Modified Euler's Method for Numerical Differentiation

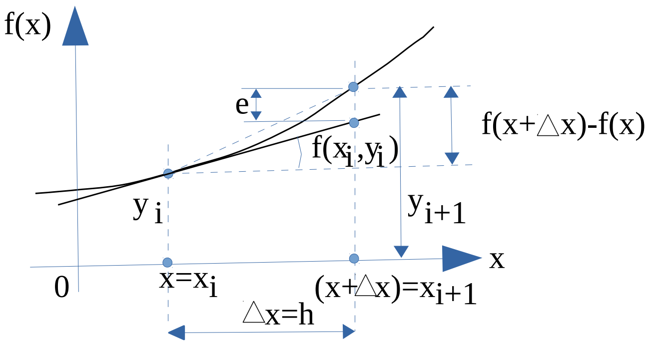

Figure- 6.1

In Note-5 we have seen how to calculate y_{i+1} i.e., $f(x_{i+1})$ by using the Euler's method where $y_{i+1}=y_i+f(x_i,y_i)h$ is the key equation. From the figure 6.1 we can also see the error $'e'$ occur in getting the value of y_{i+1} through this formula.
One way to reduce this error is to reduce the step length or the step size h.

This error can also be reduced by adopting the ***Modified Euler's method*** which is explained below:

In the **Modified Euler's method**, after calculating y_{i+1} through the equation,
$y_{i+1}=y_i+f(x_i,y_i)h$ ....(1)
its value is modified iteratively through the equation,
$y_{i+1}^{(n+1)}=y_i+\left(\frac{h}{2}\right)\left[f(x_i,y_i)+f\left(x_{i+1},y_{i+1}^{(n)}\right)\right]$ ....(2)
when, $h=x_{i+1}-x_i$ ....(3)
The criteria for terminating this iteration is,
$\left|y_{i+1}^{(n+1)}-y_{i+1}^{(n)}\right|<\epsilon$ ....(4)
where, ϵ is a small predifined value which is fixed based on the required accuracy.

Example-1:

Given the differential equation, $\frac{dy}{dx}=\frac{2y}{x}$ with initial condition, $y(1)=2$ find out $y(2)$ by taking step size $h=0.25$. Use

a) Euler's method.
b) Modified Euler's method by considering $\epsilon=1xe^{-3}$, and
c) Compare the result with the exact solution.

Solution:

a) Applying the Euler's method

Step (i)	x_i	y_i	$f(x_i,y_i)=\frac{2y_i}{x_i}$	$y_{i+1}=y_i+f(x_i,y_i)h$
0	1	2	4	3
1	1.25	3	4.8	4.2
2	1.5	4.2	5.6	5.6
3	1.75	5.6	6.4	7.2

Hence, we obtain $y(2)=7.2$

b) Applying the Modified Euler's method

Step (i)	x_i	y_i	y_{i+1}^0	$f(x_i, y_i)$	$f(x_{i+1}, y_{i+1}^n)$	y_{i+1}^n
0	1	2	3	4	4.8	3.10
					4.96	3.12
					4.992	3.124
					4.9984	3.1248
					4.9997	3.1249
1	1.25	3.1249	4.3748	4.9998	5.833	4.479
					5.972	4.4964
					5.9952	4.4993
					5.999	4.4997
2	1.5	4.4997	5.9996	5.9997	6.8567	6.1067
					6.9791	6.1220
					6.9966	6.1242
					6.9991	6.1245
					6.9995	6.1246
3	1.75	6.1246	7.8744	6.9995	7.8745	7.9838
					7.9838	7.9975
					7.9975	7.9992
					7.9992	7.9994

Hence, we obtain $y(2)=7.9994$

c) Applying the exact method:

Given the equation:

$$\frac{dy}{dx}=\frac{2y}{x}$$

$$\frac{1}{2}\int\frac{dy}{y}=\int\frac{dx}{x}$$

$$\frac{1}{2}\log_e y=\log_e x+\log_e C$$

$$\frac{1}{2}\log_e y=\log_e Cx$$

$$y^{\frac{1}{2}}=Cx$$

Now, given the initial condition, $y(1)=2$ i.e., at $x=1$, $y=2$

Hence, $C=2^{\frac{1}{2}}=1.4142$

Therefore, we obtain, $y^{\frac{1}{2}}=x\sqrt{2}$

$y=2x^2$ and hence, $y(2)=8$ is the required exact value.

Assignments

Problem 1:

Given the differential equation, $\frac{df}{dx}=-y$ subject to the initial condition: $y(0)=1$ compute $y(0.05)$ by using the Modified Euler's method. Take a step size $h=0.025$ and show atleast 3 iterations in each steps.

Problem 2:

Using the Modified Euler's method, find out the solution of the equation $\frac{df}{dx}=\frac{y-x}{y+x}$ with $y(0)=1$, at $x=0.1$ by taking step size, $h=0.05$ and considering $\epsilon=1xe^{-3}$

Note: You can present the solution by tabular form only.