⇒ Meaning of Impulse Invariant in IIT :-

$(s+p_i) \rightarrow (1 - e^{p_i T} z^{-1})$

1. After transformation [redacted]

$$\left(\frac{1}{s-p_i}\right) \xrightarrow[\text{into}]{\text{Converts}} \left(\frac{1}{1-e^{p_i T} z^{-1}}\right) \quad \text{i.e.,}$$

after transformation of analog filter into digital filter, shape of impulse response (shape of o/p for impulse i/p) remain same i.e., impulse response becomes invariant.

Justification Of Above Statement

$$\text{i/p} \xrightarrow{x_a(t) = \delta(t)} \boxed{H_a(s) = \sum_{i=1}^{N} \frac{A_i}{s-p_i}} \xrightarrow{} \text{o/p } y_a(t)$$

↓ Analog filter

$$x_a(t) = \delta(t) \xrightarrow{L \cdot T} X(s) = 1.$$

$$y(t) \xrightarrow{L \cdot T} Y(s)$$

Since $Y(s) = X(s) \cdot H_a(s)$ [redacted]

$$\Rightarrow Y(s) = 1 \cdot \sum_{i=1}^{N} \frac{A_i}{s-p_i} \quad \left\{\text{since } X(s) = 1\right\}$$

$\Rightarrow y_a(t)$ = Inverse L·T of $Y(s)$

$$y_a(t) = \sum_{i=1}^{N} A_i e^{p_i t} u(t).$$

So, sampled o/p = $y_a(t) \big|_{\text{at } t = nT}$

$$\text{Sampled o/p} = \sum_{i=1}^{N} A_i e^{p_i nT} u(nT) \quad \text{—— (1)}$$

Now find o/p of [redacted] digital filter when i/p is [redacted] impulse $x(n) = \delta(n)$

i/p $x(n) = \delta(n) \longrightarrow$ $\boxed{H(z) = \sum_{i=1}^{N} \frac{A_i}{1 - e^{p_i T} z^{-1}}}$ $\longrightarrow$ o/p $y(n)$

↓

Digital filter

$$x(n) = \delta(n) \xrightarrow{Z.T} X(z) = 1$$

$$y(n) \xrightarrow{Z.T} Y(z)$$

So, $Y(z) = X(z)\, H(z)$

$$\Rightarrow Y(z) = 1 \cdot \sum_{i=1}^{N} \frac{A_i}{1 - e^{p_i T} z^{-1}}$$

$y(n) =$ inverse Z.T of $Y(z)$.

$$y(n) = \sum_{i=1}^{N} A_i \cdot e^{p_i T n}\, u(n).$$

$$\Rightarrow \left(\begin{array}{l} \text{o/p of digital filter} \\ \text{when ip = impulse } \delta(n) \end{array} \right) = \sum_{i=1}^{N} A_i\, e^{p_i T n}\, u(n) \quad \text{—— (II)}$$

using eq^n (I) & (II)

(sampled o/p of analog filter when i/p is impulse) = (digital o/p of digital filter when i/p is impulse.)

⇓

Impulse response is invariant after transformation of

$$\frac{1}{s - p_i} \xrightarrow{\text{into}} \frac{1}{1 - e^{p_i T} z^{-1}}$$

Why not step invariant ?

$$x(t) = u(t) \xrightarrow{i/p} \boxed{H_a(s) = \sum_{i=1}^{N} \frac{A_i}{s - p_i}} \longrightarrow y_a(t)$$

$$x(t) = u(t) \xrightarrow{L \cdot T} X(s) = \frac{1}{s}$$

$$y_a(t) \xrightarrow{L \cdot T} Y(s)$$

So, $Y(s) = X(s) \cdot H_a(s)$.

$$Y(s) = \frac{1}{s} \cdot \sum_{i=1}^{N} \frac{A_i}{s - p_i}$$

$$Y(s) = \sum_{i=1}^{N} \frac{A_i}{s(s - p_i)}$$

After partial fraction.

$$Y(s) = \sum_{i=1}^{N} \frac{A_i}{p_i} \left(\frac{1}{s - p_i} - \frac{1}{s} \right)$$

$y_a(t)$ = inverse L·T of $Y(s)$

$$y_a(t) = \sum_{i=1}^{N} \frac{A_i}{p_i} \left(e^{p_i t} u(t) - u(t) \right)$$

Sampled o/p = $y_a(t) \Big|_{\text{at } t = nT}$

$$\boxed{\left(\begin{array}{l} \text{Sampled o/p} \\ \text{for step input} \end{array} \right) = \sum_{i=1}^{N} \frac{A_i}{p_i} \left(e^{p_i nT} - 1 \right) u(nT).} \quad \text{--- (1)}$$

Now find digital o/p when i/p. $x(n)$ is step i/p i.e. when $x(n) = u(n)$

$$\xrightarrow{i/p} \boxed{H(z) = \sum \frac{A_i}{1 - e^{p_i T} z^{-1}}} \longrightarrow y(n)$$

$x(n)$
$\Downarrow$
$u(n)$

$$x(n) = u(n) \Rightarrow X(z) = \frac{1}{1 - z^{-1}}.$$

So, $Y(z) = X(z) \cdot H(z).$

$$= \left(\frac{1}{1 - z^{-1}}\right) \sum_{i=1}^{N} \frac{A_i}{1 - e^{p_i T} z^{-1}}$$

After partial fraction and inverse z.T.

$$y(n) = \sum_{i=1}^{N} \frac{A_i}{1 - e^{p_i T}} \left(1 - e^{p_i (n+1) T}\right) u(n).$$

$$\Rightarrow \left(\begin{array}{l}\text{digital o/p of}\\ \text{digital filter.}\\ \text{when i/p is step}\end{array}\right) = \sum_{i=1}^{N} \left(\frac{A_i}{1 - e^{p_i T}}\right)\left(1 - e^{p_i (n+1) T}\right) u(n) \quad \text{(ii)}$$

using equation (1) and (ii)

$$\left(\begin{array}{l}\text{Sampled o/p of}\\ \text{analog filter for step i/p}\end{array}\right) \neq \begin{array}{l}\text{o/p of digital filter}\\ \text{when i/p is step}\end{array}$$

So step response has changed
$\Rightarrow$ step response is not invariant

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Mapping of S plane into Z plane For Impulse-Invariant Transformation (IIT)

For impulse invariant transformation

$$\left(\frac{1}{s-p_i}\right) \xrightarrow{\text{has transformed into}} \left(\frac{1}{1-e^{p_i T} z^{-1}}\right)$$

Analog pole

$\Downarrow$

Analog pole given by $(s-p_i)=0$ i.e. $s=p_i$ has transformed into digital pole given by $\left(1-e^{p_i T} z^{-1}\right)=0$ i.e. $z=e^{p_i T}$

digital pole

Now take $z=e^{p_i T}$

$$\Rightarrow \boxed{z=e^{sT}} \quad \{\text{since } s=p_i\}$$

Relation between digital pole and analog pole.

$z=re^{i\theta}$

$s=\sigma+j\omega$

Now take $z=e^{sT}$

$$\Rightarrow r e^{j\omega} = e^{(\sigma+j\Omega)T}$$

$\left\{\begin{array}{l}\text{since } z=re^{j\omega} \\ \text{in polar plane} \\ \text{and } s=\sigma+j\Omega\end{array}\right\}$

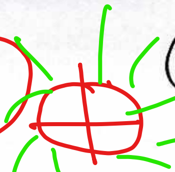
$$\Rightarrow r\, e^{j\omega} = e^{\sigma T}\, e^{j\Omega T}$$

After Comparing real and imaginary part, we get

$$\Rightarrow \boxed{r=e^{\sigma T}} \quad \text{and} \quad \boxed{\omega=\Omega T}$$

Relation between digital frequency ω and analog frequency Ω.

(6)



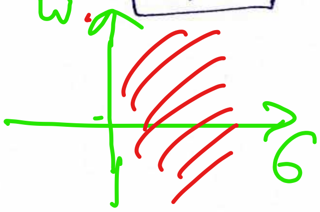
Now take $\boxed{\gamma = e^{\sigma T}}$

$e^{(+)}$

$e^{0} = 1$

$e^{(+)} > 1$

$\boxed{\text{Case I}}$ when $\underbrace{\sigma > 0}$ $\Rightarrow$ $\underbrace{(\gamma > 1)}$

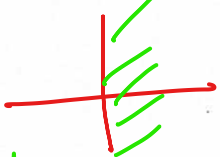


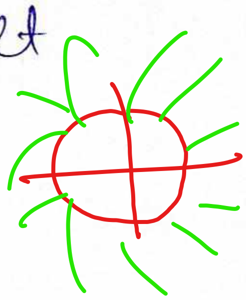
$\Downarrow$ region in the right-hand side of $j\omega$-axis on S plane $\rightarrow$ $\Downarrow$ $r>1$ is region outside unit circle

So region on the right hand side of S plane has been mapped into region outside unit circle

$\Downarrow$

For an astable analog filter, we will get an astable digital filter.





$\boxed{\text{Case II}}$ when $\underbrace{\sigma < 0}$ $\Rightarrow$ $\underbrace{\gamma < 1}$

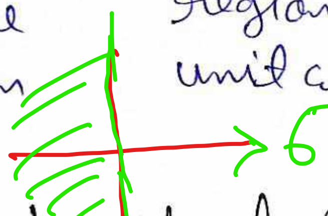
negative

$\gamma = e^{\sigma T}$

$e^{-4} = \frac{1}{e^{4}} < 1$

$\Downarrow$ region in the left hand side of $j\omega$-axis on S plane

$\Downarrow$ region inside unit circle.



So region on the left hand side of S plane has been mapped into region inside unit circle.

$\Downarrow$

For an stable analog filter, we will get stable digital filter

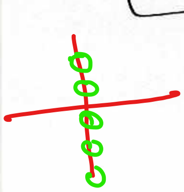
$\boxed{\text{Case III}}$ when $\underbrace{\sigma = 0}$ then $\underbrace{\gamma = 1}$

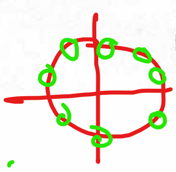
$\gamma = e^{\sigma T}$

$\sigma = 0 \rightarrow \gamma = e^{0} = 1$

$\Downarrow$ points on $j\Omega$-axis

$\Downarrow$ points on unit circle.





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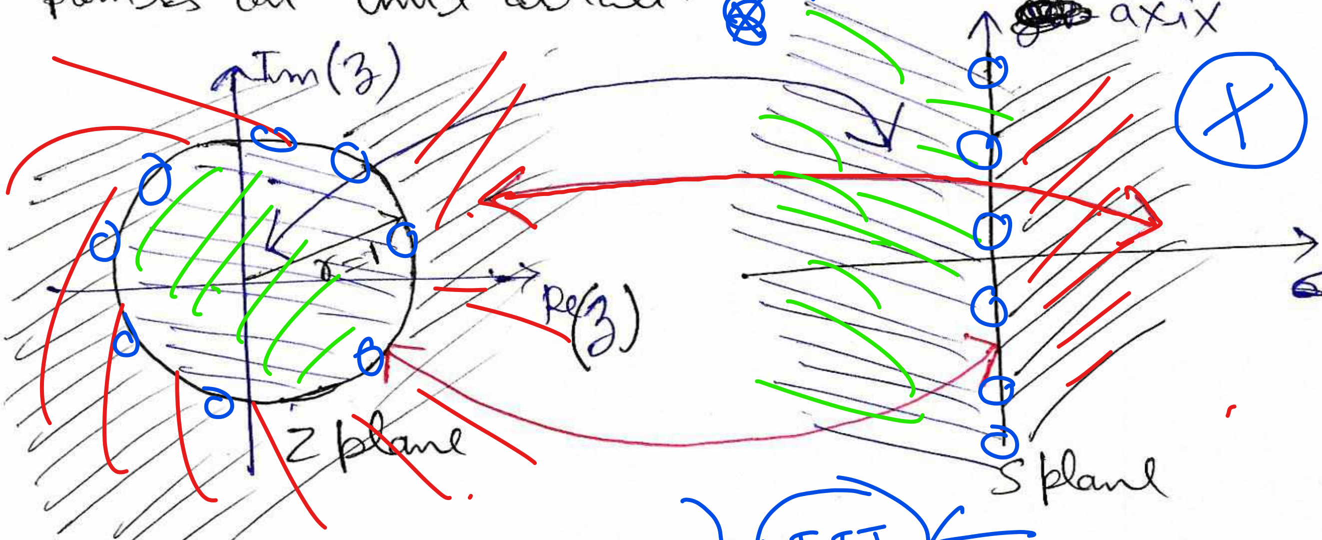
So points on $j\Omega$ axis ($\to j\Omega$-axis) has been mapped into points on unit circle.



IIT

Mapping of s plane into z-plane.

Mapping of s-plane into z plane for Bilinear Transformation (BLT)

For Bilinear transformation (BLT), $\boxed{s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)}$

Now since $s = \sigma + j\Omega$ and $z = r e^{j\omega}$

so $\sigma + j\Omega = \frac{2}{T}\left(\frac{r e^{j\omega} - 1}{r e^{j\omega} + 1}\right)$

$$\frac{x+iy}{x-iy}$$

$\Rightarrow \sigma + j\Omega = \frac{2}{T}\left\{\frac{r(\cos\omega + j\sin\omega) - 1}{r(\cos\omega + j\sin\omega) - 1}\right\}$

After seperating real and imaginary part of RHS

$$\sigma + j\Omega = \frac{2}{T}\left(\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega} + j\,\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right)$$

After Comparing real and imaginary part on both side, we get

$$\sigma = \frac{2}{T}\left(\frac{\gamma^2-1}{1+\gamma^2+2\gamma\cos\omega}\right) \text{ and } \Omega = \frac{2}{T}\left(\frac{2\gamma\sin\omega}{1+\gamma^2+2\gamma\cos\omega}\right)$$

Now take $\sigma = \frac{2}{T}\left(\frac{\gamma^2-1}{1+\gamma^2+2\gamma\cos\omega}\right)$

$\gamma^2-1=0$

$\gamma=1 \Rightarrow \sigma=0$

$(\gamma^2-1)>0$

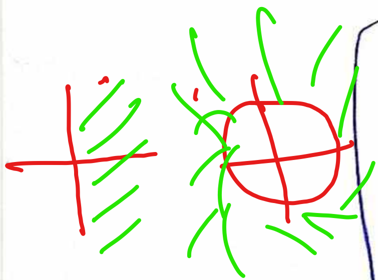
$\gamma^2>1$

$\gamma>1$

Case I when $\underbrace{\gamma>1}_{\Downarrow} \Rightarrow \underbrace{\sigma>0}_{\Downarrow}$

~~region in the right hand side of $j\Omega$-axix on s-plane~~

region in the right hand side (RHS) of $j\Omega$-axix on s-plane.



$(\gamma^2-1)>0 \Rightarrow (\gamma^2-1)=+$

region outside unit circle

So region on the right hand side of s-plane has been mapped into region outside unit circle.

Case II when $\underbrace{\gamma<1}_{\Downarrow} \Rightarrow \underbrace{\sigma<0}_{\Downarrow}$

$(\gamma^2-1)<0 \rightarrow (-)$

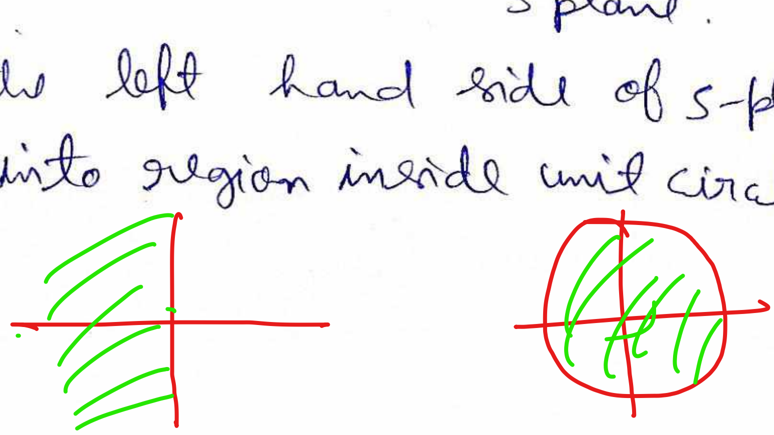
$\gamma^2<1$

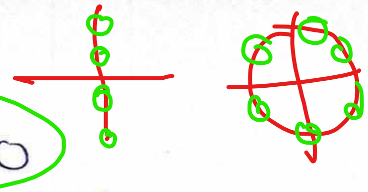
$\gamma<1 \longrightarrow$

region inside unit circle

region on the left hand side (LHS) of $j\Omega$-axix on s plane.

So region on the left hand side of s-plane has been mapped into region inside unit circle.



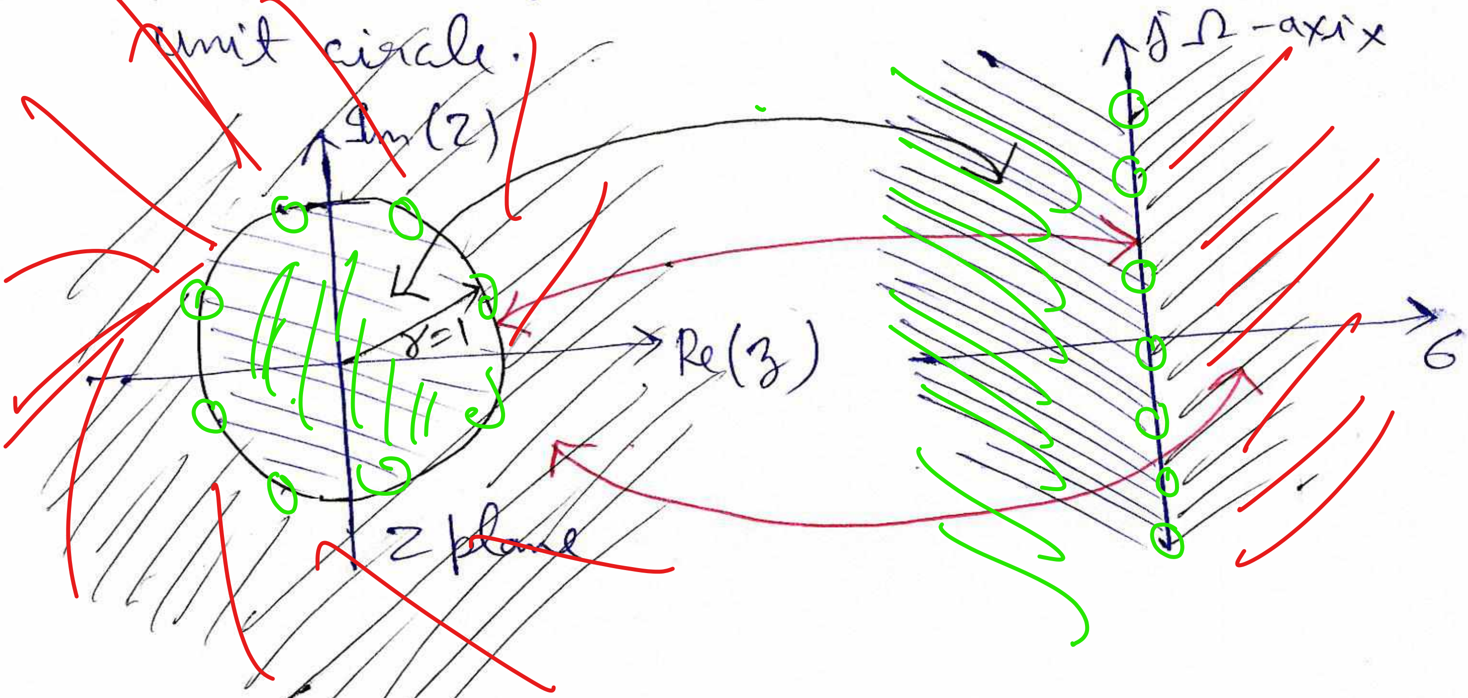


$\boxed{\text{Case III}}$ when $r = 1$ $\Rightarrow$ $\sigma = 0$

$\Downarrow$ points on the unit circle

$\Downarrow$ points on the $j\Omega$ axis

So points on $j\Omega$-axis has been mapped onto unit circle.



Mapping of S plane into z plane.

Relation Between Ω and w for Bilinear Transformation (BLT)

we know that for Bilinear Transformation,

$$\Omega = \frac{2}{T} \frac{2r \sin w}{1+r^2+2r\cos w}$$

when $r=1$ then

$$\Omega = \frac{2}{T}\left(\frac{2\sin w}{1+1+2\cos w}\right)$$

$\cos\theta = (2\cos^2\frac{\theta}{2} - 1)$ (10) $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$$\Rightarrow \Omega = \frac{2}{T}\,\frac{2\sin w}{2+2\cos w} = \frac{2}{T}\left(\frac{2\sin w}{2(1+\cos w)}\right)$$

$$\Rightarrow \Omega = \frac{2}{T}\left(\frac{\sin w}{1+\cos w}\right) = \frac{2}{T}\cdot\left(\frac{2\sin\frac{w}{2}\cos\frac{w}{2}}{1+2\cos^2\frac{w}{2}-1}\right)$$

$$\Rightarrow \Omega = \frac{2}{T}\,\frac{\cancel{2}\sin\frac{w}{2}\,\cancel{\cos\frac{w}{2}}}{\cancel{2}\cos^{\cancel{2}}\frac{w}{2}}$$

$$\Rightarrow \Omega = \frac{2}{T}\left(\frac{\sin\frac{w}{2}}{\cos\frac{w}{2}}\right) \Rightarrow \boxed{\Omega = \frac{2}{T}\tan\frac{w}{2}}$$

$$\Rightarrow \frac{\Omega T}{2} = \tan\frac{w}{2} \Rightarrow \frac{w}{2} = \tan^{-1}\left(\frac{\Omega T}{2}\right)$$

$$\Rightarrow w = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$$

since range of $\tan^{-1}(\ )$ is from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$

So range of $2\tan^{-1}(\ )$ will be from $-\pi$ to π.

$\tan^{-1}(\ ) \longrightarrow -\frac{\pi}{2}$ to $+\frac{\pi}{2}$

$2\tan^{-1}(\ ) \longrightarrow -\pi$ to $+\pi$

Disadvantage (11) of IIT

Aliasing Problem In Impulse Invariant Transformation :-

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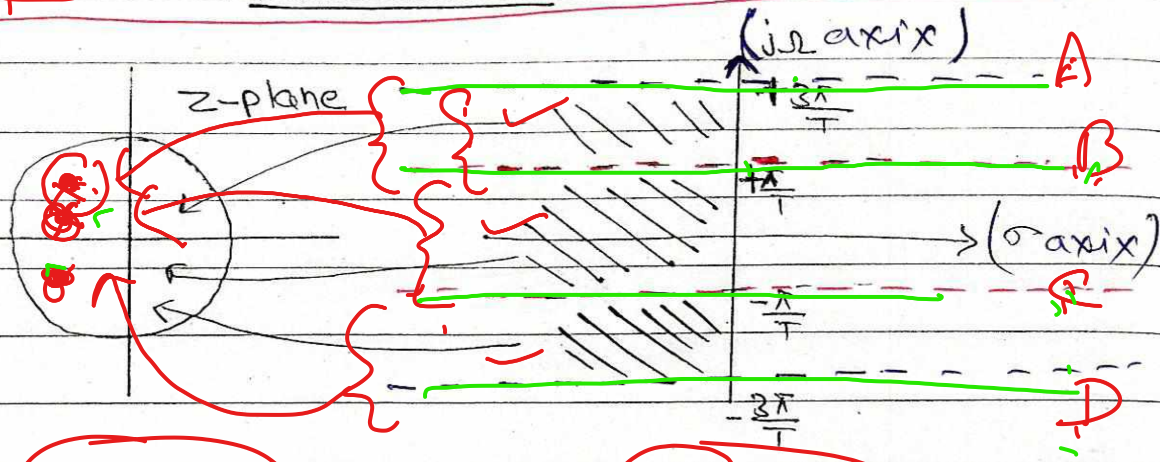

z-plane

s-plane

In case of Impulse Invariant transformation

$\omega = \Omega T$

So when $-\frac{\pi}{T} \leq \Omega \leq +\frac{\pi}{T}$ $\qquad \omega = \Omega T$

then $-\pi \leq \omega \leq +\pi$

when $+\frac{\pi}{T} \leq \Omega \leq +\frac{3\pi}{T}$

then $+\pi \leq \omega \leq +3\pi$

$-\pi \leq \omega \leq +\pi$ (How)

Similarly

when $-\frac{3\pi}{T} \leq \Omega \leq -\frac{\pi}{T}$

then $-3\pi \leq \omega \leq -\pi$

$-\pi \leq \omega \leq +\pi$ (How)

So mapping of analog frequency Ω to variable ω in digital domain is many to one which reflects effect of aliasing

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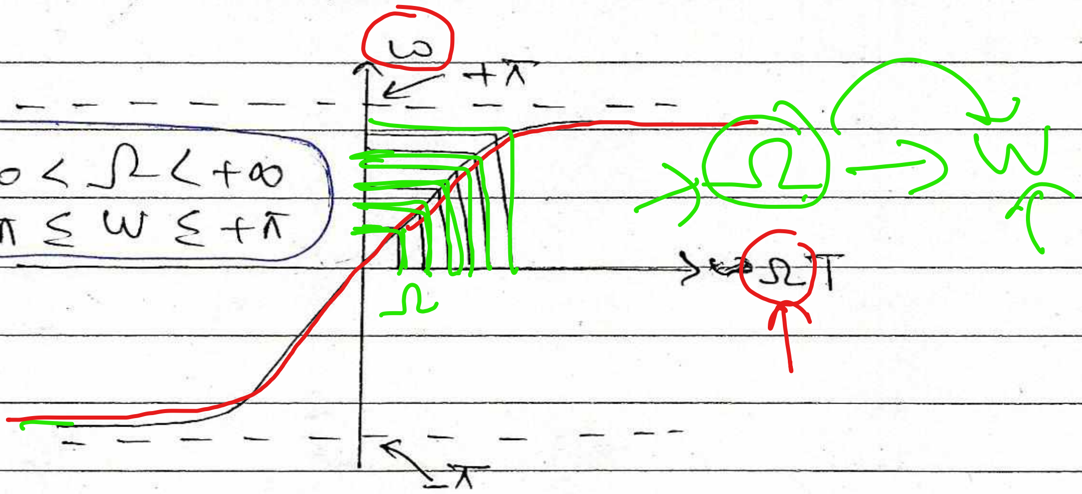
So because of many to one mapping, there is effect of aliasing in IIT method.

Absence of Aliasing In Bilinear Transformation :-

In case of Bilinear Transformation since $\omega = 2\tan^{-1}\frac{\Omega T}{2}$, so it's graph is given as below.

Here $-\infty < \Omega < +\infty$ and $-\pi \leq \omega \leq +\pi$



Entire range of Ω is mapped into range $\longrightarrow -\pi \leq \omega \leq +\pi$

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

$$\Rightarrow \sigma + j\Omega = \frac{2}{T}\left(\frac{e^{j\omega}-1}{e^{j\omega}+1}\right)$$

So for one value of ω, there is only one value of Ω. So there is one to one mapping $\Rightarrow$ No effect of aliasing

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Disadvantage

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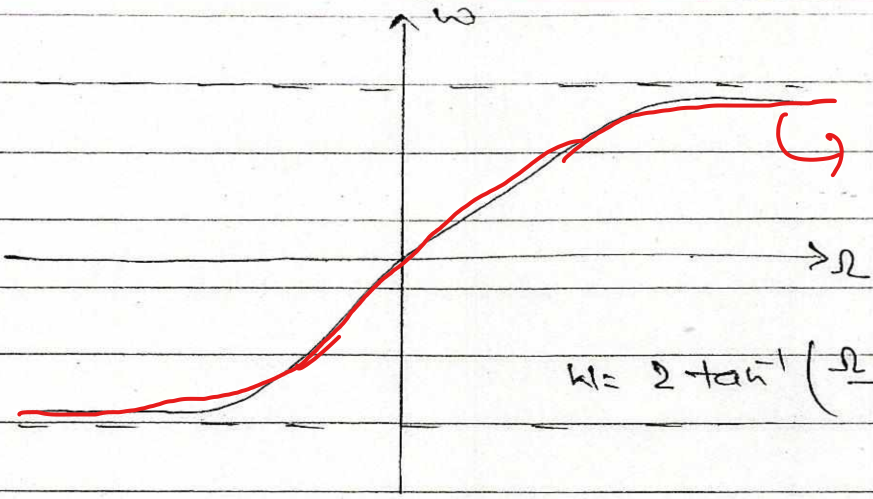
Effect of Warping In Bilinear Transformation

Wood Door



$\tan^{-1}()$

$w = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$

WARPING

(Warping ⇒ something in twisted form.)

Here mapping is highly non-linear because of $\tan^{-1}$ function.

So there is warping due to nature of non-linearity in $\tan^{-1}$ function.

Matched z-transform

If $z_1, z_2, ----, z_m$ are zeroes of analog filter and $p_1, p_2, --- p_N$ are poles of analog filter then.

$$H_a(s) = \frac{(s-z_1)(s-z_2)-----(s-z_m)}{(s-p_1)(s-p_2)----(s-p_m)}$$

$$H_a(s) = \frac{\prod_{i=0}^{M}(s-z_i)}{\prod_{j=0}^{N}(s-p_i)}$$

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In matched z- transform replace $\frac{1}{s-p_i}$ by $\frac{1}{1-e^{p_i T} z^{-1}}$ and $(s-z_i)$ = by $(1-e^{z_i T} z^{-1})$.

Que: $H_a(s) = \frac{2(s+3)}{(s+1)(s+2)}$.

Convert it into transfer function of digital filter. $H(z)$, using matched z- transform. Take $T = 1$ sec.

soln: replace $\frac{1}{s+1}$ by $\frac{1}{1-e^{-1} z^{-1}}$

$\frac{1}{s+2}$ by $\frac{1}{1-e^{-2} z^{-1}}$

$s+3$ by $1-e^{-3} z^{-1}$

and calculate $H(z)$.

So $H(z) = \frac{2(1-e^{-3} z^{-1})}{\left(\frac{1}{1-e^{-1} z^{-1}}\right)\left(\frac{1}{1-e^{-2} z^{-1}}\right)}$

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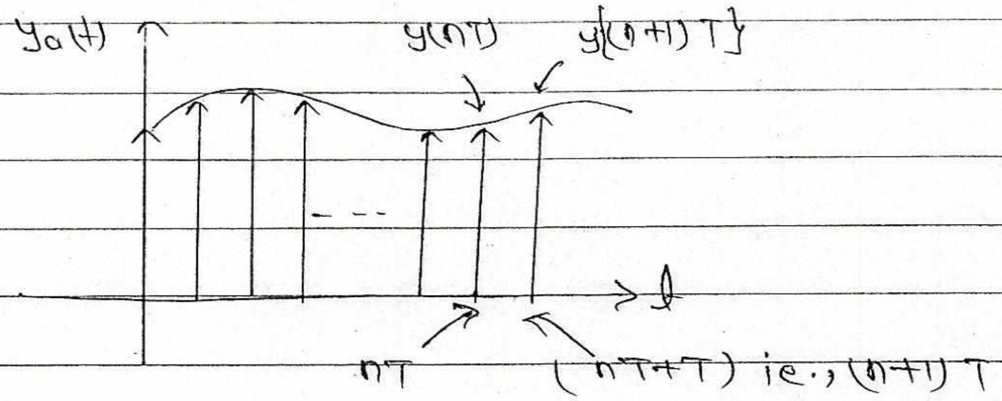
⇒ Approximation of derivative using forward Difference Method :-



using forward difference method.

$$\frac{d}{dt} y_a(t) = \frac{y\{(n+1)T\} - y(nT)}{T}$$

↓

can be obtained using following block

$$y_a(t) \longrightarrow \boxed{s} \longrightarrow \frac{d y_a(t)}{dt}$$

$\frac{y(n+1)T - y(nT)}{T}$ can be obtained using following block.

$$y(n) \longrightarrow \boxed{\frac{z-1}{T}} \longrightarrow \frac{y\{(n+1)T\} - y(nT)}{T}$$

$\downarrow$ z-T

$$Y(z) \qquad\qquad \frac{Y(z)\{z-1\}}{T}$$

So, Transfer function of digital block.

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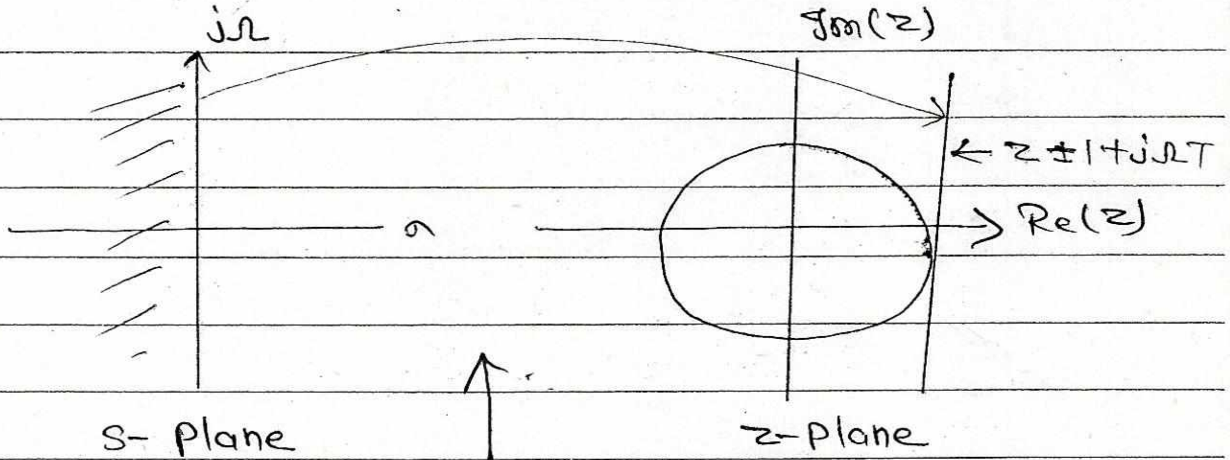
$$= \frac{Y(z)\{(z-1)/T\}}{Y(z)}$$

So, put $\boxed{s = \frac{z-1}{T}}$ in $H_a(s)$ to convert it into $H(z)$ using forward difference method in approximation of derivative method.

Since $s = \frac{z-1}{T} \Rightarrow z = 1 + sT$

when $s = j\Omega$ $\quad z = 1 + j\Omega T.$



mapping of S plane into z-plane for approximation of derivative using forward difference method

Que:- $H(z) = 3 + \frac{4z}{(z-\frac{1}{2})} - \frac{2}{(z-\frac{1}{4})}$

Is it

(a) IIR or FIR filter.

(b) Find the difference equation.

(c) Direct form-1 structure using difference eqⁿ

(d) make DF2 using DF 1

Solⁿ:- $H(z) = \frac{3(z-\frac{1}{2})(z-\frac{1}{4}) + 4z(z-\frac{1}{4}) - 2(z-\frac{1}{2})}{(z-\frac{1}{2})(z-\frac{1}{4})}$

$$= \frac{(3z - \frac{3}{2})(z-\frac{1}{4}) + 4z^2 - z - 2z + 1}{(z-\frac{1}{2})(z-\frac{1}{4})}$$

$$= \frac{3z^2 - \frac{3z}{4} - \frac{3z}{2} + \frac{3}{8} + 4z^2 - z - 2z + 1}{z^2 - \frac{1}{4}z - \frac{1}{2}z + \frac{1}{8}}$$

$$H(z) = \frac{7z^2 - \frac{21z}{4} + \frac{11}{8}}{z^2 - \frac{1}{4}z - \frac{1}{2}z + \frac{1}{8}}$$

a) IIR.

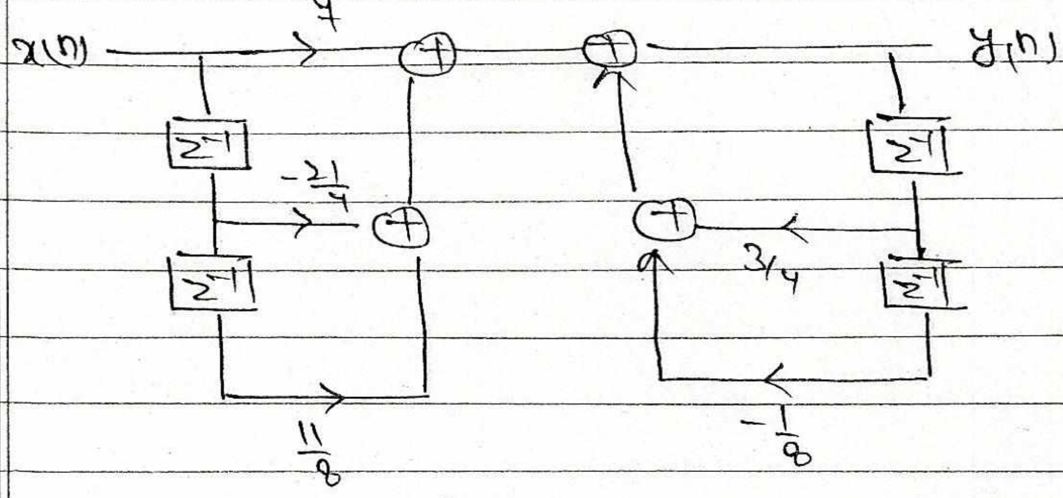
b). $\frac{Y(z)}{X(z)} = \frac{7z^2 - \frac{21z}{4} + \frac{11}{8}}{z^2 - \frac{3}{4}z + \frac{1}{8}} = \frac{7 - \frac{21}{4}z^{-1} + \frac{11}{8}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$

$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = 7X(z) - \frac{21}{4}z^{-1}X(z) + \frac{11}{8}z^{-1}X(z)$

$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = 7x(n) - \frac{21}{4}x(n-1) + \frac{11}{8}(x(n-2))$

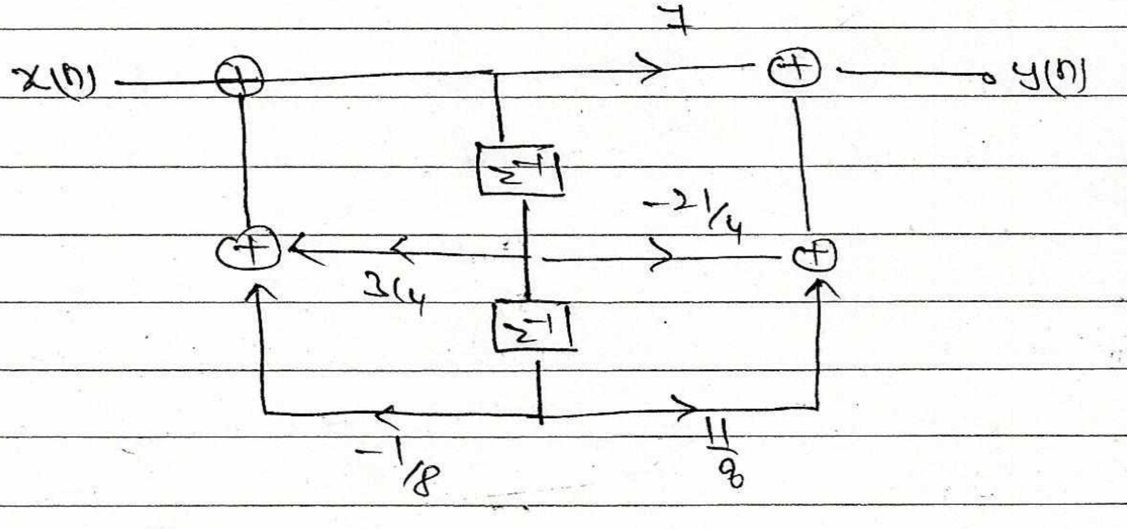
$$y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + 7x(n) - \frac{21}{4}x(n-1) + \frac{11}{8}x(n-2)$$

(c)



DF-1

(d)



DF-2