Automatic Control Laboratory Task
Control System Design of Coupled Tanks

Ashwin Suresh Nedungadi

Faculty of Electrical Engineering

Czech Technical University, Prague

nedunash@fel.cvut.cz

Öykü Tansel

Faculty of Electrical Engineering

Czech Technical University, Prague

e216728@metu.edu.tr

Abstract—This laboratory assignment aims to analyze and implement a control system for coupled water tanks. The first part of the assignment consists of analyzing the system and modeling it mathematically, then designing a suitable controller for it.

Keywords—Control Engineering, Controller Design, Linearization, Parameter Estimation, State-Feedback.

I. Introduction

THE Coupled water tank system consists of two water tanks and a gear pump that pumps water to the first water tank on the left. A valve V_t connects the two tanks and another valve V_{o2} connects the second tank to the water reservoir. There is also another valve V_{1o} which connects the first tank to the water reservoir.
The coupled water tank system is a non-linear stable system which has one control input u[V] which is the voltage applied on the gear pump and three disturbing inputs, positions V_{o1}, V_t and V_{o2} respectively.
The value of these inputs varies from 0 when the valves are fully opened to 1 when the values are fully closed. The system has two outputs, the water levels of both the tanks h_1 [m] and h_2 [m].

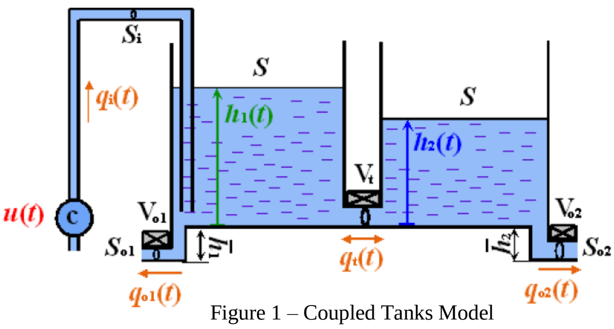

Figure 1 – Coupled Tanks Model

II. Modelling

In this model of the water tanks, we will assume that the change of water level is slower than the rate of water flowing out by ignoring the internal dynamics of our pump.
Then, the mathematical model of the water tank system can be represented by two non-linear equations:

$$S\dot{h_1} = k_c u(t) - v_t\, S_t sgn\big(h_1(t) - h_2(t)\big)\sqrt{2g|h_1(t) - h_2(t)|} - v_{01}\, S_{01}\sqrt{2g(h_1(t) + \overline{h_1})} \quad (1)$$

$$S\dot{h_2} = v_t\, S_t sgn\big(h_1(t) - h_2(t)\big)\sqrt{2g|h_1(t) - h_2(t)|} - v_{02}\, S_{02}\sqrt{2g(h_2(t) + \overline{h_2})} \quad (2)$$

Where S [m²] is the surface area of both tanks, S_{o1} & S_{o2} and S_t [m²] are the surface area of the valve intakes and V_{o1}, V_{o2} and V_t are the valve constants respectively, k_c [$m^3\, s^{-1}\, V^{-1}$] is the parameter of the pump. g [$m\, s^{-2}$] is the gravitational acceleration and $\overline{h_1}$, $\overline{h_2}$ are the distance errors between the bottom of the tanks and the individual sensors measuring water level in the tanks. We choose to neglect them as they are very small.

A. State Space Representation

$$x = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \quad \dot{x} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \quad y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x$$

B. Linearization with Jacobian Matrix

We can linearize the non linear equations (1) & (2) by getting rid of some non-linear functions like the Sgn and mod by keeping constraints like $h_1>h_2$ and then taking it's jacobian matrix to obtain the linearized equations as below.

$$\dot{h1} = \alpha\, u(t) - \varphi\, Vt\, \sqrt{2g(h1 - h2)} - \beta\, V01\, \sqrt{2gh1}$$

$$\dot{h2} = \varphi\, Vt\, \sqrt{2g(h1 - h2)} - \gamma\, V02\sqrt{2gh2}$$

Where $\alpha = \frac{Kc}{S}$; $\beta = \frac{S01}{S}$; $\gamma = \frac{S02}{S}$; $\varphi = \frac{St}{S}$

A=

$$\begin{bmatrix} -\frac{v_t \cdot S_t \cdot g}{S.\sqrt{2g(h_1-h_2)}} - \frac{v_{01} \cdot S_{01} \cdot g}{S.\sqrt{2g(h_1+\bar{h}_1)}} & \frac{v_t \cdot S_t \cdot g}{S.\sqrt{2g(h_1-h_2)}} \\ \frac{v_t \cdot S_t \cdot g}{S.\sqrt{2g(h_1-h_2)}} & -\frac{v_t \cdot S_t \cdot g}{S.\sqrt{2g(h_1-h_2)}} - \frac{v_{02} \cdot S_{02} \cdot g}{S.\sqrt{2g(h_2+\bar{h}_2)}} \end{bmatrix}$$

$$B = \begin{bmatrix} \alpha & \varphi\sqrt{2gh1} & 0 & -\gamma\sqrt{2g(h1-h2)} \\ 0 & 0 & -\beta\sqrt{2gh2} & \gamma\sqrt{2g(h1-h2)} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

C. Static Nonlinearities & Deadbands

Our model contains static nonlinearities such as the signum function which can be eliminated by taking h_1 & h_2 always greater than zero. We also have the absolute function which can be eliminated with the condition $h_1>h_2$.

It is also obvious that there are dead bands in our system due to the pumps. Any input pump voltage less than 1.0 V has no effect on the output and greater than 8.0V.

The Proportional valve can take any value between 0 & 1.

D. Parameter Estimation

There are different methods for estimating parameters in a control model.

The one we used was to adjust the inputs such that the height of our second tank would not change much. This way we can eliminate certain parts of our linearized $\dot{h}1$ & $\dot{h}2$ equations.

In our setup the values of the inputs were as follows.

$V_t = 1$; $V_{01} = 0.5$; $V_{02} = 0$; $u(t) = 1.2$ V

We can numerically calculate the values of α & φ using this method.

For β & γ we can measure the values of S, S_{01} and S_{02} from the lab apparatus using a caliper and scale.

Below is the measurement graph corresponding to the control inputs above.

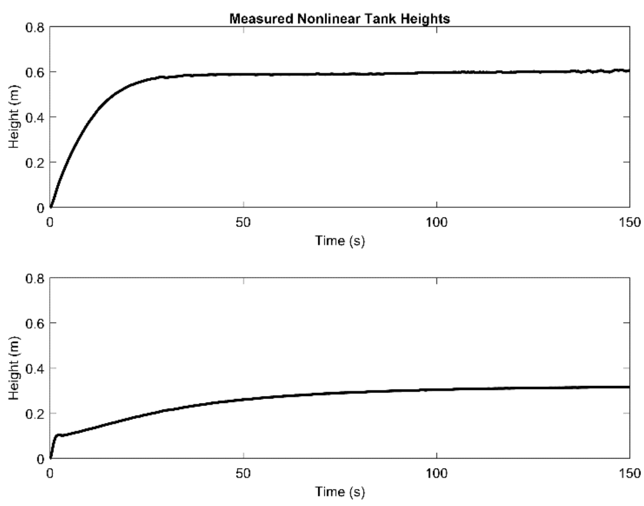

Figure 2 – Nonlinear Tank Heights

We used Matlab to automate numerical calculations for us and received the following parameters from the linearized matrices.

$$\alpha = 0.0592 \,;\, \beta = 0.0673$$

$$\gamma = 0.0911 \,;\, \varphi = 0.0069$$

When we plug in these parameters and calculate the linearized matrices using Matlab again, we obtain the linearized state space representation as below.

$$A= \begin{bmatrix} -0.1244 & 0.0341 \\ 0.0341 & -0.0625 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0592 & 0.0237 & 0 & -0.2168 \\ 0 & 0 & -0.1667 & 0.2168 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The linearized model is stable with eigenvalues of A matrix being **-0.1395** and **-0.0474** respectively.

We can also see the stability of the linearized model from the step responses from the next part of the assignment.

III. Responses

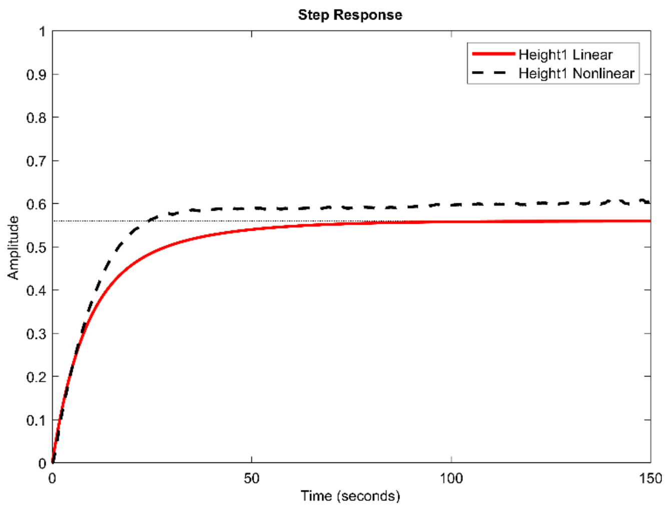

Figure 3 – Comparison of Responses for h1

As we can see from the graph, the linearized model closely approximates the nonlinear real world model.

However, the gain of the modeled first tank is slightly lesser but it’s the closest we can get using our approximation method.

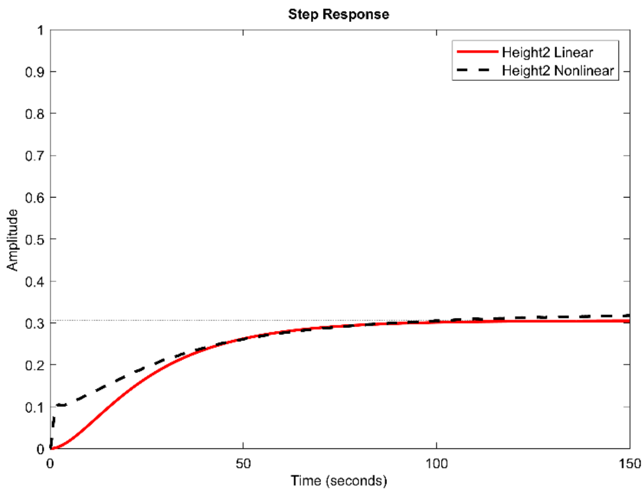

Figure 4 – Comparison of Responses for h2

From the step response of the second tank, we can see that we have closely modeled the real world system while getting rid of the nonlinear bump.
Although it's worth noting that this non-linear bump is entirely caused by $\overline{h_1}$ and $\overline{h_2}$ which we could have compensated for but didn't.

From both our responses we can be confident that we have modeled the system correctly and continue to design a suitable controller for our control model.

IV. CONTROLLER DESIGN

For designing a suitable controller for our task, we determined that a state-feedback controller would be the simplest and optimal choice of controller as it can control the input of the pump which in turn can regulate the heights of the tanks.

The first thing we need to check is weather our system is controllable. Our A & B controllability matrices have a rank of 3 which is greater than our A matrix. So we are good to go.

Next, we have to find the vector of state-feedback control gain K. For this, we can use the linear quadratic regulation method using the lqr function in MATLAB.

The LQR method takes matrices A, B, Q, and R as inputs.
Q and R determine the penalty and control expenditure matrices for the controller.
With trial and error and by experimenting with our matrices we found the optimal Q and R matrices to be:

$$Q = \begin{bmatrix} 100000 & 0 & 0 \\ 0 & 100000 & 0 \\ 0 & 0 & 100000 \end{bmatrix}$$

$$R = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$$

We then obtain our K vector to be,

$$K = [\ 3.1623 \quad 3.7631 \quad 25.9446]$$

Our new closed loop system becomes:

$$Ar = \begin{bmatrix} 0 & 0 & 1 \\ -0.1873 & -0.3473 & -1.5025 \\ 0 & 0.0341 & -0.0625 \end{bmatrix}$$

$$Br = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$Cr = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

We can then analyze the response of our new CL system,

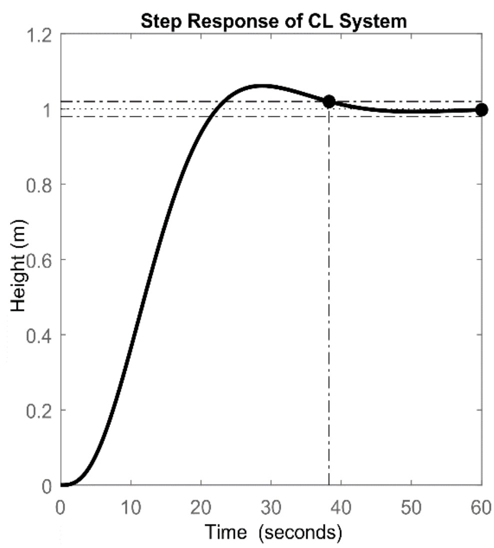

Figure 5 – Closed Loop Response

From the CL response, we can see that the overshoot is around 2% and settling time for the system is 38.8 s.

The modified Simulink model corresponding to our new CL system is shown below:

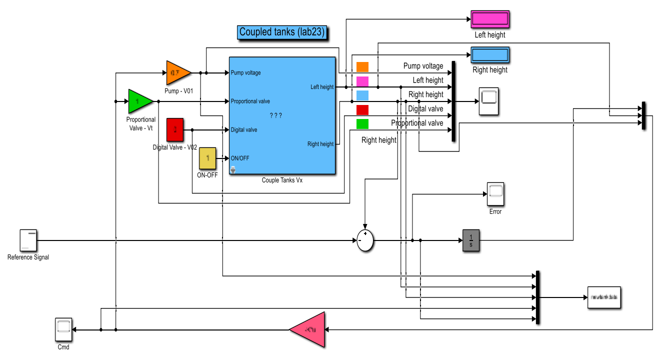

Figure 6 – Simulink Model

Running our experiment with the controller hooked up, we collected some measurements and plotted the results.

The input voltage increases steadily and then decreases until our h_2 is within our set reference limit.

As the controller is not perfect due to our eigenvalues being places close to the imaginary axis, it oscillates in the order of 0.01 m of water height.

But otherwise our error signal achieves zero error in around 50 s.

The overshoot of our second tank height h_2 is approximately 5% but the settling time is high at around 50 s to settling to the final value of 0.18 m.

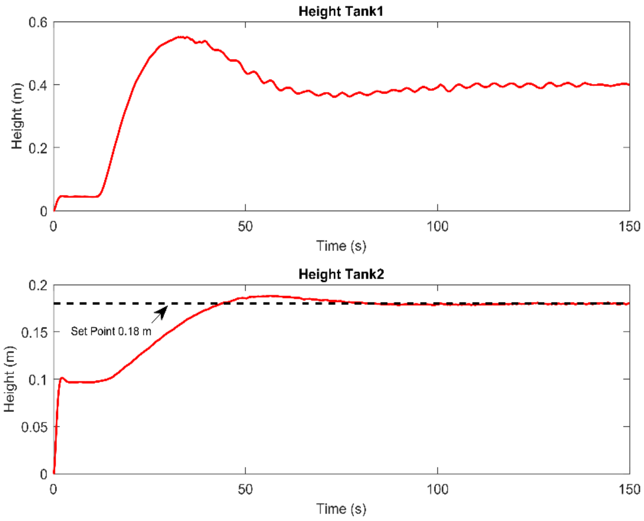

Figure 7 – Controlled Heights

Another important observation is that our system's transfer function contains a positive zero and hence shows the non-minimum phase behavior.

As this in no way affects our model as the input can only be a positive voltage of the pump, It's perfectly fine and just contributes a little towards the delay in settling time.

However, we wouldn't want to design an aircraft controller with such behavior.

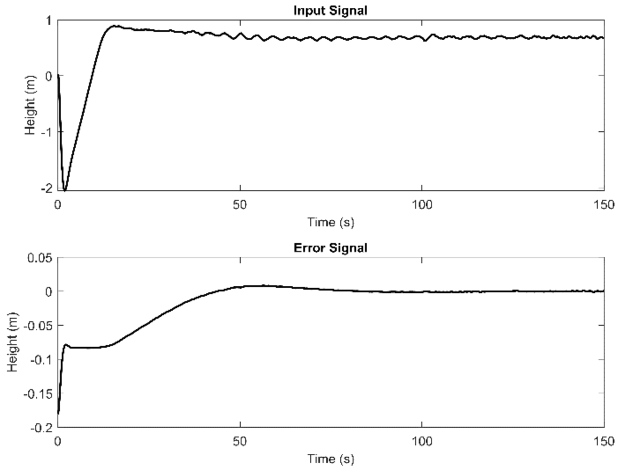

Figure 8 – Input & Error Signal

To conclude this project assignment, we have successfully modeled the coupled tank, linearized the mathematical model and then using suitable methods, estimated its parameters to then design a state-feedback controller with an overshoot of 5% and settling time of approximately 50s in Matlab.

We decided that the optimal controller to design would be a state-feedback controller that controls the pump in order to control the height of the first tank h_1 which would then control the second tank h_2 and stabilize it around our set point or reference signal.

The only drawback with this controller is a higher settling time due to the cascaded control hierarchy. We also didn't estimate the state space matrix perfectly and this added to the settling time. Moreover, the input is oscillatory as the controller switches the pump on and off to compensate for the drop in the tank heights and overflow, Hence the output also has a oscillatory behavior in the order of 0.01m .

Our final results are better than the design requirements of 20% overshoot and considering that we are not directly controlling the height of h_2 50s of settling time seems reasonable. In the future it would be better to model the system more accurately as it is the hardest to get right and goes a long way in how the final system works.