Numerical Methods

→ Solution of Algebraic & transcendental Equations (exponential or logarithmic or trigonometric eqns)

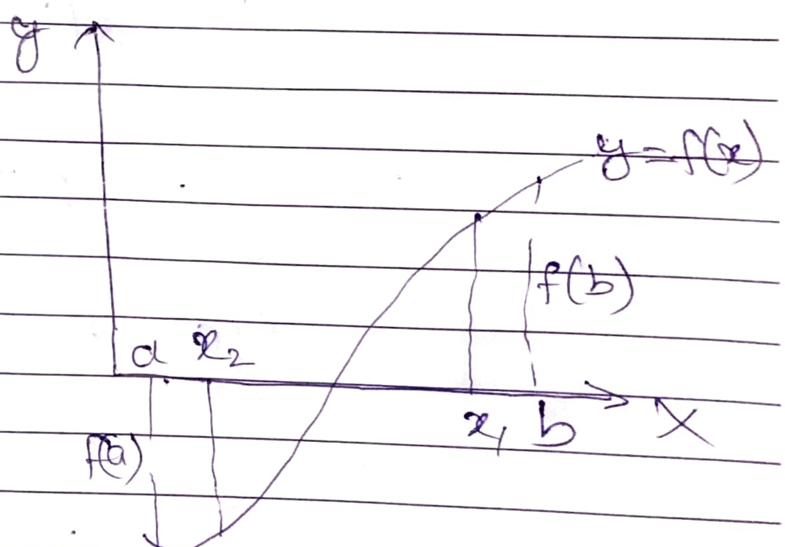
* Bisection Method :

This method is based on the repeated application of intermediate value property.

Let $f(x)$ be continuous between a & b & let
$f(a)$ be $-ve$
& $f(b)$ be $+ve$.

Then, the first approximation of the root is
$x_1 = \frac{a+b}{2}$, if $f(x_1) = 0$, then x_1 is root of $f(x) = 0$

Otherwise the root lies between a & x_1, or x_1 & b according as $f(x_1)$ is positive or negative.
Then we bisect the interval as before & continue the process until the root is found accurately.



Q. Find the real root of the eqn
$$f(x) = x^3 - x - 1 = 0$$

Soln: ~~$f(-1) = +1+1 = -1$~~
$f(0) = -1$
$f(1) = -1$

~~$f(2) = 8-2-1 = 5$~~
So root lies betn 1 & 2.

$f(1) = -1$ $\quad f(1.5) = 7/8$ $\quad f(2) = 5$

1 $\quad$ 1.25 $\quad$ 1.375 $\quad$ 1.5 $\quad$ 2

Here root will lie.

$x_1 = \frac{1+2}{2} = 1.5$

$f(1.5) = (1.5)^3 - 1.5 - 1 = 7/8$

$x_2 = \frac{1+1.5}{2} = 1.25$

$f(1.25) = (1.25)^3 - 1.25 - 1 = \frac{-19}{64}$

$x_3 = \frac{1.25+1.5}{2} = 1.375$, $\quad f(1.375) = +ve$

$x_4 = \frac{1.25+1.375}{2} = 1.3125$,

Q Find the real root of the eqn $x^3 - 2x - 5 = 0$

⊖ 2 $\quad$ ⊕ 2.5 $\quad$ ⊕ 3

Soln:

$f(x) = x^3 - 2x - 5$

$f(0) = -5$

$f(1) = -6$

$f(2) = -1$

$f(3) = 16$

$x_1 = \frac{2+3}{2} = 2.5$

$f(2.5) = 5.6250$

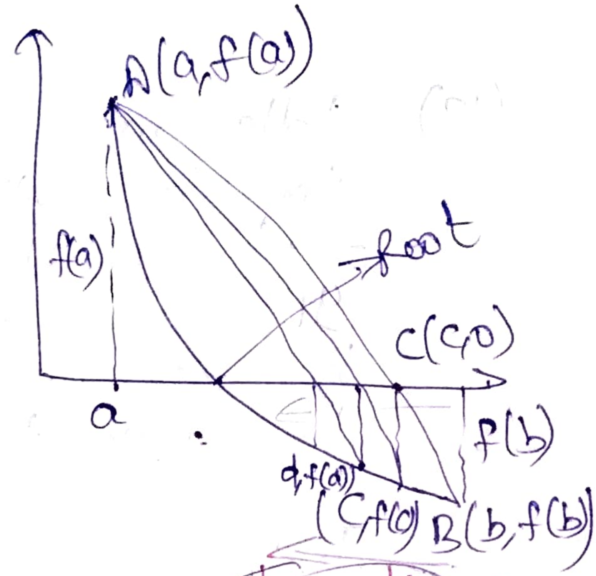
x	a	b	$x = \frac{a+b}{2}$	$f(x)$
1	2	3	2.5	$5.6250 > 0$
2	2	2.5	2.25	$1.8906 > 0$
3	2	2.25	2.125	$0.3457 > 0$
4	2	2.125	2.0625	$-0.3513 < 0$
5	2.0625	2.125	2.09375	$-0.0089 < 0$
6	2.09375	2.125	2.10928	$0.1668 > 0$
7	2.09375	2.10938	2.10156	$0.07856 > 0$
8	2.09375	2.10156	2.09766	$0.03471 > 0$
9	2.09375	2.09766	2.09570	$0.01286 > 0$
10	2.09375	2.09570	2.09473	$0.00019 > 0$
11	2.09375	2.09473	2.09424	$-0.00034 < 0$

repeated to 3 decimal

root ≈ 2.094

* Regula falsi method :-

(False position method)



Slope of AB = Slope of AC

$$\frac{f(b)-f(a)}{b-a} = \frac{0-f(a)}{c-a}$$

$$\Rightarrow c-a = \frac{-f(a)(b-a)}{f(b)-f(a)}$$

$$\Rightarrow c = a - \frac{f(a)(b-a)}{f(b)-f(a)}$$

$$\Rightarrow c = \frac{af(b) - \cancel{af(a)} - bf(a) + \cancel{af(a)}}{f(b)-f(a)}$$

$$\Rightarrow \boxed{c = \frac{af(b) - bf(a)}{f(b)-f(a)}}$$

towards root

rate of convergence is better than bisection

Working procedure

(i) find the interval (a, b) such that $f(a)f(b) < 0$

(ii) find $c = \frac{af(b)-bf(a)}{f(b)-f(a)}$

(iii) $f(a).f(c) < 0$, root lies betn (a, c)

if $f(b).f(c) < 0$, root lies in (c, b)

(iv) repeat step (i) (2)

Q find a real root of $x^3 - 2x - 5 = 0$ using the method false position upto 4 iterations.

Soln:-

$f(2) = -1$

$f(3) = 16$

$\therefore a = 2,\ b = 3$

$f(a) = -1,\ f(b) = 16$

$$c = \frac{2\times16 - 3\times(-1)}{16-(-1)} = \frac{32+3}{17} = \frac{35}{17} = 2.0588$$

$$f\left(\frac{35}{17}\right) = \left(\frac{35}{17}\right)^3 - 2\times\frac{35}{17} - 5 = -0.3908 < 0$$

Now, root lies betn 2.0588 & 3 i.e, (c, b)

2nd Iteration $a = 2.0588$, $b = 3$

$f(a) = -0.3908$, $f(b) = 16$

$$c = \frac{2.0588 \times 16 - 3(-0.3908)}{16 - (-0.3908)} = 2.0812$$

$$f(c) = f(2.0812) = -0.1479 < 0$$

3rd iteration $a = 2.0812$, $b = 3$

$f(a) = -0.1479$, $f(b) = 16$

$$c = \frac{2.0812 \times 16 - 3(-0.1479)}{16 - (-0.1479)} = 2.0896$$

$$f(c) = f(2.0896) = -0.0551$$

IV Iteration $a = 2.0896$, $b = 3$

$f(a) = -0.0551$, $f(b) = 16$

$$c = \frac{2.0896 \times 16 - 3(-0.0551)}{16 - (-0.0551)} = 2.0927$$

~~f(c)~~ Hence, the required root is 2.0927

Q Using False-position method find a real root of the equation $x \log_{10} x - 12 = 0$ in 3 steps.

Soln: ~~f(2) = 2 log10 2~~ $f(1) = -12 < 0$

$f(2) = 2\log_{10} 2 - 12 = -0.6 < 0$

root lies betn this $f(3) = 0.23 > 0$

Now, this is a transedental eqn ∴ choose mid value to get accurate result.

∴ $f(2.5) = -ve$ $\quad f(2.8) = +ve$

$f(2.6) = -ve$

$f(2.7) = -ve$ choose this for 1st iteration.

1st iteration $a = 2.7$ $b = 2.8$

$f(a) = -0.0353$ $f(b) = 0.052$

$$c = \frac{2.7 \times 0.052 - 2.8(-0.0353)}{0.052 - (-0.0353)} = 2.7404$$

$f(c) = f(2.7404) = -0.00021 < 0$

2nd iteration $a = 2.7404$ $b = 2.8$

$f(a) = -0.00021$ $f(b) = 0.052$

$$c = \frac{2.7404 \times 0.052 - 2.8(-0.00021)}{0.052 - (-0.00021)}$$

$c = 2.7406$

$f(c) = f(2.7406) = -0.00004 < 0$

3rd iteration $a = 2.7406$ $b = 2.8$

$f(a) = -0.00004$ $f(b) = 0.052$

$$c = \frac{2.7406 \times 0.052 - 2.8(-0.00004)}{0.052 - (-0.00004)}$$

$$\boxed{c = 2.7406}$$

* Itegration method:

Itegration method (fixed point iteration method)

Suppose we have equation $f(x) = 0$

The eqn can be expressed as $x = \phi(x)$

$$|\phi'(x)|_{at\ x = x_0} < 1$$

Then iteration method applied.

The successive approximation is given by

$x_n = \phi(x_{n-1})$

$x_1 = \phi x_0$

$x_2 = \phi x_1$

Q find a real root of eqn $f(x) = x^3 + x^2 - 1 = 0$.

Soln:- $f(0) = -1$, $x_0 = 0.5$
$f(1) = 1$

(I) $x^3 = 1 - x^2$

$x = (1-x^2)^{1/3}$

$\phi'(x) = \frac{1}{3}\frac{2x}{(1-x^2)^{2/3}}$

$|\phi'(x)|_{at\ x_0=0.5} = \left|\frac{1}{3}\frac{1}{(1-\frac{1}{4})^{2/3}}\right| < 1$

(II) $x^2 = 1 - x^3$

$x = (1-x^3)^{1/2}$

$\phi'(x) = \frac{1}{2}\frac{3x^2}{(1-x^3)^{1/2}}$

$|\phi'(x)|_{x_0=0.5} = < 1$

(III) $x^2(1+x) = 1$

$x = \frac{1}{\sqrt{1+x}}$

$\phi'(x) = \frac{1}{2(1+x)^{3/2}}$

$\phi'(x)_{x=0.5} < 1$

→ all are giving value less than 1 so we will choose that case which give smallest value.

So, if we choose this (Here this give smallest value) we get accurate result and in less no. of iterations.

Now, $x_n = \frac{1}{\sqrt{1+x_{n-1}}}$

Put $n=1$, $x_1 = \frac{1}{\sqrt{1+x_0}} = \frac{1}{\sqrt{1+0.5}} = 0.81649$

$x_2 = \frac{1}{\sqrt{1+x_1}} = \frac{1}{\sqrt{1.81649}} = 0.74196$

$x_3 = \frac{1}{\sqrt{1+x_2}} = \frac{1}{\sqrt{1.74196}} = 0.75767$

$x_4 = \frac{1}{\sqrt{1+x_3}} = \frac{1}{\sqrt{1.75767}} = 0.75427$

$x_5 = \frac{1}{\sqrt{1+x_4}} = \frac{1}{\sqrt{1.75427}} = 0.75500$

Simillarly, $x_6 = 0.75488$, $x_7 = 0.75487$ Ans =

Q Find the roots of $\cos x = 3x-1$, correct to 4 decimal places by using iteration method.

Soln:

$f(x) = \cos x - 3x + 1$

$f(0) = 1 - 0 + 1 = 2$ → It is more close to 0. ∴ choose this as initial approximation. $x_0 = 0$

$f(\pi/2) = 0 - 3\pi/2 + 1 < 0$
$= -3....$

This form. $x = \phi(x)$

$3x = 1 + \cos x$

$x = \frac{1+\cos x}{3}$

$\therefore \boxed{\phi(x) = \frac{1+\cos x}{3}}$

$\cos x = 3x-1$

$x = \cos^{-1}(3x-1)$

$\phi(x) = \cos^{-1}(3x-1)$

danger ∴ reject it

$|\phi'(x)| = \left|\frac{-\sin x}{3}\right|$

$|\phi'(x)|_{x_0=0} = \left|\frac{\sin 0}{3}\right| < 1$ condn satisfied ∴ we can take this as our $\phi(x)$

Now,

$$x_n = \frac{1+\cos(x_{n-1})}{3}$$

Put $n=1$,

$$x_1 = \frac{1+\cos x_0}{3} = \frac{1+\cos 0}{3} = \frac{2}{3} = 0.66667$$

$n=2$,

$$x_2 = \frac{1+\cos x_1}{3} = \frac{1+\cos(0.66667)}{3} = 0.595296$$

$$x_3 = \frac{1+\cos(0.595296)}{3} = 0.609328$$

$$x_4 = \frac{1+\cos(0.609328)}{3} = 0.606678$$

$$x_5 = \frac{1+\cos(0.606678)}{3} = 0.607182$$

$$x_6 = \frac{1+\cos(0.607182)}{3} = 0.607086$$

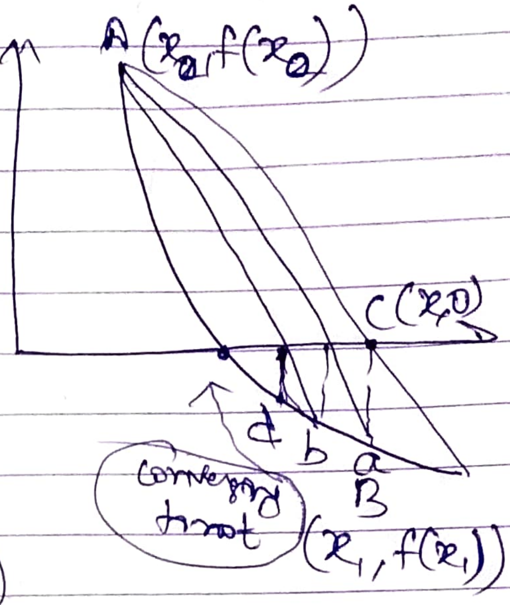
$$x_7 = \frac{1+\cos(0.607086)}{3} = 0.607105$$

$$x_8 = \frac{1+\cos(0.607105)}{3} = 0.607101$$

∴ The correct root upto 4 decimal places $= 0.6071$

* Secant Method :-

⇒ Secant Method (Chord method).
This method is quite similar to Regula-falsi method except for condition $f(x_1)f(x_2)<0$



Slope of AB = Slope of AC

$$\frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{0-f(x_0)}{x-x_0}$$

$$x-x_0 = \frac{-f(x_0)(x_1-x_0)}{f(x_1)-f(x_0)}$$

$$x = x_0 + \frac{f(x_0)(x_0-x_1)}{f(x_1)-f(x_0)}$$

$$x = \frac{x_0 f(x_1) - \cancel{x_0 f(x_0)} + \cancel{x_0 f(x_0)} - x_1 f(x_0)}{f(x_1)-f(x_0)}$$

$$x = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1)-f(x_0)}$$

$$\boxed{x_{n+2} = \frac{x_n f(x_{n+1}) - x_{n+1} f(x_n)}{f(x_{n+1}) - f(x_n)}}$$

→ This method fails when $f(x_n) = f(x_{n+1})$.

$$\boxed{x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}}$$

Q A real root of the eqn $x^3-5x+1=0$ lies in the interval $(0,1)$. Perform four iteration of the secant method.

Soln: $f(x)=x^3-5x+1$

$f(0)=1 \quad \therefore x_0=0 \quad f(x_0)=1$

$f(1)=-3 \quad x_1=1 \quad f(x_1)=-3$

$$x_{n+1}=\frac{x_{n-1}\cdot f(x_n)-x_n f(x_{n-1})}{f(x_n)-f(x_{n-1})}$$

Put $n=1$,

$$\therefore x_2=\frac{x_0\cdot f(x_1)-x_1 f(x_0)}{f(x_1)-f(x_0)}=\frac{0(-3)-1(1)}{-3-1}$$

$$=\frac{-1}{-4}=\frac{1}{4}=0.25$$

$x_2=0.25$

$\therefore f(x_2)=f(0.25)=(0.25)^3-5(0.25)+1$
$=-0.234375$

Put $n=2$, $x_3=\frac{x_1\cdot f(x_2)-x_2 f(x_1)}{f(x_2)-f(x_1)}$

$$=\frac{1\cdot(-0.234375)-0.25(-3)}{-0.234375-(-3)}$$

$=0.18644$

$f(x_3)=f(0.18644)=0.007428$

Put $n=3$, $x_4=\frac{x_2\cdot f(x_3)-x_3 f(x_2)}{f(x_3)-f(x_2)}$

$$x_4=\frac{0.25(0.007428)-(0.18644)(-0.234375)}{0.007428-(-0.234375)}$$

KHUSHI
Date
Page

$\therefore x_4 = 0.20174$

~~Put cos~~ $f(x_4) = f(0.20174) = -0.00048$

Put $n=4$

$$x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)}$$

$$\boxed{x_5 = 0.20081}$$

Q Estimate the root of the eqn $\cos x - xe^x = 0$ using the secant method with initial estimate of $x_1 = 0.5$, $x_2 = 1$.

Soln:-

$f(x_1) = f(0.5) = \cos(0.5) - 0.5\,e^{0.5} = 0.0532$

$f(x_2) = f(1) = \cos(1) - 1 \cdot e^1 = -2.17798$

$$x_{n+1} = \frac{x_{n-1} \cdot f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Put $n=2$.

$$x_3 = \frac{x_1 \cdot f(x_2) - x_2 \cdot f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{0.5 \cdot (-2.17798) - 1 \cdot (0.0532)}{-2.17798 - 0.0532}$$

$x_3 = 0.5119$

$f(x_3) = f(0.5119) = \cos(0.5119) - 0.5119 \cdot e^{0.519}$
$= 0.01773$

$y - y_1 = \frac{dy}{dx}(x - x_1)$

Put $n=3$, $x_4 = \dfrac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$

$x_4 = 0.5158$

$f(x_4) = f(0.5158) = 0.00594.$

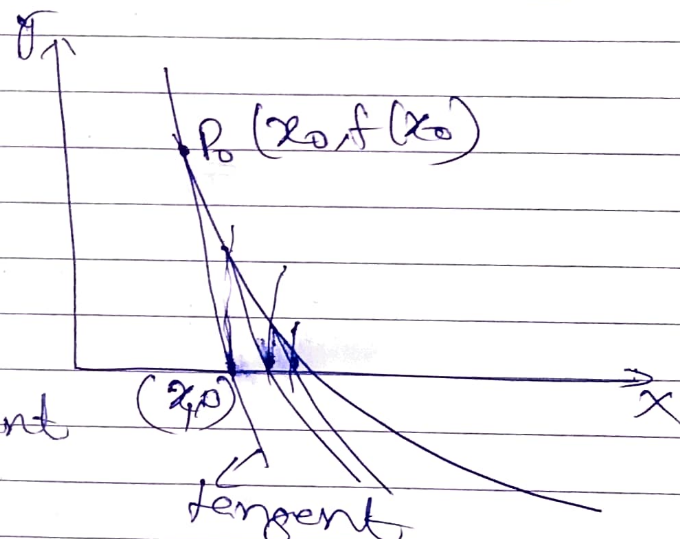
Put $n=4$, $x_5 = 0.5178$

$f(x_5) = f(0.5178) = -0.00012$

Put $n=5$, $x_6 = 0.5178$

* Newton Raphson Method :- (fast convergence)

Eq^n of tangent



$y - f(x_0) = \frac{dy}{dx}(x - x_0)$

eq^n of tangent ← $y - f(x_0) = f'(x_0)(x - x_0)$

∴ $(x_1, 0)$ point lies on tangent so, will satisfy it.

$0 - f(x_0) = f'(x_0)(x_1 - x_0)$

$x_1 - x_0 = \dfrac{-f(x_0)}{f'(x_0)}$ ⇒ $\boxed{x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}}$

$\boxed{x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}}$ $f'(x_n) \neq 0$

KHUSHI
Date :
Page:

Q find by N.R.M a root of eqn $x^3-3x-5=0$

soln:-

$f(0) = -5$ $\quad x_0 = 2$

$f(1) = -7$

$f(2) = -3 \rightarrow$ closer to 0. $\therefore x_0$.

$f(3) = 16$

$\therefore f(x) = x^3-3x-5$

$f'(x) = 3x^2-3$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^3-3x_n-5}{3x_n^2-3}$$

Put $n=0$,

$$x_1 = x_0 - \frac{x_0^3-3x_0-5}{3x_0^2-3}$$

$$= 2 - \frac{2^3-3\times2-5}{3\times2^2-3} = 2 - \frac{(-3)}{9}$$

$$x_1 = 2+\frac{1}{3} = \frac{7}{3} = 2.3333$$

Put $n=1$, $\quad x_2 = x_1 - \frac{x_1^3-3x_1-5}{3x_1^2-3}$

$$= 2.333 - \frac{(2.333)^3-3(2.333)-5}{3(2.333)^2-3}$$

$$x_2 = 2.2805$$

$\therefore x_4 = 2.2790$ Ans

Put $n=2$,

$$x_3 = x_2 - \frac{x_2^3-3x_2-5}{3x_2^2-3} = 2.2790$$

repeated

Put $n=3$, $\quad x_4 = x_3 - \frac{x_3^3-3x_3-5}{3x_3^2-3} = 2.2790$

In transcendental eqn usually take the mid ~~va~~ value as your x_0.

Q Use NRM to find a real root of $\cos x - xe^x = 0$ correct to four decimal places.

Soln:- $f(x) = \cos x - xe^x$

$$\begin{cases} f(1) = -2.1779 \\ f(0) = 1 \end{cases}$$

$$f'(x) = -\sin x - e^x - xe^x$$
$$= -\sin x - e^x(x+1)$$

$$x_0 = \frac{0+1}{2} = 0.5$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{\cos x_n - x_n e^{x_n}}{-\sin x_n - e^{x_n}(x_n+1)}$$

$$x_{n+1} = x_n + \frac{\cos x_n - x_n e^{x_n}}{\sin x_n - e^{x_n}(x_n+1)}$$

Put $n=0$,

$$x_1 = x_0 + \frac{\cos x_0 - x_0 e^{x_0}}{\sin x_0 - e^{x_0}(x_0+1)}$$

$$x_1 = 0.5 + \frac{\cos(0.5) - (0.5)e^{0.5}}{\sin(0.5) - e^{0.5}(0.5+1)}$$

$$x_1 = 0.51802$$

Put $n=1$,
$$x_2 = x_1 + \frac{\cos x_1 - x_1 e^{x_1}}{\sin x_1 - e^{x_1}(x_1+1)}$$

$$x_2 = 0.51802 + \frac{\cos(0.51802) - 0.51802 \cdot e^{0.51802}}{\sin(0.51802) - e^{0.51802}(0.51802+1)}$$

$$x_2 = 0.518$$ repeated

$\therefore$ Ans

Q. Apply NRM to solve the eqn $2(x-3) = \log_{10} x$

Soln: $f(x) = 2x - 6 - \log_{10} x$

$$\begin{cases} f(3) = 6 - 6 - \log_{10} 3 = -0.47712 \\ f(4) = 8 - 6 - \log_{10} 4 = 1.39794 \end{cases}$$

for our convenience

$\therefore x_0 = 3.5$ → for our convenience as it will give accurate result faster.

$$\therefore f(x) = 2x - 6 - 0.4343 \log_e x$$

$$f'(x) = 2 - \frac{0.4343}{x}$$

$$\therefore x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2x_n - 6 - 0.4343 \log_e x_n}{2 - \frac{0.4343}{x_n}}$$

Put $n = 0$,

$$x_1 = x_0 - \frac{2x_0^2 - 6x_0 - 0.4343\, x_0 \log_e x_0}{2x_0 - 0.4343}$$

$$x_1 = 3.25696$$

Put $n=1$,

$$x_2 = 3.256366$$

Put $n=2$, $\boxed{x_3 = 3.256}$ → repeated

* Modified Newton-Raphson method :-

A/c NRM,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Let $U(x) = \frac{f(x)}{f'(x)}$

f(x) goes to zero faster than f'(x). ∴ so, use f(x).

$$U'(x) = \frac{f'(x) \cdot f'(x) - f(x) \cdot f''(x)}{[f'(x)]^2}$$

$$\therefore x_{n+1} = x_n - \frac{U(x_n)}{U'(x_n)}$$

$$\therefore x_{n+1} = x_n - \frac{\frac{f(x)}{f'(x)} \cdot (f'(x))^2}{f'(x) \cdot f'(x) - f(x) \cdot f''(x)}$$

$$\therefore \boxed{x_{n+1} = x_n - \frac{f(x_n)\cdot f'(x_n)}{[f'(x_n)]^2 - f(x_n) f''(x_n)}}$$

Modified newton raphson

* Normally (when there is no multiple root), newton-raphson method gives better result result than modified newton raphson.

(∵ In modified there is more no. of steps involved, so lengthy process to find root but when there are multiple roots modified is better).

A/C sir,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}} \rightarrow \text{M.N.R method}$$

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Date

Page

* Gauss Elimination Method:-

Solution of linear algebraic methods!

(I) Direct method (II) Indirect method

→ Gauss elimination method

→ Gauss - Jordon method

Q. Solve the following eqn by gauss elimination method:-

$$x - y + 2z = 3$$
$$x + 2y + 3z = 5$$
$$3x - 4y - 5z = -13$$

Soln:

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 3 \\ 3 & -4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} 3 \\ 5 \\ -13 \end{pmatrix}$$

$$(A:B) = \left[\begin{array}{ccc:c} 1 & -1 & 2 & 3 \\ 1 & 2 & 3 & 5 \\ 3 & -4 & -5 & -13 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 3R_1$

$$= \left[\begin{array}{ccc:c} 1 & -1 & 2 & 3 \\ 0 & 3 & 1 & 2 \\ 0 & -1 & -11 & -22 \end{array}\right]$$

$R_3 \to 3R_3 + R_2$

$$= \left[\begin{array}{ccc:c} 1 & -1 & 2 & 3 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & -32 & -64 \end{array}\right]$$

$\therefore$ $x - y + 2z = 3$ $\Rightarrow x = -1$

$3y + z = 2$ $\Rightarrow y = 0$

$-32z = -64$ $\Rightarrow z = 2$

$\therefore$ $x = -1$

$y = 0$

$z = 2$

Q Solve the system of eqn : $2x+y+z=10$

$3x+2y+3z=18$

$x+4y+9z=16$

using (i) Gauss elimination method → upper triangular matrix

(ii) Gauss Jordan method → diagonal matrix

Soln :

$$x+4y+9z=16$$
$$2x+y+z=10$$
$$3x+2y+3z=18$$

$$\therefore \begin{bmatrix} 1 & 4 & 9 \\ 2 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \\ 18 \end{bmatrix}$$

$AX = B$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 4 & 9 & 16 \\ 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$= \left[\begin{array}{ccc:c} 1 & 4 & 9 & 16 \\ 0 & -7 & -17 & -22 \\ 0 & -10 & -24 & -30 \end{array}\right] \quad R_3 \to 7R_3 - 10R_2$$

$$= \left[\begin{array}{ccc:c} 1 & 4 & 9 & 16 \\ 0 & -7 & -17 & -22 \\ 0 & 0 & 2 & 10 \end{array}\right]$$

$\therefore x+4y+9z=16$

$-7y-17z=-22$

$2z=10$

$\therefore z=5$ $\quad y=-9$ $\quad x=7$

Ans : $\therefore x=7$, $y=-9$, $z=5$

(11) By gauss Jordan method (convert into diagonal matrix)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 4 & 9 & 16 \\ 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 4 & 9 & 16 \\ 0 & -7 & -17 & -22 \\ 0 & -10 & -24 & -30 \end{array}\right] \quad \begin{array}{l} R_1 \to 7R_1 + 4R_2 \\ R_3 \to 7R_3 - 10R_2 \end{array}$$

$$= \left[\begin{array}{ccc|c} 7 & 0 & -5 & 24 \\ 0 & -7 & -17 & -22 \\ 0 & 0 & 2 & 10 \end{array}\right] \quad \begin{array}{l} R_1 \to 2R_1 + 5R_3 \\ R_2 \to 2R_2 + 17R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 14 & 0 & 0 & 98 \\ 0 & -14 & 0 & 126 \\ 0 & 0 & 2 & 10 \end{array}\right]$$

$\therefore 14x = 98 \Rightarrow x = 7$

$-14y = 126 \Rightarrow y = -9$

$2z = 10 \Rightarrow z = 5$

* Gauss Jordan method :-

This method is modification of gauss elimination method.

Q Apply gauss-Jordan method to solve the equation :-

$x + y + z = 9$

$2x - 3y + 4z = 13$

$3x + 4y + 5z = 40$

Sol^n:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

$AX = B$

$$\therefore [A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array}\right]$$

we can apply row transformation but we can't apply column transformation as it would make x as y & y as z & so on.

Calculation:
~~Loss of Energy~~:

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{array}\right] \quad \begin{array}{l} R_1 \to 5R_1 + R_2 \\ R_3 \to 5R_3 + R_2 \end{array}$$

$$= \left[\begin{array}{ccc:c} 5 & 0 & 7 & 40 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & 12 & 60 \end{array}\right] \quad \begin{array}{l} R_2 \to 6R_2 - R_3 \\ R_1 \to 12R_1 - 7R_3 \end{array}$$

$$= \left[\begin{array}{ccc:c} 60 & 0 & 0 & 60 \\ 0 & -30 & 0 & -90 \\ 0 & 0 & 12 & 60 \end{array}\right]$$

$$\therefore \begin{pmatrix} 60 & 0 & 0 \\ 0 & -30 & 0 \\ 0 & 0 & 12 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 60 \\ -90 \\ 60 \end{pmatrix}$$

$\therefore$ $60x = 60$

$-30y = -90$

$\therefore$ $12z = 60$

$\therefore$ $x = 1$

$y = 3$

$z = 5$

④ Apply gauss-jordon method & solve the system of equation:

$10x + y + z = 12$

$2x + 10y + z = 13$

$x + y + 5z = 7$

Soln:

$$\begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 7 \end{bmatrix}$$

$[A:B] =$

$AX = B$

$$[A:B] = \left[\begin{array}{ccc:c} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 5 & 7 \\ 2 & 10 & 1 & 13 \\ 10 & 1 & 1 & 12 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 10R_1 \end{array}$$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 5 & 7 \\ 0 & 8 & -9 & -1 \\ 0 & -9 & -49 & -58 \end{array}\right] \quad \begin{array}{l} R_1 \to 8R_1 - R_2 \\ R_3 \to 8R_3 + 9R_2 \end{array}$$

$$= \left[\begin{array}{ccc:c} 8 & 0 & 49 & 57 \\ 0 & 8 & -9 & -1 \\ 0 & 0 & -473 & -473 \end{array}\right] \quad R_3 \to \frac{R_3}{-473}$$

$x = 1$
$y = 1$
$z = 1$

$$= \left[\begin{array}{ccc:c} 8 & 0 & 49 & 57 \\ 0 & 8 & -9 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - 49R_3 \\ R_2 \to R_1 + 9R_3 \end{array}$$

$$= \left[\begin{array}{ccc:c} 8 & 0 & 0 & 8 \\ 0 & 8 & 0 & 8 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\therefore$ $8x = 8 \Rightarrow x = 1$
$8y = 8 \Rightarrow y = 1$
$1z = 1 \Rightarrow z = 1$

Lower → Upper → matrix

CLASSMATE
Date :
Page :

* LU Decomposition method :- (factorization method)
also known as Choleskey's method.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$AX = B$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = LU$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\therefore A = LU$$

Now, $LUX = B$ let $UX = Y$

$$LY = B$$

On computing element of L & U.

(I) first row of U i.e, U_{11}, U_{12}, U_{13}.
(II) first column of L. i.e., L_{21}, L_{31}.
(III) Second row of U i.e, U_{22}, U_{23}.
(IV) Second column of L i.e, l_{32}
(V) Third row of U i.e, U_{33}.

→ factorisation method is also called Choleskey's method.

Q. Solve the following system of equation by LU decomposition method:-

$$x + 5y + z = 14$$
$$2x + y + 3z = 13$$
$$3x + y + 4z = 17$$

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 17 \end{pmatrix}$$

Soln:-

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} = LU$$

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ l_{21}U_{11} & l_{21}U_{12} + U_{22} & l_{21}U_{13} + U_{23} \\ l_{31}U_{11} & l_{31}U_{12} + l_{32}U_{22} & l_{31}U_{13} + l_{32}U_{23} + U_{33} \end{bmatrix}$$

$U_{11} = 1$, $\quad l_{21}U_{11} = 2$, $\quad l_{31}U_{11} = 3$

$U_{12} = 5$ $\quad l_{21} = 2$ $\quad l_{31} = 3$

$U_{13} = 1$

$L_{21}U_{12} + U_{22} = 1$

$2 \times 5 + U_{22} = 1$

$\therefore U_{22} = -9$

$L_{21}U_{13} + U_{23} = 3$

$2 \times 1 + U_{23} = 3$

$U_{23} = 1$

$L_{31}U_{12} + L_{32}U_{22} = 1$

$3 \times 5 + L_{32} \times -9 = 1$

$-9L_{32} = -14$

$L_{32} = \frac{14}{9}$

$L_{31}U_{13} + L_{32}U_{23} + U_{33} = 4$

$3 \times 1 + \frac{14}{9} \times 1 + U_{33} = 4 \Rightarrow U_{33} = 1 - \frac{14}{9} = \frac{-5}{9}$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -5/9 \end{bmatrix}$$

$AX = B$

$\underbrace{LUX}_{} = B$

Let $UX = Y$,

$LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

$y_1 = 14$

$2y_1 + y_2 = 13$

$y_2 = -15$

$3y_1 + \frac{14}{9}y_2 + y_3 = 17$

$42 - \frac{14 \times 15}{9} + y_3 = 17$

$y_3 = -5/3$

Now, $UX = Y$

$$\begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -5/9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ -15 \\ -5/3 \end{bmatrix}$$

$x + 5y + z = 14$

$-9y + z = -15$

$-5/9\, z = -5/3$

$\therefore z = 3$

$y = 2$

$x = 1$

* ~~Ga~~ Jacobi method (Iterative method)

Jacobi iteration method:-

① $27x + 6y - z = 85$ $\quad |27| > |6| + |-1|$

$6x + 15y + 2z = 72$ $\quad |15| > |6| + |2|$

$x + y + 54z = 110$ $\quad |54| > |1| + |1|$

soln:- so we can use Jacobi's method for given system of equation.

$$\therefore x = \frac{1}{27}(85 - 6y + z)$$

$$y = \frac{1}{15}(72 - 6x - 2z)$$

$$z = \frac{1}{54}(110 - x - y)$$

Now,

$x_0 = 0, y_0 = 0, z_0 = 0$

First Iteration, $x' = \frac{85}{27} = 3.148$

$$y' = \frac{72}{15} = 4.8$$

$$z' = \frac{110}{54} = 2.037$$

2nd iteration

$$x^{(2)} = \frac{1}{27}(85 - 6 \times 4.8 + 2.037) = 2.157$$

$$y^{(2)} = \frac{1}{15}(72 - 6 \times 3.148 - 2 \times 2.037) = 3.269$$

$$z^{(2)} = \frac{1}{54}(110 - 3.148 - 4.8) = 1.89$$

3rd iteration $x^{(3)} = \frac{1}{27}(85 - 6 \times 3.269 + 1.89) = 2.492$

$$y^{(3)} = \frac{1}{15}(72 - 6 \times 2.157 - 2 \times 1.89) = 3.685$$

$$z^{(3)} = \frac{1}{54}(110 - 2.157 - 3.269) = 1.937$$

4th iteration

$$x^{(4)} = \frac{1}{27}(85 - 6(3.685) + 1.937) = 2.401$$

$$y^{(4)} = \frac{1}{15}(72 - 6\times 2.492 - 2\times 1.937) = 3.545$$

$$z^{(4)} = \frac{1}{54}(110 - 2.492 - 3.685) = 1.925$$

5th iteration

$$x^{(5)} = \frac{1}{27}(85 - 6\times 3.545 + 1.925) = 2.432$$

$$y^{(5)} = \frac{1}{15}(72 - 6\times 2.401 - 2\times 1.925) = 3.583$$

$$z^{(5)} = \frac{1}{54}(110 - 2.401 - 3.545) = 1.927$$

6th iteration

$$x^{(6)} = 2.423$$
$$y^{(6)} = 3.57$$
$$z^{(6)} = 1.926$$

7th iteration

$$x^{(7)} = 2.426$$
$$y^{(7)} = 3.574$$
$$z^{(7)} = 1.926$$

Sixth & seventh iteration give ~~pract~~ same values ∴ we can stop the iteration.

Hence, soln of eqn is $x = 2.426$
$y = 3.574$
$z = 1.926$

there's a drawback in this method as it takes much time, so a new method i.e, gauss siedel method is there.

(In this when we found x', we can use it to find y' as it is the modified value now but here in Jacobi we are still using x^0 ~~for~~ for finding y' & z'.)

* Gauss siedel Method :- (Iterative method)

Q Solve the system of eqn using Gauss-seidel Iterative method.

$$2x_1 - x_2 + 0x_3 = 7$$
$$-x_1 + 2x_2 - x_3 = 1$$
$$0x_1 - x_2 + 2x_3 = 1$$

$|2| > |-1| + |0|$
$|2| \geq |-1| + |-1|$
$|2| > |0| + |-1|$

We can use this method.

Soln:

$$x_1 = \frac{1}{2}(7 + x_2)$$
$$x_2 = \frac{1}{2}(1 + x_1 + x_3)$$
$$x_3 = \frac{1}{2}(1 + x_2)$$

initial approximation

$x_1^{(0)} = 0 \quad x_3^{(0)} = 0$
$x_2^{(0)} = 0$

1st iteration

$$x_1^1 = \frac{1}{2}(7+0) = 3.5$$
$$x_2^1 = \frac{1}{2}(1 + 3.5 + 0) = 2.25$$
$$x_3^1 = \frac{1}{2}(1 + 2.25) = 1.625$$

2nd iteration :-

Here, latest updated value is used.

$$x_1^2 = \frac{1}{2}(7+2.25) = 4.625$$

$$x_2^2 = \frac{1}{2}(1+4.625+1.625) = 3.625$$

$$x_3^2 = \frac{1}{2}(1+3.625) = 2.3125$$

3rd iteration :-

$$x_1^3 = \frac{1}{2}(7+3.625) = 5.3125$$

$$x_2^3 = \frac{1}{2}(1+5.3125+2.3125) = 4.3125$$

$$x_3^3 = \frac{1}{2}(1+4.3125) = 2.65625$$

4th iteration :-

$$x_1^4 = \frac{1}{2}(7+4.3125) = 5.65625$$

$$x_2^4 = \frac{1}{2}(1+5.65625+2.65625) = 4.65625$$

$$x_3^4 = \frac{1}{2}(1+4.65625) = 2.828125$$

5th iteration :-

$$x_1^{(5)} = 5.8281$$

$$x_2^{(5)} = 4.8281$$

$$x_3^{(5)} = 2.9106$$

6th iteration

$$x_1^{(6)} = 5.9140$$

$$x_2^{(6)} = 4.9140$$

$$x_3^{(6)} = 2.9570$$

7th iteration :-

$$x_1^7 = 5.9570 \simeq 6$$

$$x_2^7 = 4.9570 \simeq 5$$

$$x_3^7 = 2.9785 \simeq 3$$

roots

$$\therefore x_1 = 6$$

$$x_2 = 5$$

$$x_3 = 3$$

Ans!