Runge-Kutta (R-K) Methods for Numerical Differentiation

In the **Modified Euler's method**, after calculating y_{i+1} through the equation,

$$y_{i+1} = y_i + f(x_i, y_i) h \quad \text{.........(1)}$$

its value is modified iteratively through the equation,

$$y_{i+1}^{(n+1)} = y_i + \left(\frac{h}{2}\right)\left[f(x_i, y_i) + f\left(x_{i+1}, y_{i+1}^{(n)}\right)\right] \quad \text{.........(2)}$$

when, $h = x_{i+1} - x_i$ ....(3)

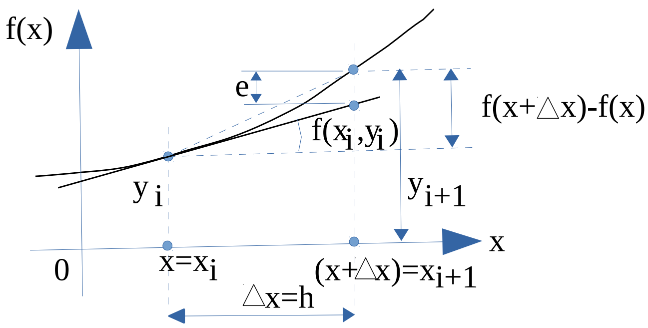

Figure- 7.1

Euler's metod is less efficient in practical problems since it requires 'h' to be small for obtaining reasonable accuracy.

The *Runge-Kutta (R-K) methods* are designed to obtain greater accuracy.
The modified Euler's method as presented through the equation (1) and (2) can be combined and rewritten as,

$$y_{i+1} = y_i + \left(\frac{1}{2}\right)\left[hf(x_i, y_i) + hf\left(x_i + h, y_i + hf(x_i, y_i)\right)\right]$$

$$y_{i+1} = y_i + \left(\frac{h}{2}\right)\left[f_i + f\left(x_i + h, y_i + hf_i\right)\right] \quad \text{.........(4)}$$

where, $f_i = f(x_i, y_i)$

Now let, $k_1 = f_i$ and $k_2 = f(x_i + h, y_i + hk_1)$ then, equation (4) can be expressed as,

$$y_{i+1} = y_i + \left(\frac{h}{2}\right)(k_1 + k_2) \quad \text{.........(5)}$$

Equation (5) presents the formula for the **2nd order R-K method.**
The formula presented in equation (5) ca be generalized as,

$$y_{i+1} = y_i + h(w_1 k_1 + w_2 k_2) \quad \text{.........(6)}$$

Where, $w_1 = w_2 = \frac{1}{2}$

The 2nd order R-K method has been modified further to achieve more accuracy through the **4th order R-K method**. The formula is,

$$y_{i+1} = y_i + h(w_1 k_1 + w_2 k_2 + w_3 k_3 + w_4 k_4) \quad \text{.........(7)}$$

Where, $w_1 = w_4 = \frac{1}{6}$ and $w_2 = w_3 = \frac{1}{3}$

Equation (7) is expressed as,

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad \text{.........(8)}$$

Where, $k_1 = f(x_i, y_i)$ ....(8a)

$$k_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{hk_1}{2}\right) \quad \text{.........(8b)}$$

$$k_3=f\left(x_i+\frac{h}{2},y_i+\frac{hk_2}{2}\right) \quad \text{.......(8c)}$$

$$k_4=f\left(x_i+h,y_i+hk_3\right) \quad \text{.......(8d)}$$

Example-1:

Find 'y' at x=1.0 by solving the initial value problem expressed by the differential equation, $\frac{dy}{dx}=-2xy^2$ with initial condition, $y(0)=1$ by applying the fourth order Runge-Kutta method. Take step size $h=0.2$. Also, compare the result with the exact solution.

Solution:

$$y_{i+1}=y_i+\frac{h}{6}(k_1+2k_2+2k_3+k_4)$$

Where, $k_1=f(x_i,y_i)$, $k_2=f\left(x_i+\frac{h}{2},y_i+\frac{hk_1}{2}\right)$, $k_3=f\left(x_i+\frac{h}{2},y_i+\frac{hk_2}{2}\right)$ and

$k_4=f\left(x_i+h,y_i+hk_3\right)$

Given, $\frac{dy}{dx}=f(x,y)=-2xy^2$ and $h=0.2$.

Step (i)	x_i	y_i	k_1	k_2	k_3	k_4	y_{i+1}
0	0	1	0	$f(0.1,1)$ $=-0.2$	$f(0.1,0.98)$ $=-0.192080$	$f(0.2,0.9616)$ $=-0.369858$	0.961533
1	0.2	0.961533	$f(0.2,0.961533)$ $=-0.369818$	$f(0.3,0.92455)$ $=-0.512877$	$f(0.3,0.91024)$ $=-0.497128$	$f(0.4,0.8621)$ $=-0.594583$	0.862052
2	0.4	0.862052	$f(0.4,0.862052)$ $=-0.594508$	$f(0.5,0.8026)$ $=-0.644170$	$f(0.5,0.79763)$ $=-0.636222$	$f(0.6,0.73481)$ $=-0.647931$	0.735278
3	0.6	0.735278	$f(0.6,0.735278)$ $=-0.648761$	$f(0.7,0.6704)$ $=-0.629215$	$f(0.7,0.67235)$ $=-0.632889$	$f(0.8,0.6087)$ $=-0.592826$	0.609752
4	0.8	0.609752	$f(0.8,0.609752)$ $=-0.594876$	$f(0.9,0.55026)$ $=-0.545023$	$f(0.9,0.55525)$ $=-0.554944$	$f(1.0,0.49876)$ $=-0.497529$	0.500007

Hence, we get $y(1)=0.500007$

Exact solution:

$$\frac{dy}{dx}=f(x,y)=-2xy^2$$

$$\int\frac{dy}{y^2}=-2\int x\,dx$$

$$-\frac{1}{y}=-x^2+c$$

Putting the given initial condition, $y(0)=1$ we get, $c=-1$

Therefore, we obtain , $y=\frac{1}{1+x^2}$

Hence, **the exact solution**, $y(1)=0.5$

Assignments

Problem 1:

Given the differential equation, $\frac{dy}{dx}=-y$ subject to the initial condition: $y(0)=1$ compute $y(0.05)$ by using the 4th order Runge-Kutta method. Take a step size $h=0.025$. Compare the result with the exact solution.

Problem 2:

Using the 4th order Runge-Kutta method, find out the solution of the equation $\frac{df}{dx}=\frac{y-x}{y+x}$ with $y(0)=1$, at $x=2$ by taking step size, $h=0.5$.

Note: You can present the solution by tabular form only.