MTH 102: Linear Algebra
Department of Mathematics and Statistics
Indian Institute of Technology - Kanpur

Problem Set 2

Problems marked **(T)** are for discussions in Tutorial sessions.

1. Find two 2×2 invertible matrices A and B such that $A \neq cB$, for any scalar c and $A + B$ is not invertible.

2. **(T)** Let
$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$
Write down the permutation matrix P such that PA is upper triangular. Which permutation matrices P_1 and P_2 make $P_1 A P_2$ lower triangular?

3. If A and B are symmetric matrices, which of these matrices are necessarily symmetric?
   (a) $A^2 - B^2$
   (b) $(A + B)(A - B)$
   (c) ABA
   (d) $ABAB$

4. **(T)** Let $P_n(\mathbb{R})$ be the set of vectors of polynomials with real coefficients and degree less than or equal to n. Show that $P_n(\mathbb{R})$ is a vector space over $\mathbb{R}$ with respect to the usual addition and scalar multiplication.

5. Show that the space of all real $m \times n$ matrices is a vector space over $\mathbb{R}$ with respect to the usual addition and scalar multiplication.

6. Let S be the set of all $n \times n$ symmetric matrices. Check whether S is a real vector space under usual addition and scalar multiplication of matrices.

7. In $\mathbb{R}$, consider the addition $x \oplus y = x + y - 1$ and $a.x = a(x - 1) + 1$. Show that $\mathbb{R}$ is a real vector space with respect to these operations with additive identity 1.

8. **(T)** Which of the following are subspaces of $\mathbb{R}^3$:
$$(a)\ \{(x, y, z) \mid x \geq 0\},\ (b)\ \{(x, y, z) \mid x + y = z\},\ (c)\ \{(x, y, z) \mid x = y^2\}.$$

9. Find the condition on real numbers a, b, c, d so that the set $\{(x, y, z) \mid ax + by + cz = d\}$ is a subspace of $\mathbb{R}^3$.

10. **(T)** Let W_1 and W_2 be subspaces of a vector space V such that $W_1 \cup W_2$ is also a subspace. Prove that one of the spaces $W_i,\ i = 1, 2$ is contained in the other.

11. Suppose S and T are two subspaces of a vector space V. *Define* the **sum**

$$S + T = \{s + t : s \in S, t \in T\}.$$

Show that $S + T$ satisfies the requirements for a vector space.

12. Let $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ be n vectors from a vector space V over $\mathbb{R}$. Define **span** of this set of vectors as

$$\operatorname{span}(\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}) = \{c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots c_n\mathbf{v_n} : c_1, c_2, \ldots, c_n \in \mathbb{R}\},$$

that is, the set of all linear combinations of vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$. Show that $\operatorname{span}(\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\})$ is a subspace of V.

13. **(T)** Show that $\{(x_1, x_2, x_3, x_4) : x_4 - x_3 = x_2 - x_1\} = \operatorname{span}(\{(1, 0, 0, -1),\ (0, 1, 0, 1),\ (0, 0, 1, 1)\}$ and hence is a subspace of $\mathbb{R}^4$.

14. **(T)** The column space of an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

defined as

$$C(A) = \operatorname{span}\left(\left\{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}\right\}\right)$$

Clearly, C(A) is a subspace of $\mathbb{R}^m$. Suppose B and D are two $m \times n$ matrices and $S = C(B)$ and $T = C(D)$, then $S + T$ is a column space of what matrix M?

15. Suppose A is an $m \times n$ matrix and B in an $n \times p$ matrix. Show that matrices A and $\begin{bmatrix} A & AB \end{bmatrix}$ (with extra columns) have the same column space. Next, find a square matrix A with $C(A^2) \subsetneq C(A)$.