Università della Svizzera italiana

Stochastic Methods **Academic Year 2020/2021**

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Assignment 5

Exercise 1: Irreducible and Aperiodic Markov Chains *(1.5 Points)*

Consider the three following Markov Chains:

$$\text{(a)}\ P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}, \qquad \text{(b)}\ P = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 \\ 0 & \frac{1}{5} & \frac{4}{5} \end{bmatrix}, \qquad \text{(c)}\ P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

Are they irreducible? Are they aperiodic? For each one of them, clearly motivate your answers.

Exercise 2: Tossing a Fair Coin *(2.5 Points)*

We consider a fair coin that is tossed several times. All tosses are independent from each other and can either result in *head* (H) or *tail* (T). Model the process as a Markov Chain and compute the expected number of tosses required to achieve the pattern HTHTH (**hint**: You could consider a chain based on the sequences of heads and tails, starting from a state called *nothing* – i.e., no correct element in the sequence – up to the required result HTHTH). Carefully explain your answer, by providing a graphical representation of the Markov chain and all the relevant computations.

Exercise 3: Markov Process and Stationary Distribution *(1.5 Points)*

We consider a Markov process with transition probability matrix $P \in \mathbb{R}^{n\times n}$:

$$\boldsymbol{\pi}^{(t)} = \boldsymbol{\pi}^{(t-1)} P,$$

with $\boldsymbol{\pi}^{(t)} \in \mathbb{R}^{1\times m}$, for every t. The stationary distribution describes the behaviour of $\boldsymbol{\pi}^{(t)}$ as $t \to \infty$. Thus, for a given Markov process with transition matrix P, the stationary distribution $\boldsymbol{\pi}^*$ is:

$$\boldsymbol{\pi}^* = \boldsymbol{\pi}^* P.$$

That is $\boldsymbol{\pi}^*$ is the left-eigenvector of the matrix P corresponding to the eigenvalue $\lambda = 1$. Let us consider a simple Markov chain with three states and transition matrix given by:

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0 & 0.9 \end{bmatrix}.$$

Compute the stationary distribution $\boldsymbol{\pi}^*$ for P.

Exercise 4: Time to Absorption and Absorption Probabilities *(2.5 Points)*

Consider a Markov Chain with states s_0, s_1, s_2 and s_3, and transition probability matrix P:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

(a) Is this transition probability matrix acceptable? If no, modify it and clearly motivate your answer, then use the correct form to answer the remaining questions.

(b) Is the Markov Chain irreducible? Is it periodic? Motivate your answers.

(c) Is the Markov Chain absorbing? Why? Which are the absorbing states?

(d) Compute the time to absorption vector $\boldsymbol{\tau}$. Starting from state s_2, how many steps occur before absorption? How does your answer change if we start from state s_3?

(e) Compute the absorption probability matrix B and give an interpretation of the results obtained.

Exercise 5: The Winner Takes It All *(2 Points)*

At each round of a gambling game, a gambler can either win and receive 1 CHF with probability p or lose the game and everything he has with probability $q = 1 - p$. Suppose that the gambler starts with 1 CHF in his pocket and that at least 1 CHF is required to take part in the game.

(a) Suppose that $p = \frac{1}{4}$ and that the gambler decides to stop playing after 5 rounds. Model the situation with a Markov chain and represent it graphically. Write explicitly the entries of the transition probability matrix P. What is the stationary distribution $\boldsymbol{\pi}^*$?

(b) Explain the structure of matrix P for generic p, q and number of rounds n. Write a Matlab function called `gambler()`, which takes as input p and n and returns the expected winning of the gambler. In a script `ex5b.m`, test it for: (I) $p = \frac{1}{5}$ and $n = 100$; (II) $p = \frac{4}{5}$ and $n = 1000$.

Please write a detailed report with your solutions using the LaTeX template provided on iCorsi.

!!! The code has to be well commented !!!