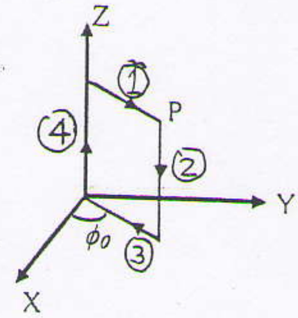
PHY 103 - MID-SEM SOL^n

1. Consider a vector field $\vec{A}(\vec{r}) = z\sin^2\phi\,\hat{s} - s\cos^2\phi\,\hat{k}$ in cylindrical polar coordinates. Find the line integral of this field over a square loop P (shown in the figure) of unit side perpendicular to the xy-plane and making ϕ_0 angle with the x-axis. Verify that the result is consistent with the Stokes' theorem. (6+4=10)



Along (1)

$$z = 1,\ \phi = \phi_0,\ s: 0 \to 1$$
$$\vec{dl} = \hat{s}\,ds$$
$$\int \vec{A}\cdot\vec{dl} = \int_0^1 \sin^2\phi_0\,ds = \sin^2\phi_0$$

Along (2)

$$s = 1,\ z: 1 \text{ to } 0\ \ \phi = \phi_0,\ \vec{dl} = -\hat{k}\,dz$$
$$\int \vec{A}\cdot\vec{dl} = -\int_1^0 \cos^2\phi_0\,dz = \cos^2\phi_0$$

Along (3)

$$z = 0,\ s: 1 \text{ to } 0\ \ \phi = \phi_0,\ \vec{dl} = -\hat{s}\,ds$$
$$\Rightarrow \int \vec{A}\cdot\vec{dl} = \int (-s\cos^2\phi_0\,\hat{k})\cdot(\hat{s}\,ds) = 0$$

Along (4)

$$s = 0;\ z: 0 \text{ to } 1\ \ \phi = \phi_0\ \ \vec{dl} = \hat{k}\,dz$$
$$\int \vec{A}\cdot\vec{dl} = \int (z\sin^2\phi\,\hat{s})\cdot(\hat{k}\,dz) = 0$$

$$\Rightarrow \oint_P \vec{A}\cdot\vec{dl} = \cos^2\phi_0 + \sin^2\phi_0 = 1$$

$$\vec{\nabla}\times\vec{A} = \frac{1}{s}\begin{vmatrix} \hat{s} & s\hat{\phi} & \hat{k} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ z\sin^2\phi & 0 & -s\cos^2\phi \end{vmatrix} = \frac{1}{s}\left[-\hat{s}\,2s\cos\phi\sin\phi\right] + \hat{\phi}\left[\sin^2\phi + \cos^2\right. + \frac{\hat{k}}{s}\left[-2z\sin\phi\cos\phi\right]$$

$$= -\hat{s}\sin 2\phi + \hat{\phi} - \frac{z}{s}\sin 2\phi\,\hat{k}$$

$$\underbrace{\vec{da} = ds\,dz\,\hat{\phi}}_{1} \Rightarrow \int(\vec{\nabla}\times\vec{A})\cdot\vec{da} = \int ds\,dz = 1$$

2. (a) Calculate the divergence of the following vector field **(4)**

$$\vec{A}(\vec{r}) + \frac{1}{4\pi}\vec{\nabla}\int \frac{\vec{\nabla}'\cdot\vec{A}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\tau'$$

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{4\pi}\nabla^2\int \frac{[\vec{\nabla}'\cdot\vec{A}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\tau'$$

$$= \vec{\nabla}\cdot\vec{A} + \frac{1}{4\pi}\int \vec{\nabla}'\cdot\vec{A}(\vec{r}')\,\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)d\tau'$$

$$\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -4\pi\,\delta(\vec{r}-\vec{r}')$$

$$= \vec{\nabla}\cdot\vec{A} - \int [\vec{\nabla}'\cdot\vec{A}(\vec{r}')]\,\delta(\vec{r}-\vec{r}')\,d\tau'$$

$$= \vec{\nabla}\cdot\vec{A} - \vec{\nabla}\cdot\vec{A} = 0$$

2. (b) Prove that for a scalar field $T(x,y,z)$, $\int_S \vec{\nabla} T \times d\vec{s} = -\oint_C T d\vec{l}$ where the surface S has the boundary C. **(6)**

$$\int (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \oint \vec{A} \cdot d\vec{l}$$

Let $\vec{A} = \vec{C}\,T$ where $\vec{C}$ is an arbitrary constant vector

$$\vec{\nabla} \times (\vec{C}\,T) = T\,\vec{\nabla} \times \vec{C} - \vec{C} \times \vec{\nabla} T$$

$$= -\vec{C} \times \vec{\nabla} T$$

$$\Rightarrow \quad -\int (\vec{C} \times \vec{\nabla} T) \cdot d\vec{a} = \int \vec{C}\,T \cdot d\vec{l}$$

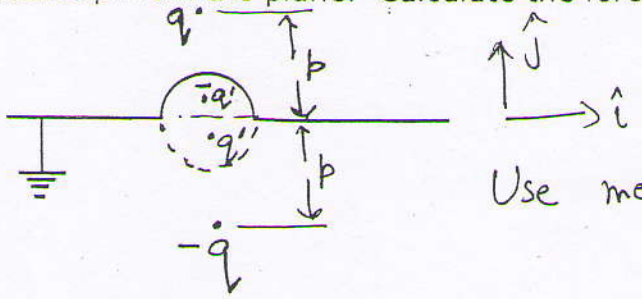
Using $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{C} \times \vec{A})$

$$-\int (\vec{C} \times \vec{\nabla} T) \cdot d\vec{a} = -\int (\vec{\nabla} T \times d\vec{a}) \cdot \vec{C} = \vec{C} \cdot \oint T\,d\vec{l}$$

Since $\vec{C}$ is arbitrary,

$$\int (\vec{\nabla} T \times d\vec{a}) = -\oint T\,d\vec{l}$$

3. (a) An infinite conducting plane sheet has a hemispherical bulge of radius a on it and is grounded (see figure). A charge is placed symmetrically above the bulge at a distance p from the plane. Calculate the force on the charge. **(5)**



Use method of images.

* Hemispherical bulge equipotential $\Rightarrow$ image

$$-q' = -q\frac{a}{p}$$

* In order to keep plane at constant potential, necessary to add q' & $-q$ charges at distances $-a^2/p$ & p respectively.

$\therefore$ Force on q is given by

$$\vec{F} = \frac{q^2}{4\pi\epsilon_0}\left[\frac{-a/p}{(p-a^2/p)^2} + \frac{a/p}{(p+a^2/p)^2} - \frac{1}{4p^2}\right]\hat{j}$$

$$= -\frac{q^2}{4\pi\epsilon_0}\hat{j}\left[\frac{4a^3p^3}{(p^4-a^4)^2} + \frac{1}{4p^2}\right]$$

3. (b) A point charge of magnitude $+q$ is at the origin. A negative point charge $-nq$ ($n>1$) is at $(0,d,0)$. Find the equation for the zero potential surface specifying its geometric features clearly. **(5)**

$$\phi(x,y,z) = \frac{+q}{(x^2+y^2+z^2)^{1/2}} - \frac{nq}{(x^2+(y-d)^2+z^2)^{1/2}} = 0$$

Squaring both sides

$$\frac{1}{x^2+y^2+z^2} = \frac{n^2}{(x^2+(y-d)^2+z^2)}$$

$$\Rightarrow \quad x^2+(y-d)^2+z^2 = n^2(x^2+y^2+z^2)$$

$$(n^2-1)(x^2+y^2+z^2) = d^2 - 2yd$$

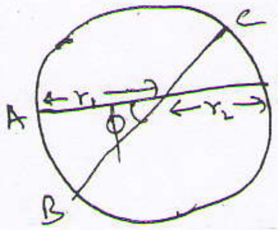
$$(x^2+y^2+z^2) = \frac{d^2}{n^2-1} - \frac{2yd}{n^2-1}$$

or $$x^2+z^2+\left(y+\frac{d}{n^2-1}\right)^2 = \frac{d^2}{n^2-1} + \frac{d^2}{(n^2-1)^2}$$

$$x^2+z^2+\left(y+\frac{d}{n^2-1}\right)^2 = \frac{n^2d^2}{(n^2-1)^2}$$

Equipotential surface with potential zero is a sphere with center at $\left(0, \frac{-d}{n^2-1}, 0\right)$ and radius $= \frac{dn}{(n^2-1)}$

4. Imagine a two dimensional world. In this world it is found that the electric field at any point inside a uniformly charged ring is always zero. (i) Find the dependence of the field $f(s)$ due to a point charge on the distance s from the charge. Show how this dependence ensures that the field inside the ring is zero. (ii) What will be the Gauss' law in this world? **(6+4=10)**



Let the dependence of the field on distance be $f(r)$. Since field at any pt. inside vanishes, we can say that field due to two segments on opposite sides of a pt. cancel out because the field due to pt. charge is along the line joining the charge to the point.

$$\Rightarrow \quad (r_1 \phi) f(r_1) = (r_2 \phi) f(r_2)$$

$$\Rightarrow \quad r_1 f(r_1) = r_2 f(r_2)$$

$$\Rightarrow \quad r f(r) = \text{Const} \quad \text{or} \quad \boxed{f(r) = \frac{\text{Constant}}{r}}$$

(ii)

Consider a pt. charge and draw a circle of radius r around it

Then :-

$$\sum E_\perp \Delta l = \frac{\text{Const}}{r} \times 2\pi r$$

$$= 2\pi(\text{Const})$$

But constant $\propto$ charge enclosed

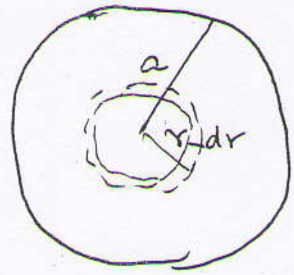
$\Rightarrow$ Gauss's law is $\sum E_\perp \Delta l = \text{Constant} \times \text{charge enclosed}$

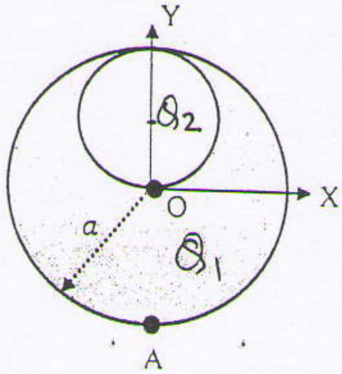
5. Consider a spherical charge distribution which has a constant charge density ρ from $r = 0$ out to $r = a$, and zero beyond. **(3+2+3=8)**

(a) Calculate the energy of this sphere i.e. the work done in assembling it.

(b) A smaller sphere of radius $a/2$ is removed from the large sphere, as shown in the figure. Determine the electric field at the origin and at point A.

(a)





Consider a shell of radius r & thickness dr.

Charge of shell $dQ = 4\pi r^2 dr \rho$

(q) Amount of charge in a sphere of radius $r = \frac{4}{3}\pi r^3 \rho$

Work done $\equiv dW = \frac{1}{4\pi\epsilon_0}\frac{q\,dQ}{r} = \frac{(4\pi\rho)^2}{3}\frac{r^4 dr}{4\pi\epsilon_0}$ —

$W = \frac{1}{4\pi\epsilon_0}\int_0^a \frac{(4\pi\rho)^2}{3} r^4 dr = \frac{1}{4\pi\epsilon_0}\frac{(4\pi\rho)^2}{3}\frac{a^5}{5}$ —

If Q is total charge, then $\boxed{W = \frac{3}{5}\frac{Q^2}{a}\left(\frac{1}{4\pi\epsilon_0}\right)}$

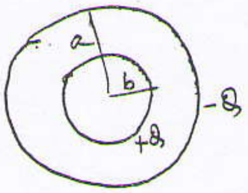
(b) At pt. O field due to $Q_1 = 0$

$\vec{E_0} = \frac{Q_2}{(a/2)^2}\left(\frac{1}{4\pi\epsilon_0}\right)(-\hat{j}) = \boxed{\frac{\rho a}{6\epsilon_0}\hat{j}}$ $Q = \frac{4}{3}\pi\left(\frac{a}{2}\right)^3(-\rho)$

For pt. A $\vec{E_A} = \left[\frac{Q_1}{a^2}(-\hat{j}) + \frac{Q_2}{(3a/2)^2}(-\hat{j})\right]\frac{1}{4\pi\epsilon_0}$ —

$= \frac{1}{4\pi\epsilon_0}\left[\frac{\frac{4}{3}\pi a^3\rho}{a^2} + \frac{-\pi a^3\rho}{6}\frac{4}{9a^2}\right](-\hat{j}) = \boxed{\frac{-\cancel{34}\;17}{54\cancel{27}}\pi a\rho\hat{j}}$

6. We want to design a spherical vacuum capacitor with a given radius a for the outer sphere. The capacitor should be able to store the greatest amount of electrical energy subject to the condition that the electrical field strength at the inner sphere is maintained at E_0. What radius b should be chosen for the inner conductor, and how much energy is stored in such a capacitor? **(6+2=8)**



$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$$

$$\phi = -\int_a^b \frac{Q}{4\pi\epsilon_0 r^2}dr = Q\left(\frac{a-b}{ab}\right)\frac{1}{4\pi\epsilon_0}$$

$$\Rightarrow \quad C = 4\pi\epsilon_0\left(\frac{ab}{a-b}\right)$$

Energy $U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\left(\frac{a-b}{ab}\right)\frac{Q^2}{4\pi\epsilon_0}$

Energy depends on charge on the capacitor. For a given 'b', max^m field near the inner sphere puts a limit on the allowed charge.

$$E_0 = \frac{Q_{max}}{b^2}\frac{1}{4\pi\epsilon_0}$$

$$\Rightarrow \quad U = \frac{1}{2}\left(\frac{a-b}{ab}\right)E_0^2 b^4 4\pi\epsilon_0$$

For maximum U as a f^n of b

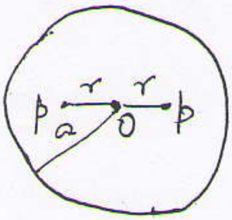
$$\Rightarrow \frac{\partial U}{\partial b} = 0 \Rightarrow \boxed{b_{max} = \frac{3}{4}a}$$

$$U_{max} = \frac{1}{2}\frac{\left(a - \frac{3}{4}a\right)}{a\left(\frac{3}{4}a\right)}E_0^2\left(\frac{3}{4}a\right)^4 = \boxed{\frac{27}{512}E_0^2 a^3 \times 4\pi\epsilon_0}$$

$$= \frac{27\, E_0^2 a^3 \epsilon_0}{128}$$

7. Imagine a sphere of radius a filled with a negative charge of uniform density, the total charge being equivalent to that of two electrons. Imbed two protons in this jelly of negative charge. Assuming that the negative charge distribution remains uniform, where should the protons be located so that the force on each one of them is zero?

(4)



* Forces on protons due to each other equal & opposite
* Forces from −ve charges must also be equal & opposite & hence they should be equidistant from the center and lie on a line through the center.

Force on proton at a distance r from center is the force exerted by a sphere of −ve charge of radius r.

Charge in sphere of radius $r = -\frac{r^3}{a^3} 2e$

Net force on the proton must be zero

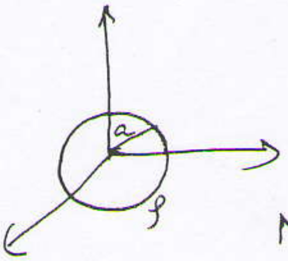
$$\Rightarrow \frac{1}{4\pi\epsilon_0}\left[\frac{e^2}{(2r)^2} + \frac{e}{r^2}\left(-2e\frac{r^3}{a^3}\right)\right] = 0$$

$$\Rightarrow \frac{1}{4r^2} = \frac{2r}{a^3}$$

$$\Rightarrow \boxed{r = a/2}$$

Protons must be located at a distance of a/2 from the center.

8. Linear charge density on a ring of radius a in the xy plane with its center at the origin is given by $\lambda(\theta)=\frac{q}{a}(\cos\theta-\sin 2\theta)$, where θ is the polar angle. Find the monopole moment Q and the dipole moment $\vec{p}$ of the system and calculate the potential at a point far away from the ring. **(2+4+1=7)**



$$\rho = \frac{q}{a}\left(\cos\theta - \sin 2\theta\right)$$

Monopole : $Q = \int_0^{2\pi} \cancel{a}\, d\theta\, \frac{q}{\cancel{a}}(\cos\theta - \sin 2\theta)$

i e $Q = q\int_0^{2\pi} d\theta\,(\cos\theta - 2\sin\theta\cos\theta) = 0$

Dipole $\vec{p} = \int_0^{2\pi} a\,d\theta\,(a\hat{r})\,\frac{q}{a}(\cos\theta - \sin 2\theta)$

$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\therefore p_x = qa\int_0^{2\pi}(\cos^2\theta - \sin 2\theta\cos\theta)\,d\theta$$

$$= \pi q a$$

$$p_y = qa\int_0^{2\pi} d\theta\,(\cos\theta\sin\theta - 2\sin^2\theta\cos\theta) = 0$$

$$\therefore \vec{p} = \pi q a\,\hat{i}$$

$$\phi = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\pi q a\cos\theta\sin\phi}{r^2},$$

9. A dielectric sphere is kept in an otherwise uniform electric field $\vec{E} = E_0\hat{z}$. Relative permittivity of the material is ε_r. The field gives rise to a uniform polarization $\vec{P}$ along the field direction.

(2+4+2+2=10)

(a) What is the resultant electric field inside the sphere?

Find in terms of E_0 and ε_r the following:

(b) polarization $\vec{P}$ in the sphere;

(c) bound volume and surface charge densities in the material; and

(d) the electric field outside the sphere that is produced by the induced polarization.

(a) $$\vec{E}_{Inside} = E_0\hat{z} - \frac{\vec{P}}{3\epsilon_0}$$

(b) $$\vec{P} = \epsilon_0\chi\left(E_0\hat{z} - \frac{\vec{P}}{3\epsilon_0}\right)$$

$$\vec{P} + \frac{\chi\vec{P}}{3} = E_0\hat{z}\,\epsilon_0\chi$$

$$\vec{P} = \frac{3\epsilon_0\chi E_0\hat{z}}{3+\chi} = \boxed{3\epsilon_0\left(\frac{\epsilon_r-1}{\epsilon_r+2}\right)E_0\hat{z}}$$

(c) $$\vec{\nabla}\cdot\vec{P} = 0 \quad \text{—(1)}$$

Volume bound charge

$$\vec{P}\cdot\hat{n} = \underbrace{3\epsilon_0\left(\frac{\epsilon_r-1}{\epsilon_r+2}\right)}_{P_0}\cos\theta$$

(d) Field outside is that of a dipole

$$\vec{E}_{out} = \frac{\cancel{3}\cancel{\epsilon_0}E_0}{\cancel{3}\cancel{\epsilon_0}}\left(\frac{\epsilon_r-1}{\epsilon_r+2}\right)\frac{R^3}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$= \boxed{E_0\left(\frac{\epsilon_r-1}{\epsilon_r+2}\right)\frac{R^3}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)}$$

10. A layer of uniform surface charge density σ is placed in the entire xy plane near an unknown charge distribution. Consider points P(0,0,ε) and P'(0,0,-ε) where ε is an infinitesimally small. If the electric field at the point P is $E_1\hat{\imath} + E_2\hat{\jmath} + E_3\hat{k}$, find its value at P'.

(3)

E_1 & E_2 are tangential components & hence continuous

E_3 & E_3' are normal components

$$E_3 - E_3' = \sigma/\epsilon_0 \Rightarrow E_3' = E_3 - \frac{\sigma}{\epsilon_0}$$

$$\Rightarrow \vec{E}' = E_1\hat{\imath} + E_2\hat{\jmath} + \left(E_3 - \frac{\sigma}{\epsilon_0}\right)\hat{k}$$