Università
della
Svizzera
italiana

Stochastic Methods **Academic Year 2020/2021**

Prof. Dr. Illia Horenko (illia.horenko@usi.ch) Edoardo Vecchi (edoardo.vecchi@usi.ch)

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Assignment 1

Exercise 1: Mean, Covariance and Correlation in Matlab *(3 Points)*

(a) You are interested in estimating the empirical mean value, covariance and correlation of a given sample set. Write two functions to perform these computations: in the function `estFor()` you will use *for loops*, while in `estVect()` you will vectorise the operations (no loops are allowed). Then, create a script `ex1a.m` in which you test your two functions on the sample set X_i for $i = 1, ..., N$ uploaded on iCorsi (`X.mat`) and compare the performance of both variants using the `tic` and `toc` commands already implemented in Matlab. Provide also a plot of the results.

(b) Use the command `mvnrnd()` to generate random $X_i \sim \mathcal{N}(\mu, \Sigma)$ for $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$. Write a script `ex1b.m`, in which you compute the empirical mean value, covariance and correlation for the sample set $(X_i)_{i=1,...,N}$ with $N = 20, 50, 100, 200, 500, 3000, 10^4, 10^5, 10^6, 10^7, 10^8$ by using the function `estVect()` implemented in the previous point. Compare the results obtained with the exact values (i.e., the ones you used to generate the random series) and plot the norm of the difference (select the most appropriate type of plot to visualise your results).

Exercise 2: Application of the Central Limit Theorem *(2 Points)*

We consider the experiment of flipping a fair coin 800 times. Use the CLT[1] to estimate:

[1] **Central Limit Theorem (CLT)**: Let $X_1, X_2, \ldots, X_n$ be a sequence of i.i.d. random variables with finite expected value $E(X_i) = \mu < \infty$ and variance $\sigma^2 > 0$. If we define a new variable Z_n as:

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}, \text{ with } S_n = \sum_{i=1}^{n} X_i,$$

then $Z_n \xrightarrow{d} \mathcal{N}(0, 1)$ or, equivalently, $\lim_{n\to\infty} P(Z_n \leq z) = \Phi(z)$, where $\mathcal{N}(0, 1)$ is a normal distribution with mean 0 and variance 1 (while $\Phi(z)$ is its cumulative distribution).

(a) the probability of obtaining more than 415 heads.

(b) the probability of obtaining between 230 and 255 heads.

Exercise 3: Law of the Total Probability *(1.5 Points)*

A student realises during the class that the charger for his laptop is missing. With a $\frac{1}{4} : \frac{1}{2} : \frac{1}{4}$ ratio the charger is at home, at the library, or was left in the train on the way to USI, respectively. Furthermore, the chances to find the charger at the library are 90% and to find the charger in the train on the way home are 50%. Please estimate:

(a) the probabilities of finding the charger at the library and in the train.

(b) the probability that the charger was left at home.

(c) the probability that the charger will not be found.

Exercise 4: Bayes' Theorem *(2 Points)*

Solve the following two problems concerning the application of Bayes' Theorem.

(a) 0.5% of a given population is infected with a dangerous virus. A diagnostic test for the identification of the virus is positive in 99% of the cases for infected people and in 2% of the cases for not infected people. Please estimate the probability that a person whose test was positive is infected with the virus.

(b) Three vaccines (V_1, V_2 and V_3) against a certain disease are currently available. A person which takes the vaccine will not develop the symptoms in 95% of the cases with V_1, in 89% of the cases with V_2 and in 83% of the cases with V_3. Let us suppose that all the population gets vaccinated and the vaccines used are V_1 (84%), V_2 (5%) and V_3 (11%). Knowing that a person exhibits symptoms compatible with the disease, what is the probability he received vaccine V_1?

Exercise 5: Fixed-Point Iteration *(1.5 Points)*

(a) We consider the function $f(x) = \exp(-x) - 0.5x$ for $x \in [0, 1]$. We want to use Banach Theorem to approximate the root of $f(x)$: what is an appropriate iterative sequence?

(b) Using the iterative formulation in (a), we want to compute the approximation of the root with an accuracy of 0.015. Starting from the point $x_0 = 0.2$, how many iterations are necessary to get the required accuracy?

Please write a detailed report with your solutions using the LaTeX template provided on iCorsi.

!!! The code has to be well commented !!!