & [\ldots].\, x_1^0\, x_2^0\, x_3^0 \ldots \\ 1 & \leftrightarrow & [\ldots].\, x_1^1\, x_2^1\, x_3^1 \ldots \\ 2 & \leftrightarrow & [\ldots].\, x_1^2\, x_2^2\, x_3^2 \ldots \\ \vdots & & \vdots \end{array}$$

x_n^m: the n-th decimal place of the real number paired off with m.

E.g.: if $0 \quad \leftrightarrow \quad [\ldots].\, 3\, 3\, 3 \ldots$

then $x_1^0 = 3,\ x_2^0 = 3,\ x_3^0 = 3, \ldots$

$$
\begin{array}{ccl}
0 & \leftrightarrow & [\ldots].x_1^0\, x_2^0\, x_3^0 \ldots \\
1 & \leftrightarrow & [\ldots].x_1^1\, x_2^1\, x_3^1 \ldots \\
2 & \leftrightarrow & [\ldots].x_1^2\, x_2^2\, x_3^2 \ldots \\
\vdots & & \qquad\quad \vdots
\end{array}
$$

$$0.y_1\, y_2\, y_3 \ldots$$