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Q6. We define left inverse of a matrix A to be any matrix X such that $XA = I$.

Note that if $A \in R^{m \times n}$, then $X \in R^{n \times m}$, where I is the $n \times n$ identity matrix.

The left inverse of a matrix exists iff the columns of the matrix are linearly independent.

(a) $A = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

For this matrix, since there is only 1 column and it is not all zeros, the columns are linearly independent trivially.

Thus, left inverse exists for this matrix A.

Let X be a left-inverse of A.

So, $XA = I$

Now take another matrix Y, such that,

$YA = 0$,

then the family of left-inverses can be written as $X + \alpha Y$ ($\alpha \in R$).

Because $(X + \alpha Y)A = XA + \alpha YA$

$= I + 0 = I$.

In this case to find out X, let's consider

$X = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5]$

$XA = I \Rightarrow x_1 + x_4 = 1 \Rightarrow x_4 = 1 - x_3$

Thus X is of the form $[x_1, x_2, x_3, 1 - x_1, x_5]$

where $x_1, x_2, x_3, x_5 \in R$.

Now, assume $Y = [y_1 \ \ y_2 \ \ y_3 \ \ y_4 \ \ y_5]$

$YA = 0 \Rightarrow y_1 + y_4 = 0 \Rightarrow y_4 = -y_1.$

So, Y is of the form $[y_1 \ \ y_2 \ \ y_3 \ \ -y_1 \ \ y_5]$

Then, the family of left inverses $X + \alpha Y$ is :-

$[x_1 + \alpha y_1 \ \ \ x_2 + \alpha y_2 \ \ \ x_3 + \alpha y_3 \ \ \ 1 - x_1 - \alpha y_1 \ \ \ x_5 + \alpha y_5]$

Thus, in this case, we can characterize all left-inverses.

(b) $A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \\ 3 & 3 \end{bmatrix}$

To show columns of A are linearly independent :-

$$\alpha \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \ 2\alpha = 0, \ -2\beta = 0, \ 3\alpha + 3\beta = 0$

$\Rightarrow \ \alpha = \beta = 0.$

Thus, columns of A are linearly independent.

Similar to the previous case, if X is a left inverse of A, then, if we find a Y such that $YA = 0$, then the set of all left-inverses will be $X + \alpha Y \ (\alpha \in \mathbb{R})$.

Consider $X = \begin{bmatrix} x_1 & x_3 & x_5 \\ x_2 & x_4 & x_6 \end{bmatrix}$

$XA = I$

$$\Rightarrow \begin{bmatrix} x_1 & x_3 & x_5 \\ x_2 & x_4 & x_6 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 2x_1 + 3x_5 = 1 \quad , \quad -2x_3 + 3x_5 = 0$$

So, $x_5 = \dfrac{1-2x_1}{3}$, $x_3 = \dfrac{1-2x_1}{2}$

and, $2x_2 + 3x_6 = 0$, $-2x_4 + 3x_6 = 1$

So, $x_6 = \dfrac{-2x_2}{3}$, $x_4 = \dfrac{-1-2x_2}{2}$

So, $$X = \begin{bmatrix} x_1 & \dfrac{1-2x_1}{2} & \dfrac{1-2x_1}{3} \\ x_2 & \dfrac{-1-2x_2}{2} & \dfrac{-2x_2}{3} \end{bmatrix}$$

To get $Y = \begin{bmatrix} y_1 & y_3 & y_5 \\ y_2 & y_4 & y_6 \end{bmatrix}$

$YA = 0$

$$\Rightarrow \begin{bmatrix} y_1 & y_3 & y_5 \\ y_2 & y_4 & y_6 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solving this, we get,

$$Y = \begin{pmatrix} y_1 & -y_1 & \dfrac{-2y_1}{3} \\ y_2 & -y_2 & \dfrac{-2y_2}{3} \end{pmatrix}$$

So, the family of all left-inverses $X + \alpha Y$ can be characterized as:-

$$\begin{bmatrix} x_1 + \alpha y_1 & \dfrac{1-2x_1}{2} - \alpha y_1 & \dfrac{1-2x_1-2\alpha y_1}{3} \\ x_2 + \alpha y_2 & \dfrac{-1-2x_2}{2} - \alpha y_2 & \dfrac{-2x_2-2\alpha y_2}{3} \end{bmatrix}$$

$x_1, x_2, y_1, y_2, \alpha \in R$.