A Mini-Project Report

On

"Transformation of Random Variables"

Submitted to

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TABLE OF CONTENT

Project Synopsis

The sole focus of this report was to give a detailed explanation of "Monotonic and Non-monotonic Transformation of single Random Variable while discussing its applications". Every topic mentioned has been explained by taking few examples. A brief and comprehensible explanation of every example has been given. Monotonic and Non-Monotonic behaviour of a function have also been discussed.

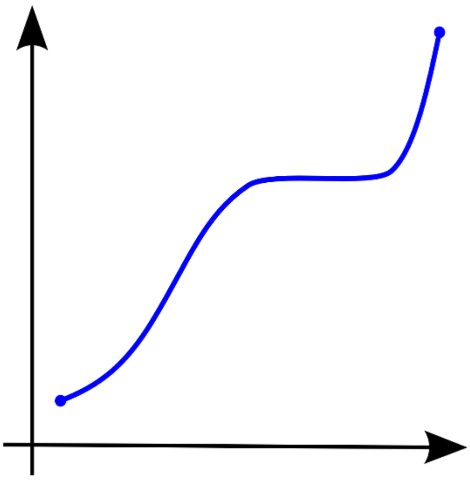
Abbreviations used:

1. r.v. = Random Variable
2. pmf = Probability Mass Function
3. pdf = Probability Density Function
4. cdf = Cumulative Distribution Function

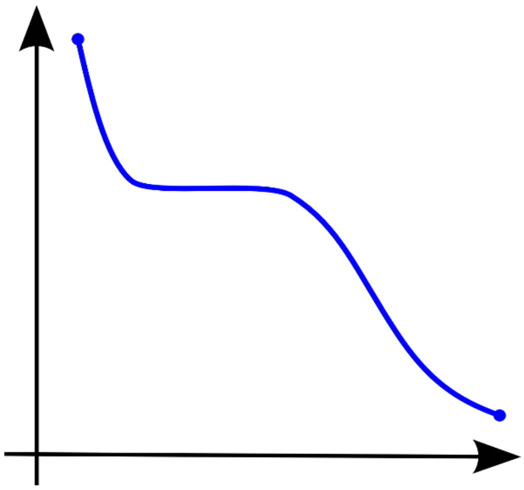
Monotonic and Non-Monotonic Function

Monotonic Function:

A function which is either non-increasing or non-decreasing throughout its domain is called a monotonic function. A monotonic function's first derivate (need not to be continuous) does not change sign.



A monotonically increasing function
Image source: Inkspace



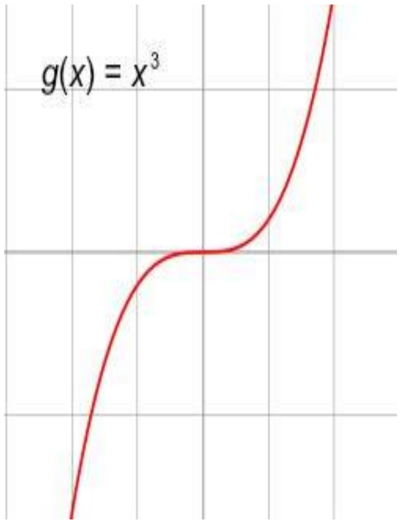
A monotonically decreasing function
Image source: Inkspace

#Example:

g(x)= x^3, *where x* ∈ *(-∞, ∞)*

Thing to notice here is that, the first derivative of *g(x)* is x^2
i.e., $\frac{d(g(x))}{dx} = x^2$

implies that no matter what sign is *g(x)*, its first derivative will always be positive, therefore always non-decreasing and hence a **monotone.**



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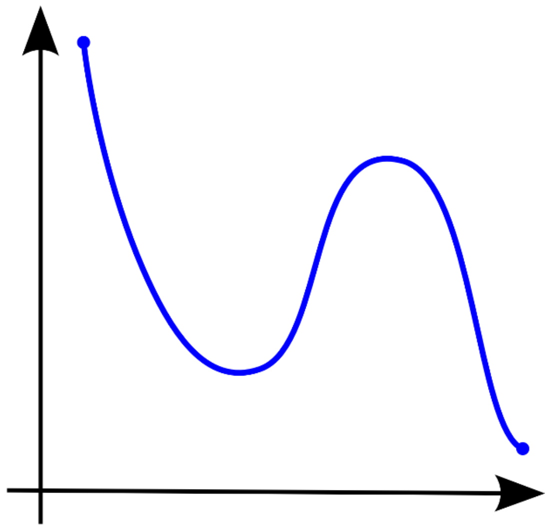
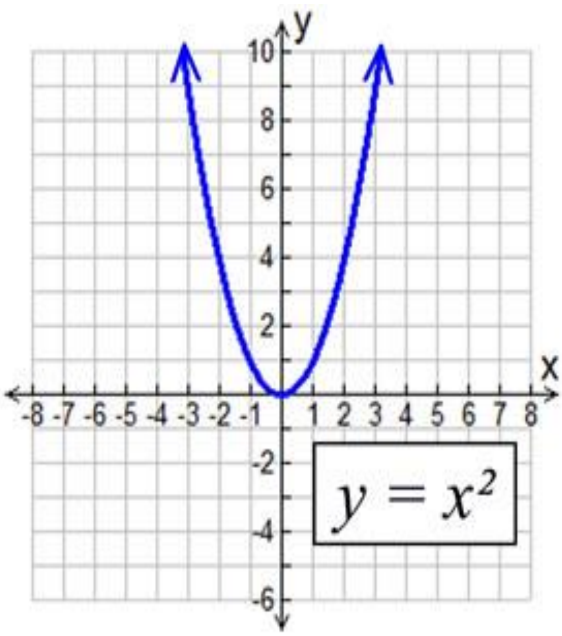
Non-Monotonic Function:

A non-monotonic function is increasing and decreasing on different intervals throughout its domain. Its first derivative changes sign.



A non-monotonic function
Image source: Inkspace



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#Example:

h(x)=x^2, where x ∈ (-∞, ∞)

its first derivative changes sign at the mid of the graph and hence not a **monotone**.

Transformation of Single Random Variables

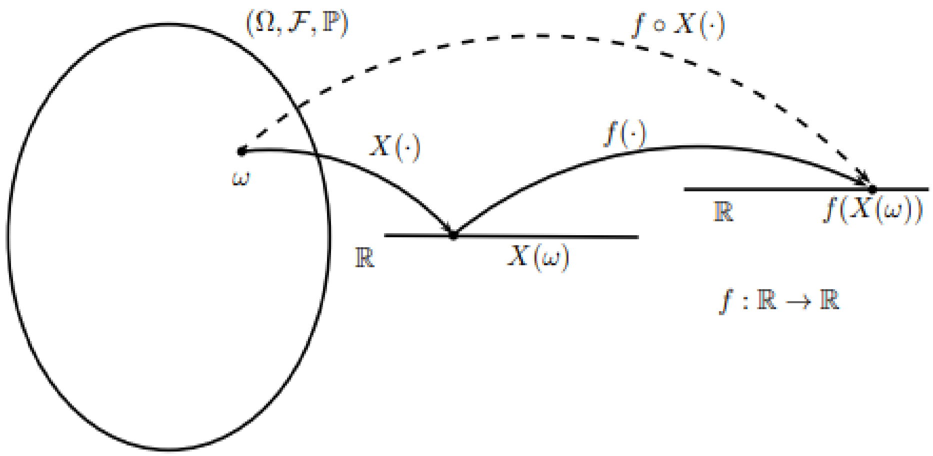
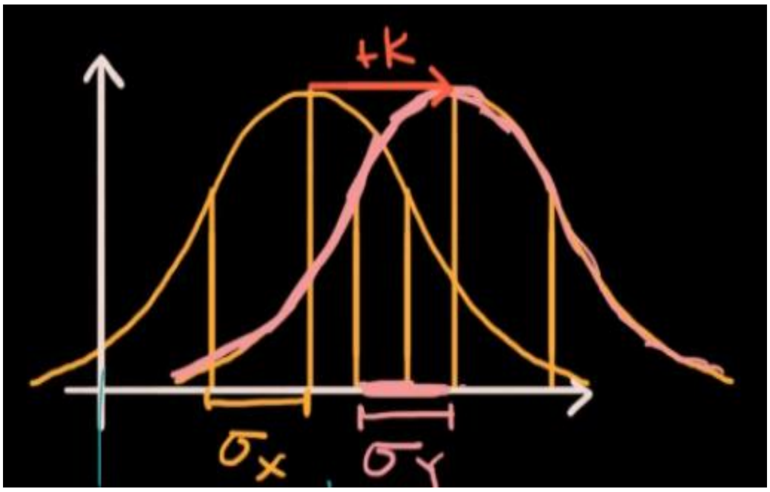


Fig: Transformation of Random Variable
Image source: [Here]

Impact of transforming (scaling and shifting) on random variables:

1. Suppose ***X*** be a random variable having a bell like distribution and ***Y*** be a new random variable obtained by shifting ***X*** by adding a non-zero constant ***k***.
   Such that, $Y = X + k$



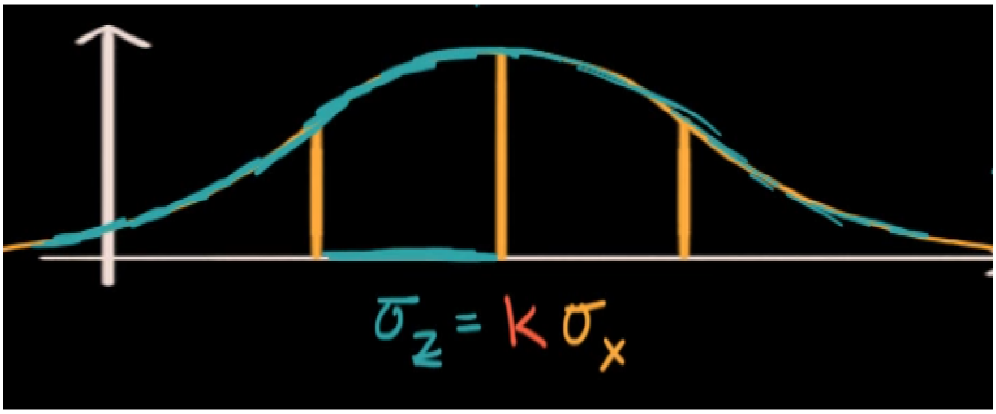
Distribution of X before and after shifting (Y)

Consequently, mean (μ_x) will also shift by the value k and the new mean will be $\mu_y = \mu_x + k$

and the standard deviation (σ_x) will remain unchanged,
i.e., $\sigma_y = \sigma_x$

2. Now, consider a new random variable ***Z*** which is obtained by scaling ***X*** by the factor ***k*** (non-zero)

   such that, $Z = kX$



Distribution of X after scaling (Z)

The curve has flattened because the area of the graph will still be **1**.

The new mean will be increased by k times
i.e., $\mu_z = k\mu_x$

and the new standard deviation will also be increased by k times
i.e., $\sigma_z = k\sigma_x$

For Discrete Random Variable:

X is a discrete random variable then the transformation is given by, $Y=T(X)$.
$p_x(x_i)=P(x=x_i)$ pmf of a discrete random variable.

1. **If T is monotonic**:
   Let X takes the value x_n.
   Then, $p_y(y_n) = p_x(x_n)$ as $P(y=y_n) = P(x=x_n)$.

2. **If T is non-monotonic:**
   Then, $P(y=y_n) = \sum P_x(x_n)$, $y_n=T(x_n)$

#Example 1:

Consider a r.v. X represents the number of heads obtained in 4 consecutive tosses of a coin then we need to find the pmf of $Y=\frac{1}{1+x}$.

Solution: Here X is a binomial r.v. with parameters (4,1/2)

$$\therefore p_x(k)={}^nC_k p^k q^{n-k}$$

k	0	1	2	3	4
$p_x(k)$	$(1/2^4)$	$(4/2^4)$	$(6/2^4)$	$(4/2^4)$	$(1/2^4)$

Since for every value x_n of X, we get different values y_n of Y
$Y=T(X)=\frac{1}{1+x}$
Therefore, ***T*** is a **monotone**.

X	0	1	2	3	4
Y	1	1/2	1/3	1/4	1/5
$P(y=y_n)$	1/16	1/4	3/8	1/4	1/16

Pmf of Y

#Example 2:

Consider a r.v. X can have the values $\{-3, -2, -1, 1, 2\}$, then find the pmf of $Y=X^2$.

$P(X=x_n) = \frac{1}{5}$ (For each value)

Solution: We have, $Y=T(X)=X^2$

$X=\{-3, -2, -1, 1, 2\}$
And $Y=X^2$

$\therefore Y=\{9,4,1\}$

$P(Y=1)=P(X=-1)+P(X=1)=\frac{1}{5}+\frac{1}{5}=\frac{2}{5}$

$P(Y=4)=P(X=-2)+P(X=2)=\frac{1}{5}+\frac{1}{5}=\frac{2}{5}$

$P(Y=9)=P(X=3)=\frac{1}{5}$

For different values of X, we get the same value of Y

Hence $\boldsymbol{T}$ is not a **monotone.**

y_n	1	4	9
$P(Y=y_n)$	$2/5$	$2/5$	$1/5$

pmf of Y

For Continuous Random Variable:

1. **Monotonic:**

   $F_y(y)=F_x(x) \Longrightarrow P(y \leq y_o)=P(x \leq x_o)$ therefore, increasing function

   $F_y(y)=1-F_x(x) \Longrightarrow P(y \leq y_o)=1-P(x \leq x_o)$ therefore, decreasing function

   Now,

$$f_y(y)=\frac{d}{dy}F_y(y)=\frac{d}{dy}F_x(x)=\frac{d}{dx}F_x(x)\,\frac{dx}{dy}$$

$$\therefore \quad \boxed{f_y(y)=f_x(x)\left|\frac{dx}{dy}\right|}$$

The transformation of a random variable with a function of monotonic nature adds up to calculating the inverse function (x as a function of y, If $Y=h(X)$ then h^{-1}) taking its derivative, putting in everything to a known formula, and simplifying it to get the PDF of the transformed random variable.

#Example:

Consider, $Y=2X+3$, find pdf of Y if X is uniform r.v. with parameters (-1,2)

Solution: Since X is a uniform r.v. (-1,2)

Here, $a=-1$ and $b=2$

therefore, $f(x)= \begin{cases} \frac{1}{3}, & -1 < x < 2 \\ 0, & otherwise \end{cases}$

and $Y=2X+3$ is linear and a monotone.

$$\therefore f_y(y)=f_x(x)\left|\frac{dx}{dy}\right|$$

$$f_y(y)=f_x\left(\frac{y-3}{2}\right)\left|\frac{d\left(\frac{y-3}{2}\right)}{dy}\right|$$

on solving this gives,

$$f_y(y)=\begin{cases} 1/6, & 1 < x < 7 \\ 0, & otherwise \end{cases}$$

NOTE: Monotonic linear transformation of a uniform r.v. is also a uniform r.v.

NOTE: Monotonic linear Transformation of a normal r.v. is also a normal r.v.

2. **Non-monotonic:**
   X is a continuous r.v. having pdf $f_x(x)$ and the transformation is $Y=T(X)$ then the pdf of Y is given by

$$f_y(y) = \sum \frac{f_x(x_n)}{\left|\left[\frac{dT(x)}{dx}\right]_{x_n}\right|}$$

   Here the sum is taken so as to include all the real roots x_n of $Y=T(X)$. cdf can also be calculated by integrating the pdf.

#<u>Example</u>:

If X is a continuous r.v. with pdf $f_x(x)$ and $Y=aX^2, a>0$. Find the pdf of Y.

<u>Solution</u>: We have, $y=ax^2$ $\qquad x= \pm\sqrt{\frac{y}{a}}$

$\therefore$ Real roots, $x_1=+\sqrt{\frac{y}{a}}$ and $x_2=-\sqrt{\frac{y}{a}}$

$$\frac{dT(x)}{dx}=\frac{d(ax^2)}{dx}=2ax$$

At $x=x_1$, $\frac{dT(x)}{dx}=2a\sqrt{y/a}=2\sqrt{ay}$

At $x=x_2$, $\frac{dT(x)}{dx}=2a(-\sqrt{\frac{y}{a}})=-2\sqrt{ay}$

$$f_y(y)= \frac{f_x(x_1)}{\left|\left[\frac{dT(x)}{dx}\right]_{x_1}\right|} + \frac{f_x(x_2)}{\left|\left[\frac{dT(x)}{dx}\right]_{x_2}\right|}$$

$$= \frac{f_x(x_1)}{2\sqrt{ay}} + \frac{f_x(x_2)}{2\sqrt{ay}}$$

$$\therefore\ f_y(y) = \frac{1}{2\sqrt{ay}}\{f_x(x_1) + f_x(x_2)\}$$

There are various other techniques to our avail to do the transformation:

1. **The cumulative distribution function (cdf) technique:**

Let y be a continuous r.v. with cdf $F_y(y) \equiv P(Y \le y)$ and let $U = g(Y)$, a function of Y and we are supposed to find the distribution of U. And this technique is useful when the cdf has closed-form analytical expression.

Steps of the cdf technique:

1. Determine the domain of Y and U.
2. Write $F_U(u) = P(U \le u)$, the cdf of U, in terms of $F_Y(y)$, the cdf of Y.
3. Take the derivative of $F_U(u)$ to get the pdf of U, $f_U(u)$.

#Example1:

Suppose that $Y \sim U(0,1)$. Find the distribution of $U = g(Y) = -\ln Y$.

Solution. The cdf of $Y \sim U(0,1)$ is given by

$$F_Y(y) = \begin{cases} 0, & y \le 0 \\ y, & 0 < y \le 1 \\ 1, & y \ge 1 \end{cases}$$

The domain for $Y \sim U(0,1)$ is $R_Y = \{y: 0 < y < 1\}$; thus, because $u = -\ln y > 0$, it follows that the domain for U is $R_U = \{u: u > 0\}$. The cdf of U is:

$$\begin{aligned} F_U(u) = P(U \le u) &= P(-\ln Y \le u) \\ &= P(\ln Y > -u) \\ &= P(Y > e^{-u}) = 1 - P(Y \le e^{-u}) \\ &= 1 - F_Y(e^{-u}) \end{aligned}$$

Because $F_Y(y) = y$ for $0 < y < 1$; i.e., for $u > 0$, we have

$$F_U(u) = 1 - FY(e^{-u}) = 1 - e^{-u}$$

Taking derivatives, we get, for $u > 0$,

$f_U(u) = \frac{d}{du}F_U(u) = \frac{d}{du}(1 - e^{-u}) = e^{-u}$

Summarizing,

$f_U(u) = \begin{cases} e^{-u}, & u > 0 \\ 0, & otherwise \end{cases}$

This is an exponential pdf with mean $\frac{1}{\lambda} = 1$; that is, U ~ exponential ($\lambda = 1$).

For more examples refer to, Transformations.

NOTE: Recall from calculus that if g is one-to-one, it has an unique inverse g^{-1} . Also recall that if g is increasing (decreasing), then so is g^{-1} .

#Derivation of the pdf technique formula using the cdf method:

Let $g(y)$ be a strictly increasing function of y defined over R_Y .

Then, it follows that $u = g(y) \Leftrightarrow g - 1\ (u) = y$ and

$F_U(u) = P(U \leq u) = P[g(Y) \leq u] = P[Y \leq g^{-1}(u)] = F_Y[g^{-1}(u)]$

Differentiating $F_U(u)$ with respect to u, we get

$f_U(u) = \frac{d}{du}F_U(u) = \frac{d}{du}F_Y[g^{-1}(u)]$

$= f_Y[g^{-1}(u)]\frac{d}{du}g^{-1}(u)$ (by chain rule)

Now as g is increasing, so is g^{-1}; thus, $\frac{d}{du}g^{-1}(u) > 0$. If $g(y)$ is strictly decreasing, then

$F_U(u) = 1 - F_Y[g^{-1}(u)]$ and $d\ du\ g^{-1}(u) < 0$, which gives

$f_U(u) = F_U(u) = \frac{d}{du}\{1 - F_Y[g^{-1}(u)]\}$

$$= -f_Y[g^{-1}(u)]\,\frac{d}{du}g^{-1}(u)$$

Combining both cases, we have shown that the pdf of U, where non-zero, is given by

$$f_U(u) = f_Y[g^{-1}(u)]\left|\frac{d}{du}g^{-1}(u)\right|$$

It is again important to keep track of the domain for U. If R_Y denotes the domain of Y, then R_U , the domain for U, is given by $R_U = \{u\colon u = g(y);\ y \in R_Y\}$.

For more details, refer to, Transformations.

Applications of transformations of random variable:

1. The inspiration behind the transformation of a random variable is depicted in the following example:
   Imagine a situation where the velocity of a particle is distributed according to a random variable V. Based on a particular value of the velocity, there will be a corresponding value of kinetic energy E and we are asked the distribution of kinetic energy. Undoubtedly, this is a scenario where we are asking for the distribution of a new random variable, which depends on the original random variable through a transformation. Such circumstances occur oftentimes in practical applications.

2. Statistical Application:
   Often transforming a random variable can lead to pre-existing important results.

   Example:
   Suppose that Y ~ exponential(β); i.e., the pdf of Y is

$$f_Y(y) = \begin{cases} \frac{1}{\beta}e^{-\frac{y}{\beta}}, & y > 0 \\ 0, & otherwise \end{cases}$$

   Let $U = g(Y) = \sqrt{Y}$, Use the method of transformations to find the pdf of U.

Solution: First, we note that the transformation $g(Y) = \sqrt{Y}$ is a continuous strictly increasing function of y over $R_Y = \{y: y > 0\}$, and, thus, $g(Y)$ is one-to-one. Next, we need to find the domain of U. This is easy since y > 0 implies $u = \sqrt{y} > 0$ as well.

Thus, $R_U = \{u: u > 0\}$. Now, we find the inverse transformation:

$$g(y) = u = \sqrt{y} \Leftrightarrow y = g^{-1}(u)$$

$$= u^2 \qquad \text{(by inverse transformation)}$$

and its derivative:

$$\frac{d}{du}g^{-1}(u) = \frac{d}{du}u^2 = 2u$$

Thus, for u > 0,

$$f_U(u) = f_Y[g^{-1}(u)]\left|\frac{d}{du}g^{-1}(u)\right|$$

$$= \frac{1}{\beta}e^{-\frac{u^2}{\beta}}\,|2u| = \frac{2u}{\beta}e^{-\frac{u^2}{\beta}}$$

Summarizing,

$$f_U(u) = \begin{cases} \frac{2u}{\beta}e^{-\frac{u^2}{\beta}}, & u > 0 \\ 0, & otherwise \end{cases}$$

This is a **Weibull distribution**. The Weibull family of distributions is common in life science (survival analysis), engineering and actuarial science applications.

CONCLUSION:

1. Behaviour of a function (Monotonic or Non-Monotonic) plays a role in transforming a random variable.
2. Type of the random variable (Discrete or Continuous) also plays the role.
3. There are different techniques to our avail to carry out the transformation.
4. This topic has some real-world applications.

References:

1. Non-Monotonic Transformations of Random Variables Nick McMullen, Daniel Ochoa Macalester College Math 354 December 9, 2016.

2. EE5110: Probability Foundations for Electrical Engineers, July-November 2015, Lecture 16: General Transformations of Random Variables, Lecturer: Dr. Krishna Jagannathan, Scribe: Ajay and Jainam.

3. Transformation.