Gauss Elimination Method for Solution of a system of linear Algebraic equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n$$

Two steps in the solution: i) Forward Elimination
ii) Back substitution

Forward Elimination

The unknowns are eliminated to obtain a upper-triangular system.

To eliminate 'x_1' from eqn (2) we multiply 1st. eqn by a factor, $-\left(\frac{a_{21}}{a_{11}}\right)$ and add the resulting equation ~~with~~ to eqn (2) to get

$$\left(a_{22} - a_{12}\frac{a_{21}}{a_{11}}\right)x_2 + \left(a_{23} - a_{13}\frac{a_{21}}{a_{11}}\right)x_3 + \cdots + \left(a_{2n} - a_{1n}\frac{a_{21}}{a_{11}}\right)x_n = \left(b_2 - b_1\frac{a_{21}}{a_{11}}\right)$$

Similarly, to eliminate 'x_1' from eqn (3) multiply 1st equation by a factor $-\left(\frac{a_{31}}{a_{11}}\right)$ and add the resulting equation eqn (3) ~~and~~ get

$$\left(a_{32} - a_{12}\frac{a_{31}}{a_{11}}\right)x_2 + \left(a_{33} - a_{13}\frac{a_{31}}{a_{11}}\right)x_3 + \cdots + \left(a_{3n} - a_{1n}\frac{a_{31}}{a_{11}}\right)x_n = \left(b_3 - b_1\frac{a_{31}}{a_{11}}\right)$$

- Continuing ~~for~~ the same operation upto nth eqn to eliminate 'x_1' from eqn (2) to (n). From nth eqn we get.

$$\left(a_{n2} - a_{12}\frac{a_{n1}}{a_{11}}\right)x_2 + \left(a_{n3} - a_{13}\frac{a_{n1}}{a_{11}}\right)x_3 + \cdots + \left(a_{nn} - a_{1n}\frac{a_{n1}}{a_{11}}\right)x_n = \left(b_n - b_1\frac{a_{n1}}{a_{11}}\right)$$

Hence, we can write that after the elimination of 'x_1' from equation 2 to n ~~any~~ the new co-efficients

$$\boxed{a_{jk} = a_{jk} - a_{1k}\frac{a_{j1}}{a_{11}}} \quad \left[\begin{array}{l} k : 1 \text{ to } n \\ j : 2 \text{ to } n \end{array}\right.$$

$$\boxed{b_j = b_j - b_1\frac{a_{j1}}{a_{11}}}$$

After ~~elimnt~~ eliminating 'x_1' from 2^{nd} to n^{th} eqn we obtain the system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots\cdots + a_{1n}x_n = b_1$$
$$a'_{22}x_2 + a'_{23}x_3 + \cdots\cdots + a'_{2n}x_n = b'_2$$
$$a'_{32}x_2 + a'_{33}x_3 + \cdots\cdots + a'_{3n}x_n = b'_3$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$a'_{n2}x_2 + a'_{n3}x_3 + \cdots\cdots + a'_{nn}x_n = b'_n$$

It is to be noted that for eliminating 'x_i' from j^{th} eqn we have multiplied the 1^{st} eqn by $\left(-\frac{a_{j1}}{a_{11}}\right)$.
Hence, a_{11} should be non-zero. The 'a_{11}' is called the pivot or pivot element.

For eliminating 'x'_2' from 3^{rd} eqn multiply 2^{nd} eqn by $\left(-\frac{a'_{32}}{a'_{22}}\right)$ and add the resulting eqn to eqn (3)
Repeat the process from 3^{rd} to n^{th} eqn to eliminate 'x'_2' from 3^{rd} to n^{th} eqn.

Hence $$a_{jk} = a_{jk} - a_{2k}\frac{a_{j2}}{a_{22}} \qquad \left[\begin{array}{l} k \rightarrow 2 \text{ to } n \\ j \rightarrow 3 \text{ to } n. \end{array}\right.$$
$$b_j = b_j - b_2\frac{a_{j2}}{a_{22}}$$

Therefore, for completion of the 1st step i.e. forward elimination we need to eliminate x_i (i, 1 to $n-1$) from eqn $i+1$ to n by

$$a_{jk} = a_{jk} - a_{ik}\frac{a_{ji}}{a_{ii}}$$
$$b_j = b_j - b_i\frac{a_{ji}}{a_{ii}}$$

$i = 1, n-1$
$j = 1;\ i+j = n-1$, factor;
$k = 0, k = n-1,$
$$a_{(i+j)(i+k)} = a_{(i+j)(i+k)} - a_{i(i+k)}\frac{a_{(i+j)i}}{a_{ii}}$$
$$b_{(i+j)} = b_{(i+j)} - b_i\frac{a_{(i+j)i}}{a_{ii}}$$

Step-2

Back substitution from the upper-triangular matrix & modified 'b' matrix.

$$x_n = \frac{b_n}{a_{nn}}$$

$$a_{(n-1)(n-1)} x_{n-1} = b_{n-1} - a_{(n-1)(n)} x_n$$

$$\cancel{= b_{n-1} - a_{(n-1)(n)} \frac{b_n}{a_{nn}}}$$

$$x_{n-2} = b_{n-2} - a_{(n-2)(n-1)} x_{n-1} - a_{(n-2)(n)} x_n$$

$$\cancel{= b_{n-2} - a_{(n-2)(n-1)}}$$

$$x_{n-i} = b_{n-i} - a_{(n-i)n} x_n - a_{(n-i)(n-1)} x_{n-1} - a_{(n-i)(n-2)} x_{n-2}$$

$$- \cdots\cdots - a_{(n-i)(n-i+1)} x_{(n-i+1)}$$

$\therefore$ ~~for~~ $x_n = b_n / a_{nn}$;

for (i = n-1 ~~; i = 1~~ to 1)

factor = 1/aii;

x_i = bi/aii;

for (j = n to i+1)

$x_i = x_i - a_{ij} * x_j *$ factor;