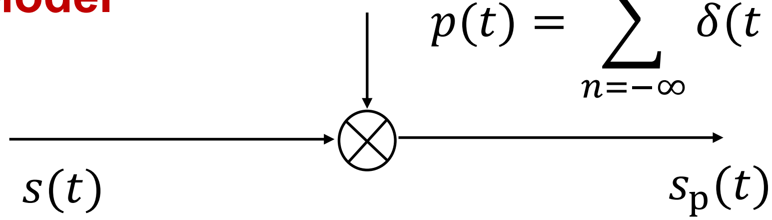
2 Multi-Dimensional Sampling

2.1 Sampling Theorem Revisited

Model

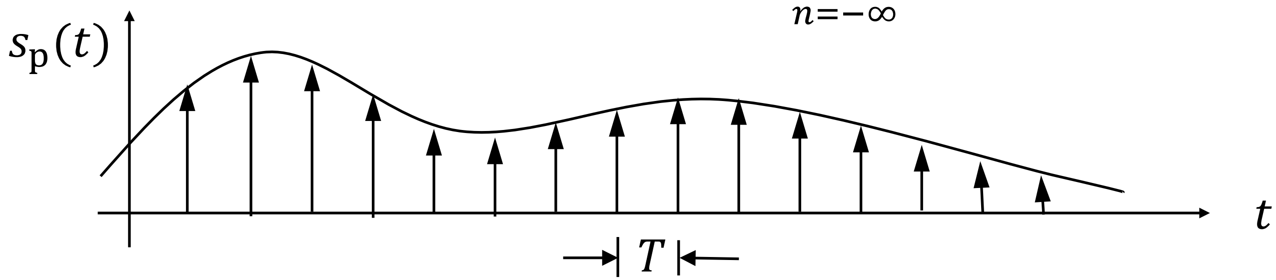
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



Sampling frequency

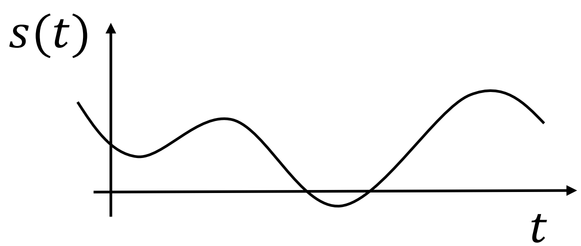
$$\omega_{\mathrm{S}} = \frac{2\pi}{T}$$

Sampled signal: $s_{\mathrm{p}}(t) = s(t)p(t) = \sum_{n=-\infty}^{\infty} s(nT)\delta(t - nT)$

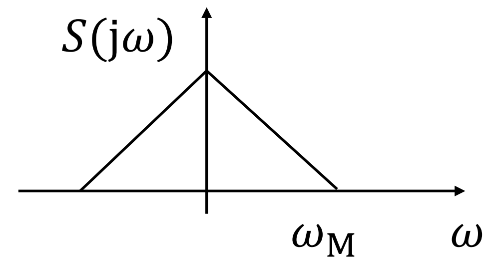


Frequency Domain Description of 1D Sampling

Time domain

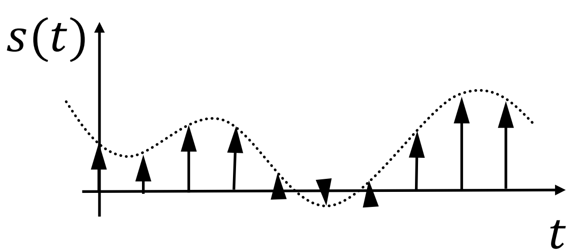


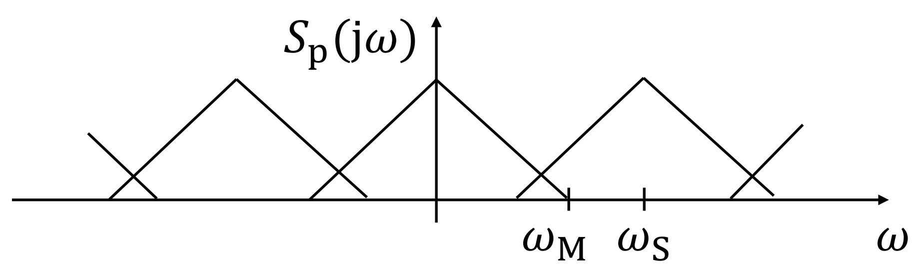
Frequency domain



Sampling function $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ $\qquad P(\mathrm{j}\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k \underbrace{\frac{2\pi}{T}}_{\omega_\mathrm{S}})$

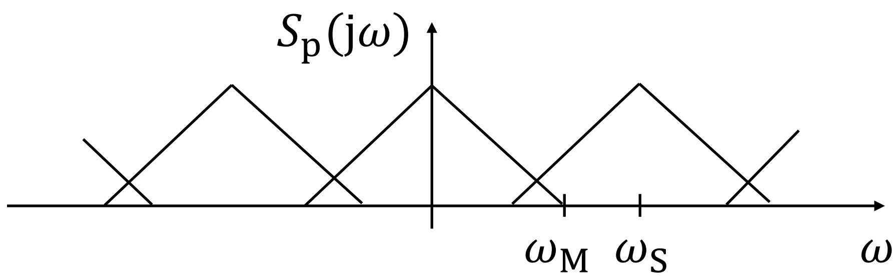
Sampled signal $s_\mathrm{p}(t) = s(t) \cdot p(t)$ $\qquad S_\mathrm{p}(\mathrm{j}\omega) = S(\mathrm{j}\omega) * P(\mathrm{j}\omega)$



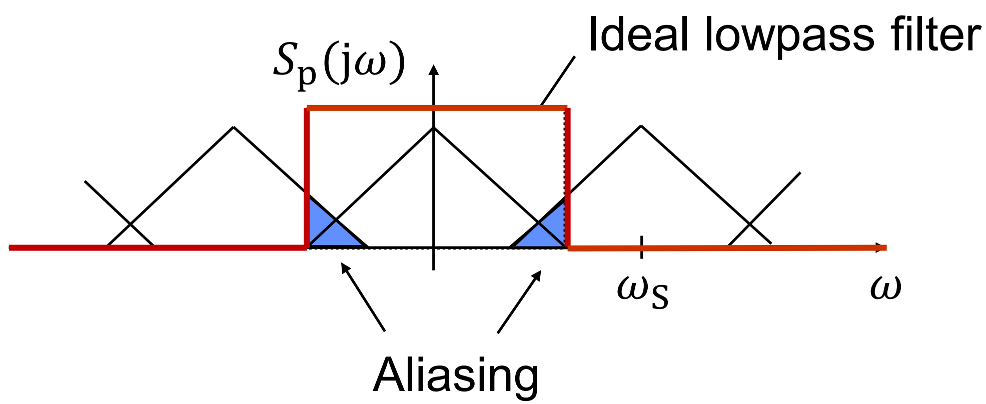


Reconstruction the Signal from its Samples

Spectrum of sampled signal

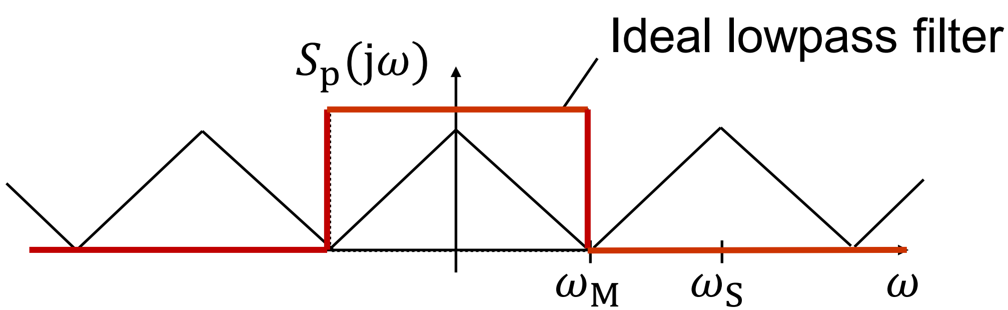


Reconstruction of $S(\mathrm{j}\omega)$ from $S_\mathrm{p}(\mathrm{j}\omega)$



1D Sampling Theorem

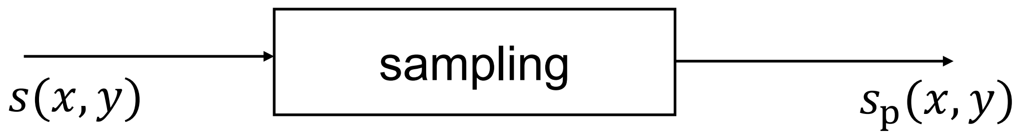
If a band limited 1D signal is sampled at a sufficiently high rate such that its spectral replicas do not overlap, it can be reconstructed without loss by ideal lowpass filtering.



Band limited signal: $S(\mathrm{j}\omega) = 0 \;\text{ for }\; |\omega| \geq \omega_{\mathrm{M}}$

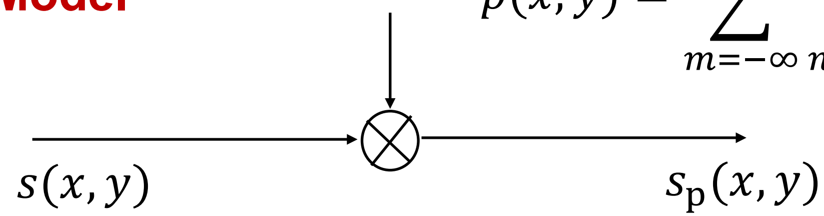
Minimum sampling rate: $\omega_{\mathrm{S}} = \dfrac{2\pi}{T} \geq 2\omega_{\mathrm{M}}$ "Nyquist rate"

2.2 2D Sampling

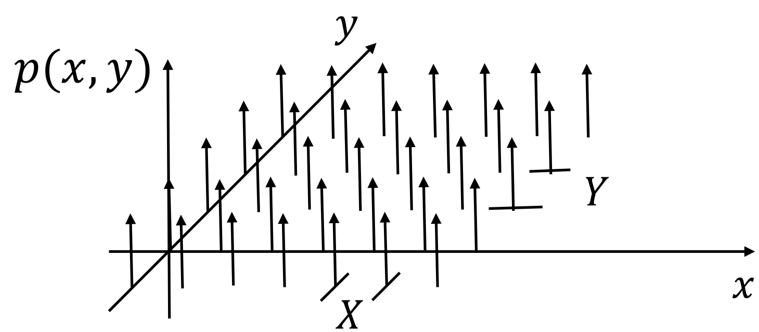


Model

$$p(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - mX, y - nY)$$



Sampling function



Sampling frequencies

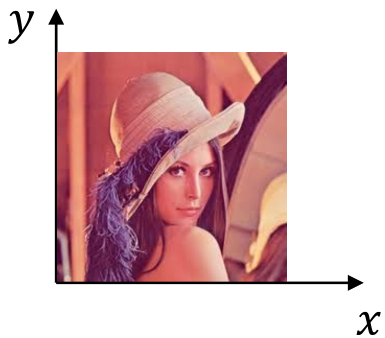
$$\omega_{\mathrm{Sx}} = \frac{2\pi}{X}$$

$$\omega_{\mathrm{Sy}} = \frac{2\pi}{Y}$$

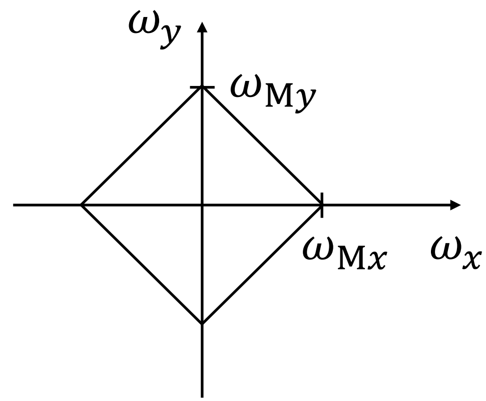
$$s_{\mathrm{p}}(x,y) = s(x,y)p(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s(mX, nY)\delta(x - mX, y - nY)$$

2D Sampling in the Frequency Domain

Spatial domain $s(x,y)$



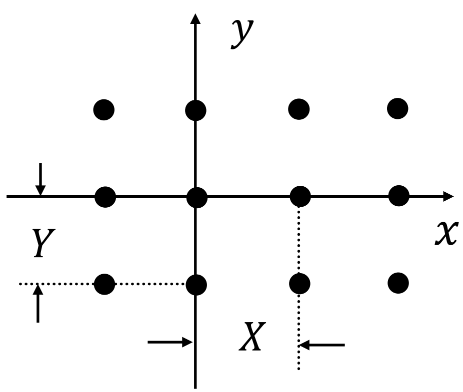
Frequency domain $S(\mathrm{j}\omega_x, \mathrm{j}\omega_y)$

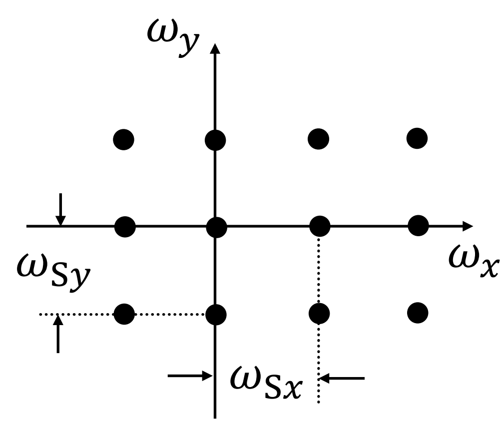


Sampling function

$$p(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - mX, y - nY)$$

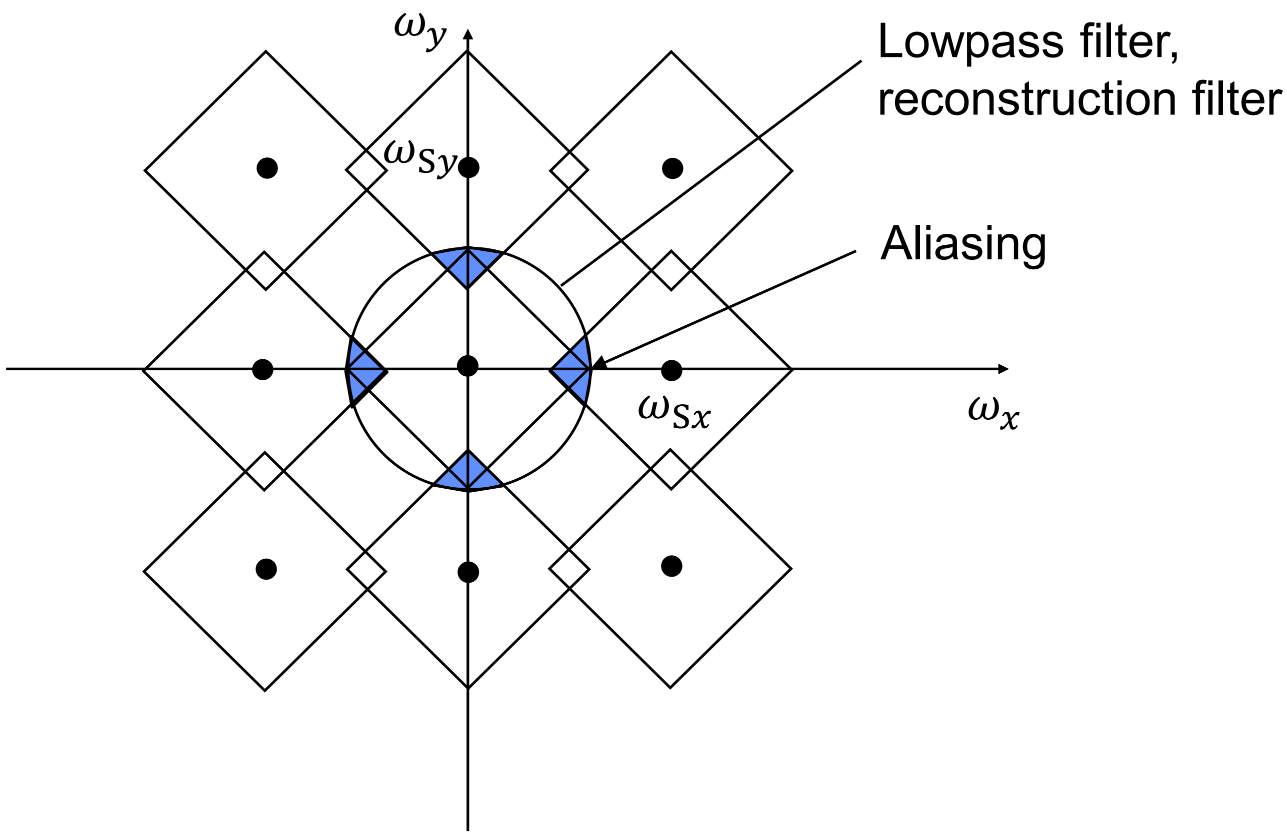
$$P(\mathrm{j}\omega_x, \mathrm{j}\omega_y) = \frac{4\pi^2}{XY} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta(\omega_x - k \underbrace{\frac{2\pi}{X}}_{\omega_{\mathrm{S}x}}, \omega_y - l \underbrace{\frac{2\pi}{Y}}_{\omega_{\mathrm{S}y}})$$





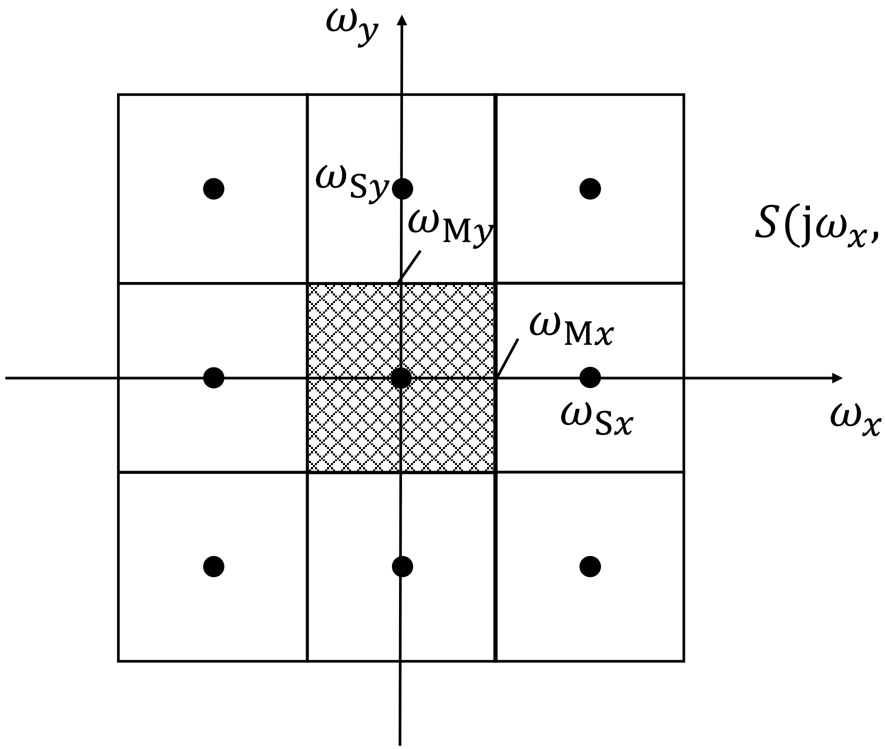
Reconstructing a 2D Signal from its Samples

Spectrum of sampled signal $S_\mathrm{p}(\mathrm{j}\omega_x, \mathrm{j}\omega_y) = S(\mathrm{j}\omega_x, \mathrm{j}\omega_y) * P(\mathrm{j}\omega_x, \mathrm{j}\omega_y)$



2D Sampling Theorem

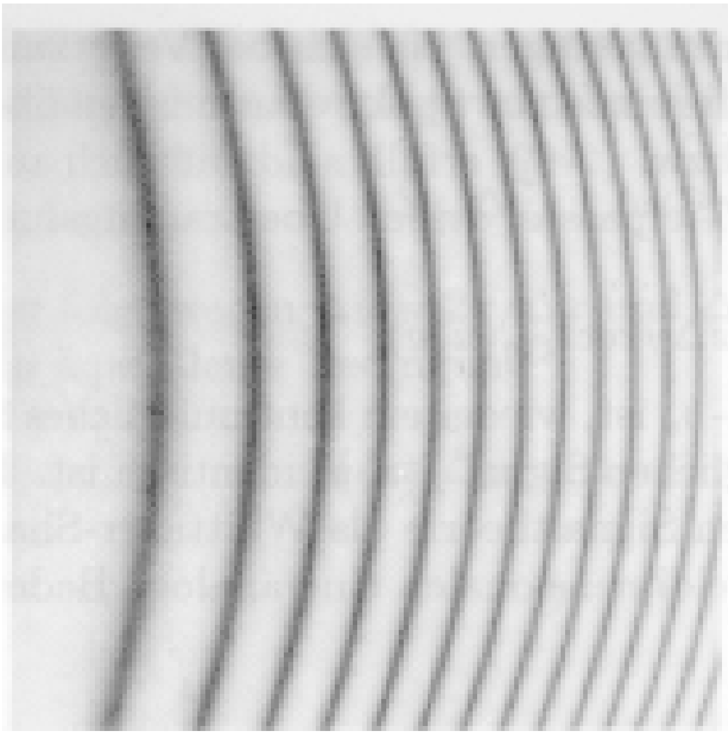
If a band limited 2D signal is sampled at a sufficiently dense grid such that its spectral replicas do not overlap, it can be reconstructed without loss by linear shift-invariant filtering.



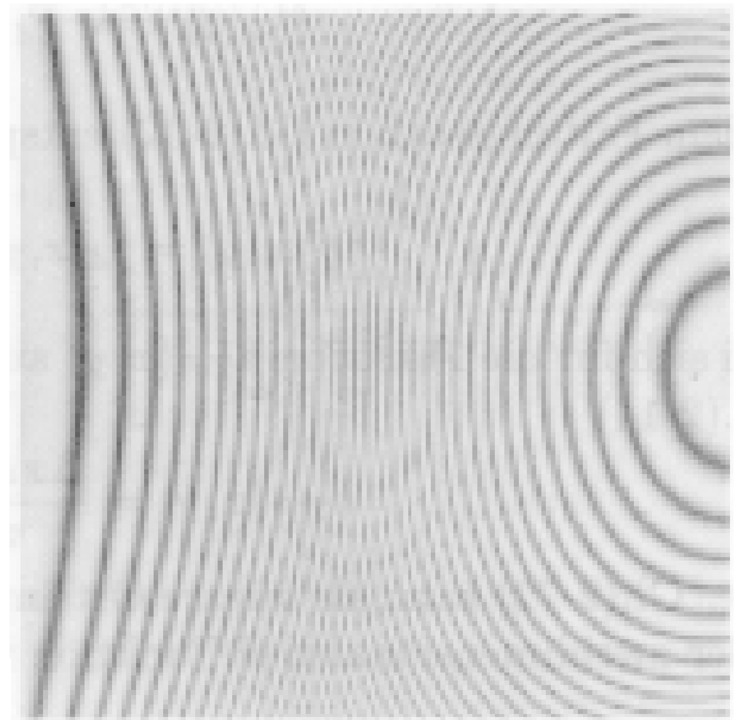
$$S(j\omega_x, j\omega_y) = 0 \ \text{ for } \ |\omega_x| \geq \omega_{Mx}, |\omega_y| \geq \omega_{My}$$

Minimum sampling rate: $\omega_{Sx} = \frac{2\pi}{X} \geq 2\omega_{Mx}$ $\quad$ $\omega_{Sy} = \frac{2\pi}{Y} \geq 2\omega_{My}$

Example for Aliasing in 2D Signal Sampling

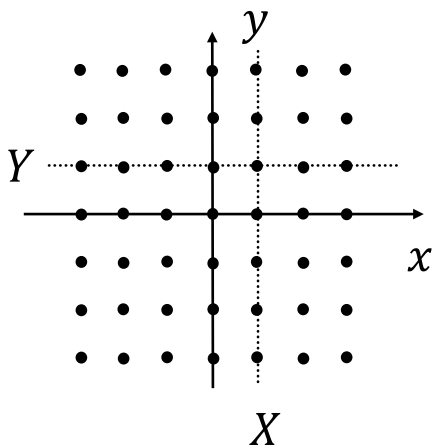


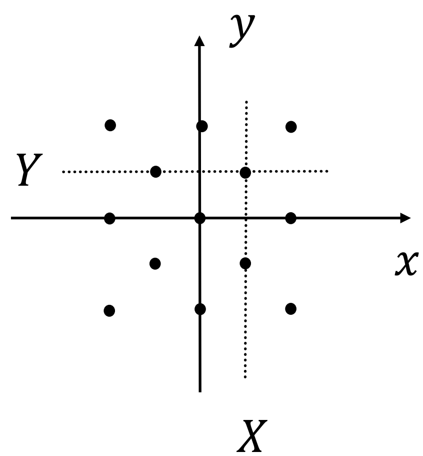
2D signal sampled without aliasing

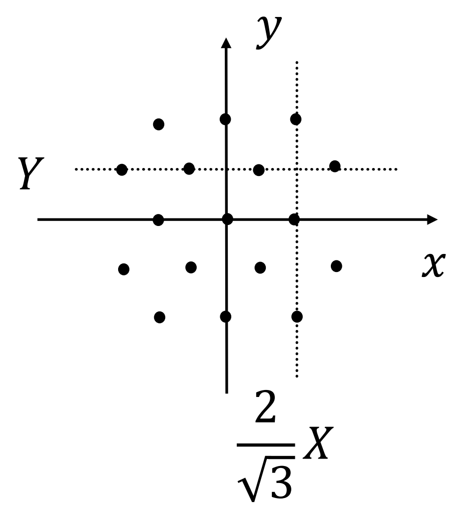


Aliasing in sampled 2D signal
"Moiré pattern"

Rectangular, Quincunx, and Hexagonal Sampling







Rectangular

Equally dense grid
in both directions

Quincunx

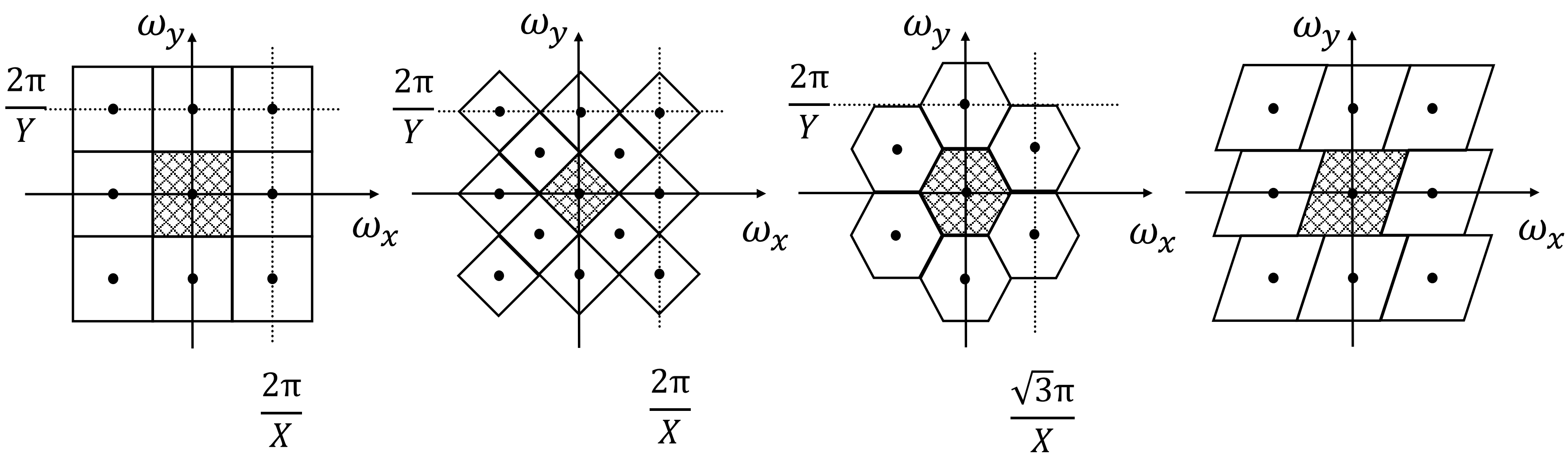
Each sample has
same distance to
four nearest neighbors

Half of the samples as
compared to a
rectangular grid

Hexagonal

Each sample has
same distance to
six nearest neighbors

Sampling in Frequency Domain



Rectangular

Highest maximum frequency in diagonal direction

Quincunx

Half of the samples compared to rec.

Same maximum frequency for x, y

Lower in diagonal

Hexagonal

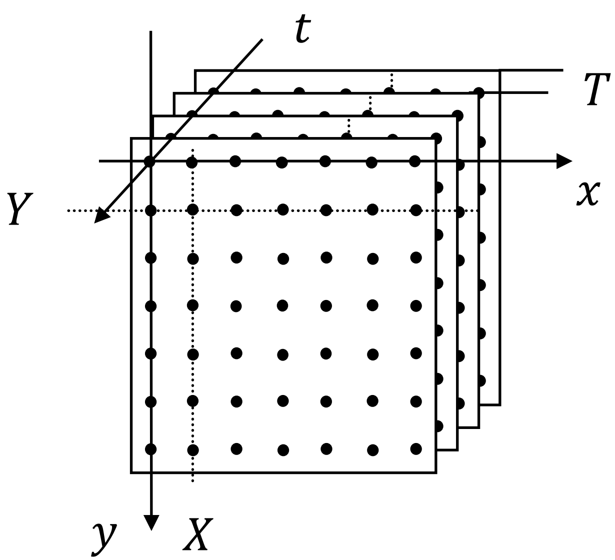
Almost same maximum freq. for all orientations

Rectangular sampling with different tiling / baseband

Demo 2 „Image Sampling“

2.3 Spatiotemporal Sampling

Simplest case: rectangular sampling in horizontal, vertical, and temporal directions



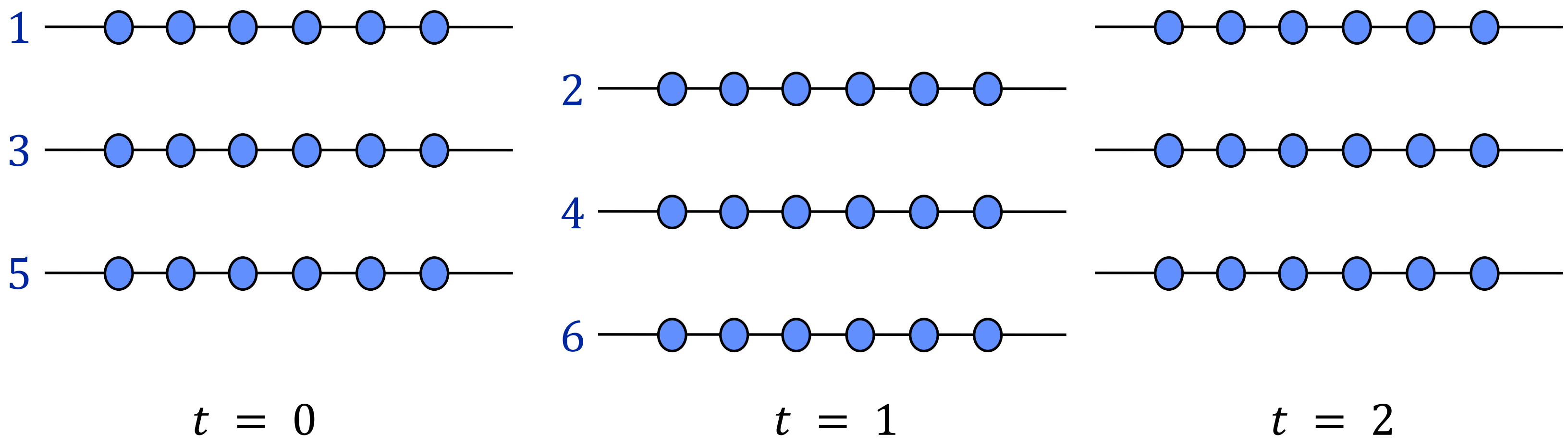
Sampling rates T, X, Y

Also known as “progressive” scanning

Can be found in
- Video display on computer monitors
- Coding of video using MPEG-x, H.26x

Interlaced Spatiotemporal Sampling

Idea: skip every second line and alternate skipping over time to save bandwidth



Corresponds to quincunx sampling in joint temporal and vertical direction

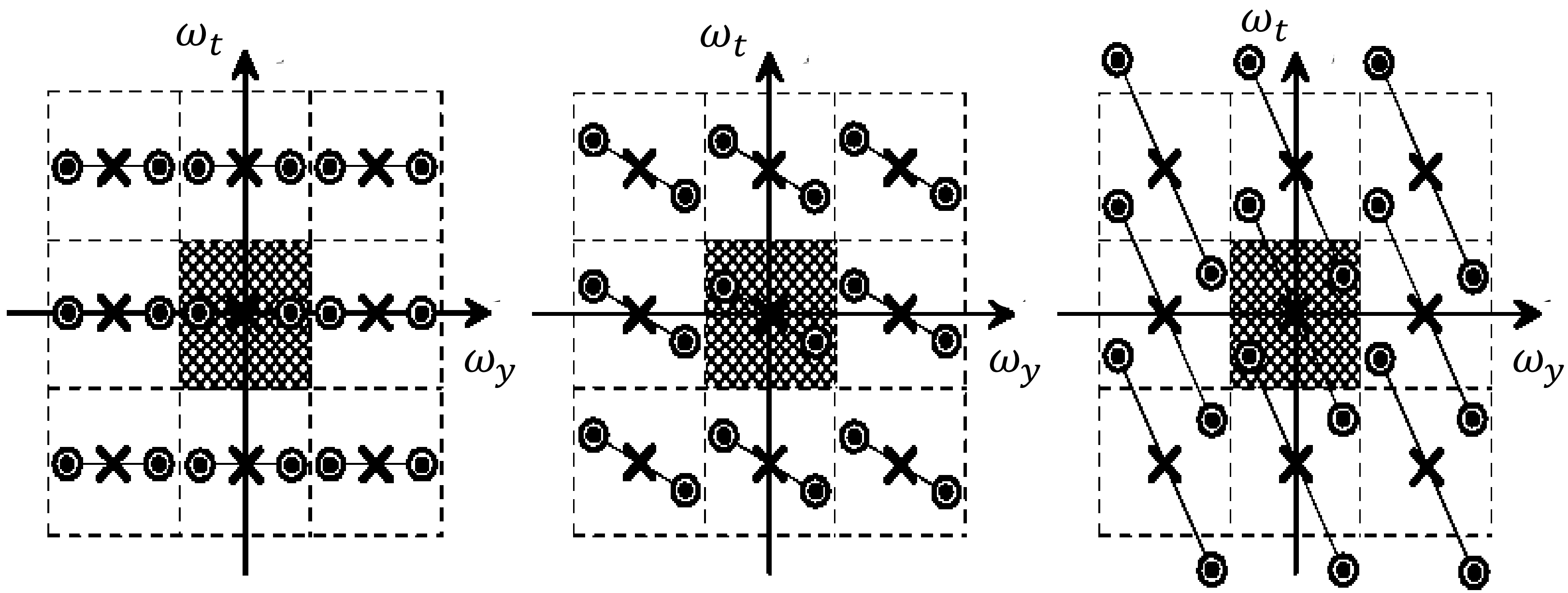
Also known as "interlaced" scanning

Can be found in
- Video display on (analog) TV systems
- Coding of video with MPEG-2 (e.g. for DVB-T), H.264 (ISDB-T)

2.4 Motion in 3D Sampling

Sampled signal: cosine with almost maximum vertical frequency moving down

Spectral replica in vertical / temporal frequency plane for



a) No translatory motion b) Small translatory motion c) Temporal aliasing

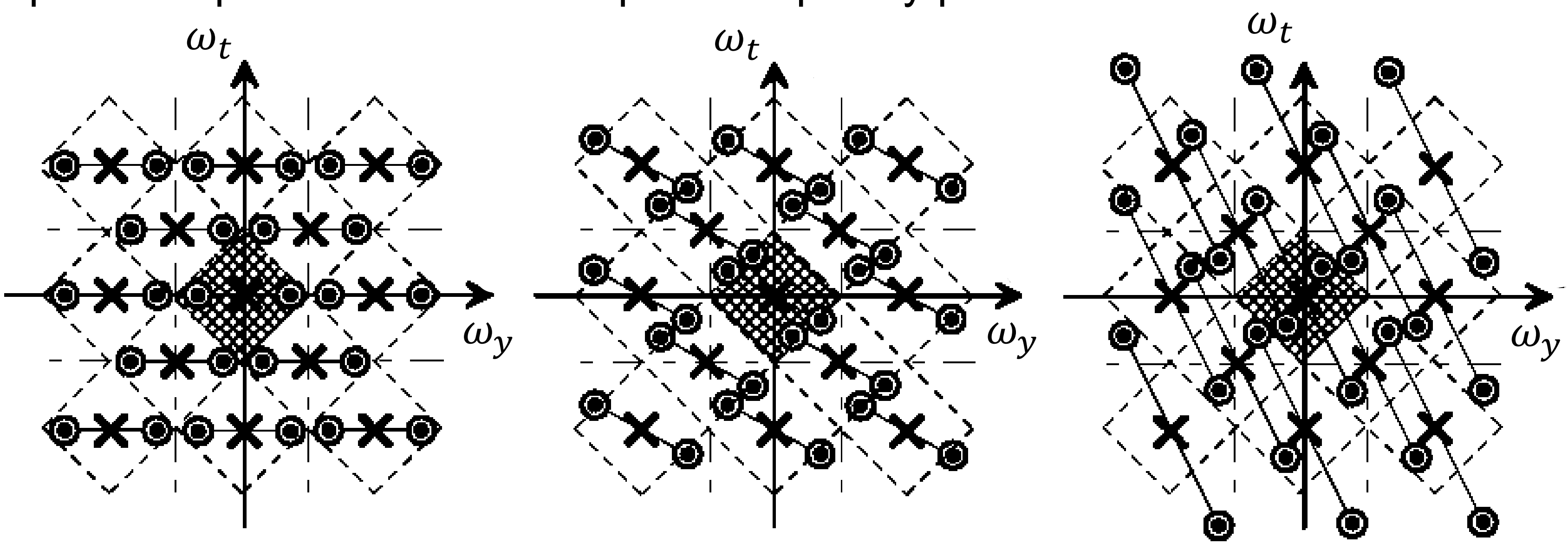
$|v_y| < \frac{Y}{T}$ $|v_y| \geq \frac{Y}{T}$

$\Rightarrow$ reversely spinning stagecoach wheel

Interlaced Sampling With Translatory Motion

Sampled signal: cosine with almost maximum vertical frequency moving down

Spectral replica in vertical / temporal frequency plane for



a) No translatory motion b) Small translatory motion c) Large translatory motion

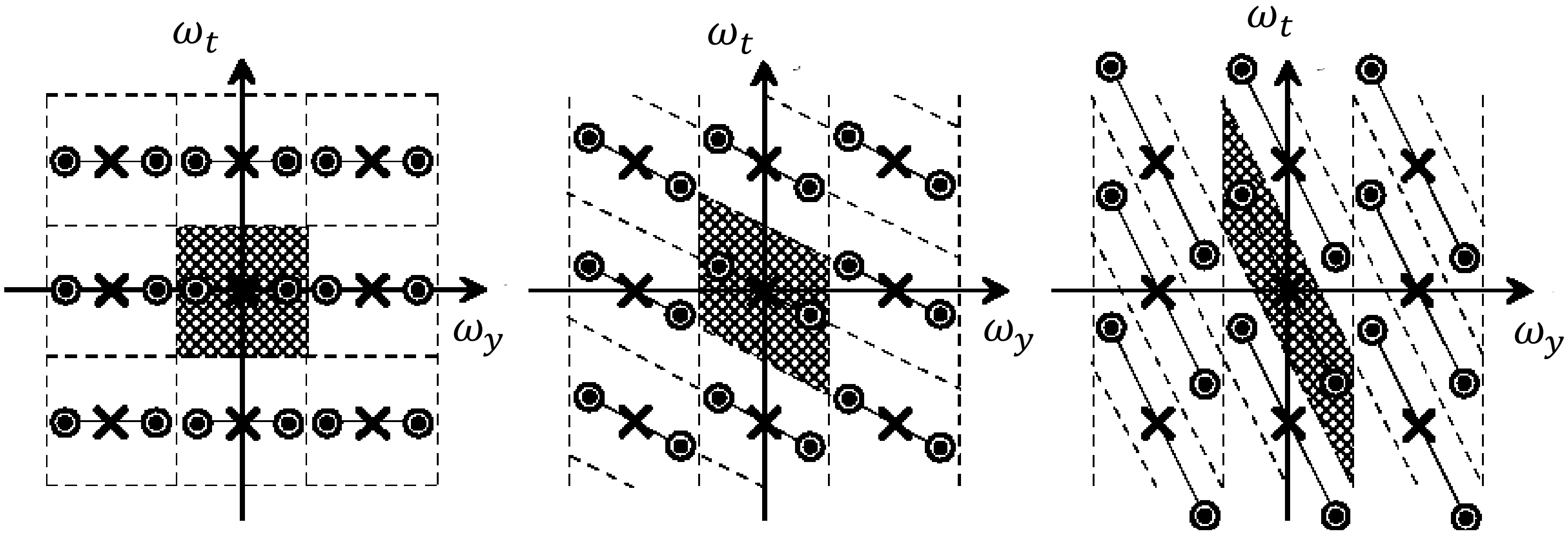
Effect: interlaced sampling leads to reconstruction errors in

- Magnitude of translatory motion
- Vertical frequency, spatial Moiré effect

Elimination of Temporal Alias By Motion Compensation

Human visual system is capable of tracking image objects

- Visual reconstruction in “temporally shifted” base band



a) No translatory motion b) Small translatory motion c) Large translatory motion

Motion compensation of visual system

⇨ Large motion magnitudes can be reconstructed without aliasing

Multi-Dimensional Sampling - Summary

- Spectral replicas of signal due to sampling
- Aliasing: spectral replicas overlap
- Hexagonal sampling for equal frequency support
- Aliasing in signal reconstruction
- Moiré patterns for interlaced sampling
- Motion compensation by visual system