MFE 406 – Problem Set 4

Due: Fri 21 Feb 2024, 11:59 pm PDT

1. **Black-Scholes-Merton formula: Sensitivities and Hedging**

   For simplicity and clarity, please use a flat BSM volatility of 15.5% for all options.

   Additional parameters are: spot price is $S_0 = 5000$, $T = 1yr$, $r = 490bp$, *and* $y = 1.10\%$.

   NB: In parts (a) and (b), **Derive** means to begin with the BS(M) formula – or some intermediate result thereof – and thereby obtain the desired result rather than simply copying it from the lecture notes.

   (a) **Derive** expressions for the put sensitivities:

   $$\Delta_S = \frac{\partial P}{\partial S}, \Delta_K = \frac{\partial P}{\partial K};$$
   $$\Gamma_{SS} = \frac{\partial^2 P}{\partial S^2}, \Gamma_{SK} = \frac{\partial^2 P}{\partial S \partial K} = \Gamma_{KS} = \frac{\partial^2 P}{\partial K \partial S}, \Gamma_{KK} = \frac{\partial^2 P}{\partial K^2}.$$

   i. What are the units of each of these?
   What are their numerical values for a 1-year ATM ($K = 5000$), 10% ITM ($K = 5500$), and 10% OTM ($K = 4500$) European SPX put options?

   ii. How do the expressions $S\Delta_S + K\Delta_K$, $S\,\Gamma_{SS} + K\,\Gamma_{SK}$, and $S\,\Gamma_{KS} + K\,\Gamma_{KK}$ relate to (any of) the previously derived values and sensitivities?

   (b) **Derive** expressions for the sensitivities:

   $$\rho_r\,(\mathrm{rho_r}) = \frac{\partial P}{\partial r}, \rho_y\,(\mathrm{rho_y}) = \frac{\partial P}{\partial y}, v\,(\mathrm{vega}) = \frac{\partial P}{\partial \sigma}\ .$$

   What are the units of each of these?

   What are their numerical values for the vanilla 1-year ATM, 10% ITM, and 10% OTM SPX puts?

   (c) Now, calculate the ratios (both analytically and numerically):

   $$\frac{\rho_y}{S\Delta_S}, \frac{\rho_r}{K\Delta_K}, \frac{\frac{\partial P}{\partial(\sigma^2/2)}}{S^2\Gamma_{SS}} = \frac{v}{\sigma S^2\Gamma_{SS}}, \frac{v}{\sigma SK\Gamma_{SK}}, \frac{v}{\sigma K^2\Gamma_{KK}}\ .$$

   We examine the BSM replication approach in the remaining parts of the problem.

   (d) Assume that we seek to Delta-hedge/replicate the 4500 strike put in the spot market, i.e., using spot SPX and cash (a money-market account).

   i. What are the holdings Δ_S and $\Delta_\$ = K\Delta_K$?

   ii. Form a portfolio Π consisting of long 1 unit of P_{4500} and short Δ_S & $\Delta_\$$ units of spot SPX & cash, respectively. What is the (present) value of portfolio Π?

   iii. Verify that your portfolio is both SPX-Delta and Cash-Delta neutral, i.e.: $\dfrac{\partial \Pi}{\partial S} = 0$ and $\dfrac{\partial \Pi}{\partial K} = 0$.

iv. Next, determine the *approximate* sensitivities of the portfolio Π to small shifts $\delta r = 1bp$, $\delta y = 1bp$, and $\delta\sigma = 0.01$ in r, y, and σ, respectively, using your values for ρ_r, ρ_y, and $\boldsymbol{v}$.

(e) Alternatively, consider hedging the 10% OTM put in the $T-$forward market, i.e., using zero-coupon bonds (ZCBs) 1_T and SPX forwards F_T with present values $B_T = e^{-rT}$ and $B_T F_T$, respectively.

i. What holdings Δ_{F_T} and Δ_{1_T} correspond to Δ_S and $\Delta_\$$ in part (c)?

ii. Form a portfolio Π consisting of long 1 unit of P_{4500} and short Δ_{F_T} & Δ_{1_T} units of SPX forwards & ZCBs, respectively. What is the present value of portfolio Π?

iii. Verify that your portfolio is both SPX-Delta and Cash-Delta neutral, i.e.: $\dfrac{\partial\Pi}{\partial S} = 0$ and $\dfrac{\partial\Pi}{\partial K} = 0$.

iv. Once more, determine the *approximate* sensitivities of the portfolio Π to small shifts $\delta r = 1bp$, $\delta y = 1bp$, and $\delta\sigma = 0.01$ in r, y, and σ, respectively, using your values for ρ_r, ρ_y, and $\boldsymbol{v}$.

How do your results compare to those in part (d)?

(f) Finally, consider a gamma hedging strategy for the 10% OTM put, using the ATM put in addition to the $T-$forward instruments in part (e).

i. What $\Delta_{P_{atm}}$ – using a slight abuse of notation – matches the Γ_{SS} sensitivity of the 10% OTM put?

ii. What holdings Δ_{F_T} and Δ_{1_T} match the Δ_S and $\Delta_\$$ sensitivities of a $\{$long P_{4500}, short $\Delta_{P_{atm}} \cdot P_{atm}\}$ position?

iii. Form a portfolio Π consisting of long 1 unit of P_{4500} and short $\Delta_{P_{atm}}$, Δ_{F_T}, and Δ_{1_T} units of ATM puts, SPX forwards, and ZCBs, respectively.
What is the present value of portfolio Π?

iv. Verify that your portfolio is SPX-Delta, Cash-Delta, and SPX-Gamma neutral, i.e.: $\dfrac{\partial\Pi}{\partial S} = 0$, $\dfrac{\partial\Pi}{\partial K} = 0$, and $\dfrac{\partial^2\Pi}{\partial S^2} = 0$.

v. Again, determine the *approximate* sensitivities of the portfolio Π to small shifts $\delta r = 1bp$, $\delta y = 1bp$, and $\delta\sigma = 0.01$ in r, y, and σ, respectively, using your values for ρ_r, ρ_y, and $\boldsymbol{v}$.
How do your results compare to those in parts (d) & (e)?
What do you conclude about how one might optimize the robustness of a hedging strategy? How are the part (c) relationships relevant to that conclusion?

vi. *Qualitatively*, how do you *expect* your part (f)v results to change if you were to gamma hedge with an ATM put with a maturity $T' \lesseqgtr T$ instead of $T' = T$?

2. Binary Option Pricing & Replication

Consider the SPX options based on the market data we have examined in class, for which the time horizon is $T = 1.0$ years. Assume, as in the previous problem, that the ATM volatility $\sigma^0_{imp} = 15.5\%$. Furthermore, near the money, implied volatility behaves as:

$$\sigma_{imp} \approx 15.5\% + 30.0\% \ln(S/K)\,.$$

For the SPX price and dividend yield, as well as the risk-free interest rate, please use: spot SPX price is $S_0 = 5000$, $r = 490$bp, and $y = 1.10\%$.

(a) Value in the Black-Scholes-Merton world (i.e., assuming that $\sigma = 15.5\%\,\forall K$):

  i. a vanilla ATM European put.

  ii. an ATM binary cash-or-nothing put, paying out \$1 if $S_T \geq S_0$;

  iii. an ATM binary asset-or-nothing put;

(b) Using the Breeden-Litzenberger representation of a cash-or-nothing binary put value as dP/dK, value using the form of σ_{imp} above:

  i. a vanilla ATM European put.

  ii. an ATM binary cash-or-nothing put, paying out \$1 if $S_T \geq S_0$;

  iii. an ATM binary asset-or-nothing put;

  What can you say about the impact of volatility skew on your answers in (b) vs. those in (a)? Is this a general property?

(c) Assuming that European options (both calls and puts) are traded with infinite liquidity at discrete strikes corresponding to multiples of 25, e.g. {..., 4950, 4975, 5000, 5025, 5050, ...}, value using the form of σ_{imp} above:

  i. the portfolio of vanilla options that minimally super-replicates the cash-or-nothing put (i.e., the portfolio of options that costs least but still pays out at least as much as the binary option in all states of nature);

  ii. the portfolio of vanilla options that maximally sub-replicates the cash-or-nothing put (i.e., the portfolio of options that costs most but still pays out no more than the binary option in all states of nature).

  How tight are these bounds?

(d) Repeat part (c) assuming that the set of available strikes corresponds to multiples of 5, e.g. {..., 4990, 4995, 5000, 5005, 5010, ...}, as would be typically be the case for listed near-the-money SPX options once T $\lesssim$ 3 months or so (after which the discretization would be 5 or 10 points out to T $\lesssim$ 6 months or so).

  i. the minimally super-replicating portfolio;

  ii. the maximally sub-replicating portfolio

  Discuss your results in relation to your answers to parts (a)ii, (b)ii, and (c).

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3. **Listed Options Data, Put-Call Parity, and Implied Volatility**

The original data set (spx_quotedata-2024-02-08_NoZerosNoW.csv) for SPX European options with monthly expiries as of 08-Feb-2024's close is being provided to you along with this problem statement.

The first two lines of the csv file contain information about the SPX close, while the third line provides labels for each column.

Your work group should examine one (and only one) expiry's data as follows. Your index value is: $\mathrm{mod}\,(2 \times \text{your cohort number} + \text{your team number}, 5)$. Then, your expiry is:

Index	Expiry
0	21-Jun-2024
1	19-Jul-2024
2	20-Sep-2024
3	18-Oct-2024
4	15-Nov-2024

(a) Determine T, the time to expiry (from 08-Feb-2024), in years. NB: It doesn't matter much ***how*** you calculate T, as long as you use that value consistently throughout.

(b) Fit the unweighted (homoskedastic) parity least-squares regression $C_{K,mid} - P_{K,mid} = R_T(F_T - K) + \epsilon_K$ for all T-expiry data points, where $C_{K,mid}$ and $P_{K,mid}$ are each defined as the arithmetic mean of the corresponding bid and ask values and ϵ_K are assumed to be i.i.d. random, zero-mean, normal samples.

   i. Report R_T, F_T, and the regression statistics as shown on page 21 of Part 3 of the lecture notes.

   ii. Plot your regression residuals vs. K and comment on any apparent heteroskedasticity.

   iii. What values of r_{imp} and y_{imp} correspond to $\{R_T, F_T\}$ given your value of T?

   iv. What implied volatilities $\sigma_{imp,atm,T}$ and total variances $\Sigma_{imp,atm,T} = \sigma^2_{imp,atm,T}T$ for ATM calls and puts correspond to the respective mid prices, given your estimated values for r_{imp} and y_{imp}?
   Hint: use your results from question 1(b) as a first guess for numerical solution using Newton's method (i.e., Newton-Raphson iteration).

   v. What uncertainties around the mid-point call and put ATM implied volatilities result from the corresponding bid/ask spreads?

(c) It can reasonably be argued that option prices in the vicinity of ATM embed the most information for a given maturity T, given that they are the most liquid and also have the highest vegas.
Repeat your part (b) estimation, restricting K values to $5000 \times (1 \pm 0.2\sqrt{T})$.

   i. Report R_T, F_T, and the regression statistics as shown on page 21 of Part 3 of the lecture notes.

   ii. Plot your regression residuals vs. K and comment on any apparent heteroskedasticity.

iii. What values of r_{imp} and y_{imp} correspond to $\{R_T, F_T\}$ given your value of T?

iv. What implied volatilities $\sigma_{imp,atm,T}$ and total variances $\Sigma_{imp,atm,T} = \sigma^2_{imp,atm,T}T$ for ATM calls and puts correspond to the respective mid prices, given your estimated values for r_{imp} and y_{imp}?

v. What uncertainties around the mid-point call and put ATM implied volatilities result from the corresponding bid/ask spreads?

vi. Compare your results to those for part (b).

(d) Now consider the data model for a particular K level:

$\{C_K \sim N\left[C_{K,mid}, (C_{K,ask} - C_{K,bid})^2\right], P_K \sim N\left[P_{K,mid}, (P_{K,ask} - P_{K,bid})^2\right]\}$,

with errors assumed uncorrelated, subject to the additional constraint:

$C_K - P_K = R_T(F_T - K)$, conditional on specified values for $\{R_T, F_T\}$.

Form the weighted least-squares problem, minimizing the inverse-variance-weighted sum-of-squared errors between the call and put values and their corresponding mid-values, subject to the parity constraint.

Solve the minimization problem in closed form, thereby determining the parity-corrected call and put values $\{C^*_{K,mid}, P^*_{K,mid}\}$.

(e) Substitute your $\{C^*_{K,mid}, P^*_{K,mid}\}$ results from part (d) into the minimization problem and simplify, thereby obtaining the (weighted) squared error for a given K.

What weight(s) do you obtain for the resulting $\{C^*_{K,mid}, P^*_{K,mid}\}$ pair?

Sum these weighted squared errors over all K to obtain a weighted least squares regression for $\{R_T, F_T\}$. Note that the ϵ_K can now be considered to be inversely proportional to the square-root of the weight(s) you just derived; that is, this is now a *heteroskedastic* least-squares model.

Repeat part(s) (b)/(c), taking into account these weights/heteroskedasticity.

i. Report R_T, F_T, and the regression statistics as shown on page 21 of Part 3 of the lecture notes.

ii. Plot your regression residuals vs. K and comment on any apparent heteroskedasticity.

iii. What values of r_{imp} and y_{imp} correspond to $\{R_T, F_T\}$ given your value of T?

iv. What implied volatilities $\sigma_{imp,atm,T}$ and total variances $\Sigma_{imp,atm,T} = \sigma^2_{imp,atm,T}T$ for ATM calls and puts correspond to the respective mid prices, given your estimated values for r_{imp} and y_{imp}?

v. What uncertainties around the mid-point call and put ATM implied volatilities result from the corresponding bid/ask spreads?

vi. Again, compare your results to those from parts (b) and (c).

vii. Are all your resulting adjusted mid-point option values and regression residuals within the bid-ask spreads?

(f) Extra credit: Plot the call and put implied volatilities for all strikes.

Are your results consistent with those shown in the lecture notes?

Are there any noteworthy outliers?