Assignment 1: Suggested solutions

The final answers in the computations below are exact (using the accompanied Python code). Note that it is easy to get rounding errors if computed in steps.

1.

   (a) If the 4% is an EAR, in two years you will have:

   $$\$200,000 \times (1+4\%)^2 = \$216,320.$$

   (b) If the 4% is a quarterly APR, the quarterly rate is

   $$\frac{4\%}{4} = 1\%.$$

   In two years, you will have:

   $$\$100,000 \times (1+1\%)^8 = \$216,571.$$

   (c) If the 4% is a monthly APR, the monthly rate is:

   $$\frac{4\%}{12} = 0.33\%.$$

   In two years, you will have:

   $$\$200,000 \times (1+0.33\%)^{24} = \$216,629.$$

2.

   (a) The amount you will have saved at the end of the 35th year is:

   $$\begin{aligned} & \$45,000 \times (1+2\%)^{34} + \$45,000 \times (1+2\%)^{33} + \cdots + \$45,000 \\ = \; & \$45,000 \times [(1+2\%)^{34} + (1+2\%)^{33} + \cdots + 1] \\ = \; & \$45,000 \times (1+2\%)^{35} \times \left[\frac{1}{1+2\%} + \frac{1}{(1+2\%)^2} + \cdots + \frac{1}{(1+2\%)^{35}}\right]. \end{aligned}$$

   The term in brackets is the PV of an annuity which pays \$1 each year for the next 35 years:

   $$\text{PV} = \frac{1}{2\%}\left[1 - \frac{1}{(1+2\%)^{35}}\right] = \$24.99.$$

Therefore, the amount you will have saved at the end of the 35th year is:

$$\$45,000 \times (1 + 2\%)^{35} \times 24.99 = \$2,249,751.$$

(b) The amount you can consume, c, is determined by the requirement that the PV of consumption, evaluated at the end of the 35th year, is equal to the amount you will have saved, that is, \$2,249,751. The PV of consumption is:

$$\begin{aligned} & \frac{c}{1+2\%} + \frac{c}{(1+2\%)^2} + \cdots + \frac{c}{(1+2\%)^{20}} \\ = \; & \frac{c}{1\%}\left[1 - \frac{1}{(1+1\%)^{20}}\right] = c \times 16.35. \end{aligned}$$

Therefore, we have:

$$c \times 16.35 = \$2,249,751,$$

or

$$c = \$137,587.$$

3.

(a) The EAR is:

$$\text{EAR} = \left(1 + \frac{\text{APR}}{12}\right)^{12} = 3.04\%.$$

(b) The monthly rate is:

$$\frac{\text{APR}}{12} = 0.25\%.$$

The monthly payment c can be computed by the annuity formula:

$$\$500,000 = \frac{c}{0.25\%}\left[1 - \frac{1}{(1+0.25\%)^{360}}\right].$$

Solving for c, we find that $c = \$2,108$.

(c) What you owe the bank immediately after the twentieth monthly payment is the PV of the remaining 340 payments. This PV can be computed by the annuity formula:

$$\frac{\$2,108}{0.25\%}\left[1 - \frac{1}{(1+0.25\%)^{340}}\right] = \$482,426.$$

4.

(a) The price of a zero-coupon bond with 3 years to maturity is:

$$\frac{100}{(1+3\%)^3} = 91.514.$$

(b) The price of a bond with a coupon rate of 1% and 2 years to maturity is:

$$\frac{1}{1+2\%} + \frac{101}{(1+2.5\%)^2} = 97.114.$$

(c) The price of a bond with a coupon rate of 13% and 4 years to maturity is:

$$\frac{13}{1+2\%} + \frac{13}{(1+2.5\%)^2} + \frac{13}{(1+3\%)^2} + \frac{113}{(1+3.5\%)^4} = 135.488.$$

Hence, the price of the three bonds are \$91.514, \$97.114, and \$135.488.

5. Consider two alternative approaches.

Approach 1: Matrix form Set up the cash flows in a matrix and solve for the discount values that make the values of the cash flows equal to the prices:

$$\begin{bmatrix} 101.5 & 0.0 \\ 2.0 & 102.0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 98.98 \\ 98.59 \end{bmatrix}$$

Solve for d_1 and d_2. This can be done in one step or recursively. Recall that $d_j = 1/(1+r_j/2)^j$ with semi-annual compounding. Then, the 6-month rate is $2[1/d_1 - 1] = 5.09\%$ and the 1-year rate is $2[(1/d_2)^{1/2} - 1] = 5.47\%$.

Approach 2: Algebra and formulas The 6-month spot rate solves:

$$98.98 = \frac{100 + \frac{3}{2}}{1 + \frac{r_1}{2}},$$

and the 1-year spot rate r_1 solves:

$$98.59 = \frac{\frac{4}{2}}{1 + \frac{r_1}{2}} + \frac{100 + \frac{4}{2}}{\left(1 + \frac{r_2}{2}\right)^2}.$$

Solving these equations, we find that $r_1 = 5.092\%$ and $r_2 = 5.472\%$.

6.

(a) The bond's price is:

$$P = \frac{5}{1+5\%} + \frac{5}{(1+6\%)^2} + \frac{105}{(1+7\%)^3} = 94.923.$$

The YTM is given by:

$$94.923 = \frac{5}{1+y} + \frac{5}{(1+y)^2} + \frac{105}{(1+y)^3}.$$

Solving for y, we find that it is 6.932%.

(b) Using the three spot rates, we can determine the forward rates $f_{1,2}$, $f_{2,3}$, $f_{1,3}$. They are given by

$$1 + f_{1,2} = \frac{(1+r_2)^2}{1+r_1} \Rightarrow f_{1,2} = 7.010\%,$$

$$1 + f_{2,3} = \frac{(1+r_3)^3}{(1+r_2)^2} \Rightarrow f_{2,3} = 9.028\%,$$

$$(1 + f_{1,3})^2 = \frac{(1+r_3)^3}{1+r_1} \Rightarrow f_{1,3} = 8.014\%.$$

(c) To get a guaranteed 3-year return, we must eliminate the uncertainty associated with reinvesting the coupons. This can be done by investing the coupons at the forward rates. If we have one unit of the bond today, our investment in 3 years will be worth:

$$\text{FV} = 105 + 5 \times (1 + f_{2,3}) + 5 \times (1 + f_{1,3})^2$$

and the return on our investment will be given by

$$94.923 \times (1 + R_3)^3 = \text{FV}.$$

To determine R_3, we can do the calculations above, or simply note that by the absence of arbitrage, the return has to equal the 3-year spot rate, that is, 7%.