

Isomorphic graph and planar
graph

Isomorphic Graph

Graphs $G = (V, E)$ and $H = (U, F)$ are **isomorphic** if we can set up a bijection $f : V \rightarrow U$ such that

x and y are adjacent in G

$\Leftrightarrow f(x)$ and $f(y)$ are adjacent in H

Function f is called isomorphism

1. Same no. of vertices
2. Same no. of edges
3. Equal no. of vertices with a given degree
4. Adjacency of vertices

Graph - Isomorphism

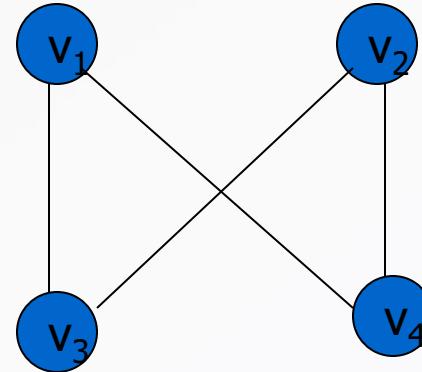
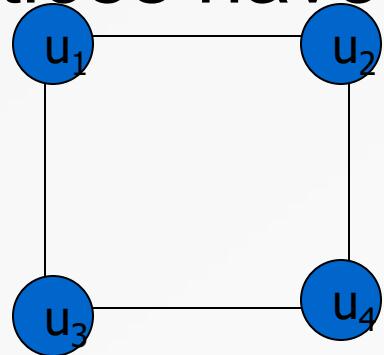
Representation example: $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$

$f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, $f(u_4) = v_2$

No. of vertices:4

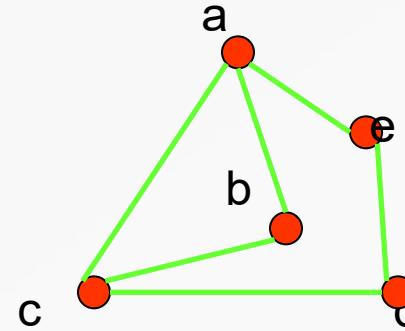
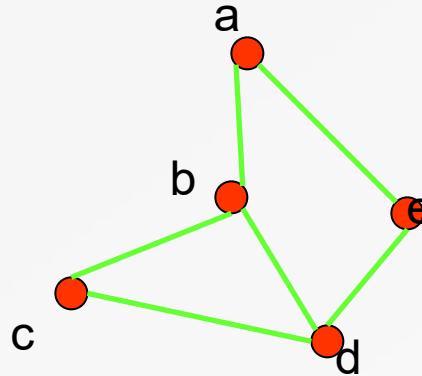
No. of edges:4

All vertices have degree 2



Isomorphism of Graphs

Example I: Are the following two graphs isomorphic?



Solution: No. of vertices: 6, No. of edges: 6

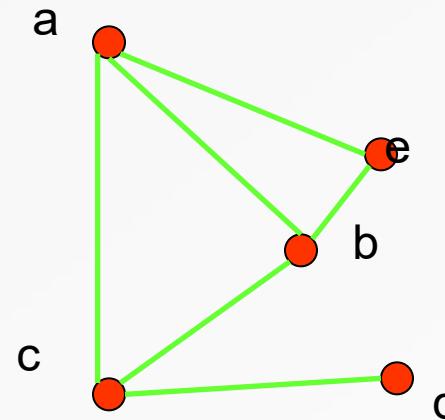
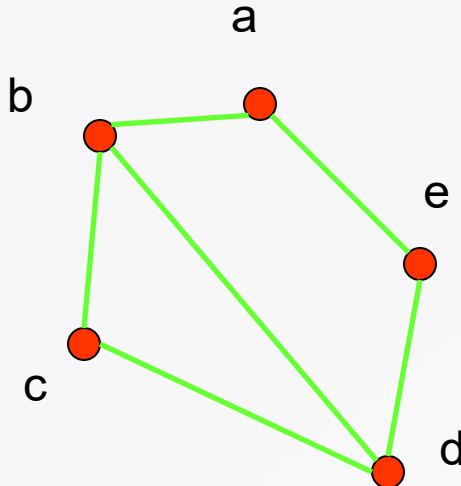
3 vertices with degree 2, 2 vertices with degree 3

Then the isomorphism f from the left to the right graph is:

$$f(a)=e, f(b)=a, f(c) = b, f(d) = c, f(e) = d.$$

Isomorphism of Graphs

Example II: How about these two graphs?



Solution: No. of vertices: 5, No. of edges: 6

No, they are not isomorphic, because they differ in the degrees of their vertices. Vertex d in right graph is of degree one, but there is no such vertex in the left graph.

Isomorphism of Graphs

Example III: Are the following two graphs isomorphic?

Solution:

Both graphs contain

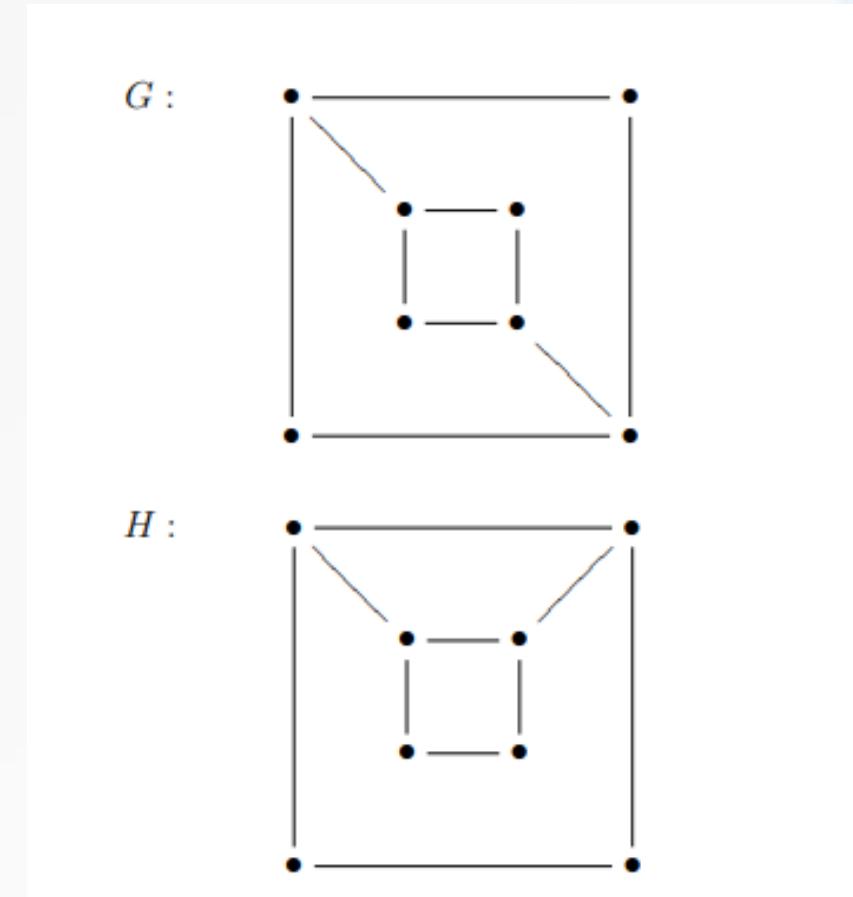
8 vertices and 10 edges

Nos of vertices of degree 2 = 4

Nos of vertices of degree 3 = 4

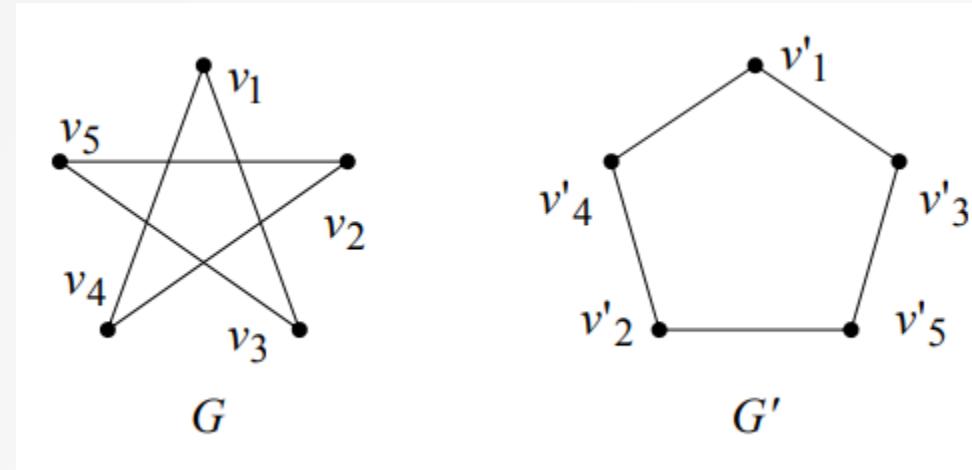
Adjacency : There exists no vertex of degree 3 whose adjacent vertices have same degree in both graphs

So its not ISOMORPHIC



Isomorphism of Graphs

Example IV: Are the following two graphs isomorphic?

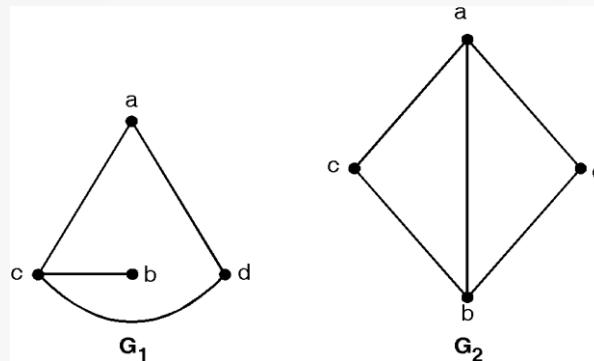


Solution: Both graphs have 5 vertices and 5 edges. All vertices have degree 2.

$f : V \rightarrow V'$	
V	V'
v_1	v'_1
v_2	v'_2
v_3	v'_3
v_4	v'_4
v_5	v'_5

Isomorphism of Graphs

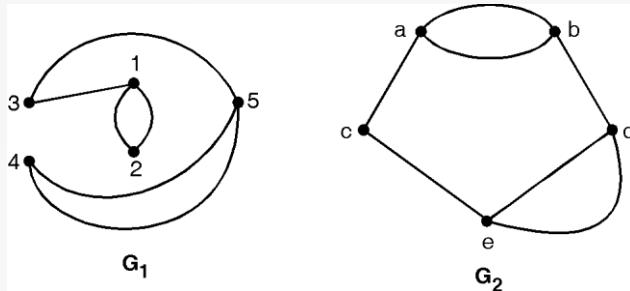
Example V: Are the following two graphs isomorphic?



Solution: Here G_1 and G_2 both have 4 vertices but G_1 has 4 edges and G_2 has 5 edges. Hence G_1 is not isomorphic to G_2 .

Isomorphism of Graphs

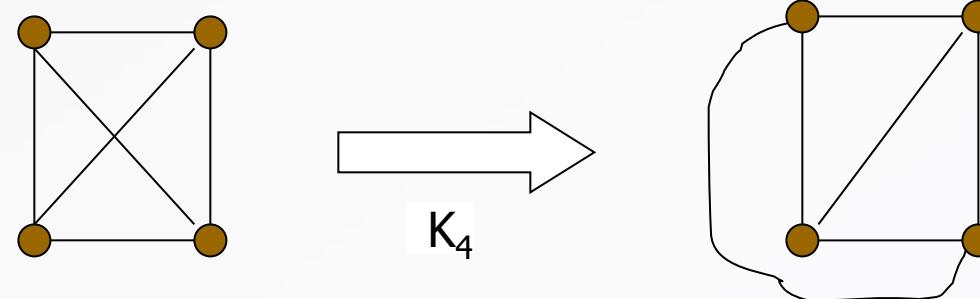
Example VI: Are the following two graphs isomorphic?



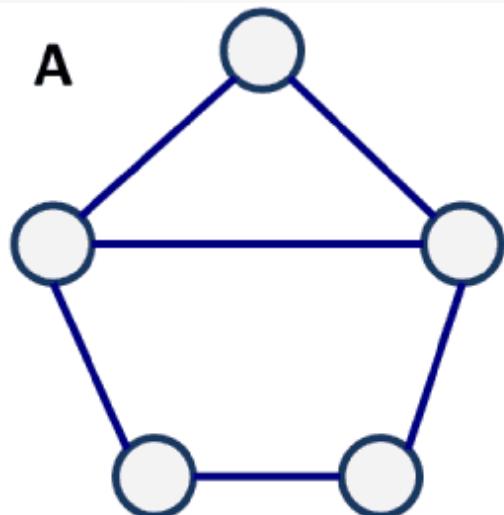
Solution: G_1 and G_2 both have 5 vertices but G_1 has 6 edges while G_2 has 7 edges. Hence $G_1 \not\cong G_2$. That is G_1 is not isomorphic to G_2 .

Planar Graphs

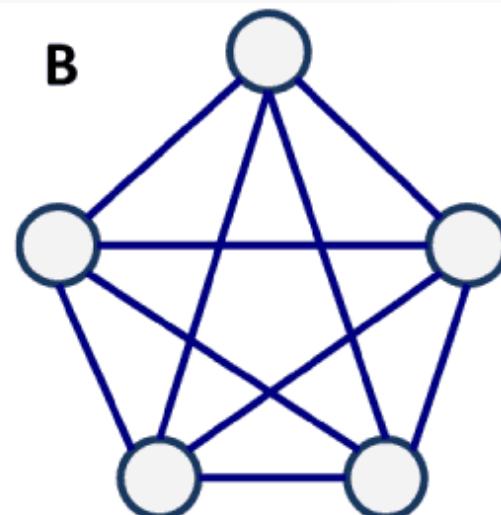
- A graph (or multigraph) G is called planar if G can be drawn in the plane with its edges intersecting only at vertices of G , such a drawing of G is called an embedding of G in the plane.
- Application Example: VLSI design (overlapping edges requires extra layers), Circuit design (cannot overlap wires on board)
- Representation examples: K_1, K_2, K_3, K_4 are planar, K_n for $n > 4$ are non-planar



Examples

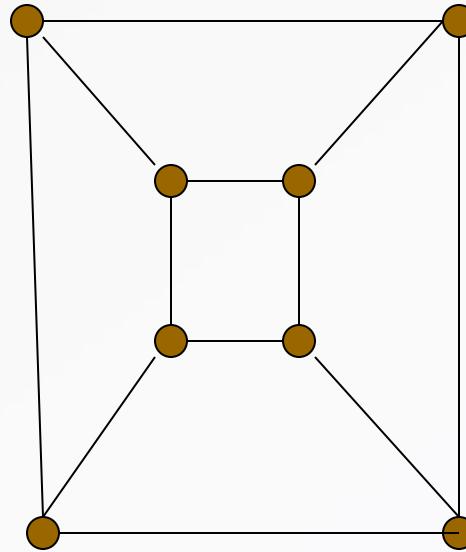
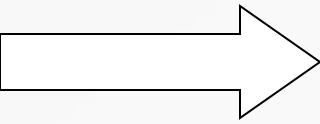
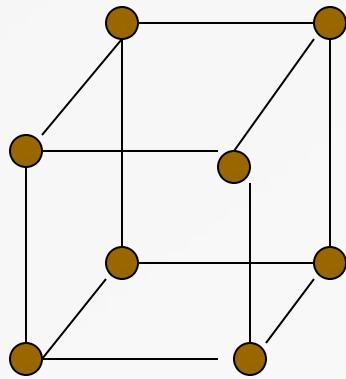


Planar



Non-Planar

Planar Graphs Example

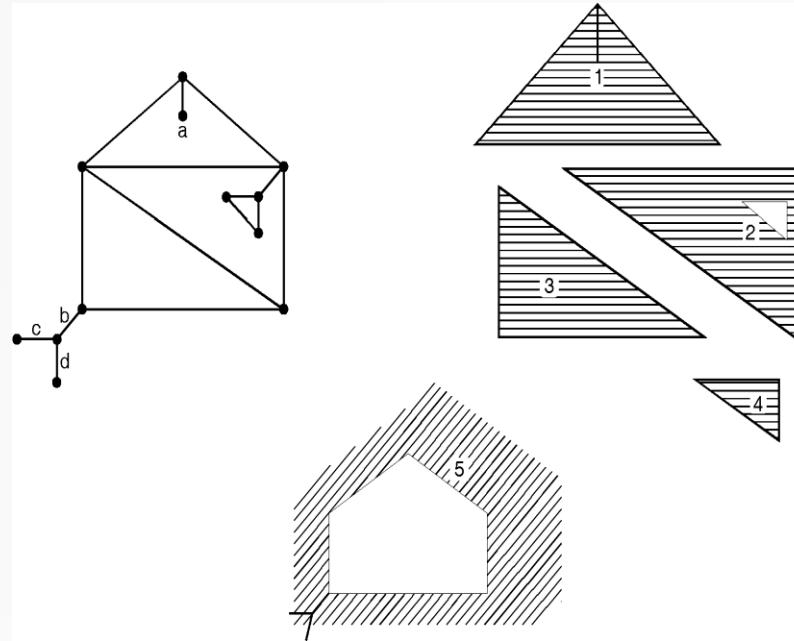


Planer Graph

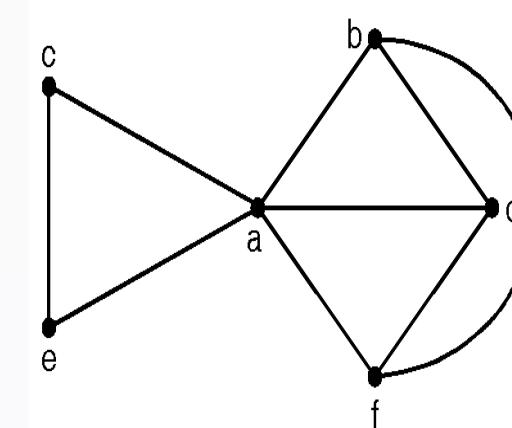
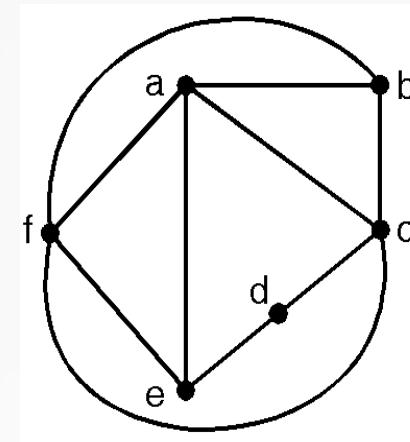
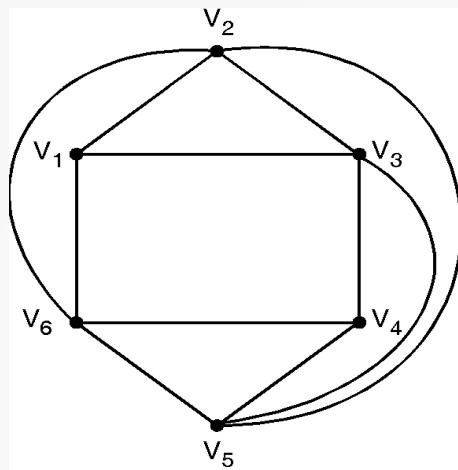
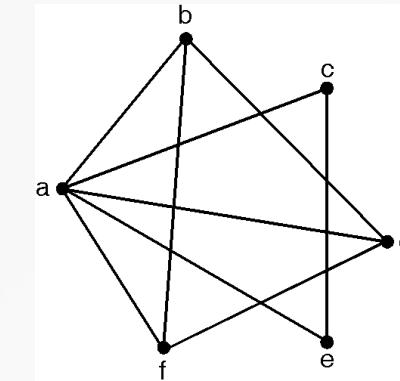
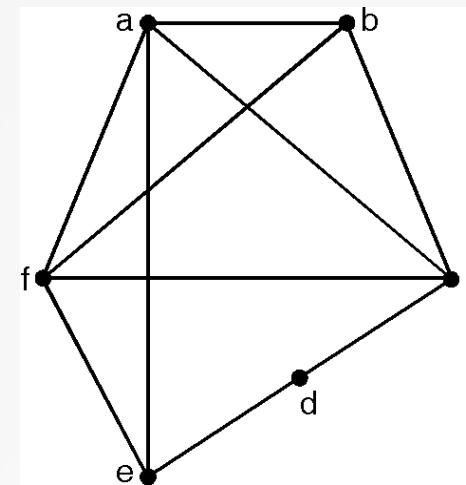
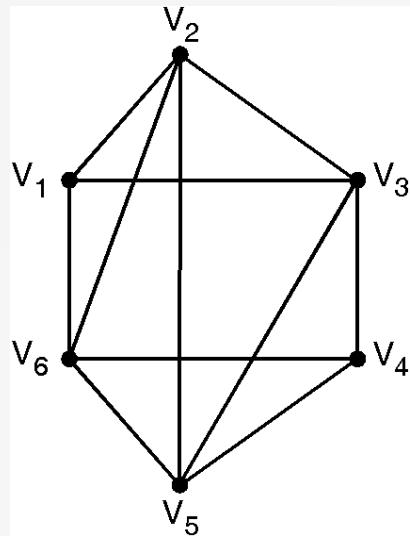
- **Theorem :** *Euler's connected planar graph theorem*

$$v - e + r = 2$$

number of vertices number of edges number of regions



Q. 1) By drawing the graph, show that following graphs are planar graphs



Q. 2 : How many edges must a planar graph have if it has 7 regions and 5 nodes.
Draw one such graph.

Soln. :

According to Euler's formula, in a planar graph

$$v - e + r = 2$$

where v, e, r are the number of vertices, edges and regions in a planar graph.

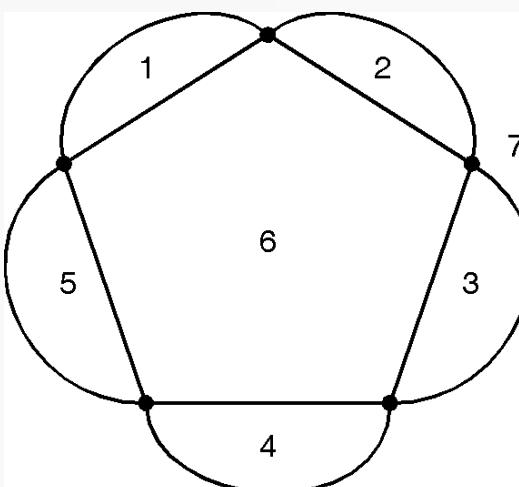
Here $v = 5$, $r = 7$, $e = ?$

$$v - e + r = 2$$

$$5 - e + 7 = 2$$

$$e = 10$$

Hence the given graph must have 10 edges.



Q. 3 : Determine the number of regions defined by a connected planar graph with 6 vertices and 10 edges. Draw a simple and a multi-graph.

Soln. :

Given $v = 6, e = 10$

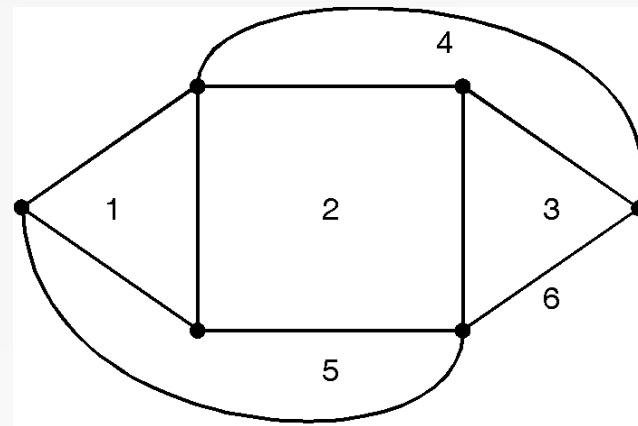
Hence by Euler's formula for a planar graph

$$v - e + r = 2$$

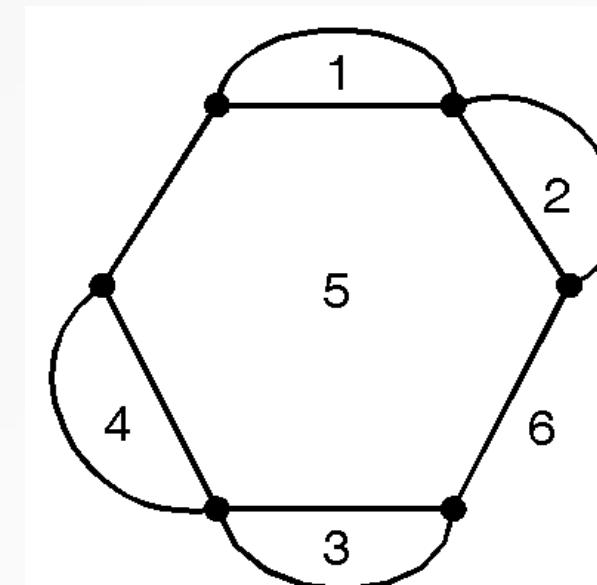
$$6 - 10 + r = 2$$

$$r = 6$$

Hence the graph should have 6 regions.



(a) Simple Graph



(a) Multi-Graph

Q. 4 : A connected planar graph has 9 vertices having degrees 2, 2, 2, 3, 3, 3, 4, 4 and 5. How many edges are there ?

Soln. :

By handshaking lemma

$$\sum d(v_i) = 2e$$

where $d(v_i)$ = degree of i th vertex
 e = number of edges

For given graph

$$2 + 2 + 2 + 3 + 3 + 3 + 4 + 4 + 5 = 2e$$
$$28 = 2e$$
$$e = 14$$

\ There are 14 edges.

Ex. 5 : Suppose that a connected planer graph has 20 vertices, each of degree 3
into how many regions does a representation of this plan graph split the plane ?

Soln. :

$$|V|=20 = \text{number of vertices}$$
$$\text{degree of each vertex} = 3$$

By hand shaking Lemma

$$\sum d(V_i) = 2e$$
$$20 \times 3 = 2e$$
$$\Rightarrow e = 30$$

By Euler's theorem,

$$|V| - |E| + |R| = 2$$
$$20 - 30 + |R| = 2$$
$$|R| = 12$$

Planar graph will split the plane in 12 regions.