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Q1. $P_n(R)$ = set of all polynomials in x with real coefficients

(a) To prove that $P_n(R)$ is a vector space:-

Addition:-

Consider $u = a_0 + a_1 x + \dots + a_n x^n$

and $v = b_0 + b_1 x + \dots + b_n x^n$

① $u + v = (a_0 + b_0) + (a_1 + b_1)x + \dots + (a_n + b_n)x^n$

$= v + u$

Also, $u + v = v + u \in P_n(R)$.

② consider $w = c_0 + c_1 x + \dots + c_n x^n$

$(u+v) + w = \left((a_0 + b_0) + (a_1 + b_1)x_1 + \dots + (a_n + b_n)x^n\right)$

$+$

$(c_0 + c_1 x + \dots + c_n x^n)$

$= (a_0 + b_0 + c_0) + (a_1 + b_1 + c_1)x_1 + \dots + (a_n + b_n + c_n)x^n$

$= u + (v + w)$

③ Consider $0 \in P_n(R)$ where,

$0 = (0) + (0)x' + \dots + (0)x^n$

Now,

$u + 0 = a_0 + a_1 x + \dots + a_n x^n = 0 + u = u$

Hence, additive identity exists.

④ For $u = a_0 + a_1 x + \dots + a_n x^n$,

consider $p = (-a_0) + (-a_1)x + \dots + (-a_n)x^n \in P_n(R)$

Then,

$u + p = p + u = 0$

So, additive inverse also exists.

Scalar Multiplication

Consider $u = a_0 + a_1x + \cdots + a_nx^n$
and $v = b_0 + b_1x + \cdots + b_nx^n$

Also, take $\alpha, \beta \in R$.

① $\alpha(\beta u) = \alpha\left((a_0\beta) + (a_1\beta)x + \cdots + (a_n\beta)x^n\right)$

$= (a_0\alpha\beta) + (a_1\alpha\beta)x + \cdots + (a_n\alpha\beta)x^n$

Also, note that,

$(\alpha\beta)u = (a_0\alpha\beta) + (a_1\alpha\beta)x + \cdots + (a_n\alpha\beta)x^n$.

So, $\alpha(\beta u) = (\alpha\beta)u$.

② for $1 \in R$,

$1 \cdot u = a_0 + a_1x + \cdots + a_nx^n = u$

③ $(\alpha+\beta)u = (\alpha+\beta)(a_0 + a_1x + \cdots + a_nx_n^n)$

$= (\alpha+\beta)a_0 + (\alpha+\beta)a_1x + \cdots + (\alpha+\beta)a_nx^n$

$= (\alpha a_0 + \alpha a_1x + \cdots + \alpha a_nx^n) +$
$(\beta a_0 + \beta a_1x + \cdots + \beta a_nx^n)$

$= \alpha u + \beta u$

④ $\alpha(u+v) = \alpha\left((a_0+b_0) + (a_1+b_1)x + \cdots + (a_n+b_n)x^n\right)$

$= (\alpha a_0 + \alpha\beta_0) + (\alpha a_1 + \alpha b_1)x + \cdots +$
$(\alpha a_n + \alpha b_n)x^n$

$= \alpha(a_0 + a_1x + \cdots + a_nx_n) +$
$\alpha(b_0 + b_1x + \cdots + b_nx_n)$

$= \alpha u + \alpha v$

So, we can say that $P_n(R)$ is a vector space.

(b) $F : P_n(R) \to R$

$F(p(x)) = \frac{d}{dx} p(x)\Big|_{x=0}$

Considering $p(x) = a_0 + a_1 x + \dots + a_n x^n$,

we can see that $F(p(x)) = a_1$

Take $q(x) = b_0 + b_1 x + \dots + b_n x^n$,

then $F(q(x)) = b_1$

$$F(\alpha p(x) + \beta q(x)) = \frac{d}{dx}\Big((\alpha a_0 + \alpha a_1 x + \dots + \alpha a_n x^n) + (\beta b_0 + \beta b_1 x + \dots + \beta b_n x^n)\Big)\Big|_{x=0}$$

$$= \alpha a_1 + \beta b_1.$$

Now,

$\alpha F(p(x)) + \beta F(q(x)) = \alpha a_1 + \beta b_1$

So, $F(\alpha p(x) + \beta q(x)) = \alpha F(p(x)) + \beta F(q(x))$

As superposition property holds, we can say that F is a linear functional.

(c) Any general $p(x) = a_0 + a_1 x + \dots + a_n x^n$

So, the corresponding vector will be:-

$$p = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}, \quad p \text{ is a } (n+1) \text{ dimensional vector.}$$

$F(p) = a_1$

So, if we need to find vector q such that

$q^T p = a_1$

Take $q = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} \to (n-1)$ zeroes.

More formally, considering 1-indexing, we can say,

$q = e_2 \Rightarrow F(p(x)) = e_2^T p.$