MTH 102: Linear Algebra
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Problem Set 1

Problems marked **(T)** are for discussions in Tutorial sessions.

1. **(T)** If A is an $m \times n$ matrix, B is an $n \times p$ matrix and D is a $p \times s$ matrix, then show that $A(BD) = (AB)D$.
2. If A is an $m \times n$ matrix, B and C are $n \times p$ matrices and D is a $p \times s$ matrix, then show that
   (a) A(B+C) = AB + AC.
   (b) (B+C)D = BD + CD.
3. **(T)** Let A, B be 2×2 real matrices such that $A \begin{bmatrix} x \\ y \end{bmatrix} = B \begin{bmatrix} x \\ y \end{bmatrix}$ for all $(x, y) \in \mathbb{R}^2$. Prove that $A = B$.
4. **(T)** The parabola $y = a + bx + cx^2$ goes through the points $(x, y) = (1, 4)$ and $(2, 8)$ and $(3, 14)$. Find and solve a matrix equation for the unknowns (a, b, c).
5. Apply Gauss elimination to solve the following system

$$\begin{aligned} 2x + y + 2z &= 3 \\ 3x - y + 4z &= 7 \\ 4x + 3y + 6z &= 5 \end{aligned}$$

6. Let A and B be two $n \times n$ invertible matrices. Show that $(AB)^{-1} = B^{-1}A^{-1}$.
7. **(T)** Using Gauss Jordan method, find the inverse of

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}.$$

8. For two matrices A and B show that
   (a) $(A + B)^T = A^T + B^T$ if $A + B$ is defined.
   (b) $(AB)^T = B^T A^T$ if AB is defined.
9. **(T)** Let A and B be two $n \times n$ matrices.
   (a) If $AB = BA$ then show that $(A + B)^m = \sum_{i=0}^{m} \binom{m}{i} A^{m-i} B^i$.

(b) Show by an example that if $AB \neq BA$ then (a) need not hold.

(c) If

$$\text{Tr }(A) = \sum_{i=1}^{n} [A]_{ii},$$

then show that Tr $(AB) =$ Tr (BA). Hence show that if A is invertible then Tr $(ABA^{-1}) =$ Tr (B).

10. Give examples of 3×3 nonzero matrices A and B such that

    (a) $A^n = 0$, for some $n > 1$.

    (b) $B^3 = B$.

11. **(T)** For a matrix $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, find A^2, A^3, A^4. Find a general formula for A^n for any positive integer n.

12. Let A be a nilpotent matrix. Show that $I + A$ is invertible.

13. If an $n \times n$ real matrix A satisfies the relation $AA^T = 0$ then show that $A = 0$. Is the same true if A is a complex matrix? Show that if A is a $n \times n$ complex matrix and $A\bar{A}^T = 0$ then $A = 0$.

14. **(T)** Find the numbers a and b such that

$$\begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}$$