Algebric Structure

Module 7

Binary operation

What is an Algebraic Structure?

An algebraic structure means:

A set + one or more binary operations defined on it.

Example:

Set of integers (ℤ) with addition (+).

Here, “+” is the binary operation.

Ex: (N, +), (Z, +, –), (R, +, . , **–**) are algebraic systems.

Algebraic systems

- **N = {1,2,3,4,…..∝ } = Set of all natural numbers.**

  Z = { 0, ± 1, ± 2, ± 3, ± 4 , ….. ∝} = Set of all integers.

  Q = Set of all rational numbers, R = Set of all real numbers.

Binary Operation: Binary Operation

A binary operation combines two elements of a set and gives another element of the same set.

Example:

For integers under addition:

2+3=5 (and 5 is also an integer)

So “+” is a binary operation on integers.

Properties

Closure= Algebric structure (a,b in A, then a*b in A)

Closure + Associative =semigroup (a*(b*c)=(a*b)*c)

Closure + Associative+Identity = monoids (a*e=a)

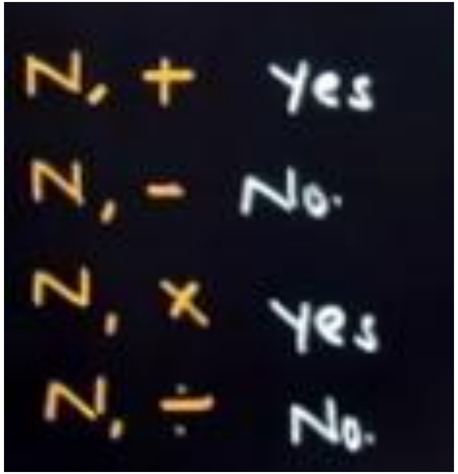
Closure + Associative+Identity + Inverse = Groups (a * a-1 =e)

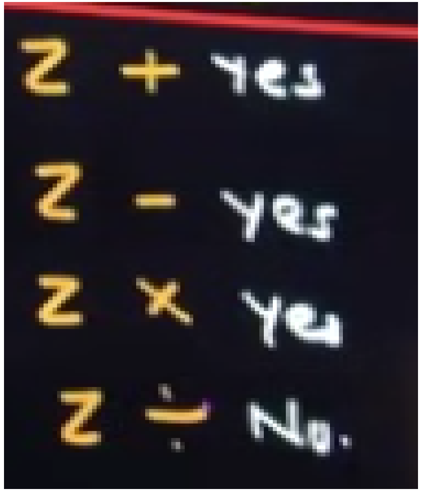
Closure + Associative+Identity + Inverse + commutative =Abelian group (a*b=b*a)

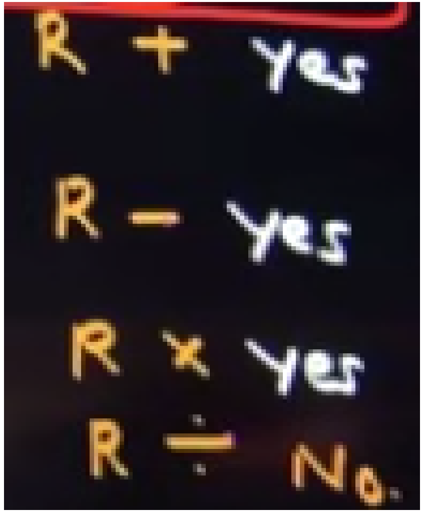
Examples -closure property

A set is said to have the closure property (or is closed) under a particular operation if performing that operation on elements of the set always produces a result that also belongs to the same set.

$a * b \in A$ for all $a, b \in A$





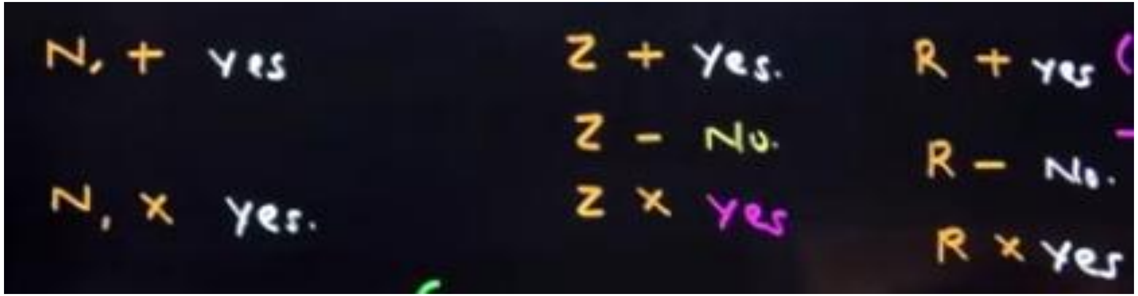


Properties

- **Associativity:** Let * be a binary operation on a set A.

  The operation * is said to be associative in A if

  (a * b) * c = a *(b * c) for all a, b, c in A



Semi group

Semi Group: An algebraic system (A, *) is said to be a semi group if

1. *** is closed operation on A.**
2. *** is an associative operation, for all a, b, c in A.**

- Ex. (N, +) is a semi group.
- Ex. (N, .) is a semi group.

The semigroup (A, *) is said to be commutative if * is a commutative operation.

Example

Set S={1,2,3,...} (positive integers) with addition (+).

Check associativity:

(2+3)+4=5+4=9

2+(3+4)=2+7=9

So it’s associative.

Hence, (S,+) is a semigroup.

Example

- The set P(S) where S is a set, together with the operation of union is a commutative semigroup
- S={1,2}
- P(S)={∅, {1},{2},{1,2}}
- (p(s),union)=closed? Yes

Associative? Yes ---Semigroup

for any a,b belonging to P(S)

a union b =b union a

Commutative semigroup

Idempotent property

- Let $*$ be a binary operation on a set A.
  The operation $*$ is said to be idempotent in A if
  $a * a = a$

Example:

logical AND is idempotent.

a	$a \wedge a$
0	$0 \wedge 0 = 0$
1	$1 \wedge 1 = 1$

similarly, Logical OR ($\vee$), Set Union ($\cup$), Set Intersection ($\cap$) opeartions are also idempotent.

Identity property

An identity element is a special element in a set that does not change any other element when used in a binary operation.

If both left and right identities are same, we call it simply the identity element.

For a binary operation * on a set S:

Left Identity → $e_L * a = a$ for all $a \in S$

Right Identity → $a * e_R = a$ for all $a \in S$

If both are true and same element, then it’s a true identity.

If only one side works, then that element is only left or only right identity.

Example

Right identity check:

- Let e_R be right identity.
- Then $a * eR = a \Rightarrow a - eR = a \Rightarrow eR = 0$
- So 0 is right identity only.

So, subtraction has right identity (0) but no left identity.

Addition on integers

- **Operation: $a*b=a+b$**
- **Check:**
- **$e+a=a$ and $a+e=a$ both hold when $e=0$**
- **So here 0 is both left and right identity, hence identity element.**

Identity

Multiplication on integers

- **Operation: $a * b = a \cdot b$**
- **Check:**
- **$e.a=a$ and $a.e=a$ both hold when $e=1$**
- **So here 1 is both left and right identity, hence identity element.**

Monoid

Monoid: An algebraic system (A, *) is said to be a **monoid** if the following conditions are satisfied.

1) *** is a closed operation in A.**
2) *** is an associative operation in A.**
3) **There is an identity in A.**

Example:

Set of natural numbers (including 0): N={0,1,2,3,...} with addition.

Check:

Associative

Identity element = 0 (because a+0=0+a=a)

So (N,+) is a monoid.

Monoid

- Ex. Show that the set 'N' is a monoid with respect to multiplication.

Solution: Here, $N = \{1,2,3,4,\ldots\ldots\}$

1. Closure property : We know that product of two natural numbers is again a natural number.

i.e., $a.b \in N$ for all $a,b \in N$

$\therefore$ Multiplication is a closed operation.

2. Associativity : Multiplication of natural numbers is associative.

i.e., $(a.b).c = a.(b.c)$ for all $a,b,c \in N$

3. Identity : We have, $1 \in N$ such that

$a.1 = 1.a = a$ for all $a \in N$.

$\therefore$ Identity element exists, and 1 is the identity element.

Hence, N is a monoid with respect to multiplication.

Inverse property

If you have a set S with a binary operation $*$ and an identity element e,

then for every element a∈S, ,

there must exist another element a-1∈S such that:

$a * a{-}1 = a{-}1 * a = e$

Here, a-1 is called the inverse of a (written as a−1).

This is called the **Inverse Property**.

Example

For each a, inverse is -a, since

$a+(-a)=0 \Rightarrow (-a)+a = 0$

Example

Let (Z,+) is inverse,

E.g $-3 + a^{-1} = e$ (for addition e=0)

$-3 + a^{-1}=0$

$a^{-1}=3$

Group

Group: An algebraic system (G, *) is said to be a **group** if the following conditions are satisfied.

1) * is a closed operation.
2) * is an associative operation.
3) There is an identity in G.
4) Every element in G has inverse in G.

Commutative Property

- Let * be a binary operation on a set A.

  The operation * is said to be commutative in A if

  a * b= b * a for all a, b in A

- The order of the elements does not matter in the operation.

Abelian Group

A **group** that also satisfies the commutative property is called an **Abelian group.**

So,

A group = Closure + Associativity + Identity + Inverse

If it also has → Commutativity, then it’s an Abelian Group.

Example

Let’s take (Z,+)

Property	**Check**
Closure	a+b ∈ ℤ
Associative	(a+b)+c = a+(b+c)
Identity	0, because a+0=0+a=a
Inverse	-a, because a+(-a)=0
Commutative	a+b = b+a

Ex. Show that, the set of all integers is an abelian group with respect to **addition**.

Solution: Let Z = set of all integers. Let a, b, c are any three elements of Z.

1. **Closure property** : We know that, Sum of two integers is again an integer.

   i.e., $a + b\ Z$ for all $a,b\ Z$

2. **Associativity**: We know that addition of integers is associative.

   i.e., $(a+b)+c = a+(b+c)$ for all $a,b,c\ Z$.

3. **Identity** : We have $0\ Z$ and $a + 0 = a$ for all $a\ Z$.

   Identity element exists, and ‘0’ is the identity element.

4. **Inverse**: To each $a\ Z$, we have $-a\ \ Z$ such that

   $a + (-a) = 0$ Each element in Z has an inverse

5. **Commutativity:** We know that addition of integers is commutative.

   i.e., $a + b = b + a$ for all $a,b\ Z$.

Hence, (Z , +) is an abelian group.

Example

Prove that the set Z of all integers with binary operation * defined by a*b=a+b+1 such that for all a,b **a * b ∈ Z is an abelian group.**

Solution:

Given, a*b=a+b+1

1. Closure property

   a,b ∈ Z,

a*b = a+b+1 ∈ Z

Hence, it satisfies the closure property.

2. Associative property:

Compute $(a*b)*c$ and $a*(b*c)$:

$(a*b)*c=(a+b+1)*c=(a+b+1)+c+1=a+b+c+2$,

$a*(b*c)=a*(b+c+1)=a+(b+c+1)+1=a+b+c+2$.

Since both give the same result for all a,b,ca,b,ca,b,c, the operation is associative.