

Questions for Tutorial-7: Fourier Series Using SageMath

Q.1 Find all the Fourier Coefficients and Fourier Series for the following functions. Also plot the graph of the function and the Fourier series

(i) $f(x) = \left(\frac{\pi-x}{2}\right)^2$ in $(0, 2\pi)$ for $n = 10$ and $n = 20$

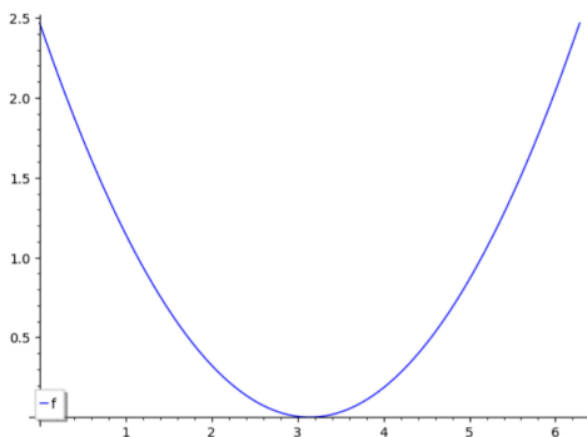
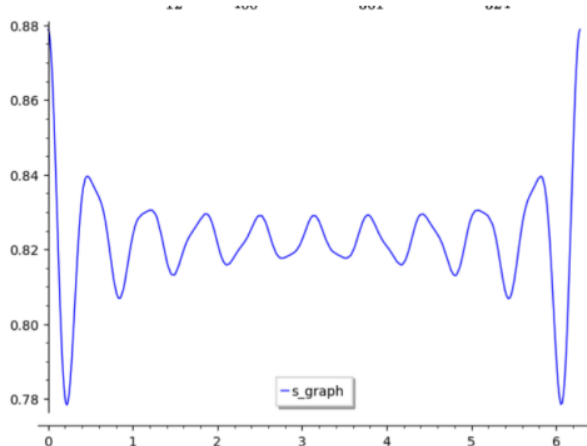
```
In [ ]: var('x')
var('n')
assume(n, 'integer')
f(x) = ((pi-x)/2)^2
a0 = (1/pi)*integrate(f,x,0,2*pi)
an = (1/pi)*integrate(f*cos(n*x),x,0,2*pi)
bn = (1/pi)*integrate(f*sin(n*x),x,0,2*pi)
s = a0/2 + sum(an*cos(n*x) + bn*sin(n*x),n,10,20)
show("Value of a0 is ",a0)
show("Value of an is ",an)
show("Value of bn is ",bn)
show("Fourier series is ",s)
show(plot(s,0,2*pi,color='blue',legend_label='s_graph'))
show(plot(f,0,2*pi,color='blue',legend_label='f'))
```

Value of a0 is $\frac{1}{6} \pi^2$

Value of an is $\frac{1}{n^2}$

Value of bn is 0

Fourier series is $\frac{1}{12} \pi^2 + \frac{1}{400} \cos(20x) + \frac{1}{361} \cos(19x) + \frac{1}{324} \cos(18x) + \frac{1}{289} \cos(17x) + \frac{1}{256} \cos(16x) + \frac{1}{225} \cos(15x) + \frac{1}{196} \cos(14x) + \frac{1}{169} \cos(13x) + \frac{1}{144} \cos(12x) + \frac{1}{121} \cos(11x) + \frac{1}{100} \cos(10x)$



(ii) $f(x) = x^5$ in $(-\pi, \pi)$ for $n = 5$ and $n = 15$

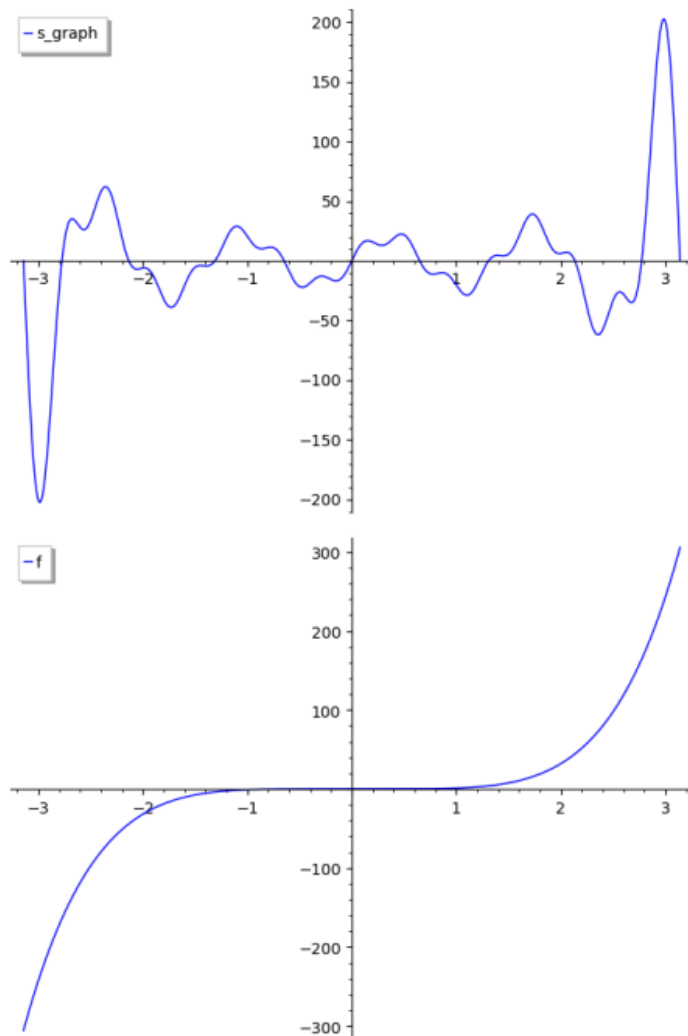
```
In [ ]: var('x')
var('n')
assume(n, 'integer')
f(x) = x^5
a0 = (1/pi)*integrate(f,x,-pi,pi)
an = (1/pi)*integrate(f*cos(n*x),x,-pi,pi)
bn = (1/pi)*integrate(f*sin(n*x),x,-pi,pi)
s = a0/2 + sum(an*cos(n*x) + bn*sin(n*x),n,5,15)
show("Value of a0 is ",a0)
show("Value of an is ",an)
show("Value of bn is ",bn)
show("Fourier series is ",s)
show(plot(s,-pi,pi,color='blue',legend_label='s_graph'))
show(plot(f,-pi,pi,color='blue',legend_label='f'))
```

Value of a0 is 0

Value of an is 0

Value of bn is $-\frac{2(120\pi + \pi^5 n^4 - 20\pi^3 n^2)(-1)^n}{\pi n^5}$

Fourier series is $\frac{2}{50625} (3375\pi^4 - 300\pi^2 + 8) \sin(15x) - \frac{1}{33614} (4802\pi^4 - 490\pi^2 + 15) \sin(14x) + \frac{2}{371293} (28561\pi^4 - 3380\pi^2 + 120) \sin(13x) - \frac{1}{5184} (864\pi^4 - 120\pi^2 + 5) \sin(12x) + \frac{2}{161051} (14641\pi^4 - 2420\pi^2 + 120) \sin(11x) - \frac{1}{1250} (250\pi^4 - 50\pi^2 + 3) \sin(10x) + \frac{2}{19683} (2187\pi^4 - 540\pi^2 + 40) \sin(9x) - \frac{1}{2048} (512\pi^4 - 160\pi^2 + 15) \sin(8x) + \frac{2}{16807} (2401\pi^4 - 980\pi^2 + 120) \sin(7x) - \frac{1}{162} (54\pi^4 - 30\pi^2 + 5) \sin(6x) + \frac{2}{625} (125\pi^4 - 100\pi^2 + 24) \sin(5x)$



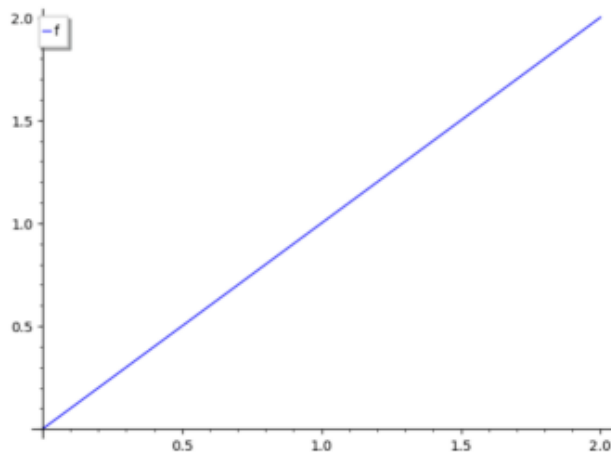
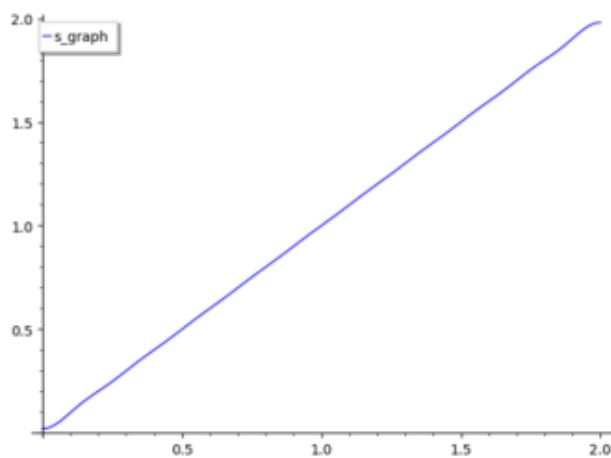
Q.2 Find the Half range cosine series for $f(x) = x$ $0 < x < 2$ for $n=20$. Also plot the graph of the function and the cosine series.

```
In [ ]: var('x')
var('n')
assume(n, 'integer')
f(x) = x
L=2
a0 = (2/L)*integrate(f,x,0,L)
an = (2/L)*integrate(f*cos(n*pi*x),x,0,L)
bn=0
s = a0/2 + sum(an*cos(n*pi*x/L),n,1,20)
show("Value of a0 is ",a0)
show("Value of an is ",an)
show("Fourier series is ",s)
show(plot(s,-L,L,color='blue',legend_label='f_graph'))
show(plot(f,-L,L,color='red',legend_label='S'))
```

Value of a0 is 2

Value of an is $\frac{2n \sin(2n) + \cos(2n)}{n^2} - \frac{1}{n^2}$

$$\begin{aligned} \text{Fourier series is } & \frac{1}{400} (\cos(40) + 40 \sin(40) - 1) \cos(10\pi x) + \frac{1}{361} (\cos(38) + 38 \sin(38) - 1) \cos\left(\frac{19}{2}\pi x\right) + \frac{1}{324} \\ & (\cos(36) + 36 \sin(36) - 1) \cos(9\pi x) + \frac{1}{289} (\cos(34) + 34 \sin(34) - 1) \cos\left(\frac{17}{2}\pi x\right) + \frac{1}{256} (\cos(32) + 32 \sin(32) - 1) \cos(8\pi x) + \frac{1}{225} \\ & (\cos(30) + 30 \sin(30) - 1) \cos\left(\frac{15}{2}\pi x\right) + \frac{1}{196} (\cos(28) + 28 \sin(28) - 1) \cos(7\pi x) + \frac{1}{169} (\cos(26) + 26 \sin(26) - 1) \cos\left(\frac{13}{2}\pi x\right) + \frac{1}{144} \\ & (\cos(24) + 24 \sin(24) - 1) \cos(6\pi x) + \frac{1}{121} (\cos(22) + 22 \sin(22) - 1) \cos\left(\frac{11}{2}\pi x\right) + \frac{1}{100} (\cos(20) + 20 \sin(20) - 1) \cos(5\pi x) + \frac{1}{81} \\ & (\cos(18) + 18 \sin(18) - 1) \cos\left(\frac{9}{2}\pi x\right) + \frac{1}{64} (\cos(16) + 16 \sin(16) - 1) \cos(4\pi x) + \frac{1}{49} (\cos(14) + 14 \sin(14) - 1) \cos\left(\frac{7}{2}\pi x\right) + \frac{1}{36} \\ & (\cos(12) + 12 \sin(12) - 1) \cos(3\pi x) + \frac{1}{25} (\cos(10) + 10 \sin(10) - 1) \cos\left(\frac{5}{2}\pi x\right) + \frac{1}{16} (\cos(8) + 8 \sin(8) - 1) \cos(2\pi x) + \frac{1}{9} \\ & (\cos(6) + 6 \sin(6) - 1) \cos\left(\frac{3}{2}\pi x\right) + \frac{1}{4} (\cos(4) + 4 \sin(4) - 1) \cos(\pi x) + (\cos(2) + 2 \sin(2) - 1) \cos\left(\frac{1}{2}\pi x\right) + 1 \end{aligned}$$



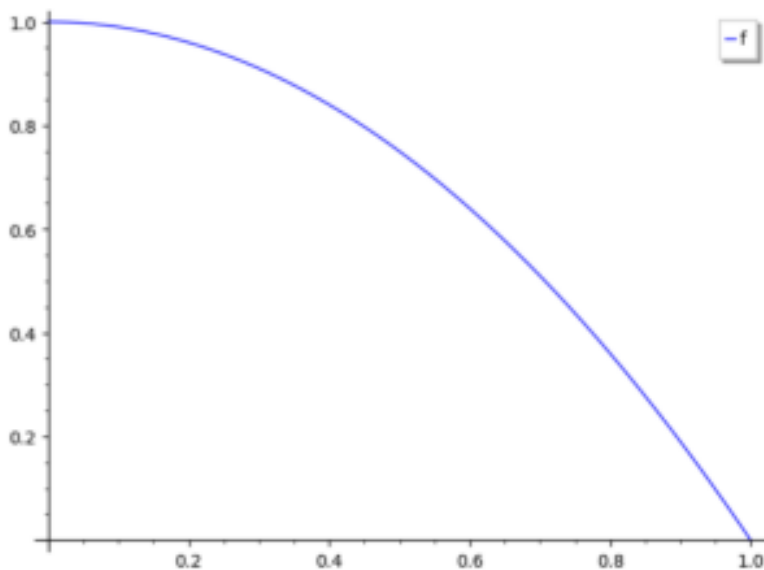
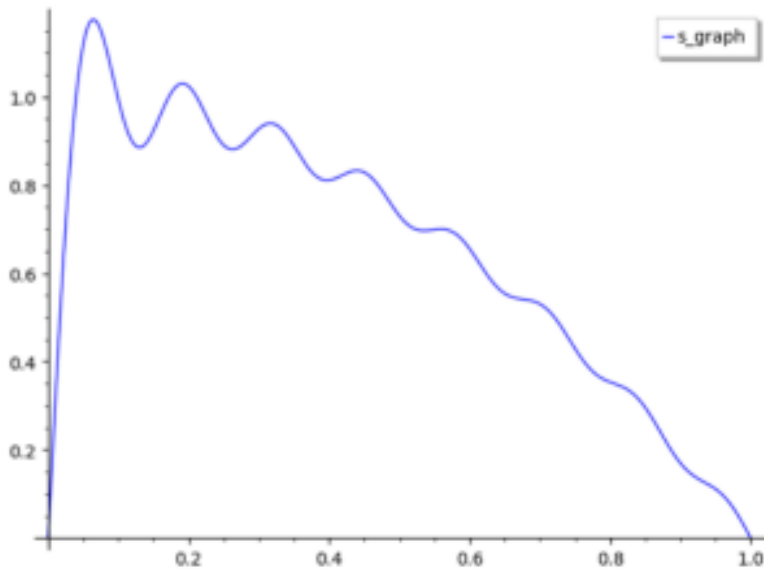
Q.3 Find the Half range sine series for $(x) = 1 - x^2$ in $(0,1)$ for $n=15$. Also plot the graph of the function and the sine series.

```
In [ ]: var('x')
var('n')
assume(n, 'integer')
L = 1
f(x) = 1 - x^2
a0 = 0
an = 0
bn = (2/L)*integrate(f*sin(n*x*pi/L),x,0,L)
s = sum(bn*sin(n*x),n,1,15)
show("Value of bn is ",bn)
show("Fourier series is ",s)
show(plot(s,-L,L,color='blue',legend_label='s_graph'))
show(plot(f,-L,L,color='blue',legend_label='f'))
```

Value of bn is $\frac{2(\pi^2 n^2 + 2)}{\pi^3 n^3} - \frac{4(-1)^n}{\pi^3 n^3}$

$$52227799123500 \pi^2 \sin(14x) + 60932432310750 \pi^2 \sin(12x) + 73118918772900 \pi^2 \sin(10x) + 91398648466125 \pi^2 \sin(8x) + 121864864621500 \pi^2 \sin(6x) + 182797296932250 \pi^2 \sin(4x) + 365594593864500 \pi^2 \sin(2x) + 216648648216 (225 \pi^2 + 4) \sin(15x) + 332812557000 (169 \pi^2 + 4) \sin(13x) + 549353259000 (121 \pi^2 + 4) \sin(11x) + 1003003001000 (81 \pi^2 + 4) \sin(9x) + 1003003001000 (49 \pi^2 + 4) \sin(7x) + 5849513501832 (25 \pi^2 + 4) \sin(5x) + 27081081027000 (9 \pi^2 + 4) \sin(3x) + 731189187729000 \sin(x)$$

Fourier series is $\frac{365594593864500 \pi^3}{\pi^3 n^3}$



Q.4 Find the Fourier series ($n=15$), a_{10} and b_{15} for $f(x) = x(\pi - x)$ in $(-\pi, \pi)$.

```
In [ ]: var('x')
var('n')
assume(n,'integer')
f(x) = x*(pi-x)
L=pi
a0 = (1/L)*integrate(f,x,-L,L)
an = (1/L)*integrate(f*cos(n*pi*x/L),x,-L,L)
bn = (1/L)*integrate(f*sin(n*pi*x/L),x,-L,L)
s = a0/2 + sum(an*cos(n*x) + bn*sin(n*x),n,1,15)
show("Value of a0 is ",a0)
show("Value of an is ",an)
show("Value of bn is ",bn)
show("Fourier series is ",s)
show(plot(s,-L,L,color='blue',legend_label='f_graph'))
show(plot(f,-L,L,color='red',legend_label='S'))
an = (1/L)*integrate(f*cos(10*x*pi/L),x,-L,L)
bn = (1/L)*integrate(f*sin(15*x*pi/L),x,-L,L)
show("Value of a10 is",a10)
show("Value of b15 is",b15)
```

Value of a_0 is $-\frac{2}{3}\pi^2$

Value of a_n is $-\frac{4(-1)^n}{n^2}$

Value of b_n is $-\frac{2\left(\frac{(\pi^2 n^2 - 1)(-1)^n}{n^3} + \frac{(-1)^n}{n^3}\right)}{\pi}$

Fourier series is $-\frac{1}{3}\pi^2 + \frac{2}{15}\pi \sin(15x) - \frac{1}{7}\pi \sin(14x) + \frac{2}{13}\pi \sin(13x) - \frac{1}{6}\pi \sin(12x) + \frac{2}{11}\pi \sin(11x) - \frac{1}{5}\pi \sin(10x) + \frac{2}{9}\pi \sin(9x)$
 $-\frac{1}{4}\pi \sin(8x) + \frac{2}{7}\pi \sin(7x) - \frac{1}{3}\pi \sin(6x) + \frac{2}{5}\pi \sin(5x) - \frac{1}{2}\pi \sin(4x) + \frac{2}{3}\pi \sin(3x) - \pi \sin(2x) + 2\pi \sin(x) + \frac{4}{225}\cos(15x) - \frac{1}{49}\cos(14x)$
 $+ \frac{4}{169}\cos(13x) - \frac{1}{36}\cos(12x) + \frac{4}{121}\cos(11x) - \frac{1}{25}\cos(10x) + \frac{4}{81}\cos(9x) - \frac{1}{16}\cos(8x) + \frac{4}{49}\cos(7x) - \frac{1}{9}\cos(6x) + \frac{4}{25}\cos(5x) - \frac{1}{4}$
 $\cos(4x) + \frac{4}{9}\cos(3x) - \cos(2x) + 4\cos(x)$

