Università della Svizzera italiana

Stochastic Methods **Academic Year 2020/2021**

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Due date: Tuesday, April 27, 2021, 11.59pm

Assignment 3

Exercise 1: Unconstrained Optimization of a Quadratic Problem *(1 Point)*

Find the extrema of the function:[1]

$$f(\mathbf{x}) = \mathbf{x}^T H \mathbf{x} + \mathbf{c}^T \mathbf{x} \quad \text{with } \mathbf{x}, \mathbf{c} \in \mathbb{R}^n \text{ and } H \in \mathbb{R}^{n \times n}.$$

Exercise 2: Constrained Optimization with Parametrization *(1 Point)*

Solve the following constrained optimization problem with parametrization:

$$\begin{aligned} \max_{x_1, x_2} \quad & f(x_1, x_2) = 5 - x_1^2 - \frac{1}{2} x_2^2, \\ \text{s.t.} \quad & x_1 + x_2 = 2. \end{aligned}$$

Exercise 3: Optimization on the Unit Circle *(2.5 Points)*

Consider a generic vector $\mathbf{x} \in \mathbb{R}^2$ and the unit circle defined as:

$$S_p = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_p = 1\}, \tag{1}$$

where the p-norm, with real number $p \geq 1$, is computed as follows:[2]

$$\|\mathbf{x}\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

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[1] *Hint:* Use the results for "Derivatives of Matrices, Vectors and Scalar Forms" in The Matrix Cookbook.

[2] In particular, remember that the maximum norm is defined as:

$$\|\mathbf{x}\|_\infty := \max(|x_1|, \ldots, |x_n|).$$

(a) Write a Matlab function `drawCircle()` which takes as input p and plots the unit circle for the given norm. Test your function by writing a script `ex3a.m` which plots, in the same figure, the unit circle for $p = 1, 2, 4, 6, 8, \infty$ and comment on the results obtained.

(b) We are now interested in finding the extrema of the following function in the first quadrant:

$$f(x_1, x_2) = x_1 x_2,$$

subject to the constraint that the solution lies on the unit circle S_p of Eq. (1) for $p = 1, 2, \infty$. Use the method of Lagrange multipliers to find the optimal solution in the three cases.

(c) Finally, write a script `ex3c.m`, in which you plot the function $f(x_1, x_2)$ (*hint*: use the function `contourf()` already implemented in Matlab) and the three constraints in the same figure, along with the optimal solutions in the first quadrant. How does the optimal solution change with respect to the unit circle? Comment on the results obtained.

Exercise 4: Lagrange Multipliers and Bordered Hessian *(3 Points)*

Use the method of Lagrange multipliers to find the constrained extrema of the following functions and use the bordered Hessian to determine whether they are maxima, minima or saddle points:

(a) Case with $n = 2$ variables and $m = 1$ constraint:

$$\begin{aligned} & f(x_1, x_2) = 3x_1^2 - 2x_2^2, \\ \text{s.t.} \quad & g(x_1, x_2) = x_1 - 2x_2 + 6 = 0. \end{aligned}$$

(b) Case with $n = 3$ variables and $m = 1$ constraint:

$$\begin{aligned} & f(x_1, x_2, x_3) = (x_1 - 10)^2 + (x_2 - 3)^2 + (x_3 - 3)^2, \\ \text{s.t.} \quad & g(x_1, x_2, x_3) = 2x_1^2 + x_2^2 + x_3^2 - 10 = 0. \end{aligned}$$

(c) Case with $n = 3$ variables and $m = 2$ constraints:

$$\begin{aligned} & f(x_1, x_2, x_3) = (x_1 - 3)^2 + (x_2 + 1)^2 - (x_3 - 2)^2, \\ \text{s.t.} \quad & g_1(x_1, x_2, x_3) = 3x_1 - 2x_2 + 4x_3 - 9 = 0, \\ \text{and} \quad & g_2(x_1, x_2) = x_1 + 2x_2 - 3 = 0. \end{aligned}$$

Exercise 5: An Application to Economics *(1.5 Points)*

Economists are used to measure how much weight or importance an individual gives to certain circumstances or goods with the concept of *utility.* Let us suppose that a mother, who has won 3000 CHF at the lottery, is looking for the best way to split them between her 3 daughters. Each daughter, however, gives a different value to money, as we can notice in the following utility functions:

- 1st daughter: $U_1(x) = 3\ln(x)$,
- 2nd daughter: $U_2(x) = \ln(x)$,
- 3rd daughter: $U_3(x) = 2\ln(x)$,

where x represents the amount of money received by each daughter. The objective of the mother is to maximise the overall happiness of her children, and she wants to achieve this by maximising the total utility. Considering the information above, solve the following tasks:

(a) Express the situation described above as a constrained optimization problem.

(b) Use the method of Lagrange multipliers and the bordered Hessian to find the optimum.

(c) Briefly comment the results you obtained. Would the optimal solution change significantly if we now assume that the utility of the 1st daughter is $U_1(x) = 3x$? Why? Justify your answer.

Exercise 6: Maximization Using the KKT Conditions *(1 Point)*

Find the maxima of the function $f(x_1, x_2) = x_1x_2$ subject to the constraint $g(x_1, x_2) = x_1^2 + x_2^2 \leq 2$, by writing the KKT conditions and evaluating the different cases.

Please write a detailed report with your solutions using the LaTeX template provided on iCorsi.

!!! The code has to be well commented !!!