

Relations and Their Properties

Practice Question

DEFINITION 1 Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

In other words, a binary relation from A to B is a set T of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B . We use the notation $a T b$ to denote that $(a, b) \in T$ and $a \mathbf{f} b$ to denote that $(a, b) \notin T$. Moreover, when (a, b) belongs to T , a is said to be **related to** b by T .

Binary relations represent relationships between the elements of two sets.

EXAMPLE 1 Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B . This means, for instance, that $0 T a$, but that $1 \mathbf{f} b$. Relations can be represented graphically, as shown in Figure 1, using arrows to represent ordered pairs.

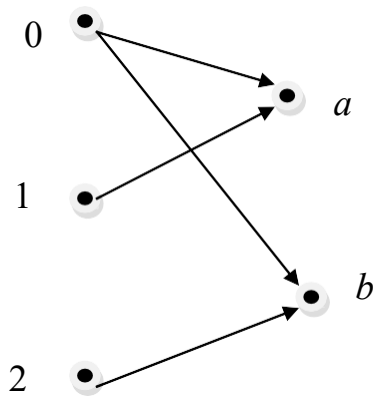


FIGURE 1 Displaying the Ordered Pairs in the Relation T

Relations on a Set

Relations from a set A to itself are of special interest.

DEFINITION 2 A *relation on a set A* is a relation from A to A . In other words, a relation on a set A is a subset of $A \times A$.

EXAMPLE 2 Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b , we see that

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}.$$

The pairs in this relation are displayed graphically in Figure 2.

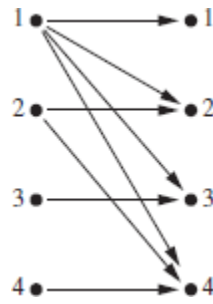


FIGURE 2 Displaying the Ordered Pairs in the Relation R from Example 2.

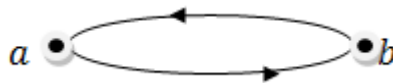
Properties of Relations

DEFINITION 3 A relation R on a set A is called *reflexive* if
 $(a, a) \in R$ for every element $a \in A$. $\forall a \in A, aRa$



EXAMPLE 3 $R = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$ is *reflexive*

DEFINITION 4 A relation R on a set A is called *symmetric*
if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$. $\forall a, b \in A, aRb \Rightarrow bRa$



EXAMPLE 4 $R = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$ is *symmetric*

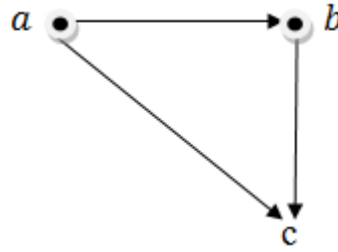
DEFINITION 5 A relation R on a set A such that for all $a, b \in A$,
if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

EXAMPLE 5 $a \leq b \wedge b \leq a \Rightarrow a = b : a, b \in A \quad \therefore \leq$ is *antisymmetric*.

DEFINITION 6 A relation R on a set A is called *transitive*
if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

$$aRb \ \& \ bRc \Rightarrow aRc$$

$$(a, b) \in R \ \& \ (b, c) \in R \Rightarrow (a, c) \in R \ , \forall a, b, c \in A$$



EXAMPLE 6 $a|b \ \wedge \ b|c \Rightarrow a|c$, $\therefore |$ is *transitive*

Combining Relations

EXAMPLE 7 Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$.

The relations $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ can be combined to obtain

$$R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\},$$

$$R_1 \cap R_2 = \{(1, 1)\},$$

$$R_1 - R_2 = \{(2, 2), (3, 3)\},$$

$$R_2 - R_1 = \{(1, 2), (1, 3), (1, 4)\}.$$

$$R_1 \oplus R_2 = R_1 \cup R_2 - R_1 \cap R_2 = \{(1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\}$$

DEFINITION 7 Let R be a relation from a set A to a set B and S a relation from B to a set C . The *composite* of R and S is the relation consisting of ordered pairs (a, c) , where $a \in A, c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

EXAMPLE 8 What is the composite of the relations R and S , where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Solution: $S \circ R$ is constructed using all ordered pairs in R and ordered pairs in S , where the second element of the ordered pair in R agrees with the first element of the ordered pair in S . For example, the ordered pairs $(2, 3)$ in R and $(3, 1)$ in S produce the ordered pair $(2, 1)$ in $S \circ R$. Computing all the ordered pairs in the composite, we find

$$S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}.$$

DEFINITION 8 Let R be a relation on the set A . The powers R^n , $n = 1, 2, 3, \dots$, are defined recursively by $R^1 = R$ and $R^{n+1} = R^n \circ R$.
The definition shows that $R^2 = R \circ R$, $R^3 = R^2 \circ R = (R \circ R) \circ R$, and so on.

EXAMPLE 9 Let $R = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$. Find the powers R_n , $n = 2, 3, 4, \dots$.

Solution: Because $R^2 = R \circ R$, we find that $R^2 = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$.

Furthermore, Because $R^3 = R^2 \circ R$, $R^3 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$.

Additional computation $R^4 = R^3 \circ R$, so $R^4 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$.

It also follows that $R^n = R^3$ for $n = 5, 6, 7, \dots$. The reader should verify this.

THEOREM 1

The relation R on a set A is *transitive* if and only if $R_n \subseteq R$ for $n = 1, 2, 3, \dots$

DEFINITION 9 Let A and B be sets. R is a *binary relation from A to B* : $R \subseteq A \times B$.

Domain of R is $D_R = \{a: a \in A \wedge \exists b \in B (aRb)\}$, $D_R \subseteq A$

Range of R is $Im R = \{b: b \in B \wedge \exists a \in A (aRb)\}$, $Im R \subseteq B$

EXAMPLE 10 Let $A = \{0, 1, 2, 3\}$ and $B = \{a, b, c\}$.

$R = \{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B .

$$D_R = \{0, 1, 2\} \subseteq A$$

$$Im R = \{a, b\} \subseteq B$$

Representing Relations Using Matrices

EXAMPLE 11 Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) , Where $a \in A, b \in B$,

$$R = \{(2, 1), (3, 1), (3, 2)\}$$

What is the matrix representing R ?

Solution:

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

The 1s in M_R show that the pairs $(2, 1)$, $(3, 1)$, and $(3, 2)$ belong to R . The 0s show that no other pairs belong to R .

#

EXAMPLE 12 Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix :

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \quad ?$$

Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}.$$

#

The matrix of a relation on a set, which is a square matrix, can be used to determine whether the relation has certain properties. Recall that a relation R on A is reflexive if $(a, a) \in R$ whenever $a \in A$.

Thus, R is reflexive if and only if $(a_i, a_i) \in R$ for $i = 1, 2, \dots, n$. Hence, R is reflexive if and only if $m_{ii} = 1$, for $i = 1, 2, \dots, n$. In other words, R is reflexive if all the elements on the main diagonal of M_R are equal to 1, as shown in Figure 1. Note that the elements off the main diagonal can be either 0 or 1.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

FIGURE 3 The Zero-One Matrix for a Reflexive Relation. (Off Diagonal Elements Can Be 0 or 1.)

The relation R is symmetric if $(a, b) \in R \Rightarrow (b, a) \in R$. Consequently, the relation R on the set $A = \{a_1, a_2, \dots, a_n\}$ is symmetric if and only if $(a_j, a_i) \in R$ whenever $(a_i, a_j) \in R$. In terms of the entries of M_R , R is symmetric if and only if $m_{ji} = 1$ whenever $m_{ij} = 1$. This also means $m_{ji} = 0$ whenever $m_{ij} = 0$. Consequently, R is symmetric if and only if $m_{ij} = m_{ji}$, for all pairs of integers i and j with $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$.

R is symmetric if and only if $M_R = (M_R)^t$, that is, if M_R is a symmetric matrix. The form of the matrix for a symmetric relation is illustrated in Figure 3(a).

The relation R is antisymmetric if and only if $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$.

Consequently, the matrix of an antisymmetric relation has the property that if $m_{ij} = 1$ with $i \neq j$, then $m_{ji} = 0$. Or, in other words, either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$. The form of the matrix for an antisymmetric relation is illustrated in Figure 3(b).

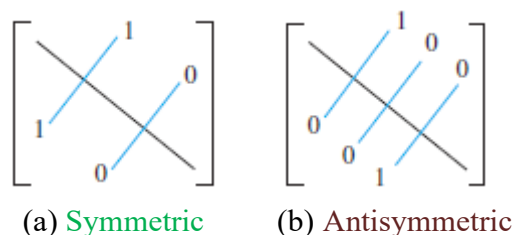


FIGURE 4 The Zero–One Matrices for Symmetric and Antisymmetric Relations.

EXAMPLE 13 Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution: Because all the diagonal elements of this matrix are equal to 1, R is reflexive. Moreover, because M_R is symmetric, it follows that R is symmetric. It is also easy to see that R is not antisymmetric.

Suppose that R_1 and R_2 are relations on a set A represented by the matrices M_{R_1} and M_{R_2} , respectively. The matrix representing the union of these relations has a 1 in the positions where either M_{R_1} or M_{R_2} has a 1. The matrix representing the intersection of these relations has a 1 in the positions where both M_{R_1} and M_{R_2} have a 1.

Thus, the matrices representing the union and intersection of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} \quad \text{and} \quad M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$$

EXAMPLE 14 Suppose that the relations R_1 and R_2 on a set A are represented by the

$$\text{matrices } M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Solution: The matrices of these relations are

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

EXAMPLE 15 Find the matrix representing the relations $S \circ R$, where the matrices representing R and S are

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Solution: The matrix for $S \circ R$ is

$$M_{S \circ R} = M_R \odot M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

#

The matrix representing the composite of two relations can be used to find the matrix for M_{R^n} . In particular,

$$M_{R^n} = M_R^{[n]}$$

EXAMPLE 16 Find the matrix representing the relation R^2 , where the matrix representing R is

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Solution: The matrix for R^2 is

$$M_{R^2} = M_{R^{[2]}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

#

Representing Relations Using Digraphs

DEFINITION 10 A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a, b) , and the vertex b is called the terminal vertex of this edge.

EXAMPLE 17 The directed graph with vertices a, b, c , and d , and edges (a, b) , (a, d) , (b, b) , (b, d) , (c, a) , (c, b) , and (d, b) is displayed in Figure 5.

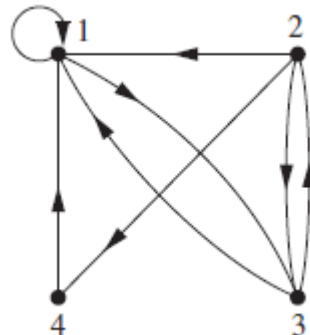


FIGURE 5 A Directed Graph.

EXAMPLE 18 The directed graph of the relation

$$R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$$

on the set $\{1, 2, 3, 4\}$ is shown in Figure 6.

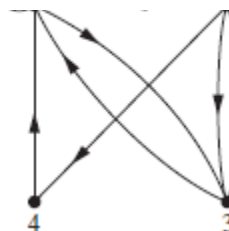


FIGURE 6 A Directed Graph of the relation R .

EXAMPLE 19 What are the ordered pairs in the relation R represented by the directed graph shown in Figure 7?

Solution: The ordered pairs (x, y) in the relation are

$$R = \{(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)\}.$$

Each of these pairs corresponds to an edge of the directed graph, with $(2, 2)$ and $(3, 3)$ corresponding to loops.

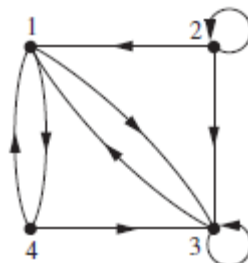


FIGURE 7 A Directed Graph of the relation R

#

EXAMPLE 20 Determine whether the relation for the directed graphs shown in Figure 8 is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution: Because there are loops at every vertex of the directed graph of R , it is *reflexive*. R is *neither symmetric nor antisymmetric* because there is an edge from a to b but not one from b to a , but there are edges in both directions connecting b and c . Finally, R is *not transitive* because there is an edge from a to b and an edge from b to c , but no edge from a to c .

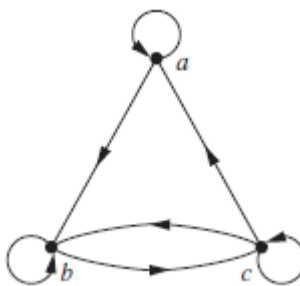


FIGURE 8 A Directed Graph of the relation R

#

EXAMPLE 21 Determine whether the relation for the directed graphs shown in Figure 9 is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

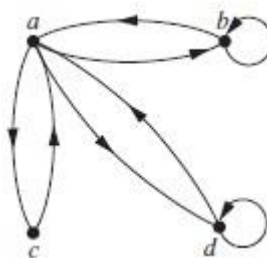


FIGURE 9 A Directed Graph of the relation R

Solution: Because loops are not present at all the vertices of the directed graph of S , this relation is *not reflexive*.

It is *symmetric* and *not antisymmetric*, because every edge between distinct vertices is accompanied by an edge in the opposite direction. It is also not hard to see from the directed graph that S is *not transitive*, because (c, a) and (a, b) belong to S , but (c, b) does not belong to S .

EXERCISES

1. Let R be a relation defined on the set $A = \{-2, -1, 0, 1, 2, 3, 4\}$

$$m, n \in A, \quad m R n \Leftrightarrow n = m^2$$

- (i) What are the ordered pairs in the relation R ?
- (ii) Find the domain and the image of R ?
- (iii) Represent R by the directed graph (diagraph) ?

Solution:

2. Let R be a relation defined on the set $A = \{1,2,3,4,5\}$

$$x, y \in A, \quad x R y \Leftrightarrow xy \leq 9$$

- (i) What are the ordered pairs in the relation R ?
- (ii) Find the domain and the image of R ?
- (iii) Draw the directed graph (diagraph) that represents R ?

Solution:

3. Let R be a relation defined on the set $A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

$$x, y \in A, \quad x R y \Leftrightarrow y = 2x$$

(i) What are the ordered pairs in the relation R ?

(ii) Find the domain and the image of R ?

Solution:

4. Let R be a relation defined from the set $A = \{1,2,3,4\}$ to the set $B = \{2,3,4,5\}$

as: $a R b \Leftrightarrow a + b = 5$

(i) What are the ordered pairs in the relation R ?

(ii) Find the domain and the image of R ?

(iii) Represent R with a matrix ?

Solution:

5. Suppose R is a relation defined on the set $A = \{-2, -1, 0, 1, 2\}$, as

$$x, y \in A, \quad x R y \Leftrightarrow |x - y| < 2$$

- (i) What are the ordered pairs in the relation R ?
- (ii) Draw the directed graph (diagraph) that represents R
- (iii) Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

6. Let R be a relation defined on the set $A = \{1,3,4,6\}$

$$a, b \in A, \quad a R b \Leftrightarrow a - b = 1$$

- (i) What are the ordered pairs in the relation R ?
- (ii) Find the domain and the image of R ?
- (iii) Draw the directed graph (diagraph) that represents R ?

Solution:

7. Let R be a relation defined on the set $A = \{0,1,2,3\}$

$$a, b \in A, \quad a R b \Leftrightarrow a + b = 4$$

- (i) What are the ordered pairs in the relation R ?
- (ii) Find the domain and the image of R ?
- (iii) Draw the directed graph (diagraph) that represents R ?

Solution:

8. Let R be a relation defined on the set $A = \{2,3,4,5,6\}$

$$a, b \in A, \quad a R b \Leftrightarrow a.b < 10$$

- (i) What are the ordered pairs in the relation R ?
- (ii) Find the domain and the image of R ?
- (iii) Draw the directed graph (diagraph) that represents R ?
- (iv) Find M_R .

Solution:

9. Let R be a relation defined on the set $A = \{-2, -1, 0, 1, 2, 3, 4\}$

$$a, b \in A, \quad a R b \Leftrightarrow a^2 = b^2$$

- (i) What are the ordered pairs in the relation R ?
- (ii) Find the domain and the image of R ?
- (iii) Draw the directed graph (diagraph) that represents R ?
- (iv) Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

10. Let R be a relation defined on the set $A = \{-2, -1, 0, 1, 2\}$

$$a, b \in A, \quad a R b \Leftrightarrow a \cdot b < 0$$

- (i) What are the ordered pairs in the relation R ?
- (ii) Find the domain and the image of R ?
- (iii) Draw the directed graph (diagraph) that represents R ?
- (iv) Find R^2 .
- (v) Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

11. Let R be a relation defined on the set $A = \{-2, -1, 0, 1, 2\}$

$$a, b \in A, \quad a R b \Leftrightarrow a \cdot b \geq 2$$

- (i) What are the ordered pairs in the relation R ?
- (ii) Find the domain and the image of R ?
- (iii) Draw the directed graph (diagraph) that represents R ?
- (iv) Find R^2 .
- (v) Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

12. Let $S = \{(1,1), (1,2), (1,3), (2,2), (3,1), (3,3)\}$ be a relation on the set $B = \{1, 2, 3\}$

- (i) Draw the directed graph (diagraph) that represents S ?
- (ii) Find S^2 , S , $So S$
- (iii) Find M_S
- (iv) Determine whether the relation S is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*

Solution:

13. Let $S = \{(a, c), (b, a), (c, b)\}$ be a relation on the set $B = \{a, b, c\}$.

- (i) Find M_S ?
- (ii) Find $\bar{S} - S^{-1}$
- (iii) Find S^2 , S^3

Solution:

14. Let $S = \{(a, b), (b, c), (c, d), (d, a)\}$ be a relation on the set $B = \{a, b, c, d\}$.

(i) Find M_S ?

(ii) Find S^2

(iii) Find $S^{-1} \circ S$

Solution:

15. Suppose R is a relation defined on the integers set $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$
 $m, n \in \mathbb{Z}^+, \quad m R n \Leftrightarrow m + n = 20$
Determine whether the relation T is *reflexive*, *symmetric*, *antisymmetric*,
and/or *transitive*.

Solution:

1- $\therefore 5 + 5 \neq 20 \Rightarrow (5, 5) \notin R \Rightarrow \therefore R$ is *irreflexive*.

2- $m, n \in \mathbb{Z}^+, \quad m R n \Leftrightarrow m + n = 20$

comutative

$\Rightarrow n + m = 20 \Rightarrow \therefore n R m \Rightarrow \therefore R$ is *symmetric*

3- $\therefore 7 R 13 : 7 + 13 = 20 \quad \wedge \quad 13 R 7 : 13 + 7 = 20$

But $7 \neq 13 \Rightarrow \therefore R$ is not *antisymmetric*

4- $\therefore 8 R 12 : 8 + 12 = 20 \quad \wedge \quad 12 R 8 : 12 + 7 = 20$

but $(8, 8) \notin R : 8 + 8 = 16 \neq 20 \Rightarrow \therefore R$ is not *transitive*.

$\therefore R$ is *symmetric* only.

16. Suppose T is a relation defined on the integers set \mathbb{Z}

$$m, n \in \mathbb{Z}, \quad m T n \Leftrightarrow m + n \geq 2$$

Determine whether the relation T is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

17. Let T be a relation defined on the set $\mathbb{N} = \{1, 2, 3, \dots\}$,

$$m, n \in \mathbb{N}, \quad m T n \Leftrightarrow m + n > 3$$

Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

18. Let T be a relation defined on the integers set $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
 $m, n \in \mathbb{Z}, \quad m T n \Leftrightarrow m + n \text{ is odd}$

Determine whether the relation T is *reflexive*, *symmetric*, *antisymmetric*,
and/or *transitive*.

Solution:

19. Let R be a relation defined on the integers set $\mathbb{N} = \{ 1, 2, 3, \dots \}$

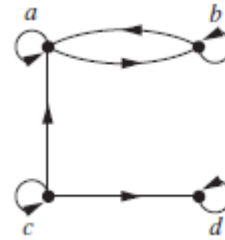
$$x, y \in \mathbb{N}, \quad x R y \Leftrightarrow x < y$$

Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

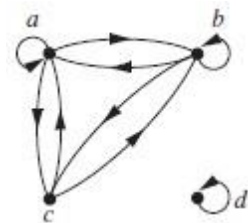
- 20.** Determine whether the relation for the directed graphs shown in the Figure is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:



- 21.** Determine whether the relation for the directed graphs shown in the Figure is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:



22. Let R be the relation represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Find the matrix representing: a) R^{-1} b) R c) R^2

Solution:

23. Let R_1 and R_2 are relations on a set A represented by the matrices

$$M_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Find the matrices representing

- a) $R_1 \cup R_2$ b) $R_1 \cap R_2$ c) $R_1 \circ R_2$ d) $R_2 \circ R_1$

Solution:

24. Let R be the relation represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Find the matrix representing:

a) R^2

b) R^3

c) R^4

Solution: