Stochastic Methods **Academic Year 2020/2021**

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Assignment 2 - Solution [Please insert your name]

Exercise 1: Lagrange Interpolation and Newton's Divided-Differences

(a) $\mathrm{P}_n(x) = \Sigma_{k=0}^{n} f(x_k) L_k(x)$

$\mathrm{L}_k(x) = \pi_{i=0}^{n} \dfrac{x - x_i}{x_k - x_i}$

$\mathrm{P}_3(x) = f(x_0).L_0(x) + f(x_1).L_1(x) + f(x_2).L_2(x) + f(x_3)L_3(x)$

$\mathrm{L}_0(x) = \dfrac{(x - x_1) * (x - x_2) * (x - x_3)}{x_0 - x_1) * (x_0 - x_2) * (x_0 - x_3)}$

$\mathrm{L}_0(x) = \dfrac{-x^3 + 12x^2 - 44x + 48}{48}$

$\mathrm{L}_1(x) = \dfrac{(x - x_0) * (x - x_2) * (x - x_3)}{x_1 - x_2) * (x_1 - x_2) * (x_1 - x_3)}$

$\mathrm{L}_1(x) = \dfrac{x^3 - 10x^2 + 24)}{16}$

$\mathrm{L}_2(x) = \dfrac{-x^3 + 8x^2 - 12x}{16}$

$\mathrm{L}_3(x) = \dfrac{x^3 - 6x^2 + 8x}{48}$

$\mathrm{P}_3(x) = \dfrac{x^3 + 2x^2 + 18x + 8}{8}$

(b) $\mathrm{P}_3(x) = f(x_0) + f[x_0, x_1].(x - x_0) + f[x_0, x_1, x2].(x - x_0).(x - x_1) + f[x_0, x_1, x2, x_3].(x - x_0).(x - x_1).(x - x_2)$

$\mathrm{f}[\mathrm{x}_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \dfrac{1}{2}$

$\mathrm{f}[\mathrm{x}_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$

$$f[x_0, x_1, x_2] = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0} = \frac{-3}{8}$$

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}$$

$$f[x_0, x_1, x_2, x_3] = \frac{\frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1} - \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}}{x_3-x_0} = \frac{1}{8}$$

$$P_3(x) = \frac{x^3-9x^2+18x+16}{8}$$

(c)

(d)

(e) I will use Newton's Method. Because it can be represented as a recursive problem and we won't have to compute previously computed results again.

Exercise 2: Implementation of Interpolation

(a)

(b)

(c)

Exercise 3: Composite Trapezoidal Rule and Composite Simpson's Rule

(a) Trapezoidal

f(x) = $x^2.e^x$

For n=4

$$\int_0^1 f(x) \approx \frac{0.25*(f(0)+2f(0.25)+2f(0.5)+2f(0.75)+f(1))}{2}$$

$$\int_0^1 f(x) \approx \frac{0.25*(2*0.25^2*e^{0.25}+2*0.5^2*e^{0.5}+2*0.75^2*e^{0.75}+e)}{2} = 0.76$$

For n=8

$$\int_0^1 f(x) \approx \frac{0.125*(f(0)+2f(0.125)+2f(0.25)+2f(0.375)+2f(0.5)+2f(0.625)+2f(0.75)+2f(0.875)}{2}$$

0.738

Simpson's Rule

f(x) = $x^2.e^x$

For n=4

h = $\frac{1}{n} = 1/4$

$\int_0^1 f(x) \approx \dfrac{h(\Sigma_{i=1}^{i=n/2}(f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})))}{3}$

$\int_0^1 f(x) \approx \dfrac{(f(x_0) + 4f(x_1) + f(x_2)) + (f(x_2) + 4f(x_3) + f(x_4))}{12}$

x$_i = a + ih; for i = 1, 2, 3, .., n-1, n$

$\int_0^1 f(x) \approx \dfrac{(f(0) + 4f(0.25) + f(0.5)) + (f(0.5) + 4f(0.75) + f(1))}{12} = 0.7189$

For n=8

h = $\dfrac{1}{n} = 1/8$

$\int_0^1 f(x) \approx \dfrac{(f(x_0) + 4f(x_1) + f(x_2)) + (f(x_2) + 4f(x_3) + f(x_4)) + (f(x_4) + 4f(x_5) + f(x_6)) + (f(x_6) + 4}{12}$

$\int_0^1 f(x) \approx \dfrac{(f(0) + 4f(0.125) + f(0.25)) + (f(0.25) + 4f(0.375) + f(0.5)) + (f(0.5) + 4f(0.625) + f(0.75)}{24}$

$\int_0^1 f(x) \approx 0.7183$

Error Bound for the Trapezoid Rule: Suppose that |f"(x)| K for some k R where a x b. Then

|E$_{trap}$|$\leq K\dfrac{(b-a)^3}{12n^2}$

f"(x) = $2e^x + 4x$e$^x + x^2$e^x

since everything is positive we know that f"(x) will at max at x=1

K = $2e$ + $4e$ + e = $7e$

n = 4

$\Rightarrow |E_{trap}| \leq \dfrac{7e}{12 * 16}$

n = 8

$\Rightarrow |E_{trap}| \leq \dfrac{7e}{12 * 64}$

Error Bound for the Composite Simpson's Rule:

Suppose that |f$^{(4)}(x)| K for some k R where a x b. Then$

|E$_{simp}$|$\leq K\dfrac{(b-a)^5}{12n^4}$

f$^{(4)}(x) = 10$e$^x + 12x$e$^x + x^2$e^x

since everything is positive we know that f"(x) will at max at x=1

K = $10e$ + $12e$ + e = $23e$

n = 4

$\Rightarrow |E_{simp}| \leq \dfrac{23e}{12 * 256} \approx 0.02$

n = 8

$\Rightarrow |E_{simp}| \leq \dfrac{7e}{12 * 4096}$

(b) $\int_0^1 x^2.\mathrm{e}^x = [x^2.\int_0^1 \mathrm{e}^x]_0^1 - [2x.\int_0^1 \mathrm{e}_0^{x1} + [2.\int_0^1 \mathrm{e}^x]_0^1$

$\int_0^1 x^2.\mathrm{e}^x$ =e−2e+2e−2 =e−2 = 0.71828

We can clearly see that the result is in range of error bound for both methods

Simpson's rule comes out to be more accurate

Exercise 4: Implementation of Numerical Quadrature

(a)

(b)

(c)