Problem Set #2

Yilu Chen, Peilin Liu, Ashutosh Ekade, Gabriel de La Noue

Problem 1

Let $f(t,x) = \frac{x^3}{3}$, then $f_t(t,x) = 0$, $f_x(t,x) = x^2$, $f_{xx}(t,x) = 2x$

From Itô's formula, $d\frac{1}{3}W_t^3 = df(t, W_t)$

$$= W_t\,dt + W_t^2\,dW_t$$

By taking the integral, we have $\int_0^t W_s^2\,dW_s = \frac{1}{3}W_t^3 - \int_0^t W_s\,ds$

Problem 2

Certainly we have $X(t)$ is adapted to $\mathcal{F}_t$ and $E[|X(t)|] < \infty$.

For $s \le t$,
$$\begin{aligned}
E[X(t) \mid \mathcal{F}(s)] &= E[W_1(t)W_2(t) \mid \mathcal{F}(s)] \\
&= E[W_1(t) \mid \mathcal{F}(s)] \cdot E[W_2(t) \mid \mathcal{F}(s)] \\
&= E[W_1(t) - W_1(s) + W_1(s) \mid \mathcal{F}(s)] \cdot E[W_2(t) - W_2(s) + W_2(s) \mid \mathcal{F}(s)] \\
&= \left(W_1(s) + E[W_1(t) - W_1(s) \mid \mathcal{F}(s)]\right) \cdot \left(W_2(s) + E[W_2(t) - W_2(s) \mid \mathcal{F}(s)]\right) \\
&= \left(W_1(s) + E[W_1(t-s)]\right)\left(W_2(s) + E[W_2(t-s)]\right) \\
&= W_1(s)W_2(s) \\
&= X(s)
\end{aligned}$$

So $X(t)$ is a martingale

Problem 3

Assume that $dZ(t) = \mu(t)Z(t)\,dt + \sigma(t)Z(t)\,dW(t)$

Let $f(t,x) = \log(x)$, then $f_t(t,x) = 0$, $f_x(t,x) = \frac{1}{x}$, $f_{xx}(t,x) = -\frac{1}{x^2}$

From Itô's formula, $d\log(Z(t)) = \left(\mu(t) - \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)\,dW(t)$

$$\Rightarrow \log Z(t) = \int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right)dt + \int_0^t \sigma(s)\,dW(s)$$

We already have $\log Z(t) = \int_0^t -\frac{\eta^2}{2}g^2(s)\,ds + \int_0^t \eta g(s)\,dW(s)$

$$\Rightarrow \begin{cases} \mu(s) - \frac{1}{2}\sigma^2(s) = -\frac{\eta^2}{2}g^2(s) \\ \sigma(s) = \eta g(s) \end{cases} \Rightarrow \begin{cases} \mu(s) = 0 \\ \sigma(s) = \eta g(s) \end{cases}$$

$$\Rightarrow dZ(t) = \eta g(t)\,dW(t)$$

As there is no drift term, $Z(t)$ is a martingale.

$$\Rightarrow E[Z(t)] = E[Z(t) \mid \mathcal{F}(0)] = Z(0) = 1$$

$$\Rightarrow E\left[e^{-\frac{\eta^2}{2}\int_0^t g^2(s)\,ds + \eta\int_0^t g(s)\,dW(s)}\right] = 1$$

$$\Rightarrow E\left[e^{\eta X(t)}\right] = E\left[e^{\eta\int_0^t g(s)\,dW(s)}\right] = e^{\frac{\eta^2}{2}\int_0^t g^2(s)\,ds}$$

Recall that the moment generating function of $X \sim N(\mu, \sigma^2)$ is $E[e^{\theta X}] = e^{\theta\mu + \frac{\theta^2}{2}\sigma^2}$

$$\Rightarrow X(t) \sim N\left(0, \sqrt{\int_0^t g^2(s)\,ds}\right)$$

Problem 4

Let $f(t,x) = x^k$, then $f_t(t,x) = 0$, $f_x(t,x) = kx^{k-1}$, $f_{xx}(t,x) = k(k-1)x^{k-2}$

From Itô's formula, $dW_t^k = \frac{1}{2}k(k-1)W_t^{k-2}\,dt + kW_t^{k-1}\,dW_t$

$$\Rightarrow W_t^k = \frac{1}{2}k(k-1)\int_0^t W_s^{k-2}\,ds + k\int_0^t W_s^{k-1}\,dW_s$$

$$\Rightarrow \beta_k(t) = E[W_t^k]$$

$$= \frac{1}{2}k(k-1)\int_0^t E[W_s^{k-2}]\,ds + kE\left[\int_0^t W_s^{k-1}\,dW_s\right]$$

$$= \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)\,ds$$