Assignment 2: Suggested solutions

The final answers in the computations below are exact (using the accompanied Python code). Note that it is easy to get rounding errors if computed in steps.

1.

   (a) The relevant rate is the two-year forward one year from today, i.e., $f_{1,3}$.

   (b) The forward rate is given by:

$$(1+f_{1,3})^2 = \frac{(1+r_3)^3}{1+r_1} \Rightarrow f_{1,3} = 4.015\%.$$

   (c) We can replicate the forward transaction by a portfolio of one- and three-year zero-coupon bonds. Suppose that we buy x_1 units of one-year bonds and x_3 units of three-year bonds. The cash flows of this portfolio are:

Year 0:	$-x_1\frac{100}{1+r_1} - x_3\frac{100}{(1+r_3)^3}$
Year 1:	$100x_1$
Year 3:	$100x_3$

   Since the cash flow from the forward transaction in year zero is zero,

$$-x_1\frac{100}{1+r_1} - x_3\frac{100}{(1+r_3)^3} = 0.$$

   Since the forward transaction generates an outflow of 1 million in year 1,

$$x_1 100 = -1,000,000.$$

   Solving this system, we find $x_1 = -10,000$ and $x_3 = 10,819$. Thus, we can replicate the forward transaction by shorting $10,000$ units of the one-year bond and buying $10,819$ units of the three-year bond.

   Note that in year 3 we receive $100x_3 = 1,081,908$, which is exactly the same as $1,000,000 \times (1+f_{1,3})^2$.

2.

   (a) For the 1% coupon bonds, $c = 0.5$, and for the 4% bonds, $c = 2$. For all the bonds the current yield is 3%, therefore $y_{10} = y_5 = 0.03$. To find the price of the 10-year, 4% coupon bond, we compute:

$$P_{10} = \frac{2}{\left(1+\frac{0.03}{2}\right)} + \frac{2}{\left(1+\frac{0.03}{2}\right)^2} + \cdots + \frac{2}{\left(1+\frac{0.03}{2}\right)^{20}} + \frac{100}{\left(1+\frac{0.03}{2}\right)^{20}} = 108.584.$$

A similar computation gives the price of the 5-year, 4% coupon bond as 104.611. One can compute the prices of the 1% coupon bonds by analogy. Thus the prices of the bonds are:

Bond A	Bond B	Bond C	Bond D
90.778	82.831	104.611	108.584

(b) Repeat the exercise above. For the 10-year, 4% coupon bond, the price is now going to be:

$$P_{10} = \frac{2}{\left(1+\frac{0.035}{2}\right)} + \frac{2}{\left(1+\frac{0.035}{2}\right)^2} + \cdots + \frac{2}{\left(1+\frac{0.035}{2}\right)^{20}} + \frac{100}{\left(1+\frac{0.035}{2}\right)^{20}} = 104.188.$$

Similar calculations apply for the other 3 bonds. The prices after a change in yields are:

Bond A	Bond B	Bond C	Bond D
88.623	79.059	102.275	104.188

To compute the price changes (in percentage form), we compute (new price–old price)/old price. For example, for bond A, the percentage change is (88.623-90.778)/90.778 = –0.023732 = –2.37%. Repeating this for the other bonds gives us:

Bond A	Bond B	Bond C	Bond D
–2.37%	–4.55%	–2.23%	–4.05%

(c) Again, we repeat the same exercise as in parts (a) and (b) and find that:

Bond A	Bond B	Bond C	Bond D
92.991	86.801	107.009	113.199

Computing the price changes, we have that:

Bond A	Bond B	Bond C	Bond D
2.44%	4.79%	2.29%	4.25%

(d) The (modifed) duration of the 10-year, 4% coupon bond is:

$$D_{10} = \frac{1}{108.584} \frac{1}{\left(1+\frac{0.03}{2}\right)} \left[\frac{2 \times 0.5}{\left(1+\frac{0.03}{2}\right)} + \frac{2 \times 1}{\left(1+\frac{0.03}{2}\right)^2} + \cdots + \frac{(100+2) \times 10}{\left(1+\frac{0.03}{2}\right)^{20}} \right] = 8.295.$$

A similar computation gives the duration of the 5-year, 4% coupon bond as 4.525. One can compute the durations of the 1% coupon bonds by analogy. Thus the durations of the bonds are:

Bond A	Bond B	Bond C	Bond D
4.810	9.342	4.525	8.295

The estimated change in price ΔP given a change in yield of Δy is given by

$$\Delta P = -D \times P \times \Delta y.$$

Therefore, if yields increase from 3% to 3.5%, the estimated new prices are:

Bond A	Bond B	Bond C	Bond D
88.594	78.962	102.244	104.081

If yields decrease from 3% to 2.5%, the estimated new prices are:

Bond A	Bond B	Bond C	Bond D
92.961	86.700	106.978	113.088

(e) What can you conclude:

- The longer the maturity, the larger the price response. Note that in both scenarios (yields going up and down), the response was greater in bond B than A and in Bond D than C. B and D have longer maturities.
- The lower the coupon (holding maturity constant), the larger the price reaction. In both cases, bond A (lower coupon than C) had a bigger response than bond C and bond B had a bigger response than D.
- Duration is a measure that takes into account coupons and times to maturity and it allows us to compare the risk in two bonds without explicit reference to the coupon and maturity structure of the bonds.
- Price changes are asymmetric for symmetric change in yields, reflecting the *convexity* of the bond price.

3.

(a) Suppose that we buy x_A units of bond A, x_B units of B, and x_C units of C. Then,

$$\begin{aligned} 100 \times x_A &= 2,000,000 \\ 4 \times x_B + 6 \times x_C &= 0 \\ 104 \times x_B + 106 \times x_C &= 1,000,000. \end{aligned}$$

Solving this system, we find $x_A = 20,000$, $x_B = 30,000$, and $x_C = -20,000$.

(b) Suppose that we buy x_D units of the 10-year bond (bond D) and x_E units of the 15-year bond (bond E). The present value of our liability is

$$P_L = \frac{1,000,000}{(1+5\%)^{30}} + \frac{2,000,000}{(1+5\%)^{31}} = 672,096.$$

The Macaulay duration of our liability is

$$D_L^{Macaulay} = \frac{\frac{1,000,000}{(1+5\%)^{30}}}{672,096} \times 30 + \frac{\frac{2,000,000}{(1+5\%)^{31}}}{672,096} \times 31 = 30.66,$$

and the modified duration is $D_L = 30.66/(1+5\%) = 29.20$. The prices of the two bonds are

$$P_D = \frac{100}{(1+5\%)^{10}} = 61.39$$

and

$$P_E = \frac{100}{(1+5\%)^{15}} = 48.10,$$

and their modified durations are

$$D_D = \frac{10}{1+5\%} = 9.52$$

and

$$D_E = \frac{15}{1+5\%} = 14.29.$$

The system of equations is

$$P_L = x_D P_D + x_E P_E$$

and

$$P_L D_L = x_D P_D D_D + x_E P_E D_E.$$

Plugging in and solving, we find $x_D = -34,279$ and $x_E = 57,722$.

(c) The hedging portfolio would change because the duration of our liability would change. The duration would increase because the 2M payment would receive higher weight. Therefore, we should rebalance our portfolio, buying more units of the longer-duration 15-year bond, and shorting more units of the 10-year bond.

(d) The advantage of synthetic replication is that it gives us perfect hedging. For example, we do not need to rebalance our portfolio. On the other hand, synthetic replication might not be feasible. For example, a 31-year bond might not be available in the market, or it might not be particularly liquid.

4.

(a) Plug into the respective formulas from the slides. For the price, we can also just notice that a risk-free bond with a coupon rate equal to the yield is issued at par.

$$
\begin{aligned}
P_1 &= \frac{100+c}{1+y} = 100 \\
D_1 &= \frac{1}{P_1} \times \frac{1}{1+y} \times \frac{100+c}{1+y} = 0.98 \\
C_1 &= \frac{1}{P_1} \times \frac{1}{(1+y)^2} \times \frac{2 \times (100+c)}{1+y} = 1.92
\end{aligned}
$$

(b) As before:

$$
\begin{aligned}
P_2 &= \frac{c}{1+y} + \frac{100+c}{(1+y)^2} = 100 \\
D_2 &= \frac{1}{P_2} \times \frac{1}{1+y} \times \left[\frac{c}{1+y} + \frac{2 \times (100+c)}{(1+y)^2}\right] = 1.942 \\
C_2 &= \frac{1}{P_2} \times \frac{1}{(1+y)^2} \times \left[\frac{2 \times c}{1+y} + \frac{6 \times (100+c)}{(1+y)^2}\right] = 5.692
\end{aligned}
$$

(c) You can replicate the 2-year FRN with two sequentially purchased 1-year par bonds. At time 0, buy a 1-year par bond for 100. (This is similar to establishing the fixed interest rate swap rate discussed in class.) At time 1 collect $100 + 100 \times y_1$. Further, buy another 1-year par bond for 100. At time 2 collect $100 + 100 \times y_2$, where y_2 is the coupon payment locked in at year 1. Thus, the price of the FRN is 100.

(d) The easiest way to answer this question is to recognize that changes in the yield at time $t = 0$ do not affect the value of the note at time $t = 1$. That is, the value after the first coupon payment is always 100. Then the duration and convexity are the same as in part (a). Thus no new calculations are required to answer this question.

If we did not realize this, we need to take the respective derivatives and plug in. Note which values are constants (locked in) versus those which change with the yield curve. Also use the solution from part (c) to denote the price as equal to the face value. The effective modified duration is then given by:

$$
\begin{aligned}
D &= \frac{-1}{P} \times \frac{d}{dy}\left[\frac{y_1 \times 100}{1+y} + \frac{100 + y100}{(1+y)^2}\right] \\
&= \frac{-1}{100} \times \left[\frac{-y_1 \times 100}{(1+y)^2} - \frac{2 \times 100 \times (1+y)}{(1+y)^3} + \frac{100}{(1+y)^2}\right] \\
&= \frac{y_1 - 1}{(1+y)^2} + \frac{2 \times (1+y)}{(1+y)^3} \\
&= \frac{1 + y_1}{(1+y)^2} \\
&= 0.98
\end{aligned}
$$

Plugging in $y = y_1$, the above expression is equivalent to that of the 1-year bond.

Take the second derivative for the convexity. Note the reversed sign relative to the duration for the first derivative.

$$
\begin{aligned}
C &= \frac{1}{P}\frac{d^2}{dy^2}\left[\frac{y_1 \times 100}{1+y} + \frac{100 + y100}{(1+y)^2}\right] \\
&= \frac{d}{dy}\left[\frac{-1 - y_1}{(1+y)^2}\right] \\
&= \frac{2 \times (1 + y_1)}{(1+y)^3} \\
&= 1.92
\end{aligned}
$$

Again, since $y = y_1$, we get the expression for the convexity of a one period bond.

(e) As discussed above, the FRN duration and convexity is the same as the one-year bond. This is because the value of the second period payments is not sensitive to changes in the yield curve.

5.

(a) Share price = 2/(0.08–0.04) = 50.

(b) Retention rate (plowback ratio) = 1 – dividend payout ratio = 1 – 2/4 = 0.5. Sustainable growth = return on equity × retention rate = 0.08×0.5 = 0.04 = 4%. So, the expected growth rate of 4% is sustainable with this level of return on equity and payout.

(c) Dividend yield = 2/50 = 4%.

(d) Dividend yield plus growth rate = 0.04 + 0.04 = 8%. So, the investor will receive her cost of equity in the form of a dividend yield of 4% plus expected growth in the share price of 4%. Note that this is the way the constant growth model always works: cost of equity = expected dividend yield + expected growth rate.

(e) With a 100% payout the retention rate is zero, so the sustainable growth rate is zero.

(f) The dividend will be 4 and the growth rate zero, so the share price will be 4/0.08 = 50.

(g) Note that the share price is the same in (a) and (f). This is because the return on equity is equal to the cost of equity. So, the NPV of any investment the firm makes using retained earnings is zero. That is, the value of the share is not increased or decreased by retaining and reinvesting earnings. This illustrates the importance that we have discussed of having positive NPV investment in order to generate true growth for shareholders.