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COMMUNICATION THEORY

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(1) $x(t) = e^{-at} u(t)$

bandwidth required to transmit 95% of $x(t)$.

$$\Rightarrow X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$\Rightarrow X(f) = \int_{-\infty}^{\infty} e^{-at} e^{-j2\pi f t} u(t)\, dt$$

$$= \int_{0}^{\infty} e^{-(a+2\pi f j)t}\, dt$$

$$= \boxed{\frac{1}{(a+2\pi f j)}}$$

$$\text{Energy of } x(t) = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2a}$$

$$\therefore \int_{-W}^{W} \left| \frac{1}{(a+(2\pi f j))} \right|^2 df = \frac{0.95}{2a}$$

$$\Rightarrow \int_{-W}^{W} \frac{1}{\left(\frac{a^2}{4\pi^2} + \frac{(2\pi f)^2}{4\pi^2}\right) 4\pi^2} \, df = \frac{19}{40a}$$

$$\Rightarrow \frac{1}{4\pi^2} \int_{-W}^{W} \frac{1}{\left(\frac{a}{2\pi}\right)^2 + f^2} \, df = \frac{19}{40a}$$

$$\Rightarrow \frac{1}{\cancel{4}\pi^2} \left[\frac{1}{\left(\frac{a}{2\pi}\right)} \tan^{-1}\left(\frac{f}{a} \times 2\pi\right) \right]_{-W}^{W} = \frac{19}{\cancel{40}a}\ {}_{10}$$

$$\Rightarrow \frac{2\pi}{a} \cdot \frac{1}{\pi^2} \cdot \left(\tan^{-1}\left(\frac{2\pi W}{a}\right) - - \tan^{-1}\left(\frac{2\pi W}{a}\right) \right)$$

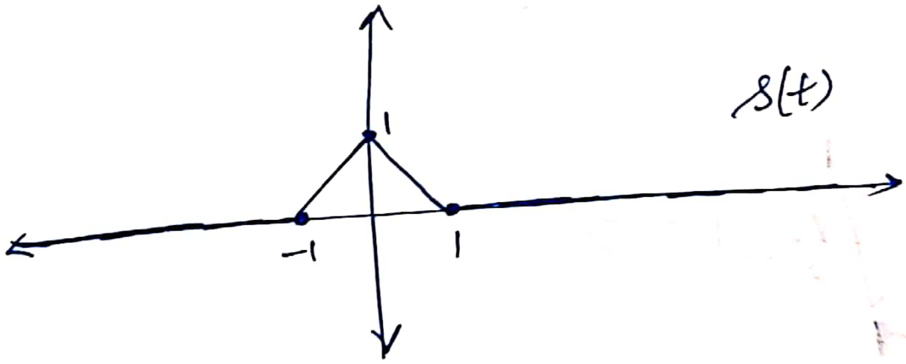
$$= \frac{19}{10a}$$

$$\Rightarrow \frac{2}{\pi \cancel{a}} \cdot 2\tan^{-1}\left(\frac{2\pi W}{a}\right) = \frac{19\pi}{40\cancel{a}}$$

$$\Rightarrow \tan^{-1}\left(\frac{2\pi W}{a}\right) = \frac{19\pi}{40}$$

$$\Rightarrow \boxed{W = \frac{a}{2\pi} \tan\left(\frac{19\pi}{40}\right)} \approx \boxed{2.022a}$$

(2) $s(t) = (1-|t|)\, I_{[-1,1]}(t)$



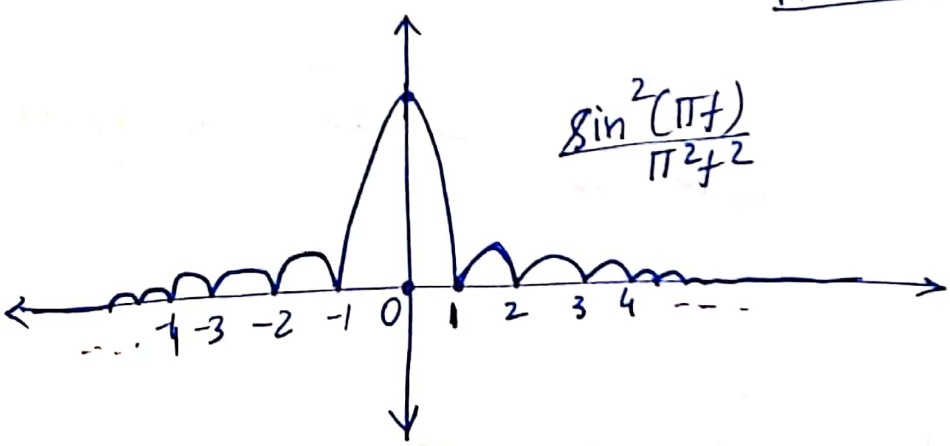
$s(t)$ is $= \left(I_{[-1/2,\,1/2]} * I_{[-1/2,\,1/2]}\right)(t)$

$\Rightarrow \mathcal{F}\{s(t)\} = \mathcal{F}\{I_{[-1/2,1/2]}(t)\} \times \mathcal{F}\{I_{[-1/2,1/2]}(t)\}$

$\mathcal{F}\{I_{[-1/2,\,1/2]}(t)\} = \mathrm{sinc}\left(\frac{2\pi f}{2}\right)$

$= \boxed{\frac{\sin(\pi f)}{\pi f}}$

$\Rightarrow \mathcal{F}\{s(t)\} = S(f) = \boxed{\frac{\sin^2(\pi f)}{\pi^2 f^2}}$



(b) Energy of $s(t)$ = $\int_{-\infty}^{\infty} |s^2(t)|\, dt$

$$= \int_{-1}^{1} (1-|t|)^2\, dt$$

$$= \int_{-1}^{1} (1+t^2-2|t|)\, dt$$

$$= \cancel{(1--1)}^{2} + \frac{1}{3}\cancel{(1--1)}^{2}$$

$$-2\left(2 \times \frac{1}{\cancel{2}} \times 1 \times 1\right)$$

$$= \frac{8}{3} - \frac{2\times 3}{3}$$

$$= \frac{2}{3}$$

$\Rightarrow$ 99% of $\frac{2}{3}$ = $\frac{\cancel{99}^{33}}{_{50}\cancel{100}} \times \frac{\cancel{2}}{\cancel{3}}$ = $\boxed{\frac{33}{50}}$ = $\boxed{0.66}$

$$\Rightarrow \int_{-W}^{W} \left|\frac{\sin^2(\pi f)}{\pi^2 f^2}\right|^2 df = \frac{33}{50}$$

and solving for W numerically,

we get $\boxed{W = 0.664 \text{ kHz}}$

③ x, y periodic with T_0, x_n, y_n are coeff.

ⓐ

T.P.

$$\frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} x(t)\, y^*(t)\, dt = \sum_{n=-\infty}^{\infty} x_n y_n^*$$

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{-jn\left(\frac{2\pi}{T_0}\right)t}$$

$$y(t) = \sum_{m=-\infty}^{\infty} y_m e^{-jm\left(\frac{2\pi}{T_0}\right)t}$$

$$y^*(t) = \sum_{m=-\infty}^{\infty} y_m^* e^{jm\frac{2\pi}{T_0}t}$$

$$x(t)\,y^*(t) = \sum_{n=-\infty}^{\infty} x_n e^{-j\frac{2\pi n t}{T_0}} \cdot \sum_{m=-\infty}^{\infty} e^{+j\frac{2\pi m t}{T_0}} y_m^*$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x_n y_m^* \, e^{j\frac{2\pi}{T_0}t(m-n)}$$

$$\int_{\alpha}^{\alpha+T_0} x(t)\,y^*(t)\,dt = \int_{\alpha}^{\alpha+T_0} \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x_n y_m^* \, e^{j\frac{2\pi t}{T_0}(m-n)}\,dt$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x_n y_m^* \left(\int_{\alpha}^{\alpha+T_0} e^{j\frac{2\pi t}{T_0}(m-n)}\,dt \right)$$

Consider $\displaystyle\int_{\alpha}^{\alpha+T_0} e^{j\frac{2\pi t}{T_0}(m-n)}\,dt \quad \frac{j2\pi(m-n)}{T_0} \cdot \frac{T_0}{j2\pi(m-n)}$

$$\frac{T_0}{j2\pi(m-n)}\left[e^{\frac{j2\pi(m-n)(\alpha+T_0)}{T_0}} - e^{\frac{j2\pi(m-n)\alpha}{T_0}} \right]$$

$$= T_0\, e^{j\frac{2\pi(m-n)\alpha}{T_0}} \left[\frac{e^{j2\pi(m-n)\frac{T_0}{T_0}} - 1}{2\pi j(m-n)} \right]$$

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$$= \begin{cases} T_0 & \text{if } m=n \\ 0 & \text{if } m \neq n \end{cases}$$

$$\Rightarrow \boxed{\int_{\alpha}^{\alpha+T_0} e^{j\frac{2\pi}{T_0}t(m-n)} dt = T_0 \, \delta(m,n)}$$

$$\Rightarrow \int_{\alpha}^{\alpha+T_0} x(t)\, y^*(t)\, dt = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x_n y_m^* \, \delta(m,n) T_0$$

$$\Rightarrow \boxed{\frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} x(t) y^*(t)\, dt = \sum_{n=-\infty}^{\infty} x_n y_n^*}$$

If we set $y = x$, then $x_n = y_n$

$$\frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} x(t)\; x^*(t)\, dt = \sum_{n=-\infty}^{\infty} (x_n \cancel{y_n}\, x_n^*)$$

$$\Rightarrow \boxed{\frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |x_n|^2}$$

Thus, Rayleigh's relation is a special case of Pareseval's theorem.

(b)

Given, $x(t)$ has finite power

$\Rightarrow$ $$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} |x(t)|^2 dt$$ exists and is finite

and $x(t \pm nT_0) = x(t)$ $\forall n \in \mathbb{N}, T_0 \in \mathbb{R}$

We know from Rayleigh's Relation that

$$\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |x_n|^2$$

and given LHS is a finite value and exists.

$\Rightarrow$ $\sum_{n=-\infty}^{\infty} |x_n|^2$ is a convergent series

$\Rightarrow$ $|x_n|^2 \to 0$ as $n \to \infty$ [True for all convergent series]

$\Rightarrow$ $x_n \to 0$ as $n \to \infty$

Hence, Proved.

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(c) $\lim_{n\to\infty}\left(1+\frac{1}{3^4}+\frac{1}{5^2}+\dots+\frac{1}{(2n+1)^4}\right)=\frac{\pi^4}{96}$

Prove the above statement.

$f(t)=t^2,\quad t\in[-\pi,\pi]$ $\qquad T_0=2\pi \Rightarrow \omega_0=1$

$\therefore$ let a_n be fourier coefficients.

$$a_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,e^{-jnt}\,dt$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} (jnt)^2\, e^{-jnt}\,\frac{dt\,(-jn)}{(-jn)(-jn)(-jn)}$$

$$=\frac{1}{2\pi(-jn)(-jn)(-jn)}\int_{-\pi}^{\pi}(-jnt)^2\,e^{(-jnt)}\,d(-jnt)$$

$$=\frac{1}{2\pi jn^3}\int_{j\pi n}^{-j\pi n} x^2 e^x\,dx$$

$$=\frac{1}{2\pi jn^3}\left[x^2e^x-2xe^x+2e^x\right]_{j\pi n}^{-j\pi n}$$

$$=\frac{1}{2\pi jn^3}\left[\begin{matrix}-(j\pi n)^2e^{-j\pi n} & +2(+j\pi n)e^{-j\pi n} & +2e^{-j\pi n}\\ +(j\pi n)^2e^{j\pi n} & +2(j\pi n)e^{j\pi n} & -2e^{j\pi n}\end{matrix}\right]$$

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$$\Rightarrow a_n = \frac{1}{2\pi j n^3}\left[(\pi n)^2\left(e^{j\pi n}-e^{-j\pi n}\right) + 2j\pi n\left[e^{-j\pi n}+e^{j\pi n}\right]\right.$$

$$\left. + (2j)2\left[\frac{-e^{-j\pi n}+e^{j\pi n}}{2j}\right]\right]$$

$$\Rightarrow a_n = \frac{1}{2\pi j n^3}\left[(2j\pi n^2)\cancel{\left(\frac{e^{j\pi n}-e^{-j\pi n}}{2j}\right)} \sin(\pi n)\right.$$

$$+ 4\pi j n \cos(\pi n)$$

$$\left. - 4j \sin(\pi n)\right]$$

$$\left[\begin{matrix}\sin(\pi n) = 0 \\ \forall n \in \mathbb{N}\end{matrix}\right]$$

$$\Rightarrow a_n = \frac{1}{2\pi j n^3}\left[0 + 4\pi j n \cos(\pi n) - 0\right]$$

$$\Rightarrow \boxed{a_n = \frac{2\cos(\pi n)}{n^2}}$$

$$\cos(\pi n) =$$

$$\boxed{a_n = \frac{2(-1)^n}{n^2}}$$

<u>By parseval's theorum</u>

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2\, dt$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} \left| \frac{2(-1)^n}{n^2} \right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} t^4 \, dt$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} \frac{4}{n^4} = \frac{1}{2\pi} \left(\frac{t^5}{5} \right)_{-\pi}^{\pi}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{8}{n^4} + |a_0|^2 = \frac{1}{\cancel{2\pi}} \frac{2\pi^{4}}{5}$$

$$a_0 = \frac{1}{T_0} \int_{-\pi}^{\pi} f(t) \, dt = \frac{1}{2\pi} \left[\frac{2\pi^2}{3} \right]$$

$$\Rightarrow |a_0|^2 = \frac{\pi^4}{9}.$$

$$\Rightarrow 2 \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\cancel{4}\pi^4}{45}$$

$$\Rightarrow 2 \left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots + \frac{1}{n^4} \right) = \frac{\pi^4}{45}$$

$$\Rightarrow 2 \left(\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots \right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \dots \right) \right) = \frac{\pi^4}{45}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} + \frac{1}{16} \left(\sum_{n=1}^{\infty} \frac{1}{n^4} \right) \rightarrow \frac{\pi^4}{90} = \frac{\pi^4}{45 \times 2}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{16\pi^4}{16\times 90} - \frac{\pi^4}{16\times 90}$$

$$= \frac{\cancel{15}\pi^4}{16\times \cancel{90}\,6}$$

$$= \frac{\pi^4}{96}$$

$$\Rightarrow \boxed{\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}}$$

(4) Determine FT of the following. **Assumption:** $\text{sinc}(t) = \frac{\sin(t)}{t}$

(a) $\text{sinc}^3(t) = f(t)$

$\therefore \mathcal{F}\{f(t)\} = ?$

$$\text{sinc}(t) \longleftrightarrow \pi \cdot I_{[-1,1]}^{(\omega)}$$

$$\Rightarrow \text{sinc}^3(t) \longleftrightarrow \left(\pi I_{[-1,1]}^{(\omega)}\right) * \left(\pi I_{[-1,1]}^{(\omega)}\right) * \left(\pi I_{[-1,1]}^{(\omega)}\right)$$

$$\Rightarrow \text{sinc}^3(t) \longleftrightarrow \pi^3 \left(\left(1 - \frac{|t|}{2}\right) I_{[-2,2]}^{(\omega)} * \left(I_{[-1,1]}^{(\omega)}\right) \right)$$

$$\Rightarrow \text{sinc}^3(t) \longleftrightarrow \pi^3 \left(\left(\frac{3}{2} - \frac{\omega^2}{6}\right) I_{[-3,3]}^{(\omega)} \right)$$

$$\boxed{\omega = 2\pi f}$$

(b) $t\,\text{sinc}(t) = g(t)$

$\mathcal{F}\{g(t)\} = ?$ let $h(t) = \text{sinc}(t)$

$$\mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} \text{sinc}(t)\, e^{-j2\pi f t}\, dt = \pi I_{[-1,1]}^{(\omega)}$$

$$= \pi I_{[-2\pi, 2\pi]}^{(f)}$$

$$\frac{\partial \mathcal{F}\{h(t)\}}{\partial f} = \int_{-\infty}^{\infty} \text{sinc}(t)\,(-j2\pi t)\, e^{-j2\pi f t}\, dt = \pi\left(\delta(f+2\pi) - \delta(f-2\pi)\right)$$

$$\Rightarrow \frac{\partial \mathcal{F}\{g(t)\}}{\partial f} = -2\pi j \int_{-\infty}^{\infty} t\,\mathrm{sinc}(t)\, e^{-j2\pi f t}\, dt$$

$$= \pi \left(\delta(t+2\pi) - \delta(t-2\pi) \right)$$

$$\Rightarrow \int_{-\infty}^{\infty} (t\,\mathrm{sinc}\,t) e^{-2\pi j f t}\, dt = \frac{\pi j}{-2\pi j} \left(\delta(t+2\pi) - \delta(t-2\pi) \right)$$

$$= \boxed{\frac{j}{2} \left(\delta(t+2\pi) - \delta(t-2\pi) \right)}$$

$$\Rightarrow \mathcal{F}\{ t\,\mathrm{sinc}(t) \} = \frac{j}{2} \left(\delta(t+2\pi) - \delta(t-2\pi) \right)$$

Interesting fact:

I realised $t\,\mathrm{sinc}(t) = \sin(t)$ only after doing it in this complicated way.

(c) $l(t) = t e^{-\alpha t} \cos(\beta t) u(t)$, $\mathcal{F}\{l(t)\} = ?$

$$\Rightarrow \int_{-\infty}^{\infty} t\, e^{-\alpha t} \cos(\beta t)\, u(t)\, e^{-j 2\pi f t}\, dt$$

$$= \frac{1}{-2\pi j} \int_{-\infty}^{\infty} \left(e^{-\alpha t} \cos(\beta t)\right) u(t) (-2\pi j t)\, e^{-j 2\pi f t}\, dt$$

$$= \frac{j}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha t} u(t) \cos(\beta t) \frac{\partial \left(e^{-j 2\pi f t}\right)}{\partial f}\, dt$$

$$= \frac{j}{2\pi} \frac{\partial}{\partial f} \left(\int_{-\infty}^{\infty} e^{-\alpha t} \cos(\beta t)\, e^{-j 2\pi f t} u(t)\, dt \right)$$

$$= \frac{j}{2\pi} \frac{\partial}{\partial f} \left[\mathcal{F}\{ e^{-\alpha t} u(t) \cos(\beta t) \} \right]$$

$$= \frac{j}{2\pi} \frac{\partial}{\partial f} \left[\frac{1}{(2\pi j f + \alpha)} * \left[\frac{\delta\left(f - \frac{\beta}{2\pi}\right) + \delta\left(f + \frac{\beta}{2\pi}\right)}{2} \right] \right]$$

$$= \frac{j}{4\pi} \frac{\partial}{\partial f} \left[\frac{1}{2\pi j f + \alpha - j\beta} + \frac{1}{2\pi j f + \alpha + j\beta} \right]$$

$$\frac{\cancel{j}}{2\,\cancel{4\pi}}\left[\frac{\cancel{2\pi j}}{(2\pi j f+\alpha-j\beta)^2}+\frac{\cancel{2\pi j}}{(2\pi j f+\alpha+j\beta)^2}\right]$$

$$\Rightarrow \frac{1}{2}\left[\frac{1}{(2\pi j f+\alpha-j\beta)^2}+\frac{1}{(2\pi j f+\alpha+j\beta)^2}\right]$$

(5)

(b) $\int_0^\infty e^{-\alpha t}\cos(\beta t)\,dt.$

$$\Rightarrow \int_{-\infty}^{\infty} \left(e^{-\alpha t}u(t)\right)\left(\cos\beta t\right) e^{-j2\pi f t}\,dt \Big|_{f=0}$$

$$\Rightarrow \mathcal{F}\left\{e^{-\alpha t}u(t) \times \cos(\beta t)\right\}(0)$$

$$\Rightarrow \left(\mathcal{F}\{e^{-\alpha t}u(t)\} * \mathcal{F}\{\cos(\beta t)\}\right)(0)$$

$$\Rightarrow \left(\frac{1}{2\pi j f+\alpha} * \frac{\delta(f-\beta/2\pi)+\delta(f+\beta/2\pi)}{2}\right)(0)$$

$$\Rightarrow \left(\frac{1}{2\pi \cancel{jf}+\alpha+j\beta} + \frac{1}{\cancel{2\pi jf}+\alpha-j\beta}\right) \cancel{(f=0)}$$

$$\Rightarrow \frac{\alpha+\alpha+j\beta-j\beta}{(\alpha+j\beta)(\alpha-j\beta)} = \boxed{\frac{2\alpha}{\alpha^2+\beta^2}}$$

(a) $$\int_0^{\infty} e^{-\alpha t}\,\mathrm{sinc}^2(t)\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{-\alpha t}u(t)}{t^2}dt\,e^{-j2\pi(0)t} - \frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{-\alpha t}u(t)\cos(2t)}{t^2}dt\,e^{-j2\pi(0)t}$$

$$= \frac{(2\pi j)^2}{2}\int_{-\infty}^{\infty}\frac{(e^{-\alpha t}u(t))\,\cancel{e}^{-j2\pi(0)t}\,dt}{\cancel{t}^2(-j2\pi)(-j2\pi)} - \frac{(2\pi j)^2}{2}\int_{-\infty}^{\infty}\frac{(e^{-\alpha t}u(t))(\cos(2t))dt\,e^{-j\ldots}}{t^2(-j2\pi)(-j2\pi)}$$

$$\boxed{\beta = 2}$$

$$\Rightarrow \cancel{\frac{-4\pi^2}{2}}\left[\cancel{\frac{1}{(2\pi j)(2\pi)}}(2\pi j f+\alpha)\log_e(2\pi j f+\alpha) - (2\pi j f+\alpha)\right]_{f=0}$$

$$\cancel{+}\ \frac{\cancel{4\pi}}{2}\left[(2\pi j f+\alpha+j\beta)\log_e(2\pi j f+\alpha+j\beta) - (2\pi j f+\alpha+j\beta)\right.$$

$$+$$

$$\left.(2\pi j f+\alpha-j\beta)\log_e(2\pi j f+\alpha-j\beta) - (2\pi j f+\alpha-j\beta)\right]_{f=0}$$

$$\Rightarrow \frac{1}{2}\left(\alpha \log_e \alpha - \alpha\right)$$

$$-\frac{1}{2}\left((\alpha+j\beta)\log_e(\alpha+j\beta) - 2\alpha \cancel{+j\beta} + (\alpha-j\beta)\log_e(\alpha-j\beta) \cancel{\alpha} + j\beta \right)$$

$$\Rightarrow +\frac{1}{2}(\alpha\ln\alpha-\alpha) - \frac{1}{2}\left(\alpha\left(-2+\log_e(\alpha^2+\beta^2)\right) + j\beta\left(\log_e\left(\frac{\alpha+j\beta}{\alpha-j\beta}\right)\right)\right)$$

$$\left|\frac{\alpha+j\beta}{\alpha-j\beta}\right| = \boxed{1}$$

$$\angle \frac{(\alpha+j\beta)^2}{\underbrace{(\alpha-j\beta)(\alpha+j\beta)}_{\alpha^2+\beta^2}} = \angle(\alpha+j\beta) - \angle(\alpha-j\beta) = 2\tan^{-1}(\beta/\alpha)$$

$$\angle\left(\frac{\alpha^2}{\alpha^2+\beta^2} + \frac{-\beta^2}{\alpha^2+\beta^2}\right) + \frac{2j\alpha\beta}{\alpha^2+\beta^2}$$

$$\Rightarrow \frac{1}{2}(\alpha\ln\alpha-\alpha) - \frac{1}{2}\left(-2\alpha + \alpha\ln(\alpha^2+\beta^2) + j\beta\left(j\, 2\tan^{-1}(\beta/\alpha)\right)\right)$$

$$\Rightarrow \int_0^\infty e^{-\alpha t}\,\mathrm{sinc}^2(t)\,dt = \frac{1}{2}(\alpha\ln\alpha-\alpha) + \alpha - \frac{\alpha}{2}\ln(\alpha^2+4) - \cancel{4}\tan^{-1}(2/\alpha)$$

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⑥ (a) $u_p(t) = \text{sinc}(2t)\cos(100\pi t)$

$v_p(t) = \text{sinc}(t)\sin\left(101\pi t + \frac{\pi}{4}\right)$

$$\Rightarrow v_p(t) = \text{sinc}(t)\sin\left(100\pi t + \left(\pi t + \frac{\pi}{4}\right)\right).$$

$$= \text{sinc}(t)\sin(100\pi t)\cos\left(\pi t + \frac{\pi}{4}\right)$$

$$+ \text{sinc}(t)\cos(100\pi t)\sin\left(\pi t + \frac{\pi}{4}\right).$$

$$= \left(\text{sinc}(t)\sin\left(\pi t + \frac{\pi}{4}\right)\right)\cos(100\pi t)$$

$$- \left(-\text{sinc}(t)\cos\left(\pi t + \frac{\pi}{4}\right)\right)\sin(100\pi t)$$

$$\Rightarrow v_p(t) = v_c(t)\cos(100\pi t) - v_s(t)\sin(100\pi t)$$

$$\Rightarrow v(t) = v_c(t) + j\, v_s(t)$$

$$\Rightarrow \boxed{v(t) = \text{sinc}(t)\sin\left(\pi t + \frac{\pi}{4}\right) - j\,\text{sinc}(t)\cos\left(\pi t + \frac{\pi}{4}\right)}$$

$$\boxed{u(t) = \text{sinc}(2t) + j(0)}$$

(b) Bandwidth of $u_p(t)$, $v_p(t)$?

$$u_p(t) = \left(\frac{\sin(2\pi t)}{2\pi t}\right)\left(\cos(100\pi t)\right)$$

$$\Rightarrow U_p(f) = \frac{1}{2} I_{[-1,1]}(f) * \frac{1}{2}\left[\delta(f-50) + \delta(f+50)\right]$$

$$= \frac{1}{4}\left(I_{[-1,1]}(f-50) + I_{[-1,1]}(f+50)\right)$$

$\therefore$ Bandwidth $= 51$ Hz

$\therefore$ Actual bandwidth $= \boxed{2}$ Hz (one sided)

$$v_p(t) = \frac{\sin(\pi t)}{(\pi t)} \cdot \sin\left(101\pi t + \frac{\pi}{4}\right)$$

$$= \frac{\sin(\pi t)}{(\pi t)} \cdot \sin\left(101\pi\left(t + \frac{1}{404}\right)\right)$$

$\therefore$ Bandwidth $= 51$ Hz

Actual Bandwidth $= 1$ Hz (consid

(c) $\langle u_p, v_p \rangle = ?$

$$\int_{-\infty}^{\infty} u_p(t)\, v_p^*(t)\, dt = ?$$

$$\langle u_p, v_p \rangle = \frac{1}{2} Re\{\langle u, v \rangle\}$$

$$u(t) = \text{sinc}(2t)$$

$$v(t) = \text{sinc}(t)\sin\left(\pi t + \frac{\pi}{4}\right) - j\,\text{sinc}(t)\cos\left(\pi t + \frac{\pi}{4}\right)$$

$$= -j\,\text{sinc}(t)\left(\cos\left(\pi t + \frac{\pi}{4}\right) + j\sin\left(\pi t + \frac{\pi}{4}\right)\right)$$

$$= +e^{\frac{6\pi j}{4}}\,\text{sinc}(t)\, e^{j\left(\pi t + \frac{\pi}{4}\right)}$$

$$= \text{sinc}(t)\, e^{j\pi t}\, e^{j\frac{7\pi}{4}}$$

$$= \boxed{\text{sinc}(t)\, e^{j\pi t}\, e^{-j\pi/4}}$$

$$\Rightarrow \langle u, v \rangle = \int_{-\infty}^{\infty} u(t)\, v^*(t)\, dt$$

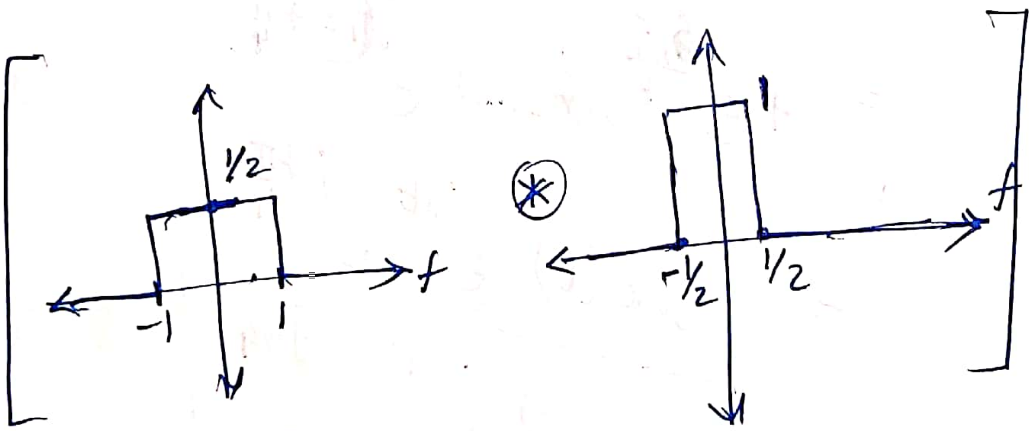
$$= \int_{-\infty}^{\infty} \text{sinc}(2t)\,\text{sinc}(t)\, e^{-j\pi t}\, e^{+j\frac{\pi}{4}}\, dt$$

$$= e^{j\frac{\pi}{4}} \int_{-\infty}^{\infty} \text{sinc}(2t)\,\text{sinc}(t)\, e^{-j\pi t}\, dt$$

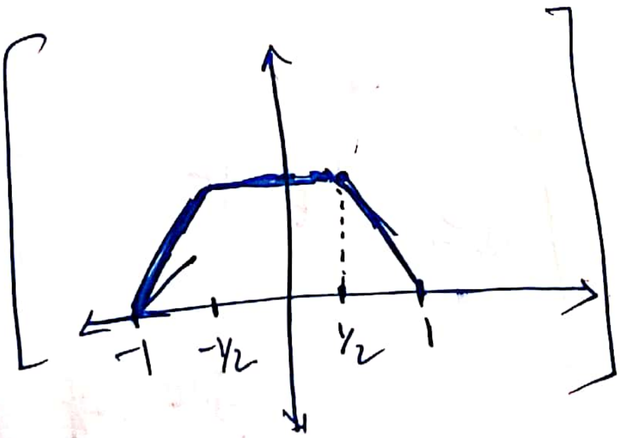
$$= e^{j\frac{\pi}{4}} \int_{-\infty}^{\infty} \text{sinc}(2t)\,\text{sinc}(t)\, e^{-2\pi j t\left(\frac{1}{2}\right)}\, dt$$

$$= e^{j\frac{\pi}{4}}\, \mathcal{F}\left\{ \text{sinc}(2t)\,\text{sinc}(t) \right\}(1/2) \Big|_{f=1/2}$$

$$= e^{j\frac{\pi}{4}} \left\{ \left(\mathcal{F}\{\text{sinc}(2t)\} * \mathcal{F}\{\text{sinc}(t)\} \right)(f) \right\} \Big|_{f=1/2}$$

$= e^{j\pi/4}$ 

$(f = 1/2)$

$\Rightarrow e^{j\pi/4}$ 

$(f = 1/2)$.

$$\Rightarrow \boxed{e^{\frac{j\pi}{4}}\left(\frac{1}{2}\right)} = \langle u, v\rangle$$

$$\Rightarrow \langle u_p, v_p\rangle = \frac{1}{2}\operatorname{Re}(\langle u,v\rangle)$$

$$= \frac{1}{4}\left(\frac{1}{\sqrt{2}}\right) = \boxed{\frac{1}{4\sqrt{2}}}$$

(d)

$$y_p(t) = (u_p * v_p)(t)$$

$$u(t) = \operatorname{sinc}(2t) + j(0)$$

$$\Rightarrow u_c(t) = \operatorname{sinc}(2t)$$

$$u_s(t) = 0$$

$$v(t) = \operatorname{sinc}(t)\sin\left(\pi t + \frac{\pi}{4}\right)$$

$$- j\operatorname{sinc}(t)\cos\left(\pi t + \frac{\pi}{4}\right)$$

$$\Rightarrow v_c(t) = \operatorname{sinc}(t)\sin\left(\pi t + \frac{\pi}{4}\right)$$

$$v_s(t) = -\operatorname{sinc}(t)\cos\left(\pi t + \frac{\pi}{4}\right)$$

$$y_c = \frac{1}{2}\left(\left(\operatorname{sinc}(2t) \circledast \operatorname{sinc}(t)\sin(\pi t + \pi/4)\right) - 0\right)$$

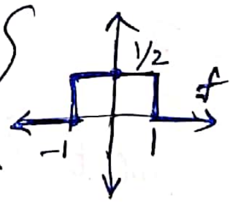
$$y_s = \frac{1}{2}\left(0 + \left(\operatorname{sinc}(2t) \circledast \left(-\operatorname{sinc}(t)\cos(\pi t + \pi/4)\right)\right)\right)$$

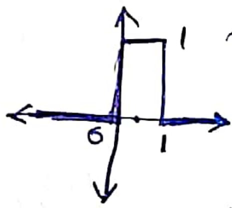
$$\Rightarrow y_p(t) = y_c(t)\cos(100\pi t) - y_s(t)\sin(100\pi t)$$

$$y_p(t) = \frac{1}{2}\,\mathrm{Re}\left\{(u * v)(t)\, e^{j100\pi t}\right\}$$

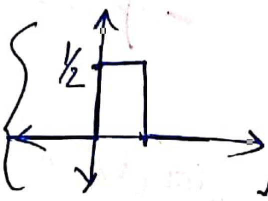
$$\Rightarrow y_p(t) = \frac{1}{2}\,\mathrm{Re}\left\{\left((\mathrm{sinc}(2t)) * \left(e^{j\pi/4} e^{j\pi t}\,\mathrm{sinc}(t)\right)\right) e^{100j\pi t}\right\}$$

$$= \frac{1}{2}\,\mathrm{Re}\left\{e^{-j\pi/4}\left((\mathrm{sinc}(2t)) * (\mathrm{sinc}(t)\, e^{j\pi t})\right) e^{100j\pi t}\right\}$$

$$= \frac{1}{2}\,\mathrm{Re}\left\{e^{j(100\pi t - \pi/4)}\ \mathcal{F}^{-1}\left\{\right.\right.$$ 

$\times$ 

$$\left.\left.\right\}\right\}$$

$$= \frac{1}{2}\,\mathrm{Re}\left\{e^{j(100\pi t - \pi/4)}\ \mathcal{F}^{-1}\left\{\right.\right.$$ 

$$\left.\left.\right\}\right\}$$

$$= \frac{1}{2}\,\mathrm{Re}\left\{\frac{e^{j(100\pi t - \pi/4)}}{2}\, e^{j\pi t}\,\mathrm{sinc}(t)\right\}$$

$$= \frac{1}{4}\,\mathrm{Re}\left\{\mathrm{sinc}(t)\, e^{j(101\pi)t}\, e^{-j\pi/4}\right\}$$

$$= \frac{1}{4}\,\mathrm{sinc}(t)\left(\frac{\cos(101\pi t)}{\sqrt{2}} - \frac{\sin(101\pi t)}{\sqrt{2}}\right)$$

(7) $u(t) = I_{[-1,1]}(t)\cos(100\pi t)$,

$h(t) = I_{[0,3]}(t)\sin(100\pi t)$

$f_c = 50Hz$

$\therefore\ y = (u * h)(t)$

$u_c(t) = \left(I_{[-1,1]}(t)\right)\cancel{\cos(100\pi t)}$

$u_s(t) = 0$

$h_c(t) = 0$

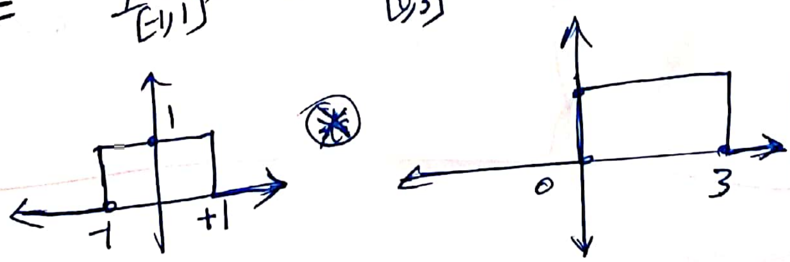
$h_s(t) = -I_{[0,3]}(t)$

$\Rightarrow y_c = \frac{1}{2}\left(\cancel{u_c \circledast h_c}^{0} - \cancel{u_s \circledast h_s}^{0}\right)$

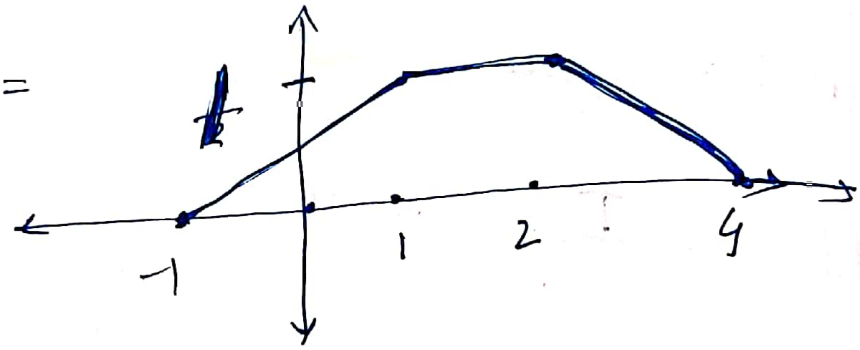
$y_s = \frac{1}{2}\left(\cancel{u_s \circledast h_c} + u_c \circledast h_s\right)$

$\Rightarrow 2y_s(t) = I_{[-1,1]}(t) \circledast -I_{[0,3]}(t)$

$\Rightarrow -2y_s(t) = I_{[-1,1]}(t) \circledast I_{[0,3]}(t)$

$=$ 

$\Rightarrow -2y_s(t) =$



$$\Rightarrow y(t) = -2\sin(100\pi t)\, y_8(t)$$

$$= \sin(100\pi t)\left(-2 y_8(t)\right)$$

$$-2y_8(t) = \begin{cases} 0 & t < -1 \\ \frac{t}{2} + 1/2 & -1 \le t < 1 \\ 1 & 1 \le t < 2 \\ -\frac{t}{2} + 2 & 2 \le t \le 4 \\ 0 & t > 4 \end{cases}$$

$$\boxed{\Rightarrow y(t) = \begin{cases} 0 & t < -1 \\ \left(\frac{t+1}{2}\right)\sin(100\pi t) & -1 \le t < 1 \\ \sin(100\pi t) & 1 \le t < 2 \\ \left(\frac{4-t}{2}\right)\sin(100\pi t) & 2 \le t \le 4 \\ 0 & t > 4 \end{cases}}$$

(2)

$$u_p(t) = a(t)\cos(2\pi(100)t)$$
$$= \text{sinc}(2t)\cos(2\pi(100)t)$$

(a) The freq band occupied by $u_p = [-101, -99] \cup [99, 101]$

(b)

$$u_p(t)\cos(199\pi t) \xrightarrow[\text{LTI, lowpass}]{h(t)} c(t)$$

$$\underset{\cos(199\pi t)}{u_p(t) \times} = \text{Re}\left\{ \text{sinc}(2t) e^{j\pi t} \cancel{e^{j200\pi t}} \cdot e^{j199\pi t} \cancel{e^{-j199\pi t}} \right\}$$

$$\Rightarrow u_p(t)\cos(199\pi t) = \text{Re}\left\{ \text{sinc}(2t) e^{j\pi t} e^{j199\pi t} \right\}$$
$$= u_{p_1}(t)$$

$$\Rightarrow u_{p_1}(t) = \text{Re}\left\{ u_1(t) e^{j199\pi t} \right\}$$

$$u_{1c}(t) = \text{sinc}(2t)\cos(\pi t)$$

$$u_{1s}(t) = \text{sinc}(2t)\sin(\pi t)$$

$$\Rightarrow b(t) = \frac{1}{2} u_c(t)$$

$$= \frac{1}{2} \sin(2t) \cos(\pi t)$$

$$\Rightarrow B(f) = \frac{1}{2}\left(\frac{1}{2} I_{[-1,1]}(f) * \left(\frac{\delta(f-1/2)+\delta(f+1/2)}{2}\right)\right)$$

$$= \frac{1}{8}\left(I_{[-\frac{3}{2},\frac{1}{2}]}(f) + I_{[-\frac{1}{2},\frac{3}{2}]}(f) \right)$$

(c) $u_p(t) \sin(199\pi t) \longrightarrow$

$$u_p(t) = Re\left\{ \sin(2t)\, e^{j(200\pi t)} \right\}$$

$$e^{j\pi t}$$

$$u_p(t)\sin(199\pi t) = Re\left\{ j \sin(2t)\, \cancel{e^{j(200\pi t)}}\, e^{j(199\pi t)}\, \cancel{e^{-j199\pi t}} \right\}$$

$$\Rightarrow u_{p_2}(t) = Re\left\{ \cancel{j}\, e^{+j\frac{3\pi}{2}} \sin(2t)\, e^{j\pi t}\, e^{j199\pi t} \right\}$$

$$\Rightarrow u_2(t) = \text{sinc}(2t)\, e^{j\pi t} e^{+j\frac{3\pi}{2}}$$

$$\Rightarrow u_{2c}(t) = \text{sinc}(2t)\cos\left(\pi t + \frac{3\pi}{2} \underset{+\frac{\pi}{2}-\frac{\pi}{2}}{}\right)$$

$$= \text{sinc}(2t)\cos\left(\pi t + \cancel{\frac{4\pi}{2}} - \frac{\pi}{2}\right)$$

$$= \text{sinc}(2t)\cos\left(\frac{\pi}{2} - \pi t\right)$$

$$= \text{sinc}(2t)\sin(\pi t)$$

$$\Rightarrow c(t) = \frac{-1}{2} u_{2c}(t) = \frac{-\text{sinc}(2t)\sin(\pi t)}{2}.$$

$$\Rightarrow C(f) = \frac{1}{4j}\left(-I_{[-\frac{3}{2},\frac{1}{2}]}(f) + I_{[-\frac{1}{2},\frac{3}{2}]}(f)\right)$$

(d)

$$u(t) = 2\left(b(t) - j\,c(t)\right)$$

$$a(t) = u(t)\, e^{-j\pi t} = 2\left(b(t) - jc(t)\right)\left(\cos(\pi t) - j\sin(\pi t)\right)$$

$$\Rightarrow a(t) =$$

$$a(t) = a^*(t)$$

$$\Rightarrow a(t) = \mathrm{Re}\left(2\left(b(t) - jc(t)\right)\left(\cos(\pi t) - j\sin(\pi t)\right)\right)$$

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$$\Rightarrow \boxed{a(t) = 2\left(b(t)\cos(\pi t) - c(t)\sin(\pi t)\right)}$$