Numerical Analysis

Interpolation: The technique or method of estimating unknown values from given set of observation is known as interpolation.

x	$f(x)$
1971	1000
1981	1025
1991	1080
2001	1120
2011	1200

1973 =?

1995 =?

2008 =?

for 2008

for 1973

Equal interval

Equal interval
- Newton forward
- Gauss forward / Gauss backward
- Newton backward
- Stirling / Bessel.

Unequal interval
- Lagrange's interpolation formula for unequal interval
- Newton divided difference formula

x	$f(x)$
75	670
80	685
87	750
90	800
94	950

77 =?

82 =?

93 =?

unequal interval

Q Estimate the population in 1895 & 1925 from following statistics.

year	1891	1901	1911	1921	1931
population (in thousand)	46	66	81	93	101

$\Delta f(x) = f(x+h) - f(x)$

Soln:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
base → 1891	46				
		20			
1901	66		−5		
		15		$-3-(-5) = 2$	
1911	81		−3		$-1-2 = -3$
		12		$-4-(-3) = -1$	
1921	93		−4		
		8			
base → 1931	101				

1895 ?

forward

Backward

→ **Newton forward:**

$$f(a+hu) = f(a) + \frac{u}{1!}\Delta f(x) + \frac{u(u-1)}{2!}\Delta^2 f(x) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(x) + \ldots$$

~~f(1895)~~ $a+hu=1895$ $\therefore a=1891,\ h=10$

$1891+10u=1895$

$u=\frac{4}{10}=0.4$

$$f(1895)=46+\frac{0.4}{1}\times 20+\frac{0.4(0.4-1)}{2}\times(-5)+\frac{0.4(0.4-1)(0.4-2)}{6}\times 2$$
$$+\frac{0.4(0.4-1)(0.4-2)(0.4-3)}{24}\times(-3)$$

$f(1895)=54.8528$

nabla $\nabla f(x)=f(x)-f(x-h)$

→ Newton backward :-

$$f(a+hu)=f(a)+\frac{u}{1!}\nabla f(a)+\frac{u(u+1)}{2!}\nabla^2 f(a)+\frac{u(u+1)(u+2)}{3!}\nabla^3 f(a)$$
$$+\frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(a)+ \text{———}$$

$a+hu=1925$ $a=1931,\ h=10$

$1931+10u=1925 \Rightarrow u=\frac{-6}{10}=-0.6$ $\therefore(u=-0.6)$

$$f(1925)=101+\frac{(-0.6)}{1}\times 8+\frac{(-0.6)(-0.6+1)}{2}\times(-4)$$
$$+\frac{(-0.6)(-0.6+1)(-0.6+2)}{6}\times(-1)+\frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{24}\times(-3)$$

$=96.8368$

→ Base इस तरह से लेना है कि u का value 1 से छोटा आए.

Q Find number of men getting wages betn Rs 10 & Rs 15 from following Data:

Wages	0–10	10–20	20–30	30–40
frequency	9	30	35	42

Wages (x)	cumulative freq. $f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
Below 10	9			
		30		
Below 20	$30+9=39$		5	
		35		2
Below 30	$39+35=74$		7	
		42		
Below 40	$74+42=116$			

Newton forward

$$f(a+hu)=f(a)+\frac{u}{1!}\Delta f(a)+\frac{u(u-1)}{2!}\Delta^2 f(a)+\frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$+ \text{---}$

$a+hu=15 \qquad a=10,\ h=10$

$10+10u=15 \Rightarrow u=\frac{5}{10}=0.5$

$$f(15)=9+\frac{0.5}{1}\times 30+\frac{0.5(0.5-1)}{2}\times 5+\frac{0.5(0.5-1)(0.5-2)}{6}\times 2$$

$= 23.5 \simeq 24$ (persons ∴ Natural no.)

Hence, Number of men getting wages between 10 & 15 is $=24-9=15$

Q find the lowest degree polynomial $y(x)$ that fit the data, find $y(5)$.

x	0	2	4	6	8
y	5	9	61	209	501

x	$f(x)$ y	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	5				
		4			
2	9		48		
		52		48	
4	61		96		0
		148		48	
6	209		144		
		292			
8	501				

$$f(a+hu)=f(a)+\frac{u}{1!}\Delta f(a)+\frac{u(u-1)}{2!}\Delta^2 f(a)+\frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$+ \text{---}$

$a + hu = x \qquad a = 0,\ h = 2$

$0 + 2u = x$

$u = \frac{x}{2}$

$$\therefore f(x) = 5 + \frac{x}{2}(4) + \frac{\left(\frac{x}{2}\right)\left(\frac{x}{2}-1\right)}{2} \times 48 + \frac{\left(\frac{x}{2}\right)\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-2\right)}{6} \times 48 + \frac{\left(\frac{x}{2}\right)\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-2\right)\left(\frac{x}{2}-3\right)}{24} \times 0$$

$y = f(x) = x^3 - 2x + 5$

$\therefore y(5) = 5^3 - 2(5) + 5 = 125 - 10 + 5 = 120$

* Central difference interpolation :-

Q find u_9 if $u_0 = 14,\ u_4 = 24,\ u_8 = 32,\ u_{12} = 35,\ u_{16} = 40$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	14 y_{-2}				
		10 Δy_{-2}			
4	24 y_{-1}		-2 $\Delta^2 y_{-2}$		
		8 Δy_{-1}		-3 $\Delta^3 y_{-2}$	
8	32 y_0		-5 $\Delta^2 y_{-1}$		10 $\Delta^4 y_{-2}$
		3 Δy_0		7 $\Delta^3 y_{-1}$	
12	35 y_1		2 $\Delta^2 y_0$		
		5 Δy_1			
16	40 y_2				

→ backward (M)

→ forward (W)

Gauss forward :-

$$f(a+hu) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2}$$

$a = 8,\ h = 4$

$8 + 4u = 9$

$u = \frac{1}{4} = 0.25$

Gauss backward :-

$$f(a+hu) = y_0 + \frac{u}{1!}\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u-1)u(u+1)}{3!}\Delta^3 y_{-2} + \frac{(u-1)u(u+1)(u+2)}{4!}\Delta^4 y_{-2}.$$

Stirling formula :- (average of gauss forward & backward)

$$\boxed{f(a+hu) = y_0 + \frac{u}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2}}$$

Soln :- By gauss forward :-

$$f(9) = 32 + \frac{0.25}{1}(3) + \frac{0.25(0.25-1)}{2}(-5) + \frac{(0.25+1)(0.25)(0.25-1)}{6} \times 7$$

$$+ \frac{(0.25+1)(0.25)(0.25-1)(0.25-2)}{24} \times 10$$

$$= 33.1162$$

By gauss backward :-

$$f(9) = 32 + \frac{0.25}{1} \times 8 + \frac{0.25(0.25+1)}{2}(-5) + \frac{(0.25-1)0.25(0.25+1)}{6} \times (-3)$$

$$+ \frac{(0.25-1)0.25(0.25+1)(0.25+2)}{24} \times 10$$

$$= 33.1162$$

By Stirling formula

$$f(9) = 32 + \frac{0.25}{1}\left(\frac{3+8}{2}\right) + \frac{(0.25)^2}{2} \times (-5) + \frac{0.25(0.25^2-1)}{6}\left(\frac{7-3}{2}\right)$$

$$+ \frac{(0.25)^2(0.25^2-1)}{24}(10)$$

$$= 33.1162$$

* Bessel's formula

$$f(a+hu) = \frac{1}{2}(y_0 + y_1) + \frac{(u-\frac{1}{2})}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right)$$

$$+ \frac{(u-\frac{1}{2})u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)(u)(u-1)(u-2)}{4!}\left(\frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right)$$

x	$f(x)$		$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	14	y_{-2}	$10\ \Delta y_{-2}$	$-2\ \Delta^2 y_{-2}$	$-3\ \Delta^3 y_{-2}$	$10\ \Delta^4 y_{-2}$
4	24	y_{-1}	$8\ \Delta y_{-1}$	$-5\ \Delta^2 y_{-1}$	$7\ \Delta^3 y_{-1}$	
8	32	y_0	$3\ \Delta y_0$	$2\ \Delta^2 y_0$		
12	35	y_1	$5\ \Delta y_1$			
16	40	y_2				

Now, $a + hu = 9$, $a = 8$, $h = 4$

$8 + 4u = 9 \Rightarrow u = \frac{1}{4} = 0.25$

$$f(9) = \frac{1}{2}(32+35) + \frac{(0.25-\frac{1}{2})}{1} \times 3 + \frac{0.25(0.25-1)}{2}\left(\frac{-5+2}{2}\right)$$

$$+ \frac{(0.25-\frac{1}{2})\,0.25\,(0.25-1)}{6} \times 7 + \frac{(0.25+1)(0.25)(0.25-1)(0.25-2)}{24} \times \left(\frac{10+0}{2}\right)$$

$= 33.1162$

* Interpolation for unequal interval:

Lagrange's interpolation for unequal interval

Q

x	5	6	9	11
$f(x) = y$	12	13	14	16

find value of y when $x = 10$ by Lagrange's interpolation formula.

Soln:

$$f(x) = \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)} \times 13$$

$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)} \times 16$$

$$f(10) = \frac{4 \times 1 \times (-1)}{(-1)(-4)(-6)} \times 12 + \frac{5 \times 1 \times (-1)}{1 \times (-3)(-5)} \times 13 + \frac{5 \times 4 \times (-1)}{4 \times 3 \times (-2)} \times 14$$

$$+ \frac{5 \times 4 \times 1}{6 \times 5 \times 2} \times 16$$

$$f(10) = 2 + \left(\frac{-13}{3}\right) + \frac{35}{3} + \frac{16}{3} = \frac{6-13+35+16}{3} = \frac{44}{3}$$
$$= 14.66$$

* <u>Newton divided difference ~~metho~~ formula for unequal interval :-</u>

	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0	5	12	$\frac{13-12}{6-5} = 1$	$\frac{\frac{1}{3}-1}{9-5} = \frac{-1}{6}$	$\frac{\frac{2}{15}-\left(\frac{-1}{6}\right)}{11-5} = \frac{1}{20}$
x_1	6	13	$\frac{14-13}{9-6} = \frac{1}{3}$	$\frac{1-\frac{1}{3}}{11-6} = \frac{2}{15}$	
x_2	9	14	$\frac{16-14}{11-9} = 1$		
x_3	11	16			

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0)$$

$$\therefore f(x) = 12 + (x-5)1 + (x-5)(x-6)\left(\frac{-1}{6}\right) + (x-5)(x-6)(x-9)\left(\frac{1}{20}\right)$$

$$\therefore f(10) = 12 + 5 + 5\times4\times\left(\frac{-1}{6}\right) + 5\times4\times1\times\frac{1}{20}$$
$$= 12+5+\frac{-20}{6}+1 = 18-\frac{20}{6} = 18-\frac{10}{3} = \frac{44}{3}$$
$$= 14.66$$

<u>Q</u> Use lagrange's formula to fit polynomial to the following data hence find $y(-2)$, $y(1)$, $y(4)$.

x	-1	0	2	3
y	-8	3	1	2

<u>Soln:</u>
$$f(x) = \frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)}(-8) + \frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)}\times 3$$
$$+ \frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2-3)}\times(1) + \frac{(x+1)(x-0)(x-2)}{(3+1)(3-0)(3-2)}\times 2$$

$$f(x) = (x-2)(x-3)\left\{\frac{-8x}{(-1)(-3)(-4)} + \frac{3(x+1)}{1(-2)(-3)}\right\} + x(x+1)\left\{\frac{x-3}{3\times2\times(-1)} + \frac{2(x-2)}{4\times3\times1}\right\}$$

$$= (x-2)(x-3)\left(\frac{2x}{3} + \frac{x+1}{2}\right) + (x^2+x)\left(-\frac{(x-3)}{6} + \frac{x-2}{6}\right)$$

$$= (x^2-5x+6)\left(\frac{4x+3x+3}{6}\right) + (x^2+x)\left(\frac{\cancel{-x}+3+\cancel{x}-2}{6}\right)$$

$$= (x^2-5x+6)\left(\frac{7x+3}{6}\right) + (x^2+x)\left(\frac{1}{6}\right)$$

$$= \frac{1}{6}\left[7x^3-35x^2+42x+3x^2-15x+18+x^2+x\right]$$

$$= \frac{1}{6}\left[7x^3-31x^2+28x+18\right]$$

Now,

$y(-2) = f(-2) = -36.33$

$y(1) = f(1) = 3.666$

$y(4) = f(4) = 13.666$

Q Apply Lagrange's formula inversely to find value of x when $y = 19$ given following

x	0	1	2
y	0	1	20

$$x = \frac{(y-1)(y-20)}{(0-1)(0-20)}\times 0 + \frac{(y-0)(y-20)}{(1-0)(1-20)}\times 1 + \frac{(y-0)(y-1)}{(20-0)(20-1)}\times 2$$

$$= 0 + \frac{y(y-20)}{1(-19)} + \frac{y(y-1)}{20\times19}\times 2$$

$$x_{\text{at } y=19} = \frac{19(19-20)}{-19} + \frac{19(19-1)}{20\times19}\times 2$$

$$= \frac{-1}{-1} + \frac{18}{10} = 1 + 1.8 = 2.8$$

Numerical Integration:- The area bounded by the curve $f(x)$ & x-axis betn limit a & b is denoted by

$$I = \int_a^b f(x)\,dx \quad \text{--- (1)}$$

Divide the interval (a,b) into n equal interval with length h (step size)

i.e., $(a,b) = (a = x_0, x_1, x_2, \text{-----}, x_{n-1}, x_n = b)$

$a = x_0$

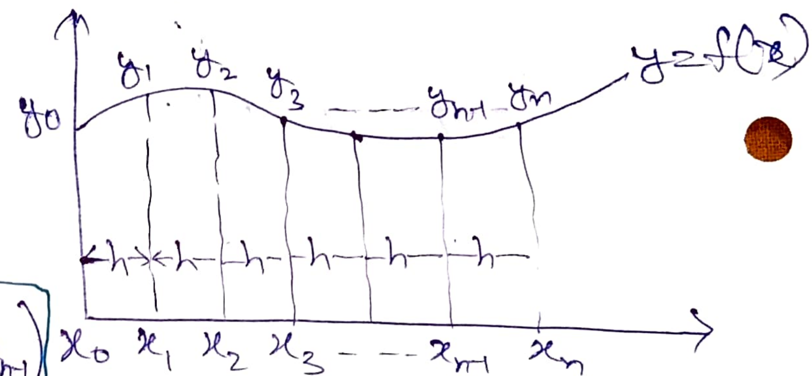
$x_1 = x_0 + h$

$x_2 = x_1 + h$

$\vdots$

$x_n = x_{n-1} + h$

$\boxed{h = \frac{b-a}{h}}$ or $\boxed{h = \frac{b-a}{n}}$



Eqn (1) can be evaluated using

(i) Trapezoidal Rule

$$\boxed{\int_a^b f(x)\,dx = h\left(\frac{y_0 + y_n}{2} + y_1 + y_2 + \text{--} + y_{n-1}\right)}$$

→ It is applicable on any no. of interval.

(ii) Simpson $\frac{1}{3}$ Rule:- (even interval)

$$\boxed{\int_a^b f(x)\,dx = \frac{h}{3}\left((y_0 + y_n) + 4(y_1 + y_3 + \text{---}) + 2(y_2 + y_4 + \text{--})\right)}$$

→ It is applicable when total no. of interval is even.

(iii) Simpson 3/8 Rule (3 multiple)

$$\boxed{\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + \text{---}) + 2(y_3 + y_6 + \text{--})\right]}$$

→ It is applicable if total no. of interval is multiple of 3.

Note:- Generally, divide in 6 intervals as we can apply all three formulas. But if the method to be used is mentioned in problem then apply that formula only.

Q Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using (i) Trapezoidal Rule (ii) Simpson $\frac{1}{3}$ Rule (iii) Simpson 3/8 Rule. Also find values of π in each case.

Soln:
$$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$$

$x_0 = 0 \qquad y_0 = \frac{1}{1+0^2} = 1$

$x_1 = \frac{1}{6} \qquad y_1 = \frac{1}{1+(\frac{1}{6})^2} = \frac{36}{37}$

$x_2 = \frac{2}{6} \qquad y_2 = \frac{1}{1+(\frac{2}{6})^2} = 0.9$

$x_3 = \frac{3}{6} \longrightarrow y_3 = \frac{1}{1+(\frac{3}{6})^2} = 0.8$

$x_4 = \frac{4}{6} \longrightarrow y_4 = \frac{1}{1+(\frac{4}{6})^2} = \frac{9}{13}$

$x_5 = 5/6 \longrightarrow y_5 = \frac{1}{1+(\frac{5}{6})^2} = \frac{36}{61}$

$x_6 = 1 \rightarrow y_6 = \frac{1}{1+1^2} = \frac{1}{2} = 0.5$

Trapezoidal formula

$$\int_0^1 \frac{1}{1+x^2}dx = h\left[\frac{y_0+y_6}{2} + y_1+y_2+y_3+y_4+y_5\right]$$
$$= \frac{1}{6}\left[\frac{1+0.5}{2} + \frac{36}{37} + 0.9 + 0.8 + \frac{9}{13} + \frac{36}{61}\right]$$
$$\int_0^1 \frac{1}{1+x^2}dx = 0.78423 \quad \text{—— (i)}$$

By Simpson $\frac{1}{3}$ rule

$$\int_0^1 \frac{1}{1+x^2}dx = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$
$$= \frac{1}{18}\left\{1+0.5+4\left(\frac{36}{37}+0.8+\frac{36}{61}\right)+2\left(0.9+\frac{9}{13}\right)\right\}$$
$$\int_0^1 \frac{1}{1+x^2}dx = 0.785396 \quad \text{—— (ii)}$$

Simpson 3/8 Rule

$$\int_0^1 \frac{1}{1+x^2}dx = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2(y_3)\right]$$
$$= \frac{1}{16}\left\{1+0.5+3\left(\frac{36}{37}+0.9+\frac{9}{13}+\frac{36}{61}\right)+2(0.8)\right\}$$
$$\int_0^1 \frac{1}{1+x^2}dx = 0.785394 \quad \text{—— (iii)}$$

By direct integration

$$\int_0^1 \frac{1}{1+x^2}dx = \left(\tan^{-1}x\right)_0^1 = \left(\tan^{-1}1 - \tan^{-1}0\right) = \pi/4 - 0 = \pi/4$$

$$\therefore \int_0^1 \frac{1}{1+x^2}dx = \frac{\pi}{4} \quad \text{--- (4)}$$

By (1) & (4) $\quad \frac{\pi}{4} = 0.78423 \Rightarrow \pi = 4(0.78423)$

$\pi = 3.1369$

By (2) & (4) $\quad \frac{\pi}{4} = 0.785396 \Rightarrow \pi = 3.14158$

By (3) & (4) $\quad \frac{\pi}{4} = 0.785394 \Rightarrow \pi = 3.14157$

Example:- Given that

x	x_0 4.0	x_1 4.2	x_2 4.4	x_3 4.6	x_4 4.8	x_5 5.0	x_6 5.2
$\log x$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

Evaluate $\int_4^{5.2} \log x \, dx$ by simpson 3/8 Rule

Soln:-

$$h = \frac{5.2-4}{6} = \frac{1.2}{6} = 0.2$$

$$\int_4^{5.2} \log_e x \, dx = \frac{3h}{8}\left[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)\right]$$

$$= \frac{3 \times 0.2}{8}\left[(1.3863 + 1.6487) + 3(1.4351 + 1.4816 + 1.5686 + 1.6094) + 2(1.5261)\right]$$

$$\int_4^{5.2} \log_e x \, dx = 1.82784$$

① Use trapezoidal rule to compute $\int_1^2 \frac{dx}{x}$ using three interval compare it with exact value.

Soln:- $h = \frac{2-1}{3} = \frac{1}{3}$

$x_0 = 1$

$x_1 = 1 + \frac{1}{3} = \frac{4}{3}$

$x_2 = \frac{4}{3} + \frac{1}{3} = \frac{5}{3}$

$x_3 = \frac{5}{3} + \frac{1}{3} = 2$

$y_0 = \frac{1}{x_0} = \frac{1}{1} = 1$

$y_1 = \frac{1}{4/3} = 3/4$

$y_2 = \frac{1}{5/3} = 3/5$

$y_3 = \frac{1}{2} = 0.5$

$$\int_1^2 \frac{1}{x}dx = h\left[\frac{y_0+y_3}{2} + y_1+y_2\right]$$

$$= \frac{1}{3}\left[\frac{1+0.5}{2} + \frac{3}{4}+\frac{3}{5}\right] = 0.70$$

<u>By integration</u>, $\int_1^2 \frac{1}{x}dx = [\log x]_1^2 = \log 2 - \log 1$

$$= \log_e 2 = 0.693$$

Q Calculate using simpson $\frac{1}{3}$ Rule the value $\int_0^{\pi/2} \sqrt{\sin x}\, dx$.

Soln: $h = \frac{\pi/2 - 0}{6} = \pi/12$

$x_0 = 0$ — $y_0 = \sqrt{\sin 0} = 0$

$x_1 = \pi/12$ — $y_1 = \sqrt{\sin \frac{\pi}{12}} = 0.5087$

$x_2 = 2\pi/12$ — $y_2 = \sqrt{\sin \frac{2\pi}{12}} = 0.7071$

$x_3 = 3\pi/12$ — $y_3 = \sqrt{\sin \frac{3\pi}{12}} = 0.8409$

$x_4 = 4\pi/12$ — $y_4 = 0.9306$

$x_5 = 5\pi/12$ — $y_5 = 0.9825$

$x_6 = 6\pi/12$ — $y_6 = 0$

$$\int_0^{\pi/2} \sqrt{\sin x}\, dx = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

$$= \frac{\pi}{36}\left[(0+1) + 4(0.5087+0.8409+0.9825) + 2(0.7071+0.9306)\right]$$

$$= \frac{1}{36}\times\frac{22}{7}\left[(0+1) + 4(\text{——}) + 2(\text{——})\right]$$

$$= 0.3779$$

* Picard method :-

Solution of ordinary differential equation of first order & 1st degree with the help of numerical analysis

(1) Picard method
(2) Taylor series method
(3) Euler's method
(4) Euler's modified method
(5) Runge-Kutta method
(6) Milne predictor & corrector method
(7) Adams-Bashforth method

* Picard method of successive assumption (Iterative method)

$$\boxed{y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\, dx}$$

Q solve by picard method (upto 3rd approximation) $\frac{dy}{dx} = x+y^2$, $y(0)=0$, also find $y(0.1)$.

Soln:- $f(x,y) = x+y^2$ | $x_0=0$, $y_0=0$

Put $n=0$, $y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$

$= y_0 + \int_{x_0}^{x} (x+y_0^2)\, dx$

$y_1 = 0 + \int_{0}^{x} (x+0^2)\, dx = \frac{x^2}{2}$

Put $n=1$, $y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx = y_0 + \int_{x_0}^{x} (x+y_1^2)\, dx$

$= 0 + \int_{0}^{x} \left(x+\left(\frac{x^2}{2}\right)^2\right) dx = \int_{0}^{x} \left(x+\frac{x^4}{4}\right) dx$

$y_2 = \frac{x^2}{2} + \frac{x^5}{20}$

Put $n=2$, $y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx = y_0 + \int_{0}^{x} (x+y_2^2)\, dx$

$= 0 + \int_{0}^{x} \left(x+\left(\frac{x^2}{2}+\frac{x^5}{20}\right)^2\right) dx$

$y_3 = \int_{0}^{x} \left(x+\frac{x^4}{4}+\frac{x^{10}}{400}+\frac{x^7}{20}\right) dx$

$$\boxed{y_3 = \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^{11}}{4400} + \frac{x^8}{160}}$$

$$y(0.1) = \frac{(0.1)^2}{2} + \frac{(0.1)^5}{20} + \frac{(0.1)^{11}}{4400} + \frac{(0.1)^8}{160} =$$

Q. Solve by picard method, find successive approximate soln upto 4th order of IVP, $y' + y = e^x$, $y(0) = 0$

boundary condn will be given.

Soln:

$$\frac{dy}{dx} + y = e^x \Rightarrow \frac{dy}{dx} = e^x - y$$

$$f(x, y) = e^x - y, \quad x_0 = 0, \; y_0 = 0$$

$$y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\,dx$$

$$y_{n+1} = y_0 + \int_{x_0}^{x} (e^x - y_n)\,dx$$

Put $n=0$, $y_1 = y_0 + \int_{x_0}^{x} (e^x - y_0)\,dx = 0 + \int_0^x (e^x - 0)\,dx = e^x - 1$

Put $n=1$, $y_2 = y_0 + \int_{x_0}^{x} (e^x - y_1)\,dx = 0 + \int_0^x (e^x - (e^x))\,dx = x$

Put $n=2$, $y_3 = y_0 + \int_{x_0}^{x} (e^x - y_2)\,dx = \cancel{0 + \int_0^x (e^x - 0)\,dx = e^x}$

$$= 0 + \int_0^x (e^x - x)\,dx = \left[e^x - \frac{x^2}{2}\right]_0^x$$

$$= e^x - \frac{x^2}{2} - 1$$

Put $n=3$, $y_4 = y_0 + \int_{x_0}^{x} (e^x - y_3)\,dx = 0 + \int_0^x (\cancel{e^x} - \cancel{e^x} + \frac{x^2}{2} + 1)\,dx$

$$\boxed{y_4 = \frac{x^3}{6} + x}$$

* Euler & Euler's modified method

→ Euler ~~modified~~ method or Runge Kutta method of 1st order

→ Euler modified method or Runge Kutta method of 2nd order

* Euler's method

$$\boxed{y_{n+1} = y_n + h(f x_n, y_n)}$$

Q. Solve $\frac{dy}{dx} = x+y$ with boundary condition $y=1$ at $x=0$ find approximate value of y for $x = 0.1$

Soln:- $h = \frac{0.1-0}{5} = 0.02$

$x_0 = 0$	$y_0 = 1$
$x_1 = 0.02$	$y_1 =$
$x_2 = 0.04$	$y_2 =$
$x_3 = 0.06$	$y_3 =$
$x_4 = 0.08$	$y_4 =$
$x_5 = 0.10$	$y_5 =$

$f(x,y) = x+y$

$$y_{n+1} = y_n + h(x_n + y_n)$$

~~Put $x=0$, $y_1 = y_0 + h(x_0 + y_0)$~~

~~$y_1 = 0 + 0.02(0+0$~~

Put $n=0$, $y_1 = y_0 + h(x_0 + y_0)$
$= 1 + 0.02(0+1) = 1.02$

Put $n=1$ $y_2 = y_1 + h(x_1 + y_1) = 1.02 + 0.02(0.02 + 1.02)$
$= 1.0408$

Put $n=3$ $y_3 = y_2 + h(x_2 + y_2) = 1.0408 + 0.02(0.04 + 1.0408)$
$= 1.0624$

Put $n=4$ $y_4 = y_3 + h(x_3 + y_3) = 1.0624 + 0.02(0.06 + 1.0624)$
$= 1.0848$

Put $n=5$ $y_5 = y_4 + h(x_4 + y_4) = 1.0848 + 0.02(0.08 + 1.0848)$
$= 1.1081$

* Euler's modified method → In each step, updated value is used.
(Runge's Kutta method ~~of~~ second order)

$$\boxed{y^*_{n+1} = y_n + h f(x_n, y_n)}$$

$$\boxed{y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})]}$$

Q Given $\frac{dy}{dx} = x^2+y$, with $y(0)=1$, find $y(0.02)$ & $y(0.04)$ by Euler's modified method.

Soln: $f(x,y) = x^2+y$

$x_0 = 0$, $y_0 = 1$

$(h=0.02)$ $x_1 = 0.02$, $y_1^* = 1.02$ $y_1 = 1.0202$

$x_2 = 0.04$, $y_2^* = 1.0406$ $y_2 = 1.0408$

Put n=0

$y_1^* = y_0 + hf(x_0, y_0)$

$y_1^* = y_0 + h[x_0^2 + y_0] = 1 + 0.02[0^2+1] = 1.02$

Put n=0 in formula

$y_1 = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^*)]$

$y_1 = y_0 + \frac{h}{2}[(x_0^2+y_0) + (x_1^2 + y_1^*)]$

$= 1 + \frac{0.02}{2}[(0^2+1) + (0.02)^2 + 1.02]$

$= 1.0202$

Again

Put n=1,

$y_2^* = y_1 + hf(x_1, y_1)$

$= 1.0202 + 0.02[x_1^2 + y_1]$

$= 1.0202 + 0.02[(0.02)^2 + 1.0202]$

$= 1.0406$

n=1

$y_2 = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^*)]$

$= 1.0202 + \frac{0.02}{2}[x_1^2 + y_1 + x_2^2 + y_2^*]$

$= 1.0202 + \frac{0.02}{2}[(0.02)^2 + 1.0202 + (0.04)^2 + 1.0406]$

$= 1.0408$

* Runge-Kutta method of 4th order (It gives most accurate result)

Consider initial value problem $\frac{dy}{dx} = f(x,y)$ when $y(x_0) = y_0$

better than Euler or Euler modified.

$K_1 = hf(x_n, y_n)$

$K_2 = hf(x_n + \frac{h}{2}, y_n + K_1/2)$

$K_3 = hf(x_n + \frac{h}{2}, y_n + K_2/2)$

$K_4 = hf(x_n + h, y_n + K_3)$

$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$\therefore y_{n+1} = y_n + K$

Q Given $\frac{dy}{dx} = x + y^2$, $y(0) = 1$, find $y(0.2)$ when $h = 0.1$ using R.K of 4th order.

Soln: $f(x, y) = x + y^2$

$x_0 = 0$, $y_0 = 1$

$x_1 = 0.1$ $y_1 = 1.1165$

$x_2 = 0.2$ $y_2 = 1.2737$

$x_3 = 0.3$ → no need

Put $n = 0$

$$K_1 = h f(x_0, y_0)$$
$$= h(x_0 + y_0^2)$$
$$= 0.1(0 + 1) = 0.1$$

$$K_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$
$$= h\left[x_0 + \frac{h}{2} + \left(y_0 + \frac{K_1}{2}\right)^2\right]$$
$$= 0.1\left[0 + \frac{0.1}{2} + \left(1 + \frac{0.1}{2}\right)^2\right]$$
$$= 0.11525$$

$$K_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$
$$= h\left[x_0 + \frac{h}{2} + \left(y_0 + \frac{K_2}{2}\right)^2\right]$$
$$= 0.1\left[0 + \frac{0.1}{2} + \left(1 + \frac{0.11525}{2}\right)^2\right]$$
$$= 0.1169$$

$$K_4 = h f(x_0 + h, y_0 + K_3)$$
$$= h\left[x_0 + h + (y_0 + K_3)^2\right]$$
$$= 0.1\left[0 + 0.1 + (1 + 0.1169)^2\right]$$
$$= 0.1347$$

$$\therefore K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$
$$= \frac{1}{6}\left(0.1 + 2(0.11525) + 2(0.1169) + 0.1347\right)$$
$$K = 0.1165$$

$$\therefore y_1 = y_0 + K$$
$$y_1 = 1 + 0.1165$$
$$y_1 = 1.1165$$

Now, put $n = 1$

$K_1 = h f(x_1, y_1) = h(x_1 + y_1^2) = 0.1\,(0.1 + (1.1165)^2) = 0.1347$

$K_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = h\left[x_1 + \frac{h}{2} + \left(y_1 + \frac{K_1}{2}\right)^2\right] = 0.1\left[0.1 + \frac{0.1}{2} + \left(1.1165 + \frac{0.1347}{2}\right)^2\right]$

$= 0.1552$

$K_3 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = h\left[x_1 + \frac{h}{2} + \left(y_1 + \frac{K_2}{2}\right)^2\right] = 0.1\left[0.1 + \frac{0.1}{2} + \left(1.1165 + \frac{0.1552}{2}\right)^2\right]$

$= 0.1576$

$K_4 = hf(x_1+h, y_1+K_3) = h\{x_1+h+(y_1+K_3)^2\}$
$= 0.1\{0.1+0.1+(1.1165+0.1576)^2\} = 0.1823$

$K = \frac{1}{6}(K_1+2K_2+2K_3+K_4) = \frac{1}{6}(0.1347+2\times0.1552+2\times0.1576+0.1823)$
$= 0.1572$

$\therefore y_2 = y_1+K = 1.1165+0.1572 = 1.2737$

* Milne Predictor & corrector method :-

Consider IVP $\frac{dy}{dx} = f(x,y)$, $y(x_0)=y_0$, $y(x_1)=y_1$, $y(x_2)=y_2$, $y(x_3)=y_3$ when $x_0, x_1, x_2, \& x_3$ are equidistant value of x with step size h.

Milne predictor formula

$$\boxed{y_4^P = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)}$$

Milne corrector formula

$$\boxed{y_4^C = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^P)}$$

Q using milne's predictor-corrector method find y when $x=0.8$, Given $\frac{dy}{dx} = x-y^2$.
$y(0)=0$, $y(0.2)=0.02$, $y(0.4)=0.0795$, $y(0.6)=0.1762$

Soln:- $f(x,y) = x-y^2$

$h=0.2$

x	y	$f = x-y^2$
$x_0=0$	$y_0=0$	$f_0 = x_0-y_0^2 = 0-0=0$
$x_1=0.2$	$y_1=0.02$	$f_1 = x_1-y_1^2 = (0.2)-(0.02)^2 = 0.1996$
$x_2=0.4$	$y_2=0.0795$	$f_2 = x_2-y_2^2 = (0.4)-(0.0795)^2 = 0.3937$
$x_3=0.6$	$y_3=0.1762$	$f_3 = x_3-y_3^2 = (0.6)-(0.1762)^2 = 0.5689$
$x_4=0.8$	$y_4^P=0.3049$	$f_4^P = x_4-(y_4^P)^2 = 0.8-(0.3049)^2 = 0.707$
	$y_4^C = 0.3046$	

$$\therefore y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 0 + \frac{4\times0.2}{3}(2\times0.1996 - 0.3937 + 2\times0.5689)$$

$$= 0.3049$$

$$y_4^c = 0.0795 + \frac{0.2}{3}(0.3937 + 4\times0.5689 + 0.707)$$

$$= 0.3046$$

* Adam-Bashforth Predictor & corrector method

$$\boxed{y_4^p = y_3 + \frac{h}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0]}$$

$$\boxed{y_4^c = y_3 + \frac{h}{24}[9f_4^p + 19f_3 - 5f_2 + f_1]}$$

Q. Given $\frac{dy}{dx} = x^2(1+y)$ by Adam bashforth method, $y(1)=1$, $y(1.1)=1.233$, $y(1.2)=1.548$, $y(1.3)=1.979$ & find $y(1.4)$

Soln: $f(x,y) = x^2(1+y)$

x	y	$f = x^2(1+y)$
$x_0 = 1$	$y_0 = 1$	$f_0 = x_0^2(1+y_0) = 1^2(1+1) = 2$
$x_1 = 1.1$	$y_1 = 1.233$	$f_1 = x_1^2(1+y_1) = (1.1)^2(1+1.233) = 2.70193$
$x_2 = 1.2$	$y_2 = 1.548$	$f_2 = x_2^2(1+y_2) = (1.2)^2(1+1.548) = 3.66912$
$x_3 = 1.3$	$y_3 = 1.979$	$f_3 = x_3^2(1+y_3) = (1.3)^2(1+1.979) = 5.03451$
$x_4 = 1.4$	$y_4^p = 2.5723$ $y_4^c = 2.5749$	$f_4^p = x_4(1+y_4^p) = (1.4)^2(1+2.5723) = 7.0017$

$$y_4^P = y_3 + \frac{h}{24}\left[55f_3 - 59f_2 + 37f_1 - 9f_0\right]$$

$$= 1.979 + \frac{0.1}{24}\left[55\times(5.03451) - 59(3.69912) + 37(2.70193) - 9(2)\right]$$

$$= 2.5723$$

$$y_4^C = y_3 + \frac{h}{24}\left[9f_4^P + 19f_3 - 5f_2 + f_1\right]$$

$$= 1.979 + \frac{0.1}{24}\left[9(7.0017) + 19(5.03451) - 5(3.69912) + 2.70193\right]$$

$$= 2.5749$$

* Curve fitting & optimisation

x	x_1	x_2	—	x_n
y	y_1	y_2	—	y_n

(I) fitting straight line by least square method

$$y = a + bx \quad \text{--- (1)}$$

or $y = ax + b$

$$\boxed{\Sigma y = na + b\Sigma x}$$

$$xy = ax + bx^2$$

$$\boxed{\Sigma xy = a\Sigma x + b\Sigma x^2}$$

find a, & b and put in eqn (1) to get eqn of line.

(II) fitting second degree Parabola by least square method

$$y = a + bx + cx^2$$

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

→ solve these & find value of a, b & c & put in eqn to get result.

Q fit a straight line curve $y = a + bx$ for following data using least square method.

x	1	2	3	4	6	8
y	2.4	3	3.6	4	5	6

$n = 6$

Soln:

x	y	xy	x^2
1	2.4	2.4	1
2	3	6	4
3	3.6	10.8	9
4	4	16	16
6	5	30	36
8	6	48	64
$\Sigma x = 24$	$\Sigma y = 24$	$\Sigma xy = 113.2$	$\Sigma x^2 = 130$

$\Sigma y = na + b\Sigma x$

$24 = 6a + b(24)$ —(I)

$\Sigma xy = a\Sigma x + b\Sigma x^2$

$113.2 = 24a + 130b$ —(II)

$\therefore a = 1.9765$

$b = 0.5059$

$\boxed{\therefore y = 1.9765 + 0.5059x}$

Q fit a curve of the form $y = a + bx + cx^2$ for following data using least square method

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

$n = 5$

Soln:

x	y	xy	x^2y	x^2	x^3	x^4
0	1	0	0	0	0	0
1	1.8	1.8	1.8	1	1	1
2	1.3	2.6	5.2	4	8	16
3	2.5	7.5	22.5	9	27	81
4	6.3	25.2	100.8	16	64	256
$\Sigma x = 10$	$\Sigma y = 12.9$	$\Sigma xy = 37.1$	$\Sigma x^2y = 130.3$	$\Sigma x^2 = 30$	$\Sigma x^3 = 100$	$\Sigma x^4 = 354$

$\Sigma y = na + b\Sigma x + c\Sigma x^2$

$\boxed{12.9 = 5a + 10b + 300c}$ — (I)

$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$

$\boxed{37.1 = 10a + 30b + 100c}$ — (II)

$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$

$\boxed{130.3 = 30a + 100b + 354c}$ — (III)

Solving (I), (II) & (III) we get,

$a = 1.42, \quad b = -1.07, \quad c = .55$

$\therefore \boxed{y = 1.42 - 1.07x + 0.55x^2}$

* Curve fitting of exponential curve by least square method:

Q. Fit the curve $y = ab^x$ by using least square method to the following data & hence find y at $x = 8$

x	1	2	3	4	5	6	7
y	87	97	113	129	202	195	193

Soln:

$y = ab^x$

$\log y = \log(ab^x)$

$= \log a + \log b^x$

$\log y = \log a + x \log b$

$\boxed{Y = A + BX}$

$Y = \log y$

$A = \log a$

$B = \log b$

$X = x$

$\Sigma Y = nA + B\Sigma X$

$\Sigma XY = A\Sigma X + B\Sigma X^2$

$X=x$	y	$Y=\log y$	XY	X^2
1	87	4.4659	4.4659	1
2	97	4.5747	9.1494	4
3	113	4.7274	14.1822	9
4	129	4.8598	19.4392	16
5	202	5.3083	26.5413	25
6	195	5.2730	831.6380	36
7	193	5.2627	36.8388	49
$\Sigma X=28$		$\Sigma Y=34.4718$	$\Sigma XY=142.2549$	$\Sigma X^2=142$

$nA+B\Sigma X=\Sigma Y$

$\Rightarrow 7A+28B=34.4718$ — (i)

$A\Sigma X+B\Sigma X^2=\Sigma XY$

$28A+142B=142.2459$ — (ii)

$A=4.3$
$B=0.15598$

$\therefore a=e^A=e^{4.3}=73.6998$

$b=e^B=e^{0.15598}=1.1688$

$\therefore y=(73.6998)(1.1688)^x$

Q Fit a least square Geometric curve $y=ax^b$ for following data

x	1	2	3	4	5
y	0.5	2	4.5	8	12.5

Soln:- $y=ax^b$

$\log y=\log a+b\log x$

$\boxed{Y=A+BX}$

$\therefore \Sigma Y=nA+B\Sigma X$

$\Sigma XY=A\Sigma X+B\Sigma X^2$

$Y=\log y \quad X=\log x$
$A=\log a$
$B=b$

x	y	$X = \log x$	$Y = \log y$	XY	X^2
1	0.5	0	−0.6931	0	0
2	2	0.6931	0.6931	0.4805	0.4805
3	4.5	1.0986	1.5041	1.6524	1.2069
4	8	1.3863	2.0794	2.8827	1.9218
5	12.5	1.6094	2.5257	4.0650	2.5902
		$\Sigma X = 4.7875$	$\Sigma Y = 6.1092$	$\Sigma XY = 9.0806$	$\Sigma X^2 = 6.1995$

$n = 5$

$$nA + B\Sigma X = \Sigma Y$$

$$5A + 4.7875B = 6.1092 \quad \text{---(i)}$$

$$A\Sigma X + B\Sigma X^2 = \Sigma XY$$

$$\cancel{4.7875A + 6.1092B =}$$

$$4.7875A + 6.1995B = 9.0806 \quad \text{---(ii)}$$

$$\therefore A = -0.6932$$

$$B = 2.0 = b$$

$$\therefore a = e^A = e^{-0.6932}$$

$$= 0.4999 \simeq 0.5$$

$$\boxed{\therefore y = (0.5)x^2}$$