Unit No. 5

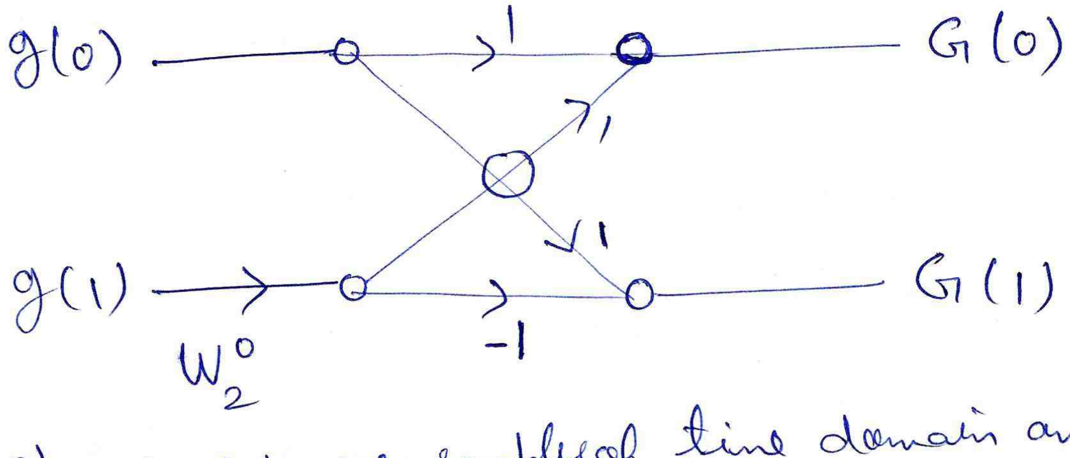
Fast Fourier Transform (FFT)

* Fast Fourier Transform (FFT) is not a new transform. FFT is a technique used for fast computation of DFT.

Algorithm of FFT

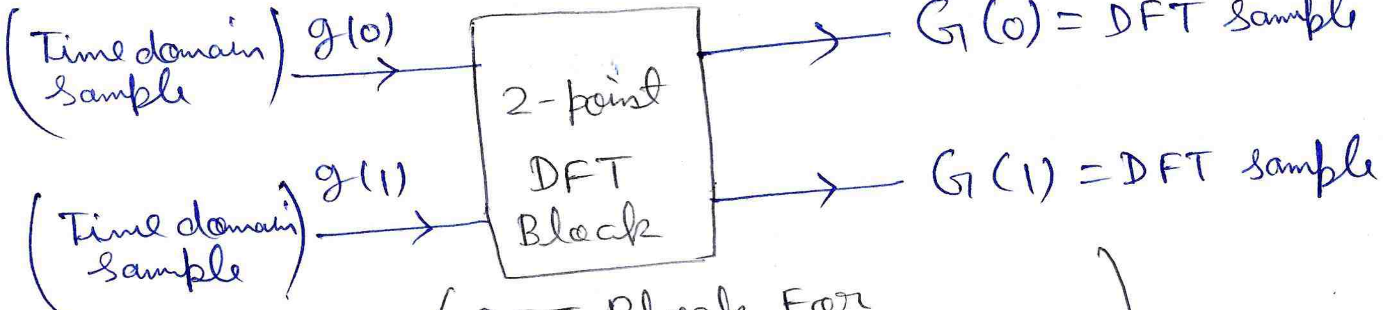
- Decimation In Time Algorithm (DIT Algorithm)
- Decimation In Frequency Algorithm (DIF Algorithm)

Butterfly Structure of 2-Point DFT For DIT Algorithm



g(0) and g(1) are samples of time domain and G(0) and G(1) are samples of frequency domain corresponding to time domain sample g(0) and g(1).

Development of Butterfly Structure For Computation of 2- Point DFT



(DFT Block For Computation of 2-point DFT)

If $g(n)$ is a sequence having two samples (i.e. $N=2$) then its DFT $G(K)$ is

$$G(K) = \sum_{n=0}^{1} g(n)\, W_2^{Kn} \quad \text{where } K = 0, 1$$

$$\Rightarrow G(K) = g(0)\, W_2^{K.0} + g(1) W_2^{K.1}, \quad K=0,1 \quad \text{——(i)}$$

$$\Rightarrow \text{for } K=0, \quad G(0) = g(0)\, W_2^{0.0} + g(1)\, W_2^{0.1}$$

$$\Rightarrow \text{for } K=0, \quad G(0) = g(0) + g(1)\, W_2^{0} \quad \text{——(ii)}$$

Now for $K=1$, (put $K=1$ in equation i)

$$\Rightarrow G(1) = g(0) + g(1)\, W_2^{1} \quad \text{——(iii)}$$

Now since $W_N = e^{-j\frac{2\pi}{N}}$

$$\Rightarrow \text{for } N=2, \quad W_2^1 = \left(e^{-j\frac{2\pi}{2}}\right)^1 = e^{-j\pi} = -1$$

So put $W_2^1 = -1$ in (iii)

So $G(1) = g(0) - g(1)$

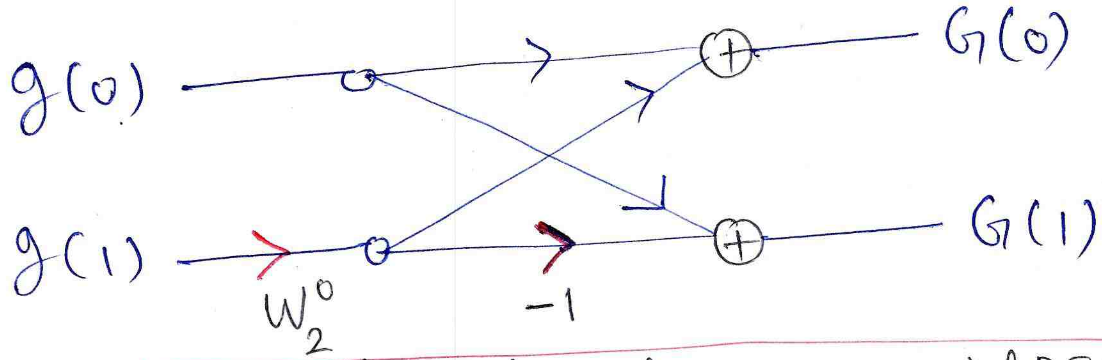
$$\Rightarrow G(1) = g(0) - g(1)\, W_2^{0} \quad \text{——(iv)}$$

So values of $G(0)$ and $G(1)$ are given using following equation

$$G(0) = g(0) + W_2^0 \cdot g(1)$$

$$G(1) = g(0) - W_2^0 \cdot g(1)$$

using above equation, we can make following Butterfly structure for computation of 2-point DFT



$\boxed{\text{Butterfly structure For-2 point DFT}}$

Development of Decimation In Time (DIT) Algorithm :-

DFT $X(K)$ of $x(n)$ is

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn}$$

Take $N = 4$ to develope algorithm

So after putting $N = 4$ in above expression

$$X(K) = \sum_{n=0}^{3} x(n)\, W_N^{Kn}$$

$$\Rightarrow X(K) = x(0)\, W_N^{K \cdot 0} + x(1)\, W_N^{K \cdot 1} + x(2)\, W_N^{K \cdot 2} + x(3)\, W_N^{K \cdot 3}$$

(4)

$$\Rightarrow X(K) = \left\{ x(0) W_N^{K\cdot 0} + x(2) W_N^{K\cdot 2} \right\} + \left\{ x(1) W_N^{K\cdot 1} + x(3) W_N^{K\cdot 3} \right\}$$

$$\Rightarrow X(K) = \sum_{m=0}^{1} x(2m) W_N^{K\cdot 2m} + \sum_{m=0}^{1} x(2m+1) W_N^{K(2m+1)}$$

$$\Downarrow$$

In general

$$X(K) = \sum_{m=0}^{\left(\frac{N}{2}-1\right)} x(2m) W_N^{K\cdot 2m} + \sum_{m=0}^{\left(\frac{N}{2}-1\right)} x(2m+1) W_N^{K(2m+1)}$$

$$\Rightarrow X(K) = \sum_{m=0}^{\frac{N}{2}-1} x(2m) \left(W_N^2\right)^{Km} + \sum_{m=0}^{\frac{N}{2}-1} x(2m+1) \left(W_N^2\right)^{Km} \cdot W_N^K$$

Now since $W_N^2 = \left(e^{-j\frac{2\pi}{N}}\right)^2 = e^{-j\frac{4\pi}{N}} = e^{-j\left(\frac{2\pi}{N/2}\right)}$

$= W_{N/2}$

So put $W_N^2 = W_{N/2}$ in above equation

$$\text{So } X(K) = \sum_{m=0}^{\frac{N}{2}-1} x(2m) W_{N/2}^{Km} + \left(\sum_{m=0}^{\frac{N}{2}-1} x(2m+1) W_{N/2}^{Km} \right) \cdot W_N^K$$

Now let $x(2m) = f_1(m)$ and $x(2m+1) = f_2(m)$

$$\text{So } X(K) = \sum_{m=0}^{\frac{N}{2}-1} f_1(m) \cdot W_{N/2}^{Km} + W_N^K \sum_{m=0}^{\frac{N}{2}-1} f_2(m) W_{N/2}^{Km}$$

$\Rightarrow X(K) = \frac{N}{2}$ point DFT of $f_1(m) + W_N^K \left(\frac{N}{2} \text{ point DFT of } f_2(m) \right)$

$$\Rightarrow X(K) = F_1(K) + W_N^K F_2(K)$$

so $X(K) = F_1(K) + W_N^K F_2(K)$

Now put $\left(K+\frac{N}{2}\right)$ in place of K on both side

so $X\left(K+\frac{N}{2}\right) = F_1\left(K+\frac{N}{2}\right) + W_N^{\left(K+\frac{N}{2}\right)} F_2\left(K+\frac{N}{2}\right)$

$$\Rightarrow X\left(K+\frac{N}{2}\right) = F_1(K) + W_N^{\left(K+\frac{N}{2}\right)} F_2(K) \quad \text{--- (1)}$$

{ because $F_1(K)$ and $F_2(K)$ are $\frac{N}{2}$ point DFT. so they are periodic after $\frac{N}{2}$
$\Rightarrow F_1\left(K+\frac{N}{2}\right) = F_1(K)$ and $F_2\left(K+\frac{N}{2}\right) = F_2(K)$ }

Now $W_N^{K+\frac{N}{2}} = W_N^K \cdot W_N^{N/2} = W_N^K \left(e^{-j\frac{2\pi}{N}}\right)^{N/2}$

$$\Rightarrow W_N^{K+\frac{N}{2}} = W_N^K \cdot e^{-j\frac{2\pi}{N}\cdot\frac{N}{2}} = W_N^K \cdot e^{-j\pi}$$

$$\Rightarrow W_N^{K+\frac{N}{2}} = -W_K^N$$

So put $W_N^{K+\frac{N}{2}} = -W_K^N$ in equation (i)

$$\text{so } X\left(K+\frac{N}{2}\right) = F_1(K) - W_K^N F_2(K) \quad \text{--- (ii)}$$

we have also prove that

$$X(K) = F_1(K) + W_N^K F_2(K) \quad \text{--- (iii)}$$

Equation (ii) and (iii) are pair of equation which will be used to compute DFT using DIT FFT algorithm.

If N = 4 then equation (ii) and (iii) becomes

$$X(K) = F_1(K) + W_4^K F_2(K) \quad \text{--- (4)}$$

$$X(K+2) = F_1(K) - W_4^K F_2(K) \quad \text{--- (5)}$$

Here K = 0, 1 because $F_1(K)$ and $F_2(K)$ are $\frac{N}{2}$ point DFT i.e. here $\frac{4}{2}$ = 2-point DFT

⇓

K=0 and K=1

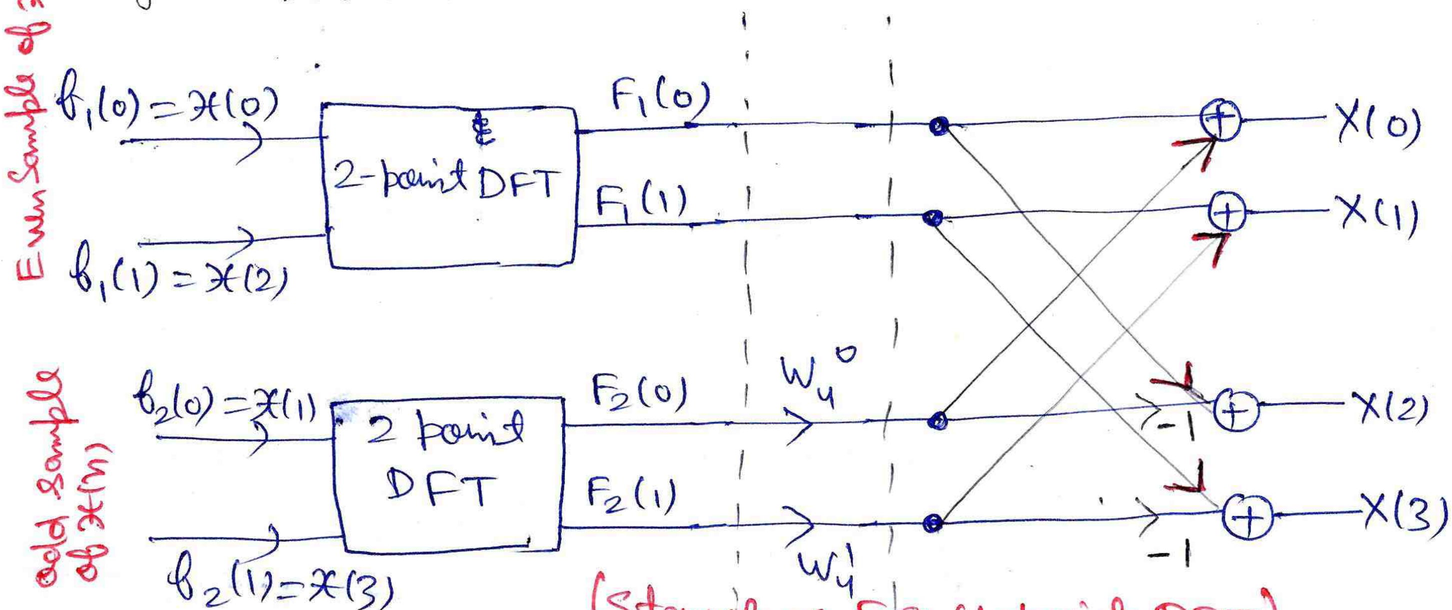
now put K=0 and K=1 in (4)

$$\left.\begin{aligned} \text{so } X(0) &= F_1(0) + W_4^0 F_2(0) \\ X(1) &= F_1(1) + W_4^1 F_2(1) \end{aligned}\right\} \longrightarrow (6)$$

Now put K=0 and K=1 in equation (5)

$$\left.\begin{aligned} \text{so } X(2) &= F_1(0) - W_4^0 F_2(0) \\ X(3) &= F_1(1) - W_4^1 F_2(1) \end{aligned}\right\} \longrightarrow (7)$$

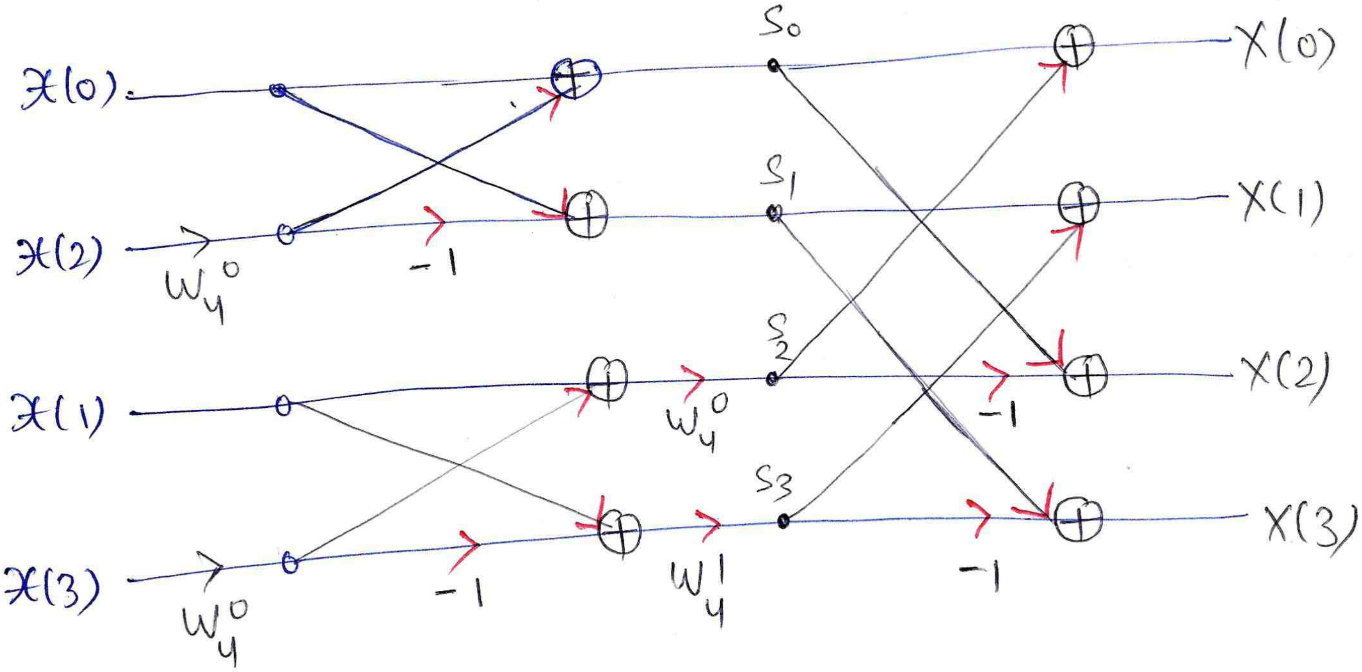
Now equation (6) and (7) can be pictorially represented as given below



(Structure For 4-point DFT)

For making Complete Butterfly structure of 4-point DFT using DIT FFT algorithm, replace 2-point DFT block by Butterfly structure of 2-point DFT.

Butterfly Structure To Compute 4-point DFT



In case of DIT FFT algorithm, order of i/p is decided by bit reversal of order of o/p.

Eg :- for $X(1) \rightarrow 1 \equiv 01 \xrightarrow[\text{reversal}]{\text{after bit}} 10 \Rightarrow 2 \Rightarrow x(2)$

for $X(2) \rightarrow 2 \equiv 10 \xrightarrow[\text{reversal}]{\text{after bit}} 01 \Rightarrow 1 \Rightarrow x(1)$

So i/p is $x(2)$ and $x(1)$ corresponding to $X(1)$ and $X(2)$. Similar is the case for $x(0)$ and $x(3)$.

In above butter fly structure, if $x(n) = \{1, 2, 3, 4\}$

then $S_0 = x(0) + x(2) = 1 + 3 = 4$

$S_1 = x(0) - x(2) = 1 - 3 = -2$

$S_2 = \{x(1) + x(3)\} W_4^0 = (2+4)\cdot 1 = 6$

$S_3 = \{x(1) - x(3)\} W_4^1 = \{2-4\}(-j)$ $\left\{\text{since } W_4^1 = -j\right\}$

$\Rightarrow S_3 = +2j$

Calculation of Samples of DFT i.e. Calculation of X(0), X(1), X(2) and X(3)

$X(0) = S_0 + S_2 = 4 + 6 = 10$

$X(1) = S_1 + S_3 = -2 + j2$

$X(2) = S_0 - S_2 = 4 - 6 = -2$

$X(3) = S_1 - S_3 = -2 - j2$

So using Butterfly structure of DIT FFT algorithm for calculation of 4-point DFT, we have shown that if $x(n) = \{1, 2, 3, 4\}$ then its 4-point DFT is

$$X(K) = \{10, -2+2j, -2, -2-j2\}$$

Meaning of Decimation in Decimation In Time (DIT)

$\Downarrow$

Input is divided as per rule of bit reversal.

Butterfly Structure For 8-Point DIT FFT Algorithm

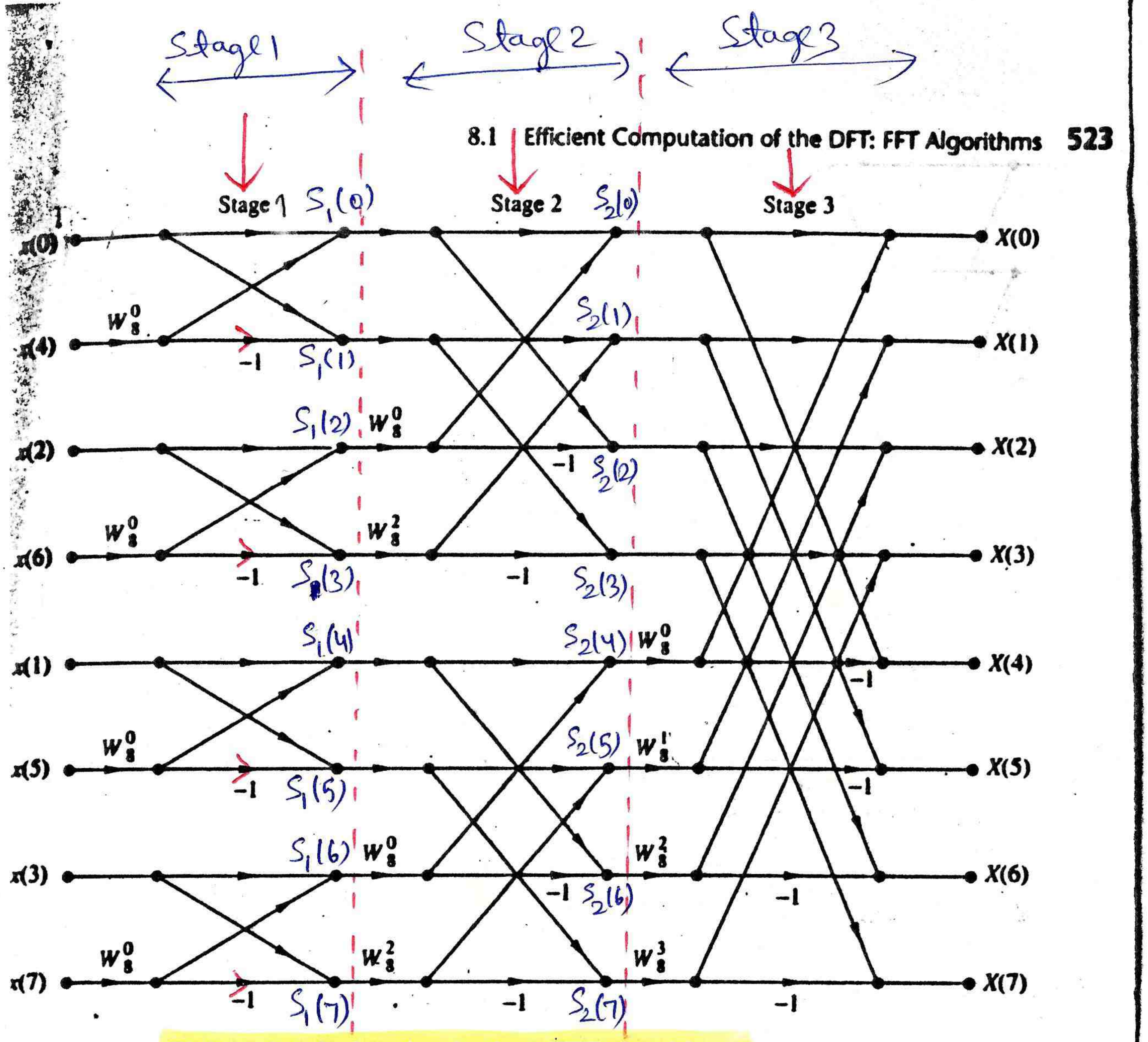


Figure 8.1.6 Eight-point decimation-in-time FFT algorithm.

Above figure is a Butterfly structure for 8-point DFT made using DIT FFT algorithm

Things To Remember For Making Butterfly Structure

(i) How to write o/p and i/p (ii) how to make stage-1, stage 2 and stage 3
(iii) How to write twiddle factor
(iv) How to multiply by (-1)

Q) using above Butterfly Structure of 8-point DIT FFT algorithm, find 8-point DFT $X(k)$ if $x(n) = \left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0 \right\}$

Solution

Step 1 → Make Butterfly Structure as given above for 8-point DFT

In above Butterfly structure, $S_1(n)$ and $S_2(n)$ represent o/p of first and second stage.

Step 2 → O/P of Stage 1

$S_1(0) = x(0) + W_8^0 x(4) = \frac{1}{2} + 1\cdot(0) = \frac{1}{2}$

$S_1(1) = x(0) - W_8^0 x(4) = \frac{1}{2} - 1\cdot(0) = \frac{1}{2}$

$S_1(2) = x(2) + W_8^0 x(6) = \frac{1}{2} + 1\cdot(0) = \frac{1}{2}$

$S_1(3) = x(2) - W_8^0 x(6) = \frac{1}{2} - 1\cdot(0) = \frac{1}{2}$

$S_1(4) = x(1) + W_8^0 x(5) = \frac{1}{2} + 1\cdot(0) = \frac{1}{2}$

$S_1(5) = S(1) - W_8^0 x(5) = \frac{1}{2} - 1\cdot(0) = \frac{1}{2}$

$S_1(6) = x(3) + W_8^0 x(7) = \frac{1}{2} + 1\cdot(0) = \frac{1}{2}$

$x_1(7) = x(3) - W_8^0 x(7) = \frac{1}{2} - 1\cdot(0) = \frac{1}{2}$

Step 3 → O/P of Stage 2

$S_2(0) = S_1(0) + W_8^0 S_1(2) = \frac{1}{2} + 1\cdot(\frac{1}{2}) = 1$

$S_2(1) = S_1(1) + W_8^2 S_1(3) = \frac{1}{2} - j\cdot\frac{1}{2}$

$S_2(3) = S_1(0) - W_8^2\, S_1(3) = \frac{1}{2} + j\cdot\frac{1}{2}$

Similarly we can show that

$S_2(4) = 1$, $S_2(5) = \frac{1}{2} - j\cdot\frac{1}{2}$, $S_2(6) = 0$

and $S_2(7) = \frac{1}{2} + j\cdot\frac{1}{2}$

Step 4 → Last Step → Final O/P.

$\rightarrow X(0) = S_2(0) + W_8^0\, S_2(4) = 1 + 1 = 2$

$\rightarrow X(1) = S_2(1) + W_8^1\, S_2(5) = \left(\frac{1}{2} - j\frac{1}{2}\right) + \left\{(.707 - j.707) \times \left(\frac{1}{2} - j\frac{1}{2}\right)\right\}$

$\Rightarrow X(1) = 0.5 - j\,1.2$

$\rightarrow X(2) = S_2(2) + W_8^2\, S_2(6) = 0 + (-j)\cdot 0 = 0$

Similarly

$X(3) = .5 - j\,0.207$, $X(4) = 0$,

$X(5) = 0.5 + j\,0.207$, $X(6) = 0$

$X(7) = 0.5 + j\,1.21$.

Computational Comparision

Normal DFT — (+) → $N^2 - N$
— (×) → N^2

DFT using FFT — (+) → $\frac{N}{2}\log_2 N$
— (×) → $N\log_2 N$

In both case, { (+) is no of adder
(×) is no of multiplier. }

(Q) If $N = 1024$ then compare Computational efficiency of DFT using FFT w.r.t. Computation of normal DFT by normal method.

Solution

Normal DFT — no of adder : $N^2 - N$
but $N = 1024$
— no of multiplier : N^2 , but $N = 1024$

DFT using FFT — no of adder : $\frac{N}{2}\log N$.
but $N = 1024$
— no of multiplier
$N\log_2 N$ but $N = 1024$