Physics 141 Problem Set 2 Corrected Solutions

Nataliya Yufa

(2.6) Gravel mixer

We want to find ω such that at least for a moment the particles are not stuck to the cylinder. This is most likely to happen at the top, where the normal force is in the direction of gravity. For the particle to leave the cylinder, N=0. Hence the y-component of the total force on the particle is

$$F_g + N = F_g = \frac{mv^2}{r}. \tag{1}$$

Using the relation $v = \omega R$ we obtain

$$mg = m\frac{v^2}{r} = m\omega^2 R, \tag{2}$$

$$\Rightarrow \omega_c = \sqrt{\frac{g}{R}}. \tag{3}$$

Thus for all values of ω less than ω_c the particles will not be stuck to the walls all the time.

(2.9) Particle in a cone

The particle happens to be moving in a horizontal circle inside a cone of half-angle θ. We want to find the radius r of the circle in terms of v_0, θ and g. Consider the x- and y-components of the net force acting on the mass.

Since there's no movement in the y-direction, F_y is zero, i.e.

$$F_y = N_y - mg = 0 \Rightarrow N_y = mg. \tag{4}$$

From the geometry we find that $\tan\theta = \frac{N_y}{N_x}$, hence

$$N_x = \frac{N_y}{\tan\theta} = \frac{mg}{\tan\theta}. \tag{5}$$

Circular motion in the horizontal direction implies that

$$F_x = N_x = \frac{mg}{\tan\theta} = \frac{mv_0^2}{r}, \tag{6}$$

and solving for r we have

$$r = \frac{v_o^2 \tan\theta}{g}. \tag{7}$$

(2.12) Pulling out the tablecloth

Initially, the glass is accelerated by the force of friction due to the tablecloth for some time t_{max}. In this time it will reach velocity v_o

$$v_o = \frac{F_f t_{max}}{m} = \mu g t_{max}. \tag{8}$$

On the other hand, the glass will be slowed down by the friction due to the table (which, because the coefficients of friction happen to be the same for the table and the tablecloth, has the same magnitude as the frictional force of the tablecloth). Therefore, during both the acceleration and the deceleration the glass will travel exactly half of the available distance.

We want the glass to slow down in a given distance $d/2$, thus

$$v_0 = \sqrt{\mu g d}. \tag{9}$$

Combine the two expressions for v_0 to get

$$v_0 = \mu g t_{max} = \sqrt{\mu g d} \tag{10}$$

$$\Rightarrow t_{max} = \sqrt{\frac{\mu g d}{\mu g}} = \sqrt{\frac{d}{\mu g}} = \sqrt{\frac{0.5ft}{0.5 \cdot 32ft/s^2}} = \frac{1}{4\sqrt{2}}s. \tag{11}$$

(2.25) Shortest possible period

Intuitively, the shortest period of rotation will occur when the two solid spheres are as close as possible to each other. Let's try to prove this. Because gravity is responsible for the circular motion, we have

$$\frac{Gm^2}{4r^2} = \frac{mv^2}{r} \tag{12}$$

$$\Rightarrow v = \sqrt{\frac{Gm}{4r}}. \tag{13}$$

The period is given by

$$T = \frac{2\pi r}{v} = \frac{4\pi}{\sqrt{Gm}} r^{3/2}. \tag{14}$$

Note that the smaller the radius, the shorter the period. However, we're limited by the radius of the spheres, as they cannot come closer than their radius R. Hence the shortest possible period would be $T = \frac{4\pi}{\sqrt{Gm}} R^{3/2}$.

(2.28) Car going around a bend

Let's choose the x- and y-axis to be horizontal and vertical respectively, since the car should be moving in a horizontal surface and therefore there should be a non-zero force solely in the horizontal direction. Writing the components of the total force we have

$$F_y = 0 = N\cos\theta - mg \pm \mu N \sin\theta \tag{15}$$

and

$$F_x = N\sin\theta \mp \mu N\cos\theta = \frac{mv^2}{R}. \tag{16}$$

Here the different signs correspond to the maximum and minimum velocities. It follows that

$$N = \frac{mg}{\cos\theta \pm \mu\sin\theta}. \tag{17}$$

Substituting this into the expression for F_x we get an equation for v^2:

$$v^2 = \frac{gR}{\cos\theta \pm \mu\sin\theta}(\sin\theta \mp \mu\cos\theta). \tag{18}$$

Writing this in a different form

$$\frac{gR(\sin\theta - \mu\cos\theta)}{\cos\theta + \mu\sin\theta} < v^2 < \frac{gR(\sin\theta + \mu\cos\theta)}{\cos\theta - \mu\sin\theta}. \tag{19}$$

(2.33) Particle on a rotating rod

a) The trick to this problem is to realize that there is no force in the radial direction as the rod is frictionless. Hence the total force is given by $\vec{F} = F\hat{\theta}$. Newton said $F = ma$, thus

$$F\hat{\theta} = m\left[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}\right], \tag{20}$$

where $\dot{\theta}$ equals ω. It must be that the radial component of the acceleration is zero since the radial part of the force is zero:

$$\ddot{r} - \omega^2 r = 0. \tag{21}$$

Now we need to find the solution to this differential equation.

Let's try the solution $r = Ae^{-\gamma t} + Be^{\gamma t}$. Then

$$\ddot{r} = \gamma^2 Ae^{-\gamma t} + \gamma^2 Be^{\gamma t} = \gamma^2 r. \tag{22}$$

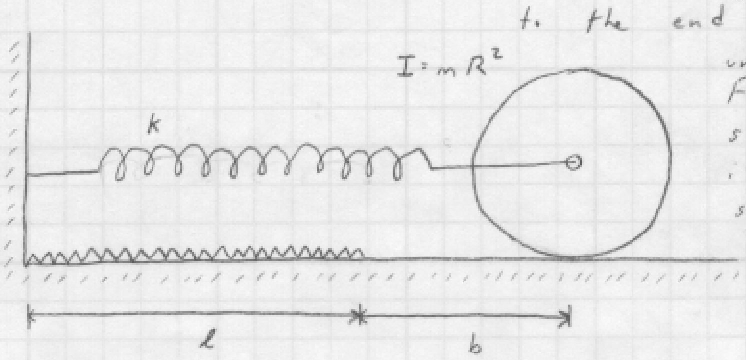
In order to satisfy our differential equation we need to have

$$\ddot{r} = \gamma^2 r = \omega^2 r, \tag{23}$$

and therefore $\gamma = \omega$.

b) By inspection, notice that $Be^{\omega t}$ will tend to positive or negative infinity, unless B is zero. Thus if we want the radius to be always decreasing without reaching zero, B must be zero. In all other cases where the particle doesn't reach the origin, we must have a positive B. In that limit, we see that $Ae^{\omega t}$ will tend to zero for large t, while $Be^{\omega t}$ will go to infinity.

6.40) ↺⊕ ←⊕



A wheel with fine teeth is attached to the end of a spring with unstretch length l. For $x > l$, the wheel slips, for $x < l$, it rolls with no slipping. All of the wheel's mass is at the rim.

a) The wheel is pulled to $l + b$ and released. How close does it come to the wall?

First we use energy conservation to go from the start to the impact at l. The wheel slips the entire distance.

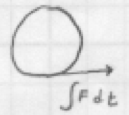
v_0: velocity of wheel before impact

v_1: velocity of wheel after impact

ω_1: angular velocity of wheel after impact; $\omega_1 R = v_1$ ← no slipping

$$\frac{1}{2} k b^2 = \frac{1}{2} m v_0^2 \quad \text{no rotational energy} \qquad ①$$

At the impact there is an impulse which changes the linear and angular velocities.



(note sign convention above)

$$p_f - p_i = -\int F\,dt$$

$$L_f - L_i = R \int F\,dt = -R(p_f - p_i)$$

$$I \omega_1 - 0 = -R(m v_1 - m v_0)$$

$$m R^2 \omega_1 = m R v_1 = -m R v_1 + m R v_0$$

$$v_1 = \frac{1}{2} v_0 \qquad ②$$

Finally, we use energy to take us in near the wall.

$$\frac{1}{2} m v_1^2 + \frac{1}{2} I \omega_1^2 = \frac{1}{2} k x_1^2$$

$$\frac{1}{2} m v_1^2 + \frac{1}{2} m r^2 \omega_1^2 = \frac{1}{2} m v_1^2 + \frac{1}{2} m v_1^2 = m v_1^2$$

$$\overset{②}{=} m \left(\frac{v_0}{2}\right)^2 = \frac{1}{2} m \left(\frac{v_0}{\sqrt{2}}\right)^2 = \frac{1}{2} k x_1^2$$

From ① we see that

$$x_1 = \frac{1}{\sqrt{2}} b$$

6.40 (cont.)

The wheel comes to

$$l - \frac{1}{\sqrt{2}} b$$

from the wall.

b) How far back out does it go?

There is no impact on the way out, so we can use energy. We can start from the point when the wheel passes equilibrium again. It will have the same velocity found before, simply reversed.

$$\frac{1}{2} m v_1^2 + \frac{1}{2} I \omega_1^2 = \frac{1}{2} k x_2^2 + \frac{1}{2} I \omega_1^2$$

the wheel slips and does not lose its rotational energy

$$\frac{1}{2} m v_1^2 = \frac{1}{2} k x_2^2$$

$$\frac{1}{2} m \left(\frac{v_0}{2}\right)^2 = \frac{1}{2} k x_2^2$$

Looking at ① we see that

$$x_2 = \frac{1}{2} b$$

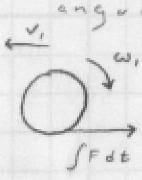
The wheel gets to

$$l + \frac{1}{2} b$$

from the wall.

c) What happens when the wheel comes back and hits the gear track?

Before the impact the wheel it moving left with velocity v_1 and rotating clockwise with angular velocity ω_1.



$$p_f - p_1 = -\int F\,dt$$

$$L_f - L_1 = R \int F\,dt = -R(p_f - p_i)$$

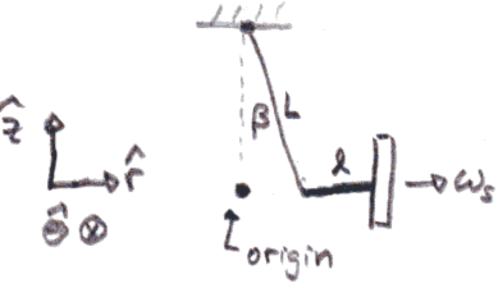
$$I\omega_2 - I(-\omega_1) = -R(m v_2 - m v_1)$$

$$m R^2 \omega_2 + m R^2 \omega_1 = -m R v_2 + m R v_1$$

$$m R v_2 + m R v_1 = -m R v_2 + m R v_1 \quad \Rightarrow \quad v_2 = 0 \;!!!$$

The wheel stops!

KK 7.3



A standard gyroscope of mass M and moment of inertia I_0 dangles at the end of a string of length L. Our goal is to find the small angle β in terms of: L, ℓ, ω_s, M, g & I_0

We choose our origin at the center of the circle at the same height as the center of mass.

Using the cylindrical coordinates as shown, the forces and their locations are:

gravity: $\vec{F}_g = -Mg\hat{z}$ @ $\vec{r}_g = (L\sin\beta + \ell)\hat{r}$

tension: $\vec{F}_T = T\cos\beta\,\hat{z} - T\sin\beta\,\hat{r}$ @ $\vec{r}_T = L\sin\beta\,\hat{r}$

The gyro will precess at some rate Ω, which we need to find. For such circular motion, $\vec{F} = m\vec{a}$ then needs

$$(T\cos\beta - Mg)\hat{z} - (T\sin\beta)\hat{r} = 0\hat{z} - M(L\sin\beta + \ell)\Omega^2\hat{r}$$

$$\Rightarrow T = Mg/\cos\beta \quad \& \quad Mg\tan\beta = M(L\sin\beta + \ell)\Omega^2$$

The torque equation reads

$$\left.\begin{aligned}\vec{\tau} &= \vec{r}_g \times \vec{F}_g + \vec{r}_T \times \vec{F}_T = Mg\ell\,\hat{\theta} \\ &= \frac{d}{dt}\vec{L} = \frac{d}{dt}\left[\vec{R}\times\vec{P} + I_0\omega_s\hat{r}\right] = I_0\omega_s\Omega\,\hat{\theta}\end{aligned}\right\} \Rightarrow \Omega = \frac{Mg\ell}{I_0\omega_s}$$

center of mass orbital L, which is constant in the $\hat{z}$ direction

So the equation for β boils down to: $\quad g\tan\beta = (L\sin\beta + \ell)\left(\frac{Mg\ell}{I_0\omega_s}\right)^2$

This is a transcendental equation, which we have to solve numerically in general. But if $\omega_s \gg \sqrt{g/\ell}$ then β will be small and we can approximate $\sin\beta \approx \beta$ etc.

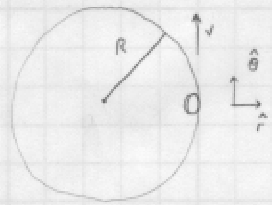
$$\Rightarrow \beta = \left(1 + \tfrac{L}{\ell}\beta\right)\left(\frac{M\ell^2}{I_0}\right)^2\frac{g}{\ell\omega_s^2} = \left(1 + \tfrac{L}{\ell}\beta\right)\alpha \quad \text{where } \alpha \equiv \left(\frac{M\ell^2}{I_0}\right)^2\frac{g}{\ell\omega_s^2}$$

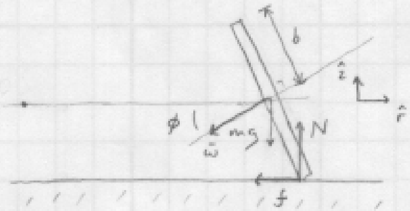
is a small expansion parameter.

To leading order, then $\quad \beta = \dfrac{\alpha}{1 - \frac{L}{\ell}\alpha} \approx \alpha + \frac{L}{\ell}\alpha^2 + \mathcal{O}(\alpha^3)$

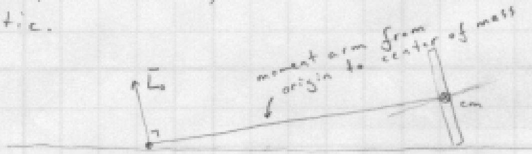
To get the α^3 term we would need to expand the $\sin\beta$ and $\tan\beta$ terms to $\mathcal{O}(\beta^3)$.

7.6) A coin is rolling in a circle. At what angle does in lean?





We will calculate the torque and change in the angular momentum, and set them equal. We can measure about the center of the circle or the center of mass of the coin. We'll use the center of mass of the coin. The only danger with not using it is that, if the center of the circle on the floor is used as an origin, the orbital angular momentum does not point in the $\hat{z}$ direction, but tilts and is thus not static.



With this origin, $\frac{d}{dt}\bar{L}_o \neq 0$
It will sweep out a cone.

You can still do the problem but it's harder.

We will use the center of mass of the coin as our origin.

7.6 (cont.)

$F_z = ma_z \Rightarrow \mathcal{N} - mg = 0$ $\qquad \mathcal{N} = mg$

$F_r = ma_r \Rightarrow -f = -m\frac{v^2}{R}$ $\qquad f = m\frac{v^2}{R}$

$\bar{\tau} = (fb\cos\phi - Nb\sin\phi)\,\hat{\theta}$

$\bar{L} = \underbrace{\bar{L}_o}_{\text{orbital}} + \underbrace{\bar{L}_s}_{\text{spin}}$

$\bar{L}_o$ has constant magnitude and, with our origin, points in the $\hat{z}$ direction. Thus,

$\frac{d}{dt}\bar{L}_o = 0$

$\frac{d}{dt}\bar{L} = \frac{d}{dt}\bar{L}_s$

$\bar{L}_s = I\omega(-\hat{r}\cos\phi - \hat{z}\sin\phi)$ $\qquad \omega = \frac{v}{b}$

$= \frac{1}{2}mb^2\frac{v}{b}(-\hat{r}\cos\phi - \hat{z}\sin\phi)$

$\frac{d}{dt}\bar{L}_s = \frac{1}{2}mbv\left(-\cos\phi\frac{d}{dt}\hat{r}\right) = -\frac{1}{2}mbv\cos\phi\left(\frac{v}{R}\hat{\theta}\right)$

$= -\frac{1}{2}m\frac{b}{R}v^2\cos\phi\,\hat{\theta}$

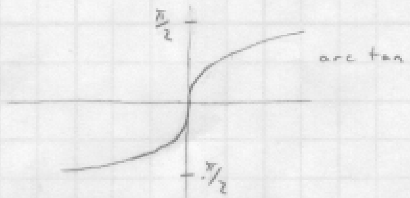
$\bar{\tau} = \frac{d}{dt}\bar{L} \Rightarrow b(f\cos\phi - \mathcal{N}\sin\phi)\,\hat{\theta}$

$= -\frac{1}{2}m\frac{b}{R}v^2\cos\phi\,\hat{\theta}$

$\frac{v^2}{R}\cos\phi - g\sin\phi = -\frac{1}{2}\frac{v^2}{R}\cos\phi$

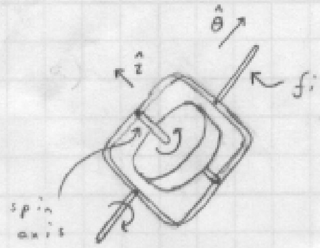
$\tan\phi = \frac{3}{2}\frac{v^2}{Rg}$

$\phi = \arctan\frac{3v^2}{2Rg}$

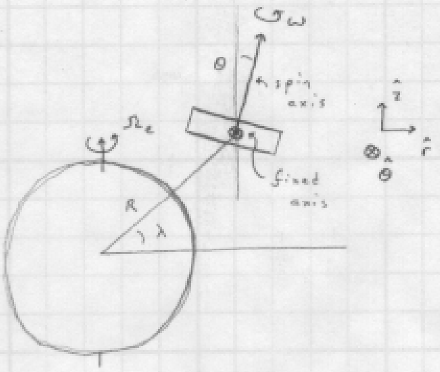


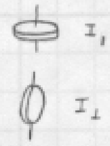
✓: A larger tilt requires a larger velocity or small radius. That makes sense!

7.10) A gyro is mounted with it's fixed axle horizontal and lying along the east-west axis.



θ: deviation around fixed axis from polar axis





We will set up the differential equation governing small deviations around the fixed axis from the polar axis.

$$\bar{L} = \bar{L}_r + \bar{L}_s + \bar{L}_o$$

small deviations $\sin\theta \to \theta$

$$= I_\perp \dot{\theta}\,\hat{\theta} + I_1 \omega \left(\hat{r} \sin\theta + \hat{z} \cos\theta \right)$$

$$+ M R^2 \cos^2 \lambda\, \Omega_e\, \hat{z}$$

$$\frac{d}{dt}\bar{L} = I_\perp \ddot{\theta}\,\hat{\theta} - I_\perp \dot{\theta}\Omega_e \hat{r} + I_1 \omega \dot{\theta}\hat{r} + I_1 \omega \theta \Omega_e \hat{\theta}$$

No torque can be applied to the gyro about its fixed axis. Thus,

$$\frac{d}{dt} L_\theta = 0$$

$$I_\perp \ddot{\theta} + I_1 \omega \Omega_e \theta = 0$$

$$\Rightarrow \quad \ddot{\theta} + \frac{I_1}{I_\perp} \omega \Omega_e \theta = 0$$

This is the equation for stable, periodic motion, with frequency

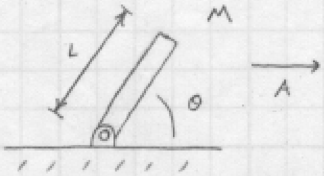
$$\omega_{osc} = \sqrt{\frac{I_1}{I_\perp} \omega \Omega_e}$$

For $\omega = 40000$ rpm $= 4188.8 \frac{rad}{s}$ $\qquad \Omega_e = \frac{2\pi}{8.64\times10^4 s} = 1$ rev/day

thin disk: $I_1 = \frac{1}{2} mr^2 \quad I_\perp = \frac{1}{4} mr^2 \qquad \frac{I_1}{I_\perp} = 2$

$$= 7.272 \times 10^{-5} \frac{rad}{s}$$

$$\omega_{osc} = 0.78 \frac{rad}{s} = 7.45 \text{ rpm}$$

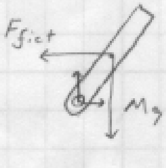
8.1) 

A bar attached to a pivot is in an accelerating frame.

a) What is the equilibrium value for θ?

We will do the problem twice, once in the accelerating frame and once in the inertial frame.

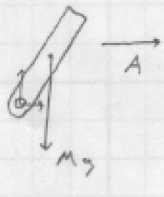
accelerating frame



In both we solve for when the rotation of the bar is zero.

$$\tau = I\alpha$$
measure about the pivot

inertial frame



There is a fictitious force $F_{fict} = MA$. Equilibrium means the sum of the torques is zero.

$$\frac{L}{2}\left(F_{fict}\sin\theta - Mg\cos\theta\right) = 0$$

$$A\sin\theta - g\cos\theta = 0$$

$$\frac{g}{A} = \tan\theta$$

There is no fictitious force but the rod is accelerating relative to our instantanious axis at the pivot.

rotation about center of mass is zero at equilibrium

$$-\frac{L}{2}Mg\cos\theta = \cancel{I\alpha}^{\,0} + M\left(\frac{L}{2}\sin\theta\right)(-A)$$

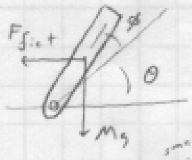
acceleration of center of mass about axis

$$g\cos\theta = A\sin\theta$$

$$\frac{g}{A} = \tan\theta$$

match!

b) What is the motion for small deviations from θ?
We'll use the accelerating frame.



$$\tau = I\alpha \Rightarrow \frac{L}{2}\left\{MA\sin(\theta+\phi) - Mg\cos(\theta+\phi)\right\} = \frac{1}{3}ML^2\ddot{\phi}$$

$$= \frac{L}{2}M\left\{A\sin\theta\cos\phi + A\cos\theta\sin\phi - g\cos\theta\cos\phi + g\sin\theta\sin\phi\right\}$$

small deviations
$$= \frac{L}{2}M\left\{A\sin\theta + A\phi\cos\theta - g\cos\theta + g\phi\sin\theta\right\}$$

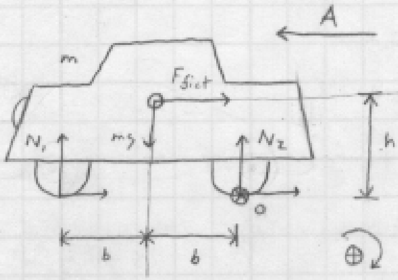
from part a)
$$\Rightarrow \cancel{(A\sin\theta - g\cos\theta)}^{\,0} + (A\cos\theta + g\sin\theta)\phi = \frac{2}{3}L\ddot{\phi}$$

$$\ddot{\phi} - \left(\frac{3}{2}\,\frac{A\cos\theta + g\sin\theta}{L}\right)\phi = 0$$

This is the equation for exponential growth. Thus, the equilibrium point is unstable.

$$\phi = \phi_0 e^{kt} \qquad k^2 = \frac{3}{2}\,\frac{A\cos\theta + g\sin\theta}{L}$$

8.4)



A car.

a) The car accelerates. At what value must it for the front wheels to lift just off the ground.

$mg = 3{,}200\ lb$
$h = 2\ ft$
$b = 4\ ft$

We will work in the accelerating frame.

$F_{fict} = mA$

Rotations will be measured about point O.

There is no rotation.

$$\tau = I\alpha \Rightarrow F_{fict}\, h + N_1 (2b) - mg\, b = 0$$

The wheels lift when $N_1 = 0$.

$$mAh - mgb = 0$$

$$A = \frac{b}{h} g$$

$\checkmark$: large b increases stability
large h reduces stability

$g = 32.2\ ft s^{-2}$

$$A = 64.4\ ft\, s^{-2} = 19.6\ m s^{-2}$$

b) The car breaks with acceleration g. What is the normal force on the wheels?

$F_{fict} = ma$

$$F_y = ma_y \Rightarrow N_1 + N_2 - mg = 0$$

$$\tau = I\alpha \Rightarrow \underbrace{-F_{fict}}_{\text{opposite fictitious force}}\, h + N_1(2b) - mgb = 0$$

$$-mgh + 2bN_1 - mgb = 0$$

$$N_1 = mg\frac{b+h}{2b}$$

$$mg\frac{b+h}{2b} + N_2 - mg = 0 \Rightarrow mg\left(\frac{2b-b-h}{2b}\right) = N_2$$

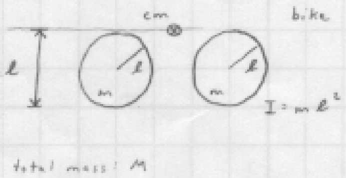
$$N_2 = mg\frac{b-h}{2b}$$

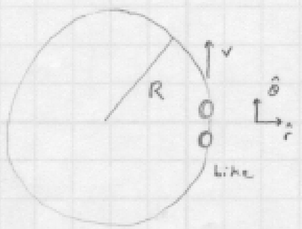
$N_1 = 2400\ lb$

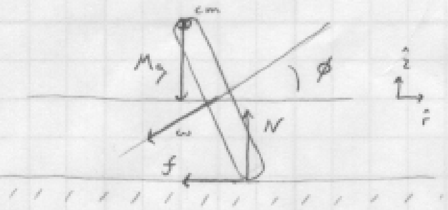
$N_2 = 800\ lb$

$\checkmark$: $N_1 > N_2$

7.9)







A bike is turning by leaning into the turn. At what angle must it lean?

We will calculate torque and the change in the angular momentum, and set them equal. We will measure about the bike's center of mass. See problem 7.6 for why this is a good idea.

$$F_z = ma_z \Rightarrow N - Mg = 0 \qquad N = mg$$

$$F_r = ma_r \Rightarrow -f = -M\frac{v^2}{R} \qquad f = M\frac{v^2}{R}$$

$$\bar{\tau} = (2\ell f\cos\phi - 2\ell N\sin\phi)\,\hat{\theta}$$

$$\bar{L} = \underset{\text{orbital}}{\bar{L}_o} + \underset{\text{spin}}{\bar{L}_s}$$

$$\frac{d}{dt}\bar{L}_o = 0 \quad \text{(because of our origin choice)}$$

$$\frac{d}{dt}\bar{L} = \frac{d}{dt}\bar{L}_s$$

(two wheels)

$$\bar{L}_s = 2I\omega(-\hat{r}\cos\phi - \hat{z}\sin\phi)$$

$$= 2m\ell^2\frac{v}{\ell}(-\hat{r}\cos\phi - \hat{z}\sin\phi)$$

$$\frac{d}{dt}\bar{L}_s = -2m\ell^2\frac{v}{\ell}\cos\phi\frac{d}{dt}\hat{r} = -2m\ell v\cos\phi\frac{v}{R}\hat{\theta}$$

$$\bar{\tau} = \frac{d}{dt}\bar{L} \Rightarrow$$

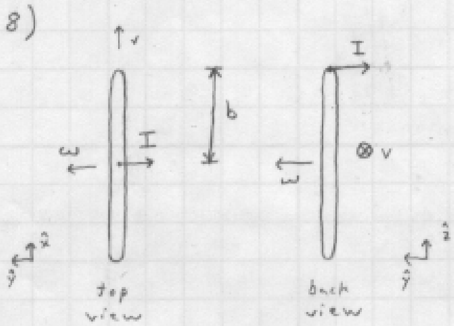
$$(2\ell f\cos\phi - 2\ell N\sin\phi)\,\hat{\theta} = -2m\ell\frac{v^2}{R}\cos\phi\,\hat{\theta}$$

$$M\frac{v^2}{R}\cos\phi - Mg\sin\phi = -m\frac{v^2}{R}\cos\phi$$

$$\tan\phi = \frac{v^2}{MRg}(m + M) = \frac{v^2}{Rg}\left(1 + \frac{m}{M}\right)$$

$\frac{m}{M}$ is small with a rider, but significant without one.

7.8)



A rolling hoop is struck with an impulse ⊥ to its velocity at its top.

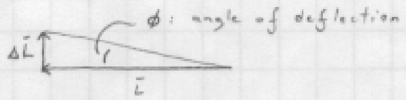
a) Through what angle is it deflected?

Assuming gyroscope approximation - that is, the relative change in the angular momentum due to the impulse is slow relative to the angular velocity.

$$\omega \gg \frac{\left|\frac{dL}{dt}\right|}{|L|}$$

$$\bar{L} = M b^2 \omega \, \hat{y}$$

$$\Delta \bar{L} = \bar{r} \times \bar{I} = b\hat{z} \times -I\hat{y} = b I \hat{x}$$



$$\tan \phi = \frac{\Delta L_x}{L_y} = \frac{bI}{Mb^2\omega}$$

(note: $\tan \phi = \frac{\Delta p_y}{p_x}$ also)

$$= \frac{I}{Mb\omega} = \frac{I}{Mv} \overset{\text{small angle}}{\approx} \phi$$

b) Show the gyroscope approximation is valid if the force F causing the impulse is small.

$$F \ll \frac{Mv^2}{b} = M\omega^2 b$$

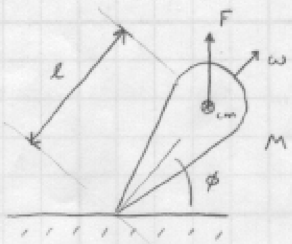
$$\frac{d}{dt}\bar{L} = \bar{r} \times \bar{F} = bF\hat{x} \ll M\omega^2 b^2 \hat{x} = \omega L_y \hat{x}$$

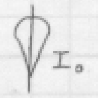
$$\left|\frac{d}{dt} L\right| \ll \omega |L|$$

or

$$\omega \gg \frac{\left|\frac{dL}{dt}\right|}{|L|}$$

8.6) A top is attached to the ground and spinning in an elevator. What is the rate of precession?





We will set torque and change in angular momentum equal.

a) The elevator is at rest.

So, $F = -W = -Mg$

$\bar{\tau} = \hat{r}(\ell \sin\phi) \times \hat{z} F$
$= Mg\ell \sin\phi\, \hat{\theta}$

$\bar{L} = I_o \omega_s (\hat{r} \sin\phi + \hat{z} \cos\phi)$ ← (assuming L due to spin is large relative to L due to precession)

$\frac{d}{dt}\bar{L} = I_o \omega_s \Omega \hat{\theta} \sin\phi$

$\bar{\tau} = \frac{d}{dt}\bar{L} \Rightarrow Mg\ell \sin\phi = I_o \omega_s \Omega \sin\phi$

$\Omega = \frac{Mg\ell}{I_o \omega_s}$ $\hat{\theta}$ direction

b) The elevator is acceleration down at 2g.

So, $F = F_{fict} - W = 2gM - gM = Mg$

$\bar{\tau} = -Mg\ell \sin\phi\, \hat{\theta}$

$\bar{\tau} = \frac{d}{dt}\bar{L} \Rightarrow -Mg\ell \sin\phi = I_o \omega_s \Omega \sin\phi$

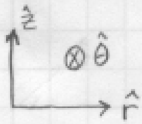
$\Omega = -\frac{Mg\ell}{I_o \omega_s}$ $-\hat{\theta}$ direction

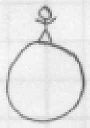
2

8.7) Find the difference in the apparent acceleration of gravity at the equator and the poles. Assume the earth is a sphere.

We will use the relationship between acceleration in a rotating vs. inertial frame.

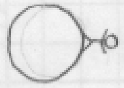
$$\bar{a}_{inertial} = \bar{a}_{rot} + \cancel{2\bar{\Omega} \times \bar{v}_{rot}} + \bar{\Omega} \times (\bar{\Omega} \times \bar{r}) \qquad \bar{v}_{rot} = 0$$





$\bar{r} = R_e \hat{z}$

$\bar{a}_{in} = -g\hat{z}$



$\bar{r} = R_e \hat{r}$

$\bar{a}_{in} = -g\hat{r}$

$$\bar{\Omega} = \Omega_e \hat{z}$$

$$\bar{a}_{rot} = -g\hat{z} + 0$$
$$= -g\hat{z}$$

$$\bar{a}_{rot} = -g\hat{r} - \Omega_e^2 R_e \hat{z} \times (\hat{z} \times \hat{r})$$
$$= -g\hat{r} - \Omega_e^2 R_e \hat{z} \times \hat{\theta}$$
$$= -g\hat{r} + \Omega_e^2 R_e \hat{r} \quad \text{of course}$$
$$= (\Omega_e^2 R_e - g)\hat{r}$$

$\Omega_e = 2\pi / 8.64 \times 10^4 s$

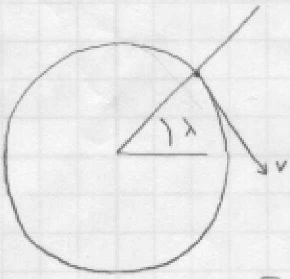
$R_e = 6.37 \times 10^6 m$

difference in magnitude is $\Omega_e^2 R_e$

$$= 0.0337 \ ms^{-2}$$

$$\frac{\Omega_e^2 R_e}{g} = 0.34\%$$

8.9) A train runs south at 60 mph at 60° latitude north. What is the magnitude and direction of the train's horizontal force on the tracks?



$mg = 400$ tons

$\lambda = 60°$

$v = 60$ mph

Ω_e : rotation of earth =

We'll find the $\hat{\theta}$ component of the fictitious force due to the rotating frame.

$$F_{fict} = -m\left\{2\bar{\Omega}\times\bar{v} + \bar{\Omega}\times(\bar{\Omega}\times\bar{r})\right\}$$

fictitious force opposite direction of frame acceleration

$$= -m\left\{2\Omega_e v\,\hat{z}\times(\hat{r}\sin\lambda - \hat{z}\cos\lambda)\right.$$

$$\left. + \Omega_e^2 R_e\,\hat{z}\times\left[\hat{z}\times(\hat{r}\cos\lambda + \hat{z}\sin\lambda)\right]\right\}$$

$$= -m\left\{2\Omega_e v\sin\lambda\,\hat{\theta} + \Omega_e^2 R_e\cos\lambda\,\hat{z}\times\hat{\theta}\right\}$$

$$= m\left\{-2\Omega_e v\sin\lambda\,\hat{\theta} + \Omega_e^2 R_e\cos\lambda\,\hat{r}\right\}$$

got this sign correct; good check

The horizontal component:

$$-2\Omega_e v m\sin\lambda\,\hat{\theta}$$

The tracks counteract this fictitious force, pushing back to keep the train on the tracks.

The train pushes on the tracks with a force

$$2\Omega_e v m\sin\lambda$$

WEST

The tracks push back to the east.

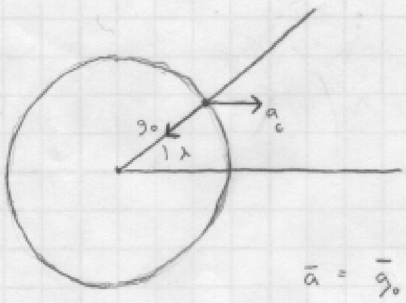
$v = 60\text{mph} = 26.8\ \text{ms}^{-1}$ $\qquad$ $m = 400$ tons (on earth) $= 363200$ kg

$\sin\lambda = 0.866$ $\qquad$ $\Omega_e = 7.27\times10^{-5}\ \text{s}^{-1}$

$$2\Omega_e v m\sin\lambda = 1226\text{N} = 276\text{ lbs}$$

$$= 0.138\text{ tons}$$

8.10) What is the effective g on a spherical rotating earth as a function of latitude?



The centrifugal acceleration causes a fictitious force. We will add this to gravity and then find the magnitude.

R_e: radius of earth
Ω_e: angular velocity of earth

$$\bar{a} = \bar{g}_0 + \bar{a}_c$$

$$= g_0(-\hat{r}\cos\lambda - \hat{z}\sin\lambda) + \Omega_e^2 R_e \cos\lambda\,\hat{r}$$

$$= \hat{r}(\Omega_e^2 R_e - g_0)\cos\lambda - \hat{z} g_0 \sin\lambda$$

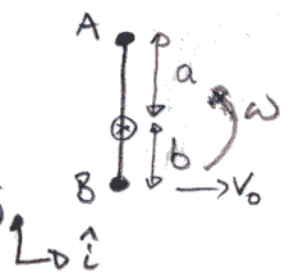
$$|\bar{a}|^2 = (\Omega_e^4 R_e^2 + g_0^2 - 2\Omega_e^2 R_e g_0)\cos^2\lambda + g_0^2\sin^2\lambda$$

$$= g_0^2\left\{\left(\frac{\Omega_e^4 R_e^2}{g_0^2} - 2\frac{\Omega_e^2 R_e}{g_0}\right)\cos^2\lambda + \cos^2\lambda + \sin^2\lambda\right\}$$

$$x \equiv \frac{\Omega_e^2 R_e}{g_0} \qquad = g_0^2\left\{(x^2 - 2x)\cos^2\lambda + 1\right\}$$

$$|\bar{a}| = g_0\left\{1 - (2x - x^2)\cos^2\lambda\right\}^{1/2}$$

KK 6.36



Two masses on a frictionless table with a string connecting them. m_A starts at rest; m_B starts with $\vec{v}_B = v_0\hat{i}$. Find the motion and the string tension.

Solution: Since there are no external forces, the center of mass will move uniformly, and the two masses will orbit the c. of mass. Specifically, the center of mass is located a distance "a" from A and "b" from B where

$$a = \frac{m_B}{m_A+m_B}\,\ell \qquad b = \frac{m_A}{m_A+m_B}\,\ell \qquad (\text{note that } a+b=\ell, \text{ as it must})$$

The center of mass moves at $\vec{V}_{cm} = V_{cm}\hat{i}$ where

$$P_{tot} = m_A \cdot 0 + m_B v_0 = (m_A+m_B)V_{cm} \Longrightarrow V_{cm} = \frac{m_B v_0}{m_A+m_B} = \frac{a}{\ell}v_0$$

But since $v_A = V_{cm} - \omega a = 0$, we learn $\omega = v_0/\ell$.

Cross check: $v_B = V_{cm} + \omega b = v_0 \Rightarrow \omega = \frac{1}{b}v_0(1-\frac{a}{\ell}) = v_0/\ell$ ✓

Since m_A moves in a circle of radius a we now get the tension:

$$T = m_A\omega^2 a = \frac{m_A m_B}{m_A+m_B}\,v_0^2/\ell$$

Cross check: $T = m_B\omega^2 b = \dfrac{m_A m_B}{m_A+m_B}\dfrac{v_0^2}{\ell}$ ✓

Note we can also get ω by thinking about angular momentum. If we pick the origin on top of the center of mass, then as we start

$$L_z = m_B v_0 b = (m_A a^2 + m_B b^2)\omega$$

$$\Longrightarrow \omega = \frac{m_B v_0 b}{m_A a^2 + m_B b^2} = \ldots = v_0/\ell$$

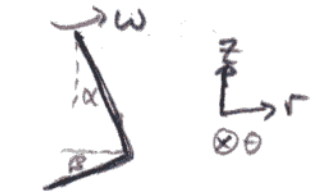
Or pick the origin at mass B, in which case L_z is zero!

$$L_z = 0 = \underbrace{-(m_A+m_B)V_{cm}b}_{\text{this is the "}\vec{R}\times\vec{P}\text{" term}} + (m_A a^2 + m_B b^2)\omega$$

$$\Longrightarrow \omega = \ldots = v_0/\ell$$

KK 7.7

Hoop of mass M, radius R is supported by a string attached to its rim. It spins at ω. Find the small angle β.



↑ hoop tilted at small angle β

Solution: Since $\vec{\omega} = \omega\hat{z}$ isn't precisely aligned with the symmetry axis of the hoop, we introduce two new basis vectors $\hat{e}_1$ and $\hat{e}_2$:

$$\hat{e}_1 = \cos\beta\,\hat{z} - \sin\beta\,\hat{r} \qquad \hat{z} = \cos\beta\,\hat{e}_1 + \sin\beta\,\hat{e}_2$$
$$\hat{e}_2 = \sin\beta\,\hat{z} + \cos\beta\,\hat{r} \implies \hat{r} = \sin\beta\,\hat{e}_1 + \cos\beta\,\hat{e}_2$$

Then picking the origin on the axis of the rotation, and assuming this is exceedingly close to the center of the hoop we go after the torque:

$$\vec{\tau} = \Sigma\,\vec{r}\times\vec{F} = \vec{0}\times(-Mg\hat{z}) + R\hat{r}\times T(\cos\alpha\,\hat{z} - \sin\alpha\,\hat{r}) = -RMg\,\hat{\theta}$$

↑ gravity force acts at zero radius

↑ tension force acts at location $\vec{r} = R\hat{r}$. Note that $T\sin\alpha$ is irrelevant, while $\vec{F} = m\vec{a} \implies T\cos\alpha = Mg$

Next get $\vec{L}$:

$$\vec{\omega} = \omega\hat{z} = \omega\cos\beta\,\hat{e}_1 + \omega\sin\beta\,\hat{e}_2$$

$$\Rightarrow \vec{L} = I_1\,\omega\cos\beta\,\hat{e}_1 + I_2\,\omega\sin\beta\,\hat{e}_2 = MR^2\omega\left[\cos\beta\,\hat{e}_1 + \tfrac{1}{2}\sin\beta\,\hat{e}_2\right]$$

↑ MR^2 through hoop's face

↑ $\frac{1}{2}MR^2$ through hoop diameter

$$= MR^2\omega\left[(\cos^2\beta + \tfrac{1}{2}\sin^2\beta)\hat{z} - \tfrac{1}{2}\sin\beta\cos\beta\,\hat{r}\right]$$

Taking $\frac{d}{dt}$ we get $\quad \frac{d}{dt}\vec{L} = -\frac{1}{2}MR^2\omega^2\sin\beta\cos\beta\,\hat{\theta}$

Equating with $\vec{\tau}$ we finally learn that $\beta \approx \sin\beta \approx \sin\beta\cos\beta = \dfrac{2g}{R\omega^2}$

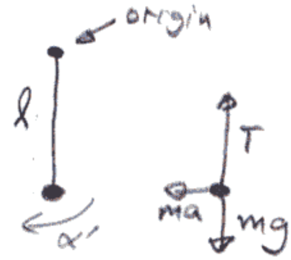
Note that β will indeed be small if ω is large (compared to $\sqrt{g/R}$)

How close is the center of the hoop to the axis? Well the radius r of its orbit satisfies $\quad M\omega^2 r = T\sin\alpha = \dfrac{Mg}{\cos\alpha}\sin\alpha \implies r = \dfrac{g}{\omega^2}\tan\alpha$

So again as long as ω is large, r is small and we were fully justified above when we kept only the MgR term in the torque, neglecting a contribution of size Mgr.

KK 8.3

A pendulum is at rest with the bob pointing at the center of the earth. Its support starts to accelerate laterally at "a". What is the instantaneous angular acceleration of the pendulum?



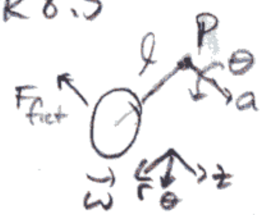
Solution: In the accelerating frame there are three forces on the bob, but only the ma force gives a torque (for a vertical pendulum), so ...

$$\tau = -ma\ell = \frac{d}{dt}L = -m\ell^2\alpha' \Rightarrow \alpha' = a/\ell$$

From the geometry, if the support (i.e. the little airplane) moves a distance x laterally, the center of earth is then at an angle $\theta \sim x/R$ relative to the old vertical. So if $\alpha' = a/R$, then the bob will always point to the center, i.e. if we choose a pendulum length $\ell = R$. Of course this is not physically practical, but we can model such a pendulum in software. Its period, for example, should be

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R}{g}} = 84 \text{ minutes}$$

KK 8.5



A gyro is attached to a frame accelerating at a in the z-direction. Find the relation between its θ coordinate and the frame velocity v.

Solution: In the gyro approximation, $\vec{L} = I\omega\hat{r}$.

The torque from the fake gravity is $\vec{\tau} = \ell\hat{r}\times(-ma\hat{z}) = ma\ell\hat{\theta}$

So ... $\vec{\tau} = \frac{d}{dt}\vec{L} \Rightarrow ma\ell\hat{\theta} = I\omega\dot{\theta}\hat{\theta}$

$$\Rightarrow \dot{\theta} = \frac{m\ell}{I\omega}\dot{v}$$

Integrating with respect to time, define $\theta = 0$ as we start, we learn $v = \frac{I\omega}{m\ell}\theta$

KK9.1 Obtain 9.7 from 9.8.

Solution: 9.8 writes down two conserved quantities, the angular momentum and the total mechanical energy:

$$0 = \frac{d}{dt} l = \frac{d}{dt}[\mu r^2 \dot\theta] = 2\mu r \dot r \dot\theta + \mu r^2 \ddot\theta$$

dividing by one power of r we get 9.7b: $\mu(r\ddot\theta + 2\dot r\dot\theta) = 0$

Similarly,

$$0 = \frac{d}{dt} E = \frac{d}{dt}\left[\tfrac{1}{2}\mu \dot r^2 + \tfrac{1}{2}\mu r^2 \dot\theta^2 + U(r)\right] = \mu \dot r \ddot r + \mu r \dot r \dot\theta^2 + \mu r^2 \dot\theta\ddot\theta + U'(r)\dot r$$

$U' = -f$

Using 9.7b to write $r^2\ddot\theta = -2r\dot r\dot\theta$ we get

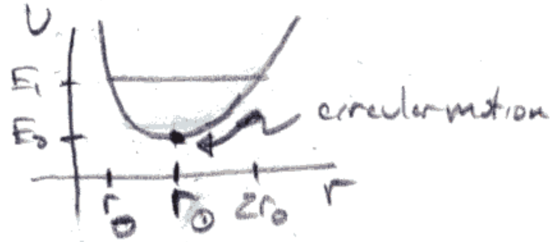
$$0 = \mu \dot r \ddot r + \mu r \dot r \dot\theta^2 + \mu\dot\theta(-2r\dot r\dot\theta) - f(r)\dot r$$

these combine

Dividing by $\dot r$ we get 9.7a: $\mu(\ddot r - r\dot\theta^2) = f$

KK9.2 $\mu = 50g$ $l = 1000\ g\,cm^2/s$ $f(r) = -4r^3\hat r \Rightarrow U(r) = r^4$

(a) $U_{eff} = \frac{l^2}{2\mu r^2} + U(r) = 10^4/r^2 + r^4$



(b) Algebraically the mininimum E_0 is achieved at $r_0 = \sqrt{2}\,5^{2/3}$ where $E_0 = 300\ 5^{2/3}$

(c) If instead the particle oscillates between some $r_{min} = r_0$ and a max $r_{max} = 2r_0$, then its energy E_1 satisfies

$$E_1 = U_{eff}(r_0) = U_{eff}(2r_0)$$

$$\Rightarrow \frac{10^4}{r_0^2} + r_0^4 = \frac{10^4}{(2r_0)^2} + (2r_0)^4 \Rightarrow r_0 = (500)^{1/6} = 2.8\,cm$$

KK 9.3 A particle moves in a circle in a $\frac{1}{r^3}$ force. Show that $r(t) = r_0 + vt$ is also possible, and find $\theta(r)$ in that case.

OK if the force is $\frac{1}{r^3}$, the potential $U(r)$ is $\frac{1}{r^2}$. But then the entire effective potential is $\frac{1}{r^2}$: $\quad U_{eff} = \frac{\ell^2}{2\mu r^2} + U(r) = \left(\frac{\ell^2}{2\mu} + k\right)\frac{1}{r^2}$

Since the particle is observed to move in a circle, we know that $r(t) = r_0$ is a possible "motion" in U_{eff}. But the only way that will be possible is if the constant k just exactly cancels the $\ell^2/2\mu$ — otherwise the $\frac{1}{r^2}$ potential has no extrema. In other words, $U_{eff} = 0$!

And if $U_{eff} = 0$, then $\mu\ddot{r} = -U_{eff}' = 0 \implies r(t) = r_0 + v_r t$ is the general solution.

To find $r(\theta)$ we refer to the discussion leading to KK's eqn 9.16:

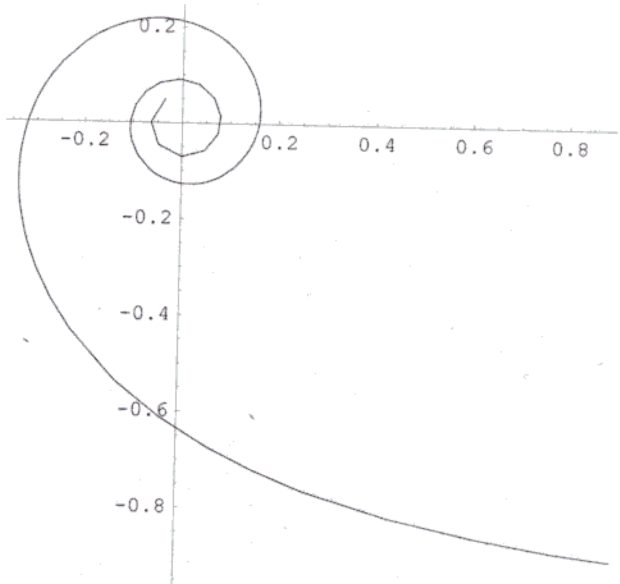
$$\frac{d\theta}{dr} = \frac{\ell}{\mu r^2}\left[\frac{2}{\mu}\left(E - U_{eff}(r)\right)\right]^{-1/2} = \frac{\ell}{\mu v_r}\frac{1}{r^2} \quad \text{if } U_{eff} = 0 \text{ and } E = \tfrac{1}{2}\mu v_r^2$$

Integrating, $\quad \theta(r) = -\frac{\ell}{\mu v_r}\frac{1}{r} + \theta_\infty$

integration constant which is also the angle as $r \to \infty$

```
In[1]:= r[t_] := 1 + v t; θ[t_] := -1/r[t];

In[2]:= v = 1; ParametricPlot[{r[t] * Cos[θ[t]], r[t] * Sin[θ[t]]}, {t, -.94, 2}, AspectRatio → 1]
```



(9.4) For what values of n are circular orbits stable with the potential energy $U(r) = \frac{-A}{r^n}$, $A > 0$?

$$U(r) = \frac{-A}{r^n} \quad \rightarrow \quad U_{eff}(r) = -Ar^{-n} + \frac{L^2}{2m} r^{-2}$$

$$\left.\frac{dU_{eff}}{dr}\right|_{r_0} = nAr_0^{-n-1} - \frac{L^2}{m} r_0^{-3} = 0$$, for circular orbit

$$\boxed{nAr_0^{-n-1} = \frac{L^2}{m} r_0^{-3}}$$

$$\left.\frac{d^2U_{eff}}{dr^2}\right|_{r_0} = n(-n-1)Ar_0^{-n-2} + \frac{3L^2}{m} r_0^{-4} > 0$$, for stability

rewrite

$$(-n-1)r_0^{-1} \cdot (nAr_0^{-n-1}) + \frac{3L^2}{m} r_0^{-4} > 0$$

$$-(n+1)r_0^{-1}\left(\frac{L^2}{m} r_0^{-3}\right) + \frac{3L^2}{m} r_0^{-4} > 0$$

simplify,

$$-(n+1)\cancel{\frac{L^2}{mr_0^4}} + 3\cancel{\frac{L^2}{mr_0^4}} > 0$$

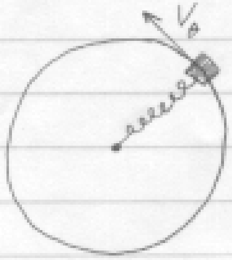
or,

$$\boxed{2 - n > 0}$$

TRUE, if $\boxed{n < 2}$

$F = kr, \; k = 3\,N/m$

(9.5) $m = 2\,kg$, $F = 3r\,N$, $E = 12\,J$



$$U(r) = \int F(r)\,dr$$

$$U(r) = \tfrac{1}{2}kr^2$$

$$U_{eff}(r) = \tfrac{1}{2}kr^2 + \frac{L^2}{2mr^2}$$

a) $\left.\dfrac{dU_{eff}}{dr}\right|_{r=r_0} = 0$ for circular motion

$$\frac{d}{dr}\left[\tfrac{1}{2}kr^2 + \frac{L^2}{2m}r^{-2}\right]_{r_0} = kr_0 - \frac{L^2}{m}r_0^{-3} = 0$$

$$kr_0 = \frac{L^2}{m}r_0^{-3}$$

or $\boxed{r_0 = \left(\dfrac{L^2}{km}\right)^{1/4}}$

if $K_{rad} = 0$, then $U_{eff}(r_0) = E_{TOT} = 12\,J$

or, $\tfrac{1}{2}kr_0^2 + \dfrac{L^2}{2mr_0^2} = E$ & $r_0^2 = \dfrac{L}{\sqrt{km}}$

$$\frac{\tfrac{1}{2}kL}{\sqrt{km}} + \frac{L^2\sqrt{km}}{2m\cdot L} = E$$

$$L\sqrt{\frac{k}{m}} = E \longrightarrow L = \sqrt{\frac{m}{k}}\,E = mV_\theta\cdot r_0$$

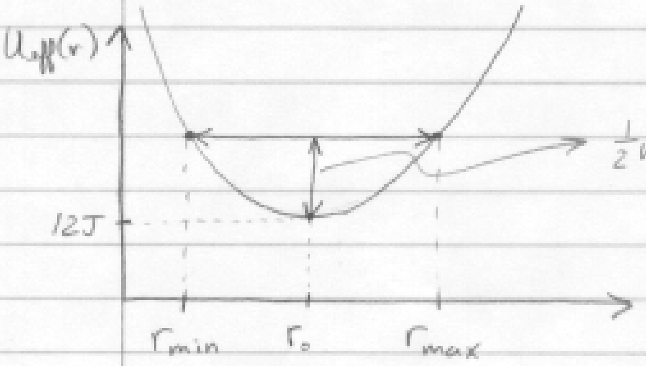
$$\boxed{V_\theta = \frac{1}{\sqrt{km}}\frac{E}{r_0}}$$

finally, $$\boxed{r_0 = \sqrt{\frac{E}{k}} = 2.00\,m}$$

&

$$\boxed{v_\theta = \frac{1}{\sqrt{km}}\frac{E}{r_0} = 2.45\ m/s}$$

b) now $E' = 12\,J + \frac{1}{2}m\dot{r}^2$



$\frac{1}{2}m\dot{r}^2 = \frac{2kg}{2}(1\,m/s)^2 = 1\,J$

$E' = 13\,J$

at r_{min} + r_{max}, $E' = U_{eff} + \cancel{0}$

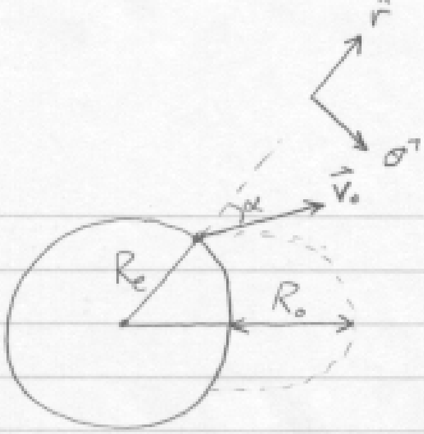
$$\frac{1}{2}kr^2 + \frac{L^2}{2mr^2} = E' \quad , L^2 = \frac{m}{k}E^2 \quad , E = 12J$$

or, $$\frac{1}{2}kr^4 - E'r^2 + \frac{E^2}{2k} = 0$$

solve, get $r^2 = 6\,m^2$ or $8/3\ m^2$

so, $$\boxed{\begin{array}{l} r_{min} = 1.63\,m \\ r_{max} = 2.45\,m \end{array}}$$

(9.8)



$$V_0 = \sqrt{\frac{GM_e}{R_e}}$$

$$R_0 = ?$$

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GM_e m}{r}$$

$$L = mV_\theta r$$

initially, $\begin{cases} \dot{r} = V_0 \cos\alpha \\ L = mV_0 \sin\alpha R_e \end{cases}$

so, $E_0 = \frac{1}{2}mV_0^2\cos^2\alpha + \frac{m^2V_0^2\sin^2\alpha R_e^2}{2mR_e^2} - \frac{GM_e m}{R_e}$

$E_0 = \frac{1}{2}mV_0^2 - \frac{GM_e m}{R_e}$ or, $\boxed{E_0 = \frac{-GM_e m}{2R_e}}$

at peak, $\dot{r} = 0$
$L = mV_0 \sin\alpha R_e$

$$E_f = \frac{m^2V_0^2\sin^2\alpha R_e^2}{2mR_0^2} - \frac{GM_e m}{R_0}$$

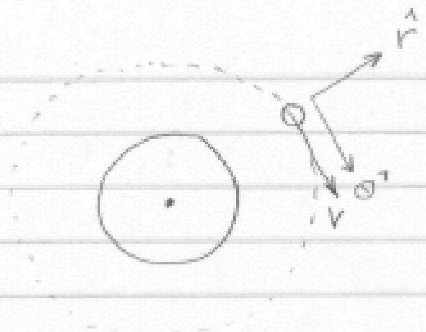
$$E_0 = E_f \longrightarrow \frac{-GM_e \cancel{m}}{2R_e} = \frac{GM_e \cancel{m} R_e \sin^2\alpha}{2R_0^2} - \frac{GM_e \cancel{m}}{R_0}$$

solve for R_0,

$$R_0^2 - 2R_eR_0 + R_e^2\sin^2\alpha = 0$$

$$\boxed{R_0 = R_e(1 + \cos\alpha)}$$

9.10



a) $$E = \frac{1}{2}m\dot{r}^2 + \underbrace{\frac{L^2}{2mr^2} - \frac{GMm}{r}}_{U_{eff}}$$

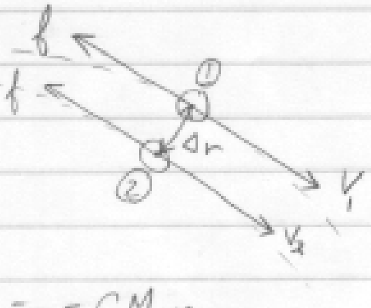
$$\left.\frac{dU_{eff}}{dr}\right|_{r_0} = \frac{-L^2}{mr_0^3} + \frac{GMm}{r_0^2} = 0 \quad , \dot{r} = 0 \quad \text{for circular orbit}$$

$$L^2 = GMm^2 r_0$$

so,

$$E = \frac{GMm^2 r_0}{2mr_0^2} - \frac{GMm}{r_0} \longrightarrow \boxed{E = \frac{-GM_e m}{2r_0}}$$

b)



after one revolution, r changes by Δr

$$E_1 = \frac{-GMm}{2r} \quad , \quad E_2 = \frac{-GMm}{2(r-\Delta r)}$$

$$E_1 - W_f = E_2$$

$$\frac{-GMm}{2r} - 2\pi r f = \frac{-GMm}{2r\left(1-\frac{\Delta r}{r}\right)} \simeq \frac{-GMm}{2r}\left(1+\frac{\Delta r}{r}\right) \quad \text{for small } \frac{\Delta r}{r}$$

$$\frac{GM\cancel{m}}{\cancel{2r}} + 2\pi r f \approx \frac{GM\cancel{m}}{\cancel{2r}} + \frac{GMm\,\Delta r}{2r^2}$$

$$\boxed{\Delta r \approx \frac{4\pi r^3 f}{GMm}}$$

c) $L_1 = m v_1 r \longrightarrow L_1^2 = m^2 v_1^2 r^2$

$L_2 = m v_2 (r - \Delta r) \longrightarrow L_2^2 = m^2 v_2^2 (r-\Delta r)^2$

$$E = \frac{L^2}{2mr^2} - \frac{GMm}{r} \quad \text{if } \dot{r} = 0$$

so,

$$E_1 = \frac{L_1^2}{2mr^2} - \frac{GMm}{r} = \frac{1}{2} m v_1^2 - \frac{GMm}{r}$$

$$E_2 = \frac{L_2^2}{2m(r-\Delta r)^2} - \frac{GMm}{r(1-\frac{\Delta r}{r})} \approx \frac{1}{2} m v_2^2 - \frac{GMm}{r} - \frac{GMm\,\Delta r}{r^2}$$

and

$$E_1 - W_f = E_2$$

$$\underbrace{\frac{1}{2} m v_1^2}_{K_1} - \cancel{\frac{GMm}{r}} - 2\pi r f = \underbrace{\frac{1}{2} m v_2^2}_{K_2} - \cancel{\frac{GMm}{r}} - \cancel{\frac{GMm\,\Delta r}{r^2}}^{\,0}$$

because $\Delta r \ll r$

so, $\boxed{K_1 - K_2 = +2\pi r f}$

Physics 141
Problem Set 7

Eduard Antonyan

1 (7.2) Rotating flywheel on a rotating table

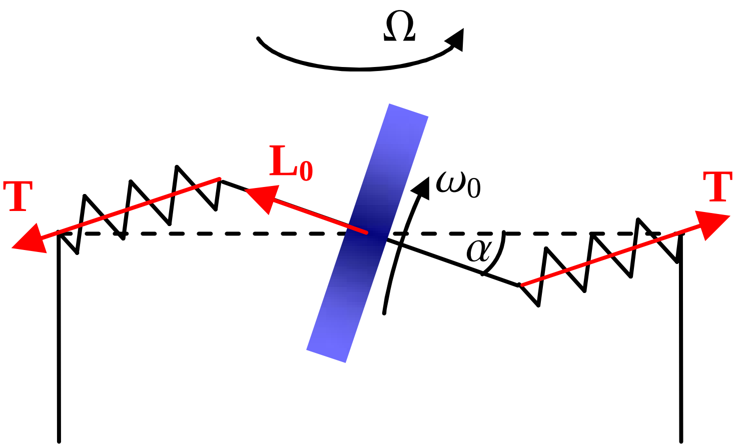


Figure 1: Side view.

As it can be seen from the picture the torque is out of the picture and its magnitude is:

$$\tau = 4T\alpha l, \tag{1}$$

where we assumed that the angle α is small. The derivative of the angular momentum $\frac{d\overrightarrow{L_0}}{dt}$ is also out the picture. That, and also its magnitude follows from:

$$\frac{d\overrightarrow{L_0}}{dt} = \overrightarrow{\Omega} \times \overrightarrow{L_0} \tag{2}$$

Thus $\tau = \frac{dL_0}{dt} = \Omega L_0 = \Omega I_0 \omega_0$, and therefore:

$$\alpha = \frac{I_0 \omega_0 \Omega}{4Tl}. \tag{3}$$

2 (7.5) Car on a curve

(a) Let us understand first why should the car tend to roll over. As one can see from the picture, $\overrightarrow{N_1}$, $\overrightarrow{f_1}$ and $\overrightarrow{f_2}$ create torque (about the center of mass), that's into the picture, and $\overrightarrow{N_2}$ creates torque out of the picture. If the car is stable, then the total torque should be zero. By Newton's second law:

$$\begin{aligned} f_1 + f_2 &= Ma = \frac{Mv^2}{r} \\ N_1 + N_2 &= Mg \end{aligned} \tag{4}$$

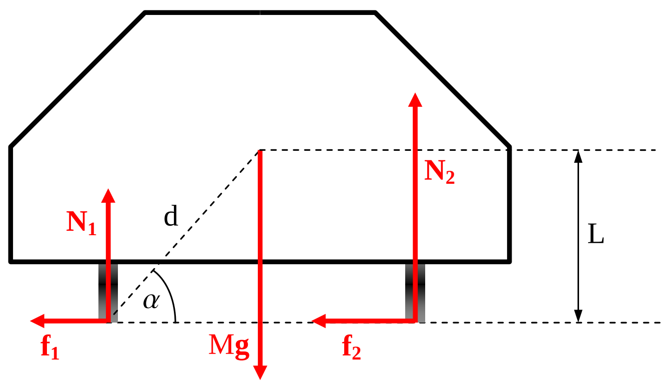


Figure 2: View from behind. The car is turning to the left.

So the faster the car is moving the larger are f_1 and f_2, and thus the torque into the page. To have equilibrium, N_1 will have to decrease and N_2 will have increase. For high enough velocities N_1 will become zero and the car will start rolling over.

Now the reason putting a spinning flywheel can help is because we can put it in such way that the torque will be used to change the angular momentum of the flywheel and not of the car!

Now let's consider the case where the car is turning to the left with angular velocity $\overrightarrow{\Omega}$ which will point vertically upward. Let's put the flywheel so that its angular momentum $\overrightarrow{L}$ is pointing *radially outward* (to the right on the picture). To make that angular momentum rotate with the car, one needs torque equal to $\frac{d\overrightarrow{L}}{dt} = \overrightarrow{\Omega} \times \overrightarrow{L}$, i.e. pointing forward with respect to the car, which is exactly what we've got.

Note, however, that if the car is turning to the right, we should not reverse the direction of the angular momentum, i.e. if the car is turning to the right the angular momentum should point *radially inward*. This is because in that case $\overrightarrow{\Omega}$ is pointing vertically downward, and the torque is pointing out of the page (backward w.r.t. the car). Thus the right choice for $\overrightarrow{L}$ would be pointing again to the right on the picture, so that $\overrightarrow{\Omega} \times \overrightarrow{L}$ is again in the direction of the torque. (More mathematically the reason the direction of $\overrightarrow{L}$ doesn't change is because torque, angular velocity and angular momentum are axial vectors, not real vectors).

(b) If the loading on the wheels is the same, then $N_1 = N_2 \equiv N$, thus $f_1 = f_2 \equiv f$. Since we've already discussed the directions of the derivative of the angular momentum and the torque, let's just write down the scalar version of the torque equation:

$$\tau = \frac{dL}{dt}. \tag{5}$$

Now

$$\tau = f_1 d \sin\alpha + N_1 d \cos\alpha + f_2 d \sin\alpha - N_2 d cos\alpha = 2fd\sin\alpha = 2fL = Mv\Omega L, \tag{6}$$

where in the last equality we used Newton's second law (4). Finally for a disk-shaped flywheel using

$$\frac{dL}{dt} = \Omega L = \Omega I\omega = \Omega\frac{mR^2}{2}\omega, \tag{7}$$

we have:

$$\omega = \frac{2MvL}{mR^2}. \tag{8}$$

3 (7.8) Deflecting hoop

(a) The torque induced by the force acting on the top of the hoop will be in the direction shown in the picture. Since $\overrightarrow{\tau} = \frac{d\overrightarrow{L}}{dt}$ that means that the angular momentum will start rotating in the horizontal plane and if we look from the top it will be rotating counterclockwise ($\overrightarrow{\omega}$ pointing vertically upward). So in the gyroscope approximation the force will not incline the plane of the hoop, but will instead deflect the line of rolling.

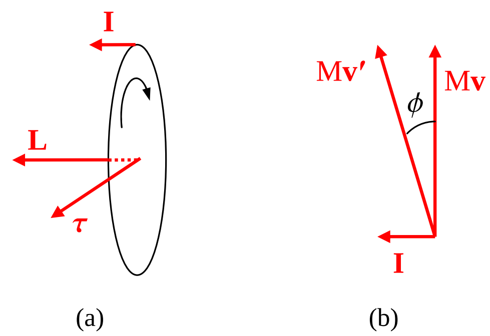


Figure 3: (a) “Normal” view. (b) View from above.

By the conservation of momentum we have:

$$M\overrightarrow{v} + \overrightarrow{I} = M\overrightarrow{v}', \tag{9}$$

where $\overrightarrow{v}'$ is the velocity of the hoop after the tap. Thus the line of the rolling of the hoop will change by an angle:

$$\phi \approx \tan\phi = \frac{I}{Mv}. \tag{10}$$

(b) The gyroscope approximation is valid if $\frac{dL}{dt} \ll L\omega$, i.e. if the change in the angular momentum is small compared to the initial angular momentum. In our case this would mean that

$$\tau_{max} = Fb \ll L\omega = Mb^2\omega^2 = Mv^2. \tag{11}$$

Thus for the gyroscope approximation to hold, we need the peak applied force to satisfy:

$$F \ll \frac{Mv^2}{b}. \tag{12}$$

4 (10.2) Oscillating spring

For the damping constant we have:

$$\gamma = \frac{\omega_1}{Q} = \frac{2\pi\, 2\,\mathrm{Hz}}{60} \approx 0.21\,\mathrm{s}^{-1}. \tag{13}$$

Then from

$$\omega_0 = \sqrt{\frac{k}{m}}, \tag{14}$$

we have:

$$k = m\omega_0^2 = m\left(\omega_1^2 + \frac{\gamma^2}{4}\right) = 0.3\mathrm{kg}\left[(2\pi\, 2\,\mathrm{Hz})^2 + \frac{(0.21\,\mathrm{s}^{-1})^2}{4}\right] \approx 47.377\,\mathrm{N/kg}. \tag{15}$$

5 (10.6) Falling masses and critical damping

(a) At the rest position we have equilibrium of forces, thus:

$$Mg = kx_0, \tag{16}$$

where m is the mass of the platform and x_0 is the distance from the initial position of the platform to its final position.
Thus

$$k = \frac{Mg}{x_0} = 980\,\mathrm{N/kg}. \tag{17}$$

(b) The velocity of the mass just before it hits the platform v is determined from energy conservation:

$$Mgh = \frac{Mv^2}{2}, \tag{18}$$

where h is the height of the mass above the platform. Then we have an inelastic collision of the mass with the platform, so by momentum conservation:

$$Mv = (M+m)v', \tag{19}$$

where v' is their velocity just after they stick together.

Since we want critical damping we need $\gamma = 2\omega_0 = 2\sqrt{\frac{k}{M+m}} \approx 18.07\,\mathrm{s}^{-1}$.

Now the general equation of motion for critical damping is given by:

$$x(t) = Ae^{-\frac{\gamma}{2}t} + Bte^{-\frac{\gamma}{2}t}. \tag{20}$$

Here $x(t)$ is the height of the platform w.r.t. the final position. Since we want $x(0) = x_0$ and $\frac{dx}{dt}|_0 = v'$, we get for the coefficients A and B:

$$\begin{aligned} A &= x_0 \\ B &= v' + \frac{\gamma}{2} = \frac{M}{M+m}\sqrt{2gh} + \sqrt{\frac{g}{x_0}\frac{M}{M+m}}. \end{aligned} \tag{21}$$

So finally the equation of motion is:

$$x(t) = x_0 e^{-\sqrt{\frac{g}{x_0}}t} + \left(\frac{M}{M+m}\sqrt{2gh} + \sqrt{\frac{g}{x_0}\frac{M}{M+m}}\right) te^{-\sqrt{\frac{g}{x_0}}t}. \tag{22}$$

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.012 **Fall Term 2005**

PROBLEM SET 8

Due: Friday, November 18 at 4:00pm.

Reading: Finish Kleppner & Kolenkow, Chapter 6

0. **Collaboration and discussion.** Please give a brief statement at the top of your homework telling us the names of all the students with whom you discussed the homework problems.
1. Kleppner & Kolenkow, Problem 6.9
2. Kleppner & Kolenkow, Problem 6.11
3. Kleppner & Kolenkow, Problem 6.13
4. Kleppner & Kolenkow, Problem 6.18
5. Kleppner & Kolenkow, Problem 6.19
6. Kleppner & Kolenkow, Problem 6.23
7. Kleppner & Kolenkow, Problem 6.29
8. Kleppner & Kolenkow, Problem 6.30
9. Kleppner & Kolenkow, Problem 6.33
10. Kleppner & Kolenkow, Problem 6.35

PHYSICS 22 PROBLEM SET 5

4 pages

PROBLEM 1

[Following Note 10.1 K&K]

(a) $\gamma^2/4 - \omega_0^2 = 16\pi^2 > 0 \Rightarrow$ Heavy Damping with

$$\alpha = -\frac{\gamma}{2} \pm \frac{\gamma}{2}\sqrt{1 - \frac{4\omega_0^2}{\gamma^2}}$$

$1 - \frac{4 \cdot 9}{100} = \sqrt{\frac{64}{100}} \cdot \frac{8}{16} \times \frac{4}{5} \times 5\pi$

$$= -5\pi \pm 4\pi$$

$$= -\pi, -9\pi$$

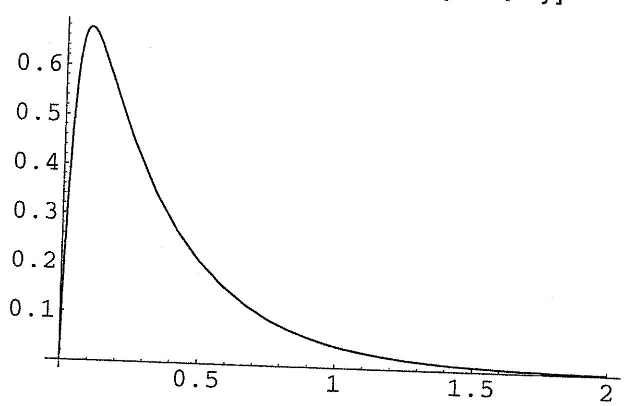
$$\Rightarrow \quad x(t) = A e^{-\pi t} + B e^{-9\pi t}$$

The initial Conditions $\Rightarrow$ $\left.\begin{aligned} 0 &= A + B \\ 8\pi &= -A\pi - 9B\pi \end{aligned}\right\}$ $A = 1,\ B = -1.$

$$\Rightarrow \quad \boxed{x(t) = e^{-\pi t} - e^{-9\pi t}}$$

PLOT $x(t)$ [Part (a)]

```
In[2]:= Plot[E^{-Pi t} - E^{-9 Pi t}, {t, 0, 2}]
```




```
Out[2]= -Graphics-
```

(b) $\gamma = 6\pi \Rightarrow \frac{\gamma^2}{4} = \omega_0^2$

$$\Rightarrow x = Ae^{-3\pi t} + Bte^{-3\pi t} \quad \text{[page 437 K\&K]}$$

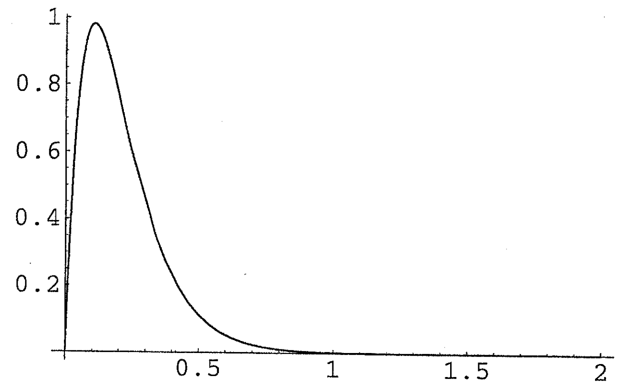
$$= (A + Bt)\, e^{-3\pi t}$$

Initial conditions $\Rightarrow A = 0,\ B = 8\pi$

$$\Rightarrow x(t) = 8\pi t e^{-3\pi t}$$

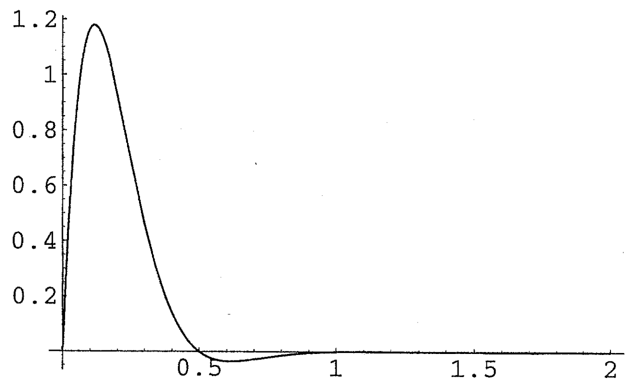
PLOT x(t) [Part (b)]

```
In[4]:= Plot[8*Pi*t*E^{-3*Pi*t}, {t, 0, 2}]
```



PLOT x(t) [Part (c)]

```
In[5]:= Plot[4*E^{-Sqrt[5]*Pi*t}*Sin[2*Pi*t], {t, 0, 2}]
```



(c) $\gamma = 2\sqrt{5}\,\pi \Rightarrow \frac{\gamma^2}{4} - \omega_0^2 = -4\pi^2 \Rightarrow$ Light damping

$$x(t) = Ae^{-\sqrt{5}\pi t}\cos(2\pi t + \phi)$$

Initial Conditions $\Rightarrow$

$$0 = A\cos\phi$$

$$8\pi = -\sqrt{5}\pi \cancelto{0}{A\cos\phi} - A \cdot 2\pi \sin\phi \Rightarrow A\sin\phi = -$$

$$\left.\begin{aligned} \cos\phi &= 0 \\ \sin\phi &= -\frac{4}{A} \end{aligned}\right] \rightarrow 1 = \frac{4^2}{A^2} \quad \text{i.e. } A = 4 \ \&\ \phi = -\pi/2$$

$$x(t) = 4e^{-\sqrt{5}\pi t}\cos\left(2\pi t - \frac{\pi}{2}\right)$$

$$\boxed{x(t) = 4e^{-\sqrt{5}\pi t}\sin(2\pi t)}$$

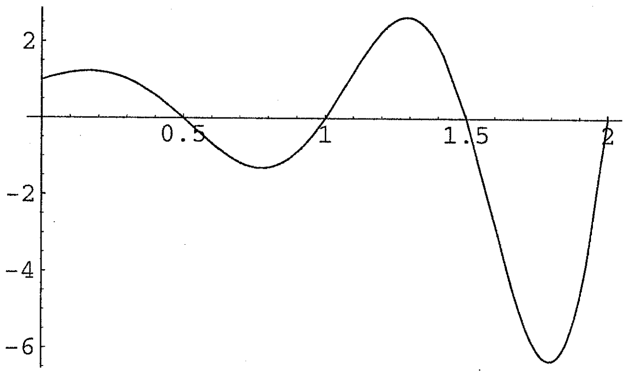
(d)

$$\frac{x(t)\big|_{\gamma=2\sqrt{5}\pi}}{x(t)\big|_{\gamma=6\pi}} = \frac{4e^{-\sqrt{5}\pi t}\sin(2\pi t)}{8\pi t e^{-3\pi t}}$$

$$= \frac{1}{2\pi}\frac{1}{t}e^{(3-\sqrt{5})\pi t}\sin(2\pi t)$$

PLOT [PART (d)]

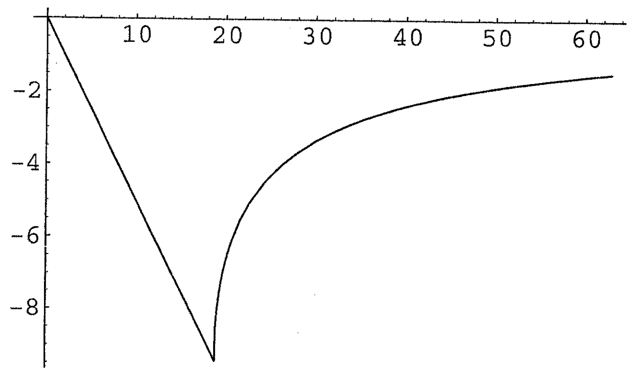
```
In[6]:= Plot[(1 / (2 * Pi)) * (1 / t) * E^{(3 - Sqrt[5]) * Pi * t} * Sin[2 * Pi * t], {t, 0, 2}]
```



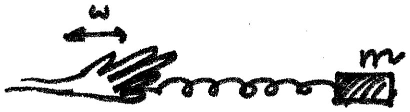
(e)

Real part has min @ $6\pi \Longleftrightarrow$ critical damping

PLOT [PART (e)]



K&K 10.7



$$F = F_{spring} + F_{friction} + F_{driving}$$

$$m\ddot{x} = -kx - b\dot{x} + F_0 \cos \omega t$$

has solution $\quad X = A\cos(\omega t + \phi)$ with

$$A = \frac{F_0}{m} \frac{1}{[(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2]}$$

$$\phi = \tan^{-1}\left(\frac{\gamma\omega}{\omega^2 - \omega_0^2}\right)$$

Want ω s.t phases of $F_{driving}$ & Velocity Match:

phase of Velocity $\sim \sin(\omega t + \phi)$

phase of driving $\sim \cos(\omega t)$

peaks & troughs match when $\phi = \pm \pi/2$.

$\rightarrow \quad \tan\phi \rightarrow \infty \Leftrightarrow \boxed{\omega \rightarrow \omega_0.}$