Orthogonality of Legendre polynomials

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2}{2m+1} & \text{if } m = n \end{cases}$$

Legendre polynomial as solution of DE:-

$$y = P_n(x).$$

where $(1-x)^2 y'' - 2xy' + m(m+1)y = 0$

$$\Rightarrow (1-x^2) P_m''(x) - 2x P_m'(x) + m(m+1) P_m(x) = 0.$$

$$\frac{d}{dx}\left[(1-x)^2 P_m'(x)\right] + m(m+1) P_m(x) = 0 \quad \text{— (i)}$$

Similarly;

$$\frac{d}{dx}\left[(1-x^2) P_n'(x)\right] + n(n+1) P_n(x) = 0. \quad \text{— (ii)}$$

Multiplying (i) by $P_n(x)$ & (ii) by $P_m(x)$ and subtracting;

$$P_n(x) \frac{d}{dx}\left[(1-x^2) P_m'(x)\right] + m(m+1) P_m(x) P_n(x)$$

$$- P_m(x) \frac{d}{dx}\left[(1-x^2) P_n'(x)\right] - n(n+1) P_m(x) P_n(x) = 0$$

$$\frac{d}{dx}\left[(1-x^2)\left[P_n(x) P_m'(x) - P_m(x) P_n'(x)\right]\right] + \left[m(m+1) - n(n+1)\right] P_m(x) P_n(x) = 0.$$

Integrating from -1 to 1;

$$\int_{-1}^{1} \frac{d}{dx}\left[(1-x^2)\left(P_n(x)P_m'(x) - P_m(x)P_n'(x)\right)\right]dx$$

$$+ \left[m(m+1) - n(n+1)\right]\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0$$

$$\Rightarrow \underbrace{(1-x^2)\left(P_n(x)P_m'(x) - P_m(x)P_n'(x)\right)\Big|_{-1}^{1}}_{0}$$

$$+ \left(m(m+1) - n(n+1)\right)\int_{-1}^{1} P_m(x)\cdot P_n(x)\,dx = 0$$

$$\Rightarrow \left(m(m+1) - n(n+1)\right)\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$$

When $m \neq n$

then;

$$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0.$$

When $m = n$

we can't say by this method.

for $m=n$,

Let $I = \int_{-1}^{1} f(x) P_n(x)\, dx$

$$= \frac{1}{2^n n!} \int_{-1}^{1} f(x) \frac{d^n}{dx^n} (x^2-1)^n$$

$$= \frac{1}{2^n n!} \left[\left. f(x) \frac{d^{n-1}}{dx^{n-1}} \cancelto{0}{(x^2-1)^n} \right|_{-1}^{1} - \int_{-1}^{1} f'(x) \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n dx \right]$$

since one x^2-1 leave & get cut.

$\vdots$

$$= \frac{1}{2^n n!} (-1)^n \int_{-1}^{1} f^{(n)}(x) (x^2-1)^n dx.$$

Scanned by CamScanner

we put $f(x) = P_m(x)$.

then;

$$\int_{-1}^{1} P_m(x)\cdot P_n(x)\,dx = \frac{(-1)^n}{2^n\cdot n!}\int_{-1}^{1} P_m^{(n)}(x)\,(x^2-1)^n\,dx.$$

$= 0$ if $m \neq n$. (proved earlier)

$$= \frac{(-1)^n\cdot 2n!}{(2^n\cdot n!)^2}\int_{-1}^{1}(x^2-1)^n\,dx \quad \text{if } m=n$$

$$= \frac{2n!\times(-1)^n}{(2^n\cdot n!)^2}\,2\int_{0}^{1}(x^2-1)^n\,dx$$

$$= \frac{2\cdot(2n)!\times(-1)^n}{(2^n\cdot n!)^2}\,I_n.$$

where $I_n = \int_0^1 (x^2-1)^n\,dx$

Now;

$$I_n = \int_0^1 (x^2-1)^n\,dx$$

$$= (x^2-1)^n x\Big|_0^1 - \int_0^1 x\cdot n\,(x^2-1)^{n-1}\cdot 2x$$

$$= -2n\int_0^1 x^2(x^2-1)^{n-1}$$

$$= -2n\left[\int_0^1 (x^2-1)^n\,dx + \int_0^1 (x^2-1)^{n-1}\,dx\right]$$

$$= -2n\left[I_n + I_{n-1}\right]$$

So;

$$I_n = \frac{-2n}{2n+1} I_{n-1}.$$

So;

$$I_n = (-1)^n \frac{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)}{3\cdot 5\cdot 7\cdots (2n+1)}$$

$$= (-1)^n \frac{\left[2\cdot 4\cdot 6\cdots (2n)\right]^2}{(2n+1)!}$$

$$= (-1)^n \frac{2^{2n}(n!)^2}{(2n+1)!}$$

So;

$$\int_{-1}^{1} P_m(x)\cdot P_n(x)\, dx = \frac{2\cdot (2n)!\cdot (-1)^n}{\cancel{(2^n\cdot n!)^2}} \times \frac{(-1)^n \cancel{2^{2n}\cdot (n!)^2}}{(2n+1)!}$$

$$= \frac{2}{2n+1}. \quad \text{when } m=n$$

$$= 0 \quad \text{When } m\neq n$$

So. 1