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Q7. $A \in \mathbb{R}^{n \times n}$, A is invertible.

We want to write $A = LU$, where L is a lower triangular matrix and U is an upper triangular matrix.

So, we can think of this as transforming A into an $n \times n$ upper-triangular matrix U by introducing zeros below the diagonal, first in column 1, then in column 2, and so on.

This is done by subtracting multiples of each row from subsequent rows (similar to Gaussian elimination). This operation is equivalent to multiplying A by a sequence of lower triangular matrices on the left such that,

$$\underbrace{L_{n-1} L_{n-2} \cdots L_2 L_1}_{L^{-1}} A = U$$

We can visualize this process as follows:- Consider a 4×4 matrix as an example.

$$\underset{A}{\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix}} \xrightarrow{L_1} \underset{L_1 A}{\begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix}} \xrightarrow{L_2} \underset{L_2 L_1 A}{\begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix}}$$

$$\xrightarrow{L_3} \underset{L_3 L_2 L_1 A}{\begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{bmatrix}}$$

The k^{th} transformation L_k introduces zeros below the diagonal in column k by subtracting multiples of row k from rows $k+1, \ldots, n$

Now, we derive the general formulas for a $n \times n$ matrix. Assume x_k is the kth column of A at the beginning of step k.

$$x_k = \begin{bmatrix} x_{1k} \\ \vdots \\ x_{kk} \\ x_{k+1,k} \\ \vdots \\ x_{nk} \end{bmatrix} \xrightarrow{L_k} L_k x_k = \begin{bmatrix} x_{1k} \\ \vdots \\ x_{kk} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

For this, we subtract l_{jk} times row k from row j, where $l_{jk} = \dfrac{x_{jk}}{x_{kk}}$ $(k < j \leq n)$.

So, the matrix L_k looks like:-

$$L_k = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & -l_{k+1,k} & 1 & \\ & & \vdots & & \ddots & \\ & & -l_{nk} & & & 1 \end{bmatrix}$$

All other entries which are not shown are 0.

In this manner, we can compute $L_1, L_2, \ldots, L_{n-1}$. Then taking their inverses & multiplying them, we get L.

$$L = L_1^{-1} L_2^{-1} \cdots L_{n-1}^{-1} = \begin{bmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ l_{31} & l_{32} & 1 & & \\ \vdots & \vdots & & \ddots & \\ l_{n1} & l_{n2} & \cdots & l_{n,n-1} & 1 \end{bmatrix}$$

This is the final matrix L.

If we wish to compute U, we can do so by using $U = L^{-1} A$.

Now, the problem asks us to recognise each matrix L_{ij}.

The L_{ij} is nothing but just a decomposition of the elements of L_j into several matrices.

We can define L_{ij} as follows:-

All main diagonal elements are 1.

In the j^{th} column & i^{th} row $(i > j)$, the element is $-l_{ij}$, where $l_{ij} = \frac{x_{ij}}{x_{jj}}$

(same notation as used above)

And, all other elements in L_{ij} are zero.

Then we can write:-

$$L_j = L_{j+1,j} \cdot L_{j+2,j} \cdots L_{nj}$$

So, L_{ij} is defined only for $(j < i \leq n)$.

So,

$$L_{ij} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & 0 & 1 & & \\ & & \vdots & & \ddots & \\ & & -l_{ij} & & & \ddots \\ & & \vdots & & & \\ & & 0 & & & 1 \end{bmatrix}$$

i^{th} row, j^{th} column