Ashutosh Kumar Singh 19CS30008

2. Given $A \in R^{n \times n}$ and A is an invertible matrix.

Maximum magnification of A:-

$$\text{maxmag}(A) = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\|x\|_2 = 1} \|Ax\|_2$$

Minimum magnification of A:-

$$\text{minmag}(A) = \min_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \min_{\|x\|_2 = 1} \|Ax\|_2$$

We define the 2-norm of a matrix A as:-

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

Note that $\|A\|_2$ is same as $\text{maxmag}(A)$.

Now,

Condition number of A:-

$$\text{cond}(A) = \|A\|_2 \, \|A^{-1}\|_2$$

(a) To prove :- $\text{maxmag}(A) = \dfrac{1}{\text{minmag}(A^{-1})}$

We are given that A is non-singular (invertible)

So, $Ax = 0 \iff x = 0$.

Also, A is invertible $\Rightarrow$ A is one-one.

$$Ax = y \Rightarrow x = A^{-1}y.$$

So, $\text{maxmag}(A) = \max_{x \neq 0} \dfrac{\|Ax\|_2}{\|x\|_2}$

$$= \max_{y \neq 0} \frac{\|y\|_2}{\|A^{-1}y\|_2}$$

$$= \frac{1}{\min_{y \neq 0} \frac{\|A^{-1}y\|_2}{\|y\|_2}}$$

$$= \frac{1}{\text{minmag}(A^{-1})}$$

Hence, $\text{maxmag}(A) = \frac{1}{\text{minmag}(A^{-1})}$

(b) We know that

$$\text{cond}(A) = \|A\|_2 \, \|A^{-1}\|_2$$

Also, 2-norm of A is same as maxmag(A).

Hence,

$$\text{cond}(A) = \text{maxmag}(A) \cdot \text{maxmag}(A^{-1}) \quad \text{— (1)}$$

Now, from part (a), we saw that

$$\text{maxmag}(A) = \frac{1}{\text{minmag}(A^{-1})}$$

Replacing A by A^{-1}, we can write,

$$\text{maxmag}(A^{-1}) = \frac{1}{\text{minmag}(A)}$$

Hence, from (1),

$$\text{cond}(A) = \frac{\text{maxmag}(A)}{\text{minmag}(A)}$$