GAUSSIAN ELEMINATION METHOD

Problem-1

Solve the following system of linear equations by Gaussian elemination method

$$10x_1 + x_2 - 5x_3 = 1$$
$$-20x_1 + 3x_2 + 20x_3 = 2$$
$$5x_1 + 3x_2 + 5x_3 = 6$$

Solⁿ:-

a) Forward elemination:

$$10x_1 + x_2 - 5x_3 = 1 \quad \text{---} (1)^0$$
$$-20x_1 + 3x_2 + 20x_3 = 2 \quad \text{---} (2)^0$$
$$5x_1 + 3x_2 + 5x_3 = 6 \quad \text{---} (3)^0$$

Elemination of the co-efficients of 'x_1' from the eqⁿ $(2)^0$ & $(3)^0$:-

$$(1)' = (1)^0 \longrightarrow \quad 10x_1 + x_2 - 5x_3 = 1 \quad \text{---} (1)'$$
$$(2)' = (2)^0 - \left(\frac{-20}{10}\right)\times(1)^0 \longrightarrow \quad 5x_2 + 10x_3 = 4 \quad \text{---} (2)'$$
$$(3)' = (3)^0 - \left(\frac{5}{10}\right)\times(1)^0 \longrightarrow \quad 2.5x_2 + 7.5x_3 = 5.5 \quad \text{---} (3)'$$

Elemination of the co-efficient of 'x_2' from eqⁿ $(3)'$:-

$$(1)^2 = (1)' \longrightarrow \quad 10x_1 + x_2 - 5x_3 = 1 \quad \text{---} (1)^2$$
$$(2)^2 = (2)' \longrightarrow \quad 5x_2 + 10x_3 = 4 \quad \text{---} (2)^2$$
$$(3)^2 = (3)' - \left(\frac{2.5}{5}\right)\times(2)' \longrightarrow \quad 2.5x_3 = 3.5 \quad \text{---} (3)^2$$

In matrix form we can write $Ux = B$

or
$$\begin{bmatrix} 10 & 1 & -5 \\ 0 & 5 & 10 \\ 0 & 0 & 2.5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 4 \\ 3.5 \end{Bmatrix}$$

b) Backward substitution:-

From eqⁿ $(3)^2 \rightarrow \quad x_3 = \frac{3.5}{2.5} = 1.4$

From eqⁿ $(2)^2 \rightarrow 5x_2 + 10\times1.4 = 4$
or $x_2 = -2$

From eqⁿ $(1)^2 \rightarrow 10x_1 - 2 - 5\times1.4 = 1$
or $x_1 = 1$

The solution obtained :-

$x_1 = 1$
$x_2 = -2$
$x_3 = 1.4$

JACOBI ITERATION METHOD

General form of the i^{th} equation –

$$\sum_{j=1}^{n} a_{ij} x_j = b_i$$

From this we get $x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} x_j\right)$

By successive iteration we have

$$x_i^{(0)} \rightarrow x_i^{(1)} \rightarrow x_i^{(2)} \rightarrow \cdots\cdots$$

$$x_i^{(1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(0)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(0)}\right]$$

$$x_i^{(K+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(K)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(K)}\right]$$

By adding and substracting we get,

$$x_i^{(K+1)} = x_i^{(K)} + \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{n} a_{ij} x_j^{(K)}\right]$$

or $x_i^{(K+1)} = x_i^{(K)} + \frac{R_i^{(K)}}{a_{ii}}$ $\quad\left[R_i^{(K)} = b_i - \sum_{j=1}^{n} a_{ij} x_j^{(K)}\right]$

Prob.- 6

Solve the following system of linear equations by using Jakobi iteration method.

$$16x_1 + 4x_2 + 8x_3 = 4$$
$$4x_1 + 5x_2 - 4x_3 = 2$$
$$8x_1 - 4x_2 + 22x_3 = 5$$

Sol^{n}:-

Scaling the equations by the corresponding leading diagonal elements we can write the equations in the following form:

$$x_1 + 0.25x_2 + 0.5x_3 = 0.25$$
$$0.8x_1 + x_2 - 0.8x_3 = 0.40$$
$$0.3636x_1 - 0.1818x_2 + x_3 = 0.2273$$

$\therefore \quad x_1^{(K+1)} = 0.25 - 0.25\, x_2^{(K)} - 0.5\, x_3^{(K)}$

$x_2^{(K+1)} = 0.4 - 0.8\, x_1^{(K)} + 0.8\, x_3^{(K)}$

$x_3^{(K+1)} = 0.2273 - 0.3636\, x_1^{(K)} + 0.1818\, x_2^{(K)}$

	Iteration numbers						
	1	2	3	4	5	6	7
x_1	1	−0.5	0.12725	−0.20001	−0.087458	−0.168776	−0.157031
x_2	1	0.4	0.8364	0.683656	0.826479	0.809416	0.86250
x_3	1	0.0455	0.48182	0.333089	0.424312	0.409354	0.43582

$\therefore$ The solution obtained:

$x_1 = -0.157031$

$x_2 = 0.86250$

$x_3 = 0.43582$

GAUSS - SEIDEL METHOD

$$x_i^{(K+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}\, x_j^{(K+1)} - \sum_{j=i+1}^{n} a_{ij}\, x_j^{(K)}\right]$$

$$x_i^{(K+1)} = x_i^{(K)} + \frac{R_i^{(K)}}{a_{ii}} \qquad \left[R_i^{(K)} = b_i - \sum_{j=1}^{i-1} a_{ij}\, x_j^{(K+1)} - \sum_{j=i}^{n} a_{ij}\, x_j^{(K)}\right]$$

Prob.-7 Solve the following system of linear equations by Gauss-seidel iteration method.

$$16x_1 + 4x_2 + 8x_3 = 4$$
$$4x_1 + 5x_2 - 4x_3 = 2$$
$$8x_1 - 4x_2 + 22x_3 = 5$$

Solⁿ:- Scaling the equation by the corresponding leading diagonal elements we can write the equations in the form:

$$x_1 + 0.25x_2 + 0.5x_3 = 0.25$$
$$0.8x_1 + x_2 - 0.8x_3 = 0.40$$
$$0.3636x_1 - 0.1818x_2 + x_3 = 0.2273$$

The equations can be written in the following form:

$$x_1^{(K+1)} = 0.25 - 0.25x_2^{(K)} - 0.5x_3^{(K)}$$
$$x_2^{(K+1)} = 0.4 - 0.8x_1^{(K+1)} + 0.8x_3^{(K)}$$
$$x_3^{(K+1)} = 0.2273 - 0.3636x_1^{(K+1)} + 0.1818x_2^{(K+1)}$$

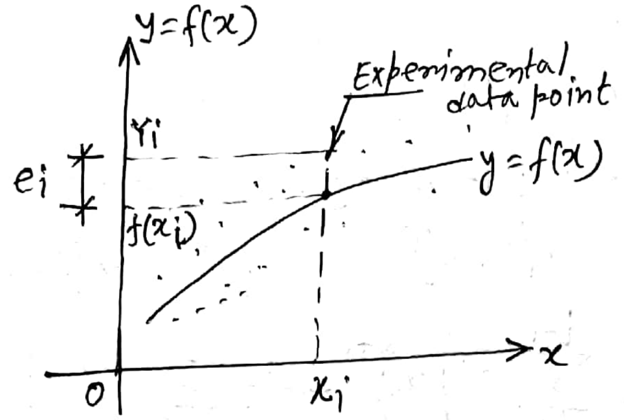
	Iteration numbers						
	1	2	3	4	5	6	7
x_1	1	−0.5	−0.49999	−0.41816	−0.36907	−0.33366	−0.30885
x_2	1	1.6	1.35997	1.2596	1.18193	1.12802	1.09003
x_3	1	0.69998	0.65634	0.60834	0.57637	0.55369	0.53776

∴ The solution obtained: $x_1 = -0.30885$

$x_2 = 1.09003$

$x_3 = 0.53776$

Least square method of curve-fitting or Least square method of Regression



Let n nos. of data points obtained from experiment. (x_i, y_i), $i = 1, 2, \dots n$:

The error of approximation at $x = x_i$

$$e_i = y_i - f(x_i)$$

$$S = e_1^2 + e_2^2 + \dots + e_n^2$$
$$= [y_1 - f(x_1)]^2 + [y_2 - f(x_2)]^2 + \dots + [y_n - f(x_n)]^2$$
$$= \sum_{i=1}^{n} [y_i - f(x_i)]^2$$

principle of Method of least squares or least square regression is to minimise 'S' i.e. sum of the squares of the errors.

Fitting a straight Line (Linear regression)

Let, $y = f(x) = a_0 + a_1 x$

$\therefore$ $S = [y_1 - (a_0 + a_1 x_1)]^2 + [y_2 - (a_0 + a_1 x_2)]^2 + \dots + [y_n - (a_0 + a_1 x_n)]^2$

i.e. $S = \phi(a_0, a_1)$

For 'S' to be minimum, we have $\frac{\partial S}{\partial a_0} = 0$, $\frac{\partial S}{\partial a_1} = 0$.

$$\frac{\partial S}{\partial a_0} = 0 = -2[y_1 - (a_0 + a_1 x_1)] - 2[y_2 - (a_0 + a_1 x_2)] - \dots - 2[y_n - (a_0 + a_1 x_n)]$$

$$\Rightarrow n a_0 + a_1 \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \quad \text{--- (1)}$$

For $\frac{\partial S}{\partial a_1} = 0 = -2x_1[y_1 - (a_0 + a_1 x_1)] - 2x_2[y_2 - (a_0 + a_1 x_2)] - \dots - 2x_n[y_n - (a_0 + a_1 x_n)]$

$$\Rightarrow a_0 \sum_{i=1}^{n} x_i + a_1 \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i \quad \text{--- (2)}$$

Equations ① & ② are called as normal equations, which (21) (33)
can be solved for two unknowns a_0 & a_1.

Again it is evident that $\frac{\partial^2 s}{\partial a_0^2}$ & $\frac{\partial^2 s}{\partial a_1^2}$ are both +ve.
Hence, for a_0 & a_1 s is minimum.

Scanned by CamScanner

Prob. 22 : Find the best fitting straight line by using the following experimental data points.

(0.5, 0.31), (1.0, 0.82), (1.5, 1.29), (2.0, 1.85), (2.5, 2.51) &

If another point $x = 3.0$, $y = 3.02$ is considered, find the effect.

Sol^n :-

POINT	0	1	2	3	4	$\Sigma n = 5$
x	0.5	1.0	1.5	2.00	2.50	$\Sigma x_i = 7.5$
y	0.31	0.82	1.29	1.85	2.51	$\Sigma y_i = 6.78$
xy	0.155	0.82	1.935	3.70	6.275	$\Sigma x_i y_i = 12.885$
x^2	0.25	1.0	2.250	4.0	6.25	$\Sigma x_i^2 = 13.75$

Scanned by CamScanner