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Stochastic Methods **Academic Year 2020/2021**

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Assignment 3 - Solution [Ashutosh Singh]

Exercise 1: Unconstrained Optimization of a Quadratic Problem

Solution

$f(x) = x^T Hx + c^T x$

$f'(x) = 2Hx + c$

For extrema of the function. Set first derivative equal to zero

$f'(x) = 0$

$2Hx + c = 0$

$\Rightarrow x = \frac{-(H^{-1}c)}{2}$

Exercise 2: Constrained Optimization with Parametrization

Solution

$\max_{x_1,x_2} f(x_1, x_2) = 5 - {x_1}^2 - \frac{1}{2}{x_2}^2$

s.t. $\mathrm{x}_1 + x_2 = 2$

Converting it to minimization problem by setting g(x) = -f(x)

$\min_{x_1,x_2} g(x_1, x_2) = -5 + {x_1}^2 + \frac{1}{2}{x_2}^2$

s.t. $\mathrm{x}_1 + x_2 = 2$

Now for parametrization, Put $x_1 = t$

$\Rightarrow x_2 = 2 - t$

$\Rightarrow g(x_1, x_2) = h(t) = -5 + t^2 + \frac{(t-2)^2}{2}$

$\Rightarrow h(t) = -3 + \frac{3t^2}{2} - 2t$

Now our minimization problem becomes
$\min_t h(t) = -3 + \frac{3t^2}{2} - 2t$
$\Rightarrow h'(t) = 3t - 2 = 0$
$\Rightarrow t = \frac{2}{3}$
$\Rightarrow x_1 = \frac{2}{3}$
$\Rightarrow x_2 = 2 - \frac{2}{3} = \frac{4}{3}$

Exercise 3: Optimization on the Unit Circle

(a) All norm balls are in the figure.

(b) Solution

$f(x_1, x_2) = x_1 x_2$

For 1-norm unit ball constraint in first quadrant

$\min_{x_1,x_2} f(x_1, x_2)$

s.t. $\mathrm{x}_1 + x_2 = 1$

Writing in Lagrange form

$L(x_1, x_2, \lambda) = x_1 x_2 + (x_1 + x_2 - 1)$

$$\nabla L(x_1, x_2, \lambda) = \begin{bmatrix} \frac{\partial L}{\partial x_1}(x_1, x_2, \lambda) \\ \frac{\partial L}{\partial x_2}(x_1, x_2, \lambda) \\ \frac{\partial L}{\partial \lambda}(x_1, x_2, \lambda) \end{bmatrix}$$

$$\nabla L(x_1, x_2, \lambda) = \begin{bmatrix} x_2 + \lambda \\ x_1 + \lambda \\ x_1 + x_2 - 1 \end{bmatrix}$$

$\Rightarrow \lambda = -\frac{1}{2}; x_1 == x_2 = \frac{1}{2}$

For 2-norm unit ball constraint in first quadrant

$\min_{x_1,x_2} f(x_1, x_2)$

s.t. $|\mathrm{x}_1|^2 + |x_2|^2 = 1$

Writing in Lagrange form

$L(x_1, x_2, \lambda) = x_1 x_2 + (x_1^2 + x_2^2 - 1)$

$$\nabla L(x_1,x_2,\lambda) = \begin{bmatrix} \dfrac{\partial L}{\partial x_1}(x_1,x_2,\lambda) \\ \dfrac{\partial L}{\partial x_2}(x_1,x_2,\lambda) \\ \dfrac{\partial L}{\partial \lambda}(x_1,x_2,\lambda) \end{bmatrix}$$

$$\nabla L(x_1,x_2,\lambda) = \begin{bmatrix} x_2 + 2x_1\lambda \\ x_1 + 2x_2\lambda \\ x_1^2 + x_2^2 - 1 \end{bmatrix}$$

$\Rightarrow (-2x_2\lambda)^2 + (-2x_1\lambda)^2 = 1$

$\Rightarrow (4\lambda^2(x_1^2 + x_2^2) = 1$

$\Rightarrow (4\lambda^2 = 1$

$\Rightarrow \lambda = \pm 1/2$

$\Rightarrow \mathrm{x}_1 = x_2$; for $\lambda = -\frac{1}{2}$,

$\Rightarrow \mathrm{x}_1 = -x_2$; for $\lambda = \frac{1}{2}$

Since we are considering only first quadrant

$\Rightarrow x_1 = \frac{1}{\sqrt{2}}, x_2 = \frac{1}{\sqrt{2}}$

For ∞-norm unit ball constraint in first quadrant

$\min_{x_1,x_2} f(x_1,x_2)$

s.t. $\max\{|x_1|,|x_2|\} = 1$

Writing in Lagrange form

$L(x_1,x_2,\lambda) = x_1x_2 + (\max\{|x_1|,|x_2|\} - 1)$

$$\frac{\partial L}{\partial x_1}(x_1,x_2,\lambda) = \begin{cases} x_2 + \lambda & \text{if } x_1 > x_2 \\ x_2 & \text{if } x_1 < x_2 \end{cases}$$

$$\frac{\partial L}{\partial x_2}(x_1,x_2,\lambda) = \begin{cases} x_1 + \lambda & \text{if } x_2 > x_1 \\ x_1 & \text{if } x_2 < x_1 \end{cases}$$

$$\frac{\partial L}{\partial l}(x_1,x_2,\lambda) = \max\{|x_1|,|x_2|\} - 1$$

if $\mathrm{x}_1 > x_2$

$$\nabla L(x_1,x_2,\lambda) = \begin{bmatrix} x_2 + \lambda \\ x_1 \\ x_1 - 1 \end{bmatrix}$$

if $\mathrm{x}_2 > x_1$

$$\nabla L(x_1, x_2, \lambda) = \begin{bmatrix} x_2 \\ x_1 + \lambda \\ x_2 - 1 \end{bmatrix}$$

if $x_2 = x_1$

$$\nabla L(x_1, x_2, \lambda) = \begin{bmatrix} x_2 + \lambda \\ x_1 + \lambda \\ x_2 - 1 (or x_1 - 1) \end{bmatrix}$$

first two conditions i.e $x_1 > x_2 and vice - versa do not yield gradient zero$

Gradient is only zero with given constraints at $x_1 = x_2 = 1$

$\Rightarrow \lambda = -1; x_1 = x_2 = 1$

(c) Solution

Optimal point moves farther away from the origin along the line $x_1 = x_2$

Exercise 4: Lagrange Multipliers and Bordered Hessian

(a) Case with n = 2 variables and m = 1 constraint:

$$\begin{aligned} \min \quad & f(x_1, x_2) = 3x_1^2 - 2x_2^2, \\ \text{s.t.} \quad & g(x_1, x_2) = x_1 - 2x_2 + 6 = 0 \end{aligned} \tag{1}$$

Writing in Lagrange form. Since there is only one constraint only one multiplier needed. Let's say λ

$$\min \quad L(x_1, x_2, \lambda) = 3x_1^2 - 2x_2^2 + \lambda(g(x_1, x_2)), \tag{2}$$

Now taking the gradient of $L(x_1, x_2, \lambda)$

$$\nabla L(x_1, x_2, \lambda) = \begin{bmatrix} \frac{\partial L}{\partial x_1}(x_1, x_2, \lambda) \\ \frac{\partial L}{\partial x_2}(x_1, x_2, \lambda) \\ \frac{\partial L}{\partial \lambda}(x_1, x_2, \lambda) \end{bmatrix}$$

$$\Rightarrow \nabla L(x_1, x_2, \lambda) = \begin{bmatrix} 6x_1 + \lambda \\ -4x_2 - 2\lambda \\ x_1 - 2x_2 + 6 \end{bmatrix}$$

For getting extreme points

$$\begin{aligned}
\nabla L(x_1, x_2, \lambda) &= 0 \\
x_1 &= -\frac{\lambda}{6}; \\
x_2 &= -\frac{\lambda}{2}; \\
&\text{and} \\
x_1 - 2x_2 + 6 &= 0 \\
\Rightarrow -\frac{\lambda}{6} + \lambda + 6 &= 0 \\
\Rightarrow \frac{5\lambda}{6} + 6 &= 0 \\
\Rightarrow \lambda &= -\frac{36}{5} \\
\Rightarrow x_1 = \frac{6}{5} \text{ and } x_2 &= \frac{18}{5}
\end{aligned}$$

We have our extrema at $(\frac{6}{5}, \frac{18}{5})$

Now calculating bordered Hessian and its determinant to check if we have a maxima or a minima

$$H(x_1, x_2, \lambda) = \begin{bmatrix} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial^2 L}{\partial^2 x_1}(x_1, x_2, \lambda) & \frac{\partial^2 L}{\partial x_1 \partial x_2}(x_1, x_2, \lambda) \\ \frac{\partial g}{\partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_2}(x_1, x_2, \lambda) & \frac{\partial^2 L}{\partial^2 x_2}(x_1, x_2, \lambda) \end{bmatrix}$$

$$H(x_1, x_2, \lambda) = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 6 & 0 \\ -2 & 0 & -4 \end{bmatrix}$$

-det$(H) > 0$

$\Rightarrow$ This is a minima

(b) Case with n = 3 variables and m = 1 constraint:

$$\begin{aligned}
\min \quad & f(x1, x2, x3) = (x_1 - 10)^2 + (x_2 - 3)^2 + (x_3 - 3)^2, \\
\text{s.t.} \quad & g(x_1, x_2, x_3) = 2x_1^2 + x_2^2 + x_3^2 - 10 = 0.
\end{aligned} \tag{3}$$

Writing in Lagrange form. Since there is only one constraint, only one multiplier needed. Let's say λ

$$\min \quad L(x_1, x_2, x_3, \lambda) = (x_1 - 10)^2 + (x_2 - 3)^2 + (x_3 - 3)^2 + \lambda(g(x_1, x_2)), \tag{4}$$

Now taking the gradient of L($\mathrm{x}_1, x_2, x_3, \lambda$)

$$\nabla L(x_1, x_2, x_3, \lambda) = \begin{bmatrix} \frac{\partial L}{\partial x_1}(x_1, x_2, x_3, \lambda) \\ \frac{\partial L}{\partial x_2}(x_1, x_2, x_3, \lambda) \\ \frac{\partial L}{\partial x_2}(x_1, x_2, x_3, \lambda) \\ \frac{\partial L}{\partial \lambda}(x_1, x_2, x_3, \lambda) \end{bmatrix}$$

$$\Rightarrow \nabla L(x_1, x_2, x_3, \lambda) = \begin{bmatrix} 2(x_1 - 10) + 4x_1\lambda \\ 2(x_2 - 3) + 2x_2\lambda \\ 2(x_3 - 3) + 2x_3\lambda \\ 2x_1^2 + x_2^2 + x_3^2 - 10 \end{bmatrix}$$

Now calculating bordered Hessian and its determinant to check if we have a maxima or a minima

$$H(x_1, x_2, \lambda) = \begin{bmatrix} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} \\ \frac{\partial g}{\partial x_1} & \frac{\partial^2 L}{\partial^2 x_1}(x_1, x_2, x_3\lambda) & \frac{\partial^2 L}{\partial x_1 \partial x_2}(x_1, x_2, x_3, \lambda) & \frac{\partial^2 L}{\partial x_1 \partial x_3}(x_1, x_2, x_3, \lambda) \\ \frac{\partial g}{\partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_2}(x_1, x_2, x_3, \lambda) & \frac{\partial^2 L}{\partial^2 x_2}(x_1, x_2, x_3, \lambda) & \frac{\partial^2 L}{\partial x_2 \partial x_3}(x_1, x_2, x_3, \lambda) \\ \frac{\partial g}{\partial x_3} & \frac{\partial^2 L}{\partial x_3 \partial x_1}(x_1, x_2, x_3, \lambda) & \frac{\partial^2 L}{\partial x_3 \partial x_2}(x_1, x_2, x_3, \lambda) & \frac{\partial^2 L}{\partial^2 x_3}(x_1, x_2, x_3, \lambda) \end{bmatrix}$$

(c) Case with n = 3 variables and m = 2 constraints:

$$\begin{aligned} \min \quad & f(x_1, x_2, x_3) = (x_1 - 3)^2 + (x_2 + 1)^2 - (x_3 - 2)^2, \\ \text{s.t.} \quad & g_1(x_1, x_2, x_3) = 3x_1 2x_2 + 4x_3 - 9 = 0, \\ \text{and} \quad & g_2(x_1, x_2) = x_1 + 2x_2 - 3 = 0. \end{aligned} \tag{5}$$

Writing in Lagrange form. Since there are two constraints we need two multipliers namely λ_1, λ_2

$$\min \quad L(x_1, x_2, x_3, \lambda_1, \lambda_2) = (x_1 - 10)^2 + (x_2 - 3)^2 + (x_3 - 3)^2 + \lambda_1 g_1(x_1, x_2, x_3) + +\lambda_2 g_2(x_1, x_2, x_3), \tag{6}$$

Now taking the gradient of L($\mathrm{x}_1, x_2, x_3, \lambda_1, \lambda_2$)

$$\nabla L(x_1, x_2, x_3, \lambda_1, \lambda_2) = \begin{bmatrix} \frac{\partial L}{\partial x_1}(x_1, x_2, x_3, \lambda_1, \lambda_2) \\ \frac{\partial L}{\partial x_2}(x_1, x_2, x_3, \lambda_1, \lambda_2) \\ \frac{\partial L}{\partial x_3}(x_1, x_2, x_3, \lambda_1, \lambda_2) \\ \frac{\partial L}{\partial \lambda_1}(x_1, x_2, x_3, \lambda_1, \lambda_2) \\ \frac{\partial L}{\partial \lambda_2}(x_1, x_2, x_3, \lambda_1, \lambda_2) \end{bmatrix}$$

$$\Rightarrow \nabla L(x_1, x_2, x_3, \lambda_1, \lambda_2) = \begin{bmatrix} 2(x_1 - 3) + 3\lambda_1 + \lambda_2 \\ 2(x_2 + 1) - 2\lambda_1 + 2\lambda_2 \\ 2(x_3 - 2) + 4\lambda_1 \\ 3x_1 2x_2 + 4x_3 - 9 \\ x_1 + 2x_2 - 3 \end{bmatrix}$$

Now calculating bordered Hessian and its determinant to check if we have a maxima or a minima

$$H = \begin{bmatrix} 0 & 0 & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ 0 & 0 & \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & 0 \\ \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_1} & \frac{\partial^2 L}{\partial^2 x_1}(x_1, x_2, x_3\lambda_1, \lambda_2) & \frac{\partial^2 L}{\partial x_1 \partial x_2}(x_1, x_2, x_3, \lambda_1, \lambda_2) & \frac{\partial^2 L}{\partial x_1 \partial x_3}(x_1, x_2, x_3, \lambda_1, \lambda_2) \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial g_2}{\partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_2}(x_1, x_2, x_3, \lambda_1, \lambda_2) & \frac{\partial^2 L}{\partial^2 x_2}(x_1, x_2, x_3, \lambda_1, \lambda_2) & \frac{\partial^2 L}{\partial x_2 \partial x_3}(x_1, x_2, x_3, \lambda_1, \lambda_2) \\ \frac{\partial g_1}{\partial x_3} & \frac{\partial g_2}{\partial x_3} & \frac{\partial^2 L}{\partial x_3 \partial x_1}(x_1, x_2, x_3, \lambda_1, \lambda_2) & \frac{\partial^2 L}{\partial x_3 \partial x_2}(x_1, x_2, x_3, \lambda_1, \lambda_2) & \frac{\partial^2 L}{\partial^2 x_3}(x_1, x_2, x_3, \lambda_1, \lambda_2) \end{bmatrix}$$

Exercise 5: An Application to Economics

(a)

$$\begin{aligned} \max \quad & f(x1, x2, x3) = 3\ln(x_1) + \ln(x_2) + 2\ln(x_3), \\ \text{s.t.} \quad & x_1 + x_2 + x_3 - 3000 = 0 \end{aligned} \tag{7}$$

(b) Writing in Lagrange form. Since there are two constraints we need two multipliers namely λ_1, λ_2

$$\min \quad L(x_1, x_2, x_3, \lambda) = 3\ln(x_1) + \ln(x_2) + 2\ln(x_3) + \lambda(x_1 + x_2 + x_3 - 3000), \tag{8}$$

$$\Rightarrow \nabla L(x_1,x_2,x_3,\lambda) = \begin{bmatrix} \frac{3}{x_1}+\lambda \\ \frac{1}{x_2}+\lambda \\ \frac{2}{x_3}+\lambda \\ x_1+x_2+x_3-3000 \end{bmatrix}$$

$$\Rightarrow x_1 = -\frac{3}{\lambda}$$

$$\Rightarrow x_2 = -\frac{1}{\lambda}$$

$$\Rightarrow x_3 = -\frac{2}{\lambda}$$

$$\Rightarrow -\frac{3}{\lambda} - \frac{1}{\lambda} - \frac{2}{\lambda} = 3000$$

$$\Rightarrow -\frac{6}{\lambda} = 3000$$

$$\Rightarrow \lambda = -\frac{1}{500}$$

$$\Rightarrow x_1 = 1500$$

$$\Rightarrow x_2 = 500$$

$$\Rightarrow x_3 = 1000$$

$$H(x_1,x_2,x_3\lambda) = \begin{bmatrix} 0 & \dfrac{\partial g}{\partial x_1} & \dfrac{\partial g}{\partial x_2} & \dfrac{\partial g}{\partial x_3} \\ \dfrac{\partial g}{\partial x_1} & \dfrac{\partial^2 L}{\partial^2 x_1}(x_1,x_2,x_3\lambda) & \dfrac{\partial^2 L}{\partial x_1 \partial x_2}(x_1,x_2,x_3,\lambda) & \dfrac{\partial^2 L}{\partial x_1 \partial x_3}(x_1,x_2,x_3,\lambda) \\ \dfrac{\partial g}{\partial x_2} & \dfrac{\partial^2 L}{\partial x_1 \partial x_2}(x_1,x_2,x_3,\lambda) & \dfrac{\partial^2 L}{\partial^2 x_2}(x_1,x_2,x_3,\lambda) & \dfrac{\partial^2 L}{\partial x_2 \partial x_3}(x_1,x_2,x_3,\lambda) \\ \dfrac{\partial g}{\partial x_3} & \dfrac{\partial^2 L}{\partial x_3 \partial x_1}(x_1,x_2,x_3,\lambda) & \dfrac{\partial^2 L}{\partial x_3 \partial x_2}(x_1,x_2,x_3,\lambda) & \dfrac{\partial^2 L}{\partial^2 x_3}(x_1,x_2,x_3,\lambda) \end{bmatrix}$$

$$H(x_1,x_2,x_3,\lambda) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -\frac{3}{x_1^2} & 0 & 0 \\ 1 & 0 & -\frac{1}{x_2^2} & 0 \\ 1 & 0 & 0 & -\frac{2}{x_3^2} \end{bmatrix}$$

$$H_{sub}(x_1, x_2, \lambda) = \begin{bmatrix} 0 & 1 & 1 \\ \\ 1 & -\frac{3}{x_1^2} & 0 \\ \\ 1 & 0 & -\frac{1}{x_2^2} \\ \\ \end{bmatrix}$$

$$-\det(H) > 0$$

Thus this can be a maximum or a minimum $-\det(H_{sub}) < 0$

$$\Rightarrow \text{We have a maxima at the point}$$

(c) Results are in the same ratio as the value of money for each girl.
No maxima would not be affected a lot since still overall happines increases if girl number 1 gets maximum money followed by three and two

Exercise 6: Maximisation Using the KKT Conditions

$$\begin{aligned} \max \quad & f(x1, x2) =, \\ \text{s.t.} \quad & g(x_1, x_2) = x_1^2 + x_2^2 \leq 2, \end{aligned} \tag{9}$$

Writing in Lagrange form.

$$\min \quad L(x_1, x_2\lambda) = x_1x_2 + \lambda(x_1^2 + x_2^2 - 2), \tag{10}$$

(a) $\dfrac{\partial^2 L}{\partial^2 x_1}$

(b) $\dfrac{\partial^2 L}{\partial^2 x_1} = 0$

(c) $\lambda(x_1^2 + x_2^2 - 2) = 0 = 0$

(d) $\lambda \leq 0$

(e) g($\mathrm{x}_1, x_2) = x_1^2 + x_2^2 \leq 2$

if $\lambda = 0 from c$

(a) $\mathrm{x}_2 ==$

(b) $x_1 = 0$

(c) $0 = 0$

(d) $0 \leq 0$

(e) $0 \leq 2$

Since all the conditions are satisfied (0,0) is a minima for f