I. 1) $C(t-\Delta t, S_0) = e^{-rT} q_d^m \sum \binom{m}{k} \left(\frac{q_u}{q_d}\right)^k C_T(S_k = ?)$

$S_0 e^{(2k-m)\sigma\sqrt{\Delta t} + m\alpha\Delta t}$

we subs. $\alpha = 0$ ..

$m = 1$,

$$C(t-\Delta t, S_0) = e^{-rT} q_d \times \left[\left(\frac{q_u}{q_d}\right)^0 C_T\left(S_0 = S_0 e^{-\sigma\sqrt{\Delta t}}\right) + \left(\frac{q_u}{q_d}\right)^1 C_T\left(S_1 = S_0 e^{\sigma\sqrt{\Delta t}}\right) \right]$$

$$C(t-\Delta t, S_0) = e^{-rT}\left[q_d C_T\left(S_0 e^{-\sigma\sqrt{\Delta t}}\right) + q_u C_T\left(S_0 e^{\sigma\sqrt{\Delta t}}\right) \right]$$

2. using Taylor's expansion for C_T,

$$C(t, S + \sigma\sqrt{\Delta t}) = C(t-\Delta t, S) + \frac{\partial C(t-\Delta t, S)}{\partial S}\sigma\sqrt{\Delta t} + \frac{\partial^2 C(t-\Delta t, S)}{\partial S^2}\frac{\sigma^2 \Delta t}{2} + O(t^{3/2})$$

$$C(t, S - \sigma\sqrt{\Delta t}) = C(t-\Delta t, S) - \frac{\partial C(t-\Delta t, S)}{\partial S}\sigma\sqrt{\Delta t} + \frac{\partial^2 C(t-\Delta t, S)}{\partial S^2}\frac{\sigma^2 \Delta t}{2} - O(t^{3/2})$$

.... as $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2} + \ldots$

& $x = S + \sigma\sqrt{\Delta t}$ & $a = S$

3)

Also from notes,

$$q_u = \frac{1}{2} + \frac{(r - y - \sigma^2/2)\sqrt{\Delta t}}{2\sigma} + O(\Delta t^{3/2})$$

$$q_d = \frac{1}{2} - \frac{(r - y - \sigma^2/2)\sqrt{\Delta t}}{2\sigma} - O(\Delta t^{3/2})$$

$$C(t-\Delta t, S_0) = \left\{ q_u \times C(t, S + \sigma\sqrt{\Delta t}) + q_d \times C(t, S - \sigma\sqrt{\Delta t}) \right\} \times e^{-r\Delta t}$$

$$= e^{-r\Delta t}\left(\frac{1}{2} + \frac{(r - y - \sigma^2/2)\sqrt{\Delta t}}{2\sigma} + O(\Delta t^{3/2}) + \cdots\right)$$

$$\left(C(t, S_0) + \frac{\partial C}{\partial S}(t, S)\,\sigma\sqrt{\Delta t} + \frac{\partial^2 C(t,S)}{\partial S^2}\,\frac{\sigma^2 \Delta t}{2} + O(\Delta t^{3/2}) \right)$$

$$+$$

$$e^{-r\Delta t}\left(\frac{1}{2} - \frac{(r - y - \sigma^2/2)\sqrt{\Delta t}}{\sigma} - O(\Delta t^{3/2}) \cdots\right)$$

$$\left(C(t,S) - \frac{\partial C}{\partial S}(t,S)\,\sigma\sqrt{\Delta t} + \frac{\partial^2 C(t,S)}{\partial S^2}\,\frac{\sigma^2\Delta t}{2} - O(\Delta t^{3/2}) \right)$$

$$= e^{-r\Delta t}\left[C(t,S) + \frac{\partial^2 C(t,S)}{\partial S^2}\,\frac{\sigma^2 \Delta t}{2} + \left(r - y - \sigma^2/2\right)\Delta t \frac{\partial C}{\partial S} + O(\Delta t^{3/2}) \right]$$

using expansion for $e^{-r\Delta t}$

$$e^{-r\Delta t} = 1 - r\Delta t + \frac{1}{2} r^2 \Delta t^2 - \frac{1}{6} r^3 \Delta t^3 + O(\Delta t^4)$$

$$C(t-\Delta t, S)$$

$$= \left(1 - r\Delta t + \frac{1}{2} r^2 \Delta t^2 - \frac{1}{6} r^3 \Delta t^3 + O(\Delta t^4) \cdots \right)$$

$$\left(C(t, S) + \frac{\partial^2 C(t,S)}{\partial^2 S^2} \frac{\sigma^2 \Delta t}{2} + (r - y - \sigma^2/2) \Delta t \frac{\partial C}{\partial S} + O(\Delta t^{3/2}) \right)$$

$$= \left[1 - r\Delta t + \frac{r^2 \Delta t^2}{2} + O(\Delta t^3)\right] \left[C_t + \frac{\sigma^2}{2} C_{SS} \Delta t + (r - y - \sigma^2/2) \Delta t C_S + O(\Delta t^{3/2})\right]$$

$$= C_t + \frac{\sigma^2 C_{SS}}{2} \Delta t + (r - y - \sigma^2/2) C_S \Delta t$$

$$- r C_t \Delta t + O(\Delta t^{3/2}) \cdots$$

$$= C_t - r C_t \Delta t + \frac{\sigma^2 C_{SS}}{2} \Delta t + (r - y - \sigma^2/2) C_S \Delta t + O(\Delta t^{3/2})$$

$$= C_t + \left(r C_S - y C_S - \frac{\sigma^2}{2} C_S + \frac{\sigma^2 C_{SS}}{2}\right) \Delta t + O(\Delta t^{3/2})$$

$$\therefore C(t-\Delta t, S) = C_t + \left(r C_S - y C_S - \frac{\sigma^2 C_S}{2} + \frac{\sigma^2 C_{SS}}{2}\right) \Delta t + O(\Delta t^{3/2})$$

4. Dividing by Δt,

$$\frac{C(t-\Delta t, S)}{\Delta t} = \frac{C(t,S)}{\Delta t} + (r - y - \sigma^2/2) \frac{\partial C(t,S)}{\partial S} + \frac{\sigma^2}{2} \frac{\partial^2 C(t,S)}{\partial S^2} + O(\Delta t^{1/2})$$

rearranging,

$$\frac{C(t-\Delta t, S) - C(t,S)}{\Delta t} = \left(r - y - \sigma^2/2\right)\frac{\partial C(t,S)}{\partial S} + \sigma^2/2\,\frac{\partial^2 C(t,S)}{\partial S^2} + o(\Delta t^{1/2})$$

$$\frac{\Delta C(t,S)}{\Delta t} = \left(r - y - \sigma^2/2\right)\frac{\partial C(t,S)}{\partial S} + \frac{\sigma^2}{2}\frac{\partial^2 C}{\partial S^2}(t,S) + o(\Delta t^{1/2})$$