Finite Differences and Interpolation

Suppose we have a function $y = f(x)$ & also the following data points are given

x	x_0	x_1	x_2	----	x_n
y	y_0	y_1	y_2		y_n

Interpolation: The process of finding out y_i corresponding to $x = x_i$, where, $x_0 < x_i < x_n$

Extrapolation: The process of finding out y_i corresponding to $x = x_i$, where x_i is out of the range x_0 to x_n.

To find out the values of $y_i = f(x_i)$ or $f'(x_i)$ where x_i is some intermediate value we can use Forward difference (FD), Backward difference (BD) or the central difference (CD).

Forward Difference (FD)

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\Delta y_2 = y_3 - y_2$$
$$\vdots$$
$$\Delta y_i = y_{i+1} - y_i$$

First Forward differences, where, 'Δ' is called as the forward difference operator.

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - y_1 + y_0 = y_2 - 2y_1 + y_0$$
$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 = y_3 - y_2 - y_2 + y_1 = y_3 - 2y_2 + y_1$$

→ second forward differences.

In general, ~~$\Delta^j y_i = \Delta^{j-1} y_{i+1} - \Delta^{j-1} y_i$~~

$$\Delta y_i^j = \Delta y_{i+1}^{j-1} - \Delta y_i^{j-1}$$

↑ j^{th} Forward Differences.

Forwar Difference Table

value of 'x'	Value of 'y'	1st. FD	2nd. FD	3rd FD	4th FD	5th FD	- - -
x_0	y_0						
		Δy_0					
x_0+h	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$	$\Delta^4 y_0$		
x_0+2h	y_2		$\Delta^2 y_1$			$\Delta^5 y_0$	
		Δy_2		$\Delta^3 y_1$	$\Delta^4 y_1$		
x_0+3h	y_3		$\Delta^2 y_2$				
		Δy_3		$\Delta^3 y_2$			
x_0+4h	y_4		$\Delta^2 y_3$				
		Δy_4					
x_0+5h	y_5						

* Any higher order Forward Difference (FD) can be expressed in terms of y_i's.

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = \Delta y_2 - \Delta y_1 - \Delta y_1 + \Delta y_0$$
$$= y_3 - y_2 - 2(y_2 - y_1) + y_1 - y_0$$
$$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 = y_4 - 3y_3 + 3y_2 - y_1 - (y_3 - 3y_2 + 3y_1 - y_0)$$

$$\Rightarrow \boxed{\Delta^4 y_0 = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0}$$

$$\boxed{\Delta^n y_0 = y_n - {}^nC_1\, y_{n-1} + {}^nC_2\, y_{n-2} - \cdots\cdots\cdots + (-1)^n y_0}$$