# PBST: Homework

*B V Rao*

**Ashutosh Kumar**

# Problem 1

A poker hand means a set of five cards selected at random from usual deck of playing cards.

1. Find the probability that it is a **Royal Flush -** means that it consists of ten, jack, queen, king, ace of one suit.
2. Find the probability that it is **Four of a kind -** means that there are four cards of equal face value.
3. Find the probability that it is a **Full house -** means that it consists of one pair and one triple of cards with equal face values.
4. Find the probability that it is a **Straight -** means that it consists of five cards in a sequence regardless of suit.
5. Find the probability that it consists of three cards of equal face value and two other cards but not a full house.
6. Find the probability that it consists of two distinct pairs and another card but does not fall into previous categories
7. Find the probability that it consists of a pair and three other cards but does not fall into previous categories

## Solution

1.
2.
3.
4.
5.
6.
7.

# Problem 2

In how many ways can eight rooks be placed on a chess board so that none can take another and none is on the white diagonal

# Problem 3

A number $X$ is chosen at random from the set $\{0, 1, 2, \ldots, 10^n - 1\}$. Find the probability that $X$ is a $k$-digit number. A number $a$ is a $k$-digit number if it is of the form $a = \sum_0^{k-1} a_i 10^i$ where $a_i$ are integers from 0 to 9 and $a_{k-1}$ is not zero

# Problem 4

One mapping is selected at random from the set of all mappings of $\{1, 2, \ldots, n\}$.

1. What is the probability that the selected mapping transforms each of the $n$ elements into 1?
2. What is the probability that element $i$ has exactly $k$ pre-images (Here $i$ and $k$ are pre-assigned) ?
3. What is the probability that the element $i$ is transformed into $j$ ?
4. What is the probability that the elements $i_1, i_2$ and $i_3$ (assume distinct) are transformed into $j_1, j_2$ and $j_3$ respectively?

# Problem 5

One permutation is selected at random from the set of all permutations of $\{1, 2, \ldots, n\}$.

1. What is the probability that the identity permutation is chosen?
2. What is the probability that the selected permutation transforms $i_1, i_2, \ldots, i_k$ into $j_1, j_2, \ldots, j_k$ respectively?
3. What is the probability that the permutation keeps $i$ fixed?
4. What is the probability that the elements 1,2 and 3 form a cycle in that order? (in some order?)
5. What is the probability that all the elements form a cycle?

# Problem 6

1. Prove that in **Polya urn model**, chance of red ball is $r/(r+g)$ at any stage.
2. Show that red at stage m and green at stage $n \neq m$ equals $rg/\{(r+g)(r+g+1)\}$.
3. Do you think this will be the case in **Friedmans' model** where you add a ball of opposite colour?

# Problem 7

A **tournament** is said to have $k$-leader property if for every set of k players there is one who beats them all. Consider a random tournament with $n$ players. Fix $k$ players. (Tournament on $n$ vertices is a directed graph, having exactly one arrow between each pair of vertices.)

1. Show that the event **"No player beats all these k"**, has probability $(1-2^{-k})^{n-k}$
2. Consequently, if $\binom{n}{k}(1-2^{-k})^{n-k} < 1$, then there is a $k$-leader tournament with $n$ players.
3. Show that for all large $n$ this inequality holds and hence $k$-leader tournaments are possible
4. If $f(k)$ is the least such n, show that f(1) = 3 and f(2) = 7.

# Problem 8

Three dice are rolled. Given that no two show the same face, find the conditional probability that at least one is an ace(?)

# Problem 9

In a bolt factory; machines $A$, $B$ and $C$ manufacture 25, 35 and 40 percent of the total. Of their outputs 5, 4 and 2 percent are defective. A bolt selected at random is found to be defective. Given this what is the probability that it was manufactured by machine A? by B? by C?

# Problem 10

A die has four red faces and two white faces and a second die has two red and four white faces. A fair coin is flipped once. If it falls heads, we keep on throwing the first die and if it falls tails, we keep on throwing the second die.

1. Show that the probability of red at any throw is $\frac{1}{2}$
2. If the first two throws result in red what is the conditional probability of red at the third throw?
3. If red turns up at the first n throws, what is the conditional probability that the first die is being used?

# Problem 11

An urn contains 3 white, 5 black and 2 red balls. Two persons draw balls in turn, without replacement. The first person to draw a white ball before the appearance of a red ball wins the game. However, if a red ball is drawn by anyone before the appearance of a white ball, then the game is declared a tie. Calculate the probability that:

1. The person who beings the game wins.
2. The other person wins.
3. The game ends in a tie.

# Problem 12

I have two boxes. Box I has 4 red and 2 green balls while box II has 2 red and 4 green balls. I pick a ball at random from box I and put in box II; Then I pick a ball at random from box II and put in box I. Now I pick one ball from each box. What are the chances that they are of the same colour?

# Problem 13

I throw a fair die. If 4, 5 or 6 show up, I put $X$ as that value. If 1, 2 or 3 show up, I roll the die again and then put $X$ as the sum of the two face obtained. Calculate the distribution of $X$.

# Problem 14

For any events $A_i, \ldots, A_n$ in an experiment show $P(\bigcup A_i) \leq \sum P(A_i)$. This is called union bound (or Boole's inequality)
For any two events $A$ and $B$ show that

$$P(A \cap B) \geq P(A) + P(B) - 1$$

This is called a Bonferroni inequality.
The inclusion-exclusion formula we proved for probability of union of events is sometimes called Poincare formula.

# Problem 15

Fix $0 < p < 1$. Then $G(n,p)$ is the (undirected) random graph model where we have $n$ vertices and where edges are chosen independently each with probability $p$. Let $q_n$ be the probability that the graph is disconnected. Show

$$q_n \leq \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}(1-p)^{k(n-k)} \leq \sum_{k=1}^{[n/2]}(n(1-p)^{n/2})^k \to 0$$

Thus $1 - q_n \to 1$. This is expressed by saying that, in this model, almost every graph is connected. A better way to think is that, as $n$ becomes large, our random graph is connected with overwhelming probability.

# Problem 16

In an experiment $P(A) = 0.4$ and $P(A \cup B) = 0.7$. Determine $P(A)$ and $P(B)$ if:

1. $A, B$ are independent.
2. $A, B$ are disjoint.

## Solution

1. Since A, B are independent, we know that $P(A \cap B) = P(A)P(B)$.
   Also, we know that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

   On substituting, we get $0.7 = 0.4 + P(B) - 0.4 \times P(B)$
   $\implies 0.3 = P(B) - 0.4 \times P(B)$
   $\implies 0.3 = 0.6 \times P(B)$
   $\implies P(B) = 0.5$

2. Since A, B are disjoint, we know that $P(A \cap B) = 0$
   $0.7 = 0.4 + P(B)$
   $\implies P(B) = 0.3$

# Problem 17

A cancer diagnostic test is 95% accurate both on those who have cancer and those who do not. Assume that 0.005 of some population have cancer. A person is selected at random and test shows he has cancer. Given this, what are the chances that he actually has cancer.

# Problem 18

Banach carries (alright, assume he is alive!) two match boxes, each with 50 sticks, one in each pocket. When he needs, he selects one pocket and takes one match from that box.

1. Today for the first time he found that the box he picked is empty. What are the chances that the other box contains exactly $k$ matches?
2. Today for the first time he is emptying a box. What are the chances that the other box contains exactly $k$ matches?