Grade Table (for checker use only)

Question	Points	Score
1	4	
2	6	
3	7	
4	8	
5	8	
6	10	
7	10	
8	12	
9	12	
Total:	77	

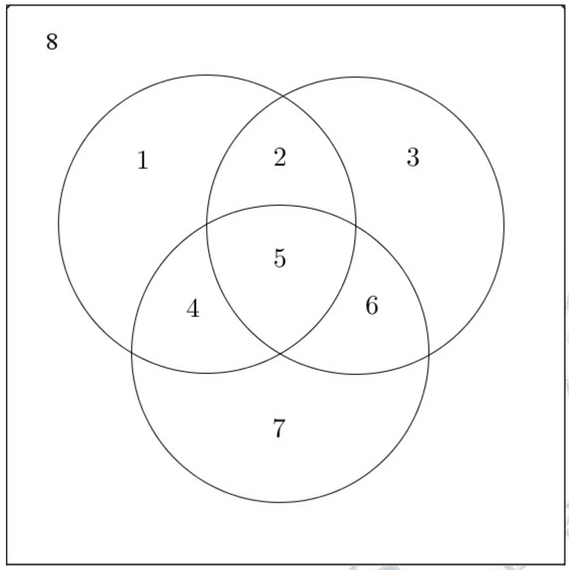
Team Members:

- ....
- ....
- ....

INSTRUCTIONS:

- **Write your team name on top of each page.**
- If you have any queries, contact an invigilator. Any sort of interaction with another team can lead to a penalty or disqualification.
- Submit any electronic devices that you possess, to one of the invigilators. You may collect them after the event. Any team caught using any electronic device will be immediately disqualified.
- Enough space has been provided in the question paper. Use it wisely. However, if you need extra sheets, contact an invigilator.

1. (4 points) A "Venn diagram" with three overlapping circles is often used to illustrate the eight possible subsets associated with three given sets:



Can the 16 possibilities that arise with four given sets be illustrated by four overlapping circles? Explain.

2. (6 points) Positive integers are written on all the surfaces of a cube, one on each face. At each corner of the cube, the product of the numbers on the faces that meet in the corner are written. It is given that the sum of all numbers written on the corners is 7785. Find all possible values of S, where S denotes sum of numbers on all faces.

3. (7 points) [5] How many pieces of cheese can you obtain from a single thick piece by making five straight slices?

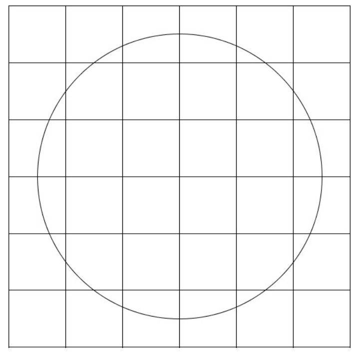
   [2] What about making 6 straight slices?

   Note: The cheese must stay in its original position while you do all the cutting, and each slice must correspond to a plane in three dimensional space.

4. (8 points) Find the largest number obtainable as the product of positive integers whose sum is 2015.

5. (8 points) Let m be a natural number with digits consisting entirely of $6's$ and $0's$. Prove that m can not be the square of a natural number.

6. (10 points) A circle, $2n-1$ units in diameter, has been drawn symmetrically on a $2n \times 2n$ grid, illustrated here for $n = 3$.



[4] (a) How many cells of the grid contain a segment of a circle?

[6] (b) Find a function $f(n, k)$ such that exactly $\sum_{k=1}^{n-1} f(n, k)$ cells of the grid lie entirely within the circle?

7. (10 points) Points D and E are chosen on the sides AB and AC of the triangle ABC in such a way that if F is the intersection point of BE and CD, then $AE+EF = AD+DF$. Prove that $AC + CF = AB + BF$.

8. (12 points) Let m be a positive integer and let

$$\omega = e^{\frac{2i\pi}{m}} = \cos\left(\frac{2\pi}{m}\right) + i\sin\left(\frac{2\pi}{m}\right)$$

be an m^{th} root of unity. Therefore, $z - \omega^k$ is a factor of the polynomial $z^m - 1$, for $0 \leq k < m$. Since these factors are distinct, the complete factorisation of $z^m - 1$ over the complex numbers must be $z^m - 1 = \prod_{0 \leq k < m}(z - \omega^k)$.

Let

$$\psi(z) = \prod_{0 \leq k < m, gcd(k,m)=1} (z - \omega^k)$$

Prove that

$$z^m - 1 = \prod_{d|m} \psi_d(z)$$

9. (12 points) Find all positive integers n such that

$$n \text{ divides } \left\lfloor \frac{(n-1)!}{n+1} \right\rfloor$$

<u>*Note:*</u> $\lfloor x \rfloor = n \implies n$ is the largest integer $\leq x$.