Ashutosh Kumar Singh 19CS30008

3. (a) $A = \begin{pmatrix} -1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \\ -1 & 1 \end{pmatrix} \in R^{3\times 2}$, $m = 3, n = 2$.

$$Ax = \begin{pmatrix} -1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2}\, x_1 \\ -1/\sqrt{2}\, x_2 \\ -x_1 + x_2 \end{pmatrix}$$

where $x_1^2 + x_2^2 = 1$ (the unit circle in R^2)

Let $-1/\sqrt{2}\, x_1 = x$, $-1/\sqrt{2}\, x_2 = y$, and, $-x_1 + x_2 = z$

Consider $x_1 = \cos t$, $x_2 = \sin t$, $t \in [0, 2\pi]$.

Then we get,

$$Ax = \begin{pmatrix} -1/\sqrt{2} \cos t \\ -1/\sqrt{2} \sin t \\ \cos t - \sin t \end{pmatrix}$$

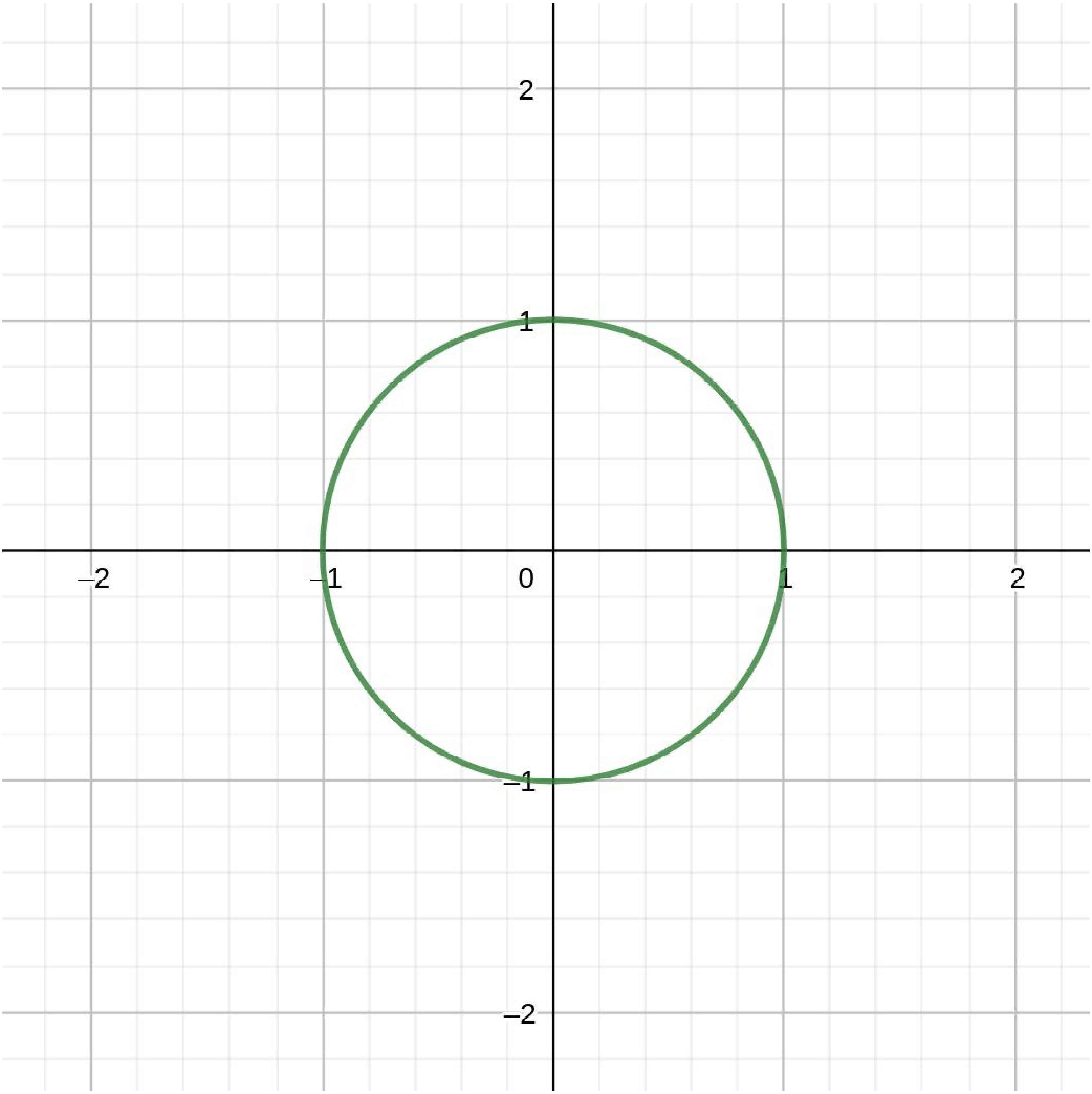
Thus, in R^2, we have the unit circle, whose image in R^3 on multiplying by A is an ellipse.

Condition number of $A = 2.2360$

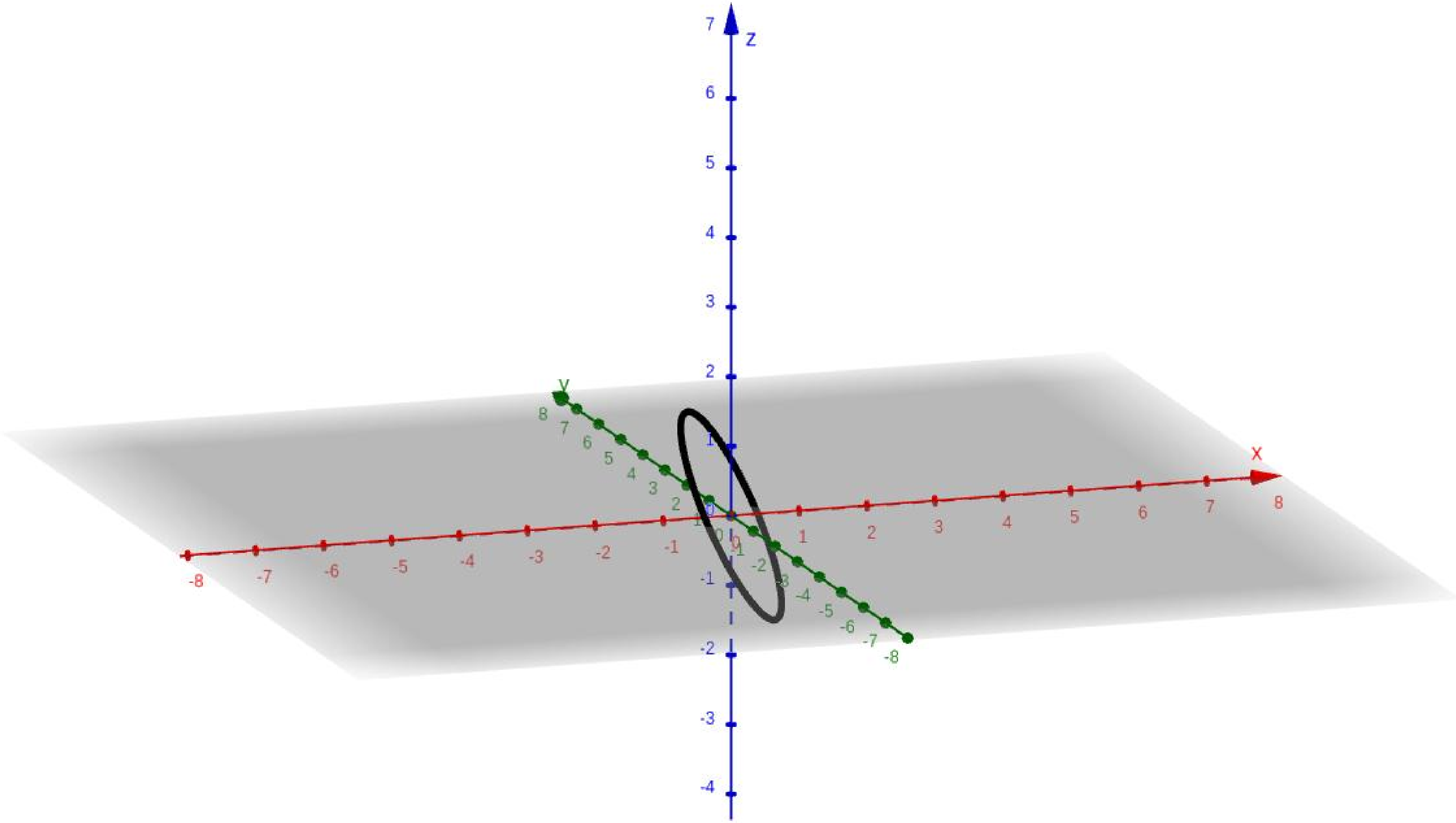
IMPORTANT NOTE:-

For parts (a), (c), (d) and (e), the input in R^n is the unit circle $x^2 + y^2 = 1$, and for simplicity, they will be shown only once in part (a).

For part (b), the input in R^n is the unit sphere $x^2 + y^2 + z^2 = 1$.



2
1
–2
–1
0
1
2
–1
–2



z
x
y

(b) $A = \begin{pmatrix} -2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix} \in R^{2\times 3}$, $m = 2$, $n = 3$

$$Ax = \begin{pmatrix} -2 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_1 + x_2 + 2x_3 \\ 2x_2 \end{pmatrix}$$

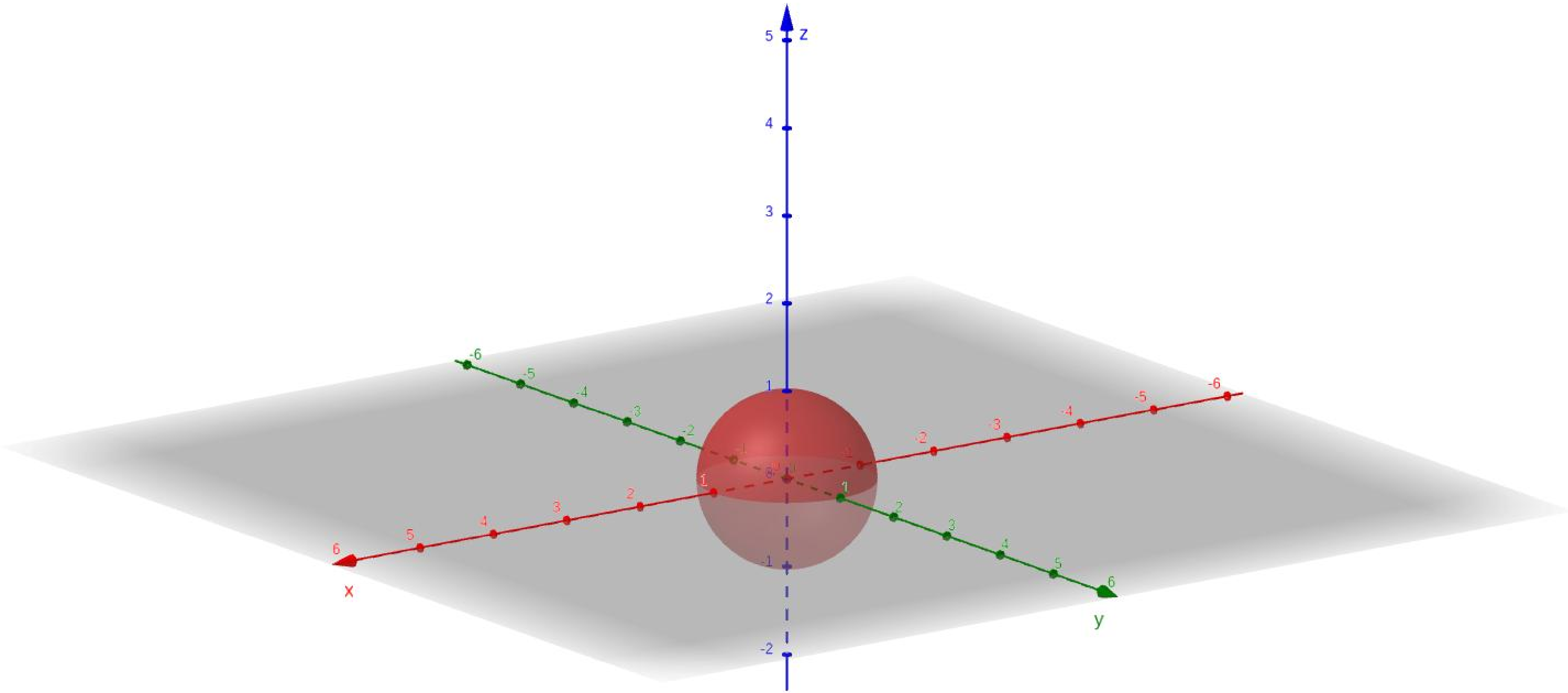
where $x_1^2 + x_2^2 + x_3^2 = 1$ (the unit sphere in R^3)

Take $-2x_1 + x_2 + 2x_3 = x$, and, $2x_2 = y$.

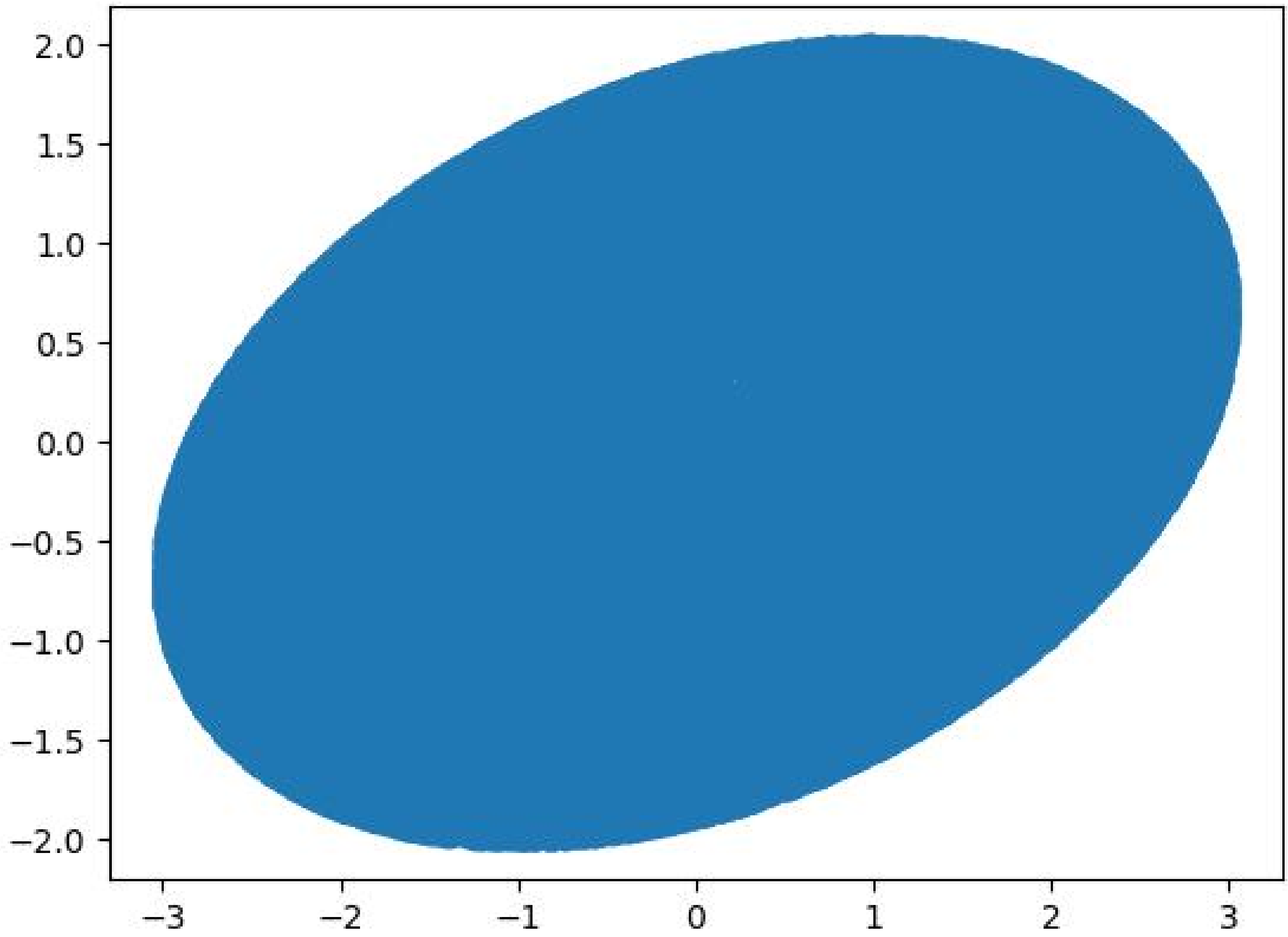
Since it is not possible to directly obtain the equation of the image ellipse or ellipsoid, we take nearly 10,000 points on the unit sphere, map them to their images in R^2, and then join them to form the ellipse.

Here, we actually get a filled ellipse in R^2, not just the boundary.

Condition number of $A = 1.7150$



z
5
4
3
2
1
-1
-2
x
6
5
4
3
2
1
-2
-3
-4
-5
-6
y
-6
-5
-4
-3
-2
1
2
3
4
5
6


2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−3
−2
−1
0
1
2
3

(c) $A = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 0.8 \end{pmatrix} \in \mathbb{R}^{2\times 2}$, $m=2, n=2$

$$Ax = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 0.8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 0.9x_2 \\ 0.9x_1 + 0.8x_2 \end{pmatrix}$$

Let $x_1 + 0.9x_2 = x$, and, $0.9x_1 + 0.8x_2 = y$

We get,

$$0.9x - y = 0.01x_2$$

and, $0.8x - 0.9y = -0.01x_1$

So, squaring and adding, we get,

$$(9x-10y)^2 + (8x-9y)^2 = \left(\frac{-x_1}{10}\right)^2 + \left(\frac{x_2}{10}\right)^2$$

And we know $x_1^2 + x_2^2 = 1$

So, $(81x^2 + 100y^2 - 180xy) + (64x^2 + 81y^2 - 144xy) = \frac{1}{100}$

$$\Rightarrow 145x^2 + 181y^2 - 324xy = \frac{1}{100} \quad \text{—(1)}$$

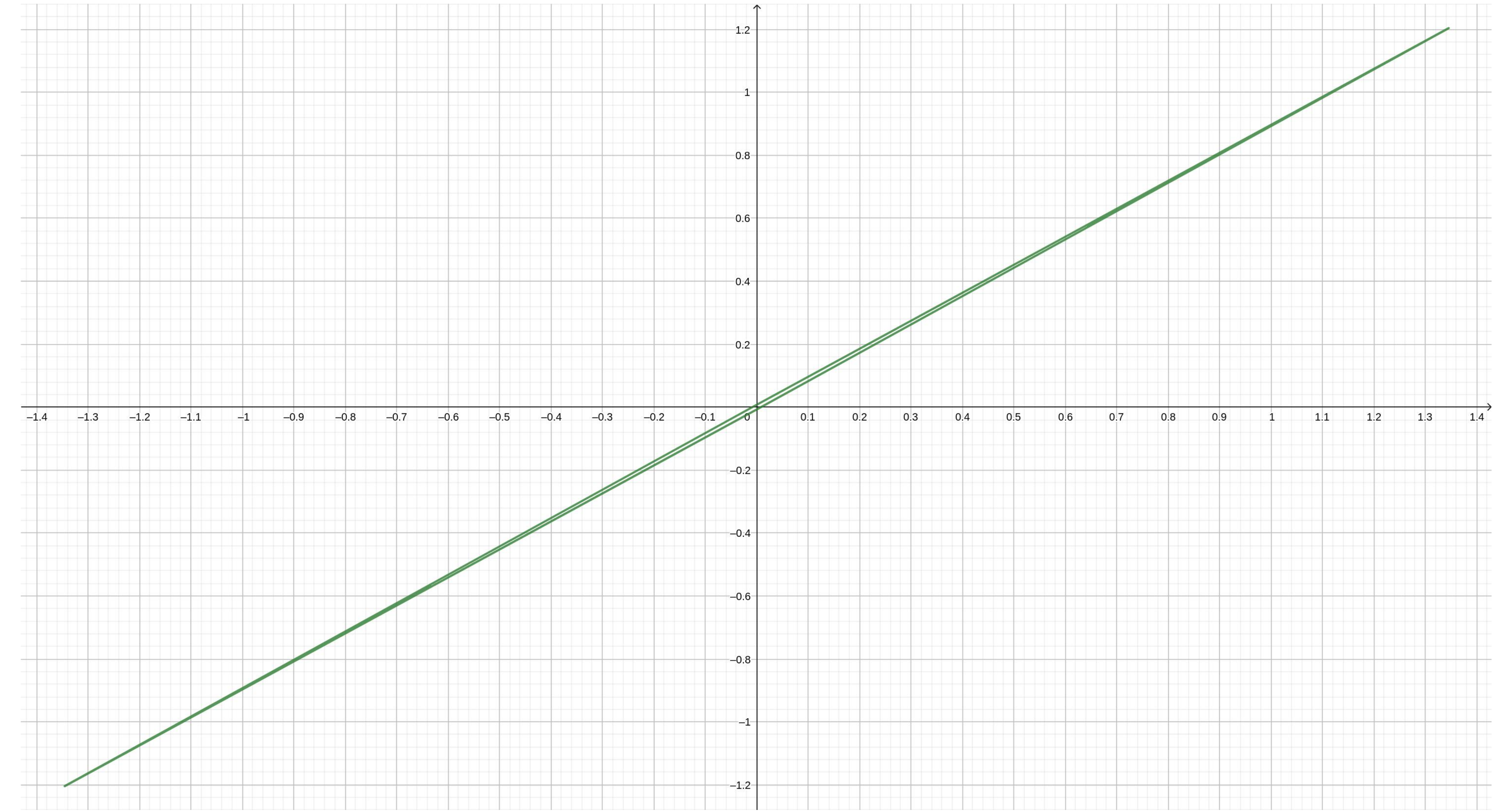
Thus, the input unit circle in $\mathbb{R}^2$ gets mapped to the ellipse with eqn (1) in $\mathbb{R}^2$.

Here, $m=n$, and the columns of A are linearly independent, so, A is invertible.

Condition number of $A = 325.9969$

Determinant of $A = 1\times 0.8 - 0.9\times 0.9$

$= -0.01$



−1.4
−1.3
−1.2
−1.1
−1
−0.9
−0.8
−0.7
−0.6
−0.5
−0.4
−0.3
−0.2
−0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
1.2
1.3
1.4
1.2
1
0.8
0.6
0.4
0.2
−0.2
−0.4
−0.6
−0.8
−1
−1.2

(d) $A = \begin{pmatrix} 1 & 0 \\ 0 & -10 \end{pmatrix} \in R^{2\times 2}$, $m=2$, $n=2$

$$Ax = \begin{pmatrix} 1 & 0 \\ 0 & -10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -10x_2 \end{pmatrix}$$

Let $x_1 = x$ and $-10x_2 = y$, then,

$$x^2 + \frac{y^2}{100} = x_1^2 + x_2^2$$

$$\Rightarrow x^2 + \frac{y^2}{100} = 1 \quad \text{— (1)}$$

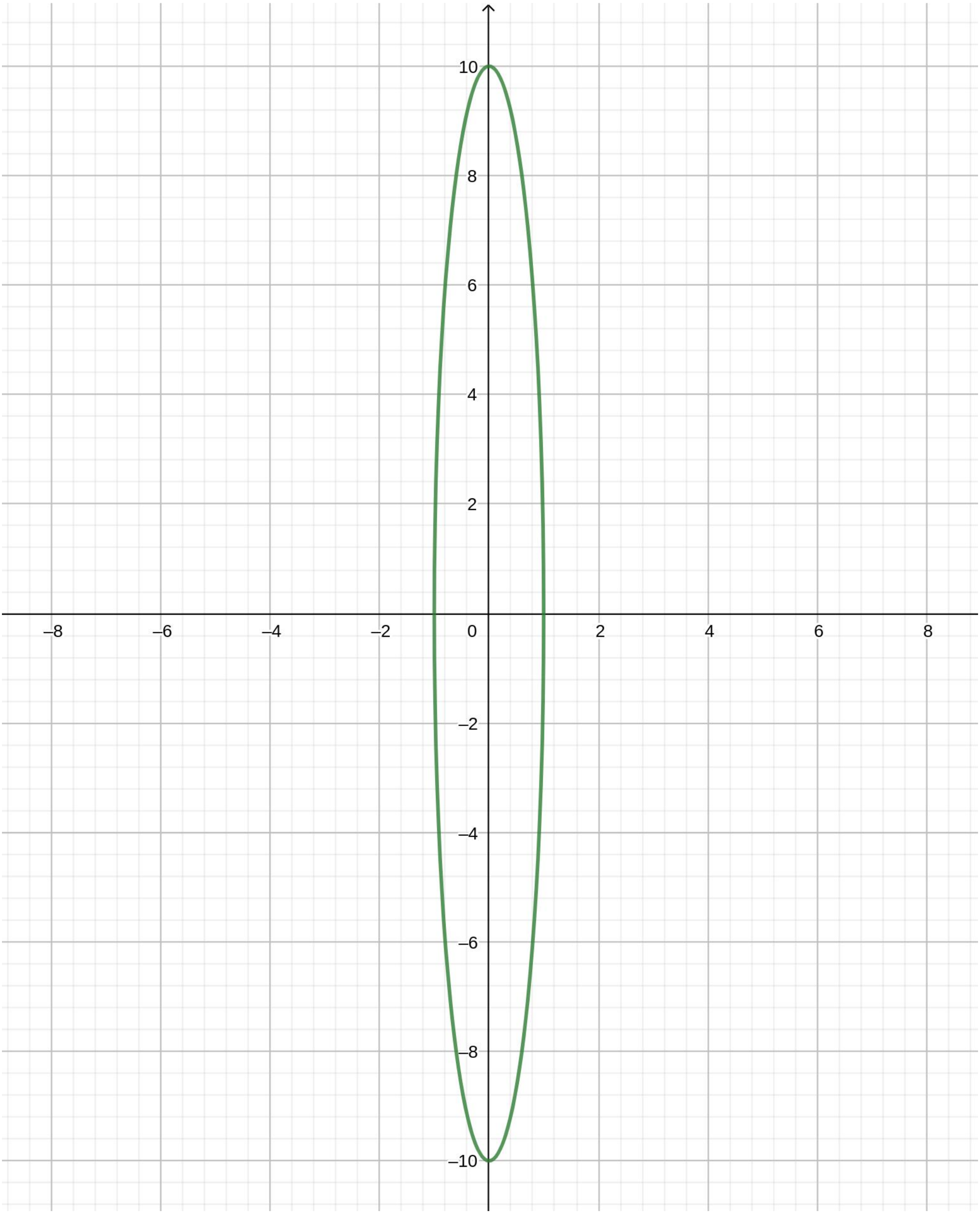
Thus, the input unit circle in R^2 gets mapped to the ellipse with eq$\underline{n}$ (1) in R^2.

Condition number of $A = 10.0$

Here $m = n$, and the columns of A are linearly independent, so, A is invertible.

Determinant of $A = 1 \times (-10) - 0 \times 0$

$= -10$


10
8
6
4
2
0
–2
–4
–6
–8
–10
–8
–6
–4
–2
2
4
6
8

(e) $A = \begin{pmatrix} 1 & 1 \\ 1 & \varepsilon \end{pmatrix} \in R^{2\times 2}$, $m=2$, $n=2$

$$Ax = \begin{pmatrix} 1 & 1 \\ 1 & \varepsilon \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 + \varepsilon x_2 \end{pmatrix}$$

Let $x_1 + x_2 = x$ and $x_1 + \varepsilon x_2 = y$.

Using the fact that $x_1^2 + x_2^2 = 1$, we get,

$$(\varepsilon^2 + 1)x^2 + 2y^2 - 2xy(1+\varepsilon) = (\varepsilon - 1)^2$$

The input vector x in all cases is the unit circle in R^2.

$\underline{\varepsilon = 10}$

$$101x^2 + 2y^2 - 22xy = 81 \quad \text{— (1)}$$

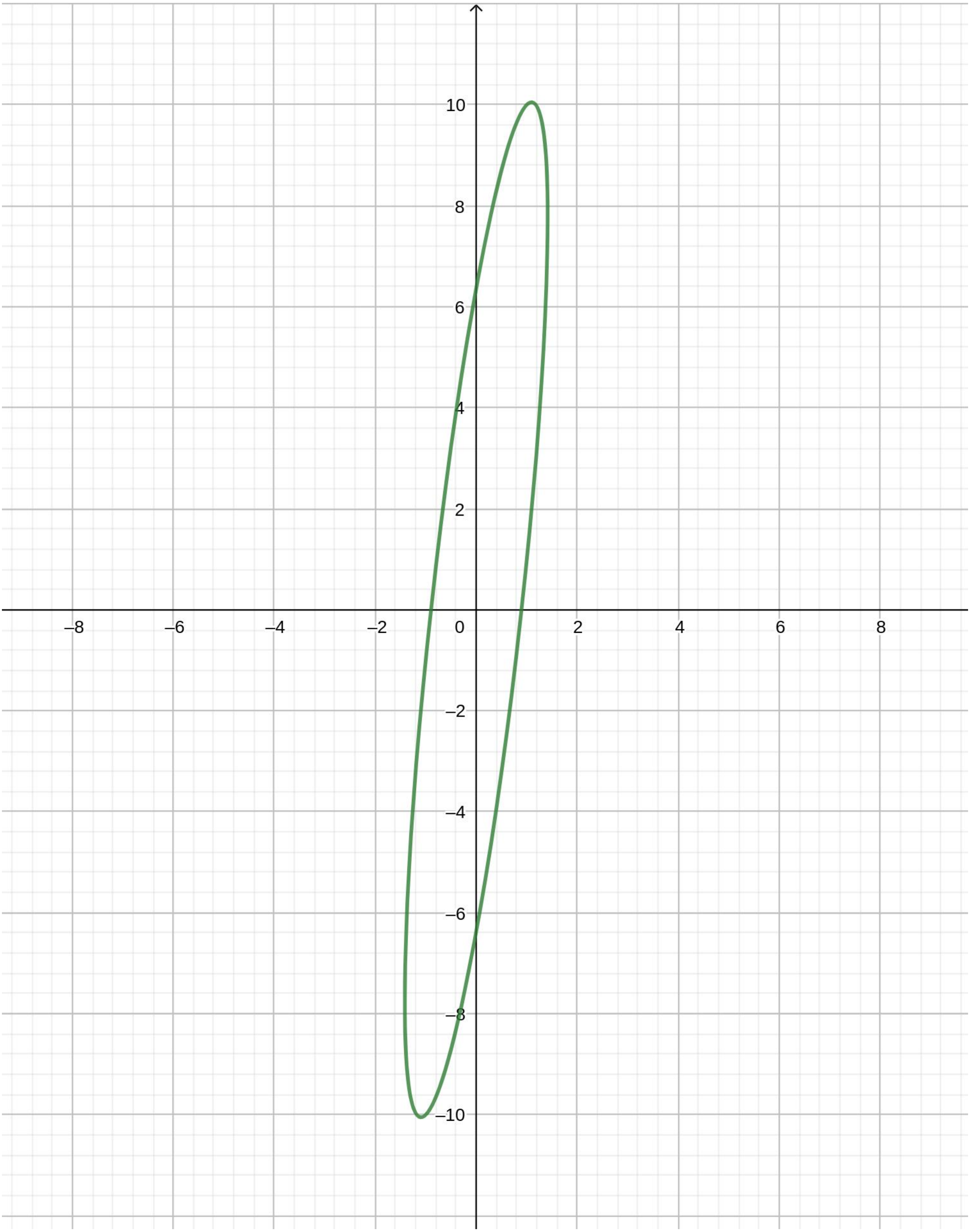
$A = \begin{pmatrix} 1 & 1 \\ 1 & 10 \end{pmatrix}$

Condition number of $A = 11.3563$

As columns of A are linearly independent, A is invertible.

Determinant of $A = \varepsilon - 1 = 10 - 1 = 9$

The image of the unit circle is the ellipse with eqn. (1).



10
8
6
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2
0
–2
–4
–6
–8
–10
–8
–6
–4
–2
2
4
6
8

$\underline{\varepsilon = 5}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}$$

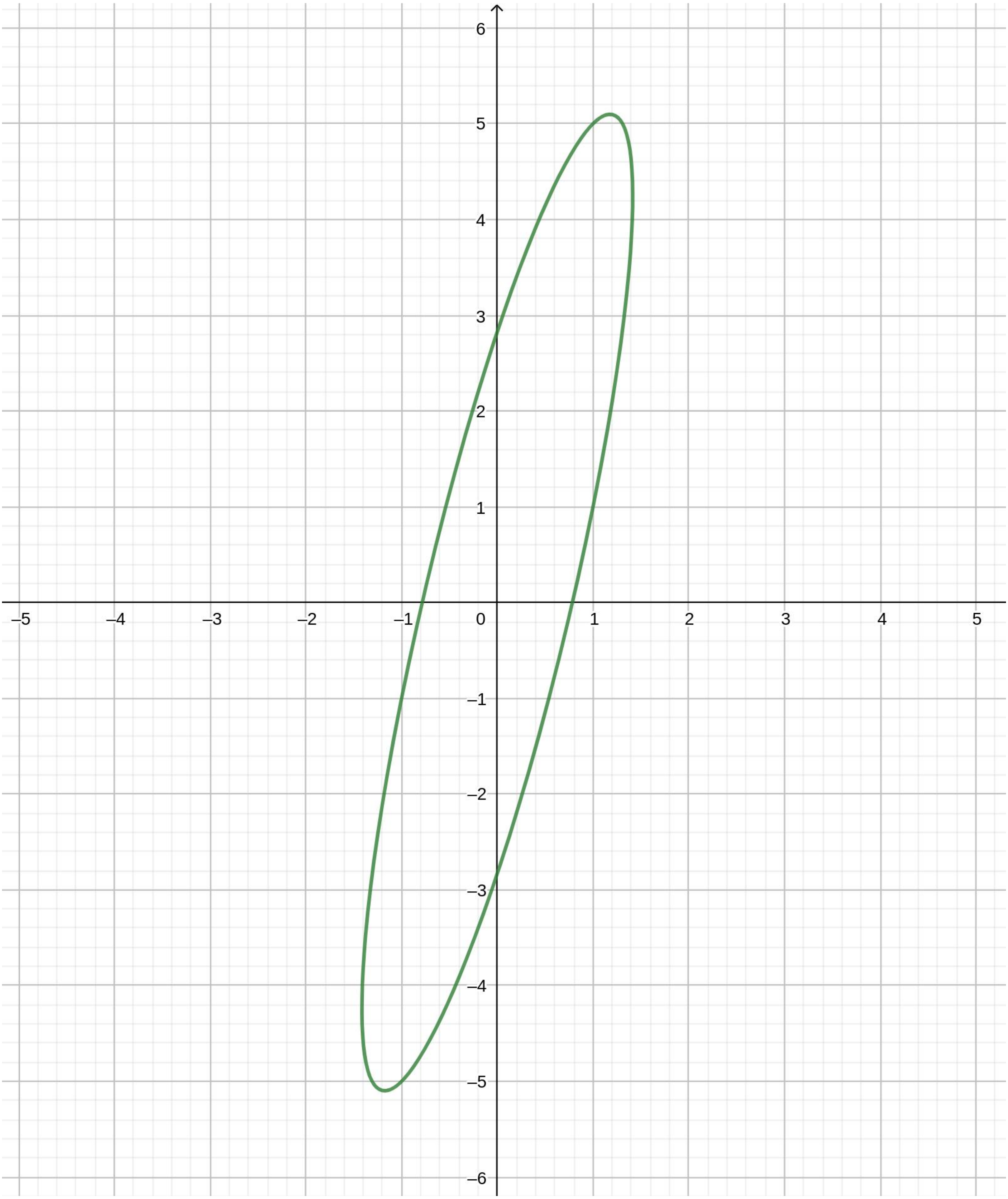
The image of the unit circle is the ellipse with equation :-

$$26x^2 + 2y^2 - 12xy = 16$$

Condition number of $A = 6.8541$

As columns of A are linearly independent, A is invertible.

Determinant of $A = \varepsilon - 1 = 5 - 1 = 4$



6
5
4
3
2
1
0
–1
–2
–3
–4
–5
–6
–5
–4
–3
–2
–1
1
2
3
4
5

$\underline{\varepsilon = 1}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

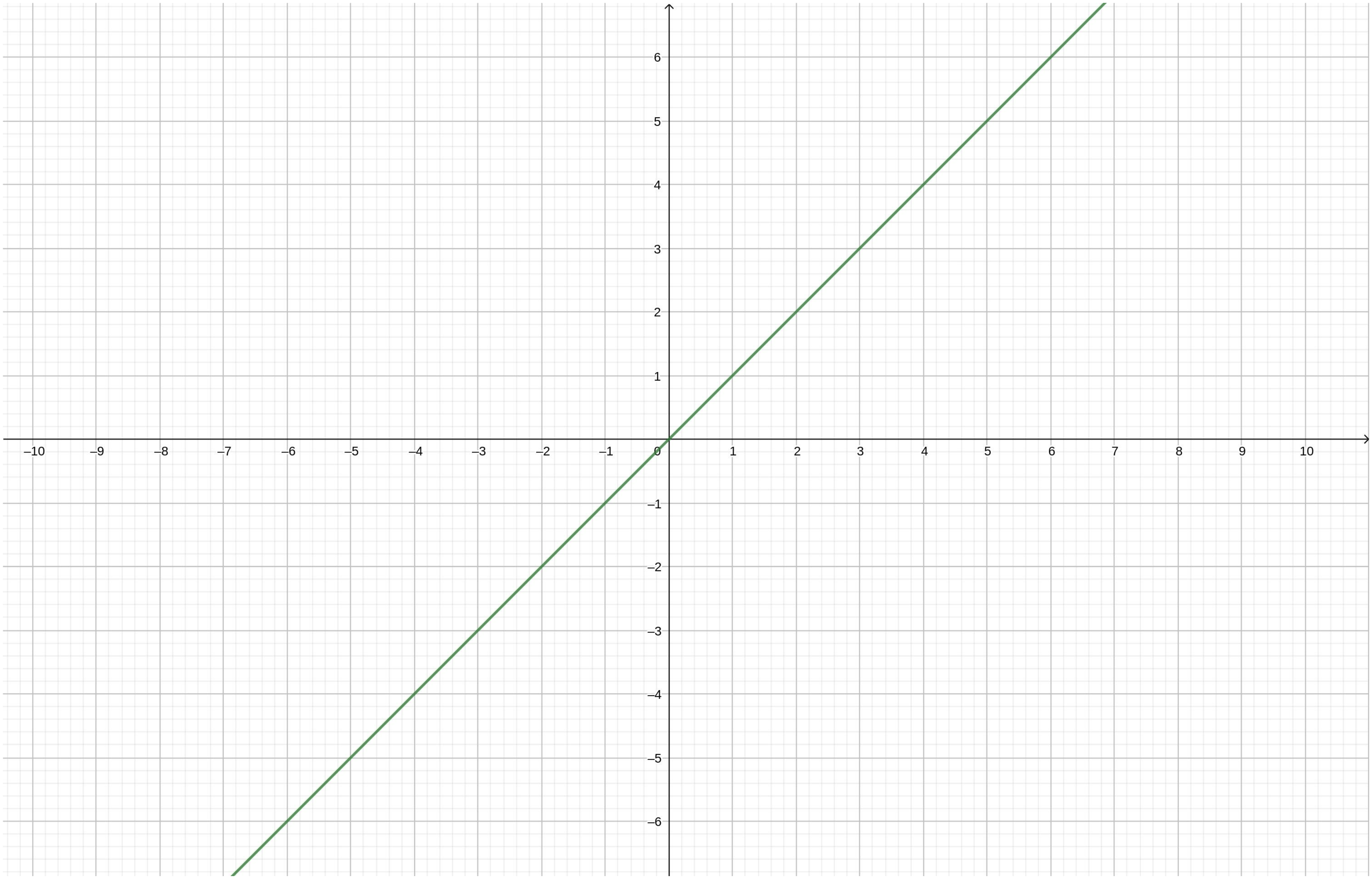
Here, the image ellipse flattens to the straight line:-

$$y = x$$

Condition number of A = infinite

Columns of A are linearly dependent, so, A is not invertible.

Determinant of $A = \varepsilon - 1 = 1 - 1 = 0$


–10
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2
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4
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8
9
10
6
5
4
3
2
1
–1
–2
–3
–4
–5
–6

$\varepsilon = 10^{-1}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 10^{-1} \end{pmatrix}$$

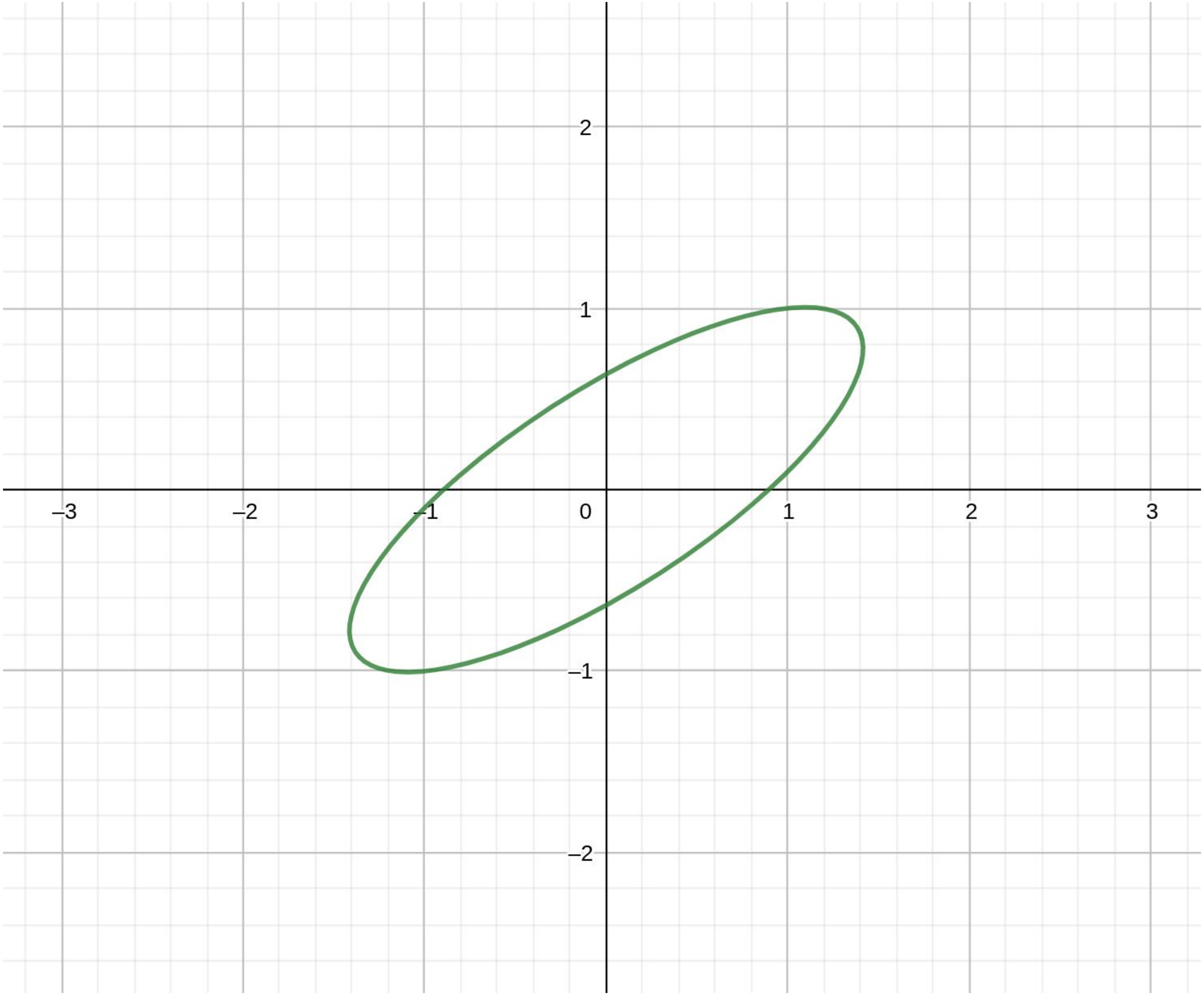
The image of the unit circle is the ellipse with the equation:-

$$1.01x^2 + 2y^2 - 2.2xy = 0.81$$

Condition number of $A = 3.0124$

As columns of A are linearly independent, A is invertible.

Determinant of $A = \varepsilon - 1 = 10^{-1} - 1 = -0.9$


2
1
–3
–2
–1
0
1
2
3
–1
–2

$\varepsilon = 10^{-2}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 10^{-2} \end{pmatrix}$$

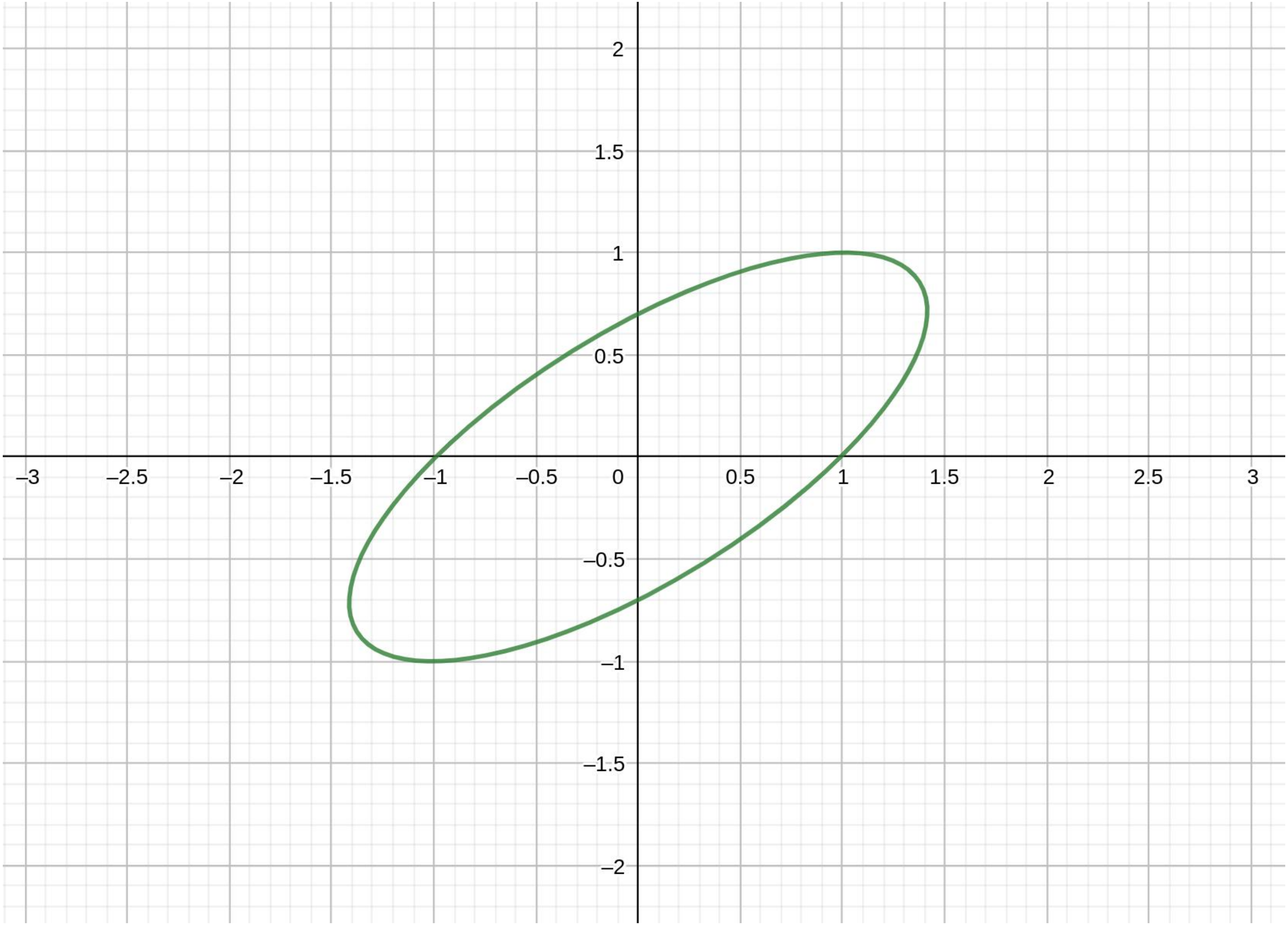
The image of the unit circle is the ellipse with the equation :-

$$1.0001\, x^2 + 2y^2 - 2.02\, xy = 0.9801$$

Condition number of $A = 2.6535$

As columns of A are linearly independent, A is invertible.

Determinant of $A = \varepsilon - 1 = 10^{-2} - 1 = -0.99$



2
1.5
1
0.5
–0.5
–1
–1.5
–2
–3
–2.5
–2
–1.5
–1
–0.5
0
0.5
1
1.5
2
2.5
3

$\underline{\varepsilon = 10^{-4}}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 10^{-4} \end{pmatrix}$$

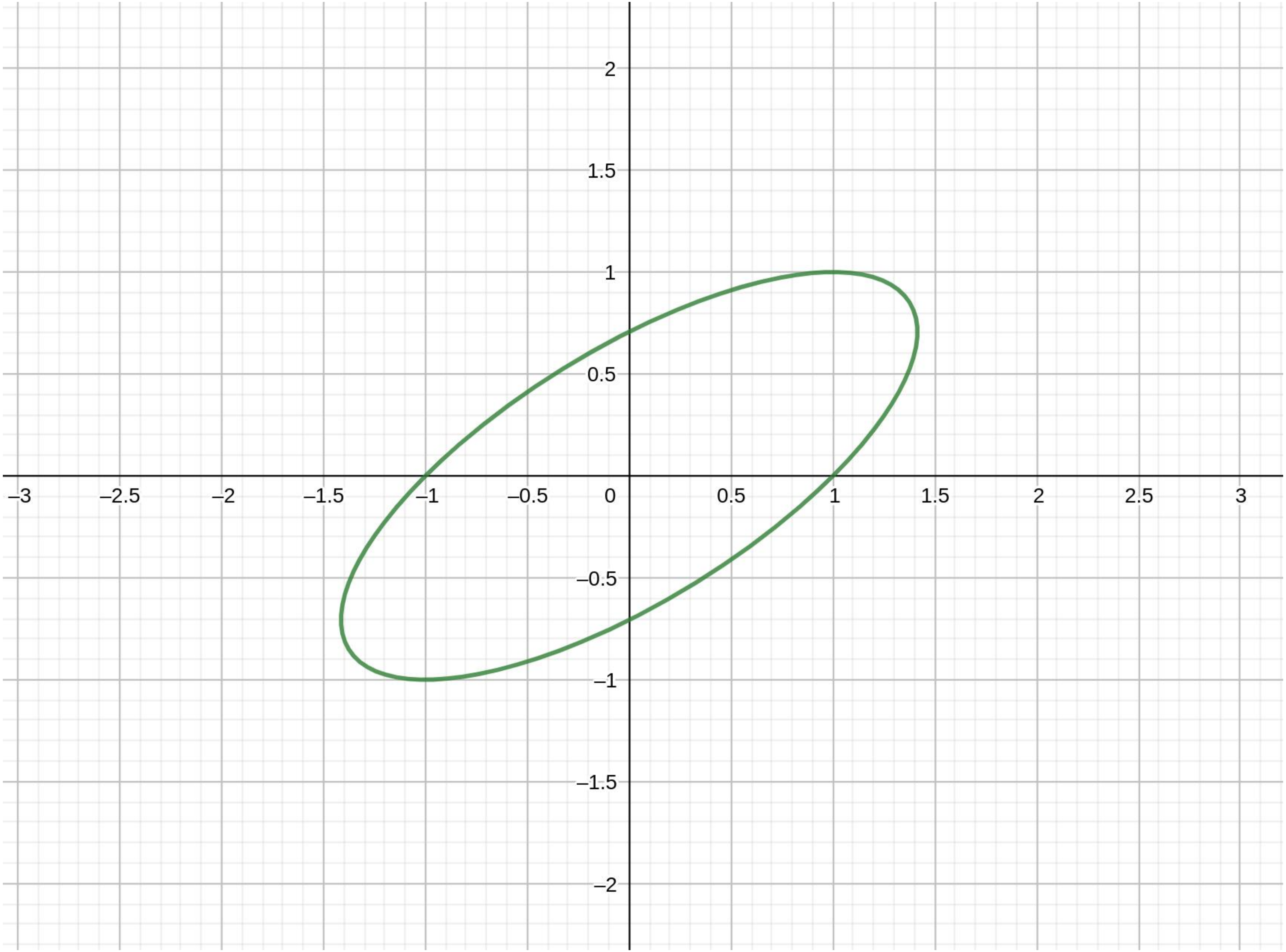
The image of the unit circle is the ellipse with the equation :-

$$1 \cdot 00000001\, x^2 + 2y^2 - 2 \cdot 0002\, xy = 0 \cdot 99980001$$

Condition number of $A = 2 \cdot 6183$

As columns of A are linearly independent, A is invertible.

Determinant of $A = \varepsilon - 1 = 10^{-4} - 1 = -0 \cdot 9999$


2
1.5
1
0.5
0
–0.5
–1
–1.5
–2
–3
–2.5
–2
–1.5
–1
–0.5
0.5
1
1.5
2
2.5
3

$\underline{\varepsilon = 0}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

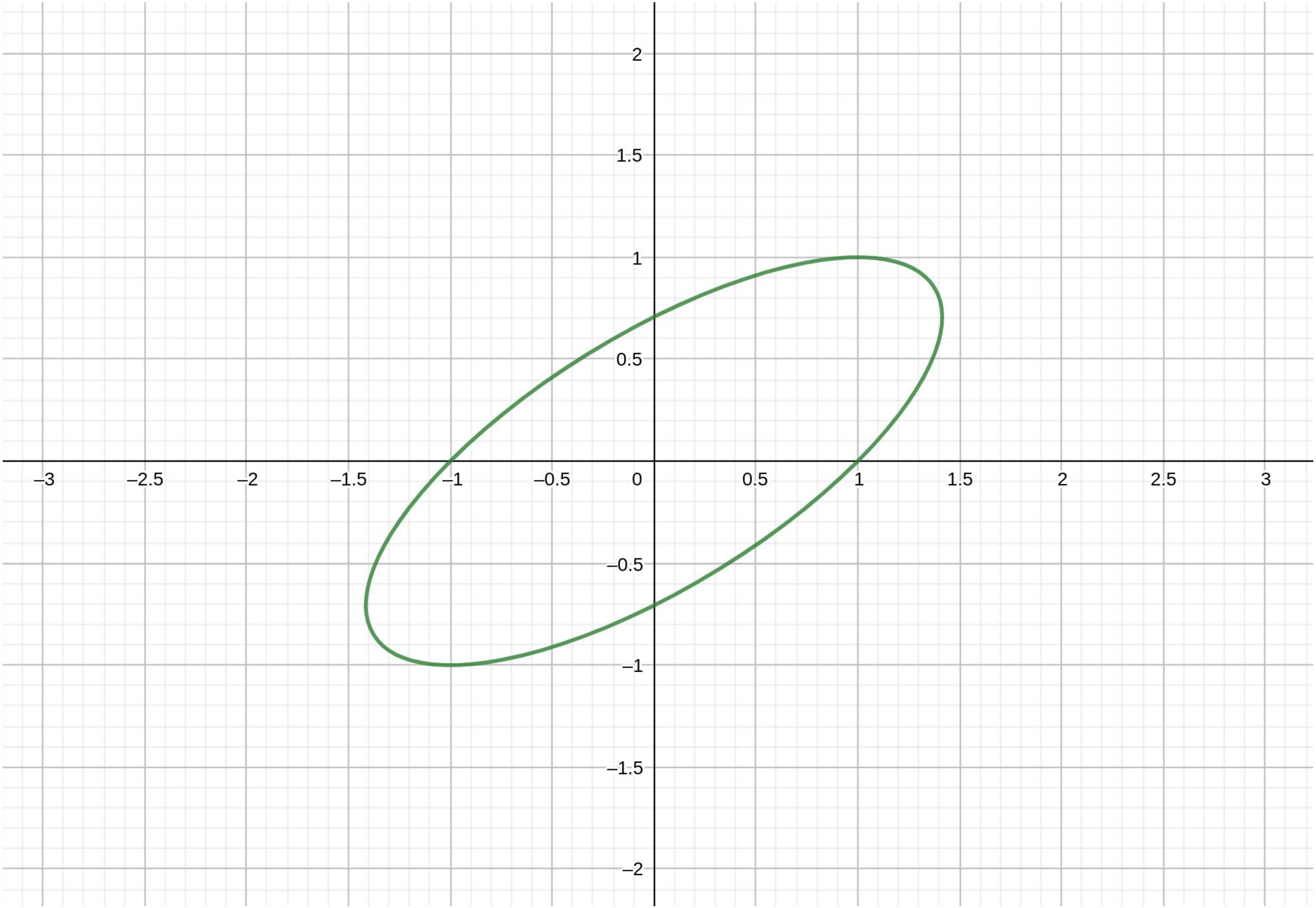
The image of the unit circle is the ellipse with the equation:-

$$x^2 + 2y^2 - 2xy = 1$$

Condition number of $A = 2.6180$

As columns of A are linearly independent, A is invertible.

Determinant of $A = \varepsilon - 1 = 0 - 1 = -1$


2
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1
0.5
0
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−1
−1.5
−2
−3
−2.5
−2
−1.5
−1
−0.5
0.5
1
1.5
2
2.5
3

Relation between determinant and condition number :-

If the determinant of A is very close to zero,
or "almost" zero,

$\Rightarrow$ matrix A is "almost singular".

$\Rightarrow$ columns of A are "almost linearly dependent".

$\Rightarrow$ Ax is "almost zero", for some x with $\|x\|_2 = 1$.

$\Rightarrow$ $\|Ax\|_2$ is "almost zero".

$\Rightarrow$ $\text{minmag}(A) \ll 1$

$\Rightarrow$ condition number of A is high.

Thus, we can say that when the determinant is close to zero, then the condition number is high.