8 Subband Coding

8.1 Principle of Subband Coding

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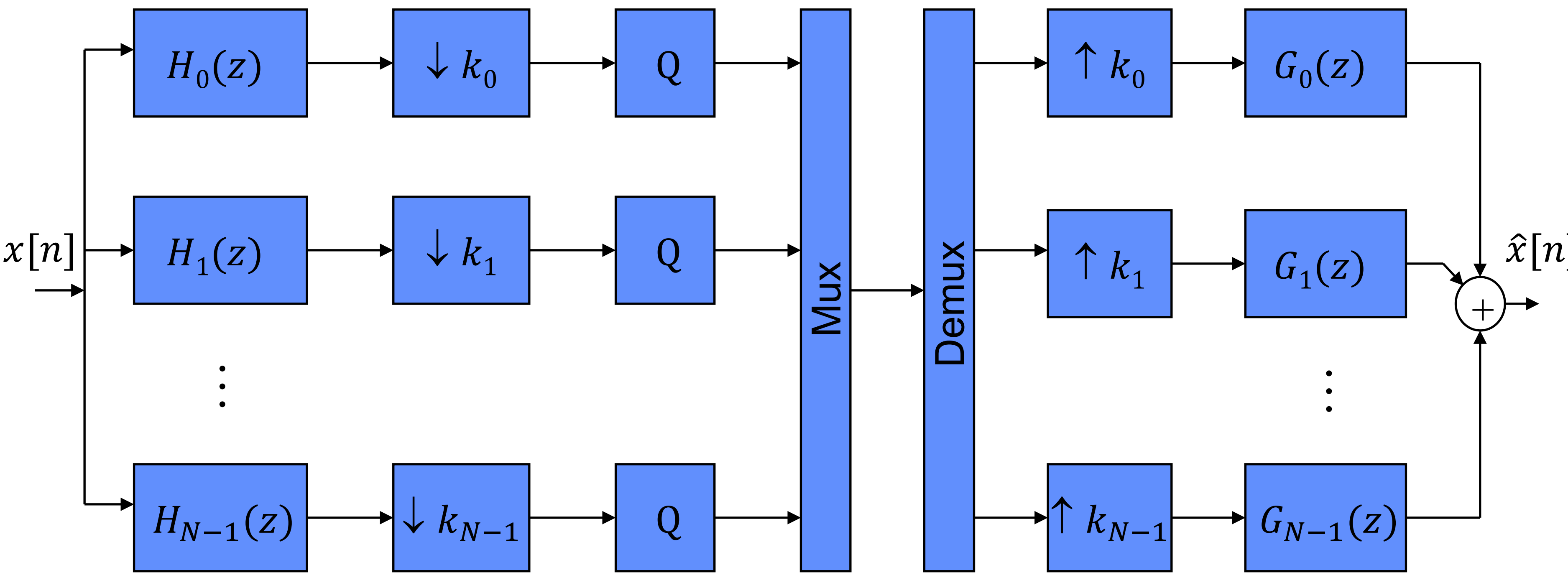
8.4 Bit Allocation for Subband Coding

8.1 Principle of Subband Coding

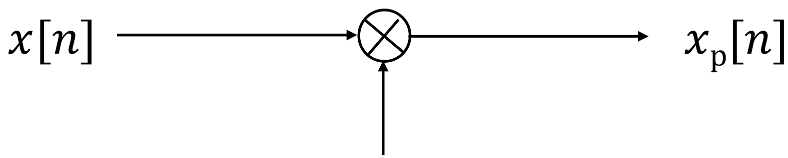


- Subsampling factor typically $k_i = N$, more general $\frac{1}{k_0} + \cdots + \frac{1}{k_{N-1}} = 1$ holds
- Perfect reconstruction filter bank required for overall distortion to be zero, i.e. $x[n] = \hat{x}[n]$

Sampling of Discrete Time Signals

Aim: generate new sequence $x_p[n]$ which is equal to original sequence $x[n]$ at integer multiples of the sampling period N and zero at intermediate samples

Model:



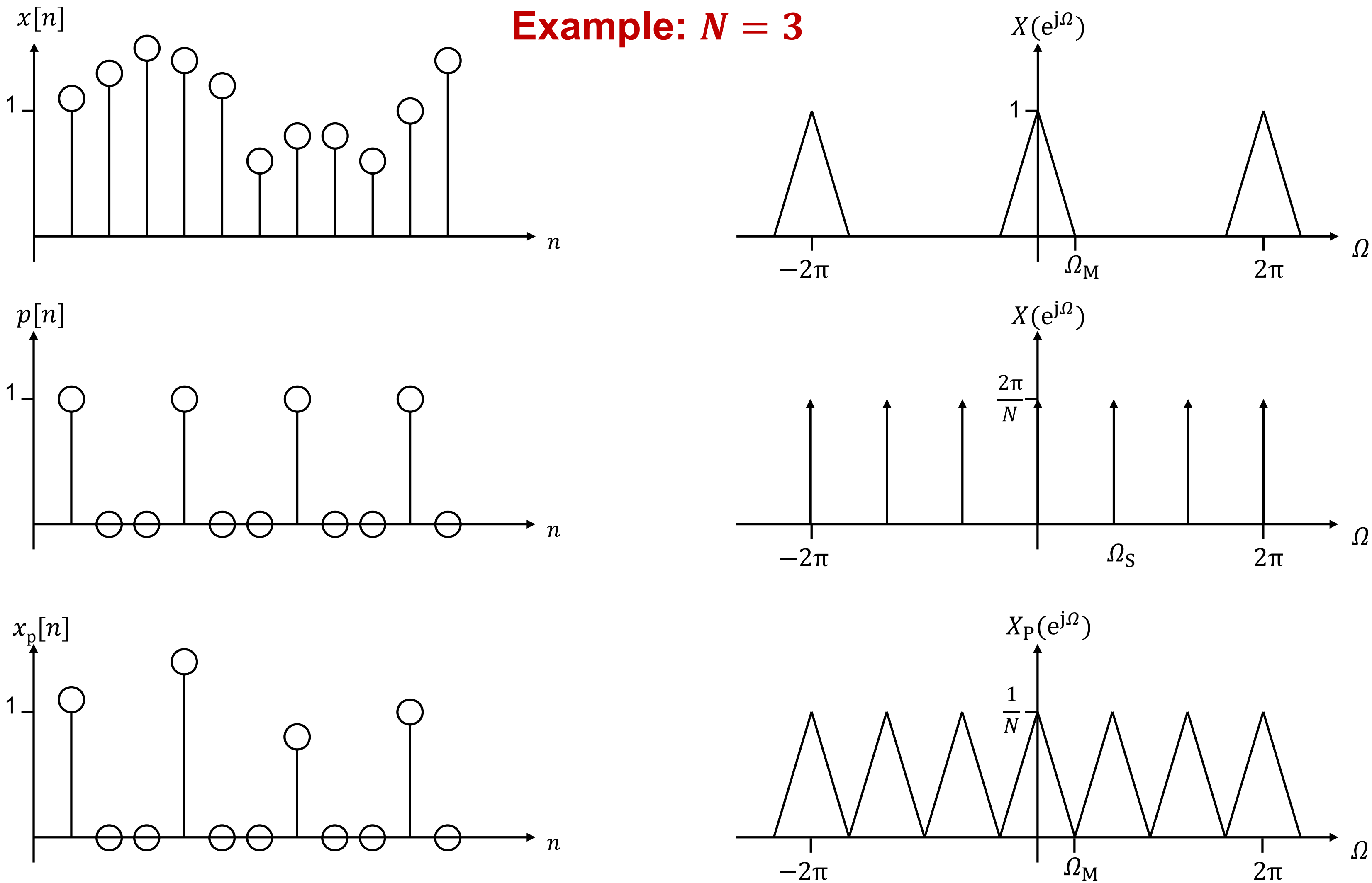
$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN] \quad \circ\!\!-\!\!\bullet \quad P(\mathrm{e}^{\mathrm{j}\Omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_{\mathrm{S}})$$

$$x_{\mathrm{p}}[n] = x[n]p[n] = \sum_{k=-\infty}^{\infty} x[kN]\delta[n-kN]$$

$$\circ\!\!-\!\!\bullet$$

$$X_{\mathrm{p}}(\mathrm{e}^{\mathrm{j}\Omega}) = \frac{1}{2\pi} X(\mathrm{e}^{\mathrm{j}\Omega}) \otimes P(\mathrm{e}^{\mathrm{j}\Omega}) = \frac{1}{N} \sum_{k=0}^{N-1} X(\mathrm{e}^{\mathrm{j}(\Omega - k\Omega_{\mathrm{S}})}) \text{ with } \Omega_{\mathrm{S}} = \frac{2\pi}{N}$$

Sampling of Discrete Time Signals (Cont.)



Discrete Time Decimation

Aim: replace the sampled sequence $x_{\mathrm{p}}[n]$ by a new sequence $x_{\mathrm{d}}[n]$, which is every N^{th} value of $x_{\mathrm{p}}[n]$, thus eliminating sampling instants known to be zero

Model: $x_{\mathrm{p}}[n] \longrightarrow$ Decimation $\longrightarrow x_{\mathrm{d}}[n]$

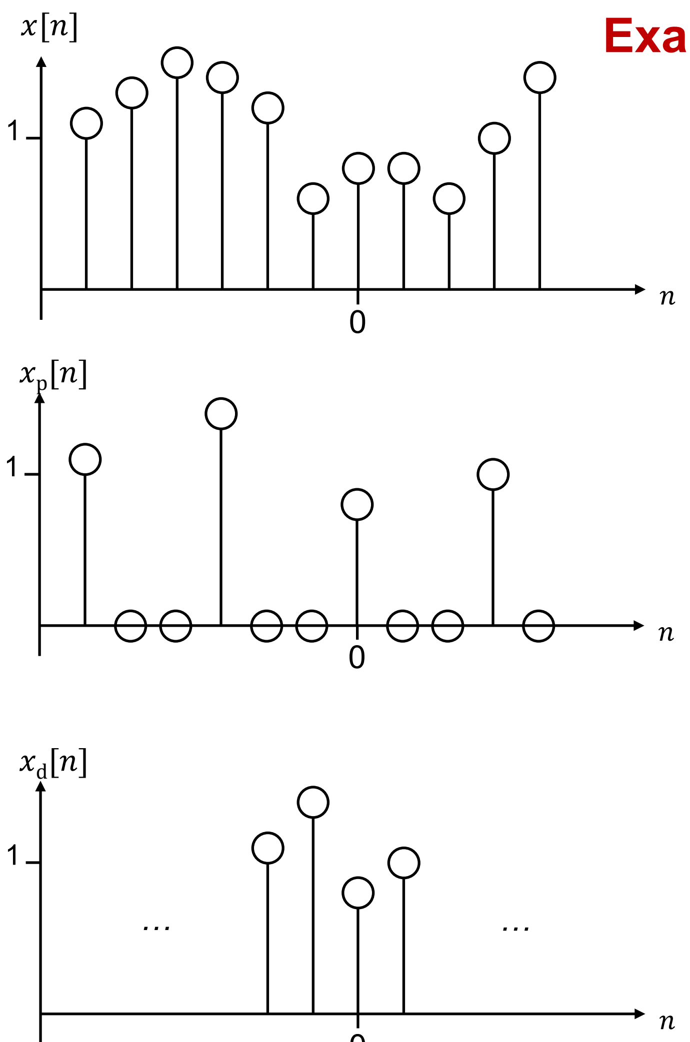
$$x_{\mathrm{d}}[n] = x_{\mathrm{p}}[nN]$$

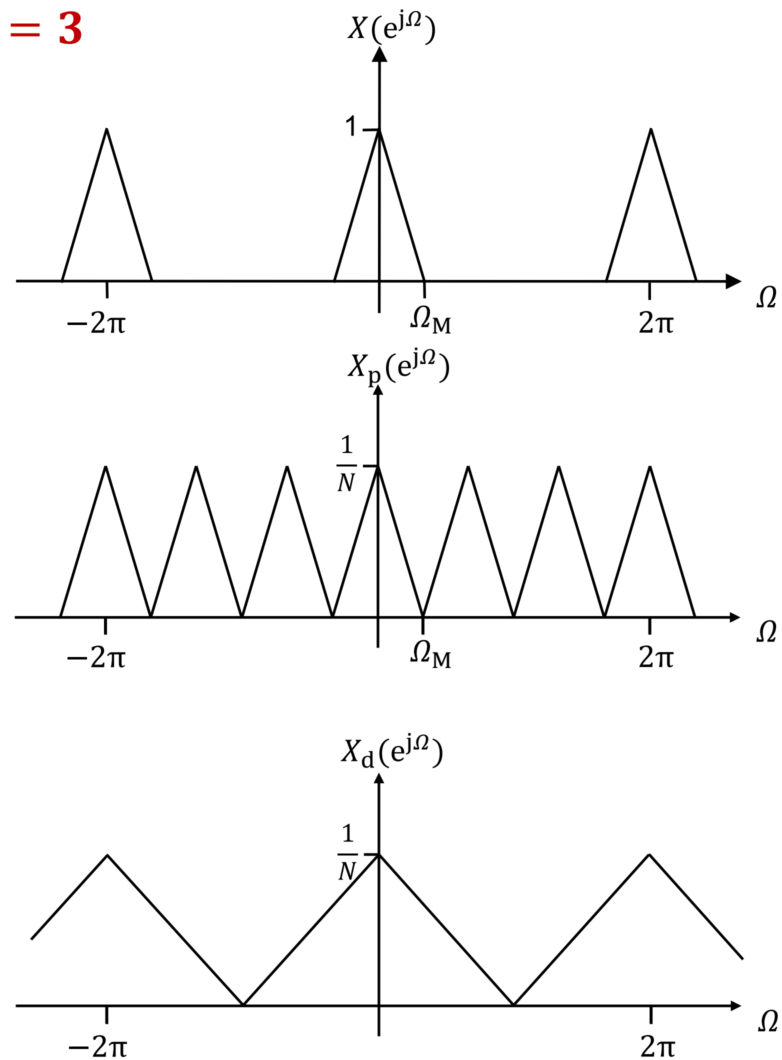
$$\circ\!\!-\!\!\bullet$$

$$X_{\mathrm{d}}(\mathrm{e}^{\mathrm{j}\Omega}) = X_{\mathrm{p}}(\mathrm{e}^{\mathrm{j}\frac{\Omega}{N}})$$

Discrete Time Decimation (Cont.)

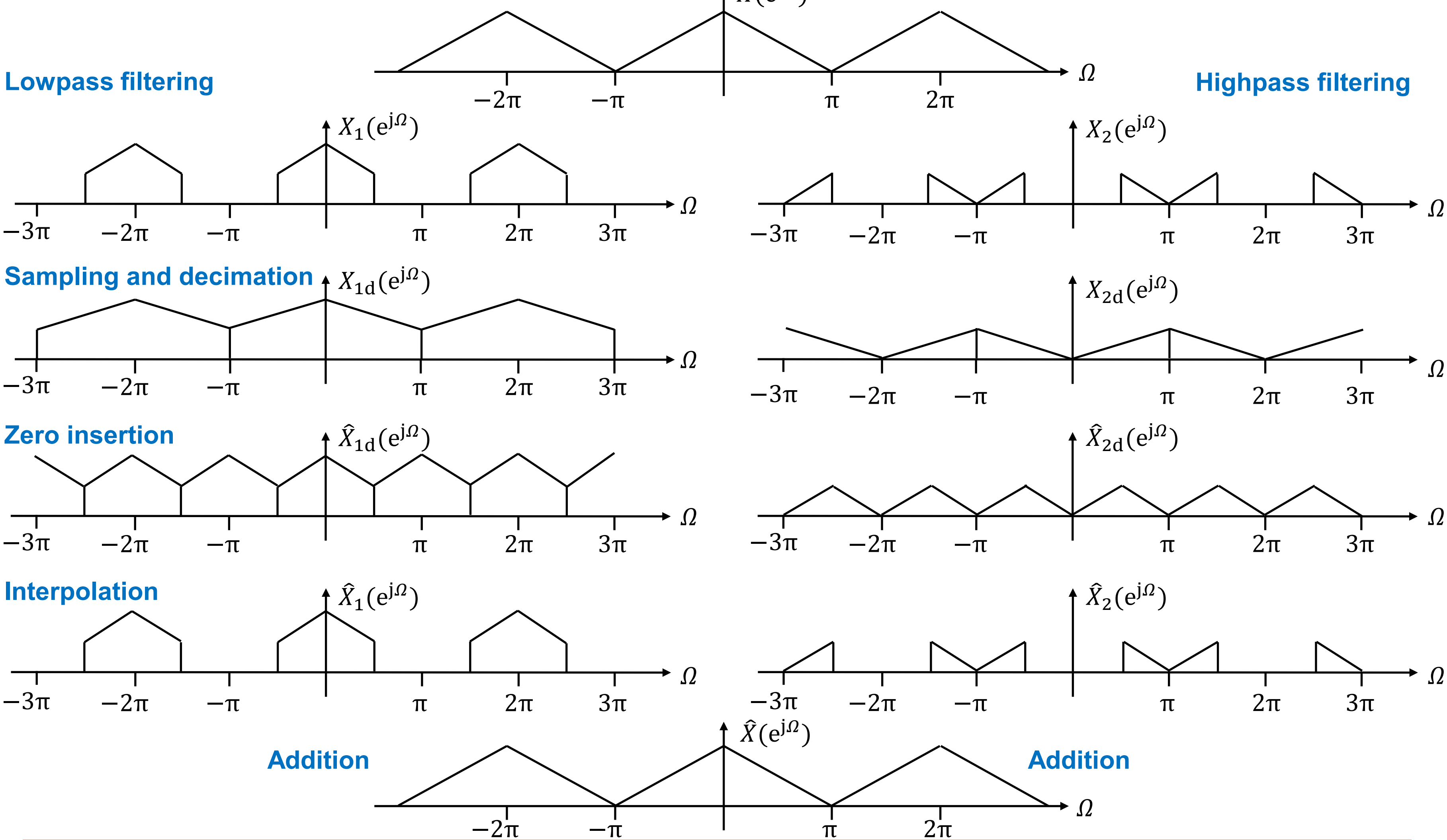
Example: $N = 3$





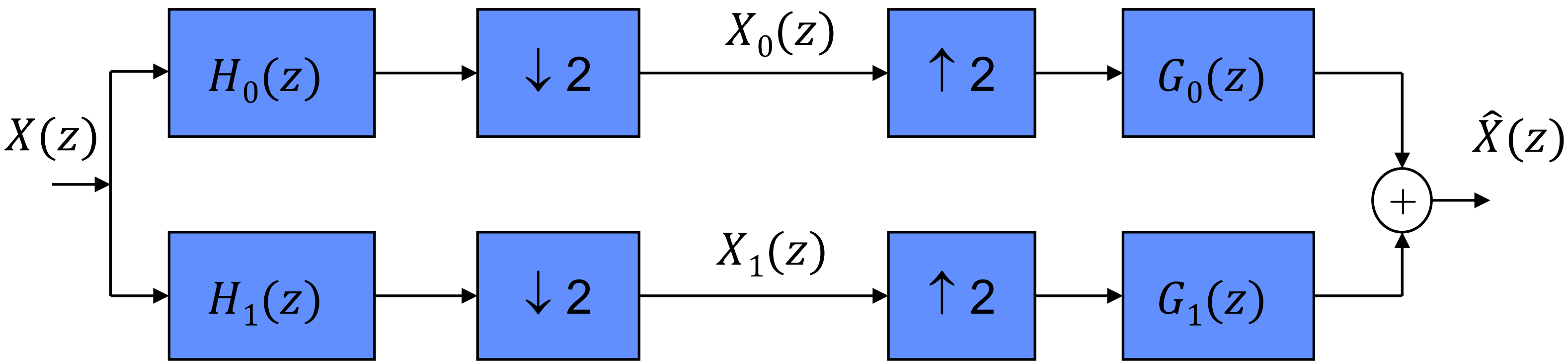
Spectral Interpretation of Subband Filtering

Assumption: ideal filters, critical sampling



8.2 Perfect Reconstruction Property

Restriction: two channel filter bank, $N = 2,\ \ k_0 = k_1 = 2$



Output signals of the two decimators for arbitrary analysis filters $H_0(z)$ and $H_1(z)$

Upper path: $$X_0(z) = \frac{1}{2}\left[X(z^{\frac{1}{2}})H_0(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})H_0(-z^{\frac{1}{2}})\right]$$

Lower path: $$X_1(z) = \frac{1}{2}\left[X(z^{\frac{1}{2}})H_1(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})H_1(-z^{\frac{1}{2}})\right]$$

Alias Cancellation

Output signals of the synthesis section with $X_0(z)$ and $X_1(z)$ as input

$$\hat{X}(z) = X_0(z^2)G_0(z) + X_1(z^2)G_1(z)$$

Substitution yields

$$\boxed{\hat{X}(z) = \frac{1}{2}[H_0(z)G_0(z) + H_1(z)G_1(z)]X(z)}$$

$$\boxed{+\frac{1}{2}[H_0(-z)G_0(z) + H_1(-z)G_1(z)]X(-z)}$$

First term: desired signal output from filter bank
Second term: effect of aliasing, should be eliminated

⇨ **Aliasing cancellation:** $H_0(-z) \cdot G_0(z) + H_1(-z) \cdot G_1(z) = 0$

is achieved by selecting $G_0(z) = H_1(-z) \bullet\!\!-\!\!\circ\ g_0[n] = (-1)^n h_1[n]$

$$G_1(z) = -H_0(-z) \bullet\!\!-\!\!\circ\ g_1[n] = -(-1)^n h_0[n]$$

Perfect Reconstruction

Output signal of filter bank with aliasing compensated:

$$\hat{X}(z) = \frac{1}{2}[H_0(z)H_1(-z) - H_1(z)H_0(-z)]X(z)$$

Perfect reconstruction, i.e. $X(z) = \hat{X}(z)$ requires that

$$H_0(z)H_1(-z) - H_1(z)H_0(-z) = 2$$

Most general formulation for filterbanks with perfect reconstruction of input signal
⇨ no unique solution, different filters may be derived which fulfill above constraint

Practical subband filter design typically requires further constraints such as

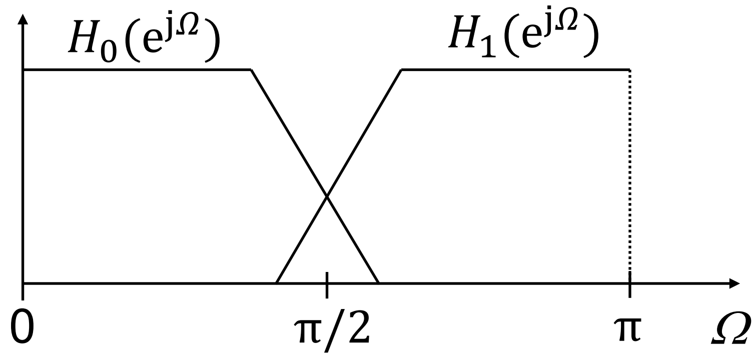
- linear phase filters
- equal bandwidth in low- and highpass components
- orthogonality of subband signals

Quadrature Mirror Filters (QMF)

Based on an arbitrary lowpass filter $F(z)$, the highpass filter is selected to have mirror-image symmetry about $\Omega = \pi/2$. For real valued symmetric filters this gives

$$H_0(z) = F(z) \quad \bullet\!\!-\!\!\circ \quad h_0[n] = f[n]$$
$$H_1(z) = F(-z) \quad \bullet\!\!-\!\!\circ \quad h_1[n] = (-1)^n f[n]$$

Spectral characteristic



Alias cancellation property gives synthesis filters

$$G_0(z) = F(z) \quad \bullet\!\!-\!\!\circ \quad g_0[n] = f[n]$$
$$G_1(z) = -F(-z) \quad \bullet\!\!-\!\!\circ \quad g_1[n] = -(-1)^n f[n]$$

Quadrature Mirror Filters (Cont.)

Perfect reconstruction properties for QMF results in constraint for lowpass filter

$$F^2(z) - F^2(-z) = 2$$

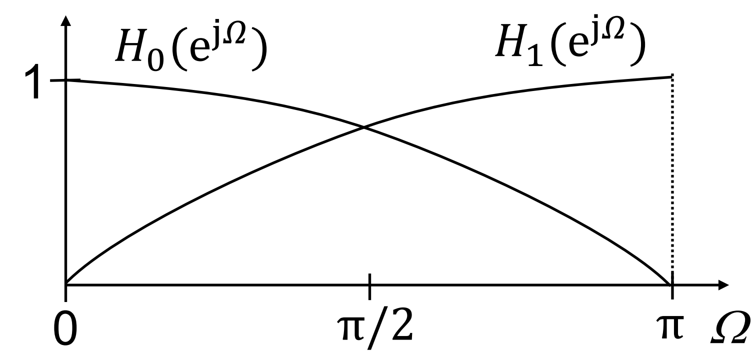
Example: Haar filter basis which is given by lowpass transfer function

$$F(z) = \frac{1}{\sqrt{2}}(1 + z^{-1})$$

1. Step: Derive filter pair using QMF constraint

$$H_0(z) = F(z) = \frac{1}{\sqrt{2}}(1 + z^{-1})$$

$$H_1(z) = F(-z) = \frac{1}{\sqrt{2}}(1 - z^{-1})$$



2. Step: Verify perfect reconstruction property

$$F^2(z) - F^2(-z) = \frac{1}{2}(1 + z^{-1})^2 - \frac{1}{2}(1 - z^{-1})^2 = 2z^{-1}$$

⇨ perfect reconstruction up to a time shift

Alias Cancellation Revisited

Elimination of alias component can also be achieved by $(-z)$ modulation of the analysis filters plus an additional shift of *k* samples

$$G_0(z) = z^k H_1(-z)$$
$$G_1(z) = -z^k H_0(-z)$$

Synthesized signal at output of two-channel filter bank has z-transform which is given by

$$\hat{X}(z) = \frac{1}{2}[H_0(z)H_1(-z) - H_1(z)H_0(-z)]X(z)z^k$$

Perfect reconstruction in this case requires that

$$P(z) - P(-z) = 2z^{-k} \quad \text{with} \quad P(z) = H_0(z)H_1(-z)$$

⇨ Term z^{-k} expresses an arbitrary phase shift occurring anywhere in the analysis / synthesis filtering process

Example: Bi-orthogonal Filter

Above relaxed conditions for alias-free and perfect reconstruction can e.g. be fulfilled with a filter pair having the following transfer functions

$$H_0(z) = \frac{1}{8}(-z^1 + 2 + 6z^{-1} + 2z^{-2} - z^{-3})$$
$$H_1(z) = \frac{1}{2}(-z^1 + 2 - z^{-1})$$

Synthesis filters are derived by applying the alias cancellation constraint with $k = 1$

$$G_0(z) = z^k H_1(-z) = \frac{1}{2}(z^2 + 2z + 1)$$
$$G_1(z) = -z^k H_0(-z) = \frac{1}{8}(-z^2 - 2z^1 + 6 - 2z^{-1} - z^{-2})$$

The above filter pair $H_0(z), H_1(z)$ is

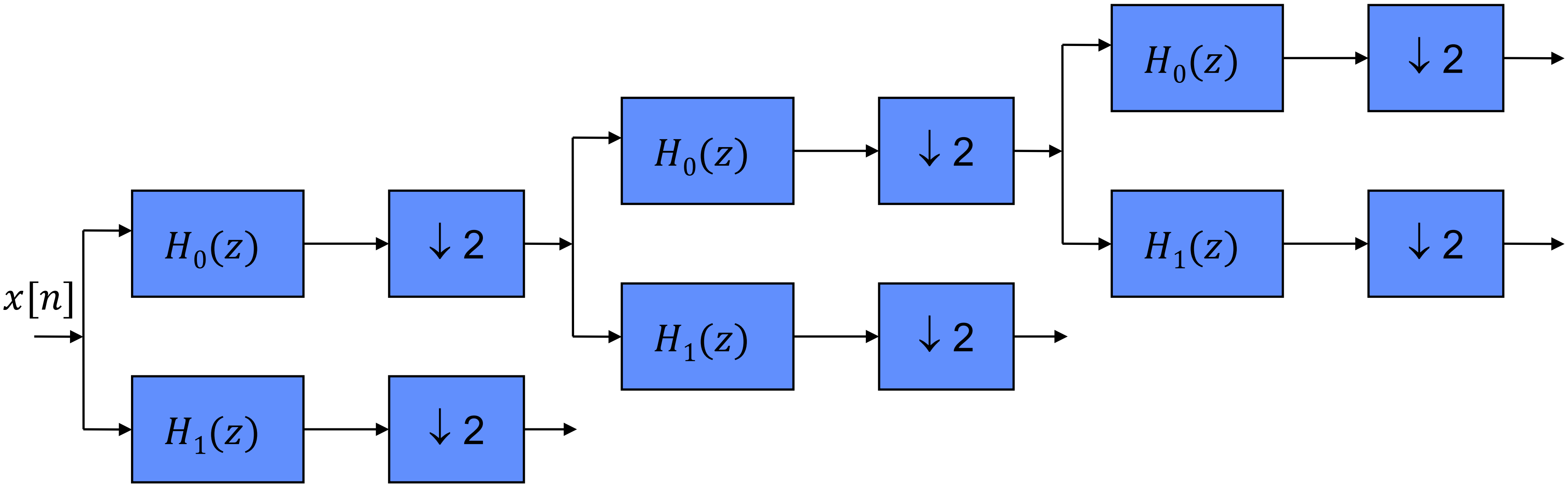
- an example of so-called **bi-orthogonal filter**
- used in discrete wavelet filtering and is also part of the JPEG2000 Standard ("LeGall 5/3 filter")

Numerical Example for Bi-orthogonal Filter

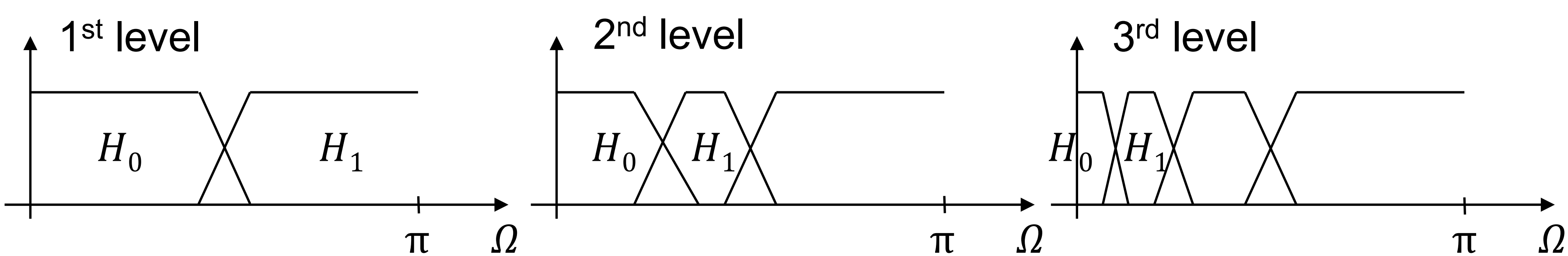
Sample signal:	...	100.00	100.00	100.00	200.00	200.00	200.00	200.00	...
LP after H_0:	...	100.00	100.00	87.50	112.50	187.50	212.50	200.00	...*)
HP after H_1:	...	0	0	-50.00	50.00	0	0	0	...
$\downarrow / \uparrow$ LP	…	100.00	0	87.50	0	187.50	0	200.00	…
$\downarrow / \uparrow$ HP:	...	0	0	-50.00	0	0	0	0	...
LP after G_0:	…	93.75	87.50	137.50	187.50	193.75	200.00	200.00	..**)
HP after G_1:	…	6.25	12.50	-37.50	12.50	6.25	0	0	…
Summation:	...	100.00	100.00	100.00	200.00	200.00	200.00	200.00	...

Note: applying symmetric filter masks H_0 and G_0, the LP signal has to be shifted right (*) and left (**) which in practice in done by shifting the down- / upsampling process of the LP signal

8.3 Discrete Wavelet Transform (DWT)



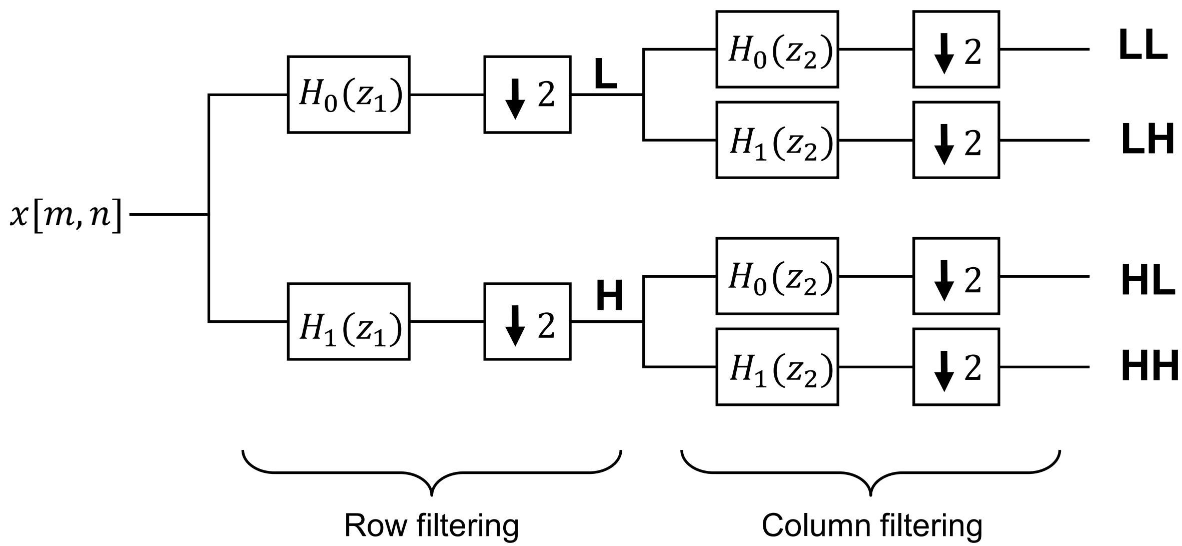
Wavelet transform: recursive application of a two-band filter bank to lowpass band of previous stage yields octave band splitting



Two-Dimensional Wavelet Transform

Separable filter bank: analysis and synthesis filters are product of horizontal and vertical filters

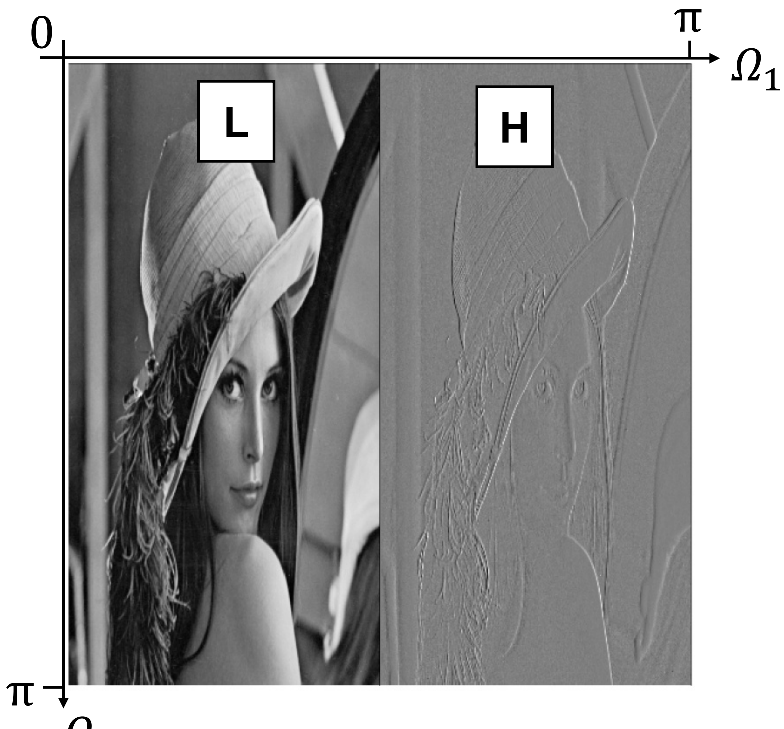
Block diagram for sequential realization



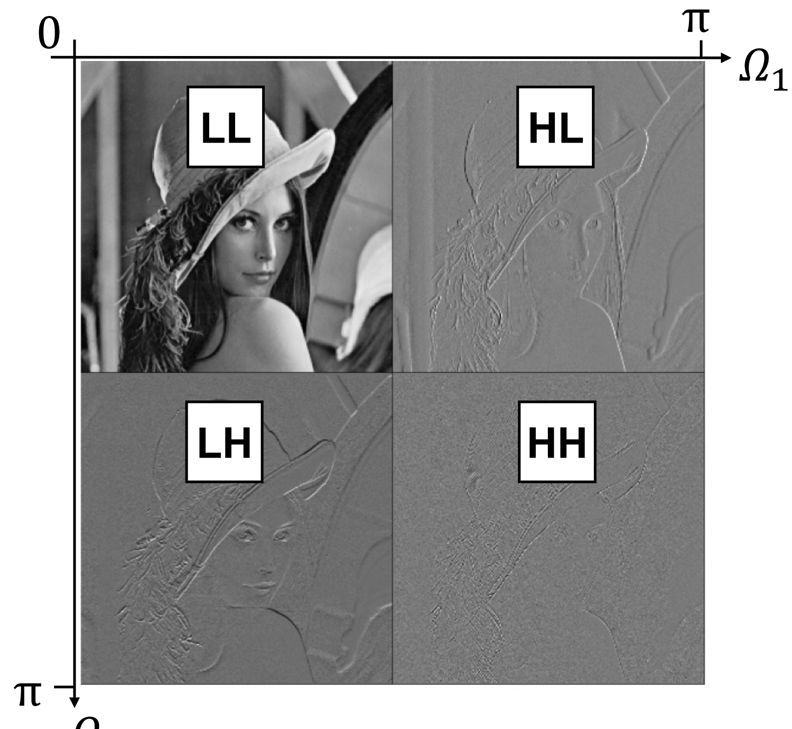
- Perform filtering and subsampling first in one dimension
- Decompose reduced number of samples into second dimension

Two-Dimensional Wavelet Transform (Cont.)

Layout of subbands in 2D frequency domain, example: Haar-Wavelet



After row filtering



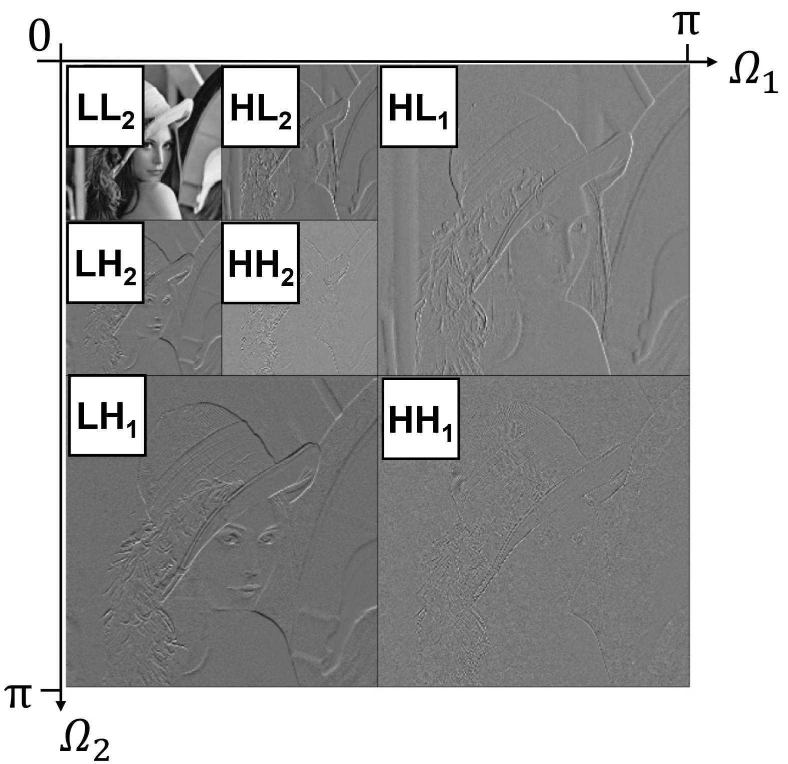
After column filtering

Demo 8 „Discrete Wavelet Transform“

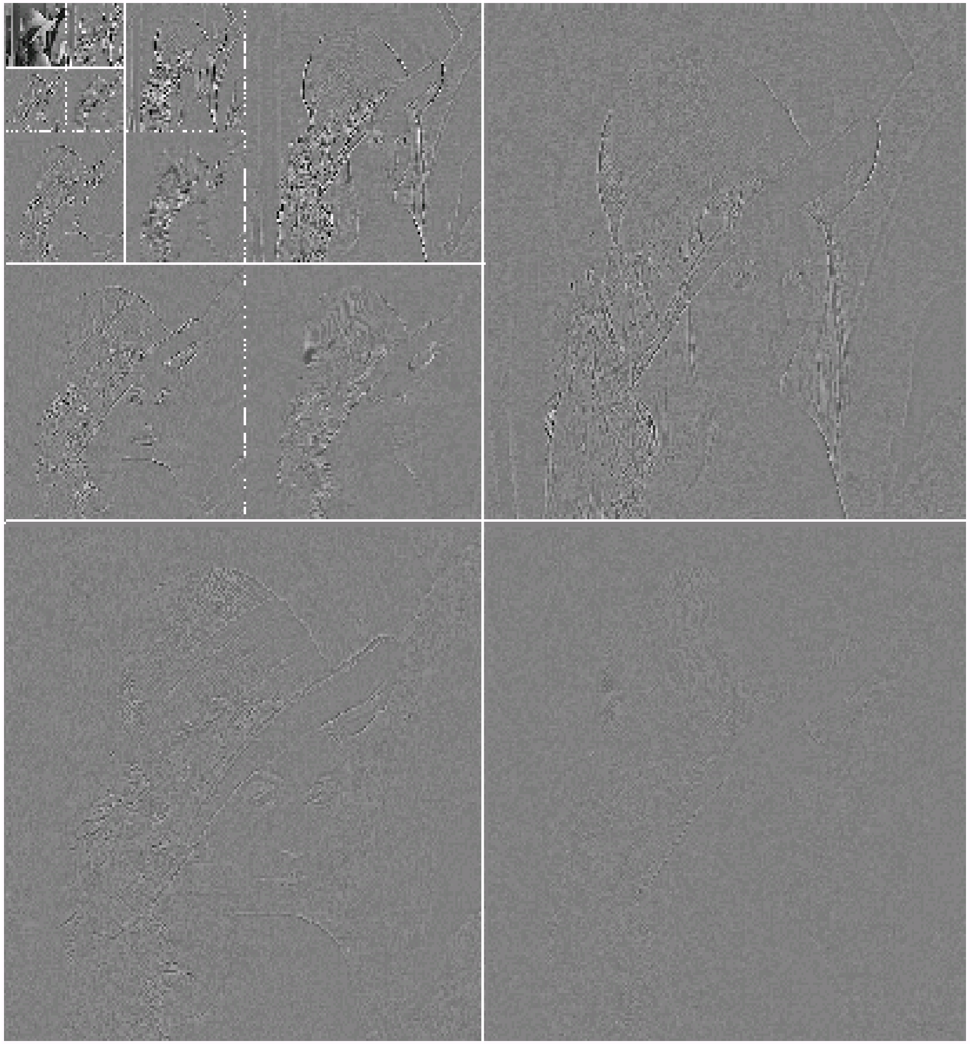
Two-Dimensional Wavelet Transform (Cont.)

Recursive application of two-band filterbank to 2D lowpass band of previous stage yields octave band splitting in 2D

Layout of subbands in 2D frequency domain for two level DWT

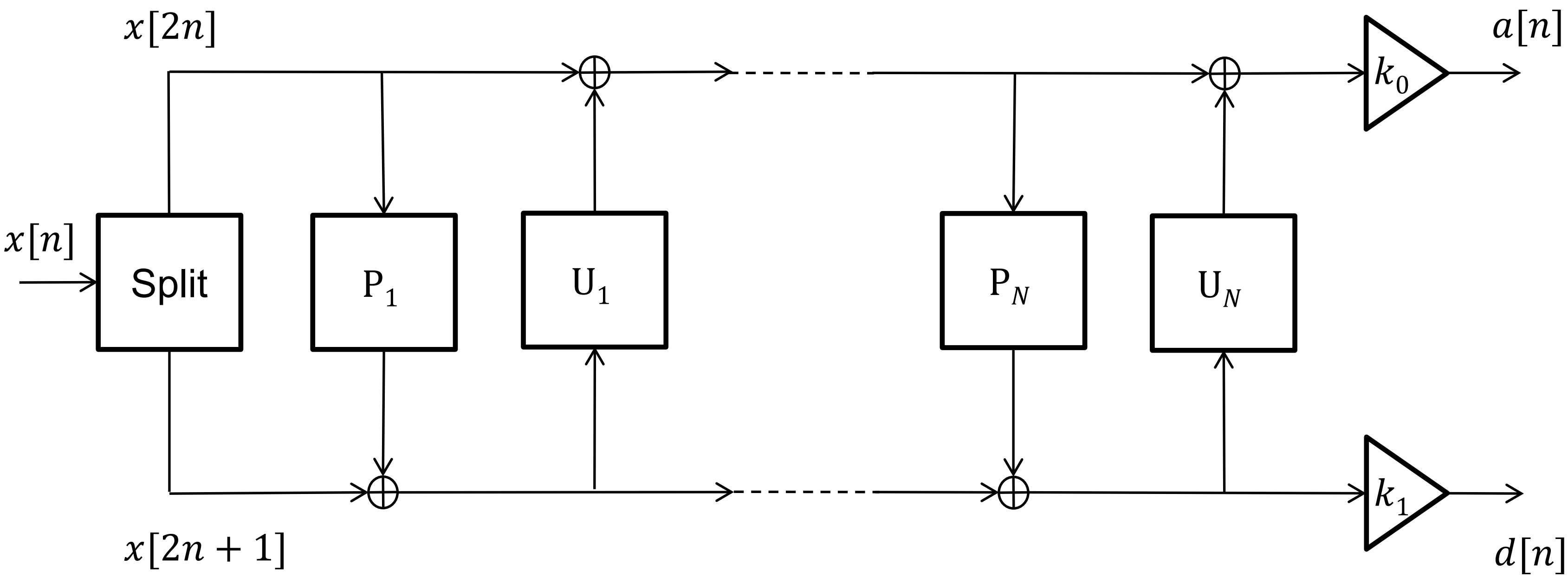


2D DWT With 4 Decomposition Levels



Lifting

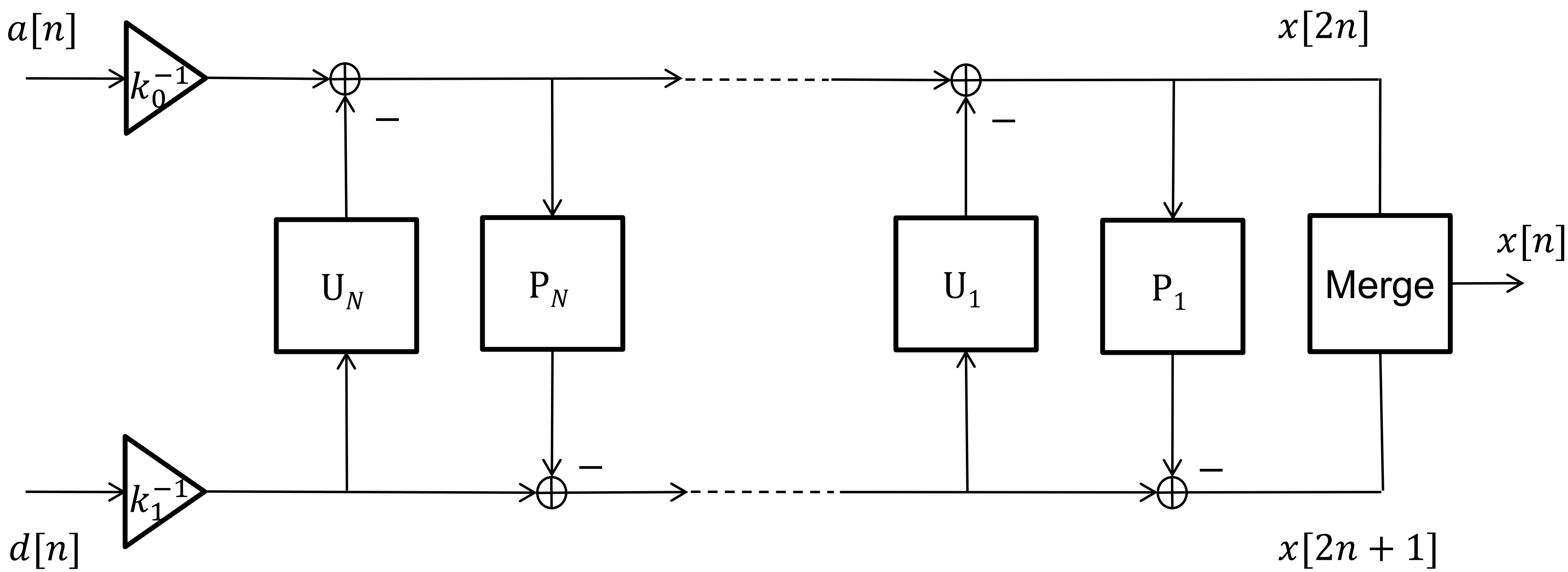
Analysis filter structure for efficient implementation of a wavelet transform proposed by Wim Sweldens in 1995:



- Separation of input signal $x[n]$ into even $x[2n]$ and odd $x[2n+1]$ samples
- Concatenation of N lifting steps working on even and odd signal samples
- P_i is called "prediction step", U_i "update step"
- Output $a[n]$ can be regarded as low band signal, $d[n]$ as high band signal

Lifting (Cont.)

Synthesis filter structure of lifting implementation:



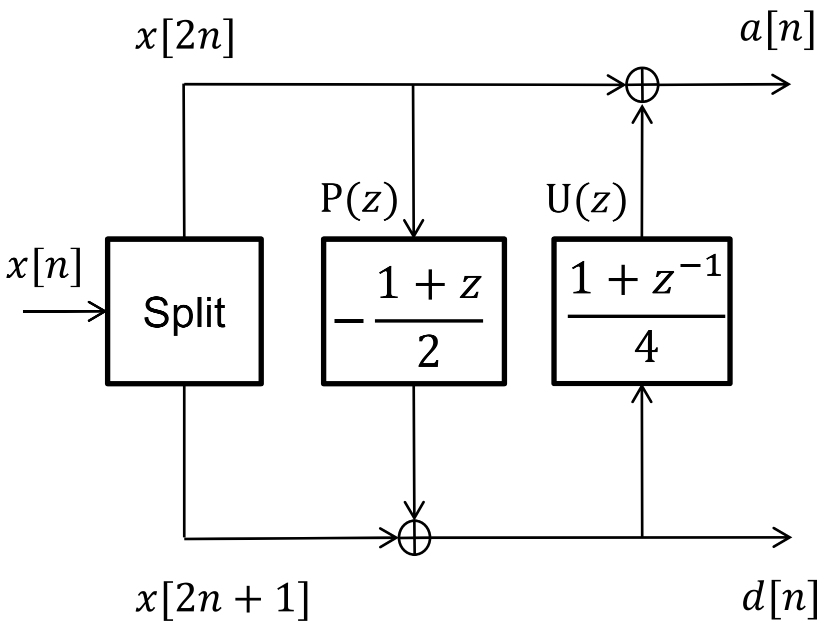
- Reverse structure as analysis part with inverted sign ("-") at summation points
- Perfect reconstruction guaranteed independent of which prediction step P and update step U are used

Example for Lifting Implementation

Consider **LeGall 5/3 filter** with impulse response functions

$$h_0[n] = \frac{1}{8}\{-1;\ 2;\ 6;\ 2;\ -1\} \quad \text{and} \quad h_1[n] = \frac{1}{2}\{-1;\ 2;\ -1\}$$

Lifting implementation is given by



Proof:

$$d[n] = x[2n+1] \overbrace{- \frac{1}{2}(x[2n] + x[2n+2])}^{P(z)}$$

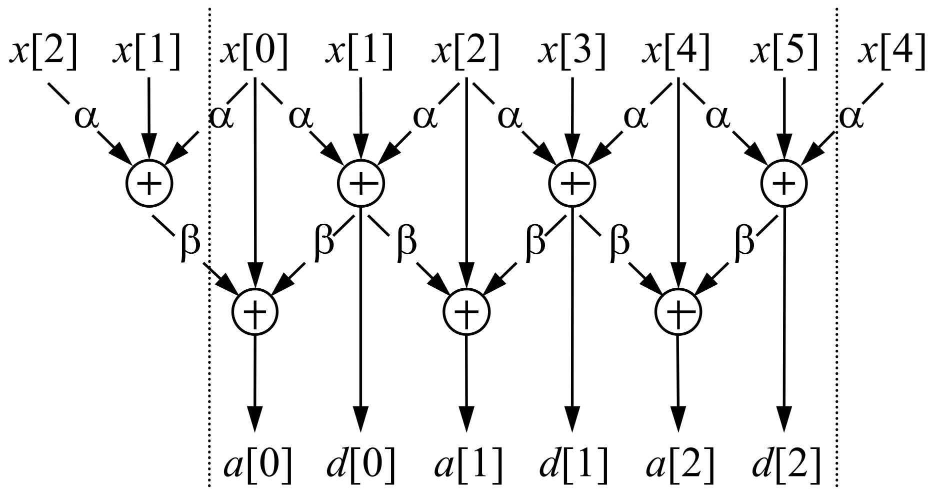
$$= -\frac{1}{2}x[2n] + x[2n+1] - \frac{1}{2}x[2n+2]$$

$$a[n] = x[2n] \overbrace{+ \frac{1}{4}(d[n-1] + d[n])}^{U(z)}$$

$$= -\frac{1}{8}x[2n-2] + \frac{1}{4}x[2n-1] + \frac{6}{8}x[2n] + \frac{1}{4}x[2n+1] - \frac{1}{8}x[2n+2]$$

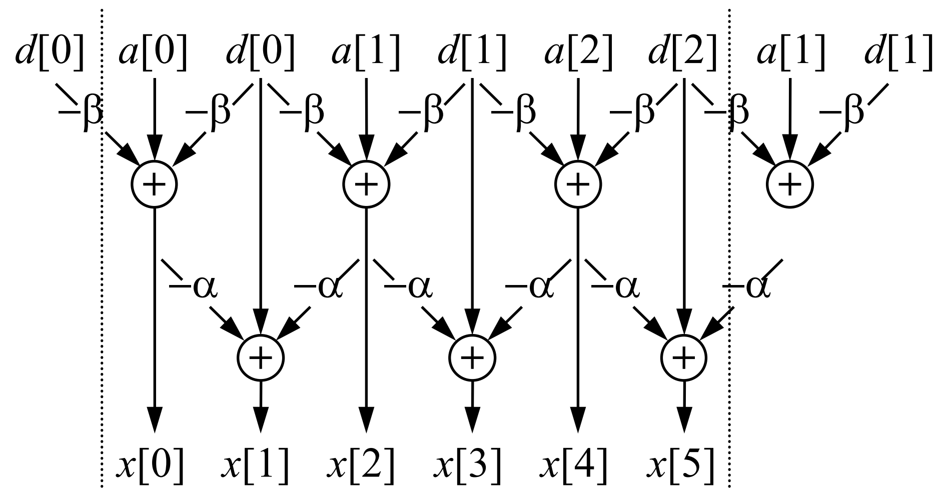
Signal Flow Graph for 5/3 Lifting Implementation

Analysis filter



Symmetrical extension,
α = -0.5, β = 0.25

Synthesis filter



8.4 Bit Allocation for Subband Coding

Coding problem: distribute a limited number of bits among B bands (each having η_b% of all samples) such that the resulting distortion d is minimized

High-rate distortion-rate function for image pixels x

$$d(R) \cong \varepsilon^2 \sigma_X^2 2^{-2R}$$

- Distortion-rate function for coding subband samples (cf. transform coding):

$$d_{\mathrm{SUB}}(R) = \sum_{b=0}^{B-1} \eta_b d_b(R_b) \cong \sum_{b=0}^{B-1} \eta_b \varepsilon^2 \sigma_{Y_b}^2 2^{-2R_b}$$

⇨ Minimize $d_{\mathrm{SUB}}(R)$ subject to rate constraint $R = \sum_{b=0}^{B-1} \eta_b R_b$

Optimum bit allocation can be found using Lagrangian formulation

$$\underset{R_0, R_1, \ldots, R_{B-1}}{\operatorname{argmin}} (J) \quad \text{with} \quad J = d_{\mathrm{SUB}}(R) + \lambda R$$

and setting the partial derivatives equal to zero:

$$\frac{\partial J}{\partial R_b} = 0 \quad \forall b$$

Subband Coding Gain

Optimum bit allocation is achieved as result of equating $\mathrm{d}J/\mathrm{d}R_b = 0$ if

- all subbands have equal distortion

$$d_b(R_b) = d_{\mathrm{SUB}}(R) \quad \forall b$$

- rate per subband sample is proportional to subband variance

$$R_b = \frac{1}{2}\log_2 \frac{\varepsilon^2 \sigma_{Y_n}^2}{d_{\mathrm{SUB}}} \quad \forall b$$

Subband coding gain defined as

$$G_{\mathrm{SUB}} = \frac{d(R)}{d_{\mathrm{SUB}}(R)} = \frac{\sigma_X^2}{\prod_{b=0}^{B-1}\left(\sigma_{Y_n}^2\right)^{\eta_b}} = \frac{\sum_{b=0}^{B-1} \eta_b \sigma_{Y_n}^2}{\prod_{b=0}^{B-1}\left(\sigma_{Y_n}^2\right)^{\eta_b}}$$

- Coding gain increases if signal variances are unequally distributed among different subbands
- No gain in case all subband variances are (almost) equal

Relation to Transform Coding

Transform coding can be regarded as special case of subband coding, block transforms have one coefficient per band, i.e.

$$\eta_b = \frac{1}{B}$$

⇨ Coding gain: G_{SUB} simplifies to G_{XFORM}

Equivalence: subband coder has an identical transform coder representation if the following conditions are fulfilled

- Number of subbands is identical to transform (block) length N
- Subsampling factor k is identical to N
- Impulse response of analysis / synthesis filters does not exceed N

Example: Haar basis functions interpreted as QMF filter bank

While transforms used in image coding are generally orthogonal, this does not necessarily hold for filters used in subband coding

Subband Coding - Summary

- Perfect reconstruction property as necessary condition
- Cascading filters leads to dyadic frequency scaling
- Efficient lifting implementation possible
- Optimum bit allocation achieves equal error variance
- Coding gain is given by ratio of arithmetic to geometric mean of weighted signal variance in subbands
- Transforms can be regarded as special case of subband signal decomposition