Università della Svizzera italiana

Stochastic Methods **Academic Year 2020/2021**

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Assignment 5 - Solution [Ashutosh Singh]

Exercise 1: Irreducible and Aperiodic Markov Chains

(a)

$$P^{(2)} = \begin{bmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{bmatrix}$$

Since $p_{ii}^{(2)} > 0$ for all i. This chain is irreducible

$$k = gcd\{t > 0 : P(X_t = s_i | X_0 = s_i) > 0\} \tag{1}$$

$$k = gcd\{2, 3, 4, 5, \}$$

$$\Rightarrow k = 1$$

This chain is aperiodic

(b) Even after a lot of steps $p_{21}^{(n)} = 0$. This chain is not irreducible

$$k = gcd\{1, 2, 3, 4, 5, \}$$

$$\Rightarrow k = 1$$

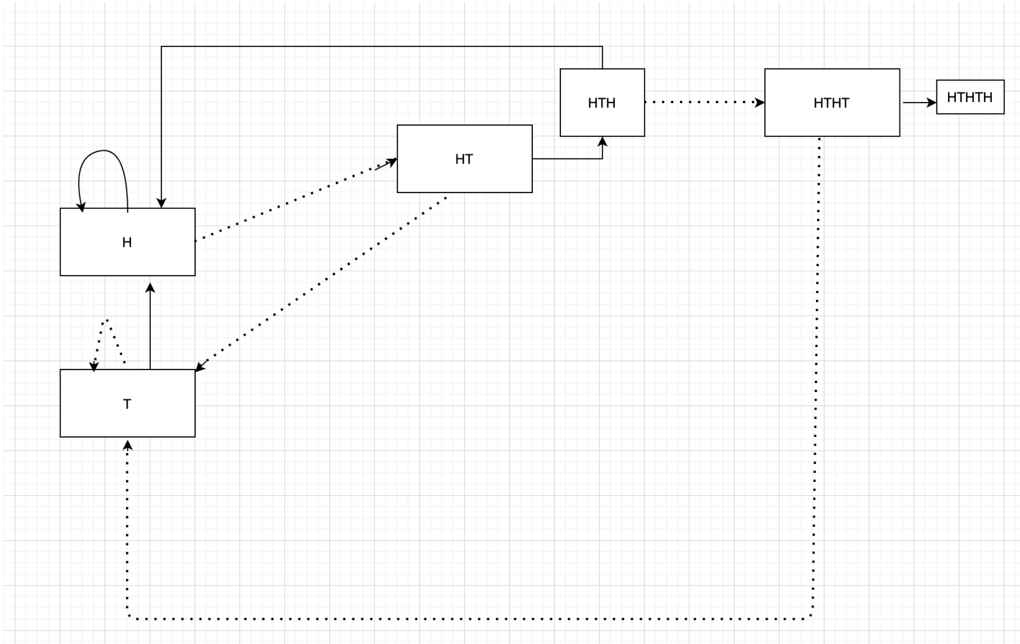
This chain is aperiodic

(c)

$$P^{(30)} = \begin{bmatrix} 0.222 & 0.3333 & 0.4444 \\ 0.222 & 0.3333 & 0.4444 \\ 0.222 & 0.3333 & 0.4444 \end{bmatrix}$$

Since $p_{ii}^{(2)} > 0$ for all i. This chain is irreducible

Exercise 2: Tossing a Fair Coin



We want HTHTH. We can model this chain with 6 states

- Bold arrow suggest movement of chain on heads
- Dashed arrow suggest movement on tails
- Each arrow has weight $\frac{1}{2}$
- Starting from S we can move to H or T
- Once at T only possible states are H or T again. Because we will have to restart for the sequence we want from H.
- From H we can move to HT(Progression) or back to H
- From HT we can move to HTH(Progression) or back to T
- So on till we reach HTHTH

States = {S, H, T, HT, HTH, HTHT, HTHTH}

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The last state is the absorbing state

$$P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}$$

$$\Phi = (I - Q)^{-1}$$

$$\tau = \Phi \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\tau = \begin{bmatrix} 42 \\ 40 \\ 42 \\ 38 \\ 32 \\ 22 \end{bmatrix}$$

Average number of steps $= mean(\tau))$

$\Rightarrow$Average number of steps $= 36$

Exercise 3: Markov Process and Stationary Distribution

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0 & 0.9 \end{bmatrix}$$

$$\pi^* = \pi^* P$$

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$$\pi^*(P - \lambda I) = 0$$

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} -1 & 0.5 & 0.5 \\ 0.2 & -0.5 & 0.3 \\ 0.1 & 0 & -0.1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$-x + 0.2y + 0.1z = 0 \tag{2}$$

$$0.5x - 0.5y = 0 \tag{3}$$

$$0.5x + 0.3y - 0.1z = 0 \tag{4}$$

$$x = y \tag{5}$$

$$8x = z \tag{6}$$

$$x + y + z = 1 \tag{7}$$

$$x + x + 8x = 1 \tag{8}$$

$$x = 0.1 \tag{9}$$

$$y = 0.1 \tag{10}$$

$$z = 0.8 \tag{11}$$

$$\pi^* = \begin{bmatrix} 0.1 & 0.1 & 0.8 \end{bmatrix}$$

Exercise 4: Time to Absorption and Absorption Probabilities

(a) No the sum of last row is greater than 1

$$P_{mod} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Even after a lot of steps $p_{21}^{(n)} = 0$. This chain is not irreducible

$$k = gcd\{1, 2, 3, ..\} \Rightarrow k = 1$$

Chain is not periodic

(c) This chain is absorbing.

- State-1 is absorbing.
- State-4 is absorbing
- State-2 and State-4 are transient.
- It is possible to go from all transient states to all absorbing states

(d)

$$P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}$$

$$R = \begin{bmatrix} \frac{1}{4} & 0 \\ \\ 0 & \frac{1}{3} \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$\Phi = (I - Q)^{-1}$$

$$\Phi = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}^{-1}$$

$$\tau = \Phi \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\tau = \begin{bmatrix} 3.67 \\ 3.33 \end{bmatrix}$$

- Starting from s2 3.67 steps on average to absorption
- Staring from s3 3.33 steps on average to absorption

(e) From part-d we get R and Φ

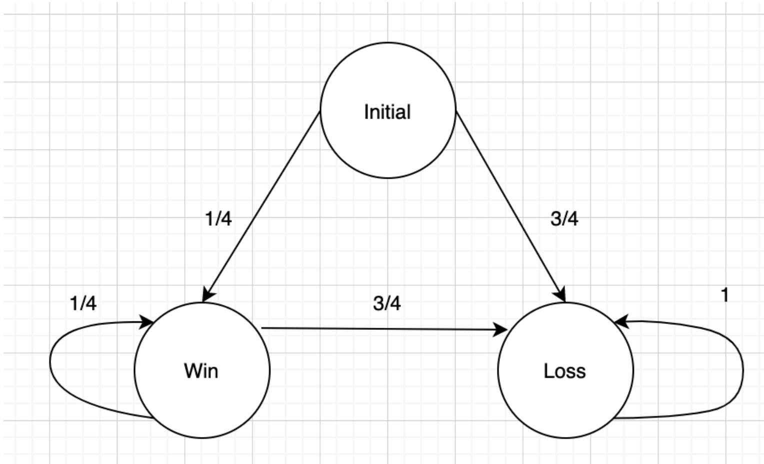
$$B = \Phi R$$

$$B = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

- Given that chain starts in s2. Probability of being absorbed in s1 is 2/3 and s4 is 1/3
- Given that chain starts in s3. Probability of being absorbed in s1 is 1/3 and s4 is 2/3

Exercise 5: The Winner Takes It All

(a)



$$P = \begin{bmatrix} 0 & 1/4 & 3/4 \\ 0 & 1/4 & 3/4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{(5)} = \begin{bmatrix} 0 & 0.000976 & 0.999 \\ 0 & 0.000976 & 0.999 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the Markov chain is Absorbing.

$$\pi^* = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

(b)

$$P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 1/4 \\ 0 & 1/4 \end{bmatrix}$$

$$\Phi = (I - Q)^{-1}$$

$$\tau = \Phi \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\tau = \begin{bmatrix} 3.67 \\ 3.33 \end{bmatrix}$$