Euler's Method for Numerical Differentiation

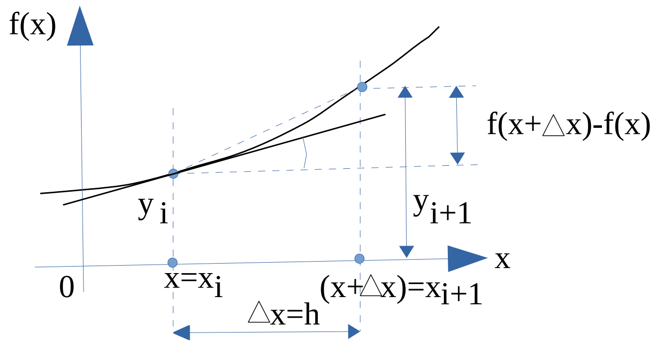

Figure- 5.1

$$\frac{df}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

for numerical calculation we have to take Δx as finite (instead of $\Delta x\to 0$).
i.e,

$$\frac{df}{dx}=\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

From figure 5.1 it is clear that we can write the above equation as,

$$\frac{df}{dx}=\frac{y_{i+1}-y_i}{x_{i+1}-x_i}$$

$$\Rightarrow y_{i+1}-y_i=\left(\frac{df}{dx}\right)(x_{i+1}-x_i) \quad \text{.......(1)}$$

When a first order linear differential equation is given in the form: $\frac{df}{dx}=f(x,y)$ subjected to the initial condition, $y(0)=y_0$ the equation (1) can further be expressed as,

$$y_{i+1}-y_i=(x_{i+1}-x_i)f(x_i,y_i)$$

$$\Rightarrow y_{i+1}=y_i+f(x_i,y_i)h \quad \text{.......(2)}$$

when, $h=x_{i+1}-x_i$ ....(3)

Equation (2) is used to obain y_{i+1} from the values y_i, the step size h and the slope, or the derivative of the function at point x_i, i.e., $f(x_i,y_i)$

Example-1:

Given the differential equation, $\frac{df}{dx}+0.4y=3e^{-x}$ with initial condition, $y(0)=5$ find out $y(3)$ by taking step size (i) $h=3$ and (ii) $h=1.5$. Use Euler's method of numerical differentiation.

Solution:

Given the differential equation,

$$\frac{df}{dx}+0.4y=3e^{-x}$$

$$\Rightarrow \frac{df}{dx}=3e^{-x}-0.4y=f(x,y)$$

(i) Step size, $h=3$
Using the Euler's iterative equation,
$y_{i+1}=y_i+f(x_i,y_i)h$

for this particular case which becomes,
$y_1=y_0+f(x_0,y_0)h$ for which, $x_0=0$ and $y_0=5$

Hence,
$y(3)=5+f(0,5)(3)=5+[3e^{-0}-0.4(5)](3)=8$

Therefore, at $x=3$ we obtain, $y=8$

(ii) Step size, $h=1.5$
In this case, we need to take two steps where, in step 1 we shall compute $y(1.5)$ by applying the Euler's equation, and then, in step 2 we shall compute $y(3)$ from $y(1.5)$.
Hence,
Step 1: $y(1.5)=y(0)+f(0,5)(1.5)=5+[3e^{-0}-0.4(5)](1.5)=6.5$

Step 2: $y(3)=y(1.5)+f(1.5,6.5)(1.5)=6.5+[3e^{-1.5}-0.4(6.5)](1.5)=3.604$

In tabular form we present these two steps as below: (here, $h=1.5$)

Step (i)	x_i	y_i	$f(x_i,y_i)$	$y_{i+1}=y_i+f(x_i,y_i)h$
0	0	5	$[3e^{-0}-0.4(5)]=1$	6.5
1	1.5	6.5	$[3e^{-1.5}-0.4(6.5)]=-1.9306$	3.604

....

Your home work is to find out the exact solution of the given differential equation of the example-1 and compare this exact solution with the obtained solutions in case (i) and (ii).

Assignment

Problem 1:

Given the differential equation, $\frac{df}{dx}=-y$ subject to the initial condition: $y(0)=1$ compute $y(0.05)$ by using the Euler's matheod. Take a step size $h=0.01$

Problem 2:

Using the Euler's method, find the solution of the equation $\frac{df}{dx}=\frac{y-x}{y+x}$ with $y(0)=1$, at $x=0.1$ by taking step sizes, $h=0.05$ and $h=0.02$.

Note: You have to present the solution by tabular form only.