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Q3. $||x||_w = \sqrt{\sum_{i=1}^{n} w_i x_i^2}$, $w_i > 0 \;\forall\; i = 1, 2, \ldots, n$

(1) Non-negative homogeneity :-

$\alpha \in R, \; x \in R^n,$

$$||\alpha x||_w = \sqrt{w_1(\alpha x_1)^2 + w_2(\alpha x_2)^2 + \ldots + w_n(\alpha x_n)^2}$$

$$= |\alpha| \sqrt{\sum_{i=1}^{n} w_i x_i^2}$$

$$= |\alpha| \, ||x||_w$$

(2) Triangle Inequality :-

$x, y \in R^n$

Note that,

$$||x||_w = \sqrt{w_1 x_1^2 + w_2 x_2^2 + \ldots + w_n x_n^2}$$

$$= \sqrt{(\sqrt{w_1}\, x_1)^2 + (\sqrt{w_2}\, x_2)^2 + \ldots + (\sqrt{w_n}\, x_n)^2}$$

$$= ||x'||_2,$$

where $x' = \begin{bmatrix} \sqrt{w_1}\, x_1 \\ \sqrt{w_2}\, x_2 \\ \vdots \\ \sqrt{w_n}\, x_n \end{bmatrix}$ $(x'_i = \sqrt{w_i}\, x_i)$

So, by triangle-inequality of 2-norm,

$$||x' + y'||_2 \leq ||x'||_2 + ||y'||_2$$

$$\Rightarrow ||x' + y'||_2 \leq ||x||_w + ||y||_w$$

Now, $x' + y' = \begin{bmatrix} \sqrt{w_1}\,(x_1 + y_1) \\ \sqrt{w_2}\,(x_2 + y_2) \\ \vdots \\ \sqrt{w_n}\,(x_n + y_n) \end{bmatrix}$

So, $||x'+y'||_2 = \sqrt{w_1(x_1+y_1)^2 + w_2(x_2+y_2)^2 + \dots + w_n(x_n+y_n)^2}$

using this

$= ||x+y||_w$

So, we had derived,

$||x'+y'|| \leq ||x||_w + ||y||_w$

$\Rightarrow ||x+y||_w \leq ||x||_w + ||y||_w$

Thus, the triangle inequality holds true.

(3) Non-negativity :-

$||x||_w = \sqrt{w_1 x_1^2 + \dots + w_n x_n^2} \geq 0$

(as square root cannot be negative)

(4) Definiteness :-

$||x||_w = 0 \Leftrightarrow x = 0$

Consider the first side :-

$||x||_w = 0 \Rightarrow x = 0$

If $||x||_w = 0$

$\Rightarrow w_1 x_1^2 + w_2 x_2^2 + \dots + w_n x_n^2 = 0$

As $w_1, w_2, \dots, w_n > 0$

$x_i^2 = 0 \quad \forall i = 1, 2, \dots, n$

$\Rightarrow x_i = 0 \quad \forall i = 1, 2, \dots, n$

$\Rightarrow x = 0$

Second side :-

$x = 0 \Rightarrow ||x||_w = 0$

This is easy to see. If $x = 0 \Rightarrow x_i = 0 \ \forall \ i = 1, 2, \dots, n$

Then, $||x||_w = \sqrt{\sum_{i=1}^{n} w_i x_i^2} = \sqrt{0} = 0$

Thus, $||\cdot||_w$ is a norm.