MTH 102: Linear Algebra
Department of Mathematics and Statistics
Indian Institute of Technology - Kanpur

Problem Set 1

Problems marked **(T)** are for discussions in Tutorial sessions.

1. **(T)** If A is an $m \times n$ matrix, B is an $n \times p$ matrix and D is a $p \times s$ matrix, then show that $A(BD) = (AB)D$.

2. If A is an $m \times n$ matrix, B and C are $n \times p$ matrices and D is a $p \times s$ matrix, then show that

   (a) A(B+C) = AB + AC.

   (b) (B+C)D = BD + CD.

3. **(T)** Let A, B be 2×2 real matrices such that $A \begin{bmatrix} x \\ y \end{bmatrix} = B \begin{bmatrix} x \\ y \end{bmatrix}$ for all $(x, y) \in \mathbb{R}^2$. Prove that $A = B$.

4. **(T)** The parabola $y = a + bx + cx^2$ goes through the points $(x, y) = (1, 4)$ and $(2, 8)$ and $(3, 14)$. Find and solve a matrix equation for the unknowns (a, b, c).

5. Apply Gauss elimination to solve the following system

$$\begin{aligned} 2x + y + 2z &= 3 \\ 3x - y + 4z &= 7 \\ 4x + 3y + 6z &= 5 \end{aligned}$$

6. Let A and B be two $n \times n$ invertible matrices. Show that $(AB)^{-1} = B^{-1}A^{-1}$.

7. **(T)** Using Gauss Jordan method, find the inverse of

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}.$$

8. For two matrices A and B show that

   (a) $(A + B)^T = A^T + B^T$ if $A + B$ is defined.

   (b) $(AB)^T = B^T A^T$ if AB is defined.

9. **(T)** Let A and B be two $n \times n$ matrices.

   (a) If $AB = BA$ then show that $(A + B)^m = \sum_{i=0}^{m} \binom{m}{i} A^{m-i} B^i$.

(b) Show by an example that if $AB \neq BA$ then (a) need not hold.

(c) If

$$\text{Tr }(A) = \sum_{i=1}^{n} [A]_{ii},$$

then show that Tr $(AB) =$ Tr (BA). Hence show that if A is invertible then Tr $(ABA^{-1}) =$ Tr (B).

10. Give examples of 3×3 nonzero matrices A and B such that

(a) $A^n = 0$, for some $n > 1$.

(b) $B^3 = B$.

11. **(T)** For a matrix $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, find A^2, A^3, A^4. Find a general formula for A^n for any positive integer n.

12. Let A be a nilpotent matrix. Show that $I + A$ is invertible.

13. If an $n \times n$ real matrix A satisfies the relation $AA^T = 0$ then show that $A = 0$. Is the same true if A is a complex matrix? Show that if A is a $n \times n$ complex matrix and $A\bar{A}^T = 0$ then $A = 0$.

14. **(T)** Find the numbers a and b such that

$$\begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}$$

Problem Set 2

Problems marked **(T)** are for discussions in Tutorial sessions.

1. Find two 2×2 invertible matrices A and B such that $A \neq cB$, for any scalar c and $A+B$ is not invertible.

2. **(T)** Let
$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$
Write down the permutation matrix P such that PA is upper triangular. Which permutation matrices P_1 and P_2 make P_1AP_2 lower triangular?

3. If A and B are symmetric matrices, which of these matrices are necessarily symmetric?
   (a) $A^2 - B^2$
   (b) $(A+B)(A-B)$
   (c) ABA
   (d) $ABAB$

4. **(T)** Let $P_n(\mathbb{R})$ be the set of vectors of polynomials with real coefficients and degree less than or equal to n. Show that $P_n(\mathbb{R})$ is a vector space over $\mathbb{R}$ with respect to the usual addition and scalar multiplication.

5. Show that the space of all real $m \times n$ matrices is a vector space over $\mathbb{R}$ with respect to the usual addition and scalar multiplication.

6. Let S be the set of all $n \times n$ symmetric matrices. Check whether S is a real vector space under usual addition and scalar multiplication of matrices.

7. In $\mathbb{R}$, consider the addition $x \oplus y = x + y - 1$ and $a.x = a(x-1)+1$. Show that $\mathbb{R}$ is a real vector space with respect to these operations with additive identity 1.

8. **(T)** Which of the following are subspaces of $\mathbb{R}^3$:
$$(a)\ \{(x,y,z) \mid x \geq 0\},\ (b)\ \{(x,y,z) \mid x+y=z\},\ (c)\ \{(x,y,z) \mid x = y^2\}.$$

9. Find the condition on real numbers a, b, c, d so that the set $\{(x,y,z) \mid ax+by+cz=d\}$ is a subspace of $\mathbb{R}^3$.

10. **(T)** Let W_1 and W_2 be subspaces of a vector space V such that $W_1 \cup W_2$ is also a subspace. Prove that one of the spaces $W_i,\ i = 1, 2$ is contained in the other.

11. Suppose S and T are two subspaces of a vector space V. *Define* the **sum**

$$S + T = \{s + t : s \in S, t \in T\}.$$

Show that $S + T$ satisfies the requirements for a vector space.

12. Let $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ be n vectors from a vector space V over $\mathbb{R}$. Define **span** of this set of vectors as

$$\operatorname{span}(\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}) = \{c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots c_n\mathbf{v_n} : c_1, c_2, \ldots, c_n \in \mathbb{R}\},$$

that is, the set of all linear combinations of vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$. Show that $\operatorname{span}(\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\})$ is a subspace of V.

13. **(T)** Show that $\{(x_1, x_2, x_3, x_4) : x_4 - x_3 = x_2 - x_1\} = \operatorname{span}(\{(1, 0, 0, -1),\ (0, 1, 0, 1),\ (0, 0, 1, 1)\}$ and hence is a subspace of $\mathbb{R}^4$.

14. **(T)** The column space of an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

defined as

$$C(A) = \operatorname{span}\left(\left\{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}\right\}\right)$$

Clearly, C(A) is a subspace of $\mathbb{R}^m$. Suppose B and D are two $m \times n$ matrices and $S = C(B)$ and $T = C(D)$, then $S + T$ is a column space of what matrix M?

15. Suppose A is an $m \times n$ matrix and B in an $n \times p$ matrix. Show that matrices A and $\begin{bmatrix} A & AB \end{bmatrix}$ (with extra columns) have the same column space. Next, find a square matrix A with $C(A^2) \subsetneq C(A)$.

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Problem Set 3

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1. Construct a matrix whose column space contains $[\ 1\ \ 1\ \ 1\ ]^T$ and whose null space is the line of multiples of $[\ 1\ \ 1\ \ 1\ \ 1\ ]^T$.

2. Compute the null space of A given by $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$.

3. **(T)** Suppose A is an m by n matrix of rank r.
   (a) If $Ax = b$ has a solution for every right side b, what is the column space of A?
   (b) In part (a), what are all equations or inequalities that must hold between the numbers m, n and r?
   (c) Give a specific example of a 3 by 2 matrix A of rank 1 with first row $[\ 2\ \ 5\ ]$. Describe the column space $C(A)$ and the null space $N(A)$ completely.
   (d) Suppose the right side b is same as the first column in your example (part c). Find the complete solution to $Ax = b$.

4. Suppose the matrix A has reduced row echelon form R :

$$A = \begin{bmatrix} 1 & 2 & 1 & b \\ 2 & a & 1 & 8 \\ & (row & 3) & \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (a) What can you say immediately about row 3 of A?
   (b) What are the numbers a and b?
   (c) Describe all solutions of $Rx = 0$. Which among row spaces, column spaces and null spaces are the same for A and for R.

5. (a) Find the number c that makes the matrix A singular (non-invertible) : $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 6 \\ 2 & 6 & c \end{bmatrix}$
   (b) If $c = 20$, what is the column space, $C(A)$ and the null space $N(A)$? Describe them in this specific case. Also describe $C(A^{-1})$ and $N(A^{-1})$ for the inverse matrix.

6. **[T]** Suppose $Ax = b$ and $Cx = b$ have same solutions for every b. Is it true that $A = C$?

7. Find matrices A and B with the given property or explain why you can not:

   (a) The only solution to $Ax = \begin{bmatrix}1\\2\\3\end{bmatrix}$ is $x = \begin{bmatrix}0\\1\end{bmatrix}$.

   (b) The only solution to $Bx = \begin{bmatrix}0\\1\end{bmatrix}$ is $x = \begin{bmatrix}1\\2\\3\end{bmatrix}$.

8. **(T)**

   (a) Suppose that A is a 3×3 matrix. What relation is there between the null space of A and the null space of A^2? How about the null space of A^3?

   (b) The set of polynomials of degree at most four ($P_4(\mathbb{R})$) in the variable x is a vector space. What is the null space of $\frac{d^2}{dx^2}$? What is the null space of $\left(\frac{d^2}{dx^2}\right)^2$?

9. Suppose R (an $m \times n$ matrix) is in reduced row echelon form $\begin{pmatrix} I & F \\ 0 & 0 \end{pmatrix}$, with r non-zero rows and first r pivot columns.

   (a) Describe the column space and null space of R.

   (b) Do the same for the $m \times 2n$ matrix $B = \begin{pmatrix} R & R \end{pmatrix}$.

   (c) Do the same for the $2m \times n$ matrix $C = \begin{pmatrix} R \\ R \end{pmatrix}$.

   (d) Finally, do the same for the $2m \times 2n$ matrix $D = \begin{pmatrix} R & R \\ R & R \end{pmatrix}$.

10. **(T)** Let $W_1 = \operatorname{span}\left\{ \begin{bmatrix}1 & 1 & 0\end{bmatrix}^T, \begin{bmatrix}-1 & 1 & 0\end{bmatrix}^T \right\}$ and $W_2 = \operatorname{span}\left\{ \begin{bmatrix}1 & 0 & 2\end{bmatrix}^T, \begin{bmatrix}-1 & 0 & 4\end{bmatrix}^T \right\}$. Show that $W_1 + W_2 = \mathbb{R}^3$. Give an example of a vector $v \in \mathbb{R}^3$ such that v can be written in two different ways in the form $v = v_1 + v_2$, where $v_1 \in W_1, v_2 \in W_2$.

11. Let M be the vector space of all 2×2 matrices and let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.

    (a) Give a basis of M.

    (b) Describe a subspace of M which contains A and does not contain B.

    (c) True (give a reason) or False (give a counter example) : If a subspace of M contains A and B, it must contain the identity matrix.

12. **[T]** Let $\{w_1, w_2, \ldots, w_n\}$ be a basis of the finite dimensional vector space V. Let v be any non zero vector in V. Show that there exists w_i such that if we replace w_i by v then we still have a basis.

Problem Set 4

Problems marked **(T)** are for discussions in Tutorial sessions.

1. Determine whether the following sets of vectors are linearly independent or not

   (a) $\{(1,0,0),(1,1,0),(1,1,1)\}$ of $\mathbb{R}^3$

   (b) $\{(1,0,0,0),(1,1,0,0),(1,2,0,0),(1,1,1,1)\}$ of $\mathbb{R}^4$

   (c) $\{(1,0,2,1),(1,3,2,1),(4,1,2,2)\}$ in $\mathbb{R}^4$.

   (d) $\{u+v,v+w,w+u\}$ in a vector space V given that $\{u,v,w\}$ is linearly independent.

2. **(T)** If $v_1,v_2,\ldots,v_d$ is a basis for a vector space V, then show that any set of n vectors in V with $n>d$, say $\{w_1,w_2,\ldots,w_n\}$, is linearly dependent.

3. Suppose V is a vector space of dimension d. Let $S=\{w_1,w_2,\ldots,w_n\}$ be a set of vectors from V. Then show that S does not span V if $n<d$.

4. **(T)** Determine if the set $T=\{x^2-x+5,4x^3-x^2+5x,3x+2\}$ spans the vector space of polynomials with degree 4 or less.

5. Let W be a subspace of V.

   (a) Show that there is a subspace U of V such that $W\cap U=\{0\}$ and $U+W=V$.

   (b) Show that there is no subspace U such that $U\cap W=\{0\}$ and $\dim U+\dim W>\dim V$.

6. **(T)** Describe all possible ways in which two planes (passing through origin) in $\mathbb{R}^3$ could intersect.

7. Construct a matrix with the required property or explain why this is impossible:

   (a) Column space contains $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\end{bmatrix}$, row space contains $\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}2\\5\end{bmatrix}$.

   (b) Column space has basis $\begin{bmatrix}1\\1\\3\end{bmatrix}$, null-space has basis $\begin{bmatrix}3\\1\\1\end{bmatrix}$.

   (c) The dimension of null-space is one more than the dimension of left null-space.

   (d) Left null-space contains $\begin{bmatrix}1\\3\end{bmatrix}$, row space contains $\begin{bmatrix}3\\1\end{bmatrix}$.

   (e) Row space and column space are same but null-space and left null-space is different.

8. Show that the system of equations $Ax = b$ given below

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 5 \\ 2x_1 + 2x_2 + 3x_3 &= 5 \\ 3x_1 + 4x_2 + 5x_3 &= 9 \end{aligned}$$

has no solution by finding $y \in N(A^T)$ such that $y^T b \neq 0$.

9. Suppose $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. Find a projection matrix P that projects b onto the column space of A, that is, $Pb \in C(A)$ and $b - Pb$ is orthogonal to $C(A)$.

10. **(T)** If a subspace S is contained in a subspace T, the show that $S^\perp$ contains $T^\perp$.

11. Suppose A is a 3 by 4 matrix and B is a 4 by 5 matrix with $AB = 0$. Show that

$$\text{rank}(A) + \text{rank}(B) \leq 4.$$

12. **(T)** Let A be an m by n matrix and B be an n by p matrix with $\text{rank}(A) = \text{rank}(B) = n$. Show that $\text{rank}(AB) = n$.

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Problem Set 5

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1. **(T)** A *permutation*, denoted by σ, is a one-to-one and onto function from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$ given in two line form as

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}.$$

   Set of all permutations of $\{1, 2, \ldots, n\}$ is denoted by S_n.

   (a) Find all elements of S_3 (the set of all permutations of the set $\{1, 2, 3\}$).

   (b) Let $\sigma \in S_5$ be given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}$$

   What does $\sigma^2 := \sigma \circ \sigma$ do to $(1, 2, 3, 4, 5)$?

2. Find the determinant of $A = [a_{ij}]$ in each of the following cases:

   (a) A is a diagonal matrix.

   (b) A is a lower triangular matrix (i.e. $a_{ij} = 0$ for all $j > i$).

   (c) A is an upper triangular matrix (i.e. $a_{ij} = 0$ for all $j < i$)

3. **(T)** For two $n \times n$ matrices A and B, show that $\det(AB) = \det(A)\det(B)$.

4. For an $n \times n$ matrix $A = [a_{ij}]$, prove that $\det(A) = \det(A^T)$.

5. Suppose the 4×4 matrix M has 4 equal rows all containing a, b, c, d. We know that $det(M) = 0$. The problem is to find by any method

$$det(I + M) = \begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix}.$$

6. **(T)** For a complex matrix $A = [a_{ij}]$, let $\bar{A} = [\ \overline{a_{ij}}\ ]$ and $A^* = \bar{A}^T$. Show that $\det(\bar{A}) = \det(A^*) = \overline{\det A}$. Therefore if A is Hermitian (that is, $A^* = A$) then its determinant is real.

7. Let $A = [a_{ij}]$ be an invertible matrix and let $B = [p^{i-j} a_{ij}]$. Find the inverse of B and also find $\det(B)$.

8. The numbers $1375, 1287, 4191$ and 5731 are all divisible by 11. Prove that the determinant of the matrix
$$\begin{bmatrix} 1 & 1 & 4 & 5 \\ 3 & 2 & 1 & 7 \\ 7 & 8 & 9 & 3 \\ 5 & 7 & 1 & 1 \end{bmatrix}$$
is also divisible by 11.

9. **(T)** A real square matrix A is said to be orthogonal if $A^T A = AA^T = I$. Show that if A is orthogonal then $\det(A) = \pm 1$.

10. Find the determinant of
$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}.$$

11. Find the determinant of
$$\begin{bmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 2 & 2 & 3 & 4 & \dots & n \\ 3 & 3 & 3 & 4 & \dots & n \\ .. & .. & .. & .. & \dots & n \\ n & n & n & n & \dots & n \end{bmatrix}.$$

12. **(T)** Let A be an invertible square matrix with integer entries. Show that A^{-1} has integer entries if and only if $\det(A) = \pm 1$.

13. We are looking for the parabola $y = C + Dt + Et^2$ that gives the least squares fit to these four measurements:
$y = 1$ at $t = -2$, $y = 1$ at $t = -1$, $y = 1$ at $t = 1$ and $y = 0$ at $t = 2$.

    (a) Write down the four equations (not solvable!) for the parabola $C + Dt + Et^2$ to go through those four points. This is the system $Ax = b$ to solve by least squares:
$$A \begin{bmatrix} C \\ D \\ E \end{bmatrix} = b.$$
    What equations would you solve to find the best C, D, E?

    (b) Compute $A^T A$. Compute its determinant. Compute its inverse.

    (c) The first two columns of A are already orthogonal. From column 3, subtract its projection onto the plane of the first two columns. That produces what third orthogonal vector v? Normalize v to find the third orthonormal vector q_3 from Gram-Schmidt.

Problem Set 6

Problems marked **(T)** are for discussions in Tutorial sessions.

1. **(T)** Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Show that $\det(A) = \lambda_1 \dots \lambda_n$ and $\text{Tr}(A) = \lambda_1 + \dots + \lambda_n$. Further show that A is invertible if and only if its all eigenvalues are non-zero.

2. Let A be an $n \times n$ invertible matrix. Show that eigenvalues of A^{-1} are reciprocal of the eigenvalues of A, moreover, A and A^{-1} have the same eigenvectors.

3. Let A be an $n \times n$ matrix and α be a scalar. Find the eigenvalues of $A - \alpha I$ in terms of eigenvalues of A. Further show that A and $A - \alpha I$ have the same eigenvectors.

4. **(T)** Let A be an $n \times n$ matrix. Show that A^T and A have the same eigenvalues. Do they have the same eigenvectors?

5. Let A be an $n \times n$ matrix. Show that:

   (a) If A is idempotent ($A^2 = A$) then eigenvalues of A are either 0 or 1.

   (b) If A is nilpotent ($A^m = 0$ for some $m \geq 1$) then all eigenvalues of A are 0.

6. **(T)** This question deals with the following symmetric matrix A:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

   One eigenvalue is $\lambda = 1$ with the line of eigenvectors $x = (c, c, 0)$.

   (a) That line is the null space of what matrix constructed from A?

   (b) Find the other two eigenvalues of A and two corresponding eigenvectors.

   (c) The diagonalization $A = S\Lambda S^{-1}$ has a specially nice form because $A = A^T$. Write all entries in the three matrices in the nice symmetric diagonalization of A.

7. Find the eigenvalues and corresponding eigenvectors of matrices

   (a) $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{bmatrix}$

8. Construct a basis of $\mathbb{R}^3$ consisting of eigenvectors of the following matrices

   (a) $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$.

9. Let A_n be an $n \times n$ tridiagonal matrix

$$A_n = \begin{bmatrix} 1 & -a & 0 & 0 & \cdots & 0 \\ -a & 1 & -a & 0 & \cdots & 0 \\ 0 & -a & 1 & -a & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -a & 1 & -a \\ 0 & 0 & \cdots & 0 & -a & 1 \end{bmatrix}.$$

(a) Show for $n \geq 3$ that

$$\det(A_n) = \det(A_{n-1}) - a^2 \det(A_{n-2}).$$

(b) Show that the equation in part (a) can equivalently be written as $x_n = Bx_{n-1}$, where

$$x_n = \begin{bmatrix} \det(A_n) \\ \det(A_{n-1}) \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -a^2 \\ 1 & 0 \end{bmatrix}.$$

(c) For $a^2 = \frac{3}{16}$, find an expression for $\det(A_n)$ for any n. (Hint: One method starts by writing $B = S\Lambda S^{-1}$, where Λ is a diagonal matrix.)

10. Show that $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$ is diagonalizable. Find a matrix S such that $S^{-1}AS$ is a diagonal matrix.

11. Let $A = \begin{bmatrix} 7 & -5 & 15 \\ 6 & -4 & 15 \\ 0 & 0 & 1 \end{bmatrix}$. Find a matrix S such that $S^{-1}AS$ is a diagonal matrix and hence calculate A^6.

12. Consider the 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ 1 & d & e \\ 0 & 1 & f \end{bmatrix}.$$

Determine the entries a, b, c, d, e, f so that:

- the top left 1×1 block is a matrix with eigenvalue 2;
- the top left 2×2 block is a matrix with eigenvalue 3 and -3;
- the top left 3×3 block is a matrix with eigenvalue 0, 1 and -2.

13. (a) Find the eigenvalues and eigenvectors (depending on c) of

$$A = \begin{bmatrix} 0.3 & c \\ 0.7 & 1-c \end{bmatrix}.$$

For which value of c is the matrix A not diagonallizable (so $A = S\Lambda S^{-1}$ is impossible)?

(b) What is the largest range of values of c (real number) so that A^n approaches a limiting matrix A^∞ as $n \to \infty$?

(c) What is that limit of A^n (still depending on c)? You could work from $A = S\Lambda S^{-1}$ to find A^n.

14. (a) If B is invertible, prove that AB has the same eigenvalues as BA.

    (b) Find a diagonalizable matrix $A \neq 0$ that is similar to $-A$. Also find a non-diagonalizable matrix A that is similar to $-A$.

15. **(T)** Find all linear transformations from $\mathbb{R}^n \longrightarrow \mathbb{R}$.

16. Let $V,\ W$ be vector spaces and let $L(V, W)$ be the vector space of all linear transformations from V to W. Show that $\dim L(V, W) = \dim V . \dim W$.

17. Show that a linear transformation is one-one if and only if null-space of T is $\{0\}$.

18. Describe all 2×2 orthogonal matrices. Prove that action of any orthogonal matrix on a vector $v \in \mathbb{R}^2$, is either a rotation or a reflection about a line.

19. **(T)** Let $v, w \in \mathbb{R}^n$, $n \geq 2$, with $\|v\| = \|w\| = 1$. Prove that there exist an orthogonal matrix A such that $A(v) = w$. Prove also that A can be chosen such that $\det(A) = 1$. (*This is why orthogonal matrices with determinant one are called rotations.*))