2018

Maths B Notes

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Maths Notes : Michaelmas

1 : Vectors

Vectors are objects with both magnitude and direction => displacement vectors are path insensitive. They are commutative ($\underline{a} + \underline{b} = \underline{b} + \underline{a}$), and they have an additive inverse ($\underline{a} + (-\underline{a}) = 0$). $\underline{0}$ is a zero vector with length 0 and undefined direction. It is also associative ($\underline{a} + (\underline{b} + \underline{c}) = (\underline{a}+\underline{b}) + \underline{c} = \underline{a} + \underline{b} + \underline{c}$)

Multiplication by a vector: vector $\underline{a}$ × scalar λ, form a vector parallel to $\underline{a}$ with magnitude $\lambda|\underline{a}|$. Distributive over addition $\lambda(\underline{a}+\underline{b}) = \lambda\underline{a} + \lambda\underline{b}$

Kinematics

① Velocity

$$\underline{v} = \lim_{dt\to 0} \frac{\underline{r}(t+dt) - \underline{r}(t)}{dt} = \lim_{dr\to 0} \frac{d\underline{r}}{dt} = \frac{d\underline{r}}{dt} = \dot{\underline{r}}$$

② Speed

$$s = \lim_{dt\to 0} \frac{|d\underline{r}|}{dt} = \lim_{dt\to 0} \frac{|\delta t \underline{v}|}{dt} = |\underline{v}|$$

③ Acceleration

$$\underline{a} = \lim_{dt\to 0} \frac{\underline{v}(t+dt) - \underline{v}(t)}{dt} = \frac{d\underline{v}}{dt} = \dot{\underline{v}} = \ddot{\underline{r}}$$

Scalar Product

$$\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}| \cos\theta$$

↳ commutative

$\underline{a} \cdot \underline{a} = |\underline{a}|\,|\underline{a}| \cos 0 = |\underline{a}|^2$

$\underline{a} \cdot \underline{b} = 0$ => $\underline{a}$ and $\underline{b}$ are perpendicular (or $\underline{a}$ or $\underline{b}$ is a zero vector)

We can resolve a vector ($\underline{a}$) into parallel and perpendicular vectors to $\hat{\underline{n}}$

Parallel: $|\underline{a}|\cos\theta\,\hat{\underline{n}} = (\underline{a}\cdot\hat{\underline{n}})\,\hat{\underline{n}}$
Perpendicular: $\underline{a} - (\underline{a}\cdot\hat{\underline{n}})\hat{\underline{n}}$

$$\underline{a} = \underline{a} - (\underline{a}\cdot\hat{\underline{n}})\hat{\underline{n}} + (\underline{a}\cdot\hat{\underline{n}})\hat{\underline{n}}$$

Scalar Product Algebra:
1. Commutative: $\underline{a}\cdot\underline{b} = \underline{b}\cdot\underline{a}$
2. Multiplication by Scalar:
   ↳ $(\lambda\underline{a})\cdot\underline{b} = \lambda(\underline{a}\cdot\underline{b}) = \underline{a}\cdot\lambda\underline{b}$
3. Distributive over addition
   ↳ $\underline{a}\cdot(\underline{b}+\underline{c}) = \underline{a}\cdot\underline{b} + \underline{a}\cdot\underline{c}$

Vector Product (Cross Product, Wedge Product)
$\underline{a}\times\underline{b}$ is a vector of magnitude $|a||b|\sin\theta$ in direction perpendicular to both $\underline{a}$ and $\underline{b}$ $\mid (0 \leq \theta \leq \pi)$ given by right-hand rule going from $\underline{a}$ to $\underline{b}$ through the smallest angle.

$\underline{a}\times\underline{b} = |a||b|\sin\theta\,\hat{\underline{n}} = -(\underline{b}\times\underline{a})$
$\underline{a}\times\underline{b} = 0 \Rightarrow a \parallel b$ (or $\underline{a} = \underline{0}$ or $\underline{b} = \underline{0}$)
$\underline{a}\times\underline{a} = 0$
$\underline{a}\cdot(\underline{a}\times\underline{b}) = 0$
↳ $\underline{b}\cdot(\underline{a}\times\underline{b}) = 0$
$|\underline{a}\times\underline{b}|$ is area of parallelogram formed by $\underline{a}$ and $\underline{b}$
Vector Product does not generalise to spaces with other dimension

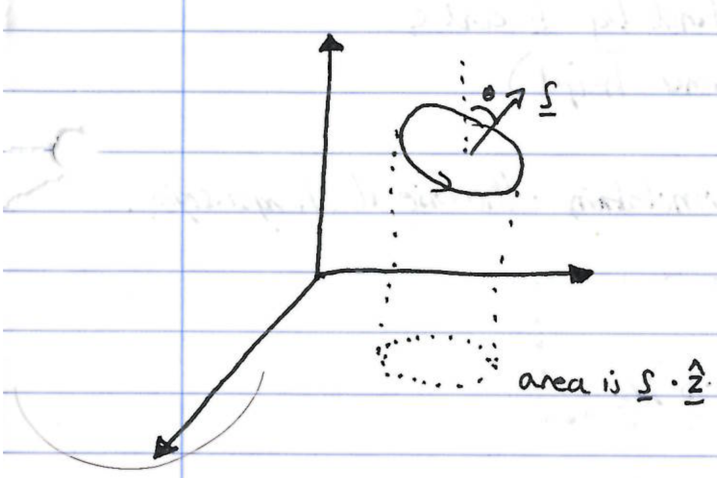
1. Anti-commutative
2. Multiplication by scalar
   ↳ $(\lambda\underline{a})\times\underline{b} = \lambda\,\underline{a}\times b$
3. Distributive over addition
   ↳ $\underline{a}\times(\underline{b}+\underline{c}) = \underline{a}\times\underline{b} + \underline{a}\times\underline{c}$
4. Not associative

Vector Area

↳ Vector Area $\underline{S}$ (of a finite plane surface) is a vector of magnitude $|S|$ equal to area and direction normal to surface.

↳ ambiguous direction - this is fixed by right hand rule fixed by the direction to traverse the perimeter

↳ $\underline{a} \times \underline{b}$ is vector area of parallelogram formed by vectors $\underline{a}$ and $\underline{b}$



The area projected onto x-y plane is $= S\cos\theta$
$= \underline{S} \cdot \hat{\underline{z}}$

For non-planar surfaces, we can find the area of the projection in terms of small (planar) parts

Therefore, for a general surface, we can do this for indefinitely small pieces

$$\therefore \underline{S} = \int d\underline{S}$$

It is clear that it only depends on the rim, all the surfaces spanning the given rim have the same vector area
↳ a closed surface has $\underline{S} = 0$

Distributive Law

Top and bottom are $\frac{1}{2}(\underline{b} \times \underline{c})$ and $\frac{1}{2}(\underline{c} \times \underline{b})$, therefore cancel
Vector area of rectangular faces are $\underline{b} \times \underline{a}$, $\underline{c} \times \underline{a}$ and $\underline{a} \times (\underline{b} + \underline{c})$

$$\underline{a} \times (\underline{b} + \underline{c}) + \underline{c} \times \underline{a} + \underline{b} \times \underline{a} = 0$$

$$\therefore \underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$$

(Distributive Law Proof)

<u>Scalar Triple Product</u> : $[a,b,c] = \underline{a}\cdot(\underline{b}\times\underline{c})$

$= \underline{c}\cdot(\underline{a}\times\underline{b}) = [c,a,b]$

$= \underline{b}\cdot(\underline{c}\times\underline{a}) = [b,c,a]$

- Volume of parallelepiped defined by $\underline{a}$, $\underline{b}$ and $\underline{c}$

$\Downarrow$

volume is $|\underline{b}\times\underline{c}|$ times projection of $\underline{a}$ onto normal to plane defined by $\underline{b}$ and $\underline{c}$

(base area times height)

This is invariant under cyclic permutations otherwise it changes sign.

<u>Vector Triple Product</u>

$\underline{a}\times(\underline{b}\times\underline{c}) = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}$

<u>Vector Geometry</u>

① Line : $\underline{r} = \underline{a} + \lambda\hat{\underline{t}}$ $\quad (\lambda \in \mathbb{R})$

$\Downarrow$ distance from a

unit vector $\hat{t}$

Can also use a non-unit vector for direction which makes $\lambda \propto$ distance along line

$\underline{r}\times\hat{\underline{t}} = \underline{a}\times\hat{\underline{t}}$

② Plane (through origin) : $\underline{r} = \lambda\underline{p} + \mu\underline{q}$ $\quad (\lambda, \mu \in \mathbb{R})$

↳ Plane is a 2D subspace in 3D space

or : $\underline{r} = \underline{a} + \lambda\underline{p} + \mu\underline{q}$

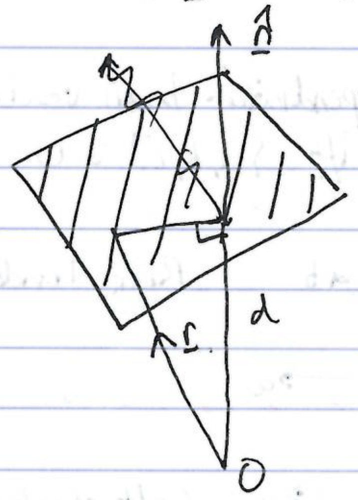
(Plane containing three points $\underline{a}, \underline{b}, \underline{c}$)

- $\underline{r} = \underline{a} + \lambda(\underline{b}-\underline{a}) + \mu(\underline{c}-\underline{a})$

$\underline{r} \cdot \hat{\underline{n}} = \underline{a} \cdot \hat{\underline{n}} = d$ (scalar product with unit normal to plane $\hat{\underline{n}} \propto \underline{p} \times \underline{q}$)

For unit normal $\hat{\underline{n}}$, $|d|$ is the perpendicular distance of the plane from the origin

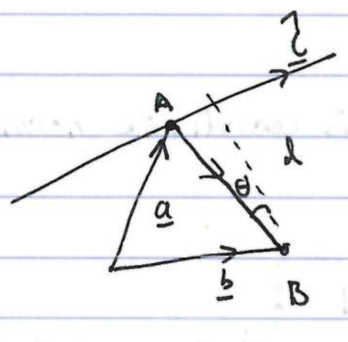
$$d = \underline{r} \cdot \hat{\underline{n}} = |\underline{r}||\hat{\underline{n}}| \cos\theta = |\underline{r}| \cos\theta$$



Example: Distance of point from line

Line: $\underline{r} = \underline{a} + \lambda \hat{\underline{l}}$

point $= \underline{b}$



$$d = |\underline{b} - \underline{a}| \sin\theta$$
$$= |\hat{\underline{l}} \times (b - a)|$$

Orthogonal Bases and Cartesian Coordinates

In 3D space, any three non-coplanar vectors constitute a basis. The basis is a set of vector that spans the space - any vector can be expressed as a linear ~~concatenation~~ combination of the basis vectors.

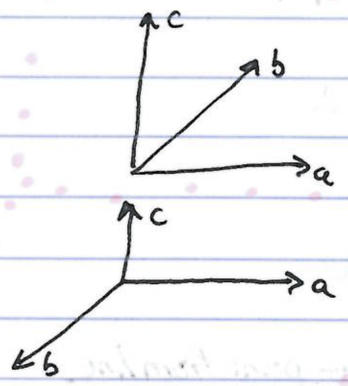
$$\underline{r} = \lambda \underline{a} + \mu \underline{b} + \nu \underline{c}$$

$$\underline{a}^{-1} = \frac{\underline{b} \times \underline{c}}{[a,b,c]}, \quad \underline{b}^{-1} = \frac{\underline{c} \times \underline{a}}{[a,b,c]}, \quad \underline{c}^{-1} = \frac{\underline{a} \times \underline{b}}{[a,b,c]}$$

$$\left\{ \begin{aligned} \underline{a}^{-1} \cdot \underline{r} &= \lambda(\underline{a}^{-1} \cdot \underline{a}) + \mu(\underline{a}^{-1} \cdot \underline{b}) + \nu(\underline{a}^{-1} \cdot \underline{c}) \\ &= \lambda \end{aligned} \right\}$$

<u>Orthogonal Bases</u> : Mutually perpendicular basis vectors
↳ right handed iff $[a,b,c] \geq 0$

Reciprocal basis $(\underline{a}^{-1}, \underline{b}^{-1}, \underline{c}^{-1})$ is just a scaled version of $\{\underline{a}, \underline{b}, \underline{c}\}$



We have a right-handed orthonormal basis, where $|a| = |b| = |c| = 1$
↳ Reprocal basis is same as original basis

<u>Cartesian Basis Vectors</u>

↳ Basis vectors along axis of Cartesian coordinate normally denoted by $\underline{i}, \underline{j}, \underline{k}$

$\underline{i} \cdot \underline{i} = 1 \qquad \underline{j} \cdot \underline{j} = 1 \qquad \underline{k} \cdot \underline{k} = 1$

$\underline{i} \cdot \underline{j} = 0 \qquad \underline{j} \cdot \underline{k} = 0 \qquad \underline{k} \cdot \underline{i} = 0$

$\underline{i} \times \underline{j} = \underline{k} \qquad \underline{j} \times \underline{k} = \underline{i} \qquad \underline{k} \times \underline{i} = \underline{j}$ (Cyclic)

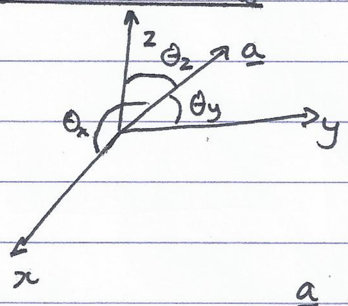
$\underline{j} \times \underline{i} = -\underline{k} \qquad \underline{k} \times \underline{j} = -\underline{i} \qquad \underline{i} \times \underline{k} = -\underline{j}$ (Anti-cyclic)

$$\underline{a} = a_x\underline{i} + a_y\underline{j} + a_z\underline{k}$$

The components are projections of $\underline{a}$ on the x, y and z axes

$a_x = \underline{a}\cdot\underline{i}$, $a_y = \underline{a}\cdot\underline{j}$, $a_z = \underline{a}\cdot\underline{k}$

Direction Cosines



Since $a_x = \underline{a}\cdot\underline{i} = |a|\cos\theta_x$

$a_y = \underline{a}\cdot\underline{j} = |a|\cos\theta_y$

$a_z = \underline{a}\cdot\underline{k} = |a|\cos\theta_z$

$$\underline{a} = |a|(\cos\theta_x, \cos\theta_y, \cos\theta_z)$$
$$\hat{\underline{a}} = (\cos\theta_x, \cos\theta_y, \cos\theta_z)$$

Scalar Product by Components

① $\lambda\underline{a} = \lambda(a_x\underline{i} + a_y\underline{j} + a_z\underline{k}) = (\lambda a_x)\underline{i} + (\lambda a_y)\underline{j} + (\lambda a_z)\underline{k}$
$= (\lambda a_x, \lambda a_y, \lambda a_z)$

② $\underline{a} + \underline{b} = (a_x\underline{i} + a_y\underline{j} + a_z\underline{k}) + (b_x\underline{i} + b_y\underline{j} + b_z\underline{k})$
$= (a_x + b_x)\underline{i} + (a_y + b_y)\underline{j} + (a_z + b_z)\underline{k}$
$= (a_x + b_x, a_y + b_y, a_z + b_z)$

③ $\underline{a}\cdot\underline{b} = (a_x\underline{i} + a_y\underline{j} + a_z\underline{k})\cdot(b_x\underline{i} + b_y\underline{j} + b_z\underline{k})$
$= a_xb_x\,\underline{i}\cdot\underline{i} + a_xb_y\,\underline{i}\cdot\underline{j} + a_xb_z\,\underline{i}\cdot\underline{k}$
$+ a_yb_x\,\underline{j}\cdot\underline{i} + a_yb_y\,\underline{j}\cdot\underline{j} + a_yb_z\,\underline{j}\cdot\underline{k}$
$+ a_zb_x\,\underline{k}\cdot\underline{i} + a_zb_y\,\underline{k}\cdot\underline{j} + a_zb_z\,\underline{k}\cdot\underline{k}$
$= a_xb_x + a_yb_y + a_zb_z$

Follows that $\underline{a}\cdot\underline{a} = a_x^2 + a_y^2 + a_z^2$

Vector Product by Components

$$\underline{a}\times\underline{b} = (a_x\underline{i} + a_y\underline{j} + a_z\underline{k}) \times (b_x\underline{i} + b_y\underline{j} + b_z\underline{k})$$

$$= a_x b_x \underline{i}\times\underline{i} + \cancel{\ } a_x b_y \underline{i}\times\underline{j} + a_x b_z \underline{i}\times\underline{k}$$
$$+ a_y b_x \underline{j}\times\underline{i} + a_y b_y \underline{j}\times\underline{j} + a_y b_z \underline{j}\times\underline{k}$$
$$+ a_z b_x \underline{k}\times\underline{i} + a_z b_y \underline{k}\times\underline{j} + a_z b_z \underline{k}\times\underline{k}$$

$$= (a_y b_z - a_z b_y)\underline{i} + (a_z b_x - a_x b_z)\underline{j} + (a_x b_y - a_y b_x)\underline{k}$$

OR: $\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

<u>Triple Product</u>

$$\underline{a}\cdot(\underline{b}\times\underline{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

∴ determinants generalise to volume of parallelpiped in N dimensions.

Cyclic permutations of $\underline{a}, \underline{b}, \underline{c}$ involve even number of swaps of rows hence leave $[\underline{a}, \underline{b}, \underline{c}]$ unchanged.

$$\underline{a}\times(\underline{b}\times\underline{c}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_x & a_y & a_z \\ (b_y c_z - b_z c_y) & (b_z c_x - b_x c_z) & (b_x c_y - b_y c_x) \end{vmatrix}$$

Can use this to show $\underline{a}\times(\underline{b}\times\underline{c}) = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}$

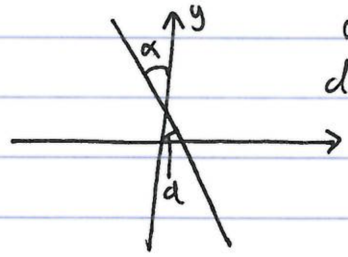
Spherical and Cylindrical Polar Coordinates

① Plane Polar

$x = r\cos\phi$, $y = r\sin\phi$

$r = \sqrt{x^2+y^2}$ $\phi = \arctan(y/x)$

(i) Straight line : $r\cos(\phi - \alpha) = d$.



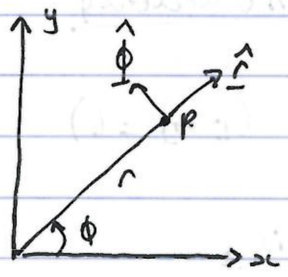
α is angle to y-axis
d is shortest distance from origin

(ii) Circle : $r = a$, radius a

<u>Coordinate Directions</u> : Directions we move in when we increase one coordinate with others fixed

↳ constant in Cartesian coordinates but vary for polar coordinates

We can introduce unit basis vectors along the coordinate directions and for orthogonal systems, they are orthonormal.

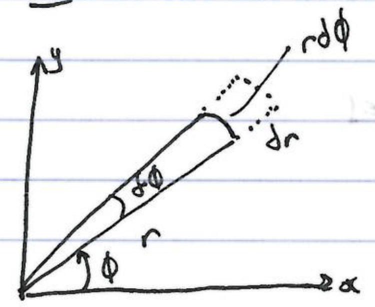


$\hat{\underline{r}} = \cos\phi\,\underline{i} + \sin\phi\,\underline{j}$

$\hat{\underline{\phi}} = -\sin\phi\,\underline{i} + \cos\phi\,\underline{j}$

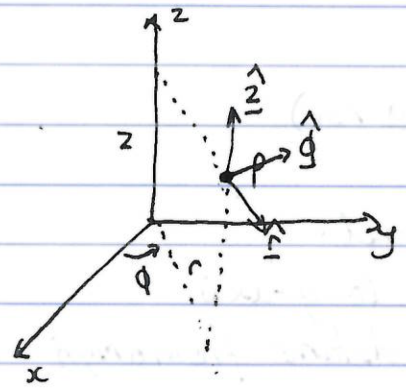
$\dot{\underline{r}} = \dot{r}\hat{\underline{r}} + r\dot{\phi}\hat{\underline{\phi}}$

<u>Area</u>



Area element $= r\,dr\,d\phi$

② Cylindrical Polar Coordinates

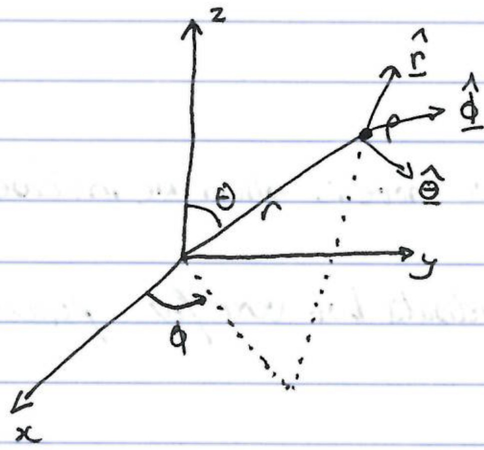


$x = r\cos\phi \quad y = r\sin\phi \quad z = z$

$0 \le r < \infty \quad 0 \le \phi < 2\pi \quad -\infty < z < \infty$

Volume Element $= r\,dr\,d\phi\,dz$

③ Spherical Polar Coordinate



$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$

$\hat{\underline{r}} = \sin\theta\cos\phi\,\underline{i} + \sin\theta\sin\phi\,\underline{j} + \cos\theta\,\underline{k}$

$\hat{\underline{\theta}} = \cos\theta\cos\phi\,\underline{i} + \cos\theta\sin\phi\,\underline{j} - \sin\theta\,\underline{k}$

$\hat{\underline{\phi}} = -\sin\phi\,\underline{i} + \cos\phi\,\underline{j}$

~~Volume~~ Displacements along coordinate directions of $\hat{r}\,dr$, ~~[illegible]~~ $r\,d\theta\,\hat{\underline{\theta}}$ and $r\sin\theta\,d\phi\,\hat{\underline{\phi}}$

so volume $= \underline{r^2\sin\theta\,dr\,d\theta\,d\phi}$

<u>Example: Distance of point from plane</u> $(\underline{r}\cdot\hat{\underline{n}} = c)$

normal to plane $\hat{\underline{n}}$ and point is $\underline{a}$

$\underline{r} = \underline{a} + \lambda\hat{\underline{n}}$

$\underline{r}\cdot\hat{\underline{n}} = \underline{a}\cdot\hat{\underline{n}} + \lambda\hat{\underline{n}}\cdot\hat{\underline{n}}$

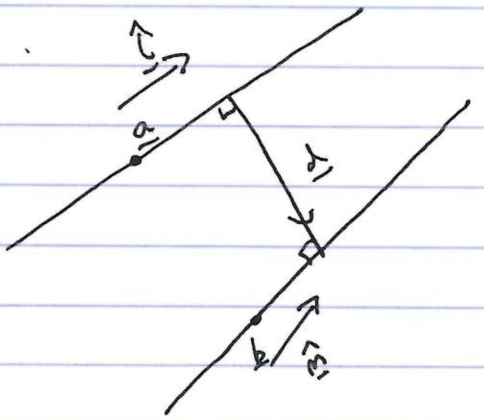
$c = \underline{a}\cdot\hat{\underline{n}} + \lambda$

$\lambda = |c - \underline{a}\cdot\hat{\underline{n}}|$

<u>Minimum distance between two skew lines</u>

line 1: $\underline{r} = \underline{a} + \lambda\hat{\underline{l}}$

line 2: $\underline{r} = \underline{b} + \mu\hat{\underline{m}}$



$$\underline{d} = (\underline{b} - \underline{a}) + (\mu \hat{\underline{m}} - \lambda \hat{\underline{l}})$$

d will be $\perp$ to $\hat{\underline{l}}$ and $\hat{\underline{m}}$

$\therefore \quad d \parallel \hat{\underline{l}} \times \hat{\underline{m}}$

$\therefore$ length $= \dfrac{\underline{d} \cdot (\hat{\underline{l}} \times \hat{\underline{m}})}{|\hat{\underline{l}} \times \hat{\underline{m}}|}$ (projection of $\underline{d}$ in its own direction)

$$= \frac{(\underline{b}-\underline{a}) \cdot (\hat{\underline{l}} \times \hat{\underline{m}}) + \mu \hat{\underline{m}} (\hat{\underline{l}} \times \hat{\underline{m}}) - \lambda \hat{\underline{l}} (\hat{\underline{l}} \times \hat{\underline{m}})}{|\hat{\underline{l}} \times \hat{\underline{m}}|}$$

$$= \frac{(\underline{b}-\underline{a}) \cdot (\hat{\underline{l}} \times \hat{\underline{m}})}{|\hat{\underline{l}} \times \hat{\underline{m}}|} \qquad \left(\begin{array}{l} \hat{\underline{m}} (\hat{\underline{l}} \times \hat{\underline{m}}) = 0 \\ \text{and } \hat{\underline{l}} (\hat{\underline{l}} \times \hat{\underline{m}}) = 0 \end{array} \right)$$

Common Point of Three Planes

A: $\underline{r} \cdot \underline{a} = l$

B: $\underline{r} \cdot \underline{b} = m$

C: $\underline{r} \cdot \underline{c} = n$

$\underline{a} \times \underline{b}$ is the line of intersection between the first two lines

$$\underline{r} \times (\underline{a} \times \underline{b}) = (\underline{r} \cdot \underline{b})\underline{a} - (\underline{r} \cdot \underline{a})\underline{b}$$
$$= m\underline{a} - l\underline{b}$$

$$\underline{c} \times (\underline{r} \times (\underline{a} \times \underline{b})) = [\underline{c} \cdot (\underline{a} \times \underline{b})]\underline{r} - \underline{c} \cdot \underline{r} (\underline{a} \times \underline{b})$$
$$= m\underline{c} \times \underline{a} - l\underline{c} \times \underline{b}$$

2: Complex Numbers

Complex number is an ordered pair of real numbers

$z = a + ib$

$i^2 = -1$

$a = \mathbb{R}(z)$

$b = \mathcal{I}(z)$

Set of complex numbers is closed.

① Addition

$\forall$ $z_1 = a + ib$ $z_1 + z_2 = (a+c) + i(b+d)$

$z_2 = c + id$

↳ commutative and associative

② Subtraction

$z_1 - z_2 = (a-c) + i(b-d)$

③ Multiplication

$z_1 z_2 = (ac - bd) + i(ad - bc)$

Inherits commutativity and associativity from real numbers. It is also distributive over addition

Complex Conjugation

↳ flipping sign of i in any expression

if ~~za~~ $z = a + ib$, $z^* = a - ib$

↳ it follows that zz^* is real and non-negative

Modulus of z - $|z| = \sqrt{zz^*} = \sqrt{a^2 + b^2}$

Also, $z + z^* = 2\mathbb{R}(z)$ and $z - z^* = 2i\mathcal{I}(z)$

Division

↳ multiplication with multiplication inverse

⇓

inverse, st $zz^{-1} = z^{-1}z = 1$

$z = a+ib \quad , z^{-1} = c+id$

$(a+ib)(c+id) = 1 = (ac-bd) + i(ad+bc)$

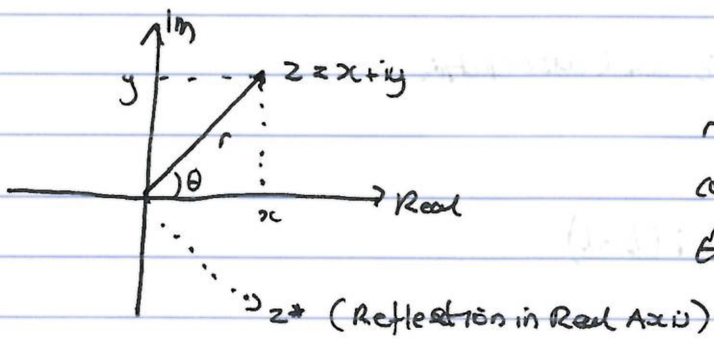
$ac - bd = 1$

$ad + bc = 0$

$$\frac{1}{z} = \frac{1}{a+ib} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2} = \frac{z^*}{|z|^2}$$

$$\therefore \frac{z_1}{z_2} = \frac{z_1 z_2^*}{|z_2|^2}$$

Complex Plane



Polar Form

$r = \sqrt{x^2+y^2} = |z|$

$\cos\theta = \frac{x}{r} \quad , \sin\theta = \frac{y}{r}$

θ is the phase $= \arg(z)$

$z = r(\cos\theta + i\sin\theta)$

$z^{-1} = r^{-1}(\cos\theta - i\sin\theta)$

Multiplication considering argand diagram

↳ Multiply moduli and add arguments

↳ If one has modulus of 1, then just rotate by the argument of it

↳ multiplication by i ≡ rotation anti-clockwise by $\frac{\pi}{2}$

De Moivre's Theorem

Let z be the unit-modulus complex number $z = \cos\theta + i\sin\theta$

$z^2 = zz = \cos 2\theta + i\sin 2\theta$

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad \forall n \in \mathbb{Z}$ (including negative integers)

Example : ~~cosθ + isin~~

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$$
$$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta$$
$$+ 5\cos\theta\sin^4\theta + i\sin^5\theta$$
$$= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) +$$
$$i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$$

$$\therefore \cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$
$$\sin 5\theta = 5\sin^5\theta - 10\sin^3\theta\cos^2\theta + 5\sin\theta\cos^4\theta$$

$$(2\cos\theta)^3 = (z + \tfrac{1}{z})^3$$
$$= z^3 + 3z + 3z^{-1} + z^{-3}$$
$$= (z^3 + z^{-3}) + 3(z + z^{-1})$$
$$= 2\cos 3\theta + 6\cos\theta$$
$$\cos^3\theta = \tfrac{1}{4}\cos 3\theta + \tfrac{3}{4}\cos\theta$$

Euler's Formula

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$ (absolutely convergent for all z)

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} \cdots$$
$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right)$$

$$\therefore \quad \underline{e^{i\theta} = \cos\theta + i\sin\theta}$$

Polar Form : $\underline{z = re^{i\theta}}$

De Moivre is reasonably obvious : $\underline{(e^{i\theta})^n = e^{in\theta}}$

Roots of Unity

- n^{th} roots of unity are the solutions of $z^n = 1$ for n a positive integer

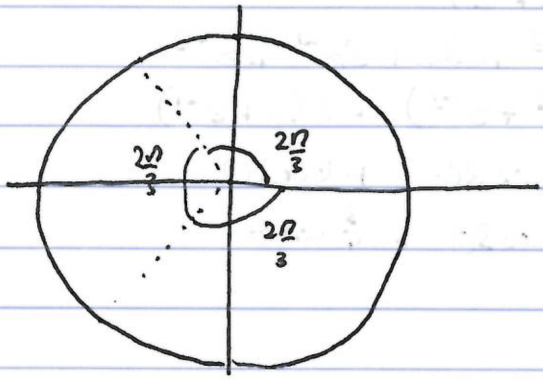
$|z| = 1$ so write $z = e^{i\Theta}$

- for n^{th} roots of unity: $e^{in\Theta} = 1$

$\therefore \Theta = \frac{2\pi h}{n}$ for $h = 0, 1, 2 \dots n-1$

If we write $\omega = e^{\frac{2\pi i}{n}}$, roots are $1, \omega, \dots, \omega^{n-1}$ and come in complex conjugate pairs

All roots fit on the circle of size 1 and fit evenly around it, for example:



Define natural log $\omega = \ln z$ as the solutions of $e^{\omega} = z$

$re^{i\Theta} = e^{\omega} = e^{\ln r} e^{i\Theta} e^{2\pi i n}$

$\therefore \omega = \ln r + i(\Theta + 2\pi n)$

$n = 0, \pm 1, \pm 2, \dots$

The principle value is $\ln r + i\Theta$ where Θ is in the principle range.

<u>Oscillations</u>

- Consider displacement of particle undergoing SHM motion at frequency $\omega = 2\pi f$

$x(t) = a\cos(\omega t + d) = a\mathbb{R}(e^{id} e^{i\omega t}) = \mathbb{R}(ae^{id} e^{i\omega t})$

↳ complex amplitude is $A \equiv ae^{id}$

Example: Particle oscillating at ω st $x(0) = \frac{b}{2}$ and $\dot{x}(0) = -\sqrt{3}\, b\omega/2$. Find amplitude:

$$x(t) = \mathbb{R}(Ae^{i\omega t}) \Rightarrow x(0) = \mathbb{R}(A) = \tfrac{b}{2}$$

$$\dot{x}(t) = \mathbb{R}(Ai\omega e^{i\omega t}) \Rightarrow \dot{x}(0) = \mathbb{R}(i\omega A) = -\omega\, \mathrm{Im}(A) = -\sqrt{3}\, b\omega/2$$

$$\therefore A = \tfrac{b}{2} + \sqrt{3}\, bi/2 = \tfrac{b}{2}(1 + i\sqrt{3}) = be^{i\frac{\pi}{3}}$$

Amplitude $= |A| = \underline{b}$ (and phase $= \delta = \frac{\pi}{3}$)

Fundamental Theorem of Algebra

The equation: $a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$, $a_n \neq 0$
has n complex roots (possibly repeated) for all possible complex coefficients

~~Proof: assume $a_0 \neq 0$ ($a_0 = 0$ trivially gives root $z=0$ which can be divided by z to give polynomial of order $n-1$)~~

Hyperbolic and Trignometric Functions

$$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}), \quad \sin z = \tfrac{1}{2i}(e^{iz} - e^{-iz})$$

For x real, these are as expected:

$$\tfrac{1}{2}(e^{ix} + e^{-ix}) = \tfrac{1}{2}(\cos x + i\sin x + \cos x - i\sin x) = \cos x$$

$$\tfrac{1}{2i}(e^{ix} - e^{-ix}) = \tfrac{1}{2i}(\cos x + i\sin x - \cos x + i\sin x) = \sin x.$$

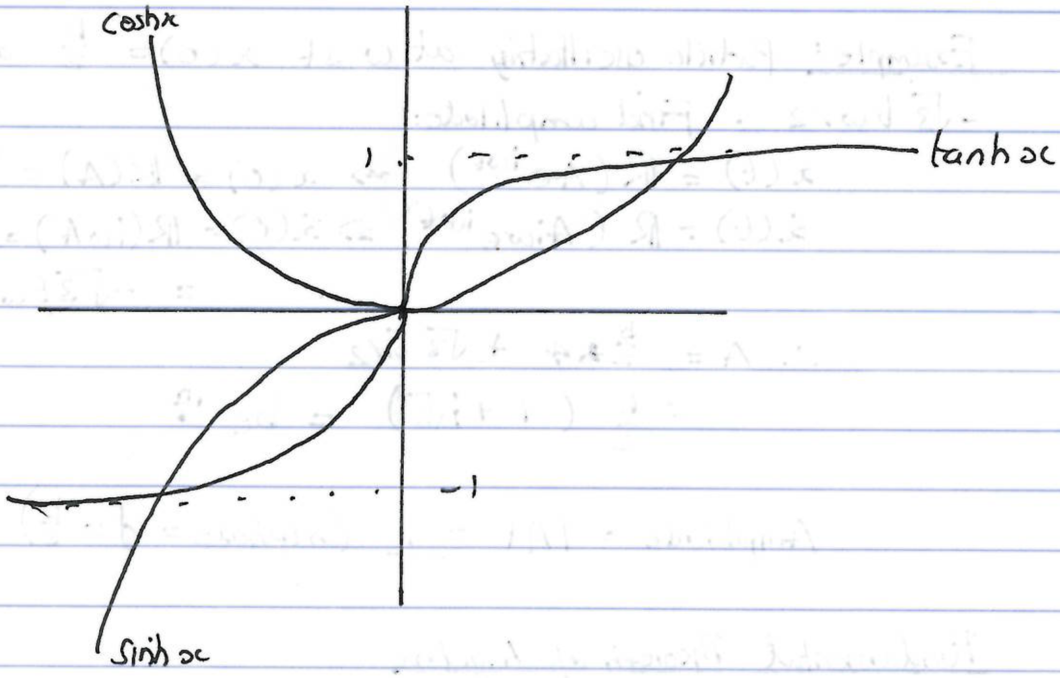
For x along imaginary axis:

$$\cos(ix) = \tfrac{1}{2}(e^{-x} + e^{x}) = \cosh x.$$

$$\sin(iy) = \tfrac{1}{2i}(e^{-y} - e^{y}) = i\sinh y$$

$$\left(\sinh y = \tfrac{1}{2}(e^{y} - e^{-y})\right)$$

We can define $\tanh x$, $\operatorname{sech} x$, $\operatorname{cosech} x$, $\coth x$ by analogy with trig functions



We can use trignometric identities to generate hyperbolic identities

$$\cos^2(x) + \sin^2(x) = 1 \quad , \text{set } x = iy$$
$$\cos^2(iy) + \sin^2(iy) = 1$$
$$\cosh^2(y) + i^2 \sinh^2(y) = 1$$
$$\underline{\cosh^2(y) - \sinh^2(y) = 1}$$

$$\cosh(y_1 \pm y_2) = \cosh(y_1)\cosh(y_2) \pm \sinh(y_1)\sinh(y_2)$$

$$\sinh(y_1 \pm y_2) = \sinh(y_1)\cosh(y_2) \pm \cosh(y_1)\sinh(y_2)$$

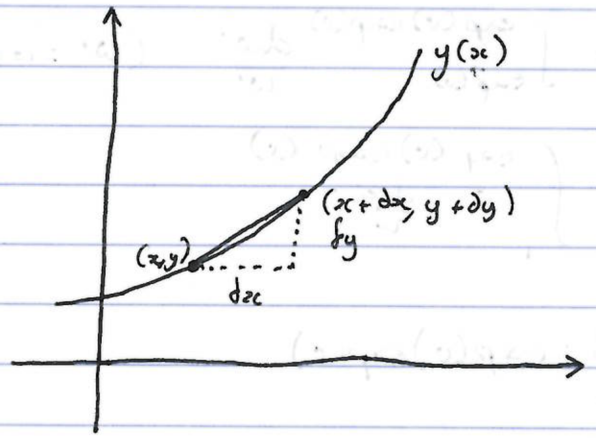
<u>Inverses</u>

$$\underline{y = \cosh^{-1}(x)}$$
$$x = \cosh(y)$$
$$x = \tfrac{1}{2}(e^{x} + e^{-x})$$
$$xe^y = \tfrac{1}{2}(e^{2y} + 1)$$
$$e^{2y} - 2xe^y + 1 = 0$$
$$e^y = x \pm \sqrt{x^2 - 1} \qquad x \geq 1.$$
$$y = \pm \ln(x + \sqrt{x^2 - 1})$$

$$\sinh^{-1}(x) = \pm \ln(x + \sqrt{x^2 + 1})$$

3: Calculus and Curve Sketching

Differentiation From First Principles



$$\frac{dy}{dx} = \lim_{dx\to 0} \frac{dy}{dx} = \tan\theta$$

Ex: $y = x^2$

$dy = (x+dx)^2 - x^2$

$$\frac{\delta y}{\delta x} = \frac{(x^2 + dx^2 + 2x\,dx - x^2)}{dx}$$

$$= dx + 2x$$

$$dx = \lim_{dx\to 0} (2x + dx) = \underline{2x}$$

Differentiability

↳ In differentiating, we assume limit exists. But, this is not always the case

↳ ① If $dy \not\to 0$ when $dx \to 0$ (the function is discontinuous)

↳ ② If dy/dx approaches different values depending on the sign of dx.

↳ Function must continuous and smooth there if it is differentiable at that point.

Elementary Results

$y = x^n$, $y' = nx^{n-1}$

$y = e^x$, $y' = e^x$ → $y = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots$

$y = \sin x$, $y' = \cos x$, $y' = 1 + \frac{2}{2!}x + \frac{3}{3!}x^2 \dots$

$y = \cos x$, $y' = -\sin x$ $= e^x$.

$y = \ln x$, $y' = \frac{1}{x}$.

Can solve $e(x)$ for any real x.
by solving $d(e^x)/dx = e^x$ with
$e^0 = 1$.

$$\int_1^{e^x} \frac{1}{w}\,dw = x,$$

Can we this to show $\exp(x) = e^x$ (behaves like a power)

RTP: $\exp(u)\exp(v) = \exp(u+v)$

$$u+v = \int_1^{\exp(u)} dw/w + \int_1^{\exp(v)} dw/w$$

$$= \int_1^{\exp(u)} \frac{dw}{w} + \int_{\exp(u)}^{\exp(u)\exp(v)} \frac{dw'}{w'} \qquad (w' = w\exp(u))$$

$$\therefore \int_1^{\exp(u+v)} \frac{dw}{w} = \int_1^{\exp(u)\exp(v)} \frac{dw}{w}$$

$$\therefore \exp(u+v) = \exp(u)\exp(v)$$

$$\exp(x) = [\exp(1)]^x = e^x$$

↑ Euler's Constant

Natural Logarithms:

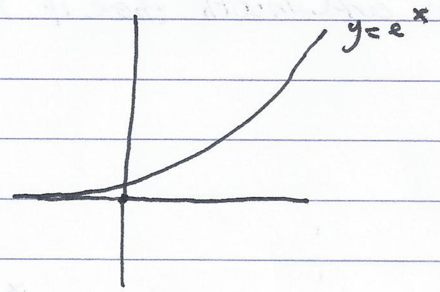
↳ number st $\exp(\ln x) = x$

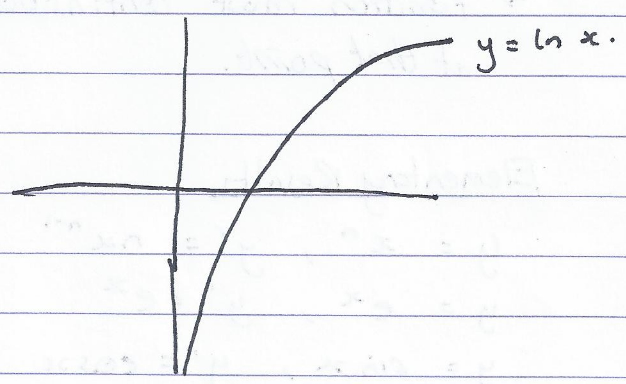
① $\ln(xy) = \ln(x) + \ln(y)$ $\quad \therefore e^{\ln(xy)} = xy = e^{\ln x} e^{\ln y} = e^{\ln x + \ln y}$

② $\ln(x^y) = y \ln x$

↳ follows from for me for rational power y

Can define logs with any base: $\log_b a = \dfrac{\log_c a}{\log_c b}$





Differentiation Rules

① Product Rule: $y(x) = f(x)g(x)$

$$\frac{d(fg)}{dx} = \frac{df}{dx} g + \frac{dg}{dx} f$$

PROOF: $\delta y = f(x+\delta x)g(x+\delta x) - f(x)g(x)$

$= (f+\delta f)(g+\delta g) - fg = \delta f g + f\delta g + \delta f \delta g$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \left[\frac{\delta f}{\delta x} g + f \frac{\delta g}{dx} + \frac{\delta f \,\delta g}{\delta x} \right]$$

$$= g\frac{df}{dx} + f\frac{dg}{dx} + \underbrace{\frac{df}{dx} \lim_{\delta x \to 0} \delta g}_{=0}$$

② Quotient Rule: $y(x) = \frac{f(x)}{g(x)}$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{df}{dx} g - \frac{dg}{dx} f}{g^2} = \frac{f'g - fg'}{g^2}$$

PROOF: $\delta y = \frac{f+\delta f}{g+\delta g} - \frac{f}{g} = \frac{\delta f g - f \delta g}{g \cdot (g+\delta g)}$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \left(\frac{\delta f}{\delta x} g - f \frac{\delta g}{\delta x} \right) \lim_{\delta x \to 0} \frac{1}{g(g+\delta g)}$$

$$= \frac{1}{g^2} \left(\frac{df}{dx} g - f\frac{dg}{dx} \right)$$

③ Chain Rule: $y(x) = f(g(x))$

$$\frac{dy}{dx} = \frac{df}{dg} \frac{dg}{dx} = f'(g(x))g'(x)$$

PROOF: $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta f}{\delta x} = \lim_{\delta x \to 0} \left(\frac{\delta f}{\delta g} \frac{\delta g}{\delta x} \right)$

$$= \lim_{\delta g \to 0} \frac{\delta f}{\delta g} \lim_{\delta x \to 0} \frac{\delta g}{\delta x}$$

$$= \frac{df}{dg} \frac{dg}{dx}.$$

④ Reciprocal Rule: $y(x) \Rightarrow \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$

Implicit Differentiation

Example: ~~...~~ $y^3 + xy + x^2 \sin y = 4$

$$\frac{dy^3}{dx} + \frac{d(xy)}{dx} + \frac{d(x^2 \sin y)}{dx} = 0$$

$$3y^2 \frac{dy}{dx} + y + x\frac{dy}{dx} + 2x\sin y + x^2 \cos y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(3y^2 + x + x^2 \cos y\right) = -y - 2x\sin y$$

$$\frac{dy}{dx} = -\frac{y + 2x\sin y}{3y^2 + x + x^2\cos y}$$

Higher Derivatives: $\dfrac{d^n y}{dx^n} = \dfrac{d}{dx}\left(\dfrac{d^{n-1}y}{dx^{n-1}}\right)$

LEIBNITZ'S FORMULA: $\dfrac{d^n(fg)}{dx^n} = \sum_{m=0}^{n} \binom{n}{m} f^{n-m} g^m$

$$= f^n g^0 + n f^{n-1} g + \tfrac{1}{2}n(n-1) f^{n-2} g^2 + \dots f^0 g^n$$

$$\left\{ \begin{array}{l} \text{Binomial Expansion: } (p+q)^n = \sum_{m=0}^{n} \binom{n}{m} p^{n-m} q^m \\ \text{Pascals Rule: } \binom{n}{m} + \binom{n}{m+1} = \binom{n+1}{m+1} \end{array} \right\}$$

Proof of Leibnitz's Formula (by Induction)

Base case $(n=1)$

↳ proposition: $\dfrac{d(fg)}{dx} = \dfrac{df}{dx} + \dfrac{dg}{dx}$ (by product rule) true

Inductive step (assume true $n=h$)

$$(fg)^{h} = \sum_{m=0}^{h} \binom{h}{m} f^{h-m} g^m$$

For $h+1$ $\quad (fg)^{h+1} = \dfrac{d}{dx} \sum_{m=0}^{h} f^{h-m} g^m$

$$= \sum_{m=0}^{h} \left[f^{(h+1-m)} g^m + f^{h-m} g^{m+1} \right]$$

$$\sum_{m=0}^{h} \binom{n}{m} f^{n-m} g^{m+1} = \sum_{m=0}^{h-1} f^{n-m} g^{m+1} + f^{0} g^{h+1}$$

$$= \sum_{m=1}^{h} \binom{n}{m-1} f^{n+1-m} g^{m} + f^{0} g^{h+1}$$

$$(fg)^{h+1} = f^{h+1} g^{0} + \sum_{m=1}^{h} \left[\binom{h}{m} + \binom{h}{m-1}\right] f^{h+1-m} g^{m} + f^{0} g^{h+1}$$

$$= f^{h+1} g^{0} + \sum_{m=1}^{h} \binom{h+1}{m} f^{h+1-m} g^{m} + f^{0} g^{h+1}$$

$$= \sum_{m=0}^{h} \binom{h+1}{m} f^{h+1-m} g^{m} \quad \therefore \text{ true for } n = h+1$$

$\therefore$ since true for base case and given true for $n=h$, true for $n=h+1$, it is true for all $n \in \mathbb{N}$ by mathematical induction.

<u>Hermite Polynomials</u> : arise in the quantum mechanics of the harmonic oscillator, defined by:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

Show: $H_{n+1} = 2x H_n - 2n H_{n-1}$

$$H_{n+1} = (-1)^{n+1} e^{x^2} \frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right)$$

$$= (-1)^{n+1} e^{x^2} \frac{d^n}{dx^n}\left(-2x e^{-x^2}\right)$$

$$= 2(-1)^n e^{x^2}\left(x \frac{d^n}{dx^n} e^{-x^2} + n \frac{d^{n-1}}{dx^{n-1}} e^{-x^2}\right)$$

$$= 2x H_n - 2n H_{n-1} \qquad \text{LEIBNITZ}$$

<u>Curve Sketching</u> : things to consider

↳ ① Symmetry : does $y(-x) = y(x)$ - even, $y(-x) = -y(x)$ - odd, or neither

② Where does it cross axes

③ Limiting behaviour (oblique assymptotes)

④ Stationary points (and their nature)

⑤ Singular points and vertical assymptots

Stationary Point: occur when $\frac{dy}{dx} = 0$

↳ can be maxima, minima or inflection point

↳ and tell nature from second derivative

↓↓

$\frac{d^2y}{dx^2} < 0 \Rightarrow$ Maximum

$\frac{d^2y}{dx^2} > 0 \Rightarrow$ Minimum

if $\frac{d^2y}{dx^2} = 0$, it could be an inflection point and should test around the point.

Point of Inflection: at point, $\frac{d^2y}{dx^2}$ changes sign.

↳ $\frac{d^2y}{dx^2} = 0$ a necessary but not a sufficient condition

↳ Always an inflection point between adjacent maxima and minima.

Asymptotes: Example: $y = \frac{x^2}{x-1}$

↳ y singular at $x = 1$

$\lim_{x \to 1^+} y = +\infty$, $\lim_{x \to 1^-} y = -\infty$, so $y = 1$ is a vertical asymptotes.

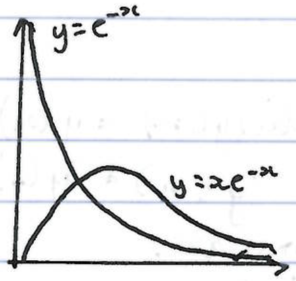
↳ As $|x| \to \infty$

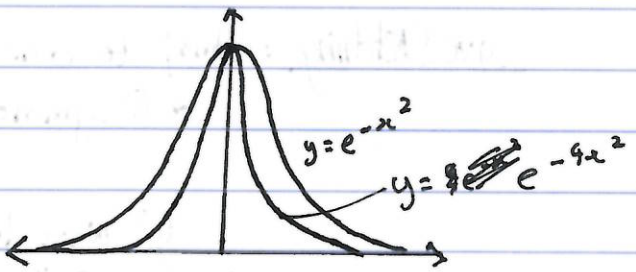
$$y = x(1 - \tfrac{1}{x})^{-1} \to x\left(1 + \tfrac{1}{x} + \tfrac{1}{x^2} \dots\right)$$

$$= x + 1 + \tfrac{1}{x} \dots$$

$\therefore$ $y = x + 1$ is an oblique asymptote.

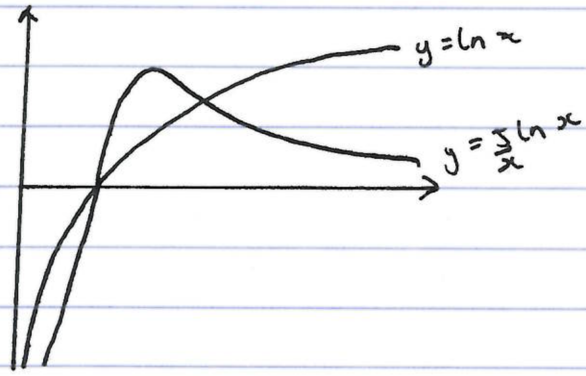
Useful Graphs

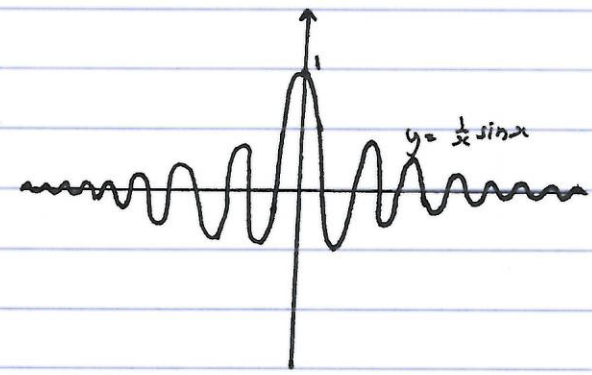




Gaussian (Normal) Distribution

$e^{-ax^2} \to \frac{1}{a}$ is proportional to width of distribution.


y = ln x
y = 5/x ln x


1
y = 1/x sin x

4: Elementary Analysis

Limits: Intuitive notion of $\lim_{x \to x_0} f(x) = K$ is that it means $f(x)$ can be made as close as we like to K by making x sufficiently close to x_0

↳ For some real function $f(x)$ defined on some open interval containing x_0 (but not necessarily at x_0) $\lim_{x \to x_0} = K$ means

↳ for any $\varepsilon > 0$, $\exists \delta > 0$ st:

$|f(x) - K| < \varepsilon \;\forall\; 0 < |x - x_0| < \delta$

↳ $f(x) \to K$ as $x \to x_0$

Limits at Infinity: Can define limits as $x \to \pm\infty$ in the same way

$\lim_{x \to \infty} f(x) = K$ iff

↳ $\forall \varepsilon > 0, \exists X > 0$ st

$|f(x) - K| < \varepsilon \;\forall x > X$

($f(x)$ needs to be defined $\forall x > X$)

Algebra of Limits

① Equality: if $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} f(x) = B \Rightarrow A = B$
(if $f(x)$ has a limit, it is unique)

$\lim_{x \to x_0} f(x) = A$

$\lim_{x \to x_0} g(x) = B$

② Addition: $\lim_{x \to x_0} (f(x) + g(x)) = A + B$

③ Multiplication: $\lim_{x \to x_0} (f(x)g(x)) = AB$

④ Division

↳(i) $\lim_{x \to x_0} (f(x)/g(x)) = \frac{A}{B}$ for $B \neq 0$

↳(ii) $\lim_{x \to x_0} (f(x)/g(x))$ does not exist for $B = 0$ and $A \neq 0$

↳(iii) $\lim_{x \to x_0} (f(x)/g(x))$ <u>may</u> exist for $A = B = 0$

↳(iv) $\lim_{x \to x_0} (f(x)/g(x))$ <u>may</u> exist for $A = B = \infty$

L'Hôpital's Rule

↳ Assuming $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ are both 0 or $\pm\infty$ AND $\lim_{x\to x_0} (f'(x)/g'(x))$ exists, then:

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$$

↳ This can be used repeatedly: if $\lim_{x\to x_0} (f'/g')$ is still indeterminate, you can try $\lim_{x\to x_0} (f''/g'')$

↳ The proof follows directly from the definition of the derivative.

Orders of Magnitude and O Notation

↳ Big-O notation expresses the manner in which a function approaches a point (can be infinity)

↳ We say $f(x) = O(g(x))$ as $x \to a$ if $f(x)$ behaves like $g(x)$ in the vicinity of $x = a$.

↳ More formally, $f(x) = O(g(x))$ as $x \to a$ iff $\exists$ constants ϵ and $K > 0$ st $|f(x)| \leq K|g(x)|$ $\forall$ $|x-a| < \epsilon$

↳ Define the order of magnitude at infinity: $f(x) = O(g(x))$ as $x \to \infty$ iff: $\exists$ constants X and $K > 0$ st $|f(x)| \leq K|g(x)|$ $\forall x > X$

Continuity

↳ Continuous function is one for which small changes in the argument produce small changes in the function

↳ Formally: real function $f(x)$ is continuous at $x = a$ iff:

(i) $f(a)$ exists

(ii) $\lim_{x\to a} f(x)$ exists and equals $f(a)$

↳ for any $\epsilon > 0$, $\exists \delta > 0$ st $|f(x) - f(a)| < \epsilon$ $\forall$ $|x-a| < \delta$

$\Downarrow$

$|x-a| < \delta$ includes the point $x = a$ (unlike the ϵ-δ condition for existence of a limit)

5: Infinite Series

n^{th} Partial Sum is: $S_n = \sum_{k=0}^{n} u_k$

↳ If the partial sums have a finite limit (S) as $n \to \infty$, we can say that $\sum_{k=0}^{\infty} u_k$ converges to S

↳ If something doesn't converge, it may diverge (if $\lim_{n\to\infty} S_n = \pm\infty$) or oscillate (eg. $S_n = \sum_{k=0}^{n} (-1)^k = 0$ or 1)

Formal Definition of Convergence

if $\lim_{n\to\infty} S_n = S$, for any ϵ, $\exists N$ s.t.

$|S - S_n| < \varepsilon \quad \forall n > N$

↳ A necessary condition for convergence is that $u_k \to 0$ as $k \to \infty$

↳ If $\underbrace{\sum_{k=0}^{\infty} u_k = A \text{ and } \sum_{k=0}^{\infty} v_k = B}_{\text{both converge}} \Rightarrow \sum_{k=0}^{\infty} u_k + v_k = A + B$

Absolute Convergence

↳ If $\sum_{k=0}^{\infty} |u_k|$ converges, the series is absolutely convergent

↳ $\Rightarrow \sum_{k=0}^{\infty} u_k$ converges

↳ If $\sum_{k=0}^{\infty} u_k$ converges but $\sum_{k=0}^{\infty} |u_k|$ does not converge, then the series is conditionally convergent

↳ For an absolutely convergent series we get the same result if we sum terms in any order

↳ This is not true for conditionally convergent series

Geometric Progression

$S_n = \sum_{k=0}^{n} r^k = \dfrac{1 - r^{n+1}}{1-r}$

Proof: $rS_n = r + r^2 \dots r^{n+1} = r^{n+1} + S_n - 1$

$S_n(r-1) = r^{n+1} - 1$

$S_n = \dfrac{r^{n+1} - 1}{r - 1}$

If $|r| < 1$, series is absolutely convergent:

$\sum_{k=0}^{\infty} r^k = \dfrac{1}{1-r}$, if $|r| \geq 1$, then the series necessarily doesn't converge.

Harmonic Series: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} \cdots$

↳ ① $\lim_{n\to\infty} u_n = 0$ (but not sufficient to ensure convergence)

↳ Can establish that the series diverges, by regrouping:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

↳ each term in brackets:

$$\frac{1}{p+1} + \frac{1}{p+2} + \cdots + \frac{1}{p+p} > \frac{p}{2p} = \frac{1}{2}$$

↳ $\therefore \sum_n \frac{1}{n} > \sum_n \frac{1}{2}$ $\therefore$ necessarily divergent

↳ Actually diverges very slowly (amongst the most slowly)

Convergence Test

↳ ① Comparison Test

↳ compare the unknown positive series $\sum u_n$ with a known series $\sum_n v_n$ of positive terms.

↳ if $u_n \leq v_n$ for all $n \geq K$ then $\sum_{n=0}^{\infty} u_n$ is convergent if $\sum_{n=0}^{\infty} v_n$ converges

↳ ~~opposite~~ if $u_n \geq v_n$ $\forall n \geq K$ and $\sum_{n=0}^{\infty} v_n$ diverges then $\sum_{n=0}^{\infty} u_n$ also diverges

↳ ② Ratio Test

↳ Given a positive series $\sum u_n$ if:

① $\lim_{n\to\infty} u_{n+1}/u_n < 1$ CONVERGENT

② $\lim_{n\to\infty} u_{n+1}/u_n > 1$ DIVERGENT

(③ $\lim_{n\to\infty} u_{n+1}/u_n = 1$ MAY CONVERGE) Requires a different test

↳ Proof: Let $\lim_{n\to\infty} u_{n+1}/u_n = L$ then if $L < 1$ then $\exists \epsilon > 0$ with $L < L + \epsilon < 1$

↳ By existence of limit: $\exists K$ s.t.

$$L - \epsilon < u_{n+1}/u_n < L + \epsilon \quad \forall n > K - 1$$

$$\Rightarrow u_{K+1} < (L+\epsilon) u_K$$

$$\& \quad u_{K+2} < (L+\epsilon)^2 u_K \quad \text{etc.}$$

Terms are a geometric progression with common ratio $(L+\epsilon) < 1$ and hence the original series converges.

Leibnitz Criterion for Alternating Series

$\sum_n (-1)^{n+1} a_n$ with $a_n > 0$ converges if a_n is (i) monotonic decreasing (for large enough n and (ii) $\lim_{n\to\infty} a_n = 0$

Proof: $S_{2n} = a_1 - a_2 + a_3 - \cdots - a_{2n}$

$S_{2(n+1)} - S_{2n} = a_{2n+1} - a_{2n+2} > 0$

↳ We also have $S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - a_{2n} < 1$ since partial sums are bounded above and monotonically increasing: therefore have a finite limit S ($< a_1$) as $n \to \infty$

↳ Alternating sums converge provided odd partial sums have same limit: $\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} (S_{2n} + a_{2n+1}) = S + 0$ since $a_n \to 0$ as $n \to \infty$

Power Series

Power Series: Specific examples of infinite series of form:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 \ldots$$

↳ This defines the function $f(x)$ provided the series converges.

⇓

↳ Generally can converge based on three possible conditions

1. Series converges for $x = 0$
2. Converges for all finite x
3. Converges for $|x| < R$, diverges $|x| > R$ and may or may not converge for $|x| = R$

↳ We can test convergence using the ratio test

Circle of Convergence

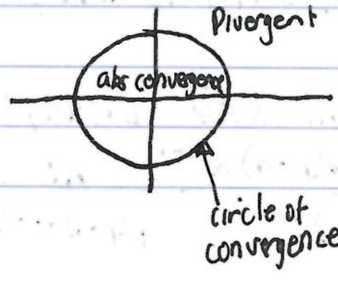
Circle of Convergence (consider complex power series)

⇓

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (a_n, z \in \mathbb{C})$$

↳ The condition for convergence is as for before

↳ $|z| < 1/\lim_{n\to\infty} |a_{n+1}/a_n|$ (defines a circle of convergence in complex plane)



$\left(\text{rad} = \dfrac{1}{\lim \left(\frac{a_{n+1}}{a_n}\right)}\right)$

Taylor Series

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f^2(a) + \frac{1}{3!}(x-a)^3 f^3(a) + \dots$$

Taylor's Theorem: provides an upper bound on the error if a function is approximated by a truncation of its Taylor series at finite n.

↳ $$f(x) = f(0) + xf'(0) + \frac{1}{2!}x^2 f^2(0) + \dots + \underbrace{\frac{1}{n!}x^n f^n(\xi)}_{R_n} \quad 0 \leq \xi \leq x$$

$$R_n = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f^n(t)\,dt$$

Examples

① $\exp(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \dots$

↳ Ratio test. $\lim_{n\to\infty}\left|\frac{x^{n+1} n!}{x^n (n+1)!}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0 \therefore$ converges $\forall x$.

↳ $R_n = \underbrace{\frac{x^n}{n!}e^{\xi}}_{0 \leq \xi \leq x} \leq \frac{x^n e^x}{n!}$

② $\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$

↳ odd function (only has odd powers)

③ $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$ → even function (only has even powers)

↳ sin and cos series are absolutely convergent $\forall x$

Binomial Theorem

$$f(x) = (1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots$$

↳ For p as an integer, series terminates and coefficient of x^n $(n \leq p)$ is $\binom{p}{n}$

↳ For general p, infinite series absolutely converges for $|x| < 1$

Ratio Test: $\lim_{n\to\infty}\left|\frac{p(p-1)\dots(p-n)x^{n+1}}{(n+1)!}\frac{n!}{p(p-1)\dots(p-n+1)x^n}\right| = |x|\lim_{n\to\infty}\left|\frac{p-n}{n+1}\right| = |x|$

④ $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

↳ Ratio Test: $\lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1} \times \frac{n}{x^n}\right| = |x|\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = |x|$

↳ $R_n = \frac{x^n (n-1)!}{n!\,(1+\xi)^n} = \frac{x^n}{n(1+\xi)^n} \leq \frac{x^n}{n}$ ∴ tends to zero for $0 \leq x \leq 1$.

↳ Series converges ~~c~~ conditionally for $x = 1$. (alternating harmonic series)

<u>Newton-Raphson Method</u>

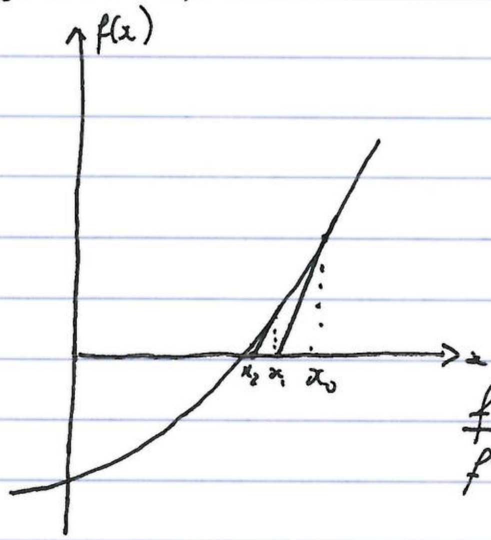
↳ Idea is that we want to find x_* s.t. $(f(x_*) = 0)$

↳ Given initial guess, x_0, need to find h s.t. $f(x_0 + h) = 0$

$$0 = f(x_0 + h) \simeq f(x_0) + hf'(x_0) \quad \left\{\begin{array}{l}\text{Approximation by}\\ \text{linear Taylor approximation}\end{array}\right\}$$

$$\Rightarrow \boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

↳ Equivalent to extrapolating the local derivative at x_n to find the next estimate of the root



<u>Convergence in Newton-Raphson</u>

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$\Rightarrow \epsilon_{i+1} = \epsilon_i - \frac{f(x_* + \epsilon_i)}{f'(x_* + \epsilon_i)}$$

$$\frac{f(x_*+\epsilon_i)}{f'(x_*+\epsilon_i)} = \frac{f(x_*) + \epsilon_i f'(x_*) + \frac{1}{2}\epsilon_i^2 f''(x_*) + \cdots}{f'(x_*) + \epsilon_i f''(x_*) + \cdots}$$

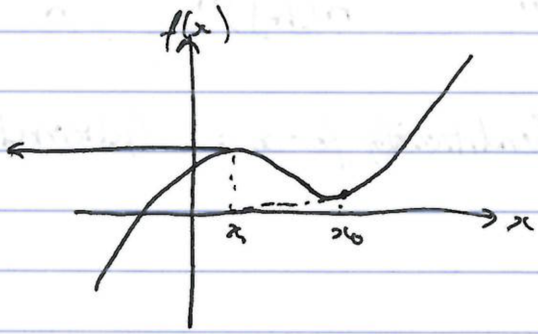
$$= \epsilon_i \left(\frac{1 + \frac{\epsilon_i}{2} f''(x_*)/f'(x_*)}{1 + \epsilon_i f''(x_*)/f'(x_*)} \right)$$

$$\simeq \epsilon_i \left(1 - \tfrac{1}{2}\epsilon_i f''(x_*)/f'(x_*)\right)$$

$$\therefore \epsilon_{i+1} \simeq \epsilon_i^2 \frac{f''(x_*)}{f'(x_*)} \quad \left(\begin{array}{l}\text{Quadratic local}\\ \text{convergence}\end{array}\right)$$

<u>Global Convergence</u>: convergence very rapid when near root but global convergence is not guaranteed

↳ can go to a different root and can go off to a very large $|x|$



6: Integration

Can be introduced in two ways – (1) Inverse of differentiation → Indefinite Integral

↳ (2) Area under the curve → Definite Integral

↳ Riemann Integral

Partition fixed interval $[a, b]$ with $N+1$ x coordinates

$a = x_0 < x_1 < x_2 < \cdots < x_N = b$

↳ We choose N points $\xi_0, \xi_1, \xi_2, \ldots, \xi_{n-1}$, one in each interval $x_i \leq \xi_i \leq x_{i+1}$

$$\int_a^b f(x)\,dx = \lim_{N \to \infty} \underbrace{\sum_{i=0}^{N-1} f(\xi_i)(x_{i+1} - x_i)}_{\text{Riemann Sum}}$$

Properties

(i) $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$

(ii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

(iii) $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

(iv) $\int_a^b k f(x) = k \int_a^b f(x)\,dx$

Fundamental Theorem of Calculus

$$F(y) = \int_a^y f(x)\,dx, \quad F(y + \delta y) = \int_a^{y+\delta y} f(x)\,dx$$

↳ over, range $[y, y+\delta y]$ $f(x) = f(y) + O(\delta y)$

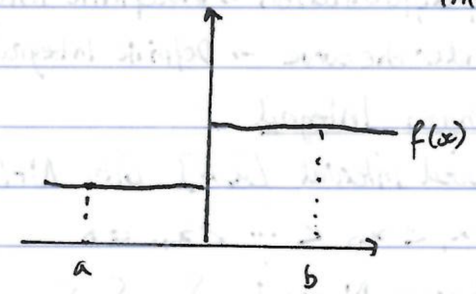
$$F(y + \delta y) = F(y) + f(y)\,\delta y + O(\delta y^2)$$

$$\frac{dF}{dy} = \lim_{\delta y \to 0} \frac{F(y + \delta y) - F(y)}{\delta y} = f(y) + \lim_{\delta y \to 0} O(\delta y) = f(y)$$

Infinite Integrals: $\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx.$

Improper Integrals: For range where integrand is singular within range of integration, define improper integral by excluding small region around singularity then taking limits

<u>Discontinuous Integrals</u>: If finite range of discontinuities, break up integrals into regions over which $f(x)$ is continuous



$$\int_a^b f(x)dx = \int_a^0 f(x)dx + \int_0^b f(x)dx$$

<u>Integration Tricks</u>

① Inspection: Spot that integrand is derivative of some function

② Substitution: Essentially formalises integration by inspection

↳ Make a substitution $u(x)$ in the integral $\int f(x)\,dx$ making it $\int f(u)\,du$

↳ We have to change the integration limits for definite integrals

③ Trigonometric Substitutions: → For $(a^2 - x^2)$ $x = a\sin\theta$ substitution works

↳ For $(a^2 + x^2)$ $x = a\tan\theta$ useful

↳ $x = a\tan(\tfrac{1}{2}\theta)$ often works

$$\left\{ \text{For } t = \tan\tfrac{1}{2}A,\ \sin A = \frac{2t}{1+t^2},\ \cos A = \frac{1-t^2}{1+t^2} \quad d\theta = \frac{2}{1+A^2}dA \right\}$$

④ Hyperbolic Substitutions: → $\sqrt{x^2-a^2}$ $(x = a\cosh y)$

↳ $\sqrt{x^2+a^2}$ $(x = a\sinh y)$

⑤ Partial Fractions: Can seperate into parts then integrate the parts seperately

⑥ Complete the square: → Useful for integrands of the form $\frac{1}{\sqrt{Q(x)}}$ or $\frac{1}{Q(x)}$ where $Q(x)$ is quadratic

↳ Therefore, we can then use trigonometric substitution

⑦ Integration by Parts: $\int f \frac{dg}{dx} dx = fg - \int g \frac{df}{dx} dx$

⑧ Trigonometric Identities: Can replace $\sin\theta$ and $\cos\theta$ with multiple angle formulae

⑨ <u>Complex Numbers</u>: → Useful for combinations of exponential and trignometric functions.

Example: $\int \cos x \, e^{ax} dx$; $\cos x = \Re(e^{ix})$

$$= \Re \int e^{ix} e^{ax} dx = \Re \int e^{x(i+a)} dx.$$

$$= \Re\left(\frac{1}{a+i} e^{(a+i)x}\right) = \Re\left(\frac{a-i}{a^2+1} e^{ax}(\cos x + i\sin x)\right)$$

$$= \underline{\frac{e^{ax}}{a^2+1}(a\cos x + \sin x)}$$

⑩ Symmetry: Even Functions → $f(x) = f(-x)$

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx.$$

Odd Function → $f(x) = -f(-x)$

$$\int_{-a}^{a} f(x)dx = 0$$

<u>Differentiation of Integrals wrt Parameters</u>

↳ Consider integral $I(a) = \int_{b(a)}^{c(a)} f(x;a)\,dx$. where a is a parameter on which the integrand and limits may depend.

$$\frac{dI}{da} = \lim_{\delta a \to 0} \frac{I(a+\delta a) - I(a)}{\delta a}$$

$$= \lim_{\delta a \to 0} \frac{1}{\delta a}\left(\int_{b(a+\delta a)}^{c(a+\delta a)} f(x;a+\delta a)dx - \int_{b(a)}^{c(a)} f(x;a)dx\right)$$

$$= \lim_{\delta a \to 0} \frac{1}{\delta a}\left(\int_{b(a)}^{c(a)} [f(x;a+\delta a) - f(x;a)]dx + \int_{c(a)}^{c(a+\delta a)} f(x;a+\delta a)dx \right.$$

$$\left. - \int_{b(a)}^{b(a+\delta a)} f(x;a+\delta a)dx\right)$$

$\frac{\partial f(x;a)}{\partial a}$ is partial derivative of $f(x;a)$ wrt a $= \lim_{\delta a \to 0} \frac{f(x;a+\delta a) - f(x;a)}{\delta a}$

$$\frac{dI}{da} = \int_{b(a)}^{c(a)} \frac{\partial f(x;a)}{\partial a}dx + \lim_{\delta a\to 0}\left(\frac{\delta c}{\delta a} f(c;a+\delta a)\right) - \lim_{\delta a\to 0}\left(\frac{\delta b}{\delta a} f(b;a+\delta a)\right)$$

∴ By intermediate value theorem.

$$= \int_{b(a)}^{c(a)} \frac{\partial f(x;a)}{\partial a}dx + \frac{dc}{da}f(c;a) - \frac{db}{da}f(b;a)$$

↳ This is very useful for evaluating integrals.

Approximating Sums as Integrals

↳ Example : Rod of length a with mass per length varying linearly from μ to 3μ. Finding total mass

Mass per length at distance x from end = $\mu(1+2x/a)$

Mass of element of length δx centred on x = $\delta m = \mu\left(1+\frac{2x}{a}\right)\delta x$

$$m = \lim_{\delta x\to 0}\sum dm = \lim_{\delta x \to 0}\sum \mu\left(1+\frac{2x}{a}\right)\delta x$$

$$= \int_0^a \mu\left(1+\frac{2x}{a}\right)dx$$

$$= \mu\left[x + \frac{x^2}{a}\right]_0^a = \underline{2\mu a}$$

Stirling's Approximation : $\ln(n!) = \sum_{k=1}^{n}\ln k$

↳ Used to approximate factorial

↳ And also gamma function

$$\Downarrow$$

$$\int_1^n \ln x \leq \sum_{k=1}^{n}\ln k \leq \int_1^{n+1}\ln x\,dx.$$

$$n\ln n - n + 1 \leq \ln n! \leq (n+1)\ln(n+1) - n$$

Approximation is : $\ln n! \approx n\ln n - n$

↳ Fractional error is $O\left(\frac{1}{n}\right)$ as $n\to\infty$

But $n! = n^n e^{-n}$ is a bad approximation $O(n)$ error

Better approximation is : $n! \approx n^n e^{-n}\sqrt{2\pi n}$

Schwartz's Inequality

$|a \cdot b| \leq |a||b|$

$(a \cdot b)^2 \leq |a|^2 |b|^2$

$\left(\sum_{i=1}^{N} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{N} a_i^2\right)\left(\sum_{i=1}^{N} b_i^2\right)$ in n dimensions

Taking sum: $\left(\int_a^b f(x)g(x)\,dx\right)^2 \leq \int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx$ } this generalises scalar product to infinite vector space

Can prove more formally by saying $\int_a^b (f+\lambda g)^2 dx \geq 0$ for all real functions $f(x)$ and $g(x)$ and real λ:

$$\therefore \int_a^b f^2(x)\,dx + 2\lambda \int_a^b f(x)g(x)\,dx + \lambda^2 \int_a^b g^2(x)\,dx \geq 0$$

Since true for all λ, $b^2 - 4ac \geq 0$:

$$4\left(\int_a^b f(x)g(x)\right)^2 \leq 4\int_a^b f^2(x)dx \int_a^b g^2(x)dx$$

↳ We get equality iff $f(x) \propto g(x)$

Multiple Integrals

↳ We can extend Riemann definition of an integral in one variable to several variables

$$\iint_S f(x,y)\,dx\,dy = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \sum_i \sum_j f(x_i, y_j)\,\delta x\,\delta y$$

↳ For double integrals wrt x and wrt y we can do it in either order (need to change limits)

Plane-Polar: Double integrals are now of form: $\iint_S f(r,\phi)\,dr\,d\phi$

↳ For some simple regions of integration, the limits of the nested integrals might not depend on the other variable in which case, the double integral reduces to the product of integrals: $\iint_S f(x)g(y)\,dx\,dy = \int f(x)dx \int g(y)\,dy$

Volume Integrals

↳ ① Cartesian: $dV = dx\,dy\,dz$

↳ ② Cylindrical Polar: $dV = r\,dr\,d\phi\,dz$

↳ ③ Spherical Polar: $dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$

Gaussian Integrals

: Integrals of the form: $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$

↳ Actually $I = \lim_{a \to \infty} \int_{-a}^{a} e^{-x^2}\,dx$

↳ We can find this using:

↳ $$I^2 = \int_{-a}^{a} e^{-x^2}\,dx \int_{-a}^{a} e^{-y^2}\,dy$$

$$= \iint_S e^{-(x^2+y^2)}\,dx\,dy$$

$$\iint_{S_-} e^{-r^2}\,r\,dr\,d\phi < I^2 < \iint_{S_+} e^{-r^2}\,r\,dr\,d\phi$$

$$2\pi \int_0^a e^{-r^2}\,r\,dr < I^2 < 2\pi \int_0^{\sqrt{2}a} e^{-r^2}\,dr$$

$$\pi(1-e^{-a^2}) < I^2 < \pi(1-e^{-2a^2})$$

$$\lim_{a \to \infty} I = \sqrt{\pi}$$

Gaussian Distribution

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(x-\mu)^2/2\sigma^2}$$

where x is a random variable with a probability density function $P(x)$ and mean of $x = \mu$ and variance $= \sigma^2$

↳ Distribution is normalised so that:

$$\int_{-\infty}^{\infty} P(x)\,dx = 1$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\,dx$$

$$= \frac{\sqrt{2\sigma^2}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-u^2}\,du = \underline{1} \text{ (QED)}$$

Error Function: For Gaussian distributions, how much probability lies between $\pm a\sigma$

↳ $= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$

ERF is defined by the integral: $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$

$\{ \hookrightarrow \mathrm{erf}(\infty) = 1 \}$

7: Probability

Want to quantify the plausibility of the various outcomes of some random experiment
↳ Must have a set of rules that define it completely and involve some chance

Sample Space: Set of all possible outcomes
↳ assumed to be discrete
↳ mutually exclusive

Event: Subset of the sample space.
↳ $P(A)$ expresses how likely an event is ⇒ relative frequency of A in N trials of the experiment as $N \to \infty$

Basic Properties → ① $0 \leq P(A) \leq 1$

② $P(\Omega) = 1$
↑ sample space

③ If events A and B have no outcomes in common, then:
$P(A|B) = P(A) + P(B)$

④ $P(A) = \sum_{i: w_i \in A} p_i \Rightarrow \sum_i p_i = 1$

Complement: Complement A' is outcomes in sample space that are not A
↳ any given outcome is either in A or A'

↳ $P(A \cap B)$ is probability of both A and B occurring
↳ $P(A \cup B)$ is probability of either A or B occurring: $= P(A) + P(B) - P(A \cap B)$
↳ ∀ events $A \cup A' = \Omega$ and $A \cap A' = \phi$

Conditional Probabilities: $P(A|B)$ = Probability of being in A given that it is in B.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\therefore P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

↳ If events are independent $(P(A \cap B) = P(A)P(B))$
↳ $P(A|B) = P(A)$

Law of Total Probability: $\{B_i\}$ = collection of mutually exclusive events.
↳ The $\{B_i\}$ is a partition of Ω. Any event C and C' all partition Ω

↳ $P(A) = \sum_i P(A \cap B_i) = \sum_i P(A|B_i)P(B_i)$

Bayes' Theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Permutations and Combinations

↳ Permutations are all possible combinations of outcomes (order-specific)

↳ Combinations are groups where the ordering doesn't matter.

↳ Total number of ways of selecting r objects from n possibilities when order matters $= {}^nP_r = \frac{n!}{(n-r)!}$

↳ Combinations: choosing r objects from n, where ordering is not important $= \frac{n!}{(n-r)!\,r!} = {}^nC_r = \binom{n}{r}$

Random Variables: Real valued variable whose value is subject to variations involving an element of chance.

↳ Generally more than one outcome can give the same value for the random variable

↳ Can think of each allowed value of the random variable, $x_j, j = 1 \dots N$, corresponding to a subset of Ω

↳ Can assign a probability distribution $P(x)$ on X by the probability of the corresponding event in Ω.

Discrete Random Variable: variable takes a (possibly countably infinite) number of discrete values.

Continuous Random Value: continuous range of values.

<u>Mean</u> : $\langle x \rangle = \sum_{j=1}^{N} x_j P(X = x_j)$
(Expected value)

<u>Variance</u> $Var(X) = \langle (X - \langle x \rangle)^2 \rangle = \sum_{j=1}^{N} (x_j - \langle x \rangle)^2 P(X = x_j)$
$= \sigma^2$

$$= \sum_{j=1}^{N} (x_j^2 - 2x_j \langle x \rangle + \langle x \rangle^2) P(X = x_j)$$
$$= \langle x^2 \rangle - 2\langle x \rangle \sum_{j=1}^{N} x_j P(X = x_j) + \langle x \rangle^2 \sum_j P(X = x_j)$$
$$= \langle x^2 \rangle - 2\langle x \rangle^2 + \langle x \rangle^2$$
$$= \langle x^2 \rangle - \langle x \rangle^2$$

<u>Binomial Distribution</u>

↳ Two possibilities, A and B.
↳ Consider number of As in n tests.
↳ Distribution is $B(n, p)$

probability of getting an A

$$P(X = r) = \binom{n}{r} p^r q^{n-r}$$

<u>Mean</u> : $\langle x \rangle = \sum_{r=0}^{n} r P(X = r) = \sum_{r=0}^{n} \frac{r n!}{(n-r)!\, r!} p^r q^{n-r}$

$$= np \sum_{r=1}^{n} \frac{(n-1)!}{(n-r)!\,(r-1)!} p^{r-1} q^{n-r}$$

$$= np (p+q)^{n-1} = \underline{np}$$

<u>Variance</u> : $\langle x^2 \rangle = \sum_{r=0}^{n} r^2 P(X = r) = n(n-1)p^2 + np$

Variance $= n(n-1)p^2 + np - n^2p^2$
$$= (n^2 - n)p^2 + np - n^2p^2$$
$$= n^2p^2 - np^2 + np - n^2p^2$$
$$= np - np^2$$
$$= np(1-p) = \underline{npq}$$

<u>Poisson Distribution:</u> ↳ Discrete distribution for a random variable X that takes $X = 1, 2 \dots$

↳ $P(X=r) = e^{-\lambda} \frac{\lambda^r}{r!}$ $(r = 0, 1, 2 \dots)$

↳ Mean $= \lambda$

↳ Variance $= \lambda$

↳ Poisson Distribution is the limit of the Binomial Distribution as $n \to \infty$ and $p \to 0$ st the mean $np = \lambda$

⇓

$$P(X=r) = \frac{n!}{(n-r)!\,r!} p^r (1-p)^{n-r}$$

$$= \frac{n(n-1)\cdots(n-r+1)}{n^r} \frac{(np)^r}{r!} \frac{(1-p)^n}{(1-p)^r}$$

$$= \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right)}{\left(1-\frac{\lambda}{n}\right)^r} \frac{\lambda^r}{r!} \left(1-\frac{\lambda}{n}\right)^n$$

Taking limits as $n \to \infty$:

$$\left(1-\frac{\lambda}{n}\right)^n = 1 - n\frac{\lambda}{n} + \frac{n(n-1)}{2}\frac{\lambda^2}{n^2} + \dots \to 1 - \lambda + \frac{\lambda^2}{2} + \dots = e^{-\lambda}$$

∴ at finite r, we get $P(X=r) \to e^{-\lambda} \frac{\lambda^r}{r!}$

<u>Continuous Random Variables</u>: Probability Density Function defined st:

$$P(x - dx/2 \le X \le x + dx/2) = f(x)dx.$$

$$P(a \le X \le b) = \int_a^b f(x)dx \qquad \{ f(x) \ge 0 \}$$

N.B. $P(X = x_0)$ is very very low. Therefore, generally, look at cumulative distribution function $F(a)$

$$F(a) = P(X \le a) = \int_{-\infty}^{a} f(x)\,dx$$

$$\left\{ \frac{dF}{da} = f(a) \right\}$$

↳ $f(x)$ has dimensions that are reciprocal of those for x

Mean and Variance

$$\langle x \rangle = \int_{-\infty}^{\infty} x f(x)\, dx.$$

$$Var(X) = \langle (X - \langle x \rangle)^2 \rangle = \langle X^2 \rangle - \langle x \rangle^2$$

Mode : Most likely value of X

~~Mode~~

Median: Value for a for which the cumulative distribution $F(a) = \frac{1}{2}$

Uniform Distribution : Consider ~~an~~ ~~[illegible]~~ continuous random variable that is distributed between $X=0$ and $X=1$

$$\text{Mean is} = \int_0^1 x\, dx = \tfrac{1}{2}\left[x^2\right]_0^1 = \underline{\tfrac{1}{2}}$$

$$Var(x) = \langle x^2 \rangle - \langle x \rangle^2 = \int_0^1 x^2 dx = \left[\tfrac{1}{3}x^3\right]_0^1$$

$$= \tfrac{1}{3} - \tfrac{1}{4} = \underline{\underline{\tfrac{1}{12}}}$$

Lifetime Distribution : Prob of a particle decaying in dt (time interval) = λ. Lifetime T is a random variable, we have to find the PDF.

$$f(t)\, dt = \lim_{N\to\infty} \left(1 - \frac{\lambda t}{N}\right)^N \lambda\, dt$$

$$= \lambda dt \left(1 - \lambda t + \frac{(\lambda t)^2}{2} + \dots\right)$$

$$= \underline{\lambda e^{-\lambda t} dt}$$

$$P(T \geq t) = \int_t^\infty \lambda e^{-\lambda t'} dt' = \underline{e^{-\lambda t}}$$

$$\langle T \rangle = \lambda \int_0^\infty t e^{-\lambda t} dt = \frac{1}{\lambda}$$

$$Var(T) = \lambda \int_0^\infty t^2 e^{-\lambda t} dt - \frac{1}{\lambda^2} = \underline{\frac{1}{\lambda^2}}.$$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

$$\langle x \rangle = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \frac{\sqrt{2\sigma^2}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (\mu + \sqrt{2\sigma^2}u) e^{-u^2} du \left[u = \frac{x-\mu}{\sqrt{2\sigma^2}}\right]$$

$$= \underline{\mu}.$$

$$\langle x^2 \rangle = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2} dx.$$

$$= \frac{(2\sigma^2)^{3/2}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} u^2 e^{-u^2} = \frac{(2\sigma^2)^{\frac{3}{2}}}{\sqrt{2\pi\sigma^2}} \frac{\sqrt{\pi}}{2} = \sigma^2.$$

Cumulative distribution: $F(a) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{a} e^{-(x-\mu)^2/2\sigma^2} dx$

$$= \frac{\sqrt{2\sigma^2}}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{(a-\mu)/\sqrt{2\sigma^2}} e^{-u^2} du$$

$$u = \frac{x-\mu}{\sqrt{2\sigma^2}}$$

$$= \frac{1}{\sqrt{\pi}}\left(\int_{-\infty}^{0} e^{-u^2} du + \int_{0}^{(a-\mu)/\sqrt{2\sigma^2}} e^{-u^2} du\right)$$

$$= \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{a-\mu}{\sqrt{2\sigma^2}}\right)\right]$$

$$\left\{ \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-u^2} du \right\}$$

F(a) usually tabulated for standard Normal deviations ($\mu = 0$ and $\sigma = 1$)

8: Ordinary Differential Equations

↳ unknown formula = $y(x)$

y ← independent variable (in dependent variable); x ← independent variable

General form of n^{th}-order ODE: $F(y^n, y^{n-1}, \ldots, y, x) = 0$

Solve in following steps:
1. Write ODE in standard form
2. Select suitable solution method
3. Find general solution
4. Find particular solution
5. Impose boundary or initial conditions
6. Check solution solves original differential equation.

Initial & Boundary Data

If differential equation is of order $n \Rightarrow$ missing integration constants required

↳ First order ODE - have a solution curve, need to identify a single point on that solution curve

↳ For order n, need n pieces of data.

① Indefinite Integral with additional data: $Y = C$, $x = 0$

$$Y = \int f(x)dx$$

② Definite integral

$$\int_C^Y dy = \int_0^x f(x)\,dx$$

Types of Integrable Equations

① Directly integrable (exact) equations:

$$\frac{dy}{dx} = f(x), \quad y = \int f(x)dx$$

② Seperable first order ODE:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}, \quad \int g(y)\,dy = \int f(x)\,dx$$

③ Homogenous Linear Equations:

$$\frac{dy}{dx} + p(x)y = 0, \quad \int \frac{dy}{y} = -\int p(x)\,dx$$

⑨ Inhomogenous Linear Equations

$$\frac{dy}{dx} + p(x)y = f(x)$$

$$\text{Integrating Factor} = \exp\left(\int^x p(t)\,dt\right) = \mu(x)$$

$$\mu(x)y(x) = \int^x \mu(t)f(t)\,dt$$

⑩ Homogeneous Differential Equation

↳ Homogeneous equation is where the equation is unchanged when y is replaced by αy and x is replaced by αx

↳ Solvable by the substitution $y = v(x)x$

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

$$= \frac{d}{dx}(vx) = \frac{dv}{dx}x + v = f(v)$$

$$\frac{dv}{dx} = \frac{f(v) - v}{x}$$

$$\int \frac{1}{f(v)-v}\,dv = \int \frac{1}{x}\,dx$$

⑪ Bernoulli differential equation

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

(i) $n = 0$, equation is linear:

$$\frac{dy}{dx} + p(x)y = q(x)$$

This is solvable using an integrating factor

(ii) $n = 1$, $\frac{dy}{dx} + (p(x) - q(x))y = 0$

Solvable by seperation of variables

(iii) $n > 1$, use substitution $z = y^{1-n}$

$$\frac{dz}{dx} + (1-n)p(x)z = (1-n)q(x)$$

Second-order equations

① Homogenous (unforced) linear second order ODE

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

② Inhomogenous (forced) linear ODE

$$Ly = \frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x) \neq 0$$

$$Ly = f(x)$$

$$L = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)$$

Principle of Superposition : → For $Ly = 0$, if u and v both satisfy $Ly = 0$, then any linear combination of solutions is also a solution (eg $\alpha v + \beta u$)

↳ For $Ly = f$

↳ Particular integrals y_p is any solution of $Ly = f$

↳ Complementary function y_c is general solution of $Ly = 0$

↳ General solution $= Ly_p + Ly_c$

↳ The solution space of a homogenous linear ODE is a vectorspace. The vectors in the space are the solution functions

↳ Dimensions of space is given by number of free parameters in the general solution

Consider : $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$

Auxiliary equation $= \lambda^2 + p(x)\lambda + q(x) = 0$

h_+, h_- (roots) $= \frac{1}{2}\left(-p(x) \pm \sqrt{(p(x))^2 - 4q(x)}\right)$

Six possibilities for the roots:

① Distinct real roots

↳ $y(x) = A\exp(h_+ x) + B\exp(h_- x)$

$$\left\{ y(x) = \exp(-ax)\left((A+B)\cosh(x\sqrt{a^2-b}) + (A-B)\sinh(x\sqrt{a^2-b})\right) \right\}$$

② Distinct real roots, one root = 0

↳ $y(x) = A\exp(h_+ x) + B$

③ Complex conjugate pair of pure imaginary roots.

↳ $h_+, h_- = \pm ai$

↳ $y(x) = A\cos(ax) + B\sin(ax)$

④ Distinct pair of complex conjugate roots:

↳ $h_+, h_- = a \pm ib$

↳ $y(x) = A\exp(ax)\cos(bx) + B\exp(ax)\sin(bx)$

⑤ Repeated real root $(\neq 0)$

↳ $h_+, h_- = a, a$

↳ $y(x) = (Ax + B)\exp(ax)$

⑥ Repeated zero root

↳ $y(x) = Ax + B$

<u>Finding Particular Integral</u> $\quad y'' + p(x)y' + q(x)y = f(x)$

① If $f(x)$ is a polynomial, try general polynomial of the same degree. If 1 is a solution of the homogeneous equation, try one degree higher. If 1 and x are both solutions, try two degrees higher.

② If $f(x) = ce^{hx}$, try $y = de^{hx}$. If e^{hx} is already in solution of homogeneous, try dxe^{hx}. If this is already part of solution, try dx^2e^{hx}

③ If $f(x) = c_1 \cos(hx) + c_2 \sin(hx)$, try $y_p = d_1 \cos(hx) + d_2 \sin(hx)$.
However, if $\cos(hx)$ and $\sin(hx)$ are solutions of the homogeneous equation, try $y_p = d_1 x \cos(hx) + d_2 x \sin(hx)$.

④ If $f(x) = c_1 \cosh(hx) + c_2 \sinh(hx)$, try $y_p = d_1 \cosh(hx) + d_2 \sinh(hx)$. But if $\cos(hx)$ and $\sin(hx)$ already solutions, try $y_p = d_1 x \cosh(hx) + d_2 x \sinh(hx)$

Damped Oscillator

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = f(t)$$

$x(t)$ = displacement
$\frac{dx}{dt}$ = velocity
$\frac{d^2x}{dt^2}$ = acceleration
γ = damping coefficient $\{\gamma \geq 0\}$
ω_0 = natural frequency $\{\omega_0 > 0\}$
$f(t)$ = force

↳ The first integral of the equation of motion gives the energy equation

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\omega_0^2x^2\right] = -2\gamma\left(\frac{dx}{dt}\right)^2 + f(t)\frac{dx}{dt}$$

↳ Damping reduces the energy (when $\gamma > 0$ and $v \neq 0$). The external force can either do positive or negative work on the system (dependent on sign of force)
↳ If there is no external force, energy can decay and $v \to 0$, therefore sustained motion only when non zero external force

In order to solve equation, need two initial conditions (values of x and $\frac{dx}{dt}$ at some time) eg $x = a$ and $\frac{dx}{dt} = 0$ at $t = 0$

AE: $\lambda^2 + 2\gamma\lambda + \omega_0^2 = 0$
$\lambda = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$

This leads to 4 possibilities

① $\gamma = 0$ (no damping)
↳ Two imaginary roots, $\lambda = \pm i\omega_0$

② $0 < \gamma < \omega_0$ (weak damping)
↳ Two complex roots with negative real parts and opposite imaginary parts

③ $\gamma = \omega_0$ (critical damping)
↳ repeated negative real roots, $\lambda = -\omega_0, -\omega_0$

④ $\gamma > \omega_0$ (strong damping)
↳ Two negative real roots

<u>Case of no damping</u> ($\gamma = 0$)

① ODE simplifies to: $\frac{d^2x}{dt^2} + \omega_0^2 x = 0$

$$x = D_1 \cos(\omega_0 t) + D_2 \sin(\omega_0 t)$$

<u>Weakly damped</u> ($0 < \gamma < \omega_0$)

② $x = e^{-\gamma t}\left(D_1 \cos(\nu t) + D_2 \sin(\nu t)\right)$ $\quad \nu = \sqrt{\omega_0^2 - \gamma^2}$
↳ amplitude decays proportional to $e^{-\gamma t}$
↳ As $\gamma \to \omega_0$, frequency tends to 0

<u>Strongly Damped</u>

④ $x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$ $\quad \lambda_1, \lambda_2 = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$

③ <u>Critical Damping</u>

$x = (C_1 + C_2 t)\, e^{-\gamma t}$

$\frac{dx}{dt} = -a\gamma^2 t e^{-\gamma t}$ → no oscillation as velocity never changes the sign, so the displacement tends monotonically to 0.

Forced Oscillations and Resonance

$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0 x = f_0 \cos(\omega t)$ $\quad$ ω is the forcing frequency
f_0 is the forcing amplitude

Particular Integral = $B\cos(\omega t) + C\sin(\omega t)$ — represents oscillation of constant amplitude and with the same frequency as the forcing, though not normally in phase.

↳ $B = \frac{(\omega_0^2 - \omega^2) f_0}{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}$, $\quad C = \frac{2\gamma\omega f_0}{(\omega_0^2 - \omega^2) + (2\gamma\omega)^2}$

↳ $= A\cos(\omega t - \phi)$

$A = \sqrt{B^2 + C^2}$, $\quad \phi = \arctan\left(\frac{C}{B}\right)$

$\phi > 0 \Rightarrow$ oscillation lags behind forcing
$\phi < 0 \Rightarrow$ oscillation leads the forcing

$$A = \frac{f_0}{\sqrt{((\omega_0^2-\omega^2)^2 + (2\gamma\omega)^2)}} = \frac{\frac{f_0}{\omega_0^2}}{\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + \left(2\frac{\gamma}{\omega_0}\frac{\omega}{\omega_0}\right)^2}}$$

$$\phi = \arctan\left(\frac{2\gamma\omega}{\omega_0^2-\omega^2}\right) = \arctan\left(\frac{\frac{2\gamma\omega}{\omega_0^2}}{1-\frac{\omega^2}{\omega_0^2}}\right)$$

↳ Oscillation amplitude is proportional to forcing amplitude (resonance)
↳ Amplitude ↑ if $f = f_0$ & if $\gamma \to 0$ ($\gamma << \omega_0$)
↳ As forcing frequency increases through natural frequency, A reaches a peak ~~and~~ as the phase lag ϕ increases through $\frac{\pi}{2}$

Undamped Resonance

↳ PI does not work if $\gamma = 0$ and $\omega = \omega_0$

$$\frac{d^2x}{dt^2} + \omega_0^2 x = f_0 \cos(\omega_0 t)$$

$$x_p = \frac{f_0}{2\omega_0} t \sin(\omega_0 t)$$ ⇒ oscillation which lags the forcing by $\frac{\pi}{2}$ and amplitude grows linearly in time

Transients

In damped cases ($\gamma > 0$), CF → 0 as $t \to \infty$. Therefore, the particular integral describes the long-term response. This doesn't depend on initial conditions

CF → transient response → depends on two arbitrary constants and does depend on initial conditions (eventually forgotten)

In undamped ($\gamma = 0$), CF is an undamped oscillation at natural frequency ω_0 - does not decay and initial conditions never forgotten.

9: Functions of more than one variable

Ordinary Derivative: $\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

↳ Local rate of change of the function f wrt x.
↳ Gradient of the straight line that is tangent to the curve at point x.

Partial Derivatives: ⇒ Firstly, specify the whole coordinate system: x, y, z

$z = f(x,y)$: $\frac{\partial f}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$

local grad of the surface $z = f(x,y)$ when travelling in the x-direction Straight lines tangent to the surface and perpendicular to the y axis.

$\frac{\partial f}{\partial y} = f_y = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$

↳ $\frac{\partial f}{\partial x}$ is the local rate of change of f with x at constant y.

⇓

∴ generally write $\left(\frac{\partial f}{\partial x}\right)_y$ or $\left.\frac{\partial f}{\partial x}\right|_y$

Gradient of f is the vector = $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

gradient operator

↳ Directional Derivative: Dot product of gradient with a unit vector in the appropriate direction
↳ Can find this also written as projection of gradient and resolving the vector.

Second Derivatives

$\frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$

etc

NB. $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

$f_{xy} = f_{yx}$

Geometric Interpretation of gradient vector to surface

Since surface given by $f = \text{constant}$, normal is the direction of fastest change in $f(x,y)$. The tangent plane at a point has the same normal as the surface at that point

$\therefore$ normal to surface $= \nabla f = \left(\underline{i}\frac{\partial f}{\partial x} + \underline{j}\frac{\partial f}{\partial y} + \underline{k}\frac{\partial f}{\partial z}\right)$

Integration: when we integrate a partial derivative, the integration constant is an arbitrary ~~const~~ function of independent variables kept constant in partial derivative

eg. if $\frac{\partial f}{\partial x} = 2xy^2$ and $\frac{\partial f}{\partial y} = 2x^2y + 2y$

$$f = x^2y^2 + g(y) \qquad f = x^2y^2 + y^2 + h(x)$$

$$\therefore g(y) = y^2 + h(x)$$
$$= y^2 + c$$
$$\therefore \underline{f = x^2y^2 + y^2 + c}$$

Generalisation: The definitions generalise to functions of any number of variables, such as $f(x_1, x_2 \dots x_i, \dots x_n)$

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \dots x_i + h, \dots x_n) - f(x_1, x_2 \dots x_n)}{h}$$

$$= \left.\frac{\partial f}{\partial x_i}\right|_{x_1, x_2 \dots x_{i-1}, x_{i+1}, x_{i+2}, \dots, x_n}$$

Differentials: For a differentiable function, by Taylor expansion

$$f(x+h) \simeq f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 \dots + \frac{1}{n!}f^n(x)h^n + R_n(x,h)$$

↳ Gives local approximation to the variation of f near the point as a polynomial in displacement.

When h is sufficiently small, first derivative dominates:

$$f(x+h) \simeq f(x) + f'(x)h.$$
$$\delta f \simeq f'(x)\delta x.$$

↳ Geometrically, this corresponds to approximating $y = f(x)$ by a straight line and in limit, this is tangent line that touches curve at point considered

↳ Approximation of exact equality of differentials.

$$df = f'(x)\,dx.$$

For two variable function $f(x,y)$:

$$\cancel{f(x,y)}\ f(x+h, y+k) \simeq f(x,y) + \alpha h + \beta k$$

$$error = f(x+h, y+k) - f(x,y) - \alpha h - \beta k$$

Geometrically, this is approximating surface $z = f(x,y)$ with tangent plane.

~~f(x+h, y+k) - β~~

We expect that $e(h,k) \to 0$ as $h,k \to 0$. If $\frac{\epsilon(h,k)}{\sqrt{h^2+k^2}} \to 0$ as $\sqrt{h^2+k^2} \to 0$ then $f(x,y)$ is said to be differentiable at (x,y)

By setting h (and then k) to 0, we can see $\alpha = \frac{\partial f}{\partial x}$ and $\beta = \frac{\partial f}{\partial y}$

$$\therefore f(x+h, y+k) \simeq f(x,y) + \frac{\partial f}{\partial x} h + \frac{\partial f}{\partial y} k$$

$$\delta f \simeq \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\Downarrow$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Taylor Series

$$f(x+h, y+k) = f(x,y) + f_x(x,y)h + f_y(x,y)k + \tfrac{1}{2} f_{xx}(x,y)h^2 + f_{xy}(x,y)hk + \tfrac{1}{2} f_{yy}(x,y)k^2 + R$$

Where $\frac{R}{h^2+k^2} \to 0$ as $h^2+k^2 \to 0$ ($h \to 0$ and $k \to 0$)

OR

$$f(x,y) = f(x_0 y_0) + f_x(x_0, y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + \tfrac{1}{2} f_{xx}(x_0,y_0)(x-x_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \tfrac{1}{2} f_{yy}(x_0,y_0)(y-y_0)^2.$$

↳ These series can be taken further if function is differentiable more than twice.

Chain Rule

$$f(x,y) = f(x(u,v), y(u,v)) = f(u,v)$$

$$dx = \left.\frac{\partial x}{\partial u}\right|_v du + \left.\frac{\partial x}{\partial v}\right|_u dv$$

$$dy = \left.\frac{\partial y}{\partial u}\right|_v du + \left.\frac{\partial y}{\partial v}\right|_u dv$$

$$df = \left.\frac{\partial f}{\partial x}\right|_y dx + \left.\frac{\partial f}{\partial y}\right|_x dy$$

$$\therefore\ df = \left(\left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial u}\right|_v + \cancel{\left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial v}\right|_v du} \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial u}\right|_v \right) du + \left(\left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial v}\right|_u + \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial v}\right|_u \right) dv$$

Since $df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv$

$$\left.\frac{\partial f}{\partial u}\right|_v = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial u}\right|_v + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial u}\right|_v$$

$$\left.\frac{\partial f}{\partial v}\right|_u = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial v}\right|_u + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial v}\right|_u$$

Special case occurs when only one of the variables is changed.

eg $(x,y) \rightarrow (x,v)$

$$\left(\frac{dx}{dx}\right)_v = 1 \text{ and } \left(\frac{\partial x}{\partial v}\right)_x = 0$$

$$\therefore \left.\frac{\partial f}{\partial x}\right|_v = \left.\frac{\partial f}{\partial x}\right|_y + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial x}\right|_v$$

$$\left.\frac{\partial f}{\partial v}\right|_x = \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial v}\right|_x$$

If both x and y are functions of one variable (t)

$$dx = \frac{dx}{dt}\,dt \quad , \quad dy = \frac{dy}{dt}\,dt$$

$$df = \left.\frac{\partial f}{\partial x}\right|_y dx + \left.\frac{\partial f}{\partial y}\right|_x dy$$

$$\frac{df}{dt} = \left.\frac{df}{dx}\right|_y \frac{dx}{dt} + \left.\frac{df}{dy}\right|_x \frac{dy}{dt}$$

<u>Reciprocity and Cyclic Relations</u>

↳ Can have $z = f(x,y)$ or $x = g(y,z)$ or $y = h(x,z)$

$$dx = \left.\frac{dx}{dy}\right|_z dy + \left.\frac{\partial x}{\partial z}\right|_y dz \qquad (1)$$

$$dy = \left.\frac{\partial y}{\partial x}\right|_z dx + \left.\frac{dy}{\partial z}\right|_x dz \qquad (2)$$

$$dz = \left.\frac{dz}{\partial x}\right|_y dx + \left.\frac{dz}{dy}\right|_x dy \qquad (3)$$

$$(2) \equiv dx = \frac{1}{\left.\frac{dy}{\partial x}\right|_z} dy - \frac{\left.\frac{dy}{\partial z}\right|_x}{\left.\frac{dy}{dx}\right|_z} dz$$

can compare this to the first one : $\left.\frac{\partial x}{\partial y}\right|_z = 1 \Big/ \left.\frac{dy}{dx}\right|_z$ (Reciprocity Relation)

And $\left(\frac{dx}{dz}\right)_y \left(\frac{dy}{dx}\right)_z \left(\frac{dz}{dy}\right)_x = -1$ (Cyclic Relation)

Exact Differentials

$\omega = P(x,y)\,dx + Q(x,y)\,dy$

↳ It is an exact differential if there is a function $f(x,y)$ st:

↳ $P(x,y)\,dx + Q(x,y)\,dy = df$

↳ iff $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ $\left(\frac{\partial f}{\partial x} = P,\ \frac{\partial f}{\partial y} = Q \text{ and } \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right)$

Integrating Factors for Differential Forms

↳ $\mu(x,y)$ is integrating factor iff

$\mu(x,y)\,[P(x,y)\,dx + Q(x,y)\,dy]$ is exact

$$\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q)$$

↳ May be integrating factor that only depends on x ($\mu(x)$)

$$\therefore \mu \frac{\partial P}{\partial y} = \frac{d\mu}{dx} Q + \mu \frac{\partial Q}{\partial x}$$

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

If right hand side is a function of x alone, then equation is self-consistent and can be integrated. If it depends on both, then no integration factor $\mu(x)$ exists → generally $\mu(x,y)$ is very hard to find.

↳ Similarly integrating factor of form $\mu(y)$

$$\frac{1}{\mu}\frac{d\mu}{dx} = -\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

Maxwell's Relations

Fundamental Thermodynamic ~~Re~~ Relation. $dU = TdS - pdV$

internal energy, temp, entropy, pressure, volume

Gas has two degrees of freedom

pV – these are mechanical variable

Ts – thermal variables

↳ Consider U as a function of S and V

$$dU = \left.\frac{\partial U}{\partial S}\right|_V dS + \left.\frac{\partial U}{\partial V}\right|_S dV$$

$\therefore \left(\frac{\partial U}{\partial S}\right)_V = T, \quad \left(\frac{\partial U}{\partial V}\right)_S = -p$

$$\frac{\partial^2 U}{\partial V \partial S} = \frac{\partial^2 U}{\partial S \partial V} \Rightarrow \left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V$$ (one of Maxwell's relations)

We can derive similar relations by making a change of variables, known as Legendre transformation.

$$F = U - TS$$
$$dF = dU - TdS - SdT$$
$$= -SdT - pdV$$
$$= \left(\frac{\partial F}{\partial T}\right)_V dT + \left(\frac{\partial F}{\partial V}\right)_T dV$$ consider F as a function of T and V

$$\left(\frac{\partial F}{\partial T}\right)_V = -S, \quad \left(\frac{\partial F}{\partial V}\right)_T = -p$$

$$\therefore \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$$ Maxwell (2)

F is known as Helmholtz free energy. We get more relation from $H = U + pV$ (enthalpy) and $G = H - TS$ (Gibbs free energy)

~~$dH = TdS$~~ $$\left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p$$ Maxwell 3.

$$\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p$$ Maxwell 4

Stationery Points

- Taylor Expansion: $f(x) \simeq f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2$
- Stationery point if $f'(x_0) = 0$
  - Local minimum if $f''(x_0) > 0$
  - Local maximum if $f''(x_0) < 0$
- If both $f'(x_0)$ and $f''(x_0)$, need to inspect higher order terms.
  - If $f''(x_0) = 0$, could be min, max or point of inflection

Functions of two or more variables

↳ If function f has $\nabla f = 0$ at a point p then f has stationary point at p.

$$f(x,y) \simeq f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

↳ If $f_x(x_0,y_0) = 0$ and $f_y(x_0,y_0) = 0$ then $f(x,y)$ is said to have a stationary point at (x_0, y_0)

↳ In terms of the gradient vector:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (f_x, f_y)$$

$$\therefore f(\underline{x}) \simeq f(\underline{x}_0) + [\nabla f(\underline{x}_0)] \cdot \delta\underline{x} \quad \text{where } \underline{x}_0 = (x_0, y_0)$$

$$\delta\underline{x} = (\delta x, \delta y)$$

$\nabla f(\underline{x}_0) = \underline{0}$ has stationary point (extends to N variables)

$$df = \nabla f \cdot d\underline{x}, \text{ when } \nabla f = \underline{0},\ df = 0\ \forall d\underline{x}$$

↳ $f(x,y) \simeq f(x_0,y_0) + \frac{1}{2} f_{xx}(x_0,y_0)(x-x_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{1}{2} f_{yy}(x_0,y_0)(y-y_0)^2$

↳ This is a constant plus homogeneous functions of δx and δy

↳ Local minimum if $f_{xx} f_{yy} > f_{xy}^2$ with $f_{xx} > 0$ and $f_{yy} > 0$

↳ Local maximum if $f_{xx} f_{yy} > f_{xy}^2$ with $f_{xx} < 0$ and $f_{yy} < 0$

↳ Saddle point if $f_{xx} f_{yy} > f_{xy}^2$

~~Conditional Stationary Values~~

Hessian Matrix

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

↳ Symmetry of mixed partial derivatives implies that the Hessian matrix is symmetric

↳ Real Eigenvalues, orthogonal eigenvectors

↳ Stationary point is a minimum if eigenvalues of Hessian matrix, maximum if they were both negative and saddle point if one negative and one positive.

Conditional Stationary Values

- Interested in problems where x and y cannot be varied independently, but are related by a condition or constraint of $g(x,y)=0$ which means (x,y) lie on a curve or surface.

- In order to find the stationary points of $f(x,y)$, subject to condition $g(x,y)=0$, solve simultaneous equations:

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g = 0$$

where λ is a Lagrange multiplier.

- We define the Lagrangian function as:

$$L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$$

- Method of Lagrange multipliers is equivalent to solving the equations:

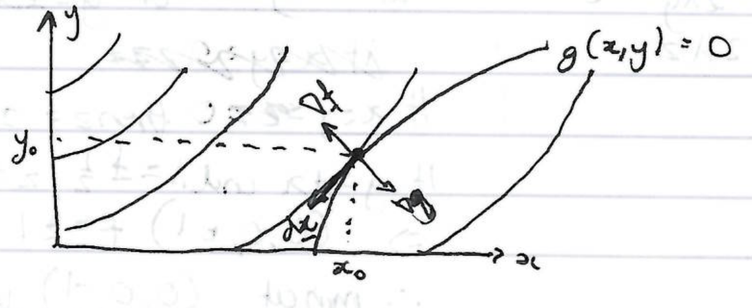
$$L_x = L_y = L_\lambda = 0$$

- We can sometimes ~~m~~ use $L = f - \lambda g$ instead of $L = f + \lambda g$

Geometrical Viewpoint

- ∇f is normal to the contour line $f =$ constant, passing through that point.
- This is because $df = \nabla f \cdot d\underline{x}$
- Direction in which f doesn't change ($df = 0$) corresponds to a displacement $d\underline{x}$ perpendicular to ∇f

- Similarly ∇g is normal to the curve $g=0$ at each point on it
  - displacements must be tangent to the curve.
- For conditional stationary point, require $df = \nabla f \cdot d\underline{x}$ not for all $d\underline{x}$ but only st $\nabla g \cdot d\underline{x} = 0$
- Therefore ~~∇f is parallel to ∇g~~ $\nabla f = \lambda \nabla g$



<u>Example</u>: Find max value of $f(x,y) = xy$ on unit circle $g(x,y) = x^2+y^2-1=0$

Suppose max at (x_0, y_0). Consider variation of $f(x,y)$ and $g(x,y)$ near this point.

$$df = f_x\,dx + f_y\,dy$$
$$dg = g_x\,dx + g_y\,dy = 0$$

$\therefore$ vector displacement $(d\underline{x}) = (dx, dy)$ is tangent to curve $g=0$

$$\therefore df = (f_x g_y - f_y g_x)\frac{dx}{g_y} = -(f_x g_y - f_y g_x)\frac{dy}{g_x}.$$

f stationary along curve when: $f_x g_y - f_y g_x = 0$.

$$\therefore \frac{f_x}{g_x} = \frac{f_y}{g_y} = \lambda$$

$$f_x = y,\ f_y = x,\ g_x = 2x,\ g_y = 2y$$
$$y = 2\lambda x$$
$$x = 2\lambda y$$
$$x^2+y^2=1$$
$$\lambda = \pm\tfrac{1}{2}$$
$$\lambda = \tfrac{1}{2} \Rightarrow y = x = \pm 1/\sqrt{2},\ f = \tfrac{1}{2}.$$
$$\lambda = -\tfrac{1}{2} \Rightarrow x = -y = \pm 1/\sqrt{2},\ f = -\tfrac{1}{2}$$

<u>More than two independent variable, one constraint</u>

<u>Example</u>: Minimize $xy+z$ to constraint $x^2+y^2+z^2=1$.

$$L = xy+z - \lambda(x^2+y^2+z^2-1)$$

$L_x = y - 2\lambda x = 0$; $y = 2\lambda x$ $\therefore x = 0$ or $y = 0$

$L_y = x - 2\lambda y = 0$; $x = 2\lambda y$ or $y = \pm x$ and $\lambda = \pm\frac{1}{2}$

$L_z = 1 - 2\lambda z$

If $x = y = 0$ then $z = \pm 1$, $\lambda = \pm\frac{1}{2}$.

If $y = \pm x$ and $\lambda = \pm\frac{1}{2}$, $z = \pm 1$

$\Rightarrow (0,0,\pm 1)\ f = \pm 1$

$\therefore$ min at $(0,0,-1)$ with $f = -1$

More than one constraint

↳ Consider ~~three~~ two surfaces as a constraint, therefore confined to intersection of these surfaces, which is a curve.

↳ $df = \nabla f \cdot d\underline{x} = 0 \qquad \nabla f \perp d\underline{x}$

$dg = \nabla g \cdot d\underline{x} = 0 \qquad \nabla g \perp d\underline{x}$

$dh = \nabla h \cdot d\underline{x} = 0 \qquad \nabla h \perp d\underline{x}$

∇f is in the plane formed by ∇g and ∇h

$\nabla f = \lambda \nabla g + \mu \nabla h$ where λ and μ are constants

$\therefore L(x,y,z,\lambda,\mu) = f(x,y,z) - \lambda g(x,y,z) - \mu h(x,y,z)$

Boltzmann Distribution: Suppose we have a system consisting of a large number of particles.

↳ Each particle is one of n states.

↳ Energy of each particle in state i is E_i

↳ N (number of particles) $= \sum_{i=1}^{n} N_i$

↳ $E_{total} = \sum_{i=1}^{n} N_i E_i$

↳ Number of ways of getting a given distribution $= \dfrac{N_{tot}!}{N_1!\,N_2!\,\cdots\,N_n!}$

↳ Each permutation is equally probable, so distribution that occurs in nature is the one with largest value of ω.

↳ Maximise value of $W(N_1, N_2, \ldots N_n)$ wrt $N_1, N_2 \ldots N_n$.

↳ Easier to maximise $\ln W = \ln(N!) - \sum_{i=1}^{n} \ln(N_i!)$

(entropy $= S = k\ln\omega$ where k is Boltzmann constant $\therefore$ = distribution of of maximum entropy)

↳ If system isolated, maximise $\ln W$ subject to constraint $N = \hat{N}$ = constant and $E = \hat{E}$ = constant

$\therefore$ use Lagrange: $L = \ln W - \alpha(N - \underset{\text{constants}}{\underset{\uparrow}{\hat{N}}}) - \beta(E - \hat{E})$

$= \ln(N!) - \sum_{i=1}^{n} \ln(N_i!) - \alpha\left(\sum_{i=1}^{n} N_i - \hat{N}\right) - \beta\left(\sum_{i=1}^{n} N_i E_i - \hat{E}\right)$

↳ can use Stirling's approximation
↳ $x! \approx \sqrt{2\pi x}\, x^x e^{-x}$

$$\frac{d}{dx}\ln(x!) \simeq \ln(x) + \frac{1}{2x} \simeq \ln(x)$$

$$\therefore \quad \frac{\partial L}{\partial N_i} = \ln N - \ln N_i - \alpha - \beta E_i$$

$\therefore$ For stationary point: $N_i = N e^{-\alpha} e^{-\beta E_i}$. This is the Boltzmann Distribution, where $\beta = \frac{1}{kT}$. Since $\beta > 0$, states of higher energy are less occupied

Value of α: $N = \sum_{i=1}^{n} N_i = N e^{-\alpha} \sum_{i=1}^{n} e^{-\beta E_i}$ *

$$\therefore N_i = \frac{N e^{-\beta E_i}}{\sum_{j=1}^{n} e^{-\beta E_j}}$$ (value of β determined by second constraint ⇒ total energy of system)

$$* \Rightarrow e^{-\alpha} = \left[\sum_{i=1}^{n} e^{-\beta E_i}\right]^{-1}$$

Note: Boltzmann Distribution doesn't predict integer values since we treat it as being a very large real number

<u>Degeneracy</u>: Label states by energy levels but multiple states share the same energy levels. The number of states share energy level E_i $= g_i$. If N_i denotes combined occupation number of degenerate states then:

$$N_i = N g_i e^{-\alpha} e^{-\beta E_i} = \frac{N g_i e^{-\beta E_i}}{\sum_{j=1}^{n} g_j e^{-\beta E_j}}$$

10: Scalar and Vector Fields

Field is a quantity that depends continuously on position (and possibly on time)
↳ Air pressure (scalar field)
↳ Electric Field (vector field)

Vector calculus: concerned with differentiation and integration of scalar and vector fields, operations required in most physical theories.

$\phi(x,y,z)$ is a scalar field $(\phi(\underline{x}))$

grad of ϕ is a ~~scalar~~ vector field:

↳ $$\text{grad}\,\phi = \nabla\phi = \underline{i}\frac{\partial\phi}{\partial x} + \underline{j}\frac{\partial\phi}{\partial y} + \underline{k}\frac{\partial\phi}{\partial z} = (\phi_x, \phi_y, \phi_z)$$

$$d\phi = \phi_x\,dx + \phi_y\,dy + \phi_z\,dz = \nabla\phi \cdot d\underline{x}$$

↳ where $d\underline{x} = \underline{i}\,dx + \underline{j}\,dy + \underline{k}\,dz = (dx, dy, dz)$

↳ This expresses the infinitesimal change in ϕ due to an infinitesimal vector displacement

Directional Derivatives

Let $\underline{t}$ be any unit vector. An infinitesimal vector displacement in direction $\underline{t}$ can

$\underline{t}\cdot\nabla\phi = \frac{d\phi}{ds}$ is the rate of change of ϕ with distance in the direction $\underline{t}$. This is a directional derivative

be written $d\underline{x} = \underline{t}\,ds$ where ds is the infinitesimal scalar displacement. Here $d\phi = \nabla\phi\cdot(\underline{t}\,ds)$
$= (\underline{t}\cdot\nabla\phi)\,ds$

$\underline{t}\cdot\nabla\phi$ is a scalar field and depends on position. For a given point $\nabla\phi$ is a fixed vector and we can consider how $\underline{t}\cdot\nabla\phi$ depends on $\underline{t}$

$$\underline{t}\cdot\nabla\phi = |\nabla\phi|\cos\theta$$

$\therefore$ direction of $\nabla\phi$ is that in which ϕ increases most rapidly; the magnitude of $\nabla\phi$ gives rate of change of ϕ with distance in that direction.

Normal to Surface

Given scalar field $\phi(\underline{x})$, $\phi(\underline{x}) = c$ defines a surface (level surface of ϕ). By varying the constant c, we generate a family of surface

Eg. $\phi = x^2+y^2+z^2$. For $c>0$, defines a sphere of radius $\sqrt{c}$ centred on origin. For $c<0$, no such surface; For $c=0$, reduces to a single point

If $\underline{t}$ is a vector tangential to a level surface $(\phi = c)$ at a given point then $\underline{t} \cdot \nabla\phi = 0$ there, since ϕ does not vary within the surface. Therefore $\nabla\phi$ normal to surface at that point.

$$n = \frac{\nabla\phi}{|\nabla\phi|}$$ (if $\nabla\phi = \underline{0}$, this fails, but then we have a stationary point, $\Rightarrow$ normal vector not defined at that point)

<u>Line Integrals</u> : Integral along a curve.

↳ Curves

↳ Can be defined most conveniently by :

$x = X(t)$, $y = Y(t)$, $z = Z(t)$

↳ $\underline{x} = \underline{X}(t)$

↳ Parameter t ~~param~~ has a range with each value corresponding to a point on the curve.

↳ Curve is <u>oriented</u>

↳ Infinitesimal displacement : $d\underline{x} = \frac{d\underline{X}}{dt}dt$

$\Rightarrow$ $\frac{d\underline{X}}{dt}$ is tangent to curve at each point

The magnitude depends on how rapidly t varies along the curve.

↳ points in the direction of increasing t.

↳ Arc length, s : $|d\underline{x}| = |ds|$

$\frac{d\underline{x}}{ds} = \underline{t}$ (unit tangent vector to the curve)

↳ Can be measured starting from any point and moving in either direction.

$\frac{ds}{dt} = \left|\frac{d\underline{X}}{dt}\right|$

line element — vector line element

$|ds| = |d\underline{x}| = \left|\frac{d\underline{X}}{dt}\right| dt$

$\therefore \frac{ds}{dt} = \frac{d\underline{x}}{dt}$.

Line Integral of a scalar field

Given a scalar field $\phi(x)$ and curve Γ parametrized by arclength s, we can define line integral:

$$\int_\Gamma \phi \, ds = \int_{s_1}^{s_2} \phi(\underline{x}(s)) ds$$

If parametrization not given by arclength but some other parameter

$$\int_\Gamma \phi \, ds = \int_{t_1}^{t_2} \phi(x(t)) \left|\frac{dx}{dt}\right| dt$$

Line integral of vector field

Given vector field $\underline{F}(x)$ and a curve Γ, parametrized by t, the most common type of line integral arises is:

$$\int_\Gamma \underline{F} \cdot d\underline{x} = \int_{t_1}^{t_2} F(x(t)) \cdot \frac{d\underline{x}}{dt} dt$$

If $\underline{t} = s$ is arclength:

$$\int_\Gamma \underline{F} \cdot dx = \int_\Gamma \underline{F} \cdot \underline{t} \, ds \quad \text{where } \underline{t} = \frac{d\underline{x}}{ds} \text{ is the unit tangent vector.}$$

$\therefore$ Integral is the line integral of the tangential component of F.

If a curve is reversed in direction, the line integral changes sign:

$$\int_{-\Gamma} F \cdot dx = -\int_\Gamma F \cdot dx.$$

Adding curves:

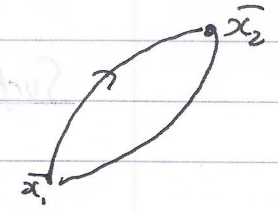
$$\int_{\Gamma_1 + \Gamma_2} \underline{F} \cdot dx = \int_{\Gamma_1} \underline{F} \cdot dx + \int_{\Gamma_2} F \cdot dx$$

also holds for line ~~scalar~~ integrals of a scalar field.

Line integral of the gradient of a scalar field =

$$\int_\Gamma \nabla\phi \cdot dx = \int_\Gamma d\phi = \phi_{x_2} - \phi_{x_1}$$

(Gradient Theorem)



Conservative Vector Fields

If $F(x)$ is the force field acting on a particle, then $F \cdot dx$ is the infinitesimal work done by force when $\underline{x}$ moves to $\underline{x} + d\underline{x}$

$$W = \int_\Gamma F \cdot dx$$

$F = -\nabla\phi$ where $\phi(x)$ is a scalar field known as a potential (minus sign is a physical convention)

$$W = -\int_\Gamma \nabla\phi \cdot dx \text{, which by the gradient theorem}$$
$$= \phi(x_1) - \phi(x_2)$$

↳ Work done is equal to potential difference between endpoints and is path independent

↳ For any closed loop: $\oint_\Gamma \overbrace{F \cdot dx}^{\text{exact differential}} = -\oint_\Gamma \nabla\phi \cdot dx = 0$

↳ Definition of conservative vector field (any)
- ↳ $F = -\nabla\phi$ for some $\phi(x)$
- ↳ $F \cdot dx$ is an exact differential
- ↳ For every pair of points, line integral is path independent

↳ We need to be careful if the potential is not single valued eg $\phi = \arctan\left(\frac{y}{x}\right)$

$$\int_\Gamma F \cdot dx = -\int \nabla\phi \cdot dx = -\int d\phi = -2\pi \neq 0$$

Because ϕ increases by 2π even though curve returns to starting point

Surface Integrals

↳ Vector Area Element

Vector area $\underline{S} = A\underline{n}$ where A = area of surface and $\underline{n}$ is unit normal vector

↳ by convention on a closed surface, n points outwards

For open surface, determined by right hand rule on the bounding oriented curve

↳ Vector area of a surface made up of contiguous plane elements is the vector sum of the areas of these planes.

↳ The vector area of a closed polyhedron = 0

↳ Curved surface S

↳ an infinitesimal element can be considered planar.

↳ $d\underline{S} = \underline{n}\, \underset{\text{area}}{dS}$

$$S = \int_S d\underline{S} = \int_S \underline{n}\, dS$$

↳ By reversing the orientation of a surface, we effectively change the direction of the normal vector

<u>Parametrization of a curved surface</u>

$\underline{x} = \underline{x}(s,t)$ where x is vector valued functions of two parameters: s, t

Starting at a given point on surface, and making infinitesimal increments, we map out an infinitesimal parallelogram (with sides $\left(\frac{\partial \underline{x}}{\partial s}\right) ds$ and $\left(\frac{dx}{dt}\, dt\right)$)

$$d\underline{S} = \left(\frac{\partial x}{\partial s} ds\right)\left(\frac{\partial x}{\partial t} dt\right)$$

$$= \left(\frac{\partial x}{\partial s} \quad \frac{\partial x}{\partial t}\right) ds\, dt$$

<u>Flux</u>

Given a vector field: $F(x)$ and surface S: $\int_S \underline{F} \cdot d\underline{S} = \int_S \underline{F} \cdot \underline{n}\, dS$ = flux of F through S

↳ Limit of sum of fluxes through infinitesimal surface elements

↳ Need to find parametrisation of surface (find ~~ex~~ expressions for $\underline{n}$ and ~~dS~~ dS. Then double integral over the surface

Divergence

Vector differential operator - ∇. operates on scalar field $\phi(x)$ to produce vector field $\nabla\phi$, the gradient of ϕ

∇ can also operate on a vector field $F(x)$

$$F(x) = (F_x(x,y,z),\ F_y(x,y,z),\ F_z(x,y,z))$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\therefore \nabla \cdot F = \text{div } F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

↳ $\nabla \cdot E = \lim_{\delta v \to 0} \frac{1}{\delta v} \int_{\delta S} \underline{E} \cdot d\underline{S}$

↳ Important to not ∇ is not commotative - $F \cdot \nabla \neq \nabla \cdot F$ & $\nabla\phi \neq \phi\nabla$

↳ $\int_V \nabla \cdot F \, dV = \int_S \underline{F} \cdot d\underline{S}$ where S is boundry surface of V

Divergence Theorem (Gauss's Theorem)

Geometric Definition of divergence: If $\nabla \cdot F > 0$, then vector field has a positive source in that region and a positive flux measured through surface enclosing it

Gauss's Theorem states: $\int_V \nabla \cdot E \, dV = \int_S \underline{E} \cdot d\underline{S}$

where V is a volume bounded by the closed surface S (∂V is equivalent).

↳ The right hand side is the flux of F through the surface S. The vector surface element is $d\underline{S} = \underline{n}\, dS$, where $\underline{n}$ is the outward unit vector.

Proof: can be prooved for a simple cuboid, can show volumes simply add for volume integrals and similarly, flux integrals combine.

∴ true for any volume made of contiguous cuboids (and can show for arbitrary tetrahedron) ⇒ true ∀ objects

A simply connected volume is one with no holes - simply an outer surface. For a multiply connected volume, all surfaces must be considered.

The Laplacian

$$\text{div}(\text{grad}\,\phi) = \nabla \cdot (\nabla \phi) = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \text{Laplacian of } \phi$$

$$\left\{ \text{Laplacian Operator} = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right\}$$

Curl

curl F ← vector field

$$\text{curl}\, F = \nabla \times F$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (F_x, F_y, F_z)$$

$$= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$\int_S (\nabla \times F) \cdot d\underline{S} = \int_C F \cdot d\underline{x}$$ where C is the bounding curve of S

{Stoke's Theorem}

$$\underline{n} \cdot (\nabla \times \underline{F}) = \lim_{\delta S \to 0} \frac{1}{\delta S} \int_{\delta C} \underline{F} \cdot d\underline{x}$$ where δS is a small rectangle centred on a point with unit normal $\underline{n}$ and δC is bounding curve

Geometric definition of curl: if $\underline{n} \cdot (\nabla \times \underline{F}) > 0$ in some region, the vector field has a positive rotation about the $\underline{n}$ axis in that region.

curl(grad ϕ) = 0 for any scalar field ϕ, $\nabla \times F = 0$ for any conservative vector field (this is a sufficient condition)

Therefore for F as a conservative vector field, these are all equivalent

1. $\underline{F} = -\nabla\phi$ for some ϕ (scalar field)
2. $\underline{F} \cdot d\underline{x}$ is an exact differential
3. $\underline{\nabla} \times \underline{F} = \underline{0}$
4. For every pair of points in region line integral is path independent
5. $\oint_{\Gamma} \underline{F} \cdot d\underline{x} = 0$ $\forall$ ~~curved~~ closed curves inside region

$\therefore$ $dw = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$ is exact iff $\text{curl}(P,Q,R) = 0$.

$$\therefore \text{iff} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) = \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0$$

↳ generalisation of exact iff $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ for ~~3D~~ two variables

Combinations

1. grad (grad) = no meaning
2. grad (div) $\neq 0$
3. grad (curl) = no meaning
4. div(div) = no meaning
5. div (grad) = Laplacian
6. div (curl) = 0
7. curl (curl) $\neq 0$
8. curl (div) = no meaning
9. curl (grad) = 0 ✗

Stokes' Theorem

$$\int_S (\nabla \times \underline{F}) \cdot d\underline{S} = \int_C \underline{F} \cdot d\underline{x}$$ where S is an $\boxed{\text{open}}$ surface bounded by C

as opposed to Gauss's Theorem

not necessarily flat

Proof: (i) Shown true for a rectangle and triangle

(ii) Two surfaces can be added together, surface integrals add together as do bounding curves,

common part of curve cancels out because of opposite ~~orientation~~ orientations

$\therefore$ Stokes' true for anything composed of triangles $\Rightarrow$ can be proved $\forall$ shapes by approximating surface (with desired accuracy) as a composite of triangles

↳ Special case occurs when applied to a planar surface

$$\iint_A \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dx\,dy = \int_C (F_x\,dx + F_y\,dy)$$

(Greene's Theorem in the plane)

$$\int_C \bar{F} \cdot d\bar{x} = \int_C F_x\,dx + F_y\,dy = \int_A (\nabla \cdot \bar{F}) \cdot d\bar{s}$$

$n = (0,0,1)$, $ds = dx\,dy$

$$= \int_A (\nabla \times \bar{F}) \cdot \bar{n}\,dS = \iint_A \nabla \times \bar{F} \cdot (0,0,1)\,dx\,dy$$

$$= \iint_A \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dx\,dy$$

11: Fourier Series

$f(x)$ is periodic iff $f(x+P) = f(x)\ \forall x$

$\therefore\ f(x+nP) = f(x)\ \forall n$

↳ The least period is the smallest positive number P, st P holds
↳ The trig functions sin and cos are simple examples of periodic functions with least period 2π
↳ Fourier series is a way of writing any reasonably well behaved periodic function as a sum of trignometric functions

Orthogonality of Vectors

↳ Two vectors are orthogonal if $f \cdot g = 0$ (perpendicular or $f \perp g = \underline{0}$)
↳ Orthonormal are orthogonal, normalised ~~bases~~ basis
  ↳ Any vector f can be written ~~using~~ ~~the~~ uniquely as a linear composition of the basis vectors.
    ↳ Any component can be found by taking scalar product with basis vector

Orthogonality of functions

↳ Consider $f(x)$ defined on interval $[a, b]$
  ↳ $\forall x$ st $a \le x \le b \Rightarrow f(x) \in \mathbb{R}$
↳ This is infinite-dimensional vector where suffix i (over integers) is replaced by real x.

↳ Scalar product (inner product)
  ↳ $= \int_a^b f(x)\,g(x)\,dx$

↳ $f(x)$ and $g(x)$ orthogonal iff $\int_a^b f(x)g(x)\,dx = 0$

Orthogonality of Trig Functions

↳ On interval $[-\pi, \pi]$; $1, \cos x, \cos 2x, \ldots, \sin x, \sin 2x, \sin 3x \ldots$ is a basis.

↳ These functions are mutually orthogonal.

$$\int_{-\pi}^{\pi} \cos nx \, dx = 0 \qquad (n \neq 0)$$

$$\int_{-\pi}^{\pi} \sin nx \, dx = 0$$

↳ $$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx$$

$$= \begin{cases} 2\pi \text{ if } m = n = 0 \\ \pi \text{ if } m = n \neq 0 \\ 0 \text{ if } m \neq n \end{cases}$$

↳ $$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0$$

↳ $$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 \text{ if } m = n = 0 \\ \pi \text{ if } m = n \neq 0 \\ 0 \text{ if } m \neq n \end{cases}$$

↳ Therefore functions are mutually orthogonal, though not normalised
↳ Can be solved by dividing by $\sqrt{\pi}$ or $\sqrt{2\pi}$ in case of function 1.

↳ These results can be generalised to any intervals $[a, b]$ of length $2L = b-a$. Can stretch the interval from $[-\pi, \pi]$ to $[-L, L]$ by $x = \pi x'/L$

↳ $$\int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} \, dx = \begin{cases} 2L & m = n = 0 \\ L & m = n \neq 0 \\ 0 & m \neq n \end{cases}$$

↳ $$\int_{-L}^{L} \cos \frac{m\pi x}{L} \sin \frac{n\pi x}{L} \, dx = 0$$

↳ $$\int_{-L}^{L} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} = \begin{cases} 0 \text{ if } m = n = 0 \\ L \text{ if } m = n \neq 0 \\ 0 \text{ if } m \neq n \end{cases}$$

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \qquad P = 2L$$

$$\int_{-L}^{L} f(x)\,dx = \frac{a_0}{2} 2L = La_0$$

To find a_m for $m \geq 1$, multiply $f(x)$ by $\cos(m\pi x/L)$

$$\int_{-L}^{L} f(x) \cos \frac{m\pi x}{L}\,dx = La_m$$

To find b_m for $m \geq 1$;

$$\int_{-L}^{L} f(x) \sin \frac{m\pi x}{L}\,dx = Lb_m$$

Even Function: $f_e(-x) = f_e(x)$
Odd Function : $f_o(-x) = -f_o(x)$

$$\int_{-L}^{L} f_e(x)\,dx = 2\int_{0}^{L} f_e(x)\,dx.$$

$$\int_{-L}^{L} f_o(x)\,dx = 0$$

↳ Product of two even functions, or two odd functions is even. Product of odd and even functions is odd.

If $f_e(x)$, then $b_n = 0$ (Cosine Fourier Series)

$$f_e(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$a_n = \frac{2}{L} \int_{0}^{L} f_e(x) \cos \frac{n\pi x}{L}\,dx$$

If $f_o(x)$, then $a_n = 0$ (Sine Fourier Series)

$$f_o(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$b_n = \frac{2}{L} \int_{0}^{L} f_o(x) \sin \frac{n\pi x}{L}\,dx$$

~~Fourier coefficients decline like $\frac{1}{n^2}$ as $n \to \infty$~~

↳ Fourier Series and Symmetries

↳ Given $f(x)$ with domain $0 \leq x \leq c$ we have options for how to define Fourier Series

↳ ① Translate the function ~~into~~ $\frac{c}{2}$ and obtain a periodic function where $P = 2L = c$

↳ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$

↳ ② Reflect function in y axis - $P = 2L = 2c$ - get a cosine Fourier series

↳ ③ Rotate function by 180 degrees about function → odd function $P = 2L = 2c$, get a sine Fourier series

↳ Discontinuity and Gibbs phenomenon

↳ Convergence particularly slow close to a discontinuity

↳ If series terminated after a finite number of terms, sum has a characteristic behaviour near discontinuity - overshoots the required value of the function near discontinuity then adjusts back to correct value through set of decaying oscillation

↳ As number of terms in the series increases, size of overshoot remains nonzero - always substantial error in function representation (tends to 18%)

↳ But scale of decaying oscillations reduce so area of region of substantial error → 0

↳ GIBBS PHENOMENON

Differentiation and Integration

↳ $f'(x)$ can be found by differentiating term for term $f(x)$.
↳ Similarly if $f'(x)$ is known, $f(x)$ can be found by integrating term for term.

↳ $f'(x)$ coefficients decline less rapidly with n than $f(x)$ ⇒ convergence is less rapid
↳ $f'(x)$ may not converge even if $f(x)$ converges

↳ Integration has opposite effect. Integral of $f(x)$ is a smoother function and Fourier series converges faster.
↳ But if $f(x)$ has a_0 term, $\int f(x)dx$ will not be periodic (have $a_0 x$)
↳ Can construct a Fourier series for integrated function but not term-by-term integral

Parseval's Theorem

Given $f(x)$ with period $2L$, mean square value of $f(x)$

$$= \frac{1}{2L}\int_{-L}^{L}[f(x)]^2dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + b_n^2$$

(Orthogonality relations remove cross terms)

This is PARSEVAL'S THEOREM

$\frac{a_0^2}{4}, \frac{1}{2}(a_1^2+b_1^2), \ldots$ represent the power spectrum of $f(x)$ - how the energy is distributed among the various harmonic components.

Parseval's Theorem states total energy / power is sum of contributions of each harmonic component.

Complex Fourier Series

$$c_n = \begin{cases} (a_{-n} + i b_{-n})/2 & n < 0 \\ a_0/2 & n = 0 \\ (a_n - i b_n)/2 & n > 0 \end{cases}$$

Then $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L} \, dx.$$

↳ Note:

$$\frac{1}{2L}\int_{-L}^{L} e^{i(n-m)\pi x/L} \, dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

↳ We can also ~~also~~ state orthogonality:

$$\int_a^b [f(x)]^* g(x) \, dx = 0$$

Linear Algebra

Linear Vector Space

↳ V : set of elements denoted by being bold
↳ K : field consisting of a set of scalar. For our purposes, this is a real or complex number

Axioms:
1. Associativity of addition and scalar multiplication
2. Commutativity
3. Exists a $\underline{0}$ st $\underline{u} + \underline{0} = \underline{u}$ $\forall u \in V$
4. Inverses exist : $\forall \underline{u} \in V$ $\exists$ $(-\underline{u}) \in V$ st $\underline{u} + (-\underline{u}) = \underline{0}$
5. $\forall a \in K$, $\underline{u}, \underline{v} \in V$, $a(\underline{u}+\underline{v}) = a\underline{u} + a\underline{v}$
6. $\forall a, b \in K$, $\underline{u} \in V$ $(a+b)\underline{u} = a\underline{u} + b\underline{u}$
7. For unit scalar $1 \in K$ and any $\underline{u} \in V$, $1\underline{u} = \underline{u}$

Examples: → Set of all n-tuples of elements of an arbitrary field with scalar multiplication and vector addition defined by:

$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1+b_1, a_2+b_2, \ldots a_n+b_n)$
$h(a_1, a_2, \ldots, a_n) = (ha_1, ha_2, \ldots, ha_n)$

↳ where the arbitrary field is $\mathbb{R}$, then this is denoted as $\mathbb{R}^n$
↳ Similarly $\mathbb{C}^n$ exists as well.

Linear Combination: $\underline{x} = a_1\underline{v_1} + a_2\underline{v_2} + \ldots + a_n\underline{v_n}$
↳ $\underline{x}$ is a linear combination of $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}$

↳ The set of all linear combinations of $\underline{v_1}, \ldots, \underline{v_n}$ is a subspace of V. This subspace is <u>spanned</u> by the $\underline{v}$'s and $\underline{v}$'s generate the subspace

Linear Independence: For: $a_1\underline{v_1} + \ldots + a_m\underline{v_m} = 0$ the vectors $\underline{v_1}, \underline{v_2}, \underline{v_n}$ are linearly independent if the only solution is $a_i = 0$ $\forall i$. Otherwise they are linearly dependent.

Dimension and basis : Vector space is of finite dimension if there are linearly independent vectors that span the space -> it is then a basis

Coordinates : n-tuple $\underset{\sim}{x} = (x_1, \dots, x_n)$ are the coordinates of a vector with respect to a given basis
$\rightarrow$ $\underset{\sim}{x}$ is a vector as $\in K^n$

Linear Maps : $\rightarrow$ maps from one vector space to a vector space
$\underset{=}{A} : \underset{\sim}{x} \rightarrow y \quad (\underset{=}{A}\underset{\sim}{x} = y)$
$\hookrightarrow$ may have an inverse

Matrices : Rectangular array of real or complex numbers
(m×n) means m rows, n columns

$$A = (a_{ij}) \quad (A)_{ij} = a_{ij}$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}$$

m×m ~~vector~~ matrix is square
m×1 is column
1×n is row

Algebra $\rightarrow$ Addition $\underset{=}{C} = \underset{=}{A} + \underset{=}{B}$, $c_{ij} = a_{ij} + b_{ij}$

$\hookrightarrow$ Multiplication by scalar : $\underset{=}{B} = \lambda \underset{=}{A}$ means $b_{ij} = \lambda a_{ij}$

$\hookrightarrow$ Matrix Multiplication $\rightarrow$ AB exists only if dimensions are of form : $\underset{=}{A}$ is a×b and $\underset{=}{B}$ is b×c

$\underset{=}{C} = \underset{=}{A}\,\underset{=}{B}$ $\qquad c_{ij} = a_{ik} b_{kj}$ (product matrix is a×c in dimension)

Definitions : (a) Transpose of an $m \times n$ matrix M is the $n \times m$ matrix M^T given by interchange of rows and columns of M.

↳ $(M^T)_{ij} = (M)_{ji}$

↳ $(M^T)^T = M$

↳ $(AB)^T = B^T A^T$ (generalises to three or more)

$$\downarrow$$

$$a_{jk} b_{ki} = b_{ki} a_{jk}$$
$$= (B^T)_{ik} (A^T)_{kj}$$

(b) Symmetric matrix satisfies $S^T = S$ $(s_{ij} = s_{ji})$
Antisymmetric matrix satisfies $S^T = -S$ $(a_{ij} = -a_{ji})$

$$S = \frac{1}{2}(B + B^T) \qquad A = \frac{1}{2}(B - B^T)$$

{Symmetric and Anti-symmetric parts of B}

(c) Diagonal matrix: only on diagonal

(d) Unit matrix : diagonal matrix ($\underline{1}$ or $\underline{I}$) with elements δ_{ij} (Kronecker Delta) $\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j$

$$A = IA = AI = A$$

(e) Orthogonal matrix : $OO^T = O^T O = I$

(f) Complex conjugation : $A = (a_{ij}) \Rightarrow A^* \ (a_{ij}^*)$

(g) Hermitian Conjugation : If $A = (a_{ij})$

$$A^\dagger = (A^T)^* = (A^*)^T = (a_{ji}^*)$$

↳ Hermitian matrix satisfies $A^\dagger = A$

(h) Trace : sum of elements on the main diagonal of the matrix

$$\text{trace}(A) = \sum_{i=1}^{m} a_{ii}$$

$\text{trace}(AB) = \text{trace}(BA)$, $\text{trace}(ABC) = \text{trace}(CAB)$ → true for any cyclic permutations

Inner Product of Vectors : → For real column vectors : $\underline{x} \cdot \underline{y} = x^T y$

↳ For complex column vectors : $\underline{x} \cdot \underline{y} = x^{+} y$

↳ Magnitude of x $\|\underline{x}\| = \sqrt{\underline{x} \cdot \underline{x}}$

↳ If $\underline{x} \cdot \underline{y}$ then orthogonal

↳ A basis $\underline{e}_1, \dots, \underline{e}_n$ which satisfies $\underline{e}_i \cdot \underline{e}_j = 1$

, $e_i \cdot e_j = 0$ $(i \neq j)$ is orthonormal

$e_i \cdot e_j = \delta_{ij}$

Linear System of Equations :

$$a_{11} x_1 + a_{12} x_2 \cdots a_{1n} x_n = y_1$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$a_{m1} x_1 + a_{m2} x_2 \cdots a_{mn} x_n = y_m$$

is equivalent to $Ax = y$

Determinants

↳ Consider A as an $n \times n$ square matrix, M_{ij} is the $(n-1)\times(n-1)$ submatrix of A obtained by deleting its i^{th} row and j^{th} column.

$|M_{ij}|$ is the minor of the element a_{ij}

Cofactor is the signed minor : $(-1)^{i+j} |M_{ij}| = A_{ij}$

↳ Matrix of cofactors is classical adjoint (adj A)

↳ $(\text{adj} A)_{ij} = A_{ji}$

↳ $|A| = \sum_{j=1}^{n} a_{ij} A_{ij}$ for some fixed i

Epsilon Tensor (Levi-Cevita Tensor)

↳ Even permutations $= +1$

Odd permutations $= -1$

Not permutation $= 0$

For 3 :

$$\varepsilon_{j_1 j_2 j_3} = \begin{cases} 0 & \text{if } j_1 = j_2 \text{ or } j_2 = j_3 \text{ or } j_1 = j_3 \\ +1 & \text{if } \sigma = j_1 j_2 j_3 \text{ is even} \\ -1 & \text{if } \sigma = j_1 j_2 j_3 \text{ is odd} \end{cases}$$

σ is pairwise permutations of neighbours to get back to $1, 2, \ldots, n$.

$$|A| = \sum_{j_1 j_2 j_3} \varepsilon_{j_1 j_2 j_3} a_{1j_1} a_{2j_2} a_{3j_3}$$

$$= \sum_{i_1 i_2 i_3} \varepsilon_{i_1 i_2 i_3} a_{i_1 1} a_{i_2 2} a_{i_3 3}$$

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \quad |A| = \underline{a} \cdot (\underline{b} \times \underline{c})$$

(Scalar triple product of its rows (or columns) treated as vectors)

If the same vector occurs twice anywhere in the sum involving the ε-tensor, then the answer is 0. Also, if matrix has any two equal rows or columns, then determinant is 0.

$\hookrightarrow \det(AB) = \det(A)\det(B)$

$\hookrightarrow |A| = |A^T|$

$$\hookrightarrow \{\underline{u} \times \underline{v}\} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\operatorname{curl} \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\hookrightarrow A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$$

$$A = \begin{pmatrix} a \rightarrow \\ b \rightarrow \\ c \rightarrow \end{pmatrix}$$

$$A^{-1} = \frac{1}{a \cdot (b \times c)} \begin{pmatrix} b \times c & c \times a & a \times b \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

since

$$\begin{pmatrix} a \rightarrow \\ b \rightarrow \\ c \rightarrow \end{pmatrix} \begin{pmatrix} b \times c & c \times a & a \times b \\ \downarrow & \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} a \cdot (b \times c) & 0 & 0 \\ 0 & b \cdot (c \times a) & 0 \\ 0 & 0 & c \cdot (a \times b) \end{pmatrix}$$

↳ If $|A| = 0$ then A^{-1} does not exist $\Rightarrow A$ is a singular matrix

↳ For an orthogonal matrix O, O^T is the inverse

Transformations

① ~~Reflection~~ Rotation (R): $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$|R| = 1$ - rotations preserve the length of the vector.

$x^T x = y^T y = x^T(R^T R)x$

True $\forall x$, therefore $R^T R = I$

② Reflection: Vector x' obtained by reflecting x in a plane with unit normal $\underline{n}$ is $\underline{x}' = \underline{x} - 2(\underline{x}\cdot\underline{n})\underline{n}$

$x' = Ox$

$O = I - 2nn^T$

↳ O is orthogonal (and has determinant $= -1$)

Cramer's Rule

If $Ax = y$ & $|A| \neq 0$

$$x = A^{-1}y = \frac{\text{adj}(A)\,y}{\det(A)}$$

$$x_i = \frac{1}{|A|}\sum_k A_{ki}\, y_k$$

$$= \frac{1}{|A|}\begin{vmatrix} a_{11} & \cdots & y_1 & \cdots \\ a_{21} & \cdots & y_2 & \cdots \\ \vdots & & \vdots & \\ a_{n1} & \cdots & y_n & \cdots \end{vmatrix}$$ where ys replace ith column

Uniqueness of Solutions: $Ax = y$

A is $m \times n$

x is $n \times 1$

y is $m \times 1$

Want to investigate solutions to the m equations for n unknowns

If there are any redundant equations, these should be omitted
If equations are inconsistent then there is no solution.

(1) If $m < n$, system is undetermined, therefore no forced (point) solution, however can find family of solutions. ($n-m$ dimension subspace)

(2) If $m > n$, LHS must be linearly dependent.
↳ ① Equations are inconsistent - no solution
↳ ② Redundant equations - discard these then reevaluate

(3) $m = n$
↳ $|A| \neq 0$ - unique solution
{easy way to solve is using Gaussian Elimination}

↳ $|A| = 0$, $y = 0$ $\quad \therefore \quad \underline{A}\,\underline{x} = \underline{0}$

$$A = \begin{pmatrix} r_1 \longrightarrow \\ r_2 \longrightarrow \\ \vdots \\ r_n \longrightarrow \end{pmatrix} \qquad \therefore \quad x \cdot r_i = 0 \quad \text{for } i = 1 \ldots n$$

We find vector z that does not lie in the ~~2D~~ $(n-1)$D basis formed by ~~two~~ $n-1$ of the vectors $(r_1, r_2, r_3, r_4 \ldots, r_{n-1})$
Say $\underline{z} = \underline{r_1} \times \underline{r_2} \times \underline{r_3} \times \cdots \times \underline{r_{n-1}}$

Clearly $\underline{z} \cdot \underline{r_j} = 0$
$\therefore \quad \underline{x} = \lambda \underline{z} = \lambda(\underline{r_1} \times \underline{r_2} \times \cdots \times \underline{r_{n-1}})$
for any value of λ.

Space spanned by $\underline{z_1}, \underline{z_2}, \ldots, \underline{z_{n-h}}$ is the kernel of A.

↳ $|A| = 0$, $y \neq 0$
↳ Column vectors of A are linearly dependent and do not form a basis

for $\mathbb{R}^n$, but only span a subspace $S_c \in \mathbb{R}^n$. If the greatest number of independent vectors is h, then S_c has $\dim S_c = h$.

If y does not lie in subspace S_c ($y \in S_c$), there is no solution for z. (S_c is called image of A since A must map every vector $x \in \mathbb{R}^n$ into S_c).

Considering the linear independent columns - say c_1 and c_2, the condition for y being in S_c is $y \cdot (c_1 \times c_2) = 0$

i.e.

$$y = p_2 c_1 + p_1 c_2 + \underset{\substack{\uparrow \\ \text{has to be } 0.}}{p_0} (c_1 \times c_2)$$

or $y \cdot w_i = 0$ for $i = 1, 2, \ldots, n-h$ in generalised case. (there are $(n-h)$ independent vectors: $w_1, w_2, \ldots, w_{n-h}$ that do not lie in S_c so that $s \in S$, then $s \cdot w_i = 0 \ \forall i$)

Clearly not a unique solution: given a solution x_0, $A(x_0 + \alpha z) = y$ where $Az = 0$

Therefore, most general solution for $Ax = y$ is:

$$x = x_0 + \sum_{i=1}^{n-h} a_i z_i$$

where z_i satisfy $A \cdot z_i = 0$, $i = 1, 2, \ldots, (n-h)$

Eigenvalues & Eigenvectors

If $Av = \lambda v$

↳ $n \times n$ matrix

↳ v is the eigenvector of A corresponding to eigenvalue λ.

↳ λ is an eigenvalue of A

(i) Acting on v with A scales it with λ, leaving direction unchanged.

(ii) If $\underline{v}$ is an eigenvector then so is $\alpha \underline{v}$

$$\underline{\underline{A}}\underline{v} = \lambda \underline{\underline{I}}\underline{v}$$

$$\underline{v}(A - \lambda I) = 0 \quad \therefore \quad \underline{v} = \underline{0} \quad \text{or} \quad \det(A - \lambda I) = 0$$

$$\det(A - \lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} \equiv P_A(\lambda) = 0$$

Determinant is polynomial of degree n in λ - characteristic polynomial.
↳ eigenvalues may be complex even if matrix entries are real.

(i) $\det A = \prod_{i=1}^{n} \lambda_i$

(ii) if $|A| = 0$ then at least eigenvalue is 0 and corresponding vectors satisfy $Av = 0$

(iii) $\text{trace } A = \sum_{i=1}^{n} \lambda_i$

(iv) For real symmetric matrices → $A = A^* = A^T$ if any λ are complex, then they must come in complex conjugate pairs

(v) Real symmetric matrix has real eigenvalues and orthogonal matrices
↳ These form an orthogonal basis

<u>Diagonalisation of real symmetric matrices</u>

Consider $n \times n$ matrix X, whose i-th column is e_i
↳ $X = \begin{pmatrix} e_1 & e_2 & \cdots & e_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$

$$X^TX = \begin{pmatrix} e_1 \rightarrow \\ e_2 \rightarrow \\ \vdots \\ e_n \rightarrow \end{pmatrix} \begin{pmatrix} e_1 & e_2 & \cdots & e_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Therefore

↳ ① $X^{-1} = X^T$

↳ ② $X^T X = X X^T = I$

↳ ③ X is orthogonal and $\det X = 1$

$$AX = A \begin{pmatrix} e_1 & e_2 & \cdots & e_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} = \begin{pmatrix} \lambda_1 e_1 & \lambda_2 e_2 & \cdots & \lambda_n e_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$$

$$A' (= X^T A X) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

(diagonal matrix with eigenvalues as elements of the diagonal).

$$Ax = y$$
$$AXX^T x = y$$
$$X^T A X \; X^T x = X^T y$$
$$A' x' = y'$$

For eigenvector e_i, $e_i' = X^T e_i$

↳ Coordinates of x in basis of eigenvectors

↳ $x = \sum_{i=1}^{n} x_i e_i$

$$x' = X^T x = \sum_{i=1}^{n} x_i X^T e_i = \sum_{i=1}^{n} x_i e_i' = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

↳ Coordinates x_i, $i = 1 \ldots$ are entries in x'

↳ Determinants are the same

p9

Partial Differential Equations

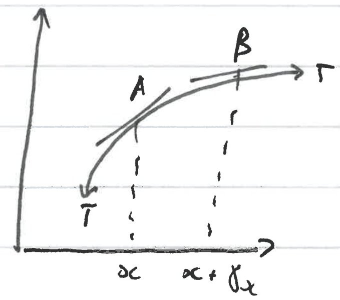
PDE is of form: $\mathbb{F}(x, y, \dots, f_x, f_y, \dots, f_{xx}, f_{xy}, f_{yy}, \dots)$

↳ order of the PDE is order of the highest derivative appear in $\mathbb{F}$

To obtain a unique solution, we need extra information in the val of f on surfaces in (x,y) space. Usually, we find solutions some region and the extra information (or boundary information) are given on all or part of the boundary ∂D of this region

↳ Generally a hard problem to workout how much information is required

Example: Wave equation:



$y(x, t)$

$$\text{Transverse force} = T(\text{slope at } B - \text{slope at } A)$$
$$= T\left(\frac{\partial y}{\partial x}(x+\delta x, t) - \frac{\partial y}{\partial x}(x, t)\right)$$
$$\simeq T \frac{\partial^2 y}{\partial x^2}\delta x = \rho\, dx \frac{\partial y^2}{\partial t^2} \quad (ma)$$
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \qquad c = \sqrt{\frac{T}{\rho}}$$

Other examples exist, but they strike me as entirely pointless (and very useless).

Classification: Consider general form in 2D:

$$a\frac{\partial^2 \psi}{\partial x^2} + 2b\frac{\partial^2 \psi}{\partial x \partial y} + c\frac{\partial^2 \psi}{\partial y^2} + f\frac{\partial \psi}{\partial x} + g\frac{\partial \psi}{\partial y} + h\psi = 0$$

This equation is elliptic if $b^2 < ac$

↳ Laplace's Equation $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

" parabolic if $b^2 = ac$

↳ Heat equation:

$$\kappa \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial \psi}{\partial t} = 0$$

" hyperbolic if $b^2 > ac$

↳ Wave equation:

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0$$

<u>Methods of solution</u>

① Some Elliptic and Hyperbolic Equations in 2D.

↳ of the form: $a\frac{\partial^2 \psi}{\partial x^2} + 2b\frac{\partial^2 \psi}{\partial x \partial y} + c\frac{\partial^2 \psi}{\partial y^2} = 0$

We look for a solution of the form $\psi(x,y) = x + py \equiv f(z)$

$\{z = x + py\}$

$$\frac{\partial f}{\partial x} = \frac{df}{dz}\frac{\partial z}{\partial x} = \frac{df}{dz}$$

$$\frac{\partial f}{\partial y} = \frac{df}{dz}\frac{\partial z}{\partial y} = p\frac{df}{dz}$$

$$a\frac{d^2 f}{dz^2} + 2bp\frac{d^2 f}{dz^2} + cp^2\frac{d^2 f}{dz^2} = 0$$

∴ solve $cp^2 + 2bp + a = 0$

↳ roots complex for elliptic equations

$u = x + p_+ y$, $v = x + p_- y$. Because equation is linear, independent combination gives general solution.

$\psi(x,y) = f(x + p_+ y) + g(x + p_- y) = f(u) + g(v)$

where f and g are arbitrary functions of a single variable

(2) Seperation of Variables , $b=0$

$$a\frac{\partial^2\varphi}{\partial x^2} + c\frac{\partial^2\varphi}{\partial y^2} + f\frac{\partial\varphi}{\partial x} + g\frac{\partial\varphi}{\partial y} + h\varphi = 0$$

Try solution of form $\varphi(x,y) = X(x)\,Y(y)$
We get :

$$aY\frac{d^2X}{dx^2} + cX\frac{d^2Y}{dy^2} + fY\frac{dX}{dx} + gX\frac{dY}{dy} + hXY = 0$$

$$\frac{1}{X}\left[a\frac{d^2X}{dx^2} + f\frac{dX}{dx} + hX\right] = -\frac{1}{Y}\left[c\frac{d^2Y}{dy^2} + g\frac{dY}{dy}\right] \boxed{=\lambda}$$

True since left independent of y and right independent of x => not equal unless both independent.

$$\therefore\ a\frac{d^2X}{dx^2} + f\frac{dX}{dx} + (h-\lambda)X = 0. \qquad (1)$$

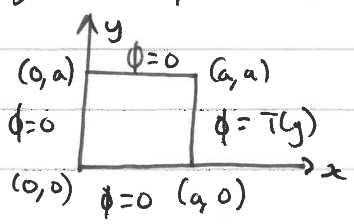
$$c\frac{d^2Y}{dy^2} + g\frac{dY}{dy} + \lambda Y = 0$$

It appears not all values of λ are allowed - eigenvalues
$\forall\ \varphi_\lambda(x,y) = X_\lambda(x)\,Y_\lambda(y)$
$\alpha\,\varphi_\lambda(x,y)$ is also a solution with α being a constant. We then choose the normalisation, with it being any linear combination.

$$\varphi(x,y) = \sum_\lambda a_\lambda \varphi_\lambda(x,y)$$

<u>Laplace's Equation</u> : $\dfrac{\partial^2\phi(x,y)}{\partial x^2} + \dfrac{\partial^2\phi(x,y)}{\partial y^2} = 0$

Question is: what is the steady state temp distribution $\phi(x,y)$ with following properties (boundary conditions)



Using separation of variables: $\phi(x,y) = X(x)\,Y(x)$

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

$$\frac{1}{X}\frac{d^2X}{d^2Y} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda$$

$\therefore$ two ODEs
$$\frac{d^2X}{dx^2} - \lambda X = 0$$
$$\frac{d^2Y}{dy^2} + \lambda Y = 0$$

We set $\lambda < 0$ $\begin{cases} X \text{ is sinusoidal} \\ Y \text{ is exponential} \end{cases}$ or $\lambda > 0$ $\begin{cases} X \text{ is exponential} \\ Y \text{ is sinusoidal} \end{cases}$

Choise is determined by boundary conditions $\therefore$ in this case choose $\lambda > 0$, say $\lambda = m^2$

$$X_m(x) = Ae^{mx} + Be^{-mx}$$
$$Y_m(y) = C\sin my + D\cos my$$

two boundary conditions $\begin{cases} \phi(x,0) = \phi(x,a) = 0 \Rightarrow X(x)Y(0) = X(x)Y(a) = 0 \\ Y(0) = Y(a) = 0 \quad \Rightarrow D = 0 \end{cases}$

$$Y(y) = Y_n(y) = C_n \sin\left(\frac{n\pi}{a}y\right)$$
$$\therefore \quad m = \frac{n\pi}{a} \quad \text{for } m = 1, 2, \ldots$$

Therefore, for X_ms we do the same.

$$\phi(x,y) = \sum_{n=1}^{\infty}\left(A_n e^{n\pi x/a} + B_n e^{-n\pi x/a}\right)\left(\sin\left(\frac{n\pi}{a}y\right)\right)$$

↳ set C_n to 1 without loss of generality

More boundary conditions: $\phi(0,y) = 0 \quad \therefore \quad A_n + B_n = 0 \;\forall n$

$\phi(a,y) = T(y)$ for some function of y.

$$\therefore \sum_{n=1}^{\infty} 2A_n \sinh(n\pi)\sin\left(\frac{n\pi}{a}y\right) = T(y)$$

$$\sum_{n=1}^{\infty} 2A_n \sinh(n\pi) \sin\left(\frac{n\pi}{a}y\right) \times \frac{2}{a}\sin\left(\frac{m\pi y}{a}\right) = \frac{2}{a} T(y) \sin\left(\frac{m\pi}{a}y\right)$$

$$\left(\int_0^a\right)$$

$$\sum_{n=1}^{\infty} 2A_n \sinh(n\pi) \frac{2}{\cancel{a}} \int_0^a dy \sin\left(\frac{n\pi}{a}y\right) \sin\left(\frac{m\pi}{a}y\right)$$
$$= \frac{2}{a}\int_0^a dy\, T(y) \sin\left(\frac{m\pi}{a}y\right)$$

$$\boxed{2A_n \sinh(n\pi) = \frac{2}{a}\int_0^a dy\, T(y) \sin\left(\frac{m\pi}{a}y\right)}$$

Not discussing the rest of the example because I see no interest in them.