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Q5. To prove :- Matrix multiplication is associative.

Let $A \in R^{m \times p}$, $B \in R^{p \times q}$, $C \in R^{q \times n}$

We need to show that $A(BC) = (AB)C$

we can see that dimensions of $A(BC)$ and $(AB)C$ are equal ($= m \times n$).

So, now we need to prove elementwise equality.

Let $M = BC$, (k,j) th element of $M = M_{k,j}$

We can write, $M_{k,j} = B_{k,1} C_{1,j} + B_{k,2} C_{2,j} + \cdots + B_{k,q} C_{q,j}$

Now, note that the (i,j) th element of $A(BC)$ can be written as :-

$$A_{i,1}(B_{1,1} C_{1,j} + B_{1,2} C_{2,j} + \cdots + B_{1,q} C_{q,j})$$
$$+ A_{i,2}(B_{2,1} C_{1,j} + B_{2,2} C_{2,j} + \cdots + B_{2,q} C_{q,j})$$
$$+ \quad \vdots$$
$$+ A_{i,p}(B_{p,1} C_{1,j} + B_{p,2} C_{2,j} + \cdots + B_{p,q} C_{q,j})$$

~~$A_{i,1}B_{1,1} + A_{i,1}B_{1,2} + \cdots$~~

$$= (A_{i,1} B_{1,1} + A_{i,2} B_{2,1} + \cdots + A_{i,p} B_{p,1}) C_{1,j}$$
$$+ (A_{i,1} B_{1,2} + A_{i,2} B_{2,2} + \cdots + A_{i,p} B_{p,2}) C_{2,j}$$
$$+ \quad \vdots$$
$$+ (A_{i,1} B_{1,q} + A_{i,2} B_{2,q} + \cdots + A_{i,p} B_{p,q}) C_{q,j}$$
$$= (a_i B_1) C_{1,j} + (a_i B_2) C_{2,j} + \cdots + (a_i B_q) C_{q,j}$$

$a_i \rightarrow i^{th}$ row of A

$B_i \rightarrow i^{th}$ column of B

Now, let $N = AB$, then;

$$N_{i,j} = a_i B_j$$

The $[(AB)C]_{i,j} = (NC)_{i,j}$

$$= N_{i,1} C_{1,j} + N_{i,2} C_{2,j} + \cdots + N_{i,q} C_{q,j}$$

$$= (a_i B_1) C_{1,j} + (a_i B_2) C_{2,j} + \cdots + (a_i B_q) C_{q,j}$$

Thus, the (i,j)th entry of $A(BC)$ is same as $(AB)C$.

Hence, matrix multiplication is associative.

To prove: matrix multiplication is not commutative.

Take an example which is :-

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Then, $AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$

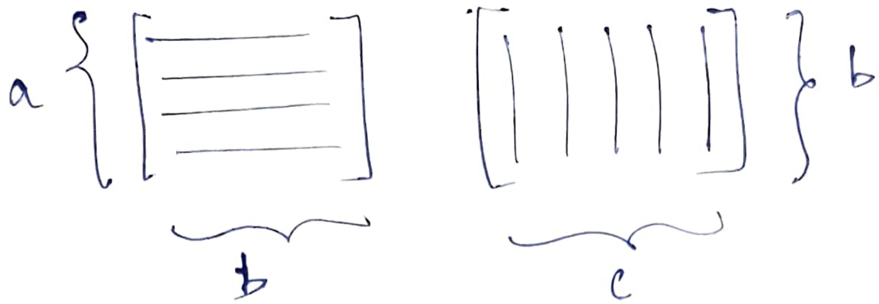
and $BA = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$

Thus, in general for 2 matrices A and B,

$$AB \neq BA.$$

Consider 2 matrices $P \in R^{a \times b}$ and $Q \in R^{b \times c}$

To compute the product PQ :-



To compute one single entry of PQ, we need b multiplications and $(b-1)$ additions.

So, for one element, computational cost $= 2b-1$

There are (ac) elements.

So, total cost $= a(2b-1)c$.

Now, given $A \in R^{p \times q}$, $B \in R^{q \times r}$, $C \in R^{r \times t}$

$A \cdot B \rightarrow p(2q-1)r$

$(AB) \cdot C \rightarrow p(2r-1)t$

So, total operations for $(AB)C = p(2q-1)r + p(2r-1)t$

$B \cdot C \rightarrow q(2r-1)t$

$A \cdot (BC) \rightarrow p(2q-1)t$

So, total operations for $A(BC) = q(2r-1)t + p(2q-1)t$.

We want,

$$p(2q-1)r + p(2r-1)t < q(2r-1)t + p(2q-1)t \quad \text{(1)}$$

Simplifying,

$2pqr - pr + 2prt - \cancel{pt} < 2qrt - qt + 2pqt - \cancel{pt}$

Dividing by $pqrt$, we get,

$$\frac{2}{t} - \frac{1}{qt} + \frac{2}{q} < \frac{2}{p} - \frac{1}{pr} + \frac{2}{r}$$

However, to obtain a simpler condition, we can neglect the 1 in the central terms, for example, we can approximate $p(2q-1)r$ as $p(2q)r$.

Thus, after doing this approximation, from equation 1, we get,

$p(2q)r + p(2r)t < q(2r)t + p(2q)t$.

Dividing both sides by $2pqrt$, we get,

$$\frac{1}{t} + \frac{1}{q} < \frac{1}{p} + \frac{1}{r}.$$