PROOF OF THE IRRATIONALITY OF $\sqrt{2} + \sqrt{3}$

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Lemma A: $\forall\ m \in$ Z, if m^2 is divisible by 6, then m is divisible by 6.

Let's prove the contrapositive: for all m $\in$ Z, if m is not divisible by 6, then m^2 is not divisible by 6.

[Proof]

Let m be a particular but arbitrarily chosen integer such that m is not divisible by 6.

By applying the Division Algorithm with d=6 ,

$\forall$ m $\in$ Z, m = 6q or m = 6q+1, or m=6q+2, or m=6q+3, or m=6q+4, or m=6q+5.

Note that the remainder (r) must be strictly less than the divisor (d). And r must be greater than or equal to 0. Therefore, these cases span all the possible values for m.

By assumption, though, m is not divisible by 6, so r ≠ 0. Therefore, for all integers, there are only 5 possible cases in which m is not divisible by 6.

Case 1:

m = 6q+1

We need to show that for this particular case, m^2 is not divisible by 6.

By substitution,

$m^2 = (6q + 1)^2$

$m^2 = 36q^2 + 12q + 1$, by algebra

$m^2 = 6(6q^2 + 2q) + 1$, by factorization

Note that $6q^2 + 2q \in$ Z since q $\in$ Z by assumption and Z is closed under multiplication and addition.

In order for m^2 to be divisible by 6, the remainder must be non-zero.

Therefore, m^2 is not divisible by 6 by the division algorithm, since when the divisor is 6, the remainder is 1.

Case 2:

m = 6q+2

We need to show that for this particular case, m^2 is not divisible by 6.

By substitution,

$m^2 = (6q + 2)^2$

$m^2 = 36q^2 + 24q + 4$, by algebra

$m^2 = 6(6q^2 + 4q) + 4$, by factorization

Note that $6q^2 + 4q \in$ Z since q $\in$ Z by assumption and Z is closed under multiplication and addition.

In order for m^2 to be divisible by 6, the remainder must be non-zero.

Therefore, m^2 is not divisible by 6 by the division algorithm, since when the divisor is 6, the remainder is 4.

Case 3:

m = 6q+3

We need to show that for this particular case, m^2 is not divisible by 6.

By substitution,

$m^2 = (6q + 3)^2$

$m^2 = 36q^2 + 36q + 9$, by algebra

$m^2 = 6(6q^2 + 6q + 1) + 3$, by factorization

Note that $6q^2 + 6q + 1 \in$ Z since q $\in$ Z by assumption and Z is closed under multiplication and addition.

In order for m^2 to be divisible by 6, the remainder must be non-zero.

Therefore, m^2 is not divisible by 6 by the division algorithm, since when the divisor is 6, the remainder is 3.

Case 4:

m = 6q+4

We need to show that for this particular case, m^2 is not divisible by 6.

By substitution,

$$m^2 = (6q + 4)^2$$
$$m^2 = 36q^2 + 48q + 16, \text{ by algebra}$$
$$m^2 = 6(6q^2 + 8q + 2) + 4, \text{ by factorization}$$

Note that $6q^2 + 8q + 2 \in$ Z since q $\in$ Z by assumption and Z is closed under multiplication and addition.

In order for m^2 to be divisible by 6, the remainder must be non-zero.
Therefore, m^2 is not divisible by 6 by the division algorithm, since when the divisor is 6, the remainder is 4.

Case 5:

m = 6q+5

We need to show that for this particular case, m^2 is not divisible by 6.

By substitution,

$$m^2 = (6q + 5)^2$$
$$m^2 = 36q^2 + 60q + 25, \text{ by algebra}$$
$$m^2 = 6(6q^2 + 10q + 4) + 1, \text{ by factorization}$$

Note that $6q^2 + 10q + 4 \in$ Z since q $\in$ Z by assumption and Z is closed under multiplication and addition.

In order for m^2 to be divisible by 6, the remainder must be non-zero.
Therefore, m^2 is not divisible by 6 by the division algorithm, since when the divisor is 6, the remainder is 1.

So, by Cases 1-5, we have shown that for every possible integer m, in which m is not divisible by 6, m^2 is not divisible by 6 also. Therefore, this statement is true by the division algorithm.

Because this statement is true, the contrapositive must also be true.

In other words, $\forall\, m \in$ Z, if m^2 is divisible by 6, then m is divisible by 6.

This was what was to be proven.

QED

2. Lemma B: $\sqrt{6}$ is irrational.

Let us prove this by contradiction.

[Proof by contradiction]

Suppose $\sqrt{6}$ is rational.

Then, $\exists\ a, b \in$ Z such that:

$$\sqrt{6} = \frac{a}{b} \text{ where } b \neq 0.$$

Also, assume a and b are coprime. In other words, the fraction $\frac{a}{b}$ is fully reduced. If it isn't then we fully reduce it so that a and b are coprime and share no non-trivial divisors.

By algebra then,

$$6 = \frac{a^2}{b^2}$$

$$6b^2 = a^2 \text{ by multiplication}$$

$$a^2 = 6b^2 \text{ by Symmetry}$$

Note that $b^2 \in$ Z since b $\in$ Z by assumption and Z is closed under multiplication.

Therefore, a^2 is divisible by 6.

By **Lemma A**, since a^2 is divisible by 6, a is also divisible by 6.

By definition of divisibility,

$$a = 6k \text{ for some } k \in Z.$$

Substituting a=6k into the equation $a^2 = 6b^2$:

$$(6k)^2 = 6b^2$$
$$36k^2 = 6b^2$$
$$6k^2 = b^2$$
$$b^2 = 6k^2$$

Note that $k^2 \in Z$ since k $\in$ Z by assumption and Z is closed under multiplication.

Therefore, b^2 is divisible by 6.

By **Lemma A**, since b^2 is divisible by 6, b is also divisible by 6.

Therefore, a and b are both divisible by 6.

However, this is a contradiction because we assumed at the beginning of the proof that a and b have no non-trivial common divisors (divisors other than 1 and -1). However, both a and b indeed share a non-trivial divisor, 6.

Therefore, the supposition is false, that $\sqrt{6}$ is rational.

Hence,

$\sqrt{6}$ is irrational.

[This was what was to be shown]

QED

3. PROOF that $\sqrt{2} + \sqrt{3}$ is irrational.

Let's prove this by contradiction.

[Proof by →←]

Suppose $\sqrt{2} + \sqrt{3}$ is *rational*.

By definition of rational, $\exists$ a, b $\in$ Z such that:

$$\sqrt{2} + \sqrt{3} = \frac{a}{b}, \text{ where } b \neq 0.$$

Squaring both sides,

$$2 + 2\sqrt{6} + 3 = \frac{a^2}{b^2}$$

By addition,

$$5 + 2\sqrt{6} = \frac{a^2}{b^2}.$$

By algebra,

$$2\sqrt{6} = \frac{a^2}{b^2} - \frac{5b^2}{b^2}$$

$$2\sqrt{6} = \frac{a^2 - 5b^2}{b^2}$$

$$\sqrt{6} = \frac{a^2 - 5b^2}{2b^2}$$

Note that $a^2 - 5b^2 \in Z$ since a, b $\in$ Z by assumption and Z is closed under multiplication and addition. Similarly, $2b^2 \in Z$ since b $\in$ Z by assumption and Z is closed under multiplication. Also $2b^2 \neq 0$, since $b \neq 0$ by assumption, so by zero-product property, $2b^2 \neq 0$.

Therefore, $\sqrt{6}$ is rational.

But, according to **Lemma B**, $\sqrt{6}$ is irrational. So there is a clear contradiction here.

Therefore, the supposition that $\sqrt{2} + \sqrt{3}$ is rational is false.

Hence,

$\sqrt{2} + \sqrt{3}$ is irrational.

QED