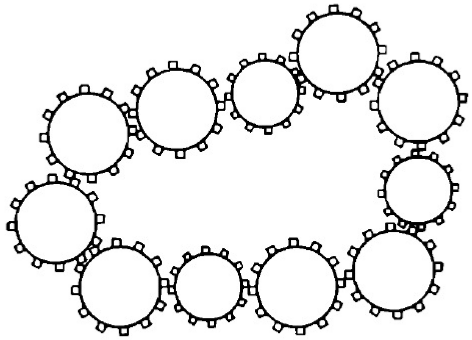
1. (6 points) Eleven gears are placed on a plane, arranged in a chain, as shown below. Can all the gears rotate simultaneously? Explain your answer.



(4 points) What if we have a chain of 572 gears?

Solution: The answer is no. Suppose that the first gear rotates clockwise. Then the second gear must rotate counter-clockwise, the third clockwise again, the fourth counter-clockwise, and so on. It is clear that the "odd" gears must rotate clockwise, while the "even" gears must rotate counter-clockwise. But then the first and eleventh gears must rotate in the same direction. This is a contradiction. The main idea in the solution to this problem is that the gears rotating clockwise and counter-clockwise alternate.

Similarly, for 572 gears, the rotation is possible.

2. (6 points) Prove that if a number has an odd number of divisors, then it is a perfect square.

Solution: If a number n is not a perfect square, its divisors can be paired up (all such pairs will have a number $> \lfloor\sqrt{n}\rfloor$ and a number $< \lfloor\sqrt{n}\rfloor$), thus making the number of divisors even. However, in case of odd number of divisors, we have an extra divisor, which is possible only if we have a square number.

3. (8 points) If p, $4p^2+1$, and $6p^2+1$ are prime numbers, find p.

Solution: The answer is $p = 5$. Analyze the remainders upon division by 5.

4. (10 points) How many ways are there to place: a) two bishops, b) two knights, c) two queens on a chessboard so that they do not attack each other?

Solution:

The answers are
a) $(28 \cdot 56 + 20 \cdot 54 + 12\ \ 52 + 4 \cdot 50)/2 = 1736$;
b) $(4\ \ 61 + 8\ \ 60 + 20 \cdot 59 + 16 \cdot 57 + 16 \cdot 55)/2 = 1848$;
c) $(28 \cdot 42 + 20 \cdot 40 + 12 \cdot 38 + 4\ \ 36)/2 = 1288$.

To demonstrate the method. we will prove part a). There are 28 squares on the border of the chessboard. and from one of these the first bishop attacks 8 squares (including the one it stands on). Therefore, there are 56 squares left for the second bishop. Further, there are 20 squares adjacent to the border squares. When positioned on these squares. the first bishop attacks 10 squares, so there are 54 squares on which to place the second bishop. Analogously, there are 12 squares from which the first bishop attacks 12 squares, and, finally, 4 central squares (standing on these, the first bishop attacks 14 squares). After adding up all the variants, we must divide the sum by two, since we counted each arrangement exactly twice (we do not distinguish the bishops).

5. (12 points) Prove that all the numbers in the series

$$10001, 100010001, 1000100010001, \ldots$$

are composite.

Solution:

First, $10001 = 73 \cdot 137$, which is not obvious but nevertheless true. Second, to prove that any other number $10001\ldots10001$ of the series is composite, we multiply it by 1111. The result has $4k$ digits ($k > 2$) and, therefore, is divisible by $x = 1000\ldots001 = 10^{2k} + 1$ (indeed,

$$\underbrace{111\ldots11}_{4k\text{ digits}} = \underbrace{1000\ldots001}_{2k+1\text{ digits}}\ \underbrace{111\ldots11}_{2k\text{ digits}})$$

Finally, we use the fact that x is greater than 1111 and less than the original number. Therefore, the original number must be divisible by $x/\gcd(x, 1111) > 1$.

6. (10 points) How can you get a fair coin toss if someone hands you a coin that is weighted to come up heads more often than tails?

Solution: If a cheat has altered a coin to prefer one side over another (a biased coin), the coin can still be used for fair results by changing the game slightly. John von Neumann gave the following procedure:

1. Toss the coin twice.
2. If the results match, start over, forgetting both results.
3. If the results differ, use the first result, forgetting the second.

The reason this process produces a fair result is that the probability of getting heads and then tails must be the same as the probability of getting tails and then heads, as the coin is not changing its bias between flips and the two flips are independent. By excluding the events of two heads and two tails by repeating the procedure, the coin flipper is left with the only two remaining outcomes having equivalent probability. This procedure only works if the tosses are paired properly; if part of a pair is reused in another pair, the fairness may be ruined.

Mathematically we can write:

Lets define an event, E as tossing the biased coin twice. The possible outcomes with probabilities is as follows

$$P(h,h) = x^2$$
$$P(h,t) = x(1-x)$$
$$P(t,h) = x(1-x)$$
$$P(t,t) = (1-x)^2$$

The event h,t or t,h are equi-likely, without any bias we can call that. If Event h,t occurs it means head, t,h means tails but if h,h or t,t occurs we repeat the experiment.

7. (12 points) Given any 9 integers show that it is possible to choose, from among them, 4 integers a, b, c, d such that $a + b - c - d$ is divisible by 20.

Solution:

Suppose there are four numbers a, b, c, d among the given nine numbers which leave the same remainder modulo 20. Then $a + b \equiv c + d \pmod{20}$ and we are done.

If not, there are two possibilities:

(1) We may have two disjoint pairs $\{a, c\}$ and $\{b, d\}$ obtained from the given nine numbers such that $a \equiv c \pmod{20}$ and $b \equiv d \pmod{20}$. In this case we get $a+b \equiv c+d \pmod{20}$.

(2) Or else there are at most three numbers having the same remainder modulo 20 and the remaining six numbers leave distinct remainders which are also different from the first remainder (i.e., the remainder of the three numbers). Thus there are at least 7 disinct remainders modulo 20 that can be obtained from the given set of nine numbers. These 7 remainders give rise to $\binom{7}{2} = 21$ pairs of numbers. By pigeonhole principle, there must be two pairs $(r_1, r_2), (r_3, r_4)$ such that $r_1 + r_2 \equiv r_3 + r_4 \pmod{20}$. Going back we get four numbers a, b, c, d such that $a + b \equiv c + d \pmod{20}$.

8. (12 points) Let $f(x) = x^3 + ax^2 + bx + c$ and $g(x) = x^3 + bx^2 + cx + a$, where a, b, c are integers with $c \neq 0$. Suppose that the following conditions hold:

(a) $f(1) = 0$ (b) The roots of $g(x)$ are squares of the roots of $f(x)$.

Find the value of $a^{2015} + b^{2015} + c^{2015}$

Solution: Note that $g(1) = f(1) = 0$, so 1 is a root of both $f(x)$ and $g(x)$. Let p and q be the other two roots of $f(x)$, so p^2 and q^2 are the other two roots of $g(x)$. We then get $pq = c$ and $p^2q^2 = a$, so $a = c^2$. Also, $(a)^2 = (p+q+1)^2 = p^2+q^2+1+2(pq+p+q) = b+2b = b$. Therefore $b = c^4$. Since $f(1) = 0$ we therefore get $1 + cc^2 + c^4 = 0$. Factorising, we get $(c+1)(c^3c^2+1) = 0$. Note that $c^3c^2+1 = 0$ has no integer root and hence $c = 1, b = 1, a = 1$. Therefore $a^{2015} + b^{2015} + c^{2015} = 1$

9. (12 points) Let ABC be a triangle with $A = 90^o$ and $AB = AC$. Let D and E be points on the segment BC such that $BD : DE : EC = 3 : 5 : 4$. Find $\angle DAE$.

Solution. Rotating the configuraiton about A by $90°$, the point B goes to the point C. Let P denote the image of the point D under this rotation. Then $CP = BD$ and $\angle ACP = \angle ABC = 45°$, so ECP is a right-angled triangle with $CE : CP = 4 : 3$. Hence $PE = ED$. It follows that $ADEP$ is a kite with $AP = AD$ and $PE = ED$. Therefore AE is the angular bisector of $\angle PAD$. This implies that $\angle DAE = \angle PAD/2 = 45°$. □

10. (12 points) 100 minions are lined up in a row by an assassin. The assassin puts red and blue hats on them. They can't see their own hats, but they can see the hats of the minions in front of them. The assassin starts in the back and says "What color is your hat?" the minion can only answer "red" or "blue." The minion is killed if he gives the wrong answer; then the assassin moves on to the next minion. The minions in front get to hear the answers of the minions behind them, but not whether they live or die. They can consult and agree on a strategy before being lined up, but after being lined up and having the hats put on, they can't communicate in any way other than those already specified. What strategy should they choose to maximize the number of minions who are guaranteed to be saved?

Solution: This is a difficult problem to solve during an interview (especially if you've already taxed the candidate's brain). Look for obvious solutions first, and the reasoning behind them and then try to lead them to the ultimate solution.

A logical answer could be all the minions would just say "red" and that way about half of them would survive on average, assuming the hats were distributed randomly.

This is a good start and should naturally lead to having every other minion say the color of the hat in front of them. The first minion would say the color of the hat in front of him, then the next minion would just say that color that was just said. So we can guarantee that half survive – the even numbered minions (since the person behind them told them the answer). And potentially if the hats were distributed randomly some of the minions would get lucky and the hat in front of them would be the same color as their own. So this strategy should save more than half, and on average 75% of them would live.

At this point, if the solution is not clear, the candidate may give answers like, "they could agree that if they said their hat color in a soft voice, it means the hat in front of them is the same color, and if they say it in a loud voice, it means the hat in front is a different color". This is definitely good and on the correct track. another option is they could say "reeeeeeeeeeed" for x number of seconds, where x represented the distribution of hats where a hat was a bit in a binary number, (red = 1, blue = 0). another interesting answer. there are many others like these that "bend" the rules and come to a solution.

But the real solution acknowledges that the minions can only say "red" or "blue" and cannot alter their voice in such a convincing way as to signal any information other than the word they said. A good way to get this point across, is simply to change the problem slightly by saying "the assassin gets to hear their plan before she puts the hats on, and so will try to thwart the plan however she can."

So if they decide to all say "red", she'll put blue hats on all of them. If they decide to all say the color of the hat in front of them, she'll alternate the hats on every head, guaranteeing half will die. Even with the assassin hearing their plan, there is still a way to save almost everyone.

We know that the first person is never going to have any information about the color of their hat, so they cannot be guaranteed to survive. But, i'll give you a hint to the solution: I can save every other person for sure.

Solution: they agree that if the number of red hats that the back person can see is even, that minion will say "red". if they add up to an odd number, they will say "blue". This way number 99 can look ahead and count the red hats. If they add up to an even number and number 100 said "red", then 99 must be wearing a blue hat. If they add up to an even number and number 100 said "blue", signalling an odd number of red hats, number 99 must also be wearing a red hat. Number 98 knows that 99 said the correct hat, and so uses that information along with the 97 hats in front to figure out what color hat is on 98's head.