An Introduction to Non-parametric Bayesian Statistics

Reza Solgi

reza.solgi@usi.ch

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- Bayesian non-parametric models provide a flexible modeling setting in which the model itself can infer about the necessary amount of flexibility for the data in hand.
- In fact a non-parametric model can be thought of the limit of a parametric model with a finite number of parameters, when we take the number of parameters to infinity.
- A suitable non-parametric model should have a large support; Ideally the support of the prior should be dense in the set of all plausible models. Moreover the hyperparameters of the model should be interpretable; so we can assign reasonable values to them, and last but not least, the model should be computationally tractable.

Example 1: Density Estimation

Consider a univariate density estimation problem:

$$x_i \overset{ind.}{\sim} F$$

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We should first restrict the family of distribution to a family of parametrized distribution, say Normal distributions:

$$\begin{aligned} \mathcal{F}_{\text{param}} &:= \{\text{all Normal distributions with mean } \mu \in \mathbb{R}, \text{ and variance } \sigma^2 \in \mathbb{R}_+\} \\ &\subset \mathcal{F} \end{aligned}$$

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Note that support of $\mathcal{F}_{\text{param}}$ is dramatically restricted w.r.t. the support of $\mathcal{F}$. In fact $\mathcal{F}_{\text{param}}$ is not dense in $\mathcal{F}$, in the sense that for any $F \in \mathcal{F}$ we can not find a member of $\mathcal{F}_{\text{param}}$ sufficiently close to F.

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$\mathcal{F}$ is an infinite dimensional space, while $\mathcal{F}_{\text{param}}$ is just a two dimensional subset of it.

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Blind Men and The Elephant

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This is where the Dirichlet Process comes into play!

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$$f \quad : \quad \mathbb{R} \rightarrow \mathbb{R}$$

In a parametric approach to this problem, we should parametrize this unknown function using a vector of parameters θ:

$$f \quad \equiv \quad f(\theta)$$

For instance, we can use a linear/polynomial function, in which θ is the vector of coefficients of the linear/polynomial function.

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This is where the Gaussian Process comes into play!

Tutorial's Outline

In this tutorial the following subjects will be covered:

- **Gaussian Process**
- **Dirichlet Process**
- **Dirichlet Process Mixture**

Remembering Some Definitions...

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Random Variable

Assume $(E, \mathcal{E})$ is a measurable space, and X is an E-valued function defined on Ω:

$$X : \Omega \to E$$

We call X a random variable on $(\Omega, \mathcal{F})$, if:

$$\sigma(X) := \{X^{-1}(A);\ A \in \mathcal{E}\} \quad \subset \quad \mathcal{F}$$

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$$\mathbb{P}(X \in A) := \mathbb{P}(\{\omega; X(\omega) \in A\})$$

is well-defined, because $\{\omega; X(\omega) \in A\} = X^{-1}(A)$ lies in the σ-algebra of our assumed probability space $\mathcal{F}$.

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If we assume $E = \mathbb{R}$, we can use the Borel set of $\mathbb{R}$ in the place of $\mathcal{E}$, and the resulting random variable is called a real-valued random variable.

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FDD of a Stochastic Process

Assume $X = (X_s)_{s \in S}$ is a stochastic process, and t_1, ..., $t_n \in S$ for some n. The probability distribution function of the vector $(X_{t_1}, ..., X_{t_n})$ is called the finite dimensional distribution (FDD) of the stochastic process $(X_s)_{s \in S}$ for $\{t_1, ..., t_n\}$

Kolmogorov Existence Theorem

Consistent Collection of Measures

Assume $(E, \mathcal{E})$ is a measurable space, and let S be a set, t_1, ..., $t_n \in S$, and assume $\mu_{t_1,\ldots,t_n}$ is a measure on $(E^n, \mathcal{E}^n)$. This collection of measures is called consistent iff for all natural n, for all t_1, ..., t_n, t_{n+1}, every permutation π of $\{1, ..., n\}$, and for all A_1, ..., $A_n \in \mathcal{E}$, we have:

$$\begin{aligned} \mu_{t_1,\ldots,t_n}(A_1 \times \cdots \times A_n) &= \mu_{\pi_1,\ldots,\pi_n}(A_{\pi_1} \times \cdots \times A_{\pi_n}) \\ \mu_{t_1,\ldots,t_n,t_{n+1}}(A_1 \times \cdots \times A_n \times E) &= \mu_{t_1,\ldots,t_n}(A_1 \times \cdots \times A_n) \end{aligned}$$

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Kolmogorov Existence Theorem

Consider the measurable space $(E, \mathcal{E})$, and the set S, and let $\mu_{t_1,\ldots,t_n}$ is a measure defined for all t_1, ..., $t_n \in S$ and $n \in \mathbb{N}$. If the collection of measures $\mu_{t_1,\ldots,t_n}$ is a consistent collection of measures, then there **exist** a **unique** stochastic process $(X_s)_{s \in S}$ which $\mu_{t_1,\ldots,t_n}$ is its FDD.

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As a consequence of this theorem, we can uniquely define a stochastic process by defining its FDDs (as long as the FDD collection is consistent).

Gaussian Process (GP) - Definition

Gaussian Process

The process $X = (X_s)_{s\in S}$ is a Gaussian Process with mean function $\mu(s)$ and the covariance function $K(s_1, s_2)$,

$$(X_s)_{s\in S} \quad \sim \quad \mathrm{GP}(\mu, K)$$

iff, for all $n \in \mathbb{N}$, and for all s_1, ..., $s_n \in S$:

$$\begin{bmatrix} X_{s_1} \\ \vdots \\ X_{s_n} \end{bmatrix} \quad \sim \quad \mathcal{N}_n\left(\begin{bmatrix} \mu(s_1) \\ \vdots \\ \mu(s_n) \end{bmatrix}, \begin{bmatrix} K(s_1, s_1) & \cdots & K(s_1, s_n) \\ \vdots & \ddots & \vdots \\ K(s_n, s_1) & \cdots & K(s_n, s_n) \end{bmatrix} \right)$$

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It is straightforward to show the FDD defined in the definition above is a consistent collection of distributions, and therefore due to the Kolmogorov Existence Theorem, the GP defined above exists and is uniquely characterized by the mean function $\mu(s)$ and the covariance function $K(s, s')$.

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Note that the mean function μ is a rather arbitrary function, while the covariance function K should be defined in such way that ensures the semi-positiveness of the covariance matrix of FDD.

Gaussian Process (GP) - Example

Example - Wiener process (Brownian Motion)
Assume $S = [0, T]$, and:

$$\begin{aligned} \mu(s) &= 0 \\ K(s_1, s_2) &= s_1 \wedge s_2 \end{aligned}$$

$(X_s)_{s \in [0,T]} \sim \mathrm{GP}(\mu, K)$ is called the Wiener process (or Brownian Motion).

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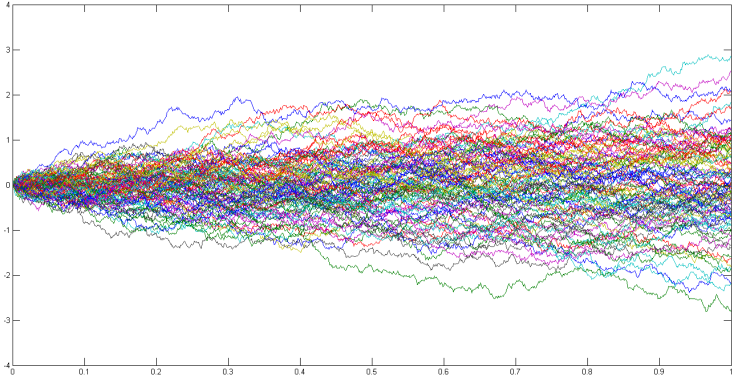

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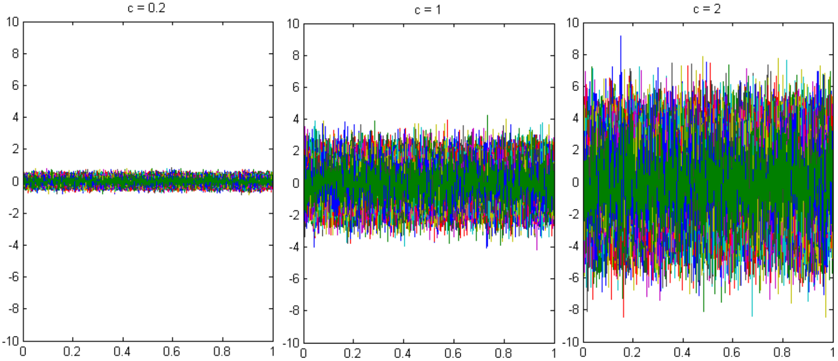

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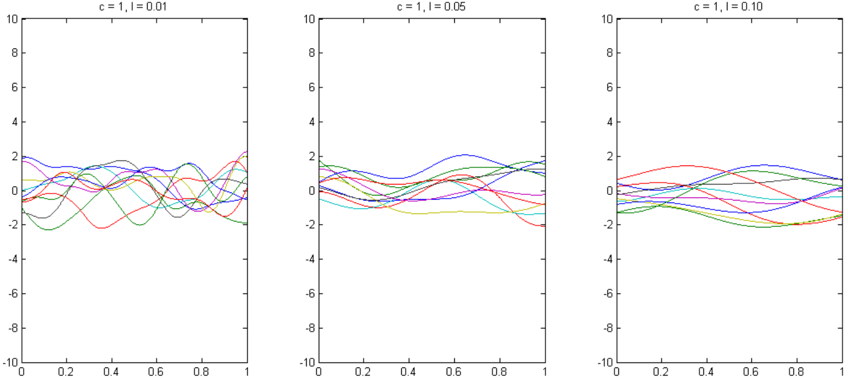

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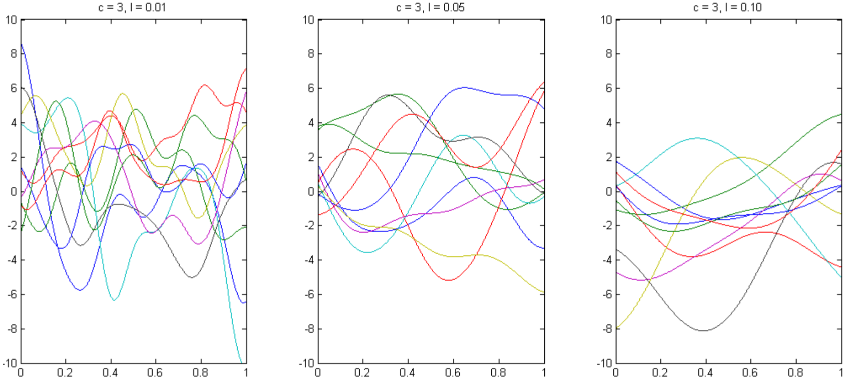

Semi-parametric Regression with GP

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Moreover define $\mathbf{f} = (f(x_1), \ldots, f(x_N))$.

From the definition of the GP:

$$
\mathbf{f} \sim \mathcal{N}(\mathbf{m}, \mathbf{S})
$$

where

$$
\mathbf{m} = (\mu(x_1), \ldots, \mu(x_N)), \qquad \text{and} \qquad \mathbf{S} = \begin{bmatrix} K(x_1, x_1) & \cdots & K(x_1, x_N) \\ \vdots & \ddots & \vdots \\ K(x_N, x_1) & \cdots & K(x_N, x_N) \end{bmatrix}
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and from this it follows that:

$$\mathbf{f}|\mathbf{y} \quad \sim \quad \mathcal{N}\left(\left[\sigma^{-2}\mathbf{I}_N + \mathbf{S}^{-1}\right]^{-1} \left[\sigma^{-2}\mathbf{y} + \mathbf{S}^{-1}\mathbf{m}\right], \left[\sigma^{-2}\mathbf{I}_N + \mathbf{S}^{-1}\right]^{-1} \right)$$

Semi-parametric Regression with GP

if we assume $\mathbf{x}^* = (x_1^*, ..., x_{N^*}^*)$, where $x_i \neq x_j^*$, and $\mathbf{f}^* = (f_1^*, ..., f_{N^*}^*)$, then:

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{f}^* \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mathbf{m} \\ \mathbf{m}^* \end{bmatrix}, \begin{bmatrix} \mathbf{S} + \sigma^2 \mathbf{I}_N & \mathbf{S}_{xx^*} \\ \mathbf{S}_{x^*x} & \mathbf{S}^* \end{bmatrix} \right)$$

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Remember if

$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{bmatrix}, \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix} \right)$$

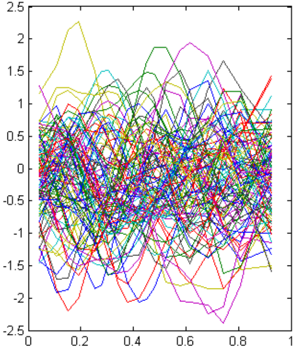
then

$$\mathbf{x}_2 | \mathbf{x}_1 \sim \mathcal{N}\left(\mathbf{m}_2 + \mathbf{S}_{21} \mathbf{S}_{11}^{-1} (\mathbf{x}_1 - \mathbf{m}_1), \mathbf{S}_{22} - \mathbf{S}_{21} \mathbf{S}_{11}^{-1} \mathbf{S}_{12} \right)$$

Semi-parametric Regression with GP: Example

prior:

$$
\begin{aligned}
\mu &= -0.16 \\
K &= 0.5 \exp\left[-\frac{(s_1 - s_2)^2}{0.2}\right]
\end{aligned}
$$



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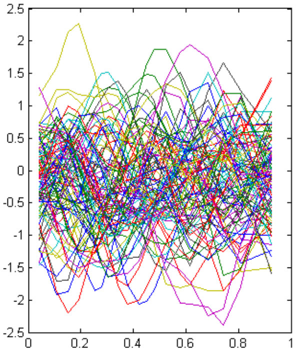

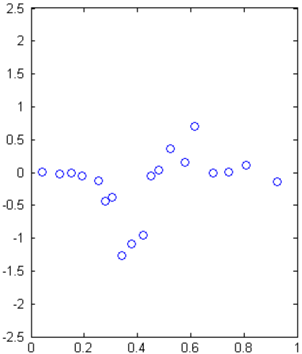

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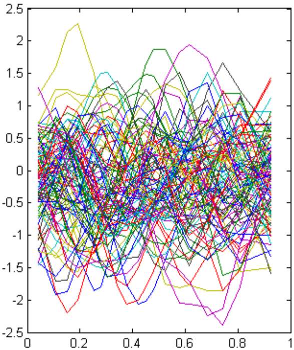
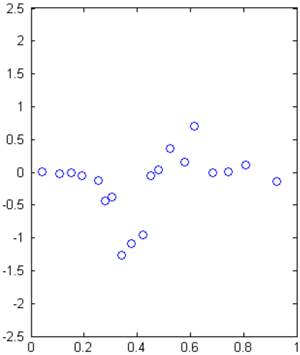
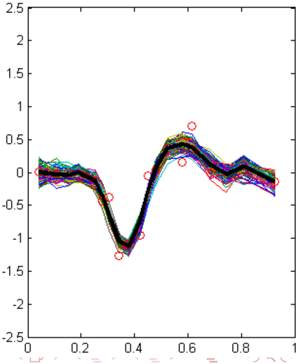
Posterior







Semi-parametric Regression with GP

Note that the value of hyperparameters (μ and K parameters) changes the posterior distribution. In the absence of prior information (on the smoothness of f and the noise level), we should put non-informative priors on them. In this case the analytical computation of posterior is not possible and we should employ the an MCMC simulation to sample from the posterior. Conditioned on the hyperparameters $\mathbf{f}$ is Gaussian, however usually the full conditional of parameters are not standard distributions, and we should use a (possibly adaptive) Metropolis-Hastings step to sample from them.

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Introduction

Binomial-Beta Conjugacy

Consider the Binomial r.v.:

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\begin{aligned}
S = \mathrm{Supp}(X) &= \{\mathbf{x} = (x_1, x_2);\; x_1, x_2 \in \{0, 1, ..., n\}, x_1 + x_2 = n\} \\
f(\mathbf{x}) &= \frac{n!}{x_1! x_2!} p_1^{x_1} p_2^{x_2} \; \mathbb{I}_S(\mathbf{x})
\end{aligned}
$$

with the parameter $p_1 \in [0, 1]$, and $p_2 = 1 - p_1$. The likelihood of this model is:

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Therefore the conjugate prior has the following kernel:

$$p(p_1) \quad \propto \quad p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1}$$

That is the Kernel of the Beta distribution:

$$f(p_1) \quad = \quad \frac{1}{\mathrm{Beta}(\alpha_1, \alpha_2)}\ p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1}$$

Binomial-Beta Conjugacy

Binomial-Beta Conjugacy

Model:

$$X \sim \text{Binomial}(p)$$

Prior:

$$p \sim \text{Beta}(\alpha_1, \alpha_2)$$

Posterior:

$$p|X \sim \text{Beta}(\alpha_1 + x_1, \alpha_2 + x_2)$$

Multinomial-Dirichlet Conjugacy

Consider the Multinomial r.v.:

$$\begin{aligned} S = \mathrm{Supp}(X) &= \{\mathbf{x} = (x_1, \ldots, x_K);\ x_1, \ldots, x_K \in \{0, 1, \ldots, n\}, x_1 + \cdots + x_K = n\} \\ f(\mathbf{x}) &= \frac{n!}{x_1! \cdots x_K!} p_1^{x_1} \cdots p_K^{x_K}\ \mathbb{I}_S(\mathbf{x}) \end{aligned}$$

with the parameter $\mathbf{p} = (p_1, \ldots, p_{K-1}) \in (K-1)$-simplex[1], and $p_K = 1 - \sum_{j=1}^{K-1} p_j$.
The likelihood of this model is:

$$L(\mathbf{p}|\mathbf{x}) \quad \propto \quad p_1^{x_1} \cdots p_K^{x_K}$$

[1] The d-simplex is $\Delta^d = \{(x_1, \ldots, x_d); \forall j\ x_j \geq 0, \quad \text{and} \quad \sum_{j=1}^{d} x_j \leq 1\}$.

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That is the Kernel of the **Dirichlet distribution**:

$$f(\mathbf{p}) \quad = \quad \frac{\prod_{j=1}^{K} \Gamma(\alpha_j)}{\Gamma\left(\sum_{j=1}^{K} \alpha_j\right)} \; p_1^{\alpha_1 - 1} \cdots p_K^{\alpha_K - 1}$$

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Dirichlet Distribution

Assume G_i are independent Gamma-distributed r.v.s:

$$G_i \quad \sim \quad \text{Gamma}(\alpha_i, 1) \qquad 1 \leq i \leq K$$

and define:

$$P_i \quad = \quad \frac{G_i}{\sum_{j=1}^{K} G_j} \qquad 1 \leq i \leq K$$

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It can be shown that the PDF of $(P_1, ..., P_{K-1})$ is:

$$f(\mathbf{p}) \quad = \quad \frac{\prod_{j=1}^{K} \Gamma(\alpha_j)}{\Gamma\left(\sum_{j=1}^{K} \alpha_j\right)} \; p_1^{\alpha_1 - 1} \cdots p_K^{\alpha_K - 1}$$

where $\mathbf{p} = (p_1, ..., p_{K-1})$ and $p_K = 1 - \sum_{j=1}^{K-1} p_j$.

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Moreover:

$$E\left(\begin{bmatrix} P_1 \\ \vdots \\ P_K \end{bmatrix}\right) = \begin{bmatrix} \frac{\alpha_1}{\sum_{j=1}^K \alpha_j} \\ \vdots \\ \frac{\alpha_K}{\sum_{j=1}^K \alpha_j} \end{bmatrix}$$

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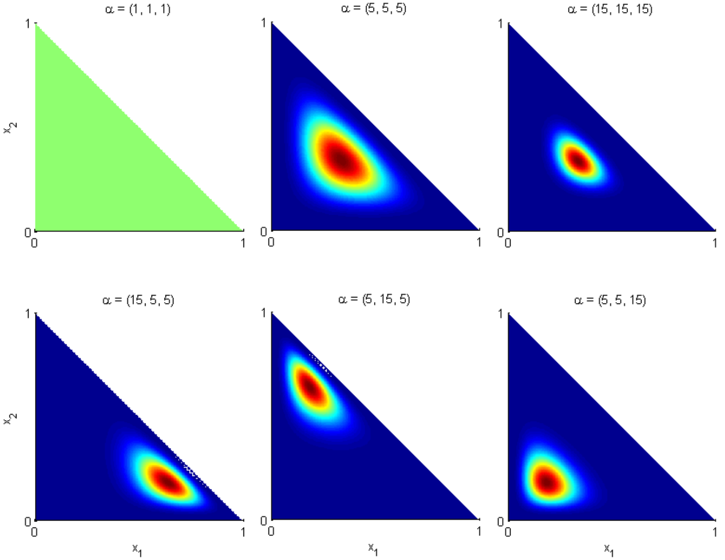
and the mode of the distribution is:

$$\begin{bmatrix} \frac{\alpha_1 - 1}{\left(\sum_{j=1}^K \alpha_j\right) - K} \\ \vdots \\ \frac{\alpha_{K-1} - 1}{\left(\sum_{j=1}^K \alpha_j\right) - K} \end{bmatrix} \qquad \forall j \; \alpha_j > 1$$

$(x_1, x_2) \sim \text{Dirichlet}(\alpha)$



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In general, we can aggregate any subset of Dirichlet variables yields to obtain a new random variable, and this new random variable is Dirichlet distributed, with corresponding aggregation of the parameters (remember that $\text{Gamma}(\alpha_1, 1) + \text{Gamma}(\alpha_2, 1) \stackrel{L}{\equiv} \text{Gamma}(\alpha_1 + \alpha_2, 1)$).

Dirichlet Distribution - Aggregation Property

Assume:

$$(P_1, ..., P_d) \quad \sim \quad \text{Dirichlet}(\alpha_1, ..., \alpha_d)$$

and let $\{A_1, ..., A_K\}$ be a partition of $\{1, ..., d\}$. Then

$$(\sum_{i \in A_1} P_i, ..., \sum_{i \in A_K} P_i) \quad \sim \quad \text{Dirichlet}(\sum_{i \in A_1} \alpha_i, ..., \sum_{i \in A_K} \alpha_i)$$

This is called the “Aggregation Property” of Dirichlet distribution.

Dirichlet Distribution - Random Number Generation

The first way to generate random numbers from the Dirichlet distribution is to rely on its definition based on normaliziation of independent Gamma random variables.

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The second approach is the stick-breaking method. First we should generate v_1, ..., v_{d-1}:

$$v_i \quad \sim \quad \text{Beta}(\alpha_i, \sum_{j=i+1}^{d} \alpha_j)$$

and set $v_d = 1$, and then build the p_is:

$$p_i \quad = \quad \begin{cases} v_1 & i = 1 \\ v_i \prod_{j=1}^{i-1}(1 - v_j) & 1 < i \leq d \end{cases}$$

It can be shown that:

$$(p_1, ..., p_d) \quad \sim \quad \text{Dirichlet}(\alpha_i, ..., \alpha_d)$$

Dirichlet Distribution - Pólya Urn Scheme

Consider the Dirichlet random variable

$$(P_1, ..., P_d) \quad \sim \quad \text{Dirichlet}(\alpha_1, ..., \alpha_d)$$

We put α_i balls of color i for $i = 1, ..., d$ in an empty urn. The initial fraction of the balls of different color in the urn are:

$$(f_1^0, ..., f_d^0) \quad = \quad (\alpha_1/\alpha, ..., \alpha_d/\alpha)$$

where $\alpha = \sum_{i=1}^{d} = \alpha_i$.

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$$(f_1^{(1)}, ..., f_d^{(1)}) \quad = \quad ((\alpha_1 + \delta_1(X_1))/(\alpha + 1), ..., (\alpha_d + \delta_d(X_1))/(\alpha + 1))$$

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If we continue with this sampling scheme, the asymptotic fraction of the balls of different colors in the urn will be a random draw from Dirichlet$(\alpha_1, ..., \alpha_d)$:

$$(f_1^{(n)}, ..., f_d^{(n)}) \quad \underset{n \to +\infty}{\overset{L}{\to}} \quad \text{Dirichlet}(\alpha_1, ..., \alpha_d)$$

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Consider the measurable space $(E, \mathcal{E})$, and assume $\mathcal{F}$ is the space of probability measures on this space.

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On this space the family of probability distributions can be formulated by a single parameter $p \in [0, 1]$:

$$F(x) \quad = \quad (1-p)\ \mathbf{1}_{[0,+\infty)}(x) + p\ \mathbf{1}_{[1,+\infty)}(x)$$

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A natural choice is the beta distribution:

$$p \quad \sim \quad \text{Beta}(\alpha_1, \alpha_2)$$

and this completes the definition of a random measure on $(E, \mathcal{E})$.

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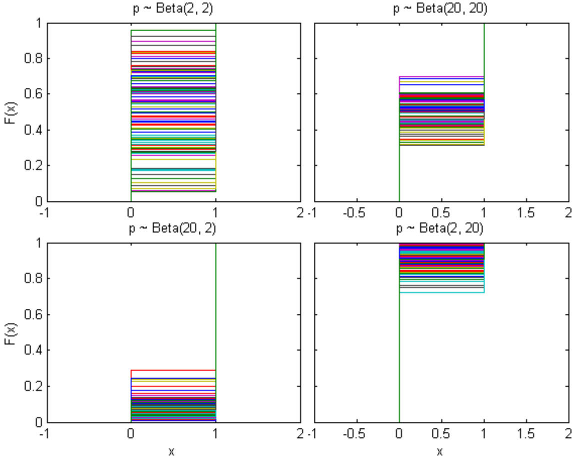

RPD

The same idea can be extended to a finite space.

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Example: $E = \{1, ..., K\}$
On this space the family of probability distributions can be formulated by the vector of parameters $\mathbf{p} \in \Delta^{K-1}$:

$$F(x) = p_1 \, \mathbf{1}_{[1,+\infty)}(x) + \cdots + p_{K-1} \, \mathbf{1}_{[K-1,+\infty)}(x) + p_K \, \mathbf{1}_{[K,+\infty)}(x)$$

where $p_K = 1 - \sum_{j=1}^{K-1} p_j$. Therefore by putting a prior on p we can define a prior on the probability distributions on this space.

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The same idea can be extended to a finite space.

Example: $E = \{1, ..., K\}$
On this space the family of probability distributions can be formulated by the vector of parameters $\mathbf{p} \in \Delta^{K-1}$:

$$F(x) \quad = \quad p_1 \ \mathbf{1}_{[1,+\infty)}(x) + \cdots + p_{K-1} \ \mathbf{1}_{[K-1,+\infty)}(x) + p_K \ \mathbf{1}_{[K,+\infty)}(x)$$

where $p_K = 1 - \sum_{j=1}^{K-1} p_j$. Therefore by putting a prior on p we can define a prior on the probability distributions on this space.

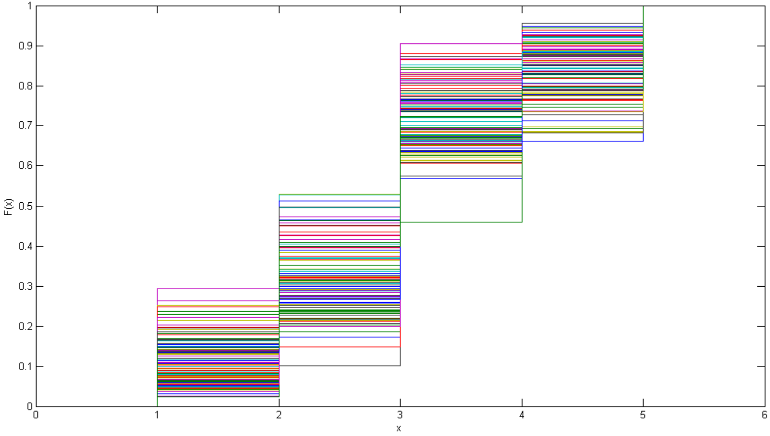
A natural choice is the Dirichlet distribution:

$$\mathbf{p} \quad \sim \quad \text{Dirichlet}(\alpha_1, ..., \alpha_K)$$

and this completes the definition of a random measure on $(E, \mathcal{E})$.

$F(x) = \sum_{j=1}^{K} p_j \mathbf{1}_{[j,+\infty)}(x)$

$p \sim \text{Dirichlet}(3, 5, 10, 3, 4)$



Dirichlet Process - Definition

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Dirichlet Process

Assume G_0 is a probability measure on $(E, \mathcal{E})$, and let c be a positive real number. A random measure F is called a Dirichlet Process (DP),

$$F \quad \sim \quad \mathrm{DP}(c, G_0)$$

iff for any finite measurable partition of E,

$$\forall k \in \mathbb{N}, \; E_1 \cup \cdots \cup E_k = E, \quad E_i \cap E_j = \emptyset, \quad E_i \in \mathcal{E}$$

we have:

$$(F(E_1), ..., F(E_k)) \quad \sim \quad \mathrm{Dirichlet}(c \;\, G_0(E_1), ..., c \;\, G_0(E_k))$$

G_0 and c are called the centering (or base) measure, and the concentration parameter, respectively.

DP: Parameter Interpretability

The existence and uniqueness of DP is ensured by the Kolmogorov Existence Theorem[2].

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Consider the measurable set $A \in \mathcal{E}$. Then (A, A^c) is a finite measurable partition of E, and by Ferguson's definition of DP we have:

$$(F(A), F(A^c)) \quad \sim \quad \text{Dirichlet}(c \; G_0(A), c \; G_0(A^c))$$

therefore:

$$\mathbb{E}\left[F(A)\right] \quad = \quad \frac{c \; G_0(A)}{c \; G_0(A) + c \; G_0(A^c)} = G_0(A)$$

$$\mathbb{V}\left[F(A)\right] \quad = \quad \frac{c^2 \; G_0(A) \; G_0(A^c)}{\left[c \; G_0(A) + c \; G_0(A^c)\right]^2 \left[c \; G_0(A) + c \; G_0(A^c) + 1\right]} = \frac{G_0(A)\left[1 - G_0(A)\right]}{c + 1}$$

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\end{aligned}$$

Therefore It can be seen that draws from DP(c, G_0) are concentrated around the centering measure G_0, with a dispersion that is inversely proportional to the concentration parameter c.

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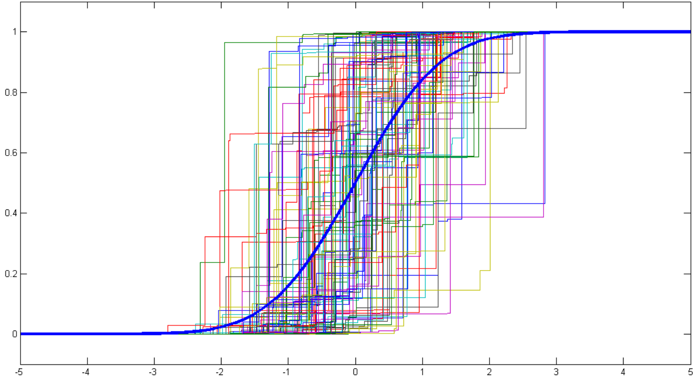
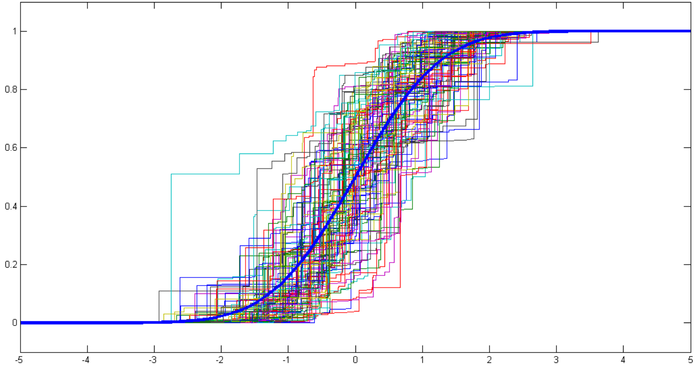
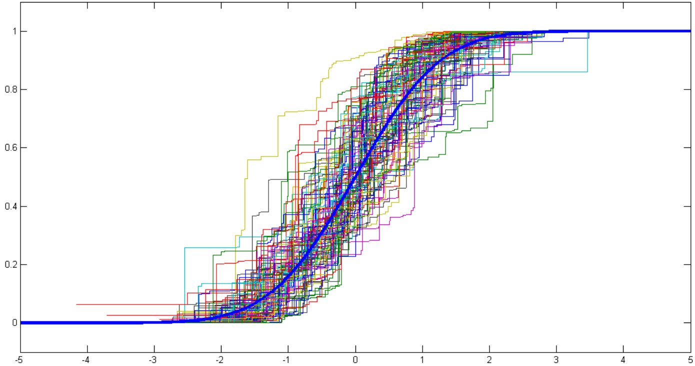


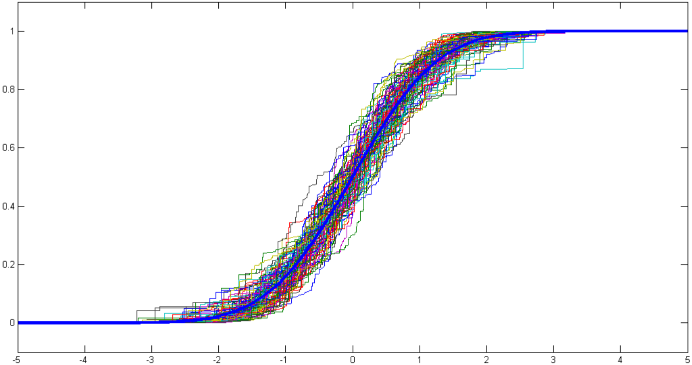
$G \sim \mathrm{DP}(G_0, 5)$, with $G_0 \equiv \mathcal{N}(0, 1)$



$G \sim \mathrm{DP}(G_0, 10)$, with $G_0 \equiv \mathcal{N}(0,1)$



$G \sim \mathrm{DP}(G_0, 50)$, with $G_0 \equiv \mathcal{N}(0, 1)$



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$$F | \theta_1 \sim \mathrm{DP}\left(c+1, \frac{c}{c+1} G_0 + \frac{1}{c+1} \delta_{\theta_1}\right)$$

Where δ_θ is a point mass centered at θ.

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By mathematical induction, this result can be extended to an iid sample of size N:

$$\theta_i | F \overset{iid}{\sim} F \qquad i = 1, 2, ..., N$$

Then

$$F|\theta_1, ..., \theta_N \sim \mathrm{DP}\left(c+N, \frac{c}{c+N} G_0 + \frac{1}{c+N}\sum_{j=1}^{N}\delta_{\theta_j}\right)$$

Stick-Breaking Representation

The Ferguson's definition of DP is not a constructive definition. The "Steak-Breaking Representation" (SB) of DP provides a constructive definition[3].

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SB Representation of DP

Assume $v_i \overset{iid}{\sim} \text{Beta}(1, c)$, and let $w = (w_1, w_2, \dots)$ be an infinite sequence of weights, obtained by the following SB process:

$$w_k = \begin{cases} v_1 & k = 1 \\ v_k \prod_{j=1}^{k-1}(1 - v_j) & k > 1 \end{cases}$$

Moreover assume $\theta_i \overset{iid}{\sim} G_0$ (and independent from v_is), and consider the following discrete random measure:

$$F(\theta) = \sum_{j=1}^{+\infty} w_j \delta_{\theta_j}(\theta)$$

This guarantees that $F \sim \text{DP}(c, G_0)$. Conversely samples from a DP are a.s. discrete measures and have a SB representation.

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Stick-Breaking Representation: Mixture Weights

The mixture weights $(w_1, w_2, \ldots)$ distribute a "stick" of length 1 between the countably infinite number of mixture components.

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In the literature it is common to denote this sequence of weights by

$$\mathbf{w} \sim \mathrm{GEM}(\alpha)$$

where GEM stands for Griffiths, Engen, and McCloskey.

Stick-Breaking Representation: Mixture Weights

The mixture weights definition implies:

$$\mathbb{E}(w_k) \quad = \quad \frac{1}{1+c}\left(\frac{c}{1+c}\right)^{k-1}$$

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For small values of c, almost all the probability mass will be assigned to the first few components of the mixture. When $c \to +\infty$, the random probability measure F goes to the centering measure G_0.

DP: Predictive Distribution

Assume $G \sim \mathrm{DP}(c, G_0)$, and $\theta_i | G \overset{iid}{\sim} G$, for $i = 1, ..., N$. Since G is almost surely discrete, with positive probability there are ties among θ_is. Let $\{\theta_1^*, ..., \theta_{N^*}^*\}$ be the set of distinct values of θ_is:

$$\forall i \in \{1, ..., N\}, \quad \exists! j \in \{1, ..., N^*\}, \quad \theta_i = \theta_j^*$$

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It can be shown that the predictive distribution of θ_{N+1} is[4]:

$$\theta_{N+1} | \theta_1, ..., \theta_N \quad \sim \quad \frac{c}{c+N} G_0 + \frac{1}{c+N} \sum_{j=1}^{N^*} n_j \delta_{\theta_j^*}$$

where n_j is the number of previous observations of θ_j^*.

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where n_j is the number of previous observations of θ_j^*.

In other words, the new realization from G, is either a new realization from the base measure G_0 with probability $\frac{c}{c+N}$, or it is equal to one of the previously observed realizations with probability $\frac{n_j}{c+N}$.

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DP's Predictive Distribution As a Pólya Urn

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Note that the obtained asymptotic distribution is discrete, and for any finite N, it provides a clustering of N observations in $K \leq N$ clusters (K is the number of distinct colors).

Moreover note that K is not fixed a priori and increases with N.

Chinese Restaurant Process

Assume $G \sim \mathrm{DP}(c, G_0)$, and for $i = 1, \ldots, n$, let $\theta_i | G \sim G$. As it has been explained earlier with positive probability there are repeated values among θ_is. Therefore DP induces a partitioning among the n observations.

[5] J. Pitman, Combinatorial stochastic processes, Technical Report 621, U. C. Berkeley Department of Statistics, August 2002.

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  - chooses one of the K already occupied tables with probability $\frac{n_k}{c+n}$, and orders the dish chosen by customers on this table θ_k.
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Number of Clusters (Tables)

Obviously the number of occupied tables (number of data clusters) increases with the number of customers (observations). In other words the model is able to self-adapt to more complex models when the new data arrives (because it is a non-parametric model!), and we do not need to set the model complexity a priori.

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It can be shown that as $N \to +\infty$, we have:

$$\mathbb{E}(K) = k(N,c) \quad \in \quad \mathcal{O}(\ln N)$$

that means the number of clusters is proportional to the logarithm of the observations.

SBP: Beyond DP

As we have seen earlier the DP has a SB representation:

$$F(\theta) \quad = \quad \sum_{j=1}^{+\infty} w_j \delta_{\theta_j}(\theta)$$

where $\theta_i \overset{iid}{\sim} G_0$, $v_i \overset{iid}{\sim} \text{Beta}(1, c)$, $w_1 = v_1$ and $w_k = v_k \prod_{j=1}^{k-1}(1 - v_j)$ for $k > 1$.

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Ishwaran and James [6] have generalized this process to $v_i \overset{iid}{\sim} \text{Beta}(a_i, b_i)$, and have shown that the process is well-defined ($\sum_{i=1}^{+\infty} w_i = 1$ a.s.) if:

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Another special case is $a_i = 1 - a$ and $b_i = c + ai$, that is called PoissonDirichlet or Pitman-Yor process.

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Finite Mixture Models: A Simple Example

Consider a mixture of two Normal distributions with unknown mixing proportion p_1, and known variances:

$$Y_i | p_1, \mu_1, \mu_2 \quad \sim \quad p_1 \, \mathcal{N}(\mu_1, 1) + (1 - p_1) \, \mathcal{N}(\mu_2, 1)$$

where the unknown parameters are p_1, μ_1 and μ_2. Moreover assume y_1, ..., y_N are the observed data.

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We may put a Beta prior on p_1, and (independent) Normal priors on μ_1 and μ_2,

$$p(p_1, \mu_1, \mu_2) \quad = \quad p(p_1) \; p(\mu_1) \; p(\mu_2)$$

and sample the posterior

$$p(p_1, \mu_1, \mu_2 | \mathbf{y}) \quad \propto \quad p(p_1, \mu_1, \mu_2) p(\mathbf{y} | p_1, \mu_1, \mu_2)$$

by MCMC.

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The model can be re-formulated by introducing indicator variables d_1, ..., d_N that are $\{1,2\}$-valued random variables:

$$\begin{aligned} Y_i|\mu_1,\mu_2,d_i &\sim \mathcal{N}(\mu_{d_i},1) \\ d_i &\sim \text{Discrete}(\{1,2\},(p_1,1-p_1)) \end{aligned}$$

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We can also generalize the model to a mixture of K components:

$$\begin{aligned} Y_i | \theta_1, ..., \theta_K, d_i &\sim H(\theta_{d_i}) \\ d_i &\sim \text{Discrete}(\{1, ..., K\}, (p_1, ..., p_K)) \end{aligned}$$

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where $\sum_{i=1}^{K} p_i = 1$, and $H(\theta)$ is a distribution parameterized by the vector of parameters θ.

Obviously a natural choice for the prior of $(p_1, ..., p_K)$ is the Dirichlet distribution.

Finite Mixture Models

Good News:

- Any distribution can be approximated by a location-scale mixture of Normals with enough number of components. This is true for both univariate and multivariate distributions.
- Univariate unimodel distributions can be approximated by a mixture of uniforms with enough number of components.
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Bayesian non-parametrics provides a solution to this problem: We put a prior on the model's complexity, and using the data do the inference on the model complexity too!

Finite Mixture Models: An Alternative Formulation

The finite mixture model

$$Y_i|\theta_1, ..., \theta_K, \mathbf{p} \quad \sim \quad \sum_{i=1}^{K} p_i \; H(\theta_i)$$

can be re-formulated in the following way:

$$\begin{aligned} Y_i|\theta_1, ..., \theta_K \quad &\sim \quad \int_{\Theta} H(\theta) \; dF(\theta) \\ F(\theta) \quad &= \quad \sum_{i=1}^{K} p_i \; \delta_{\theta_i}(\theta) \end{aligned}$$

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Our natural candidate for the prior on F is Dirichlet Process!

Dirichlet Process Mixture (DPM)

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Dirichlet Process Mixture

Consider the following hierarchical model:

$$\begin{aligned} Y_i|\theta_i &\overset{iid}{\sim} H(\theta_i) \\ \theta_i|F &\sim F \end{aligned}$$

Using a DP as the prior of the unknown distribution F,

$$F \sim DP(c, F_0)$$

results a model that is called Dirichlet Process Mixture (DPM)

DPM: Limit of Finite Mixtures

Consider the finite mixture model with K components:

$$Y_i|\theta_1, ..., \theta_K, \mathbf{p} \quad \sim \quad \sum_{i=1}^{K} p_i \ H(\theta_i)$$

with the following priors:

$$\begin{aligned} \theta_i &\sim F_0 \\ \mathbf{p} &\sim \text{Dirichlet}(c/K, ..., c/K) \end{aligned}$$

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It can be shown[7] that the limit of this finite mixture model when $K \to +\infty$, is the following DPM:

$$\begin{aligned} Y_i|\theta_i &\overset{iid}{\sim} H(\theta_i) \\ \theta_i|F &\sim F \\ F &\sim DP(c, F_0) \end{aligned}$$

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DPM - Example

Assume the mixture kernel is a univariate Normal distribution, with $\theta = (\mu, \sigma^2)$:

$$H(\theta) = \mathcal{N}(\mu, \sigma^2)$$

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Therefore the base measure of the DP should be a distribution on $\mathbb{R} \times \mathbb{R}^+$. A convenient choice is the normal inverse Gamma:

$$F_0(\theta) = \text{N-IG}(\theta; \alpha, \beta, \lambda, \nu)$$

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The inference in this model has been addressed in Escobar and West 1995[8]. It is important to mention that we may put priors on the hyperparameters of the DP (the parameters of the base measure α, β, λ and ν, and the concentration parameter c) and sample them as well.

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The inference in this model has been addressed in Escobar and West 1995[8]. It is important to mention that we may put priors on the hyperparameters of the DP (the parameters of the base measure α, β, λ and ν, and the concentration parameter c) and sample them as well.

In particular the value of the concentration parameter c affects the posterior significantly, and in the absence of prior knowledge we should put a non-informative prior on it.

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DPM - Example

In particular using results of Antoniak 1974[9], we know the PMF of the number of occupied components k:

$$P(k|c,N) = C_N(k)N!c^k \frac{\Gamma(c)}{\Gamma(c+N)}$$

and for large sample sizes, we have:

$$\mathbb{E}(k|c) \approx c \ln\left(\frac{c+N}{c}\right)$$

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Therefore with a fixed sample size, c controls the complexity (the number of occupied components) of the model.

Therefore it is important to put a rather vague prior on c (a possible choice is a Gamma distribution), and sample it in a MCMC simulation.

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DPM: CRP Representation

By Introducing indicator variables d_1, ..., and d_N, we can represent the DPM in the following way:

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\begin{aligned}
Y_i | d_i, \theta_1, \theta_2, \ldots &\sim H(\theta_{d_i}) \\
\theta_i &\sim F_0 \\
\mathbf{d} &\sim \mathrm{CRP}(c)
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Based on this representation, more efficient MCMC sampling schemes can be designed, for example see Neal 2000[10].

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Using this representation of the DPM, efficient and easy to implement slice sampler could be designed (for both conjugate and non-conjugate models). For example see Walker 2007[11] and Kalli, Griffin, and Walker 2011[12].

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