Assignment 13 - Solutions

1. Use spherical coordinates. Let $x = \rho \cos\theta \sin\phi$, $y = \rho \sin\theta \sin\phi$ and $z = \rho\cos\phi$, where $0 \le \rho \le 1$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$.

$$\iiint\limits_W \frac{dzdydx}{\sqrt{1+x^2+y^2+z^2}} = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} \frac{\rho^2 \sin\phi d\rho d\theta d\phi}{\sqrt{1+\rho^2}} = 2\pi(\sqrt{2} - \ln(1+\sqrt{2})).$$

2. In the cylinder there are three surfaces S_1, S_2 and S_3 where

   (a) S_1 : The base of the cylinder, i.e., $z = 0$,

   (b) S_2 : The top of the cylinder i.e., $z = h$,

   (c) S_3 : The curved surface of the cylinder.

   (a) On S_1, the integral is zero.

   (b) The surface integral over $S_2 = \iint\limits_{S_2} x^2 z d\sigma = \int_0^{a}\int_0^{2\pi} (r\cos\theta)^2 hrd\theta dr = \dfrac{ha^4\pi}{4}$.

   (c) A parametric representation of S_3 is

$$r(u,v) = (a\cos u, a\sin u, v), 0 \le u \le 2\pi, 0 \le v \le h.$$

   The surface integral over $S_3 = \iint\limits_{S_3} x^2 z d\sigma = \int_0^{h}\int_0^{2\pi} x^2 z \parallel r_u \times r_v \parallel dudv$

   $= \int_0^{h}\int_0^{2\pi} (a\cos u)^2 v\sqrt{EG-F^2}dudv$, where $E = r_u \cdot r_u$, $G = r_v \cdot r_v$ and $F = r_u \cdot r_v$.

   Note that $\sqrt{EG-F^2} = a$. Therefore, $\iint\limits_{S_3} x^2 z d\sigma = \frac{a^3h^2\pi}{2}$.

   Hence, the required integral is $\frac{ha^4\pi}{4} + \frac{a^3h^2\pi}{2}$.

   Over the entire volume, the integral is

$$V = \int_0^{h}\int_0^{2\pi}\int_0^{a} (r\cos\theta)^2 zrdrd\theta dz = \frac{h^2\pi a^4}{8}.$$

3. $\int\limits_C (y, -x, 1)\cdot dR = \int_0^{2\pi} ((\sin t)(-\sin t)dt - \cos t\cos t + \frac{1}{2\pi})dt.$

4. Take $C = R(t) = (cost, sint)$, $0 \le t \le 2\pi$. Then

$$\int\limits_C T\cdot dR = \int_0^{2\pi} T(t)\cdot R'(t)dt = \int_0^{2\pi} \frac{R'(t)}{\parallel R'(t) \parallel}\cdot R'(t)dt = 2\pi$$

5. If $F = yzi + (xz+1)j + xyk$, then $F = \nabla\varphi$, where $\varphi(x,y,z) = xyz + y$. Hence, by the 2nd fundamental theorem of calculus for line integrals, the problem follows.

Assignment 14 - Solutions

1. $M = 2x^2 - y^2$ and $N = x^2 + y^2$. By Green's Theorem

$$\int_C (2x^2 - y^2)dx + (x^2 + y^2)dy = \int_0^1 \int_0^{\sqrt{1-x^2}} (N_x - M_y)dy\,dx$$
$$= \int_0^1 \int_0^{\sqrt{1-x^2}} 2(x+y)dy\,dx = \tfrac{4}{3}.$$

2. Let $F = -y^3\vec{i} + x^3\vec{j} - z^3\vec{k}$. By Stoke's Theorem, $\int_{\partial S} F.dr = \int\int_S (curl\,F).\vec{n}d\sigma$.

   Note that $\nabla \times F = 3(x^2 + y^2)\vec{k}$. Hence, $\int_{\partial S} F.dr = \iint_D 3(x^2 + y^2)dxdy = \frac{3\pi}{2}$.

3. Note that $div\ F = 0$. By divergence theorem

$$\iint_S F \cdot n d\sigma = \iint_{S_\rho} F \cdot n d\sigma$$

   where S_ρ is a sphere of (small) radius ρ with center at origin. On S_ρ, $n = \frac{1}{\rho}(xi+yj+zk)$ and hence $F \cdot n = \frac{1}{\rho^2}$. Therefore,

$$\iint_{S_\rho} F \cdot n d\sigma = \frac{1}{\rho^2}\iint_{S_\rho} d\sigma = \frac{1}{\rho^2}4\pi\rho^2 = 4\pi.$$

4. $div\,F = 2x + 2y + 2z$. By the divergence theorem,

$$\int\int_{\partial D} F.\vec{n}d\sigma = \int\int\int_D 2(x+y+z)dV = 2\int\int_{x^2+y^2\leq 1} (\int_0^{x+2} (x+y+z)dz)dxdy = \frac{19\pi}{4}$$