Condensed Matter Diary

February 5, 2026

Abstract

This is just some notes I took while reading books, papers/watching lectures/doing some calculations. The purpose is just to force myself to TeX out the scrap notes I take while doing research. I also want to make it public so that I remain motivated to keep updating it. Hopefully this selfish endeavour will be useful to the curious eyes that come across it :)

Disclaimer: Pretty much everything in here is either directly copied or lightly rephrased versions of what I came across and original sources are listed whenever relevant.

List of topics to start writing about:

1. LSM, LR theorems
2. Haldane phase/conjecture, AFM O(3) NLSM paper.
3. AKLT model and properties. Identify the VBS ground state as an MPS state.
4. BLBQ models in general (Affleck's paper) understand the phase diagram and in particular the ULS model (or SU(3) point) and Takhtajan, Fateev, Zamolodchikov model. Understand the mapping to WZW theory. Understand the yang-baxter/integrability pov on these models.
5. Connection between spectrum, correlator's behaviour and order/disorder. Also on LRO vs SSB (mermin-wagner).
6. KT transformation and connection to our mapping.
7. Dualities (duality webs-witten, senthil)(boson vortex duality, flux threading/attachment)
8. Krammers Wannier, Jordan Wigner iron out some details.
9. Edge modes (gapping the bulk) (FQH-LL bulk boundary), Kitaev model+Majoranas
10. SPT phases(Projective reps/group cohomology pov- xie chen, xiao gang wen...) [see mcgreevy lecs]
11. Generalised Symmetries
12. Bosonization done right
13. QFT in 1+1d - David tong gauge theory notes
14. QFT in 2+1 d- David tong+Zohar lecs
15. G theorem, boundary RG flow

16. Hastings entropy Area law for many body , boundary term for topological order-mutual information

17. Anomalies

18. Disordered systems, localization stuff. Check to see if any of the old notes are still around for an extensive topics list. Else see reading list email/message.

1 Coherent states, segal bargmann transform, heat kernel $\leftrightarrow$ oscillator propagator

Some stuff to look at: [1]

2 Lieb-Schultz-Mattis Theorem

(Source: Hal Tasaki's lecture)

Spin rotation operator (unitary), rotates the spin state by theta about the α axis. $\exp\left[-i\theta S^{\alpha}\right]$.

LSM theorem is a no-go theorem which states that certain quantum many-body systems cannot have a unique ground state with a nonzero energy gap. This statement is interesting because it shows that symmetry of a quantum spin system places a strong constraint on its low energy properties.

The original theorem was in the context of the antiferromagnetic heisenberg chain. Consider a 1d lattice of quantum spins described by the Hamiltonian $\hat{H} = \sum_{j=1}^{L} \hat{S}_j \cdot \hat{S}_{j+1}$, with ground state energy E_{gs}, and with periodic boundary conditions. For half integer spins$\left(S = \frac{1}{2}, \frac{3}{2}, \cdots\right)$, there exists an energy eigenvalue E such that

$$E_{\text{gs}} \leq E \leq E_{\text{gs}} + \frac{C}{l} \tag{1}$$

for any $l < L$ and some constant $C = 8\pi^2 S^2$. This not only tells us that E is an excitation energy that is larger than the ground state energy, but since l can be made as arbitrarily small as you want, this also suggests that the system is gapless!

Outline of the proof

1. Invoke the Marshall 1955, Lieb, Mattis 1962 theorem, which states that the ground state $|\text{GS}\rangle$ of $\hat{H}$ is unique $\forall L$ even (finite).

2. Make a variational estimate, exploiting the above uniqueness. Using the $U(1)$ rotational invariance of $\hat{H}$, we can conclude that the ground state has to be rotationally invariant (As it is unique), that is

$$\exp\left[-\mathrm{i}\sum_{j=1}^{L} \theta_j \hat{S}_j^z\right] |GS\rangle = |\text{GS}\rangle \tag{2}$$

   We can then do a gradual *non-uniform* (site-dependent) rotation or a twist given by

$$\hat{U}_l = \exp\left[-\mathrm{i}\sum_{j=1}^{l} \theta_j \hat{S}_j^z\right], \quad \theta_j = \frac{2\pi}{l} j = \Delta\theta \cdot j,\ 0 \leq j \leq l \tag{3}$$

Note that $\theta_j = 2\pi$ for $j \geq l$, $\theta_j = 0$ for $j \leq 0$, but of course $2\pi = 0$ in this case. We use this twist operator to construct our variational state $|\Psi_l\rangle = \hat{U}_l |\mathrm{GS}\rangle$. For $0 \leq j \leq l$ we have

$$\langle \Psi_l | \hat{S}_j \cdot \hat{S}_{j+1} | \Psi_l \rangle = \frac{E_{\mathrm{gs}}}{L} + \mathcal{O}\left((\Delta\theta)^2 \right) \tag{4}$$

This tells us that within the region, the local energy density given by LHS is equal to the ground state energy density $\frac{E_{\mathrm{gs}}}{L}$ plus some corrections that go as $\Delta\theta^2 or \frac{1}{l^2}$. Thus, multiplying by l, $\langle \Psi_l | \hat{H} | \Psi_l \rangle - E_{\mathrm{gs}} \leq \frac{C}{l}$. This bound is valid for any spin S.

3. Orthogonality of the twisted state. We now need to show that this twisted state is different from the ground state. We can define a Unitary $\hat{R}$ such that

$$\hat{R}\hat{S}_j^\alpha \hat{R} = \begin{cases} \hat{S}_{l-j}^\alpha & \alpha = x \\ -\hat{S}_{l-j}^\alpha & \alpha = y, z \end{cases} \tag{5}$$

This operator allows us to verify *[Not yet understood!! - AM]* that $\langle \mathrm{GS} | \Psi_l \rangle$ is zero for half-integer spins.

In the infinite volume limit, it turns out that there is still a unique ground state with gapless excitations, and not multiple ground states instead. This theorem also provides no information for the integer spin case. In fact, the integer spin models have a unique ground state with a gap! This is related to the Haldane conjecture/gap.

2.1 LSM theorem for models with only discrete symmetries

$U(1)$ invariance is essential for the previous proof. Can we do better?

It turns out that for quantum spin chains with half-integer spins and with a short-ranged translation invariant Hamiltonian that is invariant under time-reversal symmetry i.e. $\hat{S}_j^\alpha \to -\hat{S}_j^\alpha$, $\alpha = x, y, z$ it can never be the case that the infinite volume ground state is unique and accompanied by a non-zero gap.

The proof by Y.Ogata, Y.Tachikawa and H.Tasaki relies on using the Ogata index for edge states, developed for the study of SPT phases to obtain a necessary condition for the existence of a unique gapped ground state.

3 Heisenberg model

Here is some stuff to read about this (from Fradkin chapter 5)

1. Bethe ansatz solution for the ground and excited states.
2. Mapping to the sine-gordon theory.
3. non-abelian bosonization
4. Mapping to the sigma model

Continuum description for large S by Haldane: [2]. The mappings provide us with non-perturbative probes of the ground and excited states. Read the refs (witten, polyakov and wiegmann, affleck)

4 BLBQ models

$$H_{\text{BLBQ}}(\theta) = \sum_j \cos\theta \left(\vec{S}_j \cdot \vec{S}_{j+1}\right) + \sin\theta \left(\vec{S}_j \cdot \vec{S}_{j+1}\right)^2 \tag{6}$$

4.1 AKLT point

(Source: Hal Tasaki's Textbook)

AKLT (Affleck, Kennedy, Lieb, Tasaki) is a special point in the class of Bilinear Biquadratic (BLBQ) models and shows up at the $\theta = \arctan\frac{1}{3}$ point. At this point, the model has a few properties.

1. We only have SO(3) or spin rotation symmetry.
2. The ground state is the Valence Bond Solid state. One can derive this by first re-writing the BLBQ Hamiltonian at the AKLT point in terms of the spin 2 projection operator. Depending on the type of boundary conditions imposed, we can have gapless edge modes. For open boundary conditions, the ground state will have 4 gapless edge modes and with periodic boundary conditions there are none. The VBS state can also be written as a Matrix product state *[write about this please - AM]*
3. The correlators (spin correlation functions) decay exponentially, indicating a massive excitation. This mass gap or energy gap is in line with Haldane's conjecture, which states that we should see a gapped ground state for integer spin nearest neighbour spin models in 1+1 D.
4. $\mathbb{Z}_2 \times \mathbb{Z}_2$ is the center and the gapped/SPT properties come from it.
5. It is integrable. (Check Bethe-Ansatz/Yang-Baxter) *[What does this give you? - AM]*

Spin 1 chains also have this decomposition in terms of creation/annihilation ops of U(1), and they have some connection to the chains having twisted boundary conditions.*[Vaguely recall this discussion, make it precise - AM]*

4.2 ULS point

To study the (global?) symmetry of these spin chains, it can be useful to convert to the Abrikosov fermion representation. Since we are working with the spin -1 model, we use 3 fermion channels at each site with a local fermion number constraint $\sum_{\mu=1}^{3} c^\dagger_{j,\mu} c_{j,\mu} = 1$. *[Show how there is SU(2) when the bond/pairing term is present - AM]* For a generic coupling coupling γ the model looks like:

$$H_{\text{BLBQ}}(\gamma) = \sum_j c^\dagger_{j,\alpha} c_{j,\beta}\; c^\dagger_{j+1,\beta} c_{j+1,\alpha} + (\gamma - 1) c^\dagger_{j,\alpha} c_{j,\beta} c^\dagger_{j+1,\alpha} c_{j,\beta}, \tag{7}$$

where, the first term describes a hopping between channels at adjacent sites and the second term describes a "single-bond" pairing between adjacent sites. At $\gamma = \frac{4}{3}$ we are at the AKLT point, and at $\gamma = 1$ we are at the Umini Lai Sutherland (ULS) point. Notice immediately that under a transformation of $c_{j,\mu} \to U c_{j,\mu}$ we remain invariant at the ULS point. This tells us that the ULS point has global SU(3) symmetry.

The ground state of the SU(3) point is the Trimer liquid state. *[Need to understand this construction better. - AM]*

Also understand this statement: *In general, critical phases of SU(ν) symmetric spin chains are stabilised by $\mathbb{Z}_\nu$, the center of SU(ν).*

4.3 Babujian-Takhtajan point

4.4 How does the WZW show up in spin S chains?

Sources: [3, 4].

For the case of the BLBQ model, we can show that the critical theory (CFT) at the ULS point is the SU(3) level 1 WZW model. We can obtain this by perfroming a Hubbard stratanovich transformation to get the euclidean action, and then write down the mean field description to get the IR theory. *[Complete this please! - AM]* As expected, the CFT central charge adds up correctly (i.e matches the central charge from Yang-Baxter*[how to do this? - AM]*) after we construct the theory from the lattice model.

The IR gauge-fixed action looks like

$$S = \int \frac{\mathrm{d}^2 z}{2\pi} \left(\mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{ghost}}\right) \tag{8}$$

where the lagrangians are given by

$$\mathcal{L}_{\text{matter}} = 2\bar{\psi}^{\dagger}_{L,\alpha}\bar{\partial}\psi_{L,\alpha} + 2\psi^{\dagger}_{R,\alpha}\partial\psi_{R,\alpha}, \quad \mathcal{L}_{\text{gauge}} = -\partial\phi\bar{\partial}\phi, \quad \mathcal{L}_{\text{ghost}} = 2\bar{\eta}\partial\bar{\epsilon} + 2\eta\bar{\partial}\epsilon \tag{9}$$

[Make this proper - AM] Little about WZW models. These are a simple example of solvable(in the sense of yang baxter) RCFTs. In these theories we have a gauge group describing some internal symmetry of the theory but also gets enhanced (extended chiral algebra) by the conformal algebra to form an affine lie algebra (kac moody algebra). General philosophy:more symmetry is good news! the theory will turn out to be simple (because it is highly constrained by the symmetries), i.e has few primary operators and the rest derivable using the algebra.

It is possible to write the WZW theory in terms of elements of the gauge group, and that will turn out to be purely bosonic. However for certain WZW theories like su(n), so(n) and u(n), one can equivalently describe the theory in terms of multiple species of free (Real) fermions. For example, see the action of the SU(3) level 1 WZW described above for ULS. See Lorenz Ebenhardt's notes/lectures.

A Dubious idea:*[and most probably completely false - AM]* We can get SPTs in higher dims by stacking 1d SPTs. Is this somehow connected to the fact that WZW describes the boundary of a chern-simons theory? Because we could have some 1d spin chain which has an SPT phase and has an IR description by a WZW theory, and by stacking them the bulk gets gapped (topological) and is described by a CS theory?

5 Edge modes and Kitaev chain POV

There is a direct mapping from the Kitaev chain to the spin 1 models. It allows us to see all the subtleties encountered with respect to edge modes/gapped states in the spin 1 case through a simple "correspondence" to the majoranas on the kitaev chain. Here is some calculations to work out that may give some insight.*[Vaguely remember this discussion, make this precise - AM]*

Check this out [5]

1. Strong pairing to weak pairing transition for spinless vs spinful fermions.

2. (Related) See the $\frac{t}{U}$ map of Hubabrd model to Heisenberg chain, and see what shows up in the leading order of Pert theory.

6 Bosonization

[Move refs to study doc - AM] Here is a list of topics

1. Why does bosonization work? When does it work? What is the role of linearization/1d?
2. point split vs normal ordering to regulate divergences
3. Klein factors, why is it important when dealing with multiple fermionic species?
4. How is continuum limit taken?
5. Commutators, Correlators of all the fields involved

6.1 Abelian bosonization, constructive approach

Goal: Given a **1D** lattice model of fermions, want to represent the fermion fields $\psi_\mu(x)$ at the continuum in terms of bosonic fields ϕ_μ through a relation of the form: $\psi_\mu \sim F_\mu e^{-\mathrm{i}\phi_\mu}$ where μ is the fermion species index. F_μ is the Klein factor, which *lowers* the number of μ fermions by one. This relation is an *operator identity* in Fock space. This is the general philosophy of bosonization, however in the constructive approach, we want to build the bosonic field in the continuum from "scratch", just using the ingredients from our model on a lattice. (As opposed to the field theory approach, where the existence of the bosonic fields is taken for granted)

Important developments in the constructive side where through the works of Mattis and Lieb, Luther and Peschel, Emery, Haldane.

There are many ways to see why bosonization works. Here is one to start: It turns out that $\psi_\mu(x)\,|0\rangle$ is an eigenstate of the boson annihilation operators $b_{q\mu}$ from which the boson field $\phi_\mu(x) = -\sum_{q>0}\sqrt{\frac{2\pi}{qL}}\left(e^{-q(\mathrm{i}x+\frac{a}{2})}b_{q\mu} + \text{h.c.}\right)$ is constructed. This means it must have a *coherent state* representation in terms of the boson creation operators, which is exactly the bosonization relation.

6.1.1 Subtlety regarding the boson commutator

Check [6, 7]. The bosonic mode operators can be defined in terms of the fermionic operators on the lattice (in k space) as follows (supressing free species index μ, and remember only for positive momenta q)

$$b_q^\dagger := \frac{\mathrm{i}}{\sqrt{n_q}}\sum_k c_{k+q}^\dagger c_k \quad , \; q\left(=\frac{2\pi}{L}n_q\right) > 0, n_q \in \mathbb{Z}^+. \tag{10}$$

For any $|\vec{N}\rangle$ state, the state $b_q^\dagger\,|\vec{N}\rangle$ consists of linear combinations of particle hole excitations relative to $|\vec{N}\rangle$. So these mode operators can be viewed as momentum raising/lowering operators or alternatively, identified as the fourier components of the density operator. The commutator for these operators looks like

$$[b_{q\mu}, b_{q'\mu'}] = \delta_{\mu\mu'}\delta_{qq'} \tag{11}$$

The above equation crucially relies on the fact that we normal order the operators with respect to the fermi sea. If we had normal ordered with respect to the vacuum, then the commutator would simply vanish instead!

6.1.2 Klein factors

They allow us to connect the different fermion number sectors. These operators raise/lower the total fermion number by one, this cannot be achieved by any linear combination of bosonic operators! They also have the added benefit of enforcing the anti-commutation relation between different species of fermions! They are defined with the following properties

1. $F_\mu, F_\mu^\dagger$ commute with all the bosonic operators.
2. The state $F_\mu^\dagger |\vec{N}\rangle$ (or $F_\mu |\vec{N}\rangle$) has the same set of bosonic excitations as the state $|\vec{N}\rangle$ but created on a ground state with one more (or less) μ-electron.

These two defining properties completely fix the operators, and we can give an explicit construction roughly by inverting the bosonization formula. An additional consequence of the above properties is that these operators are unitary and so can be written as $e^{\mathrm{i}\theta}, \theta = \theta^\dagger$. Ofcourse, the fact that different fermionic species anti-commute now comes baked in, and can be seen through the anti-commutation relations of the Klein factors.

6.1.3 The Bosonic field, its dual and the chiral versions

6.1.4 Comments on the bosonization formula

6.1.5 Some Green's functions

6.2 Field theory/CFT approach

6.2.1 Subtlety regarding compactness of the bosonic field

6.2.2 Point splitting, normal ordering and correct order of limits to be taken

6.2.3 Vertex Operators

6.3 Non-abelian bosonization

6.3.1 Current algebras

7 QFT in 1+1 d

List of topics

1. Luttinger model, luttinger liquid
2. spin-charge separation
3. sine-gordon RG analysis (BKT)

4. Tong 1+1 d

7.1 Luttinger Model/Liquid

The idea behind this model is to "replace" fermi liquid theory in 1 spatial dimension. One dimensional systems exhibit physics that is dramatically different compared to higher dimensions. One peculiarity is that there is no concept of "single particle" states/excitations. Everything is collective in 1d. The way to handwave this is as follows: In higher dims, you can squeeze past neighbouring species in your way, in 1 dim you can't, there is simply no space! Interactions in 1 dimension therfore always lead to collective behaviour. *All low energy states of a luttinger liquid can be described in terms of free quasiparticle excitations of the massless scalar field.*

One can see the breakdown of fermi liquid theory in many ways. One is to see how many-body perturbation theoretic (diagrams!) treatment of these systems looks like and compare with usual diagrammatics in higher dim. See Dzyaloshinskii Larkin solution for more. [8, 9]

Main features: massless/gapless, relativistic scalar fields representing density and phase fluctuations.

7.1.1 Scattering processes

7.1.2 Sine-Gordon, BKT, Massive Thirring

7.1.3 Spin Charge separation

8 A vaguely stated connection...

After seeing a collection of different (but related) ideas pop-up in multiple contexts, I wanted to summarise them in a nice way and maybe see if I can atleast give a semblance of "proof" for these statements. I am quite confident that the way in which it is written down below is completely wrong, but nevertheless I want to keep track of it and try finding counterexamples to disprove every incorrect implication.

Connection 1. *Systems with a disordered state* $\longleftrightarrow$ *Exponential Decay of correlation functions* $\longleftrightarrow$ *Gapped spectrum of the Hamiltonian* $\longleftrightarrow$ *Existence of short-range interacting/localized/massive excitations* $\longleftrightarrow$ *Propagation speed of excitations is less than a "prescribed speed limit".* $\longleftrightarrow$ *System described at some limit by a TQFT + impurity (=quasiparticles) [Check Witten's lec, also in a TQFT there are no propagating modes. - AM]*

Now here is a complementary connection

Connection 2. *System at Criticality* $\longleftrightarrow$ *Power law decay of correlation functions* $\longleftrightarrow$ *Gapless spectrum of Hamiltonian* $\longleftrightarrow$ *Massless excitations/Long Range excitation* $\longleftrightarrow$ *Propagation speed of excitation equals speed limit* $\longleftrightarrow$ *System described at some limit by a CFT.*

A related question: What sets the speed limit? I guess partially answered by Lieb-Robinson, but want some more intuition. There is also a further relation to chirality of excitations, but I am not able to make a concrete statement yet.

Regarding chirality: Basically, it makes sense to talk about chiral modes on the edge when the bulk is gapped. If you had a gapless bulk then these edge modes are not "independent" so it does not make sense to talk of the chirality of the modes. Though it is not true that chiral modes are always accompanied by a gapped bulk.*[Please find the counterexample! - AM]*

A potential counter example to connection 1: Kitaev model (in the context of kitaev spin liquids; with hexagonal lattice in 2+1d) is gapless yet does not have power-law behaviour of spin correlators. They have this weird thing where they vanish sharply beyond a point on the lattice, instead of a smooth decay.*[Check this! - AM]*

8.1 SPT vs Intrinsic topological order

Some examples for SPT: the AKLT phase; Intrinsic top order: (fractional) quantum hall phase, spin liquid phase. Thank you stackexchange! [10, 11, 12]. Here are some claims [13]:

1. STUFF (= quantum phases of matter) has either SPT or has intrinsic topological order.
2. SPT phases have short range entanglement. SPT phases are gapped.
3. There is no intrinsic Topological order in 1d (1+1d). That is, all gapped states are short range entangled.

8.2 Classification of 1d Bosonic, Fermionic SPT phases

Stuff about group cohomology classes/projective representations.

8.3 What is a spin liquid anyway??

This is the most common question I get from people when I explain what I am working on, and it is kind of sad that I can't give a simple, convincing answer. Here is an attempt to atleast list out some well known properties and give some examples so that I can work towards a proper answer.

1. It has long range entanglement. Also has intrinsic topological order *[Explain this!! - AM]*
2. It is gapped. Caveat: (non-trivial)Gapless spin-liquids are not well defined.But can be roughly thought of as a deconfined phase of a masless gauge theory
3. Fractionalized excitations: spinons/holons. *[Explain in context of FQHE and spin-charge separation in luttinger liquids - AM]*

Some vague descriptions

1. It is a phase dominated by quantum fluctuations. We see it at zero temperature.
2. "Collective excitations".
3. Non-local order parameter.

Some common questions:

1. Is it the same as spin glass/ spin ice?? A:No. But is there a connection? A:Perhaps..*[Think please! - AM]*

2. What is a phase of matter? A: Gulps..

3. What is a symmetry-broken phase of matter? *[Start here, then tell how SPT is different, then go to Top.order - AM]*

9 Monopoles, Instantons

Some sources to get started: Tong(TASI Solitons), Figuera O farill(Em duality for children), t'hooft and eta'il notes. Want to understand them form gauge theory point of view. Want to eventually understand ADHM construction.

10 Interesting stuff from Statmech

10.1 2d ising combinatorics

Kac Ward

Pfaffian approach

10.2 Kadanoff Ceva disorder operators

10.3 Star-triangle duality, Yang Baxter

10.4 Krammers-wannier duality/non-invertible symmetry

[14]

10.5 Aasen-Fendley-Mong construction (Topological defect lines)

[15]. For a CFT pov see [16]

10.6 ice-type model(6 vertex), residual entropy

10.7 8 vertex model, breakdown of universality, connection to Ashkin Teller

see papers by Fan, Wegner.

10.8 8 vertex as 2 ising + 4spin int, Kadanoff Wegner argument

is there a connection to the 4 spin interaction seen in coupled wire constructions of spin liquids?

10.9 XYZ commutes with 8v, and thus came yang baxter

See [17] for MPO perspective

10.10 More on integrability of 1d models: CBA, ABA, ...

Start here [18]

10.11 For intro to topological defects/vortices start with XY model

See Hal Tasaki's videos and also these notes from Jensen and from Levitov/dogson. See also about XY-Villain duality.

10.12 d+1 statmech = d qm, polyakov

11 Some notes on "gauging"

Start from [19]

12 Biting the bullet that is tensor networks

See Frank verstraete lecs from houches 2025

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