Finite Impulse Response (FIR) Filters

Chapter 7

7.1 INTRODUCTION

A filter is essentially a network that selectively changes the waveshape of a signal in a desired manner. The objective of filtering is to improve the quality of a signal (for example, to remove noise) or to extract information from signals.

A digital filter is a mathematical algorithm implemented in hardware / software that operates on a digital input to produce a digital output. Digital filters often operate on digitized analog signals stored in a computer memory. Digital filters play very important roles in DSP. Compared with analog filters, they are preferred in a number of applications like data compression, speech processing, image processing, etc., because of the following advantages.

(i) Digital filters can have characteristics which are not possible with analog filters such as linear phase response.
(ii) The performance of digital filters does not vary with environmental changes, for example, thermal variations.
(iii) The frequency response of a digital filter can be adjusted if it is implemented using a programmable processor.
(iv) Several input signals can be filtered by one digital filter without the need to replicate the hardware.
(v) Digital filters can be used at very low frequencies.

The following are the main disadvantages of digital filters compared with analog filters:

(i) Speed limitation
(ii) Finite word length effects
(iii) Long design and development times

A discrete-time filter produces a discrete-time output sequence $y(n)$ for the discrete-time input sequence $x(n)$. A filter may be required to have a given frequency response, or a specific response to an impulse, step, or ramp, or simulate an analog system. Digital filters are classified either as **finite duration unit pulse response** (FIR) filters or **infinite duration unit pulse response** (IIR) filters, depending on

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$$h(n) = \begin{cases} 2, & |n| \le 4 \\ 0, & \text{otherwise} \end{cases}$$

has only a finite number of non-zero terms. Thus, the system is an FIR system. The system with the impulse response $h(n) = a^n\, u(n)$ is non-zero for $n \ge 0$. It has an infinite number of non-zero terms and is an IIR system. IIR filters are usually implemented using structures having feedback (recursive structures—poles and zeros) and FIR filters are usually implemented using structures with no feedback (non-recursive structures—all zeros).

Suppose a system has the following difference equation representation with input $x(n)$ and output $y(n)$

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

An FIR filter of length M is described by the difference equation

$$\begin{aligned} y(n) &= b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + b_3 x(n-3) + \ldots \\ &\quad + b_{M-1} x(n-M+1) \\ &= \sum_{k=0}^{M-1} b_k x(n-k) \end{aligned}$$

where $\{b_k\}$ is the set of filter coefficients. As seen from the above equation, the response of the FIR filter depends only on the present and past input samples, whereas for the IIR filter, the present response is a function of the present and past N values of the excitation as well as past values of the response.

FIR filters have the following advantages over IIR filters

(i) They can have an exact linear phase.
(ii) They are always stable.
(iii) The design methods are generally linear.
(iv) They can be realised efficiently in hardware.
(v) The filter start-up transients have finite duration.

FIR filters are employed in filtering problems where linear phase characteristics within the passband of the filter is required. If this is not required, either an IIR or an FIR filter may be employed. An IIR filter has lesser number of side lobes in the stopband than an FIR filter with the same number of parameters. For this reason if some phase distortion is tolerable, an IIR filter is preferable. Also, the implementation of an IIR filter involves fewer parameters, less memory requirements and lower computational complexity.

7.2 MAGNITUDE RESPONSE AND PHASE RESPONSE OF DIGITAL FILTERS

The discrete-time Fourier transform of a finite sequence impulse response $h(n)$ is given by

The filter coefficients are then written as

$$h(n)=\frac{2}{M}\sum_{k=0}^{(M-3)/2}\text{Re}\left[\tilde{H}(k)e^{j\pi n(2k+1)/M}\right], M \text{ odd} \tag{7.33}$$

and

$$h(n)=\frac{2}{M}\sum_{k=0}^{\frac{M}{2}-1}\text{Re}\left[\tilde{H}(k)e^{j\pi n(2k+1)/M}\right], M \text{ even} \tag{7.34}$$

Example 7.5 *A low-pass filter has the desired response as given below*

$$H_d(e^{j\omega})=\begin{cases}e^{-j3\omega} & 0\le\omega<\frac{\pi}{2}\\ 0 & \frac{\pi}{2}\le\omega\le\pi\end{cases}$$

Determine the filter coefficients h(n) for M = 7, using Type-I frequency sampling technique.

Solution The samples of the given frequency response is taken uniformly at $\omega k = 2\pi k/M$. For $0\le\omega\le\frac{\pi}{2}$, the values of k = 0, 1. For $\frac{\pi}{2}\le\omega\le\frac{3\pi}{2}$, $k = 2, 3, 4, 5$ and for $\frac{3\pi}{2}\le\omega\le 2\pi, k=6$. Thus, the sampled frequency response is given by

$$\tilde{H}(k)=\begin{cases}e^{-j6\pi k/7}, & k=0,1\\ 0, & k=2,3,4,5\\ e^{-j6\pi k/7}, & k=6\end{cases}$$

The filter coefficients $h_d(n)$ are given by the inverse discrete Fourier transform,

$$h_d(n)=\frac{1}{M}\sum_{k=0}^{M-1}\tilde{H}(k)e^{j2\pi kn/M}=\frac{1}{7}\sum_{k=0}^{6}\tilde{H}(k)e^{j2\pi kn/7}$$

$$=\frac{1}{7}\left[\sum_{k=0}^{1}e^{-j6\pi k/7}e^{j2\pi kn/7}+e^{-j6\pi k/7}e^{j2\pi kn/7}\Big|_{k=6}\right]$$

$$=\frac{1}{7}\left[\sum_{k=0}^{1}e^{j2\pi k(n-3)/7}+e^{j12\pi(n-3)/7}\right]$$

$$=\frac{1}{7}\left[1+e^{j2\pi(n-3)/7}+e^{j12\pi(n-3)/7}\right]$$

Since $\tilde{H}(k)=\tilde{H}^*(M-k)$, we have $e^{j12\pi(n-3)/7}=e^{-j2\pi(n-3)/7}$.

Therefore,

$$h_d(n)=\frac{1}{7}\left[1+e^{j2\pi(n-3)/7}+e^{-j2\pi(n-3)/7}\right]$$

$$h_d(n)=0.1429+0.2857\cos[0.898(n-3)]$$

7.4.3 Window Techniques

The desired frequency response of any digital filter is periodic in frequency and can be expanded in a Fourier series, i.e.

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n)e^{-j\omega} \tag{7.35}$$

where

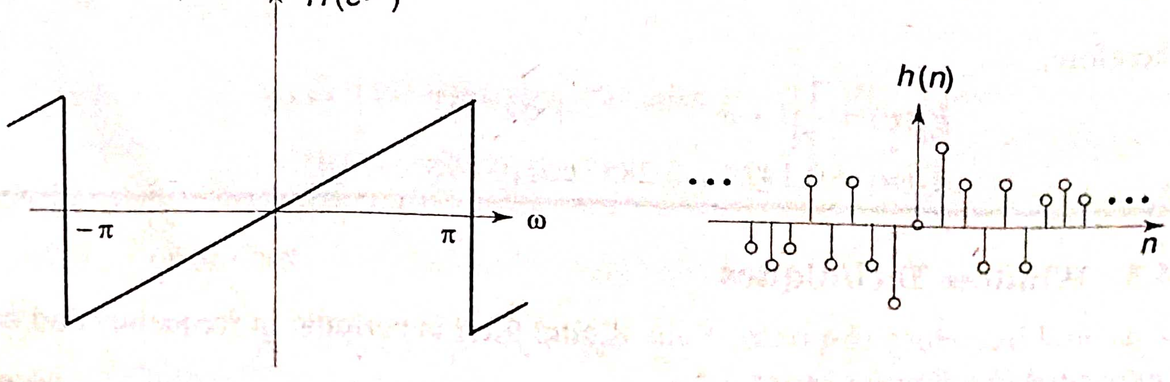
$$h(n) = \frac{1}{2\pi}\int_0^{2\pi} H(e^{j\omega})e^{j\omega n}\,d\omega \tag{7.36}$$

The Fourier coefficients of the series $h(n)$ are identical to the impulse response of a digital filter. There are two difficulties with the implementation of Eq. 7.35 for designing a digital filter. First, the impulse response is of infinite duration and second, the filter is non-causal and unrealisable. No finite amount of delay can make the impulse response realisable. Hence the filter resulting from a Fourier series representation of $H(e^{j\omega})$ is an unrealisable IIR filter.

The infinite duration impulse response can be converted to a finite duration impulse response by truncating the infinite series at $n = \pm N$. But, this results in undesirable oscillations in the passband and stopband of the digital filter. This is due to the slow convergence of the Fourier series near the points of discontinuity. These undesirable oscillations can be reduced by using a set of time-limited weighting functions, $w(n)$, referred to as window functions, to modify the Fourier coefficients. The windowing technique is illustrated in Fig. 7.1.

The desired frequency response $H(e^{j\omega})$ and its Fourier coefficients $\{h(n)\}$ are shown at the top of this figure. The finite duration weighting function $w(n)$ and its Fourier transform $W(e^{j\omega})$ are shown in the second row. The Fourier transform of the weighting function consists of a main lobe, which contains most of the energy of the window function and side lobes which decay rapidly. The sequence $\hat{h}(n) = h(n).w(n)$ is obtained to get an FIR approximation of $H(e^{j\omega})$. The sequence $\hat{h}(n)$ is exactly zero outside the interval $-N \le n \le N$. The sequence $\hat{h}(n)$ and its Fourier transform $\hat{H}(e^{j\omega})$ are shown in the third row. $\hat{H}(e^{j\omega})$ is nothing but the circular convolution of $H(e^{j\omega})$ and $W(e^{j\omega})$. The realisable causal sequence $g(n)$, which is obtained by shifting $\hat{h}(n)$, is shown in the last row and this can be used as the desired filter impulse response.

A major effect of windowing is that the discontinuities in $H(e^{j\omega})$ are converted into transition bands between values on either side of the discontinuity. The width of these transition bands depends on the width of the main lobe of $W(e^{j\omega})$. A secondary effect of windowing is that the ripples from the side lobes of $W(e^{j\omega})$



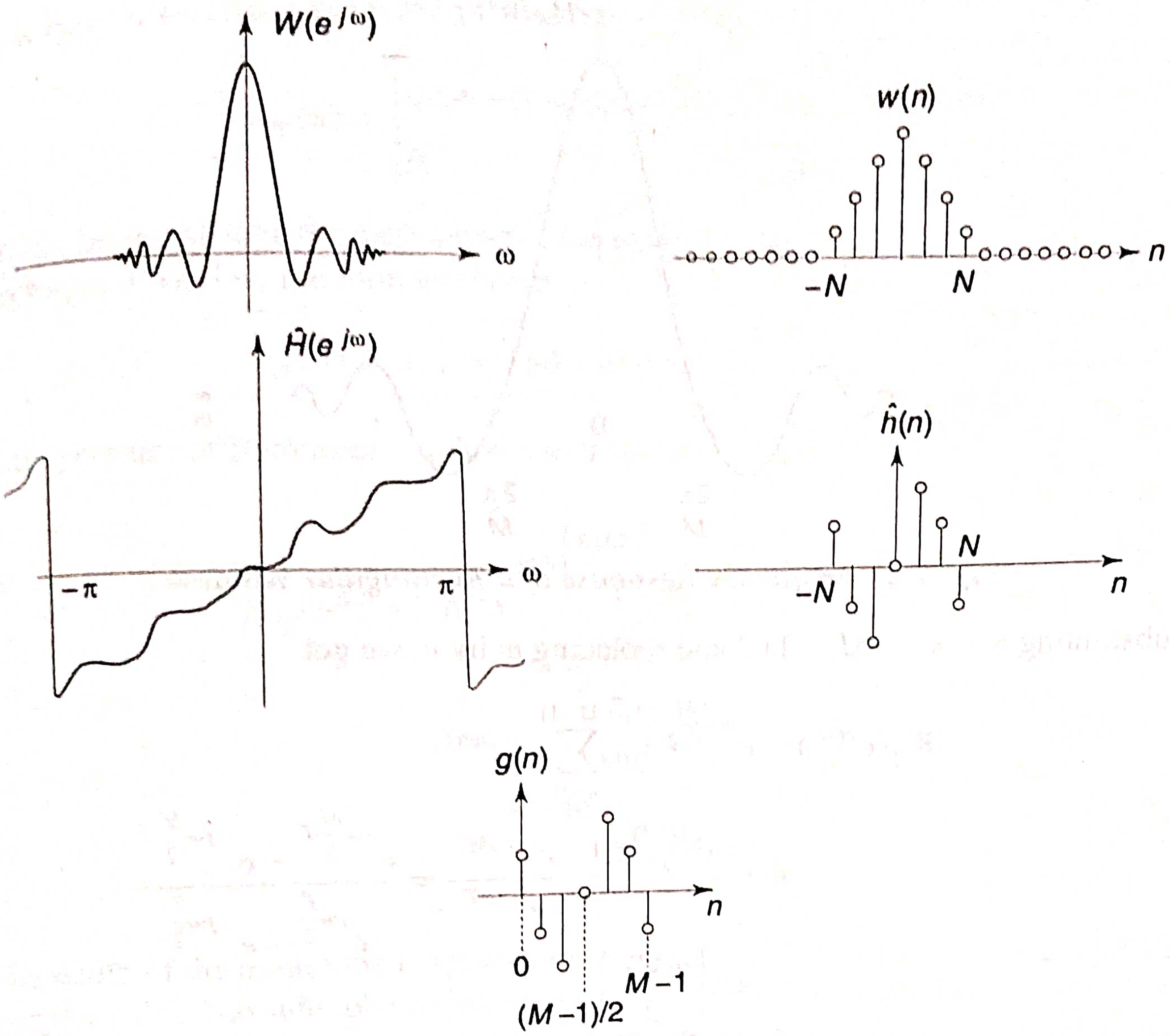


Fig. 7.1 *Illustration of the Window Technique*

produces approximation errors for all ω. Based on the above discussion, the desirable characteristics can be listed as follows

(i) The Fourier transform of the window function $W(e^{j\omega})$ should have a small width of main lobe containing as much of the total energy as possible.

(ii) The Fourier transform of the window function $W(e^{j\omega})$ should have side lobes that decrease in energy rapidly as ω tends to π. Some of the most frequently used window functions are described in the following sections.

Rectangular Window Function

The weighting function for the rectangular window is given by

$$\omega_R(n) = \begin{cases} 1, & \text{for } |n| \le \dfrac{M-1}{2} \\ 0, & \text{otherwise} \end{cases} \tag{7.37}$$

The spectrum of $w_R(n)$ can be obtained by taking Fourier transform of Eq. 7.37, as

$$W_R(e^{j\omega T}) = \sum_{n=-\frac{(M-1)}{2}}^{\frac{(M-1)}{2}} e^{-j\omega nT}$$

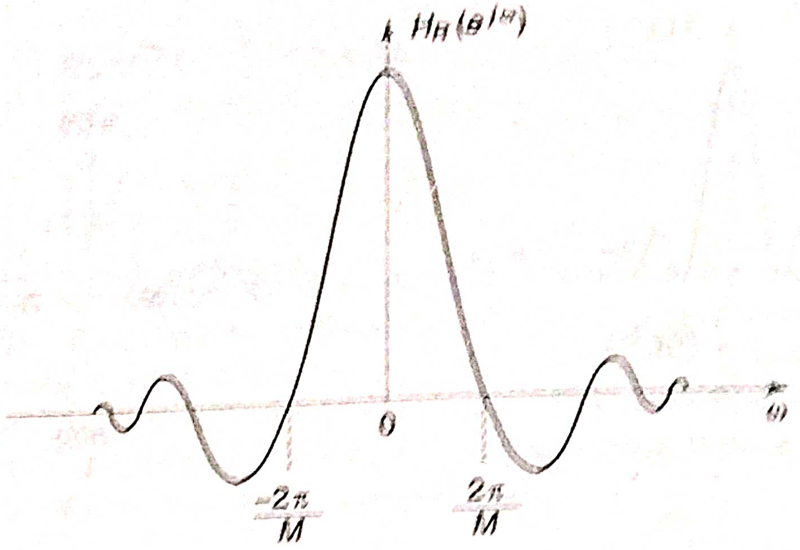


Fig .7.2 *Frequency Response of a Rectangular Window*

Substituting $n = m - (M-1)/2$ and replacing m by n, we get

$$W_R(e^{j\omega T}) = e^{j\omega\frac{(M-1)}{2}T} \sum_{n=0}^{(M-1)} e^{-j\omega nT}$$

$$= e^{j\omega\frac{(M-1)}{2}T} \frac{1-e^{-j\omega MT}}{1-e^{-j\omega T}} = \frac{e^{j\omega\frac{M}{2}T} - e^{-j\omega\frac{M}{2}T}}{e^{j\omega\frac{T}{2}} - e^{-j\omega\frac{T}{2}}}$$

$$= \frac{\sin\left(\frac{\omega MT}{2}\right)}{\sin\left(\frac{\omega T}{2}\right)} \tag{7.38}$$

A sketch of Eq. 7.38 is shown in Fig. 7.2. The transition width of the main lobe is approximately $4\pi/M$. The first sidelobe will be 13 dB down the peak of the main lobe and the rolloff will be at 20 dB per decade. For a causal rectangular window, the frequency response will be,

$$W_R(e^{j\omega T}) = \sum_{n=0}^{(M-1)} e^{-j\omega nT}$$

$$= e^{-j\omega\frac{(M-1)}{2}T} \frac{\sin\left(\frac{\omega MT}{2}\right)}{\sin\left(\frac{\omega T}{2}\right)} \tag{7.39}$$

From Eq.7.38 and 7.39, it is noted that the linear phase response of the causal filter is given by $\theta(\omega) = \omega(M-1)T/2$, and the non-causal impulse response has a zero phase shift.

Hamming Window Function

The causal Hamming window function is expressed by

$$w_H(n) = \begin{cases} 0.54 - 0.46\cos\frac{2\pi n}{M-1}, & 0 \le n < M-1 \\ 0, & \text{otherwise} \end{cases}$$

The non-causal Hamming window function is given by

$$w_H(n) = \begin{cases} 0.54 + 0.46\cos\dfrac{2\pi n}{M-1}, & \text{for } |n| \leq \dfrac{M-1}{2} \\ 0, & \text{otherwise} \end{cases} \tag{7.40}$$

It can be noted that the non-causal Hamming window function is related to the rectangular window function as shown,

$$w_H(n) = w_R(n)\left[0.54 + 0.46\cos\frac{2\pi n}{M-1}\right] \tag{7.41}$$

The spectrum of Hamming window can then be obtained as

$$W_H(e^{j\omega T}) = 0.54\frac{\sin\left(\dfrac{\omega MT}{2}\right)}{\sin\left(\dfrac{\omega T}{2}\right)} + 0.46\frac{\sin\left(\dfrac{\omega MT}{2} - \dfrac{M\pi}{(M-1)}\right)}{\sin\left(\dfrac{\omega T}{2} - \dfrac{\pi}{(M-1)}\right)} + 0.46\frac{\sin\left(\dfrac{\omega MT}{2} + \dfrac{M\pi}{(M-1)}\right)}{\sin\left(\dfrac{\omega T}{2} + \dfrac{\pi}{(M-1)}\right)} \tag{7.42}$$

The width of the main lobe is approximately $8\pi/M$ and the peak of the first side lobe is at -43 dB. The side lobe rolloff is 20 dB/decade. For a causal Hamming window the second and third terms are negative as shown.

$$\omega_H(e^{j\omega T}) = 0.54\frac{\sin\left(\dfrac{\omega MT}{2}\right)}{\sin\left(\dfrac{\omega T}{2}\right)} - 0.46\frac{\sin\left(\dfrac{\omega MT}{2} - \dfrac{M\pi}{(M-1)}\right)}{\sin\left(\dfrac{\omega T}{2} - \dfrac{\pi}{(M-1)}\right)} - 0.46\frac{\sin\left(\dfrac{\omega MT}{2} + \dfrac{M\pi}{(M-1)}\right)}{\sin\left(\dfrac{\omega T}{2} + \dfrac{\pi}{(M-1)}\right)} \tag{7.43}$$

Hanning Window Function

The window function of a causal Hanning window is given by

$$w_{Hann}(n) = \begin{cases} 0.5 - 0.5\cos\dfrac{2\pi n}{M-1}, & 0 \leq n \leq M-1 \\ 0, & \text{otherwise} \end{cases} \tag{7.44}$$

The window function of a non-causal Hanning window is expressed by

$$w_{Hann}(n) = \begin{cases} 0.5 + 0.5\cos\dfrac{2\pi n}{M-1}, & 0 < |n| < \dfrac{M-1}{2} \\ 0, & \text{otherwise} \end{cases}$$

The width of the main lobe is approximately $8\pi/M$ and the peak of the first side lobe is at -32 dB.

Blackman Window Function

The window function of a causal Blackman window is expressed by

$$w_B(n) = \begin{cases} 0.42 - 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1}, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

The window function of a non-causal Blackman window is given by

$$w_B(n) = \begin{cases} 0.42 + 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1}, & \text{for } |n| < \dfrac{M-1}{2} \\ 0, & \text{otherwise} \end{cases} \tag{7.45}$$

The width of the main lobe is approximately $12\pi/M$ and the peak of the first side-lobe is at -58 dB.

Bartlett Window Function

The window function of a non-causal Bartlett window is expressed by

$$w_{Bart}(n) = \begin{cases} 1+n, & -\dfrac{M-1}{2} < n < 1 \\ 1-n, & 1 < n < \dfrac{M-1}{2} \end{cases}$$

Table 7.1 gives the important frequency-domain characteristics of some window functions.

Table 7.1 *Frequency-domain characteristics of some window functions*

Type of Window	*Approximate Transition Width of Main Lobe*	*Minimum Stopband Attenuation (dB)*	*Peak of first Sidelobe (dB)*
Rectangular	$4\pi/M$	-21	-13
Bartlett	$8\pi/M$	-25	-27
Hanning	$8\pi/M$	-44	-32
Hamming	$8\pi/M$	-53	-43
Blackman	$12\pi/M$	-74	-58

Example 7.6 *Determine the unit sample response of the ideal low pass filter. Why is it not realizable?*

Solution The desired frequency response of the ideal low pass digital filter is given by

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}; & -\omega_c \le \omega \le \omega_c \\ 0 & ;\omega_c \le |\omega| < \pi \end{cases}$$

The frequency response is given by

$$H(e^{j\omega}) = \sum_{n=0}^{6} h(n)e^{-j\omega n}$$

$$= e^{-j3\omega}[h(3) + 2h(0)\cos 3\omega + 2h(1)\cos 2\omega + 2h(2)\cos\omega]$$

$$= e^{-j3\omega}[0.75 + 0.3466\cos\omega - 0.0988\cos 2\omega + 0.012\cos\omega]$$

Example 7.11 *Design a high pass filter using Hamming window with a cut-off frequency of 1.2 rad/sec and N = 9.*

Solution $\omega_c = 1.2$ rad/ sec and $N = 9$.

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}, & -\pi \le \omega \le -\omega_c \text{ and } \omega_c \le \omega \le \pi \\ 0, & \text{otherwise} \end{cases}$$

To find $h_d(n)$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega\alpha}e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{-j\omega\alpha}e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{-j\omega\alpha}e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{j\omega(n-\alpha)}d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{j\omega(n-\alpha)}d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{\omega_c}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_c(n-\alpha)} - e^{-\pi(n-\alpha)} + e^{j\omega(n-\alpha)} - e^{j\omega_c(n-\alpha)}}{j(n-\alpha)}\right]$$

$$= \frac{\sin(n-\alpha)\pi - \sin\omega_c(n-\alpha)}{\pi(n-\alpha)}$$

$$h_d(n) = \frac{\sin(n-\alpha)\pi - \sin\omega_c(n-\alpha)}{\pi(n-\alpha)} \text{ for } n \ne \alpha$$

$$h_d(n) = \frac{1}{\pi}\left(\underset{n\to\infty}{Lt}\frac{\sin(n-\alpha)\pi}{(n-\alpha)} - \underset{n\to\infty}{Lt}\frac{\sin(n-\alpha)\pi}{(n-\alpha)}\right)$$

$$= \frac{1}{\pi}(\pi - \omega_c) = \left[1 - \frac{\omega_c}{\pi}\right], \quad \text{for} \quad n \ne \alpha$$

$$h(n) = h_d(n)w_H(n)$$

$w_H(n)$=Window sequence for Hamming window.

$$= 0.54 - 0.46\cos\left[\frac{2\pi n}{N-1}\right] \quad \text{for} \quad n = 0 \text{ to } N-1$$

$$h(n) = \frac{1}{\pi(n-\alpha)}[\sin(n-\alpha)\pi - \sin(n-\alpha)\omega_c]$$
$$\left[0.54 - 0.46\left[\frac{2\pi n}{N-1}\right]\right]; \text{ for } n \neq \alpha$$

$$= \left(1 - \frac{\omega_c}{\pi}\right)\left[0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)\right]; \text{ for } n = \alpha$$

Here
$$\alpha = \frac{N-1}{2} = \frac{9-1}{2} = 4$$

$$h(n) = \frac{-\sin(n-4)\omega_c}{\pi(n-4)}\left[0.54 - 0.46\cos\frac{n\pi}{4}\right]; \text{ for } n \neq 4$$
$$= \left(1 - \frac{\omega_c}{\pi}\right) \times \left[0.54 - 0.46\cos\frac{n\pi}{4}\right]; \text{ for } n = 4$$
$$h(0) = \frac{-\sin(-4)\times(1.2)}{\pi\times(0-4)}[0.54 - 0.46\cos 0] = 0.0063$$

Similarly,

$h(1) = 0.0101$
$h(2) = 0.0581$
$h(3) = 0.2567$
$$h(4) = \left[\left(1 - \frac{1.2}{\pi}\right)\times(0.54 - 0.46\cos\pi)\right] = 0.6180$$
$h(5) = -0.2567$
$h(6) = -0.0581$
$h(7) = -0.0101$
$h(8) = -0.0063$

Since impulse response is symmetrical with centre of symmetry at $n = 4$.

$h(0) = h(8)$
$h(1) = h(7)$ h(1) = h(
$h(7) = h(6)$
$h(8) = h(5)$

To find magnitude response

$$|H(\omega)| = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{N-1/2} 2h\left(\frac{N-1}{2} - n\right)\cos\omega n$$
$$= h(4) + 2h(3)\cos\omega + 2h(2)\cos 2\omega + 2h(1)\cos 3\omega + 2h(0)\cos 4\omega$$
$$= 0.618 - 0.5134\cos\omega - 0.1162\cos 2\omega - 0.0202\cos 3\omega + 0.0126\cos 4\omega$$

To find the transfer function

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$
$$= h(0)[z^0 + z^{-8}] + h(1)[z^{-1} + z^{-7}] + h(2)[z^{-2} + z^{-6}] + h(3)[z^{-3} + z^{-5}] + h(4)z^{-4}$$

$W_{Bart}(0) = 0,\ W_{Bart}(1) = 0.5,\ W_{Bart}(2) = 1$

$h(0) = h(4) = 0,\ h(1) = h(3) = 0.5,\ h(2) = 1$

Step II: To find $H(e^{j\omega})$

$$H(e^{j\omega}) = e^{-j\omega(N-1/2)}\left[h\left(\frac{N-1}{2}\right) + 2\sum_{n=0}^{(N-3)/2} h(n)\cos\left(\frac{\omega}{\frac{N-1}{2}-n}\right)\right]$$

$$= e^{-j2\omega}\left[h(2) + 2\sum_{n=0}^{1} h(n)\cos(\omega(2-n))\right]$$

$$= e^{-j2\omega}\left[1 + 2[0.5\cos(\omega(1))]\right]$$

Therefore, $H(e^{j\omega}) = e^{-j2\omega}[1+\cos\omega]$

Example 7.14 *Design a bandbass filter to pass frequencies in the range 1 –2 rad/sec using Hanning window N = 5.*

Solution Given $\omega_c = 1$ to 2 rad/sec and $N = 5$.

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}; & -\omega_{c_2} \le \omega_{c_1} \text{ and } \omega_{c_1} \le \omega \le \omega_{c_2} \\ 0; & \text{otherwise} \end{cases}$$

To find (n)

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} e^{-j\omega\alpha}e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} e^{-j\omega\alpha}e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\omega_{c2}}^{-\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{\omega_{c1}}^{\omega_{c2}}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega_{c1}(n-\alpha)} - e^{-j\omega_{c_2}(n-\alpha)}}{j(n-\alpha)}\right] + \frac{1}{2\pi}\left[\frac{e^{j\omega_{c_2}(n-\alpha)} - e^{j\omega c_1(n-\alpha)}}{j(n-\alpha)}\right]$$

$$h_d(n) = \frac{1}{\pi(n-\alpha)}[\sin\omega_{c2}(n-\alpha) - \sin\omega_{c1}(n-\alpha)],\ \text{for } n \ne \alpha$$

$$h_d(n) = \frac{1}{\pi}\left(\underset{n\to\alpha}{\text{Lt}}\frac{\sin\omega_{c2}(n-\alpha)}{(n-\alpha)} - \underset{n\to\alpha}{\text{Lt}}\frac{\sin\omega_{c1}(n-\alpha)}{(n-\alpha)}\right)$$

$$h_d(n) = \frac{\omega_{c2}-\omega_{c1}}{\pi},\ \text{for } n = \alpha$$

$h(n) = h_d(n)w_{Hann}(n)$, where, $w_{Hann}(n)$ is the Hanning Window Sequence.

$$w_{Hann}(n) = \left(0.5 - 0.5\cos\frac{2\pi n}{N-1}\right),\ \text{for } n = 0 \text{ to } N-1.$$

$$h(n) = \frac{\sin[\omega_{c2}(n-\alpha)] - \sin[\omega_{c1}(n-\alpha)]}{\pi(n-\alpha)} \times \left[0.5 - 0.5\cos\frac{2\pi n}{N-1}\right],\ \text{for } n \ne \alpha$$

$$h(n) = \left[\frac{\omega_{c2}-\omega_{c1}}{\pi}\right]\left[0.5 - 0.5\cos\frac{2\pi n}{N-1}\right],\ \text{for } n = \alpha$$

Here $\alpha = \frac{N-1}{2} = \frac{5-1}{2} = 2$

$$h(0) = \frac{\sin[2\times(0-2)] - \sin[1\times(0-2)]}{\pi(0-2)} \times [0.5 - 0.5\cos 0] = 0$$

Similarly,

$$h(1) = 0.0108$$

$$h(2) = \frac{1}{\pi} = 0.3183$$

$$h(3) = 0.0108$$

$$\left.\begin{aligned} h(4) &= 0 \\ h(4) &= h(0) \\ h(3) &= h(1) \end{aligned}\right\} \text{Impulse response is symmetry with center at } n=2.$$

To find magnitude response

$$|H(\omega)| = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{n-1/2} 2h\left(\frac{N-1}{2} - n\right)\cos\omega n$$

$$= h(2) + 2h(1)\cos\omega + 2h(0)\cos 2\omega$$

$$= 0.3183 + 0.0216\cos\omega + 0$$

$$= 0.3183 + 0.0216\cos\omega$$

To find the transfer function

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n}$$

$$= h(1)[z^{-1} + z^{-3}] + h(0)[z^{-0} + z^{-4}] + h(2)z^{-2}$$

$$Y(z) = h(1)[z^{-1}X(z) + z^{-3}X(z)] + h(2)[z^{-2}X(z)]$$

7.4.4 Kaiser Window

From the frequency-domain characteristics of the window functions listed in Table 7.1, it can be seen that the width of the main lobe is inversely proportional to the length of the filter. As the length of the filter is increased, the width of the mainlobe becomes narrower and narrower, and the transition band is reduced considerably. The attenuation in the side-lobes is, however, independent of the length and is a function of the type of the window. Therefore, a proper window function is to be selected in order to achieve a desired stopband attenuation. A window function with minimum stopband attenuation has the maximum main lobe width. Therefore, the length of the filter must be increased considerably to reduce the main lobe width and to achieve the desired transition band.

A desirable property of the window function is that the function is of finite duration in the time domain and that the Fourier transform has maximum energy in the main lobe or a given peak side lobe amplitude. The prolate spheroidal functions have this desirable property; however, these functions are complicated and difficult to compute. A simple approximation to these functions have been developed by Kaiser in terms of zeroth order modified Bessel functions of the first kind. In a Kaiser window, the side lobe level can be controlled...

varying a parameter, α. The width of the main lobe can be varied by adjusting the length of the filter. The Kaiser window function is given by

$$w_K(n) = \begin{cases} \dfrac{I_0(\beta)}{I_0(\alpha)}, & \text{for } |n| \le \dfrac{M-1}{2} \\ 0, & \text{otherwise} \end{cases} \tag{7.46}$$

where α is an independent variable determined by Kaiser. The parameter β is expressed by

$$\beta = \alpha\left[1-\left(\frac{2n}{M-1}\right)^2\right]^{0.5} \tag{7.47}$$

The modified Bessel function of the first kind, $I_0(x)$, can be computed from its power series expansion given by

$$\begin{aligned} I_0(x) &= 1+\sum_{k=1}^{\infty}\left[\frac{1}{k!}\left(\frac{x}{2}\right)^k\right]^2 \\ &= 1+\frac{0.25x^2}{(1!)^2}+\frac{(0.25x^2)^2}{(2!)^2}+\frac{(0.25x^2)^3}{(3!)^2}+\cdots \end{aligned} \tag{7.48}$$

Figure 7.3 shows the idealised frequency responses of different filters with their passband and stopband specifications. Considering the design specifications of the

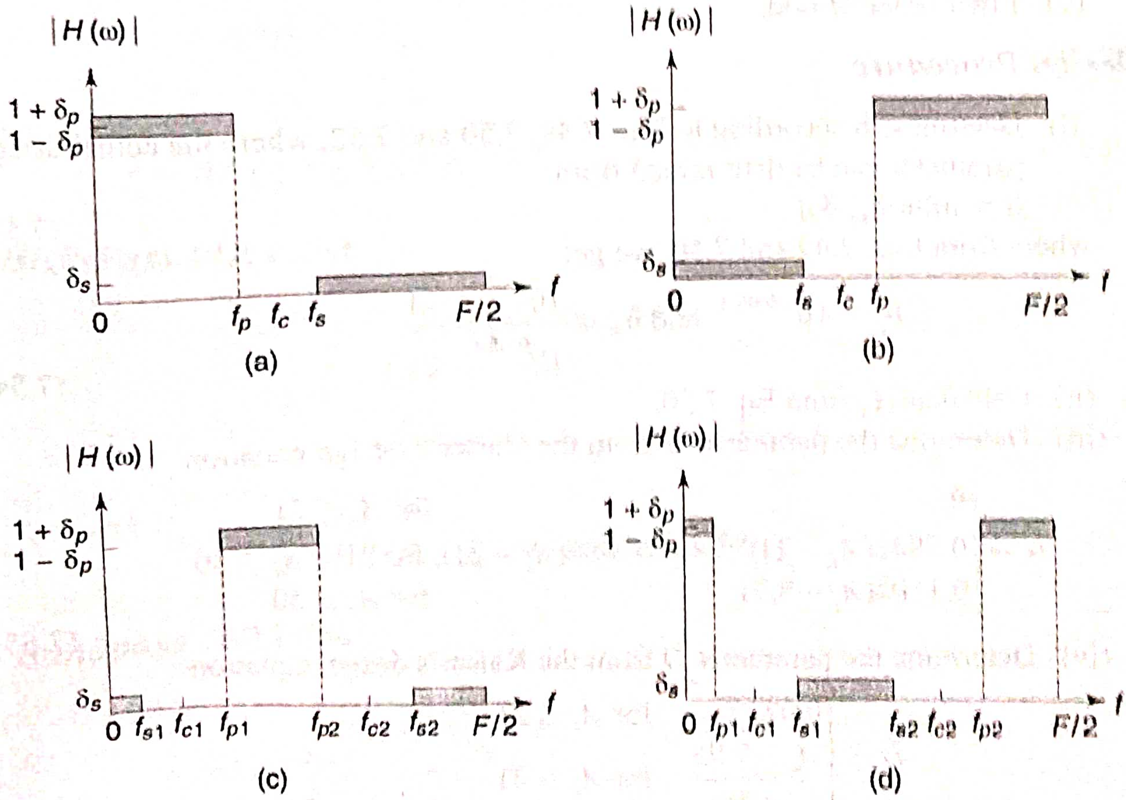


Fig 7.3 *Idealised Frequency Responses (a) Low-pass Filter; (b) High-pass Filter; (c) Bandpass Filter; (d) Bandstop Filter*

$$H(z)=\frac{T^2}{1-2z^{-1}+z^{-2}+16T^2}$$

If $T = 1$ s,

$$H(z)=\frac{1}{z^{-2}-2z^{-1}+17}$$

Example 8.3 *An analog filter has the following system function. Convert this filter into a digital filter using backward difference for the derivative.*

$$H(s)=\frac{1}{(s+0.1)^2+9}$$

Solution The system response of the digital filter is

$$H(z)=H(s)\Big|_{s=\frac{1-z^{-1}}{T}}=\frac{1}{\left(\frac{1-z^{-1}}{T}+0.1\right)^2+9}$$

$$H(z)=\frac{T^2}{z^{-2}-2(1+0.1T)z^{-1}+(1+0.2T+9.01T^2)}$$

$$H(z)=\frac{\frac{T^2}{(1+0.2T+9.01T^2)}}{1-2\frac{(1+0.1T)}{(1+0.2T+9.01T^2)}z^{-1}+\frac{z^{-2}}{(1+0.2T+9.01T^2)}}$$

If $T = 1$ s,

$$H(z)=\frac{0.0979}{1-0.2155z^{-1}+0.09792z^{-2}}$$

8.3 IIR FILTER DESIGN BY IMPULSE INVARIANT METHOD

In this technique, the desired impulse response of the digital filter is obtained by uniformly sampling the impulse response of the equivalent analog filter. That is,

$$h(n) = h_a(nT) \tag{8.17}$$

where T is the sampling interval. The transformation technique can be well understood by first considering a simple distinct pole case for the analog filter's system function, as shown below.

$$H_a(s)=\sum_{i=1}^{M}\frac{A_i}{s-p_i} \tag{8.18}$$

The impulse response of the system specified by Eq. 8.18 can be obtained by taking the inverse Laplace transform and it will be of the form

$$h_a(t) = \sum_{i=1}^{M} A_i e^{p_i t} u_a(t) \tag{8.19}$$

where $u_a(t)$ is the unit step function in continuous time. The impulse response $h(n)$ of the equivalent digital filter is obtained by uniformly sampling $h_a(t)$, i.e. by applying Eq. 8.17

$$h(n) = h_a(nT) = \sum_{i=1}^{M} A_i e^{p_i nT} u_a(nT) \tag{8.20}$$

The system response of the digital system of Eq. 8.20, can be obtained by taking the z-transform, i.e.

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

Using Eq. 8.20,

$$H(z) = \sum_{n=0}^{\infty} \left[\sum_{i=1}^{M} A_i e^{p_i nT} u_a(nT) \right] z^{-n} \tag{8.21}$$

Interchanging the order of summation,

$$H(z) = \sum_{i=1}^{M} \left[\sum_{n=0}^{\infty} A_i e^{p_i nT} u_a(nT) \right] z^{-n}$$

$$H(z) = \sum_{i=1}^{M} \frac{A_i}{1 - e^{p_i T} z^{-1}} \tag{8.22}$$

Now, by comparing Eqs. 8.18 and 8.22, the mapping formula for the impulse invariant transformation is given by

$$\frac{1}{s - p_i} \to \frac{1}{1 - e^{p_i T} z^{-1}} \tag{8.23}$$

Equation 8.23 shows that the analog pole at $s = p_i$ is mapped into a digital pole at $z = e^{p_i T}$. Therefore, the analog poles and the digital poles are related by the relation

$$z = e^{sT} \tag{8.24}$$

The general characteristic of the mapping $z = e^{sT}$ can be obtained by substituting $s = \sigma + j\Omega$ and expressing the complex variable z in the polar form as $z = re^{j\omega}$. With these substitutions, Eq. 8.24 becomes

$$re^{j\omega} = e^{\sigma T} e^{j\Omega T}$$

Clearly,

$$r = e^{\sigma T}$$
$$\omega = \Omega T \tag{8.25}$$

Consequently, $\sigma < 0$ implies that $0 < r < 1$ and $\sigma > 0$ implies that $r > 1$. When $\sigma = 0$, we have $r = 1$. Therefore, the left-half of s-plane is mapped inside the unit circle in the z-plane and the right-half of s-plane is mapped into points that fall outside the unit circle in z. This is one of the desirable properties for stability. The $j\Omega$-axis is mapped into the unit circle in z-plane. However, the mapping of the $j\Omega$-axis is not one-to-one. The mapping $\omega = \Omega T$ implies that the interval $-\pi/T \le \Omega \le \pi/T$ maps into the corresponding values of $-\pi \le \omega \le \pi$. Further, the frequency interval

The impulse response of the system specified by Eq. 8.18 can be obtained by taking the inverse Laplace transform and it will be of the form

$$h_a(t) = \sum_{i=1}^{M} A_i e^{p_i t} u_a(t) \tag{8.19}$$

where $u_a(t)$ is the unit step function in continuous time. The impulse response $h(n)$ of the equivalent digital filter is obtained by uniformly sampling $h_a(t)$, i.e. by applying Eq. 8.17

$$h(n) = h_a(nT) = \sum_{i=1}^{M} A_i e^{p_i nT} u_a(nT) \tag{8.20}$$

The system response of the digital system of Eq. 8.20, can be obtained by taking the z-transform, i.e.

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

Using Eq. 8.20,

$$H(z) = \sum_{n=0}^{\infty} \left[\sum_{i=1}^{M} A_i e^{p_i nT} u_a(nT) \right] z^{-n} \tag{8.21}$$

Interchanging the order of summation,

$$H(z) = \sum_{i=1}^{M} \left[\sum_{n=0}^{\infty} A_i e^{p_i nT} u_a(nT) \right] z^{-n}$$

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Clearly,

$$r = e^{\sigma T}$$

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Consequently, $\sigma < 0$ implies that $0 < r < 1$ and $\sigma > 0$ implies that $r > 1$. When $\sigma = 0$, we have $r = 1$. Therefore, the left-half of s-plane is mapped inside the unit circle in the z-plane and the right-half of s-plane is mapped into points that fall outside the unit circle in z. This is one of the desirable properties for stability. The $j\Omega$-axis is mapped into the unit circle in z-plane. However, the mapping of the $j\Omega$-axis is not one-to-one. The mapping $\omega = \Omega T$ implies that the interval $-\pi/T \le \Omega \le \pi/T$ maps into the corresponding values of $-\pi \le \omega \le \pi$. Further, the frequency interval frequency interval $(2k-1)\pi/T \le \Omega \le (2k+1)\pi/T$, where k is an integer, will also map into the interval $-\pi \le \omega \le \pi$ in the z-plane. Thus the mapping from the analog frequency Ω to the frequency variable ω in the digital domain is many-to-one, which simply reflects the effects of aliasing due to sampling of the impulse response. Figure 8.4 illustrates the mapping from the s-plane to the z-plane.

Some of the properties of the impulse invariant transformation are given below.

$$\frac{1}{(s+s_i)^m} \rightarrow \frac{(-1)^{m-1}}{(m-1)!}\frac{d^{m-1}}{ds^{m-1}}\left[\frac{1}{1-e^{-sT}z^{-1}}\right]; s \rightarrow s_i \quad (8.26)$$

$$\frac{s+a}{(s+a)^2+b^2} \rightarrow \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}} \quad (8.27)$$

$$\frac{b}{(s+a)^2+b^2} \rightarrow \frac{e^{-aT}(\sin bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}} \quad (8.28)$$

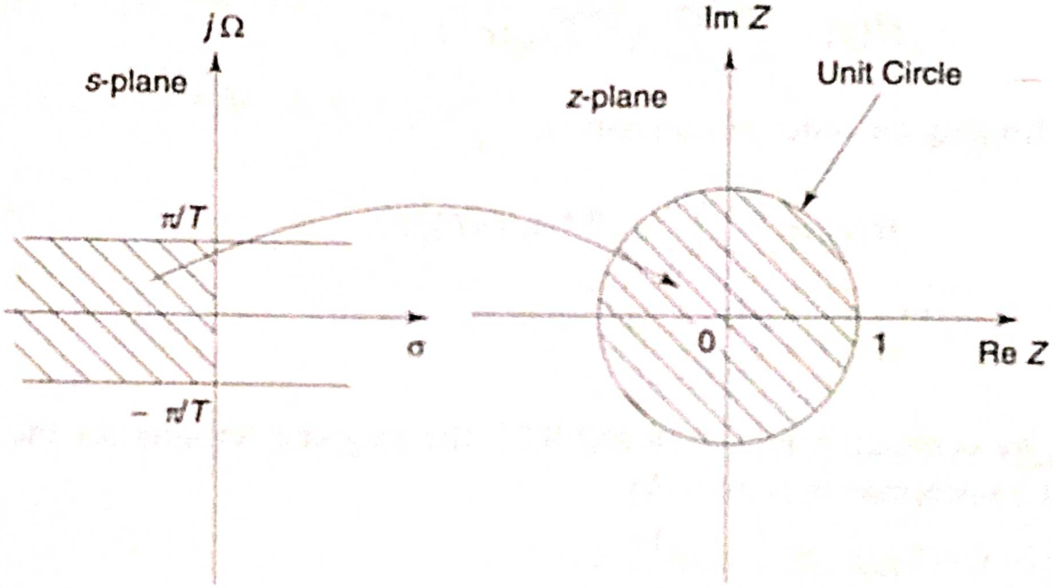

Fig. 8.4 *The Mapping of $z = e^{sT}$*

Example 8.4 *Convert the analog filter into a digital filter whose system function is*

$$H(s) = \frac{s+0.2}{(s+0.2)^2+9}$$

Use the impulse invariant technique. Assume T = 1s.

Solution The system response of the analog filter is of the standard form

$$H(s) = \frac{s+a}{(s+a)^2+b^2}$$

where $a = 0.2$ and $b = 3$. The system response of the digital filter can be obtained using Eq. 8.27.

$$H(z) = \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

$$= \frac{1-e^{-0.2T}(\cos 3T)z^{-1}}{1-2e^{-0.2T}(\cos 3T)z^{-1}+e^{-0.4T}z^{-2}}$$

Taking $T = 1$ s,

$$H(z) = \frac{1-(0.8187)(-0.99)z^{-1}}{1-2(0.8187)(-0.99)z^{-1}+0.6703z^{-2}}$$

That is,

$$H(z) = \frac{1+(0.8105)z^{-1}}{1+1.6210z^{-1}+0.6703z^{-2}}$$

Example 8.5 *For the analog transfer function*

$$H(s) = \frac{1}{(s+1)(s+2)}$$

determine $H(z)$ using impulse invariant technique. Assume $T = 1$ s.

Solution Using partial fractions, $H(s)$ can be written as

$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{A(s+2)+B(s+1)}{(s+1)(s+2)}$$

$$1 = A(s+2)+B(s+1)$$

Letting $s = -2$, we get $B = -1$ and letting, $s = -1$, we get $A = 1$. Therefore,

$$H(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

The system function of the digital filter is obtained by using Eq. 8.23.

$$H(z) = \frac{1}{1-e^{-T}z^{-1}} - \frac{1}{1-e^{-2T}z^{-1}}$$

$$= \frac{z^{-1}[e^{-T}-e^{-2T}]}{1-(e^{-T}+e^{-2T})z^{-1}+e^{-3T}z^{-2}}$$

Since $T = 1$ s,

$$H(z) = \frac{0.2326z^{-1}}{1-0.5032z^{-1}+0.0498z^{-2}}$$

Example 8.6 *Determine $H(z)$ using the impulse invariant technique for the analog system function*

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)}$$

Solution Using partial fractions, $H(s)$ can be written as

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)} = \frac{A}{s+0.5} + \frac{Bs+C}{s^2+0.5s+2}$$

Therefore,

$$A(s^2 + 0.5s + 2) + (Bs + C)(s + 0.5) = 1$$

Comparing the coefficients of s^2, s and the constants on either side of the above expression, we get

$$A + B = 0$$
$$0.5A + 0.5B + C = 0$$
$$2A + 0.5C = 1$$

Solving the above simultaneous equations, we get $A = 0.5$, $B = -0.5$ and $C = 0$. The system response can be written as,

$$H(s) = \frac{0.5}{s+0.5} - \frac{0.5s}{s^2+0.5s+2}$$

$$= \frac{0.5}{s+0.5} - 0.5\left(\frac{s}{(s+0.25)^2+(1.3919)^2}\right)$$

$$= \frac{0.5}{s+0.5} - 0.5\left(\frac{s+0.25}{(s+0.25)^2+(1.3919)^2} - \frac{0.25}{(s+0.25)^2+(1.3919)^2}\right)$$

$$= \frac{0.5}{s+0.5} - 0.5\left(\frac{s+0.25}{(s+0.25)^2+(1.3919)^2}\right) + 0.0898\left(\frac{1.3919}{(s+0.25)^2+(1.3919)^2}\right)$$

Using Eqs. 8.27 and 8.28,

$$H(z) = \frac{0.5}{1-e^{-0.5T}z^{-1}} - 0.5\left[\frac{1-e^{-0.25T}(\cos 1.3919T)z^{-1}}{1-2e^{-0.25T}(\cos 1.3919T)z^{-1}+e^{-0.5T}z^{-2}}\right] + 0.0898\left[\frac{e^{-0.25T}(\sin 1.3919T)z^{-1}}{1-2e^{-0.25T}(\cos 1.3919T)z^{-1}+e^{-0.5T}z^{-2}}\right]$$

Letting $T = 1$ s,

$$H(z) = \frac{0.5}{1-0.6065\,z^{-1}} - 0.5\left(\frac{1-0.1385\,z^{-1}}{1+0.277\,z^{-1}+0.606\,z^{-2}}\right) + 0.0898\left[\frac{0.7663\,z^{-1}}{1-0.277\,z^{-1}+0.606\,z^{-2}}\right]$$

8.4 IIR FILTER DESIGN BY THE BILINEAR TRANSFORMATION

The IIR filter design using (i) approximation of derivatives method and (ii) the impulse invariant method are appropriate for the design of low-pass filters and bandpass filters whose resonant frequencies are low. These techniques are not suitable for high-pass or band-reject filters. This limitation is overcome in the mapping technique called the ***bilinear transformation***. This transformation is a one-to-one mapping from the s-domain to the z-domain. That is, the bilinear transformation is a conformal mapping that transforms the $j\Omega$-axis into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components. Also, the transformation of a stable analog filter results in a stable digital filter as all the poles in the left half of the s-plane are mapped onto points inside the unit circle of the z-domain. The bilinear transformation is obtained by using the trapezoidal formula for numerical integration. Let the system function of the analog filter be

$$H(s) = \frac{b}{s+a} \tag{8.29}$$

The differential equation describing the analog filter can be obtained from Eq. 8.29 as shown below.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

$$sY(s) + aY(s) = bX(s) \tag{8.30}$$

Taking inverse Laplace transform,

$$\frac{dy(t)}{dt} + a\,y(t) = b\,x(t) \tag{8.31}$$

Equation 8.31 is integrated between the limits $(nT - T)$ and nT

$$\int_{nT-T}^{nT} \frac{dy(t)}{dt}dt + a\int_{nT-T}^{nT} y(t)dt = b\int_{nT-T}^{nT} x(t)dt \tag{8.32}$$

The trapezoidal rule for numeric integration is given by

$$\int_{nT-T}^{nT} a(t)dt = \frac{T}{2}[a(nT) + a(nT-T)] \tag{8.33}$$

Applying Eq. 8.33 in Eq. 8.32, we get

$$y(nT) - y(nT-T) + \frac{aT}{2}y(nT) + \frac{aT}{2}y(nT-T) = \frac{bT}{2}x(nT) + \frac{bT}{2}x(nT-T)$$

Taking z-transform, the system function of the digital filter is,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a} \tag{8.34}$$

Comparing Eqs. 8.29 and 8.34, we get

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right) \tag{8.35}$$

The general characteristic of the mapping $z = e^{sT}$ can be obtained by substituting $s = \sigma + j\Omega$ and expressing the complex variable z in the polar form as $z = re^{j\omega}$ in Eq. 8.35.

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = \frac{2}{T}\left(\frac{re^{j\omega}-1}{re^{j\omega}+1}\right) \tag{8.36}$$

Substituting $e^{j\omega} = \cos\omega - j\sin\omega$ and simplifying, we get

$$s = \frac{2}{T}\left(\frac{r^2-1}{1+r^2+2r\cos\omega} + j\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right)$$

Therefore,

$$\sigma = \frac{2}{T}\left(\frac{r^2-1}{1+r^2+2r\cos\omega}\right) \tag{8.37}$$

$$\Omega = \frac{2}{T}\left(\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right) \tag{8.38}$$

From Eq. 8.37, it can be noted that if $r < 1$, then $\sigma < 0$, and if $r > 1$, then $\sigma > 0$. Thus, the left-half of the s-plane maps onto the points inside the unit circle in the z-plane and the transformation results in a stable digital system. Consider Eq. 8.38, for unity magnitude ($r = 1$), σ is zero. In this case,

$$\Omega = \frac{2}{T}\left(\frac{\sin\omega}{1+\cos\omega}\right)$$

$$= \frac{2}{T}\left(\frac{2\sin(\omega/2)\cos(\omega/2)}{\cos^2(\omega/2)+\sin^2(\omega/2)+\cos^2(\omega/2)-\sin^2(\omega/2)}\right)$$

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2} \tag{8.39}$$

or equivalently,

$$\omega = 2\tan^{-1}\frac{\Omega T}{2} \tag{8.40}$$

Equation 8.40 gives the relationship between the frequencies in the two domains and this is shown in Fig. 8.5. It can be noted that the entire range in Ω is mapped only once into the range $-\pi \le \omega \le \pi$. However, as seen in Fig. 8.5, the mapping is non-linear and the lower frequencies in analog domain are expanded in the digital domain, whereas the higher frequencies are compressed. This is due to the non-linearity of the are tangent function and is usually called as ***frequency warping***.

Example 8.7 *Convert the analog filter with system function*

$$H(s) = \frac{s+0.1}{(s+0.1)^2+9}$$

into a digital IIR filter using bilinear transformation. The digital filter should have a resonant frequency of $\omega_r = \frac{\pi}{4}$.

Solution From the system function, we note that $\Omega_c = 3$. The sampling period T can be determined using Eq. 8.39, i.e.

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_r}{2}$$

The sampling period is obtained from the above equation using

$$T = \frac{2}{\Omega_c}\tan\frac{\omega_r}{2} = \frac{2}{3}\tan\frac{\pi}{8} = 0.276\,\text{s}$$

Using bilinear transformation,

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}}$$

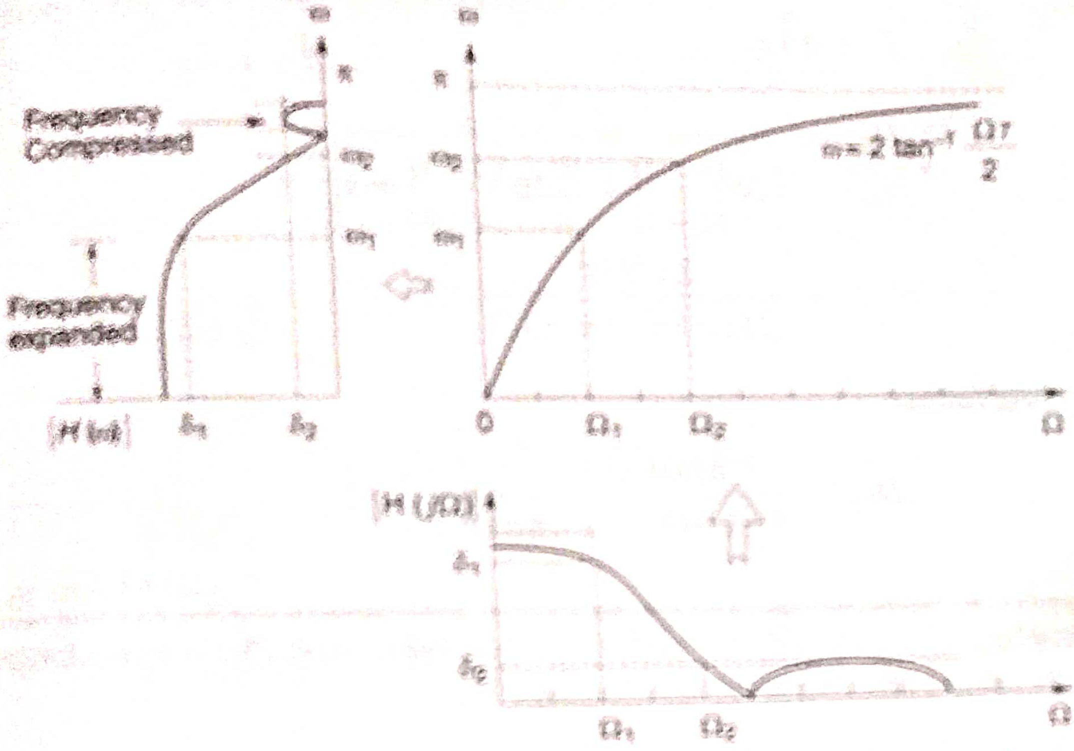

Fig. 8.5 *Relationship Between ω and Ω as Given in Eq. 8.40*

$$H(z)=\frac{\frac{2}{T}\frac{(z-1)}{(z+1)}+0.1}{\left[\frac{2}{T}\frac{(z-1)}{(z+1)}+0.1\right]^2+9}$$

$$=\frac{(2/T)(z-1)(z+1)+0.1(z+1)^2}{[(2/T)(z-1)+0.1(z+1)]^2+9(z+1)^2}$$

Substituting $T = 0.276$ s,

$$H(z)=\frac{1+0.027z^{-1}-0.973z^{-2}}{8.572-11.84z^{-1}+8.177z^{-2}}$$

Example 8.6 *Apply bilinear transformation to*

$$H(s)=\frac{2}{(s+1)(s+3)}$$

with $T = 0.1$ s.

Solution For bilinear transformation,

$$H(z)=H(s)\Big|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$=\frac{2}{\left[\frac{2}{T}\frac{(z-1)}{(z+1)}+1\right]\left[\frac{2}{T}\frac{(z-1)}{(z+1)}+3\right]}$$

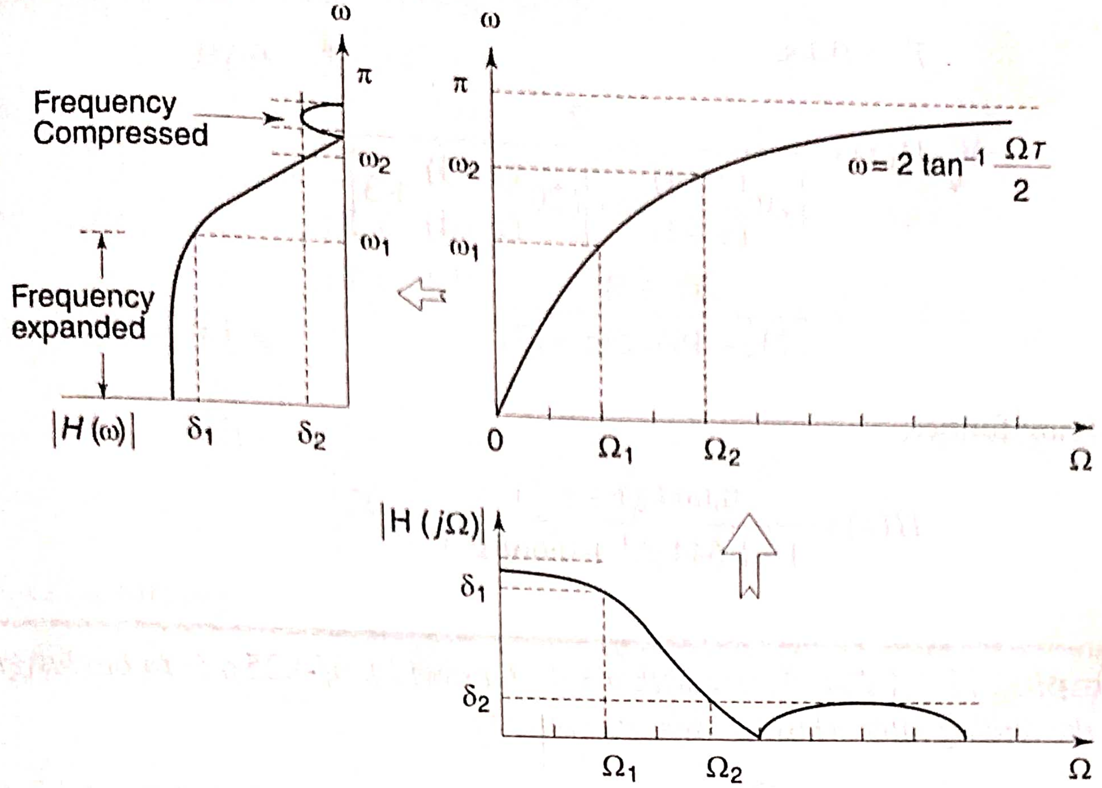

Fig. 8.5 *Relationship Between ω and Ω as Given in Eq. 8.40*

$$H(z) = \frac{\dfrac{2}{T}\dfrac{(z-1)}{(z+1)} + 0.1}{\left[\dfrac{2}{T}\dfrac{(z-1)}{(z+1)} + 0.1\right]^2 + 9}$$

$$= \frac{(2/T)(z-1)(z+1) + 0.1(z+1)^2}{[(2/T)(z-1) + 0.1(z+1)]^2 + 9(z+1)^2}$$

Substituting $T = 0.276$ s,

$$H(z) = \frac{1 + 0.027z^{-1} - 0.973z^{-2}}{8.572 - 11.84z^{-1} + 8.177z^{-2}}$$

Example 8.8 *Apply bilinear transformation to*

$$H(s) = \frac{2}{(s+1)(s+3)}$$

with $T = 0.1$ s.

Solution For bilinear transformation,

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$= \frac{2}{\left(\dfrac{2}{T}\dfrac{(z-1)}{(z+1)} + 1\right)\left(\dfrac{2}{T}\dfrac{(z-1)}{(z+1)} + 3\right)}$$

Using $T = 0.1$ s,

$$H(z) = \frac{2}{\left[20\frac{(z-1)}{(z+1)}+1\right]\left[20\frac{(z-1)}{(z+1)}+3\right]}$$

$$= \frac{2(z+1)^2}{(21z-19)(23z-17)}$$

Simplifying further,

$$H(z) = \frac{0.0041(1+z^{-1})^2}{1-1.644\,z^{-1}+0.668\,z^{-2}}$$

Example 8.9 *A digital filter with a 3 dB bandwidth of 0.25π is to be designed from the analog filter whose system response is*

$$H(s) = \frac{\Omega_c}{s+\Omega_c}$$

Use bilinear transformation and obtain H(z).

Solution Using Eq. 8.39,

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_r}{2} = \frac{2}{T}\tan 0.125\pi = 0.828/T$$

The system response of the digital filter is given by

$$H(z) = H(s)\big|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$= \frac{\Omega_c}{\frac{2}{T}\frac{(z-1)}{(z+1)}+\Omega_c} = \frac{\frac{0.828}{T}}{\frac{2}{T}\frac{(z-1)}{(z+1)}+\frac{0.828}{T}}$$

$$= \frac{0.828(z+1)}{2(z-1)+0.828(z+1)}$$

Simplifying we get further,

$$H(z) = \frac{1+z^{-1}}{3.414-1.414\,z^{-1}}$$

Example 8.10 *Using bilinear transformation obtain H(z) if*

$$H(s) = \frac{1}{(s+1)^2}$$

and $T = 0.1$ s.

Solution For the bilinear transformation,

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$= \frac{1}{\left(\frac{2}{T}\frac{(z-1)}{(z+1)}+1\right)^2}$$

Substituting $T = 0.1$ s,

$$H(z) = \frac{1}{\left(20\frac{(z-1)}{(z+1)}+1\right)^2} = \frac{(z+1)^2}{(21z-19)^2}$$

Further simplifying,

$$H(z) = \frac{0.0476(1+z^{-1})^2}{(1-0.9048\,z^{-1})^2}$$

8.5 BUTTERWORTH FILTERS

The Butterworth low-pass filter has a magnitude response given by

$$|H(j\Omega)| = \frac{A}{\left[1+(\Omega/\Omega_c)^{2N}\right]^{0.5}} \tag{8.41}$$

where A is the filter gain and Ω_c is the 3 dB cut-off frequency and N is the order of the filter. The magnitude response of the Butterworth filter is shown in Fig. 8.6(a). The magnitude response has a maximally flat passband and stopband. It can be seen that by increasing the filter order N, the Butterworth response approximates the ideal response. However, the phase response of the Butterworth filter becomes more non-linear with increasing.

The analog magnitude response of the Butterworth filter with the design parameters is shown in Fig. 8.6(b).

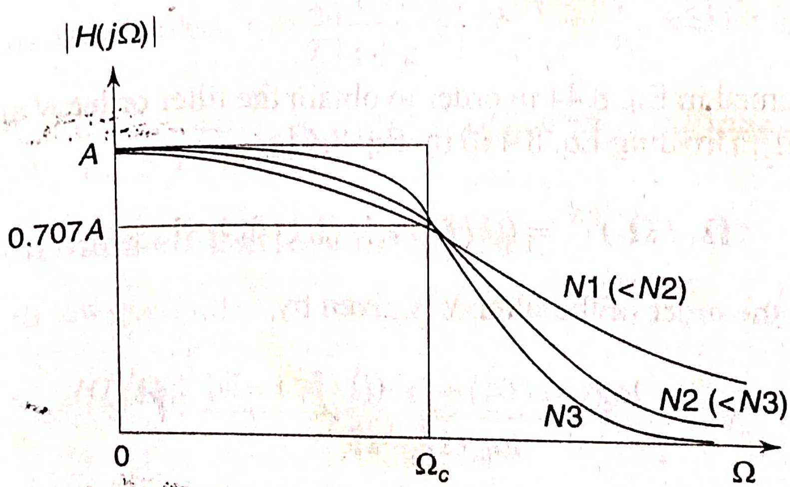

Fig. 8.6(a) *Magnitude Response of a Butterworth Low-pass Filter*

The design parameters of the Butterworth filter are obtained by considering the low-pass filter with the desired specifications as given below.

$$\delta_1 \le |H(j\Omega)| \le 1,\ 0 \le \Omega \le \Omega_1 \tag{8.42a}$$

$$|H(j\Omega)| \le \delta_2,\ \Omega_2 \le \Omega \le \pi \tag{8.42b}$$

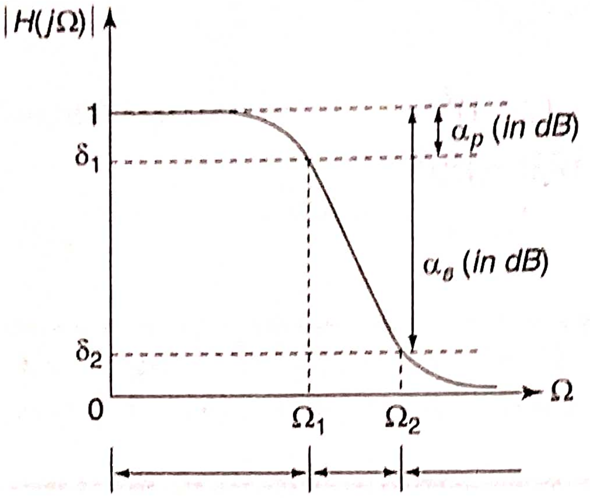

Fig. 8.6(b) *Magnitude Response of Butterworth LPF with Design specifications*

From the Fig. 8.6(b),

$$\alpha_p = -20\log\delta_1$$

$$\alpha_s = -20\log\delta_2$$

and $$\delta_1 = \frac{1}{\sqrt{1+\varepsilon^2}}$$

$$\delta_2 = \frac{1}{\sqrt{1+\lambda^2}}$$

where ε and δ_1 are the parameters specifying allowable pass-band, and λ and δ_2 are the parameters specifying allowable stop-band.

The corresponding analog magnitude response is to be obtained in the design process. Using Eq. 8.41 in Eq. 8.42 and if $A = 1$, we get

$$\delta_1^2 \le \frac{1}{1+(\Omega_1/\Omega_c)^{2N}} \le 1 \tag{8.43a}$$

$$\frac{1}{1+(\Omega_2/\Omega_c)^{2N}} \le \delta_2^2 \tag{8.43b}$$

Equation 8.43 can be written in the form

$$(\Omega_1/\Omega_c)^{2N} \le \frac{1}{\delta_1^2} - 1 \tag{8.44a}$$

$$(\Omega_2/\Omega_c)^{2N} \ge \frac{1}{\delta_2^2} - 1 \tag{8.44b}$$

Equality is assumed in Eq. 8.44 in order to obtain the filter order N and the 3 dB cut-off frequency Ω_c. Dividing Eq. 8.44b by Eq. 8.44a

$$(\Omega_2/\Omega_1)^{2N} = ((1/\delta_2^2)-1)/((1/\delta_1^2)-1 \tag{8.45}$$

From Eq. 8.45, the order of the filter N is given by

$$N = \frac{1}{2}\frac{\log\{((1/\delta_2^2)-1)/((1/\delta_1^2)-1)\}}{\log(\Omega_2/\Omega_1)} \tag{8.46}$$

(ii) $\Omega_c = \dfrac{\Omega_p}{\left[10^{0.1A_p} - 1\right]^{1/2N}}$

4. Next, we determine the poles using the following expression:

$$p_i = \pm \Omega_c\, e^{j(N+2i+1)\pi/2N}, \quad i = 0, 1, 2, \ldots N-1.$$

If the poles are complex conjugate then organize the poles (p_i) as complex conjugate pairs that means,

s_1 and s_1^*, s_2 and s_2^* etc.

5. Next, we calculate the system transfer function of analog filter using following expression:

$$H_a(s) = \frac{\Omega_c^N}{(s-p_1)(s-p_2)\ldots}$$

and if poles are complex conjugate then, we have

$$H_a(s) = \frac{\Omega_c^N}{(s-s_1)(s-s_1^*)(s-s_2)(s-s_2^*)}$$

6. Lastly, we design the digital filter using impulse invariance method or bilinear transformation method.

EXAMPLE 3.20 A digital filter has following frequency specification:

Passband frequency = $\omega_p = 0.2\pi$

Stopband frequency = $\omega_s = 0.3\pi$

What are the corresponding specifications for passband and stopband frequencies in analog domain if,

(i) Impulse invariance technique is used for designing.

(ii) Bilinear transformation is used for designing.

Solution: Here, let us assume sampling time T = 1 sec

(i) For impulse invariance method, we have

$$\Omega_p = \frac{\omega_p}{T} = 0.2\,\pi = 0.63 \text{ rad/sec.}$$

and $$\Omega_s = \frac{\omega_s}{T} = \frac{0.3\pi}{T} = 0.3\pi = 0.94 \text{ rad/sec.}$$

(ii) For bilinear transformation, we have

$$\Omega_p = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right) = 2\tan\left(\frac{0.2\pi}{2}\right) = 2\tan\left(\frac{0.2\times180}{2}\right) = 0.65 \text{ rad/sec.}$$

and $$\Omega_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right)$$

or $$\omega_s = 2\tan\left(\frac{0.3\pi}{2}\right) = 2\tan\left(\frac{0.3\times180}{2}\right) = 1.019 \text{ rad/sec.} \qquad \textbf{Ans.}$$

EXAMPLE 3.21. Design a second order discrete-time Butterworth filter with cut-off frequency of 1 kHz and sampling frequency of 10^4 samples/sec by bilinear transformation.

Solution: In this problem, the specifications of digital filter are not given directly. So, first we have to obtain required design specifications for digital filter. Then, for Butterworth approximation, we have to convert the specifications into specifications of equivalent analog filter. Finally, using bilinear transformation we have to obtain H(z).

Given that

Order of filter, N = 2

Cut-off frequency of analog filter, F_c = 1 kHz = 1000 Hz.

Sampling frequency, $F_s = 10^4$ samples/sec = 10,000 Hz.

First, let us determine the required design specifications of digital filter.

We have the equation to convert continuous frequency (F) into discrete frequency (f). It is:

$$f = \frac{F}{F_s}$$

Thus,
$$f_c = \frac{F_c}{F_s} = \frac{1000}{10,000} = 0.1 \text{ cycles/sample}$$

Now, the angular frequency (frequency of digital filter) is:

$$\omega_c = 2\pi f_c = 2\pi \times 0.1 = 0.2\pi \text{ radians/sample}$$

Now, let us determine the specifications of analog filter for Butterworth approximation.

For bilinear transformation, we have

$$\Omega = \frac{2}{T} \tan\frac{\omega}{2}$$

Thus,
$$\Omega_c = \frac{2}{T} \tan\frac{\omega_c}{2}$$

Here T = sampling time = $\frac{1}{F_s} = \frac{1}{10,000}$

or
$$\Omega_c = (2 \times 10,000) \tan\left(\frac{0.2\pi}{2}\right) = 6498.39 \text{ radians/sec.}$$

The value of N is given so it is not necessary to calculate it. Thus, the specifications of analog filter are under:

(i) Cut-off frequency = Ω_c = 6498.39 radian/sec.

(ii) Order of filter = N = 2.

Let us calculate the poles using,

$$p_i = \Omega_c e^{j(N+2i+1)\pi/2N} \quad \ldots i = 0, 1, \ldots N-1$$

Here N = 2, Thus, i = 0 to N − 1 means i = 0 and i = 1.

(i) For i = 0, we have

$$p_0 = \pm\Omega_c e^{j(N+1)\pi/2N} = \pm 6498.39\, e^{j3\pi/4} = \pm 6498.39\left[\cos\left(\frac{3\pi}{4}\right) + j\sin\left(\frac{3\pi}{4}\right)\right]$$

$$= \pm 6498.39\,[-0.707 + j\,0.707]$$

or
$$p_0 = 6498.39\,[-0.707 - j\,0.707] \text{ and } -6498.39\,[-0.707 + j\,0.707]$$

or
$$p_0 = -4595.05 + j\,4595.05 \text{ and } 4595.05 - j\,4595.05$$

(ii) For i = 1, we have

$$p_1 = \pm\Omega_c e^{j(N+2+1)\pi/2N}$$
$$= \pm 6498.39\, e^{j(5\pi/4)}$$

or
$$p_1 = \pm 6498.39\left[\cos\left(\frac{5\pi}{4}\right) + j\sin\left(\frac{5\pi}{4}\right)\right]$$

$$= \pm 6498.39\,[-0.707 - j\,0.707]$$

or $p_1 = 6498.39\,[-0.707 - j\,0.707]$

and $-6498.39\,[-0.707 - j\,0.707]$

or $p_1 = -4595.05 - j\,4595.05$

and $4595.05 + j\,4595.05$

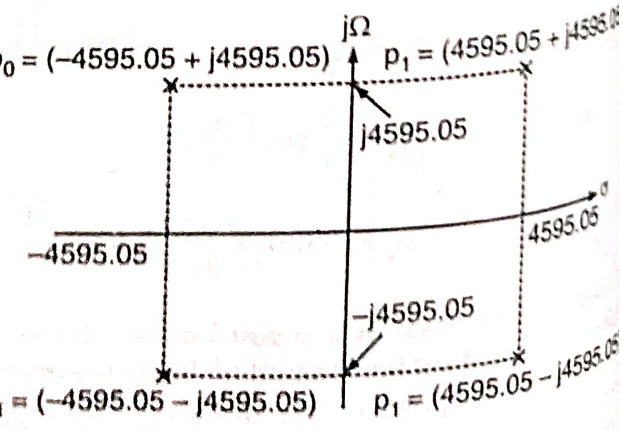

Fig. 3.16. Plot of poles, p_i.

Now, let us plot these poles as shown in figure 3.16.

> **Important Point:** *We know that analog filter is stable if the poles lie on the L.H.S. of s-plane. So, for the filter to be stable, we have to select the poles which are on the L.H.S. of s-plane. Observe figure 3.16. We shall denote these poles by s_1 and s_1^*; because these poles are complex conjugate of each other.*

Thus, $s_1 = -4595.05 + j\,4595.05$

and $s_1^* = -4595.05 - j\,4595.05$

Now, the transfer function of analog filter is obtained by using the following expression:

$$H_a(s) = \frac{\Omega_c^N}{(s-s_1)(s-s_1^*)} = \frac{(6498.39)^2}{(s+4595.05-j4595.05)(s+4595.05+j4595.05)}$$

Simplifying, we get

$$H_a(s) = \frac{(6498.39)^2}{s^2+9190.1s+42.22\times10^6}$$

Lastly, let us design digital filter using bilinear transformation.

In case of bilinear transformation, H(z) is obtained by putting $s = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$ in equation of $H_a(s)$

Here, $T = \frac{1}{10{,}000}$.

Thus, we have to put,

$$s = 2 \times 10^{+4}\left[\frac{z-1}{z+1}\right] \text{ in the equation of } H_a(s).$$

Therefore,

$$H(z) = \frac{(6498.39)^2}{\left[2\times10^4\left(\frac{z-1}{z+1}\right)\right]^2 + 183.802\times10^6\left(\frac{z-1}{z+1}\right) + 42.22\times10^6}$$

This is the required transfer function for digital filter. **Ans.**

...th filter which satisfi...