MODULE 6
GRAPH THEORY

Varying Applications (examples)

- Computer networks -
  - where nodes represent computers and edges represent connections.
- Maps and Routes-
  - find shortest or optimal routes.
- Scheduling exams and assign channels to television stations
- Solve shortest path problems between cities
- Social Networks
- Traffic Management
- Project Scheduling

Topics Covered

Graphs and Subgraphs 05 CO4

6.1 Definitions, Paths and circuits,
Types of Graphs,
Eulerian and Hamiltonian

6.2 Planer graphs

6.3 Isomorphism of graphs

6.4 Subgraph

Definitions - Graph

A **graph** G consists of a finite set v of objects called **vertices,** a finite set E of objects called **edges** and a function γ that assigns to each edge, a subset {v, w} where v and w are vertices (and may be the same).

We will write $G = (V, E, \gamma)$

Graph

Let V={1, 2, 3, 4}

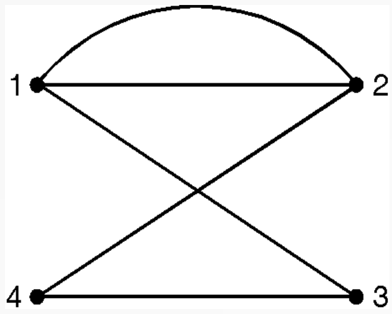
and E={ e_1, e_2, e_3, e_4, e_5}.

Let g be defined by

$\gamma(e_1) = \gamma(e_5) = \{1, 2\}$,

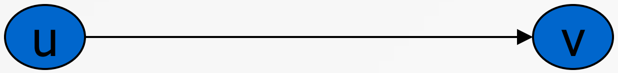
$\gamma(e_2) = \{4, 3\}$, $\gamma(e_3) = \{1, 3\}$, $\gamma(e_4) = \{2, 4\}$.

Then G={V, E, γ) is a graph as shown in

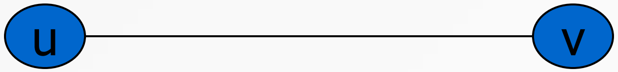


Definitions – Edge Type

Directed: Ordered pair of vertices. Represented as (u, v) directed from vertex u to v.



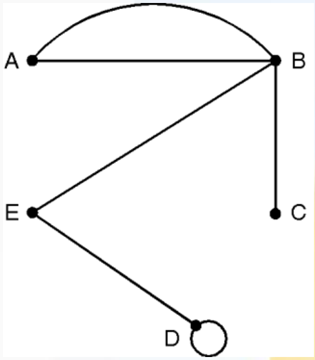
Undirected: Unordered pair of vertices. Represented as {u, v}. Disregards any sense of direction and treats both end vertices interchangeably.



Definitions

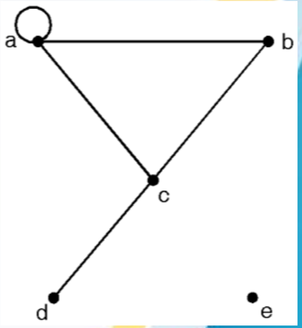
Degree :

The **degree** of a vertex is the number of edges having that vertex as an end point.



Loop :

A graph may contain an edge from a vertex to itself, such an edge is referred to as a **loop**. A loop contributes 2 to the degree of a vertex. Since that vertex serves as both end points of the loop.



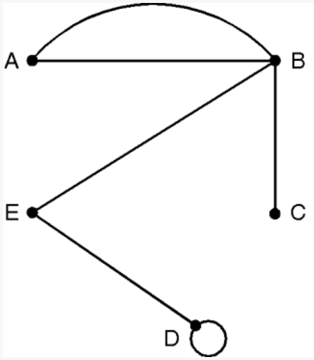
Isolated Vertex :

A vertex with degree 0 will be called an **isolated vertex**.

Adjacent Vertices :

A pair of vertices that determine an edge are **adjacent vertices**.

- Degree of each vertex:
- A=2
- B=4
- C=1
- D=3
- E=2



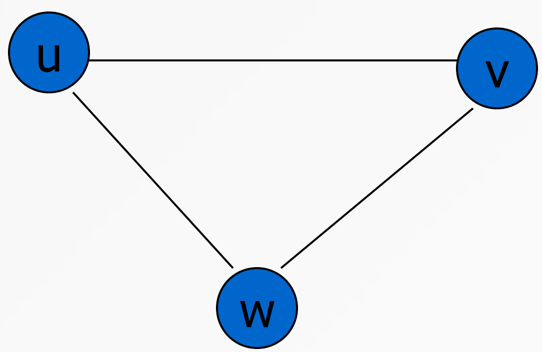
Note:

- If a vertex has one loop,then
- Degree of vertex
- =(number of non-loop edges) +2×(number of loops)

Simple undirected graph

Simple (Undirected) Graph: consists of V, a nonempty set of vertices, and E, a set of unordered pairs of distinct elements of V called edges (undirected)

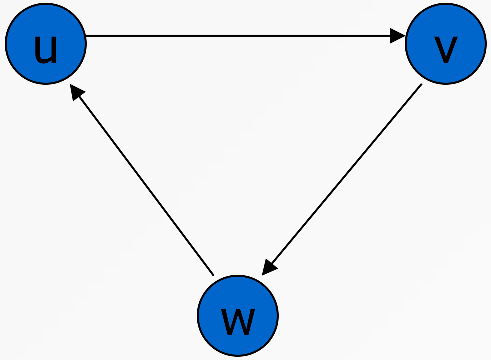
Representation Example: G(V, E), V = {u, v, w}, E = {{u, v}, {v, w}, {u, w}}



Directed graph

Directed Graph: G(V, E), set of vertices V, and set of Edges E, that are ordered pair of elements of V (directed edges)

Representation Example: G(V, E), V = {u, v, w}, E = {(u, v), (v, w), (w, u)}



Terminology – Directed graphs

In-degree (u): number of in coming edges

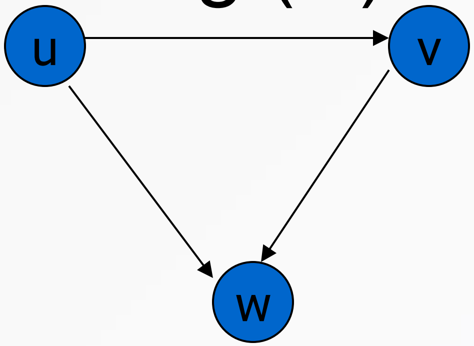
Out-degree (u): number of outgoing edges

Representation Example: For V = {u, v, w} ,
E = { (u, w), (v, w), (u, v) },

indeg(u) = 0, outdeg (u) = 2,

indeg(v) = 1,outdeg(v)=1

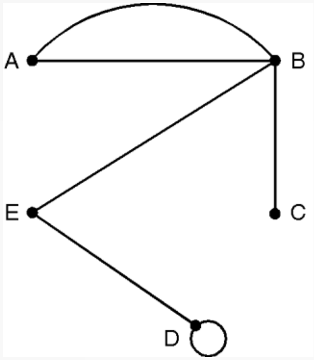
indeg(w) = 2, outdeg (w) = 0



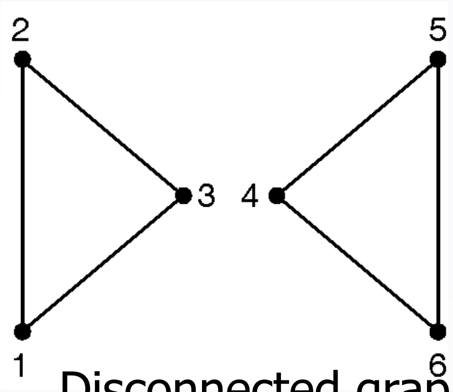
Connected graph

A graph is called **connected** if there is a path from any vertex to any other vertex in the graph.

Otherwise, the graph is **disconnected**. If the graph is disconnected, the various connected pieces are called the **components** of the graph.



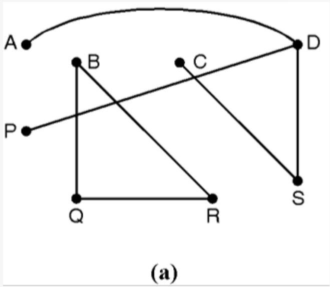
Connected graph



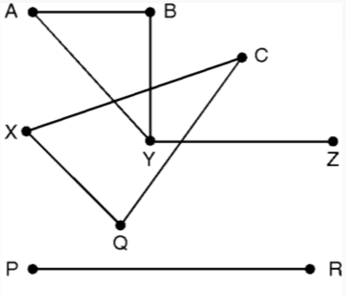
Disconnected graph

Problem

Determine whether the graph (shown in Fig.) is connected or disconnected. If disconnected find its connected component.



(a)



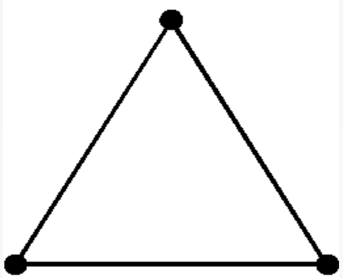
(b)

(a)Graph shown in Fig. (a) is not connected its connected components are {A, D, P, S, C} and {B, Q, R}
(b)Graph shown in Fig. (b) is not connected its connected components are {A, B, Y, Z}, {C, X, Q}, {P, R}

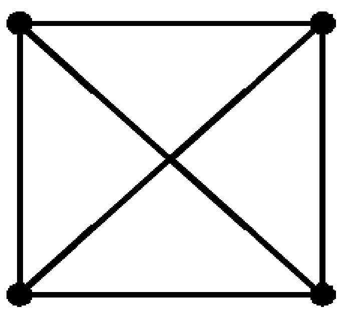
Complete Graph

For each integer $n \geq 1$, let K_n denote the graph with vertices $\{v_1, v_2, \ldots v_n\}$ and with an edge $\{v_i, v_j\}$ for every i and j.

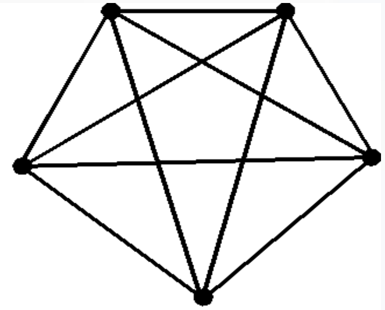
In other words, every vertex in K_n is connected to every other vertex. In



K_3



K_4



K_5

More Graphs

Linear graph:

L_2

L_4

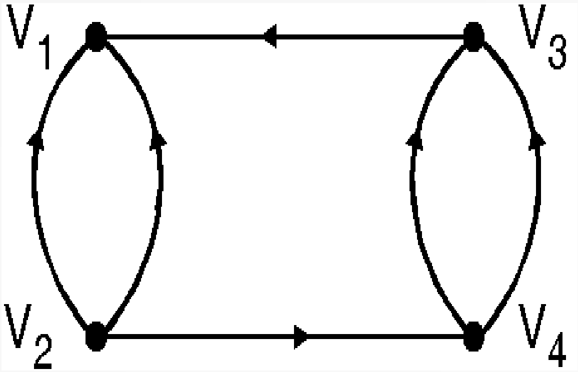
Discrete Graph:

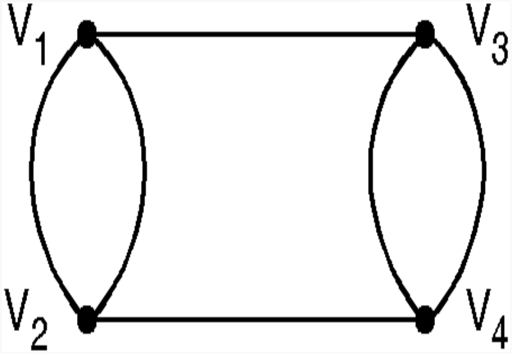
D_2

D_5

Multigraph

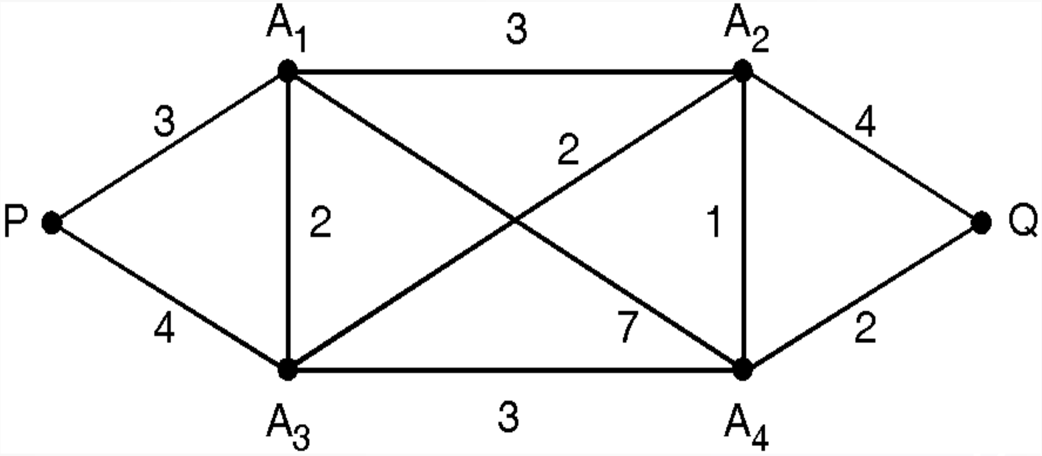
Directed graph having multiple edges between two vertices is called as **multigraph**. Undirected graph having more than one edge between two vertices is also called as **Multigraph**.





Labelled and weighted graph

A graph G is called a **labelled graph** it its edges and /or vertices are assigned data of one kind or another. In particular, G is called a **weighted graph** if each edge 'e' of G is assigned a non–negative number called the weight or length of V.



Subgraph

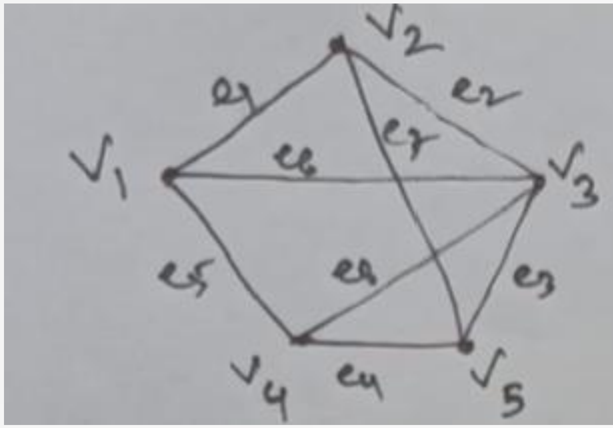
- Given two subgraphs G and G1, we say that G1 is a subgraph of G if the following conditions hold:

1. All the vertices and edges of G1 are in G,

G(V,E) and G1(V1,E1) such as v1 subset v, E1 is subset of E.

1. Each edge of G1 has the same end vertices in G as in G1.

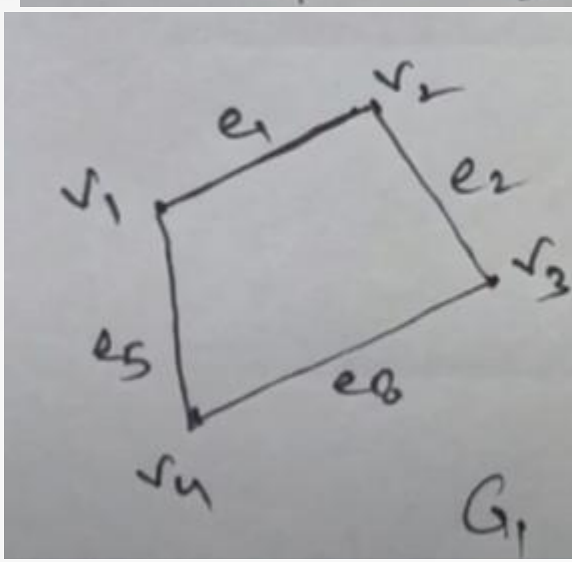
Example



$G(V, E)$

$V = \{v_1, v_2, v_3, v_4, v_5\}$

$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$



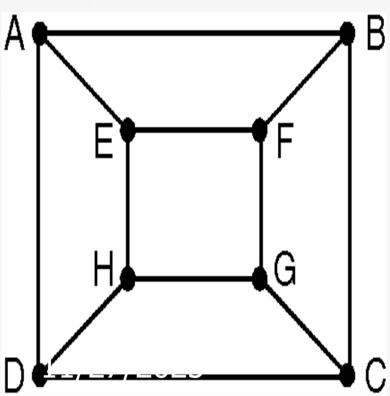
$V_1 = \{v_1, v_2, v_3, v_4\} \subseteq V$

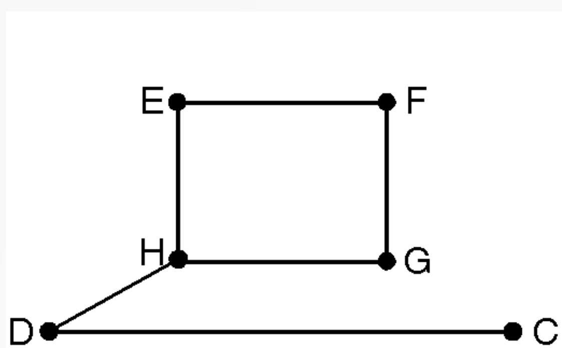
$E_1 = \{e_1, e_2, e_5, e_8\} \subseteq E$

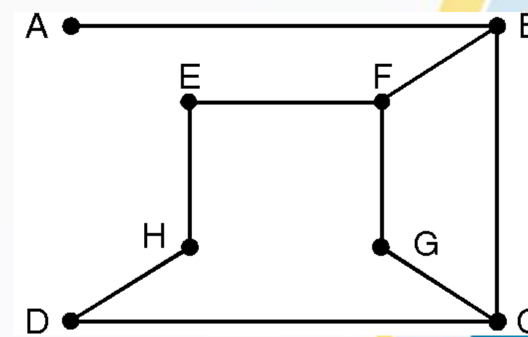
G1 is a subgraph of G

Subgraph Example

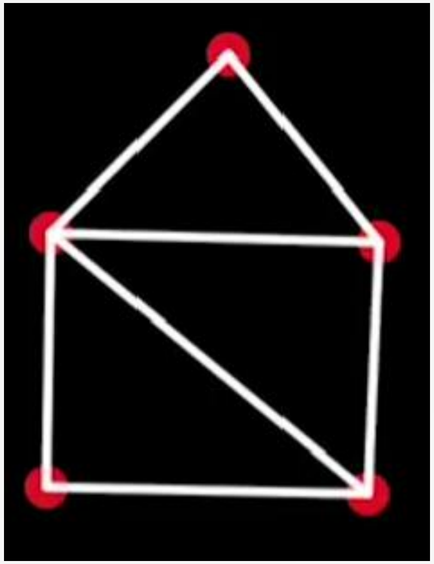
Let G = (V, E, γ) is a graph. Choose a subset E_1 of the edges in E and a subset V_1 of the vertices in V. So that V_1 contains all the end points of edges in E_1. Then H = (V_1, E_1, γ_1) is also a graph, where γ_1 is γrestricted to edges in E1. Such a graph H is called a **subgraph** of G.

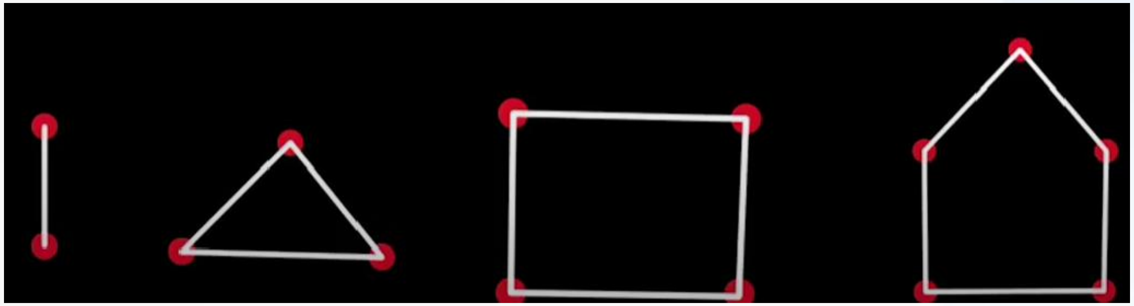


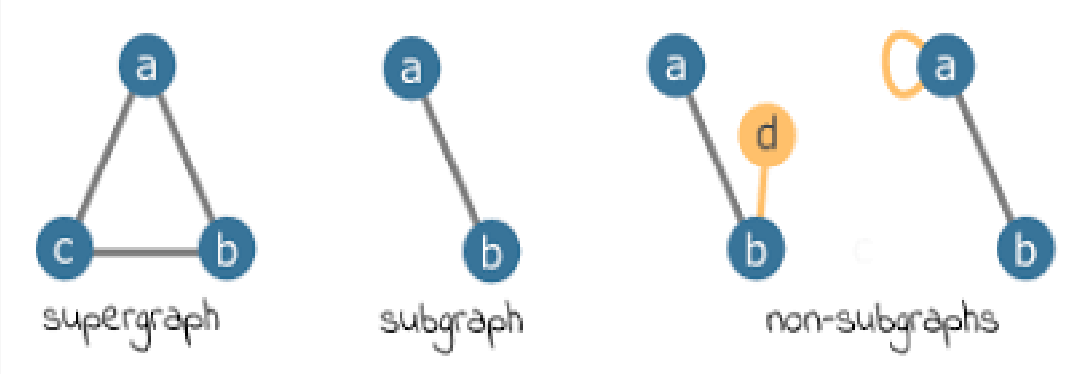




Example

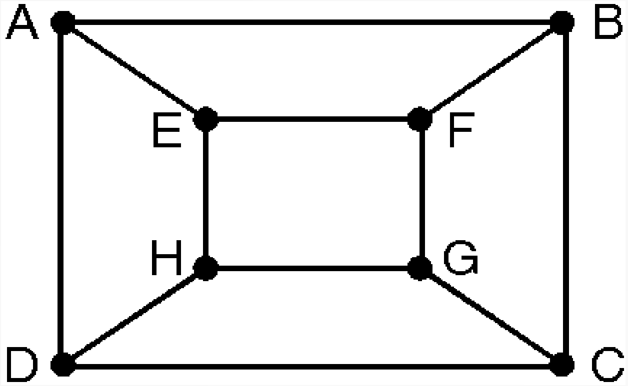


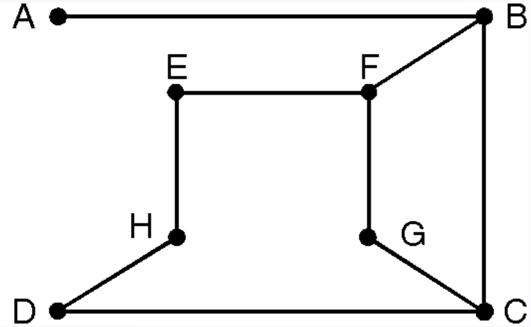




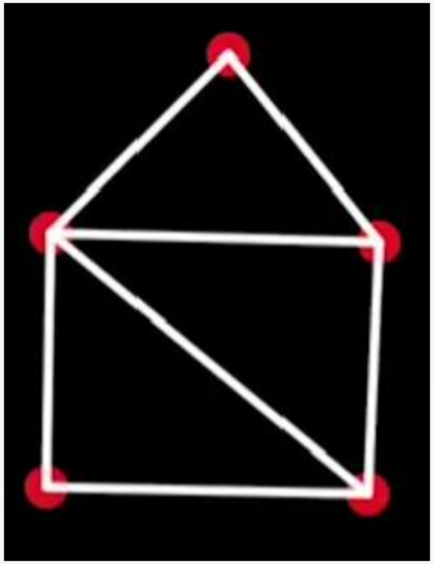
Spanning Subgraph

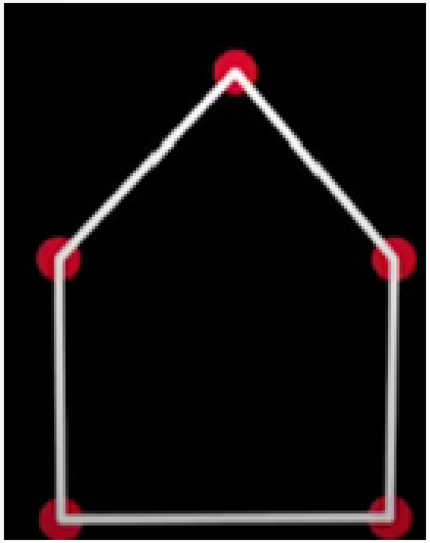
A subgraph is said to be **spanning subgraph** if it contains all the vertices of G.





Example



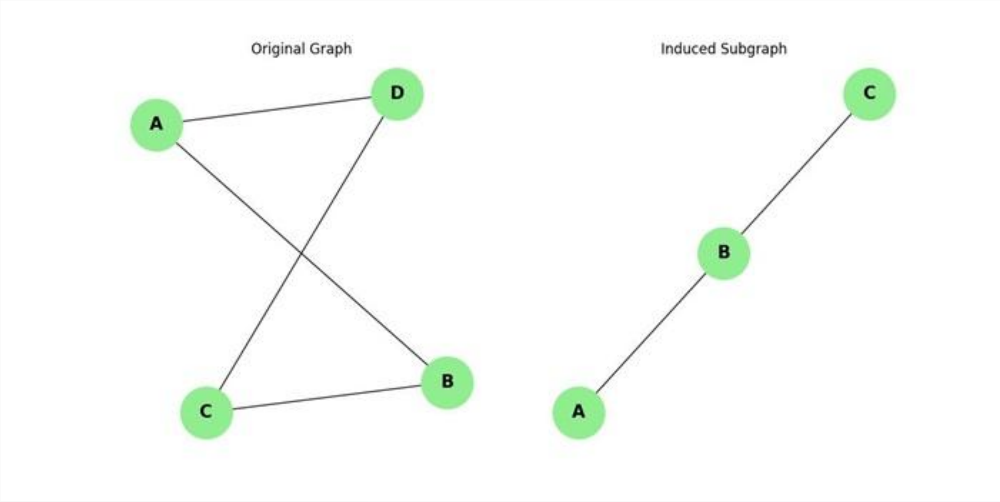


Induced Subgraph

- an **induced subgraph** is a subgraph formed by **choosing some vertices** from the original graph **and keeping all the edges that connect those chosen vertices.**
- In other words, an **induced subgraph** contains **all edges between the selected vertices** that were present in the original graph.
- **Example:**
- Suppose the main graph has
  Vertices: A, B, C, D
  Edges: AB, AC, BD, CD
- Now, if we pick the vertices **A, B, and C**,
  then the **induced subgraph** will include **all edges among A, B, and C** that exist in the original graph.

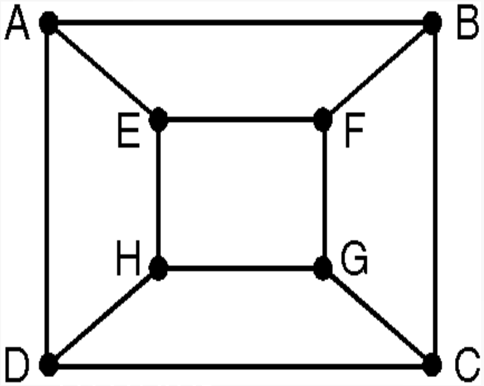
Example

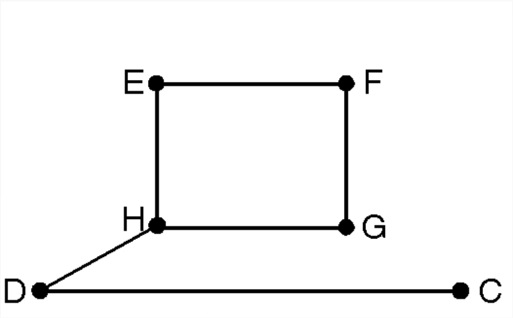
- If we have a graph with vertices
- V = {A, B, C, D} and
- edges E = {(A-B), (B-C), (C-D), (A-D)},
- selecting the vertices {A, B, C} will give the induced subgraph that contains edges {(A-B), (B-C)} (since those are the only edges connecting vertices in {A, B, C} in the original graph)

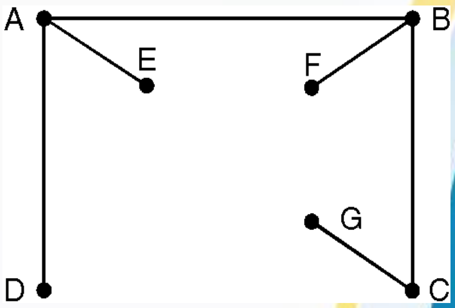


Complement of Subgraph

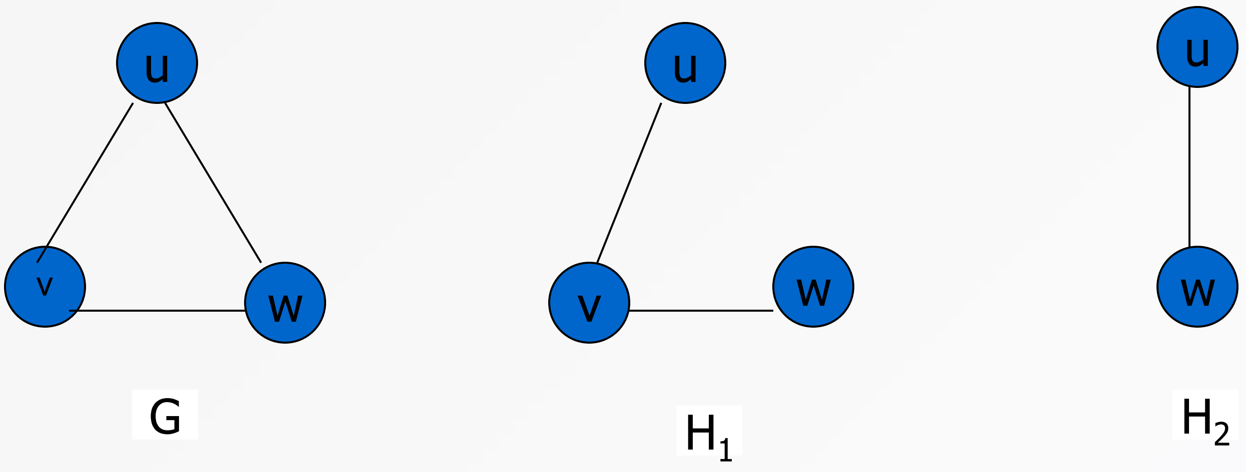
The complement of a subgraph G' = (V', E') with respect to the graph G = (V, E) is another subgraph
G" = (V", E") such that E" is equal to E – E' and V" contains only the vertices with which the edges in E" are incident.



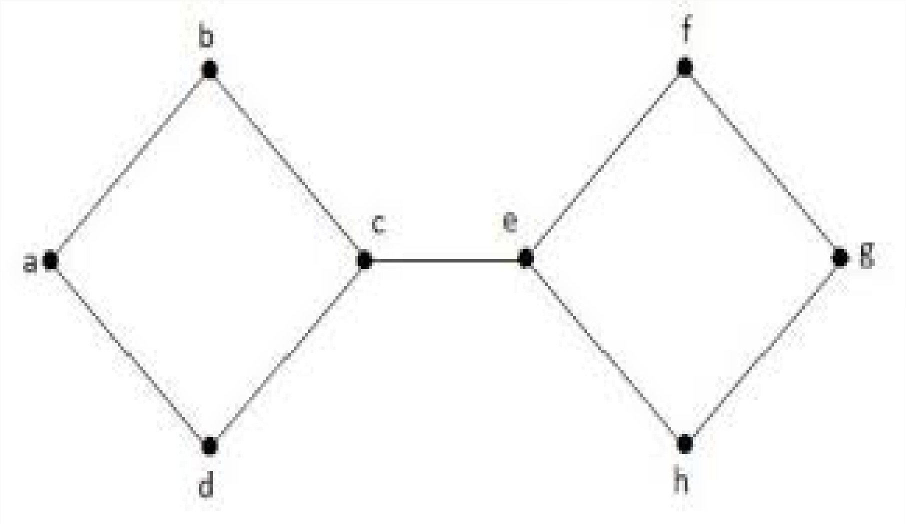




Complement of Subgraph

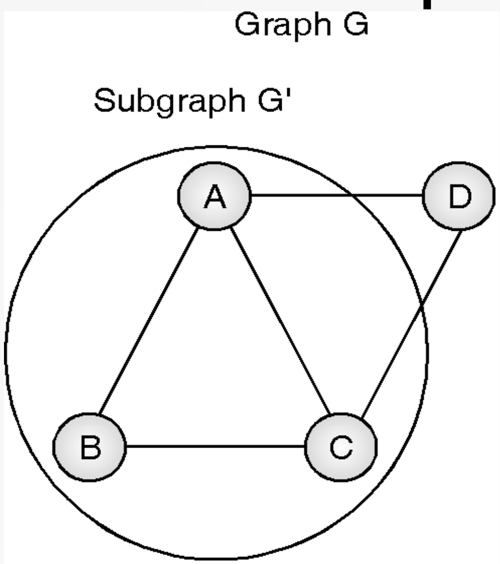


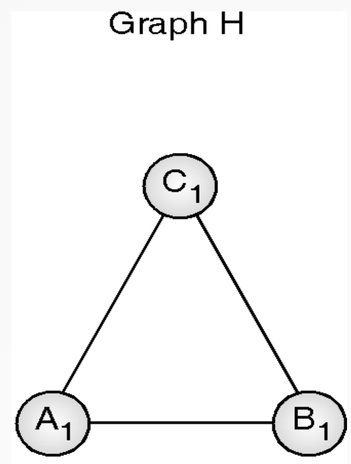
cut edge and cut vertex



Subgraph Isomorphism

The ***subgraph isomorphism*** is a computational task in which two graphs G and H are given as input, and one must determine whether G contains a subgraph that is isomorphic to H.





$f : \{(A, C_1), (B, A_1), (C, B_1)\}$

Handshaking Lemma

Consider a graph G with e number of edges and n number of vertices. Since each edge contributes two degrees, the sum of the degrees of all vertices in G is twice the number of edges in G i.e.

$$\sum \quad d(\,vi) = 2\,e$$

Problem

How many nodes are necessary to construct a graph with exactly 6 edges in which each node is of degree 2.

Soln. : Suppose there are n vertices in the graph with 6 edges. Also, given the degree of each vertex is 2. Therefore by handshaking lemma,

$$\Sigma d(v_i) = 2e = 2 \times 6$$

$$\Rightarrow \quad d(v_1) + d(v_2) + \ldots + d(v_n) = 12.$$

$$\Rightarrow \quad = \quad 12$$

$$\Rightarrow \quad 2n = 12$$

$$\Rightarrow \quad n = 6$$

Hence, 6 nodes are required to construct a graph with 6 edges in which each node is of degree 2.

Problem

Determine the number of edges in a graph with 6 nodes, 2 of degree 4 and 4 of degree 2. Draw two such graphs.

Soln: Suppose the graph with 6 vertices has e number of edges. Therefore, by handshaking lemma.

$$\Sigma d(v_i) = 2e$$

$$\Rightarrow \quad d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + d(v_6) = 2e$$

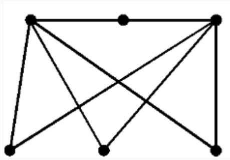
Now, given 2 vertices are of degree 4 and 4 vertices are of degree 2.

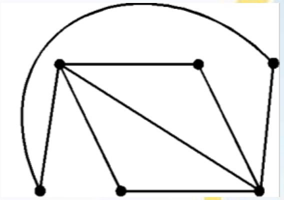
Hence from the above equation

$$\Rightarrow \quad (4 + 4) + (2 + 2 + 2 + 2) = 2e$$

$$\Rightarrow \quad 16 = 2e$$

$$\Rightarrow \quad e = 8$$





Path & Circuit

Path : A path is a sequence of vertices where no edge is chosen more than once

A path is called simple if no vertex repeats more than once

Length of Path : Number of edges in a path is called as length of path

Circuit: A circuit is a path that begins and ends with the same vertex.

Example:

In the graph below,

if we have vertices A,B,C,D

then A→B→C→D s a path.

Euler path and Euler circuit

EULER PATH

- A path in a graph G is called an Euler path if it includes every edge exactly once

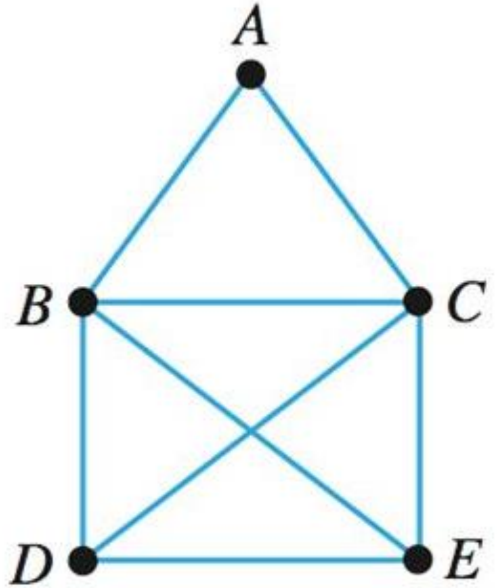
EULER CIRCUIT

- A Euler path that is a circuit

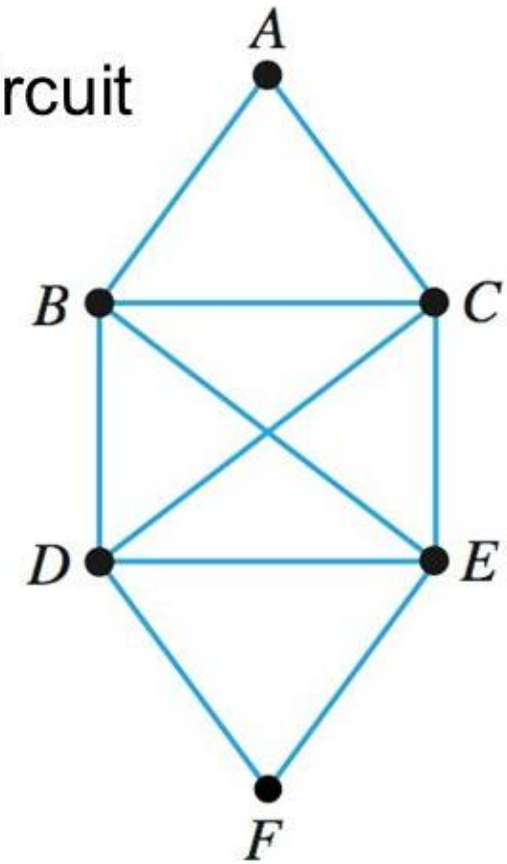
Example

- Euler path

D, E, B, C, A, B, D, C, E

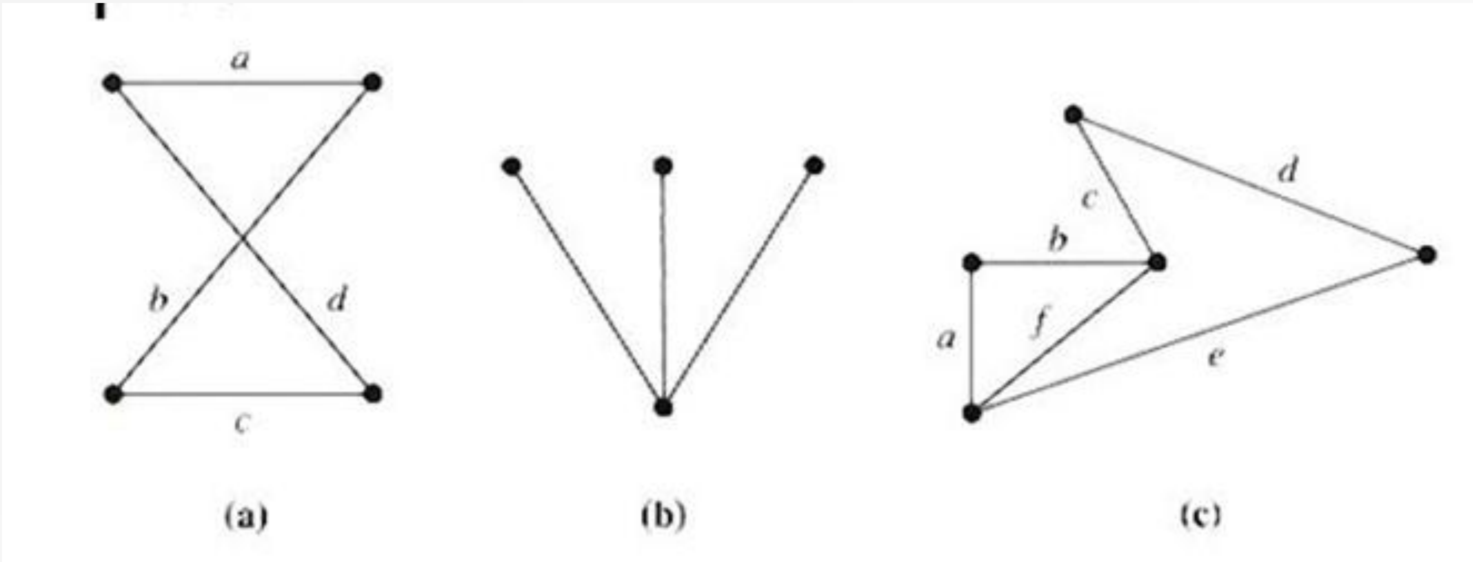


- Euler circuit



D, E, B, C, A, B, D, C, E, F, D

Identify Euler path and circuit

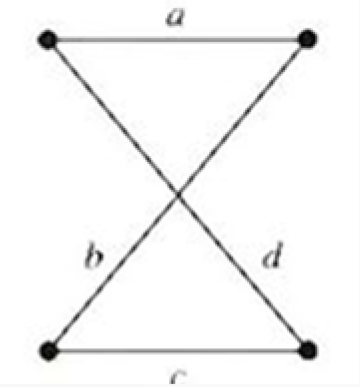


- The path a, b, c, d in (a) is an Euler circuit since all edges are included exactly once.
- The graph (b) has neither an Euler path nor circuit.
- The graph (c) has an Euler path a, b, c, d, e, f but not an Euler circuit.

Theorem: EULER CIRCUIT

A) If graph G has a vertex of odd degree , then there can be no Euler circuit in G

B) If G is a connected graph and every vertex has an even degree then there is a Euler circuit in G



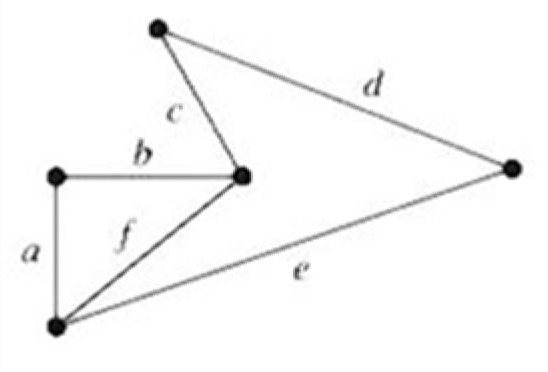
All vertices have even degree

Theorem: EULER PATH

A) If a graph G has more than two vertices of odd degree then there can be no Euler path in G

B) If G is connected and has exactly two vertices of odd degree then there is a Euler path in G

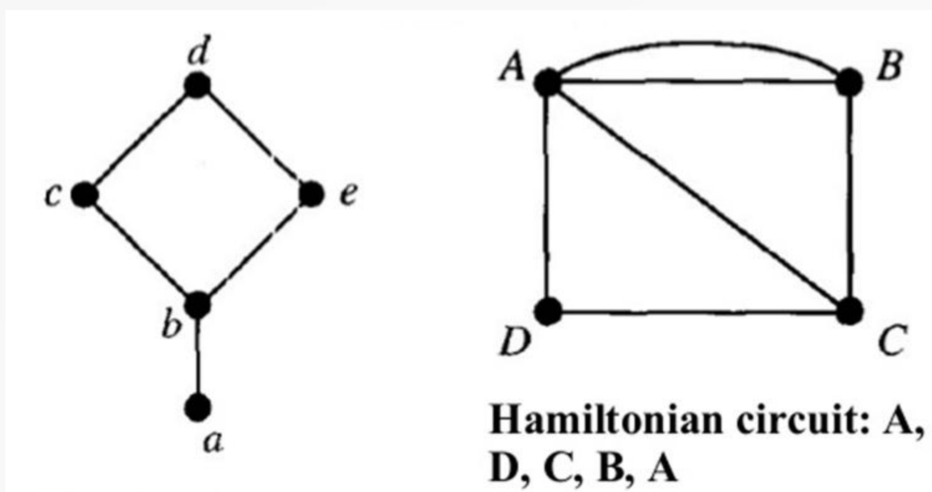
All vertices have odd degree



Two vertices have odd degree

Hamiltonian Path & Circuit

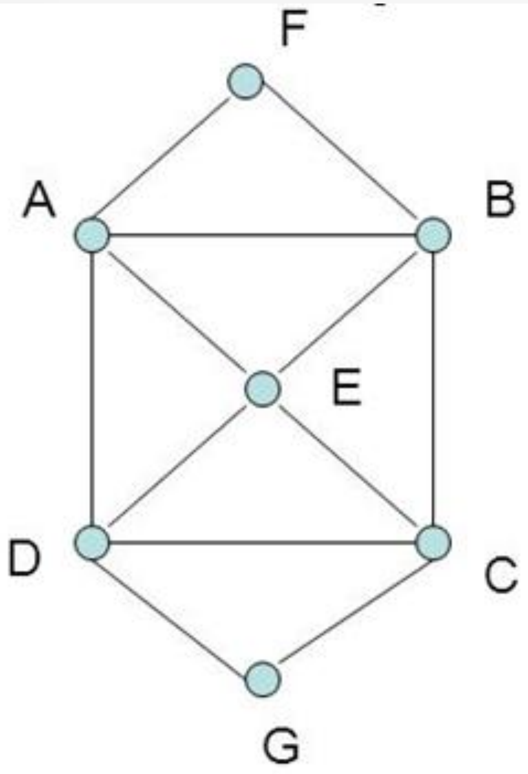
- A Hamiltonian path contains each vertex exactly once
- A Hamiltonian circuit is a circuit that contains each vertex exactly once except for the first vertex which is also the last



Hamiltonian path: a, b, c, d, e

Hamiltonian circuit: A, D, C, B, A

Example



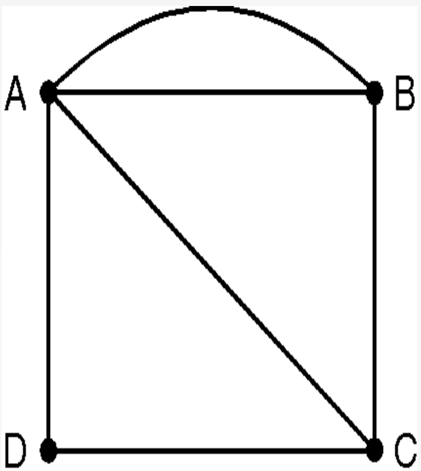
Has many **Hamilton circuits**:

1) A, F, B, E, C, G, D, A
2) A, F, B, C, G, D, E, A

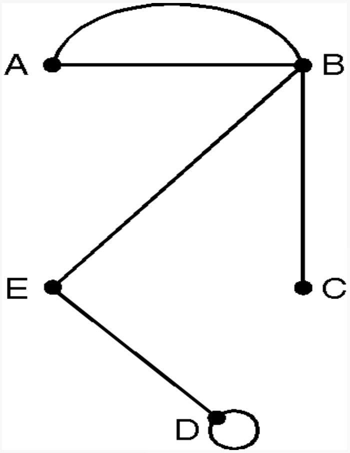
Has many **Hamilton paths**:

1) A, F, B, E, C, G, D
2) A, F, B, C, G, D, E

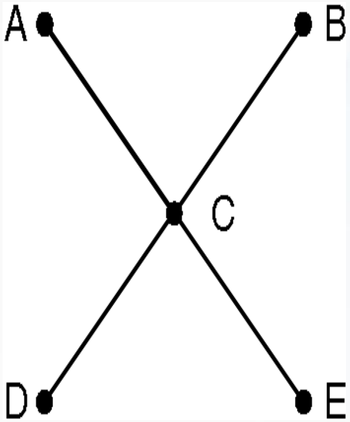
Examples



Hamiltonian
Circuit:ADCBA
Euler path:
B,A,C,D,A,B,C



Hamiltonian Path:no
Euler
path:D,D,E,B,A,B,C



Hamiltonian
Path:no
Euler path:NO

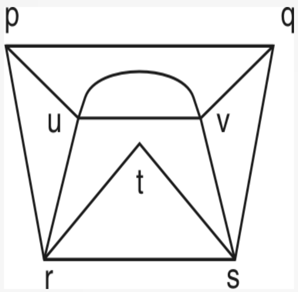
Theorem: HAMILTONIAN CIRCUIT

A) G has a Hamiltonian circuit if for any two vertices u and v of G that are not adjacent ,degree(u)+degree(v) ≥ nos of vertices

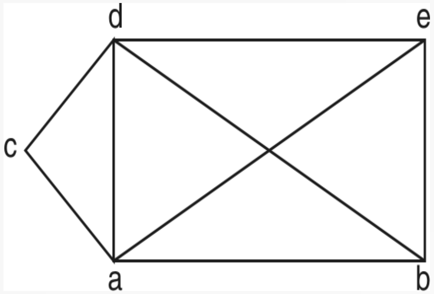
B) G has a Hamiltonian circuit if each vertex has degree greater than or equal to n/2

Problem

Determine the Eulerian and Hamiltonian path, if exists, in the following graphs.

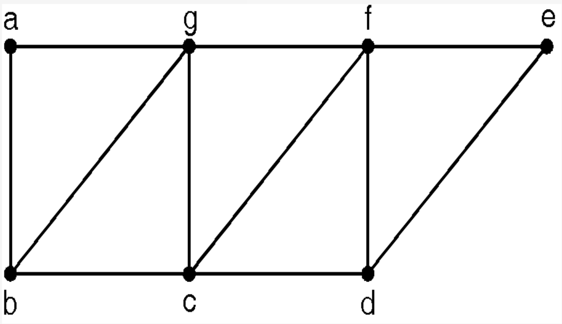


Hamiltonian path : p, u, v, q, s, t, r
Hamiltonian circuit : r, p, u, v, q, s, t, r



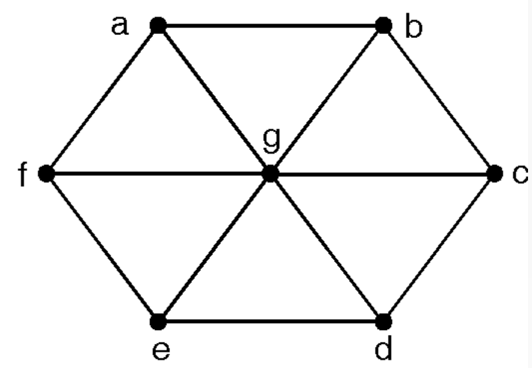
Hamiltonian path : c, d, e, b, a
Hamiltonian circuit : c, d, e , b, a, c

Identify Euler path, circuit, Hamiltonian path and circuit



(a)

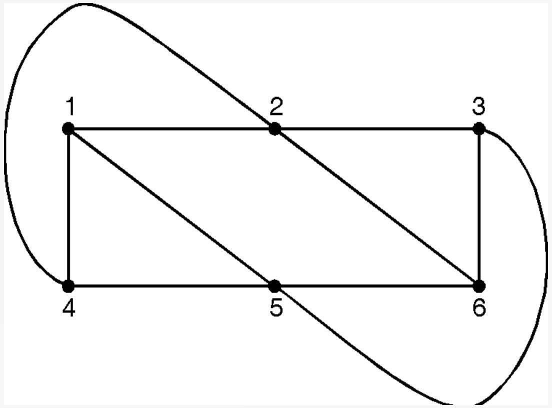
(a) two vertices b and d have odd degree.
Hence there is an Euler path.
π: b, a, g, f, e, d, c, b, g, c, f, d



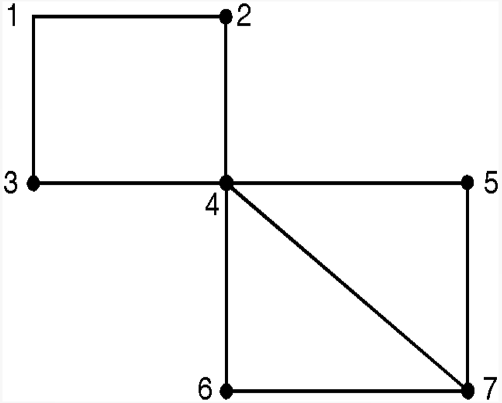
(b)

(b)6 vertices have odd degree, 3 and 1 vertex of even degree, 6.
So Euler path does not exist in this graph.

Identify Euler path, circuit, Hamiltonian path and circuit

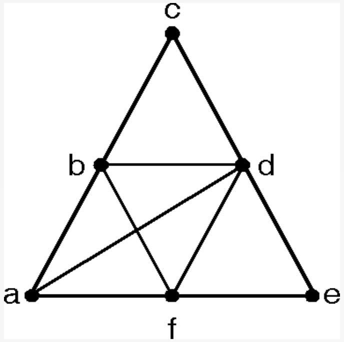


Number of vertices is 6. Each vertex has degree greater than equal to 6/2. So there is an Hamiltonian circuit.
π : 1, 4, 5, 6, 3, 2, 1

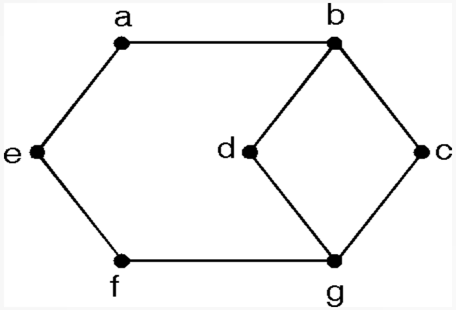


There is no Hamiltonian circuit.
But there is an Hamiltonian path
π: 3, 1, 2, 4, 6, 7, 5.

Identify Euler path, circuit, Hamiltonian path and circuit



(i) Eulerian Path : π: a, b, c, d, b, f, d, a, f, e, d
G has 2 vertices of odd degree.
Hamiltonian Circuit : a, b, c, d, e, f, a.
Hamiltonian Path : a, b, c, d, e, f



(ii) Eulerian Circuit : -
Eulerian Path : g, d, b, a, e, f, g, c, b.
Hamiltonian Path : d, b, a, e, f, g, c