Ashutosh Kumar Singh 19CS30008

6. Bi-linear Interpolation

$$P_{ij} = (x_i, y_j), \quad i = 1, 2, \ldots, M, \quad j = 1, 2, \ldots, N$$

$$f(u, v) = \theta_1 + \theta_2 u + \theta_3 v + \theta_4 uv$$

$$f(P_{ij}) = F_{ij}$$

(a) $A\theta = b$

$$\underbrace{\begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_1 & y_2 & x_1 y_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_1 & y_N & x_1 y_N \\ 1 & x_2 & y_1 & x_2 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_2 & y_N & x_2 y_N \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_M & y_1 & x_M y_1 \\ 1 & x_M & y_2 & x_M y_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_M & y_N & x_M y_N \end{bmatrix}}_{A_{MN \times 4}} \underbrace{\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix}}_{\theta_{4 \times 1}} = \underbrace{\begin{bmatrix} F_{11} \\ F_{12} \\ \vdots \\ F_{1N} \\ F_{21} \\ F_{22} \\ \vdots \\ F_{2N} \\ \vdots \\ F_{M1} \\ F_{M2} \\ \vdots \\ F_{MN} \end{bmatrix}}_{b_{MN \times 1}}$$

(b) We want a unique solution to the system of equations $A\theta = b$.

Firstly, we may expect or assume ~~the~~ atleast the existence of a solution, i.e., $b \in$ colspace (A).
Then, for the solution to be unique, the columns of A should form a basis of colspace (A).
This implies that the columns of A should be linearly independent.
Hence, no. of rows $\geq$ no. of columns, i.e., A should be either square or tall.

No. of rows $= MN$
No. of columns $= 4$

Hence, $MN \geq 4$.
Since we want to minimize M and N, we choose $MN = 4$.
Now, for M and N individually, we have three possibilities:-

1) $M = 1, N = 4$
2) $M = 2, N = 2$
3) $M = 4, N = 1$

Consider 1) $M = 1, N = 4$, then, the second column becomes all x_1 and hence, the columns become linearly dependent because now the second column is x_1 times the first column, which is all 1's. If the columns become linearly dependent, then the solution will not be unique. Hence, we discard this possibility.

If we consider 3) $M = 4$, $N = 1$, then similarly, the third column becomes all y_i, and again the columns become linearly dependent, hence, we discard this too.

Thus, the minimum values of M and N such that $A\theta = b$ may expect a unique solution are :- $\boxed{M = 2 \text{ and } N = 2}$.