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Zariski density of crystalline points

Gebhard Böckle[a,1], Ashwin Iyengar[b,1], and Vytautas Paškūnas[c,1]

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We show that crystalline points are Zariski dense in the deformation space of a representation of the absolute Galois group of a *p*-adic field. We also show that these points are dense in the subspace parameterizing deformations with the determinant equal to a fixed crystalline character. Our proof is purely local and works for all *p*-adic fields and all residual Galois representations.

Galois representations | *p*-adic Hodge theory | density

Significance

The study of Galois representations plays a significant role in the Langlands program and related topics in modern number theory. In the past two decades, there has been a whirlwind of activity toward proving a local *p*-adic version of the Langlands correspondence. While interesting in its own right, this correspondence also has implications for the global Langlands correspondence over number fields. Density of crystalline points has been used by Colmez and Kisin in the 2-dimensional case to complete the proofs of certain cases of the *p*-adic Langlands program. This article generalizes the proof of Zariski density to all dimensions and to all *p*-adic local fields.

Fix a finite extension $F/\mathbb{Q}_p$ and a further large finite extension $L/\mathbb{Q}_p$ with ring of integers $\mathcal{O} \subset L$, uniformizer ϖ, and residue field k. If $\overline{\rho} : G_F \to \mathrm{GL}_d(k)$ is a continuous representation of the absolute Galois group of F, then let $R^{\square}_{\overline{\rho}}$ denote the framed deformation ring of $\overline{\rho}$ and write $\mathfrak{X}^{\square}_{\overline{\rho}} = (\mathrm{Spf}\, R^{\square}_{\overline{\rho}})^{\mathrm{rig}}$ for its rigid generic fiber.

Theorem 1.1. *The set $\mathcal{S}_{\mathrm{cr}}$ of $x \in \mathfrak{X}^{\square}_{\overline{\rho}}$ whose associated p-adic Galois representation ρ_x is crystalline with regular Hodge–Tate weights is Zariski dense in $\mathfrak{X}^{\square}_{\overline{\rho}}$.*

By *Zariski dense*, we mean that any rigid analytic function on $\mathfrak{X}^{\square}_{\overline{\rho}}$ that vanishes at every point in $\mathcal{S}_{\mathrm{cr}}$ is identically zero.

Let us outline the proof of the theorem. By reducing the question to an explicit combinatorial problem, we show that if $\overline{\rho}$ is absolutely irreducible and $\psi : G_F \to \mathcal{O}^\times$ is a crystalline character lifting $\det \overline{\rho}$, then there is a crystalline lift ρ of $\overline{\rho}$ with regular Hodge–Tate weights and determinant ψ. For general $\overline{\rho}$ and crystalline ψ, we use an inductive procedure based on a result of Emerton–Gee (1) on extensions of crystalline representations. This procedure follows their proof that every $\overline{\rho}$ admits a crystalline lift but is refined to control the determinant. Our main innovation is in the irreducible case.

Then, using the description of the irreducible components of $\mathfrak{X}^{\square}_{\overline{\rho}}$ in our recent paper (2), we deduce that $\mathcal{S}_{\mathrm{cr}}$ meets every irreducible component of $\mathfrak{X}^{\square}_{\overline{\rho}}$ nontrivially. On the other hand, we already know that the closure of $\mathcal{S}_{\mathrm{cr}}$ is a union of some subset of the irreducible components of $\mathfrak{X}^{\square}_{\overline{\rho}}$. This is proved by infinite fern arguments by Chenevier (3) if $F = \mathbb{Q}_p$ and $\overline{\rho}$ is absolutely irreducible, Nakamura (4) if F is arbitrary and $\overline{\rho}$ has scalar endomorphisms, and AI (5) in general. The paper (5) makes use of recent work of Breuil et al. (6, 7). Hence, the closure is equal to the union of all irreducible components, and we are done.

We also prove an analog of Theorem 1.1 with a fixed determinant. If $\psi : G_F \to \mathcal{O}^\times$ is a crystalline character lifting $\det \overline{\rho}$, then we write $R^{\square,\psi}_{\overline{\rho}}$ for the quotient of $R^{\square}_{\overline{\rho}}$ parameterizing framed deformations of $\overline{\rho}$ with determinant ψ and $\mathfrak{X}^{\square,\psi}_{\overline{\rho}}$ for its rigid generic fiber.

Theorem 1.2. *The set $\mathcal{S}^{\psi}_{\mathrm{cr}}$ of $x \in \mathfrak{X}^{\square,\psi}_{\overline{\rho}}$ whose associated p-adic Galois representation ρ_x is crystalline with regular Hodge–Tate weights is Zariski dense in $\mathfrak{X}^{\square,\psi}_{\overline{\rho}}$.*

In ref. 2, we have shown that $\mathfrak{X}^{\square,\psi}_{\overline{\rho}}$ is irreducible, and as already explained above, we show that $\mathcal{S}^{\psi}_{\mathrm{cr}}$ is nonempty by building on the results of Emerton–Gee (1). Unfortunately, the infinite fern arguments are not readily available in the literature in the fixed determinant context. Instead, we prove, by checking that the argument of Iyengar (5) carries over, that if we replace $\mathcal{S}_{\mathrm{cr}}$ by a subset $\mathcal{S}^{\mathrm{cn}}_{\mathrm{cr}}$ obtained by imposing a congruence condition on the Hodge–Tate weights, then the closure of $\mathcal{S}^{\mathrm{cn}}_{\mathrm{cr}}$ is still a union of irreducible components of $\mathfrak{X}^{\square}_{\overline{\rho}}$. The set $\mathcal{S}^{\mathrm{cn}}_{\mathrm{cr}}$ is contained in the subset of $\mathfrak{X}^{\square}_{\overline{\rho}}$ corresponding to twists of ρ_x for $x \in \mathcal{S}^{\psi}_{\mathrm{cr}}$ by crystalline characters,

Author affiliations: [a]IWR and Mathematical Institute, Ruprecht-Karls-Universität Heidelberg, Heidelberg 69120, Germany; [b]Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218; and [c]Fakultät für Mathematik, Universität Duisburg-Essen, Essen 45127, Germany

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[1]To whom correspondence may be addressed. Email: gebhard.boeckle@iwr.uni-heidelberg.de, iyengar@jhu.edu, or paskunas@uni-due.de.

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which are trivial modulo ϖ. This allows us to deduce that the closure of $\mathcal{S}_{\mathrm{cr}}^{\psi}$ in $\mathfrak{X}_{\overline{\rho}}^{\square,\psi}$ cannot have a positive codimension, thus finishing the proof of Theorem 1.2.

As we explain in Section 5, density results in the rigid analytic generic fiber imply density results in the scheme theoretic generic fiber, and we obtain the following:

Corollary 1.3. *The set $\mathcal{S}_{\mathrm{cr}}$ is Zariski dense in* Spec $R_{\overline{\rho}}^{\square}[1/p]$ *and in* Spec $R_{\overline{\rho}}^{\square}$*, and the same holds for $\mathcal{S}_{\mathrm{cr}}^{\psi}$ in* Spec $R_{\overline{\rho}}^{\square,\psi}[1/p]$ *and in* Spec $R_{\overline{\rho}}^{\square,\psi}$.

Our theorems offer a possibility of first proving a statement of interest for crystalline representations lifting $\overline{\rho}$ and then extending it to all the representations by Zariski density. Such arguments were first employed by Colmez (8) and Kisin (9) to show that every 2-dimensional irreducible p-adic representation of $G_{\mathbb{Q}_p}$ lies in the image of Colmez's Montreal functor. This is an important result in the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ and has motivated further research on the question of density of crystalline points.

Let us review some of the past work on this problem. The results of refs. 8 and 9 were proved for $d=2$ under assumptions on $\overline{\rho}$ and p; the latter were subsequently removed in ref. 10 when $F=\mathbb{Q}_3$, (11) when $F=\mathbb{Q}_2$ and $\overline{\rho}$ does not have scalar semisimplification and (12) when $F=\mathbb{Q}_2$ and $\overline{\rho}$ is trivial. The argument of ref. 12 has been generalized by AI (5) when $\overline{\rho}$ is trivial and either $p>d$ or $\mu_{p^\infty}(F)$ is trivial. Moreover, if $d=2$, $p>2$ and F is a finite extension of $\mathbb{Q}_p$, then our Theorem 1.1 has been proved by GB–Juschka in ref. 13. It has already been observed by Chenevier and Nakamura in their original papers that if one imposes certain restrictions on $\overline{\rho}$, F, and d ($\overline{\rho}$ absolutely irreducible and either (1) $\overline{\rho}\not\cong\overline{\rho}(1)$ or (2) $p\nmid d$ and $\zeta_p\in F$) then either $\mathfrak{X}_{\overline{\rho}}^{\square}$ has only one irreducible component or twisting by crystalline characters lifting the trivial representation permutes the irreducible components of $\mathfrak{X}_{\overline{\rho}}^{\square}$ transitively, and their results imply that the closure of crystalline points contains all the irreducible components. Finally, we have proved Theorem 1.1 under the assumption $p\nmid 2d$ in ref. 2. The proof given there relied on patching Galois representations associated with automorphic forms, so it was global in nature; by contrast, in this work, all methods are local.

Our Theorem 1.1 settles the question completely for all $\overline{\rho}$ and all p and all F. Theorem 1.2 also settles the question in the fixed determinant setting.

1.4. Notation. Let F be a fixed finite extension of $\mathbb{Q}_p$ with ring integers $\mathcal{O}_F$ and residue field k_F. We fix a uniformizer ϖ_F of F. We also fix an algebraic closure $\overline{F}/F$ and let $G_F:=\mathrm{Gal}(\overline{F}/F)$ denote the absolute Galois group of F with its profinite topology. We let $\mu_{\mathrm{tors}}(F)$ be the group of roots of unity in F and let $\mu:=\mu_{p^\infty}(F)$ denote the group of p-power roots of unity in F.

We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and let $L\subset\overline{\mathbb{Q}}_p$ be a further finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}\subset L$, uniformizer $\varpi\in\mathcal{O}$, and residue field $k:=\mathcal{O}/\varpi$. We let $\Sigma=\Sigma_F$ be the set of embeddings $\sigma:F\hookrightarrow L$, we assume that L is large enough, so that $|\Sigma|=[F:\mathbb{Q}_p]$. For a tuple $\underline{k}=(k_\sigma)_{\sigma\in\Sigma}\in\mathbb{Z}^{\Sigma}$, we define a character $\chi_{\underline{k}}:F^\times\to\mathcal{O}^\times$ by letting $\chi_{\underline{k}}(\varpi_F)=1$ and $\chi_{\underline{k}}(x)=\prod_{\sigma\in\Sigma}\sigma(x)^{k_\sigma}$ for all $x\in\mathcal{O}_F^\times$. The Artin map of local class field theory $\mathrm{Art}_F:F^\times\to G_F^{\mathrm{ab}}$ identifies G_F^{ab} with the profinite completion of $F^\times$, which allows us to consider $\chi_{\underline{k}}$ as a character of G_F. In this case, $\chi_{\underline{k}}$ is crystalline with Hodge–Tate weights $\underline{k}$. Conversely, if $\chi:G_F\to\overline{\mathbb{Q}}_p^\times$ is a crystalline character with Hodge–Tate weights $\underline{k}$, then χ is equal to a product of $\chi_{\underline{k}}$ and an unramified character (14, Proposition B.4).

In our arguments, we will frequently replace L by a finite extension. In particular, in Section 2, we assume that $|\Sigma_E|=[E:\mathbb{Q}_p]$, where E is an unramified extension of F of degree d.

Recall that the Hodge–Tate weights $\underline{k}(\rho)$ of a Hodge–Tate representation $\rho:G_F\to\mathrm{GL}_d(\overline{\mathbb{Q}}_p)$ is a d-tuple of integers $\underline{k}(\rho)_\sigma=(k_{\sigma,1}\geq k_{\sigma,2}\geq\ldots\geq k_{\sigma,d})$ for each embedding $\sigma\in\Sigma_F$ and we say that ρ has *regular* Hodge–Tate weights if for each $\sigma\in\Sigma_F$ the inequalities are strict.

2. Crystalline Lifts with Fixed Determinant

In this section, we construct crystalline lifts of any mod ϖ representation of G_F. The absolutely irreducible representations are induced from characters over an unramified extension, so in this case, we can just lift the characters in an explicit way using some combinatorial arguments involving Hodge–Tate weights. For nonsemisimple representations, we use the Emerton–Gee stack to lift extension classes.

Lemma 2.1. *Let E/F be a finite Galois extension of degree d. Let $\rho=\mathrm{Ind}_{G_E}^{G_F}\theta$, where $\theta:G_E\to A^\times$ is a continuous character with values in the unit group of some topological ring A. Let $\psi:=\det\rho$. Then,*

$$\psi(\mathrm{Art}_F(x))=\theta(\mathrm{Art}_E(x)),\quad \forall x\in N_F^E E^\times.$$

Moreover, if $\mathrm{Gal}(E/F)$ *is cyclic and $g\in G_F$ maps to a generator of* $\mathrm{Gal}(E/F)$*, then $\psi(g)=(-1)^{d-1}\theta(g^d)$.*

Proof: We have $\rho|_{G_E}\cong\oplus_{\sigma\in\mathrm{Gal}(E/F)}\theta^\sigma$. Thus, $\psi(g)=\prod_{\sigma\in\mathrm{Gal}(E/F)}\theta^\sigma(g)$ for all $g\in G_E$. We consider ψ and θ^σ as characters of G_E^{ab}. Then, for all $y\in E^\times$, we have

$$\psi(\mathrm{Art}_E(y))=\prod_{\sigma\in\mathrm{Gal}(E/F)}\theta^\sigma(\mathrm{Art}_E(y))=\prod_{\sigma\in\mathrm{Gal}(E/F)}\theta(\sigma\mathrm{Art}_E(y)\sigma^{-1})=\prod_{\sigma\in\mathrm{Gal}(E/F)}\theta(\mathrm{Art}_E(\sigma(y)))=\theta(\mathrm{Art}_E(N_F^E(y))).$$

A justification for the second to last equality can be found at the end of (15, Section XI.3). We have a commutative diagram

$$\begin{array}{ccc} E^\times & \xrightarrow{\mathrm{Art}_E} & G_E^{\mathrm{ab}} \\ {\scriptstyle N_F^E}\big\downarrow & & \big\downarrow{\scriptstyle \iota} \\ F^\times & \xrightarrow{\mathrm{Art}_F} & G_F^{\mathrm{ab}} \end{array} \qquad [1]$$

where ι is induced by the inclusion $G_E \subset G_F$; (16, Section 2.4). Thus, if $x = N_F^E(y)$ with $y \in E^\times$, then $\psi(\mathrm{Art}_F(x)) = \psi(\mathrm{Art}_E(y)) = \theta(\mathrm{Art}_E(N_F^E(y))) = \theta(\mathrm{Art}_E(x))$.

Let us assume that the image of $g \in G_F$ generates $\mathrm{Gal}(E/F)$. Let $f \in \mathrm{Ind}_{G_E}^{G_F}\theta$ be the function with support G_E satisfying $f(1) = 1$. Then, $f, gf, \ldots, g^{d-1}f$ is a basis of $\mathrm{Ind}_{G_E}^{G_F}\theta$ as a k-vector space, and since $g^d \in G_E$, we have $g(g^{d-1}f) = \theta(g^d)f$. We may compute the determinant of the matrix of $\rho(g)$ associated with this basis to find that $\det \rho(g) = (-1)^{d-1}\theta(g^d)$. □

Lemma 2.2. *Let I and J be finite sets. Suppose that we are given integers $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ such that $\sum_{i\in I} a_i = \sum_{j\in J} b_j$. Then, there exist integers x_{ij}, for $(i,j) \in I \times J$ such that $\sum_{i\in I} x_{ij} = b_j$ for all $j \in J$ and $\sum_{j\in J} x_{ij} = a_i$ for all $i \in I$.*

Proof: Induction on the cardinality of the set I. If $|I| = 1$, then letting $x_{ij} = b_j$ for all $j \in J$ gives the unique solution. If $|I| > 1$, then we choose $i_0 \in I$ and $j_0 \in J$ and define $x_{i_0j_0} = a_{i_0}$ and $x_{i_0j} = 0$ for $j \in J$, $j \neq j_0$. By the induction hypothesis, there exist integers x_{ij} for $(i,j) \in (I \setminus \{i_0\}) \times J$ such that $\sum_{j\in J} x_{ij} = a_i$ for all $i \in I \setminus \{i_0\}$, $\sum_{i\in I\setminus\{i_0\}} x_{ij} = b_j$ for all $j \in J \setminus \{j_0\}$ and $\sum_{i\in I\setminus\{i_0\}} x_{ij_0} = b_{j_0} - a_{i_0}$. □

Lemma 2.3. *Let I and J be finite sets with $|J| > 1$, let m be a positive integer, and let C be a positive integer. Suppose that we are given integers $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ such that $\sum_{i\in I} a_i \equiv \sum_{j\in J} b_j \pmod m$. Then, there exist pairwise distinct integers x_{ij}, for $(i,j) \in I \times J$, such that $\sum_{i\in I} x_{ij} \equiv b_j \pmod m$ for all $j \in J$ and $\sum_{j\in J} x_{ij} = a_i$ for all $i \in I$ and $|x_{ij}| > C$ for all $(i,j) \in I \times J$.*

Proof: We may choose integers b'_j for $j \in J$ such that $\sum_{j\in J} b'_j = \sum_{i\in I} a_i$ and $b'_j \equiv b_j \pmod m$ for all $j \in J$. Let x_{ij} be the integers given by Lemma 2.2 for $\{a_i\}_{i\in I}$ and $\{b'_j\}_{j\in J}$. We will show that by adding suitable multiples of m, we can ensure that x_{ij} are pairwise distinct. If for a fixed i, $x_{ij} = x_{ik}$ for $j, k \in J$, $j \neq k$, then we can replace (x_{ij}, x_{ik}) with $(x_{ij} + mN, x_{ik} - mN)$ for some $N \gg 0$. Repeating the process, we can ensure that all x_{ij} are pairwise distinct for a fixed i. Let $j_0 \in J$ be such that $x_{ij_0} > x_{ij}$ for all $j \neq j_0$ and a fixed i. If C is a positive real number, then by replacing x_{ij_0} with $x_{ij_0} + m(|J| - 1)N$ and x_{ij} with $x_{ij} - mN$ for $j \neq j_0$ and $N \gg 0$, we may ensure that for a fixed i, x_{ij} are all distinct and satisfy $|x_{ij}| > C$. If we identify $I = \{1, \ldots, n\}$, where $n = |I|$, then using this argument, we may modify the integers x_{ij} so that for a fixed i, x_{ij} are all distinct and $|x_{i+1j}| > \max_{j\in J} |x_{ij}|$ for $1 \leq i \leq n-1$. Hence, all x_{ij} are pairwise distinct. □

Lemma 2.4. *Let $\overline{\theta} : k_F^\times \to k^\times$ be a character. Then, there exist uniquely determined integers $0 \leq b_\sigma \leq p-1$ indexed by the embeddings $\sigma : k_F \hookrightarrow k$ such that $\overline{\theta}(x) = \prod_\sigma \sigma(x)^{b_\sigma}$ for all $x \in k_F^\times$ and not all $b_\sigma = p-1$.*

Proof: Let us pick an embedding $\sigma_0 : k_F \hookrightarrow k$. Since the character group of $k_F^\times$ is cyclic of order $p^f - 1$ and σ_0 is a generator, there exists a unique integer $0 \leq b < p^f - 1$ such that $\overline{\theta}(x) = \sigma_0(x)^b$ for all $x \in k_F^\times$. We may write $b = \sum_{i=0}^{f-1} b_i p^i$ with $0 \leq b_i \leq p-1$ and not all $b_i = p-1$. The digits b_i are uniquely determined by b. For $0 \leq i \leq f-1$, let $\sigma_i : k_F \hookrightarrow k$ be the embedding $\sigma_i(x) := \sigma_0(x^{p^i})$. Then, $\{\sigma_i : 0 \leq i \leq f-1\}$ is a complete set of embeddings of k_F into k and $\overline{\theta}(x) = \prod_{i=0}^{f-1} \sigma_i(x)^{b_i}$. □

Lemma 2.5. *Let E be an unramified extension of F of degree $d > 1$. Let $\overline{\theta} : \mathcal{O}_E^\times \to k^\times$ be a character, and $\psi : \mathcal{O}_F^\times \to \mathcal{O}^\times$ be a character such that $\psi(x) = \prod_{\sigma\in\Sigma_F} \sigma(x)^{a_\sigma}$ for $(a_\sigma)_{\sigma\in\Sigma_F} \in \mathbb{Z}^{\Sigma_F}$. Assume that $\psi(x) \equiv \overline{\theta}(x) \pmod \varpi$ for all $x \in \mathcal{O}_F^\times$. Then, there exists a character $\theta : \mathcal{O}_E^\times \to \mathcal{O}^\times$, $\theta(x) = \prod_{\tau\in\Sigma_E} \tau(x)^{k_\tau}$, such that the following hold:*

1. The integers k_τ are pairwise distinct;
2. θ lifts $\overline{\theta}$;
3. $\theta(x) = \psi(x)$ for all $x \in \mathcal{O}_F^\times$.

Proof: Let F_0 be the maximal unramified subextension of F and let E_0 be the unramified extension of F_0 of degree d. Then, E is the compositum of F and E_0, and Σ_E is in bijection with the set of pairs $(\sigma, \tau_0) \in \Sigma_F \times \Sigma_{E_0}$ such that $\sigma|_{F_0} = \tau_0|_{F_0}$. We may further identify Σ_{E_0} with the set of embeddings of k_E into k. Since $\overline{\theta}$ is trivial on $1 + \mathfrak{p}_E$, we may consider it as a character of $k_E^\times$, and thus by Lemma 2.4, there exist integers $0 \leq b_{\tau_0} \leq p-1$ for $\tau_0 \in \Sigma_{E_0}$ such that $\overline{\theta}(x) = \prod_{\tau_0\in\Sigma_{E_0}} \tau_0(x)^{b_{\tau_0}}$ for all $x \in k_E^\times$. For $\sigma_0 \in \Sigma_{F_0}$ let $I_{\sigma_0} = \{\sigma \in \Sigma_F : \sigma|_{F_0} = \sigma_0\}$, let $J_{\sigma_0} = \{\tau_0 \in \Sigma_{E_0} : \tau_0|_{F_0} = \sigma_0\}$ and $\Sigma_{E,\sigma_0} := \{\tau \in \Sigma_E : \tau|_{F_0} = \sigma_0\}$. The map $\tau \mapsto (\tau|_F, \tau|_{E_0})$ induces a bijection between Σ_{E,σ_0} and $I_{\sigma_0} \times J_{\sigma_0}$. It follows from Lemma 2.4 that there exist uniquely determined integers $0 \leq c_{\sigma_0} \leq p-1$ indexed by $\sigma_0 \in \Sigma_{F_0}$ such that $\overline{\theta}(x) = \prod_{\sigma_0 \in \Sigma_{F_0}} \sigma_0(x)^{c_{\sigma_0}}$ for all $x \in k_F^\times$ and not all $c_{\sigma_0} = p-1$. The relation $\psi(x) \equiv \overline{\theta}(x) \pmod{\varpi}$ for all $x \in \mathcal{O}_F^\times$ and uniqueness of c_{σ_0} imply that for all $\sigma_0 \in \Sigma_{F_0}$, we have

$$\sum_{\sigma \in I_{\sigma_0}} a_\sigma \equiv c_{\sigma_0} \equiv \sum_{\tau_0 \in J_{\sigma_0}} b_{\tau_0} \pmod{p-1}.$$

Let us label the embeddings $\Sigma_{F_0} = \{\sigma_i : 1 \leq i \leq f\}$. Using Lemma 2.3 for each i with $1 \leq i \leq f$, we may find pairwise distinct integers k_τ for $\tau \in \Sigma_{E,\sigma_i}$ such that $\sum_{\tau|_F = \sigma} k_\tau = a_\sigma$ for all $\sigma \in I_{\sigma_i}$ and $\sum_{\tau|_{E_0} = \tau_0} k_\tau \equiv b_{\tau_0} \pmod{p-1}$ for all $\tau_0 \in J_{\sigma_i}$. Moreover, $\min_{\tau \in \Sigma_{E,\sigma_{i+1}}} |k_\tau| > \max_{\tau \in \Sigma_{E,\sigma_i}} |k_\tau|$ for $1 \leq i \leq f-1$. In particular, the integers k_τ for $\tau \in \Sigma_E$ are pairwise distinct, and the character θ satisfies conditions (2) and (3). □

Proposition 2.6. *If $\overline{\rho} : G_F \to \mathrm{GL}_d(k)$ is absolutely irreducible and $\psi : G_F \to \mathcal{O}^\times$ is a crystalline character lifting* $\det \overline{\rho}$*, then there is a crystalline lift $\rho : G_F \to \mathrm{GL}_d(\mathcal{O})$ of $\overline{\rho}$ with regular Hodge–Tate weights such that* $\det \rho = \psi$.

Proof: Since $\overline{\rho}$ is absolutely irreducible, we have $\overline{\rho} \cong \mathrm{Ind}_{G_E}^{G_F} \overline{\theta}$, where E is an unramified extension of F of degree d and $\overline{\theta} : G_E \to k^\times$ is a character. Since E/F is unramified $N_F^E E^\times = \varpi_F^{d\mathbb{Z}} \mathcal{O}_F^\times$. It follows from Lemma 2.1 that

$$\psi(\mathrm{Art}_F(x)) \equiv \overline{\theta}(\mathrm{Art}_E(x)) \pmod{\varpi}, \quad \forall x \in \mathcal{O}_F^\times. \qquad \textbf{[2]}$$

Let $\varphi \in G_F$ be any element which maps to $\mathrm{Art}_F(\varpi_F)$ in G_F^{ab}. Since $N_F^E(\varpi_F) = \varpi_F^d$ and ϖ_F is a uniformizer of E, it follows from the diagram Eq. **1** that there is a uniformizer ϖ_E of E such that the image of φ^d in G_E^{ab} is equal to $\mathrm{Art}_E(\varpi_E)$. Since E/F is cyclic and the image of φ generates $\mathrm{Gal}(E/F)$, we have

$$\psi(\mathrm{Art}_F(\varpi_F)) \equiv (-1)^{d-1} \overline{\theta}(\mathrm{Art}_E(\varpi_E)) \pmod{\varpi}, \qquad \textbf{[3]}$$

by the last part of Lemma 2.1.

We first lift $\overline{\theta}$ to a crystalline character θ of G_E and then induce it to G_F. Since Art_E induces an isomorphism between G_E^{ab} and the profinite completion of $E^\times$, it is enough to define $\theta(\mathrm{Art}_E(x))$ for $x \in E^\times$. We let

$$\theta(\mathrm{Art}_E(\varpi_E)) := (-1)^{d-1} \psi(\mathrm{Art}_F(\varpi_F)). \qquad \textbf{[4]}$$

Using Eq. **2** and Lemma 2.5, we can find a set of pairwise distinct integers k_τ for $\tau \in \Sigma_E$ and define

$$\theta(\mathrm{Art}_E(x)) = \prod_{\tau \in \Sigma_E} \tau(x)^{k_\tau}, \quad \forall x \in \mathcal{O}_E^\times,$$

so that $\theta(\mathrm{Art}_E(x)) \equiv \overline{\theta}(\mathrm{Art}_E(x)) \pmod{\varpi}$ for all $x \in \mathcal{O}_E^\times$ and $\theta(\mathrm{Art}_E(x)) = \psi(\mathrm{Art}_F(x))$ for all $x \in \mathcal{O}_F^\times$. It follows from Eqs. **2**, **3**, and **4** that θ lifts $\overline{\theta}$. By (14, Proposition B.4), θ is crystalline with Hodge–Tate weights $(k_\tau)_{\tau \in \Sigma_E}$. Since E/F is unramified $\rho := \mathrm{Ind}_{G_E}^{G_F} \theta$ is a crystalline representation lifting $\overline{\rho}$, and since the integers k_τ are pairwise distinct, ρ has regular Hodge–Tate weights; (17, Corollary 7.1.2). By Lemma 2.1

$$(\det \rho)(\mathrm{Art}_F(x)) = \theta(\mathrm{Art}_E(x)) = \psi(\mathrm{Art}_F(x)), \quad \forall x \in \mathcal{O}_F^\times, \qquad \textbf{[5]}$$

and

$$(\det \rho)(\mathrm{Art}_F(\varpi_F)) = \det \rho(\varphi) = (-1)^{d-1} \theta(\varphi^d) = (-1)^{d-1} \theta(\mathrm{Art}_E(\varpi_E)) = \psi(\mathrm{Art}_F(\varpi_F)). \qquad \textbf{[6]}$$

It follows from Eqs. **5** and **6** that $\det \rho = \psi$. □

Now, we use the Emerton–Gee stack defined in ref. 1 to extend Proposition 2.6 to all $\overline{\rho}$.

Proposition 2.7. *For any continuous representation $\overline{\rho} : G_F \to \mathrm{GL}_d(k)$ and a crystalline character $\psi : G_F \to \mathcal{O}^\times$ lifting* $\det \overline{\rho}$*, there is a crystalline lift $\rho : G_F \to \mathrm{GL}_d(\mathcal{O})$ of $\overline{\rho}$ (after possibly enlarging L) with regular Hodge–Tate weights and* $\det \rho = \psi$.

Proof: We argue by induction on the number of irreducible subquotients of $\overline{\rho}$ by repeatedly applying (1, Theorem 6.3.2) in the induction step. If $\overline{\rho}$ is absolutely irreducible, then we are done by Proposition 2.6. For the inductive step, we may write

$$0 \to \overline{\rho}_1 \to \overline{\rho} \to \overline{\rho}_2 \to 0,$$

with $\overline{\rho}_2$ being absolutely irreducible. By (2, Lemma 6.5), there is a crystalline character ψ_1 lifting $\det \overline{\rho}_1$. Then, $\psi_2 := \psi \psi_1^{-1}$ is a crystalline character lifting $\det \overline{\rho}_2$. By the induction hypothesis, we may assume that $\overline{\rho}_1, \overline{\rho}_2$ have crystalline lifts ρ_1, ρ_2 with regular Hodge–Tate weights satisfying $\det \rho_1 = \psi_1$ and $\det \rho_2 = \psi_2$.

Let d_1 and d_2 be the dimensions of $\overline{\rho}_1$ and $\overline{\rho}_2$, respectively. We may choose $N \gg 0$ such that the Hodge–Tate weights of $\rho_1' := \rho_1 \otimes \chi_{\mathrm{cyc}}^{d_2(p-1)N}$ are all positive, and the Hodge–Tate weights of $\rho_2' := \rho_2 \otimes \chi_{\mathrm{cyc}}^{-d_1(p-1)N}$ are all negative, where χ_{cyc} is the p-adic cyclotomic character. Since $\chi_{\mathrm{cyc}}^{p-1}$ is a crystalline character, which is trivial modulo ϖ, ρ_1' and ρ_2' are crystalline lifts of $\overline{\rho}_1$ and $\overline{\rho}_2$, respectively. Moreover, the Hodge–Tate weights of ρ_1' are *slightly less than* the Hodge–Tate weights of ρ_2', in the terminology of (1, Section 6.3). (We note that χ_{cyc} has Hodge–Tate 1 under our conventions and -1 under the convention of ref. 1. This leads to the reversal of inequalities.) Further,

$$\psi_1' := \det \rho_1' = \psi_1 \chi_{\mathrm{cyc}}^{d_1 d_2 (p-1)N}, \quad \psi_2' := \det \rho_2' = \psi_2 \chi_{\mathrm{cyc}}^{-d_1 d_2 (p-1)N}.$$

Hence, $\psi_1'\psi_2' = \psi$. Then, we are in the situation of (1, Theorem 6.3.2), which yields a lift (after possibly enlarging L) of $0 \to \overline{\rho}_1 \to \overline{\rho} \to \overline{\rho}_2 \to 0$ given by

$$0 \to \rho_1'' \to \rho \to \rho_2' \to 0,$$

such that ρ and ρ_1'' are crystalline and ρ_1'' has the same Hodge–Tate weights as ρ_1'. This implies that $\psi_1'' := \det \rho_1''$ and ψ_1' are both crystalline with the same Hodge–Tate weights, and hence $\eta := \psi_1'(\psi_1'')^{-1}$ is an unramified character by (14, Proposition B.4). Since both ψ_1' and ψ_1'' lift $\det \overline{\rho}_1$, η is is trivial modulo ϖ. After enlarging L, we may find an unramified character $\chi : G_F \to 1 + \mathfrak{p}_L$, such that $\chi^d = \eta$. Then, $\rho \otimes \chi$ is a crystalline lift of $\overline{\rho}$ with regular Hodge–Tate weights and determinant ψ. $\square$

Corollary 2.8. *Let $\chi : \mu_{p^\infty}(F) \to \mathcal{O}^\times$ be a character. Then, (after possibly enlarging L), there exists a crystalline lift $\rho : G_F \to \mathrm{GL}_d(\mathcal{O})$ of $\overline{\rho}$ with regular Hodge–Tate weights such that $(\det \rho)(\mathrm{Art}_F(x)) = \chi(x)$ for all $x \in \mu_{p^\infty}(F)$.*

Proof: By (2, Lemma 6.5), there exists a crystalline character $\psi : G_F \to \mathcal{O}^\times$ lifting $\det \overline{\rho}$ such that $\psi(\mathrm{Art}_F(x)) = \chi(x)$ for all $x \in \mu_{p^\infty}(F)$. The assertion follows from Proposition 2.7. $\square$

3. Zariski Density

Now fix a continuous representation $\overline{\rho} : G_F \to \mathrm{GL}_d(k)$ and assume that its irreducible subquotients are absolutely irreducible; this can always be achieved by replacing L with an unramified extension since the image of $\overline{\rho}$ is finite. Let $R_{\overline{\rho}}^{\square}$ denote the universal framed deformation ring of $\overline{\rho}$ with universal representation $\rho^{\mathrm{univ}} : G_F \to \mathrm{GL}_d(R_{\overline{\rho}}^{\square})$. We will first describe the irreducible components of the rigid analytic space $\mathfrak{X}_{\overline{\rho}}^{\square} := (\mathrm{Spf}\, R_{\overline{\rho}}^{\square})^{\mathrm{rig}}$ using the results of ref. 2.

By local class field theory, there is a group homomorphism

$$\mu \hookrightarrow F^\times \xrightarrow{\mathrm{Art}_F} G_F^{\mathrm{ab}} \xrightarrow{\det \rho^{\mathrm{univ}}} (R_{\overline{\rho}}^{\square})^\times,$$

which induces a homomorphism of $\mathcal{O}$-algebras $\mathcal{O}[\mu] \to R_{\overline{\rho}}^{\square}$, where $\mu := \mu_{p^\infty}(F)$. Since $\mathcal{O}[\mu][1/p] \cong \prod_{\chi:\mu\to\mathcal{O}^\times} L$, we get a decomposition

$$R_{\overline{\rho}}^{\square}[1/p] \cong \prod_{\chi:\mu\to\mathcal{O}^\times} R_{\overline{\rho}}^{\square,\chi}[1/p],$$

where $R_{\overline{\rho}}^{\square,\chi} = R_{\overline{\rho}}^{\square} \otimes_{\mathcal{O}[\mu],\chi} \mathcal{O}$ for each $\chi : \mu \to \mathcal{O}^\times$. The rings $R_{\overline{\rho}}^{\square,\chi}$ are $\mathcal{O}$-flat, normal, integral domains, and $R_{\overline{\rho}}^{\square}$ is $\mathcal{O}$-flat by (2, Theorem 1.2). It follows from (18, Theorem 2.3.1) that the irreducible components of $\mathfrak{X}_{\overline{\rho}}^{\square}$ are given by $\mathfrak{X}_{\overline{\rho}}^{\square,\chi} := (\mathrm{Spf}\, R_{\overline{\rho}}^{\square,\chi})^{\mathrm{rig}}$ for $\chi : \mu \to \mathcal{O}^\times$.

Remark 3.1: If $x \in \mathfrak{X}_{\overline{\rho}}^{\square}$ is a closed point, then its residue field $\kappa(x)$ is a finite extension of L. The reduction map $R_{\overline{\rho}}^{\square} \twoheadrightarrow \kappa(x)$ gives rise to a representation

$$\rho_x : G_F \xrightarrow{\rho^{\mathrm{univ}}} \mathrm{GL}_d(R_{\overline{\rho}}^{\square}) \to \mathrm{GL}_d(\kappa(x)).$$

If χ_x denotes the map $\mu \to G_F^{\mathrm{ab}} \xrightarrow{\det \rho_x} \kappa(x)^\times$, then χ_x actually lands in $\mathcal{O}^\times$, and we have $x \in \mathfrak{X}_{\overline{\rho}}^{\square,\chi_x}$.

Definition 3.2: A point $x \in \mathfrak{X}_{\overline{\rho}}^{\square}$ is regular crystalline if ρ_x is crystalline with regular Hodge–Tate weights.

Theorem 3.3. *The subset $\mathcal{S}_{\mathrm{cr}}$ of regular crystalline points in $\mathfrak{X}_{\overline{\rho}}^{\square}$ is Zariski dense.*

Proof: It follows from Corollary 2.8 and the description of the irreducible components of $\mathfrak{X}_{\overline{\rho}}^{\square}$ given above that every irreducible component contains a regular crystalline point. By (5, Proposition 5.10), the Zariski closure of the regular crystalline points in $\mathfrak{X}_{\overline{\rho}}^{\square}$ is a union of irreducible components. Thus, $\mathcal{S}_{\mathrm{cr}}$ is Zariski dense in $\mathfrak{X}_{\overline{\rho}}^{\square}$. $\square$

4. Deformation Rings with Fixed Determinant

Let $\psi : G_F \to \mathcal{O}^\times$ be a crystalline character lifting $\det \overline{\rho}$. It follows from Proposition 2.7 that (possibly after replacing L by a finite extension) there is a crystalline lift $\rho : G_F \to \mathrm{GL}_d(\mathcal{O})$ of $\overline{\rho}$ with regular Hodge–Tate weights $\underline{k}(\rho)$ such that $\det \rho = \psi$.

Let $\mathcal{S}_{\mathrm{cr}}^{\mathrm{cn}}$ be the subset of $\mathfrak{X}_{\overline{\rho}}^{\square}$ consisting of x such that ρ_x is crystalline with regular Hodge–Tate weights $\underline{k}(\rho_x)$ and

$$k_{\sigma,i}(\rho_x) \equiv k_{\sigma,i}(\rho) \pmod{dt},$$

for all $\sigma \in \Sigma$ and $1 \le i \le d$, where t is the order of $\mu_{\mathrm{tors}}(F)$.

Lemma 4.1. *If $x \in \mathcal{S}_{\mathrm{cr}}^{\mathrm{cn}}$, then there is a crystalline character $\theta : G_F \to \overline{\mathbb{Q}}_p^\times$ lifting the trivial character, such that $\theta(\mathrm{Art}_F(\zeta)) = 1$ for all $\zeta \in \mu_{\mathrm{tors}}(F)$ and $\det(\rho_x \otimes \theta) = \psi$.*

Proof: Since both ρ and ρ_x are crystalline, the character $\eta := \psi(\det \rho_x)^{-1}$ is crystalline with Hodge–Tate weights $k_\sigma(\eta) = \sum_{i=1}^d (k_{\sigma,i}(\rho) - k_{\sigma,i}(\rho_x))$ for $\sigma \in \Sigma$. It follows from the definition of $\mathcal{S}_{\mathrm{cr}}^{\mathrm{cn}}$ that dt divides $k_\sigma(\eta)$. Let $\underline{k} \in \mathbb{Z}^\Sigma$ be the tuple $(k_\sigma(\eta)/d)_{\sigma\in\Sigma}$, then the character $\chi_{\underline{k}}$ is crystalline and $\eta\chi_{\underline{k}}^{-d}$ is unramified. Since $\chi_{\underline{k}}$ is a t-th power of a character of G_F, it is trivial on the torsion subgroup of G_F^{ab} and hence $\chi_{\underline{k}}(g) \equiv 1 \pmod{\varpi}$ for all $g \in I_F$. Since $\chi_{\underline{k}}(\mathrm{Art}_F(\varpi_F)) = 1$ by definition, we conclude that $\chi_{\underline{k}}$ lifts the trivial character. Since both ρ_x and ρ lift $\overline{\rho}$, η lifts the trivial character. In particular, $\eta(\mathrm{Art}_F(\varpi_F)) \in 1 + \mathfrak{m}_{\overline{\mathbb{Z}}_p}$. We may choose $y \in 1+\mathfrak{m}_{\overline{\mathbb{Z}}_p}$ such that $y^d = \eta(\mathrm{Art}_F(\varpi_F))$, and let $\alpha : G_F \to 1+\mathfrak{m}_{\overline{\mathbb{Z}}_p}$ be an unramified character such that $\alpha(\mathrm{Art}_F(\varpi_F)) = y$. Then, $\theta := \alpha\chi_{\underline{k}}$ satisfies all the conditions. $\square$

Theorem 4.2. *The Zariski closure of $\mathcal{S}_{\mathrm{cr}}^{\mathrm{cn}}$ in $\mathfrak{X}_{\overline{\rho}}^{\square}$ is a union of irreducible components of $\mathfrak{X}_{\overline{\rho}}^{\square}$.*

Before proving the theorem, let us note the following consequence.

Corollary 4.3. *The subset $\mathcal{S}_{\mathrm{cr}}^{\psi}$ of $\mathfrak{X}_{\overline{\rho}}^{\square,\psi}$ consisting of x such that ρ_x is crystalline with regular Hodge–Tate weights is Zariski dense in $\mathfrak{X}_{\overline{\rho}}^{\square,\psi}$.*

Proof: Let $\mathcal{X} : \mathfrak{A}_{\mathcal{O}} \to$ Sets be the functor such that $\mathcal{X}(A)$ is the set of characters $\theta_A : G_F \to 1 + \mathfrak{m}_A$, which are trivial on $\mathrm{Art}_F(\mu_{\mathrm{tors}}(F))$. This functor is prorepresentable by $\mathcal{O}(\mathcal{X}) \cong \mathcal{O}[[x_0, \dots, x_n]]$, with $n = [F : \mathbb{Q}_p]$. The map

$$D_{\overline{\rho}}^{\square,\psi}(A) \times \mathcal{X}(A) \to D_{\overline{\rho}}^{\square}(A), \quad (\rho_A, \theta_A) \mapsto \rho_A \otimes_A \theta_A,$$

induces a homomorphism of local $\mathcal{O}$-algebras $R_{\overline{\rho}}^{\square} \to R_{\overline{\rho}}^{\square,\psi} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}(\mathcal{X})$. It follows from Corollary 5.2 and Lemma 5.3 in ref. 2 that this map is finite and becomes étale after inverting p. Thus, the rigid analytic space $\mathcal{X}^{\mathrm{rig}} := (\mathrm{Spf}\, \mathcal{O}(\mathcal{X}))^{\mathrm{rig}}$ is an open polydisc, and the induced map on rigid analytic spaces

$$\varphi : \mathfrak{X}_{\overline{\rho}}^{\square,\psi} \times_{\mathrm{Sp}\, L} \mathcal{X}^{\mathrm{rig}} \to \mathfrak{X}_{\overline{\rho}}^{\square},$$

is finite and étale. Let $\mathcal{S}'$ be the subset of $\mathfrak{X}_{\overline{\rho}}^{\square,\psi} \times_{\mathrm{Sp}\, L} \mathcal{X}^{\mathrm{rig}}$ corresponding to pairs (ρ', θ) such that $\rho' \in \mathcal{S}_{\mathrm{cr}}^{\psi}$ and θ is crystalline. We note that $\mathcal{S}'$ is nonempty since it contains a point corresponding to the pair $(\rho, \mathbf{1})$. Let $\mathfrak{V}$ be the Zariski closure of $\mathcal{S}'$ inside $\mathfrak{X}_{\overline{\rho}}^{\square,\psi} \times_{\mathrm{Sp}\, L} \mathcal{X}^{\mathrm{rig}}$. We have shown in (2, Theorem 5.6) that $R_{\overline{\rho}}^{\square,\psi}$ is a normal integral domain. Thus, if $\mathcal{S}_{\mathrm{cr}}^{\psi}$ is not Zariski dense in $\mathfrak{X}_{\overline{\rho}}^{\square,\psi}$, then $\mathfrak{V}$ has positive codimension in $\mathfrak{X}_{\overline{\rho}}^{\square,\psi} \times_{\mathrm{Sp}\, L} \mathcal{X}^{\mathrm{rig}}$. Since φ is finite, $\varphi(\mathfrak{V})$ is a closed subset of $\mathfrak{X}_{\overline{\rho}}^{\square}$ of positive codimension. Lemma 4.1 implies that $\varphi(\mathfrak{V})$ contains the set $\mathcal{S}_{\mathrm{cr}}^{\mathrm{cn}}$. Theorem 4.2 implies that $\varphi(\mathfrak{V})$ contains an irreducible component of $\mathfrak{X}_{\overline{\rho}}^{\square}$. We have shown in (2, Theorem 1.1) that $R_{\overline{\rho}}^{\square}$ is complete intersection and hence equidimensional. Thus, $\dim \varphi(\mathfrak{V}) = \dim \mathfrak{X}_{\overline{\rho}}^{\square}$, yielding a contradiction. $\square$

We will now prove Theorem 4.2. We follow the proof of (5, Proposition 5.10), where the analogous statement is shown for the set of all regular crystalline points, i.e., without imposing the congruence condition on the Hodge–Tate weights.

We recall following (19, Section 3.3.1) that a subset $\mathcal{S}$ of a rigid analytic space X accumulates at $x \in X$ if there is a basis of open neighborhoods U of x such that $\mathcal{S} \cap U$ is Zariski dense in U. Let $\mathcal{W}$ be a rigid analytic space over L parameterizing the continuous characters of $\mathcal{O}_F^\times$.

Lemma 4.4. *Let $x \in \mathcal{W}^d$ correspond to a d-tuple of characters $(\chi_{\underline{k}_1}, \dots, \chi_{\underline{k}_d})$ for some $\underline{k}_i \in \mathbb{Z}^\Sigma$ for $1 \le i \le d$. Let C be a real positive number, and let $\mathcal{S}'$ be the subset of $\mathcal{W}^d$ corresponding to d-tuples of characters $(\chi_{\underline{k}'_1}, \dots, \chi_{\underline{k}'_d})$ with $\underline{k}'_i \in \mathbb{Z}^\Sigma$ for $1 \le i \le d$ such that $k'_{\sigma,i} - k'_{\sigma,i+1} > C$ and $k'_{\sigma,i} \equiv k_{\sigma,i} \pmod{dt}$ for all $\sigma \in \Sigma$ and $i \ge 1$. Then, $\mathcal{S}'$ accumulates at x.*

Proof: This follows from the argument of (20, Lemme 2.7). $\square$

We will now prove the analog of (5, Proposition 5.9). Let $X_{\mathrm{tri}}^{\square}(\overline{\rho}) \subset \mathfrak{X}_{\overline{\rho}}^{\square} \times \mathcal{T}_L^d$ be the trianguline variety (7, Section 2.1). Let $\mathcal{S}''$ be the subset of $X_{\mathrm{tri}}^{\square}(\overline{\rho})$ such that $x \in \mathcal{S}''$ correspond to pairs (ρ_x, δ_x) with $\rho_x \in \mathcal{S}_{\mathrm{cr}}^{\mathrm{cn}}$, defined at the beginning of the section, and ρ_x is φ^f-generic and noncritical (5, Definition 5.5) for these notions as well as for the term *benign*.

Proposition 4.5. *Suppose $x = (\rho_x, \delta_x) \in X^{\square}_{\mathrm{tri}}(\overline{\rho})$ is benign. Then, $\mathcal{S}''$ accumulates at x.*

Proof: The argument of (6, Proposition 4.1.4) goes through; the only change is that in the last sentence of the proof, we use Lemma 4.4. □

Proof of Theorem 4.2: Let Z be an irreducible component of the closure of $\mathcal{S}^{\mathrm{cn}}_{\mathrm{cr}}$, and let Z^{sm} be the smooth locus inside Z. Since Z^{sm} is open and $\mathcal{S}^{\mathrm{cn}}_{\mathrm{cr}}$ is dense in Z, there is $x \in Z^{\mathrm{sm}} \cap \mathcal{S}^{\mathrm{cn}}_{\mathrm{cr}}$. Let ρ_x be the corresponding crystalline Galois representation and let $\underline{k}$ be the Hodge–Tate weights of ρ_x. Let $\mathfrak{X}^{\square,\underline{k}}_{\overline{\rho},\mathrm{cr}}$ be the rigid analytic space associated with the crystalline deformation ring of $\overline{\rho}$ with fixed Hodge–Tate weights $\underline{k}$. Let $Z^{\underline{k}}_{\mathrm{cr}}$ be an irreducible component of $\mathfrak{X}^{\square,\underline{k}}_{\overline{\rho},\mathrm{cr}}$ containing x. Then, $\mathfrak{X}^{\square,\underline{k}}_{\overline{\rho},\mathrm{cr}}$ is contained in $\mathcal{S}^{\mathrm{cn}}_{\mathrm{cr}}$ and hence $Z^{\underline{k}}_{\mathrm{cr}}$ is contained in Z. It follows from (21, Theorem 3.3.8) that irreducible components of $\mathfrak{X}^{\square,\underline{k}}_{\overline{\rho},\mathrm{cr}}$ do not intersect. Thus, $Z^{\underline{k}}_{\mathrm{cr}}$ is open in $\mathfrak{X}^{\square,\underline{k}}_{\overline{\rho},\mathrm{cr}}$. By applying (4, Lemma 4.2) with $U = Z^{\mathrm{sm}} \cap Z^{\underline{k}}_{\mathrm{cr}}$, we deduce that there is $y \in U$ such that ρ_y is benign. We note that since the Hodge–Tate weights of ρ_y are equal to $\underline{k}$, they satisfy the congruence condition imposed in the definition of $\mathcal{S}^{\mathrm{cn}}_{\mathrm{cr}}$, and hence, $y \in \mathcal{S}^{\mathrm{cn}}_{\mathrm{cr}}$. The rest of the proof of (5, Proposition 5.10) carries over verbatim by using Proposition 4.5 instead of (5, Proposition 5.9). □

5. Density in Scheme-Theoretic Generic Fiber

Let R be a complete local Noetherian $\mathcal{O}$-algebra with residue field k. Let $\mathfrak{X} = (\mathrm{Spf}\, R)^{\mathrm{rig}}$ be the rigid analytic space associated with the formal scheme $\mathrm{Spf}\, R$, (22, Section 7). The canonical map

$$R[1/p] \to \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}), \tag{7}$$

induces a bijection between the set of maximal ideals of $R[1/p]$ and the set of points of $\mathfrak{X}$. Moreover, if $x \in \mathfrak{X}$ corresponds to a maximal ideal $\mathfrak{m} \subset R[1/p]$, then there is a natural map of local rings

$$R[1/p]_{\mathfrak{m}} \to \mathcal{O}_{\mathfrak{X},x}, \tag{8}$$

compatible with Eq. **7**, which induces an isomorphism on completions (22, Lemma 7.1.9).

Lemma 5.1. *If $\mathcal{S}$ is a Zariski dense subset of $\mathfrak{X}$, then $\mathcal{S}$ is also dense in* $\mathrm{Spec}\, R[1/p]$. *Moreover, if R is $\mathcal{O}$-torsion-free, then $\mathcal{S}$ is dense in* $\mathrm{Spec}\, R$.

Proof: The map Eq. **8** is injective since it induces an isomorphism on completions. This implies that the map Eq. **7** is injective. If $f \in R[1/p]$ vanishes at all $x \in \mathcal{S}$, then by considering f as a function on $\mathfrak{X}$ via Eq. **7**, we deduce that f is zero. Hence, $\mathcal{S}$ is Zariski dense in $\mathrm{Spec}\, R[1/p]$.

If R is $\mathcal{O}$-torsion-free, then the generic points of $\mathrm{Spec}\, R$ and $\mathrm{Spec}\, R[1/p]$ coincide, and hence $\mathcal{S}$ is dense in $\mathrm{Spec}\, R$. □

By (2, Theorem 1.1 and Theorem 1.4), $R^{\square}_{\overline{\rho}}$ and $R^{\square,\psi}_{\overline{\rho}}$ are $\mathcal{O}$-torsion free, so as a direct consequence of Theorem 3.3, Corollary 4.3, and Lemma 5.1, we obtain the following:

Corollary 5.2. *The set $\mathcal{S}_{\mathrm{cr}}$ is Zariski dense in* $\mathrm{Spec}\, R^{\square}_{\overline{\rho}}[1/p]$ *and in* $\mathrm{Spec}\, R^{\square}_{\overline{\rho}}$, *and the same holds for $\mathcal{S}^{\psi}_{\mathrm{cr}}$ in* $\mathrm{Spec}\, R^{\square,\psi}_{\overline{\rho}}[1/p]$ *and in* $\mathrm{Spec}\, R^{\square,\psi}_{\overline{\rho}}$.

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1. M. Emerton, T. Gee, Moduli stacks of étale (φ, Γ)-modules and the existence of crystalline lifts. *Ann. Mat. Stud.* **215**, ix–298 (2023).
2. G. Böckle, A. Iyengar, V. Paškūnas, On local Galois deformation rings. arXiv [Preprint] (2021). http://arxiv.org/abs/2110.01638.
3. G. Chenevier, Sur la densité des représentations cristallines de $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. *Math. Ann.* **355**, 1469–1525 (2013).
4. K. Nakamura, Zariski density of crystalline representations for any p-adic field. *J. Math. Sci. Univ. Tokyo* **21**, 79–127 (2014).
5. A. Iyengar, Deformation theory of the trivial mod p Galois representation for GL_n. *Int. Math. Res. Not. (IMRN)* **22**, 8896–8935 (2020).
6. C. Breuil, E. Hellmann, B. Schraen, A local model for the trianguline variety and applications. *Publ. Math. Inst. Hautes Études Sci.* **130**, 299–412 (2019).
7. C. Breuil, E. Hellmann, B. Schraen, Smoothness and classicality on eigenvarieties. *Invent. Math.* **209**, 197–274 (2017).
8. P. Colmez, Représentations de $GL_2(\mathbb{Q}_p)$ et (ϕ, Γ)-modules. Astérisque (2010), vol. 330, pp. 281–509.
9. M. Kisin, Deformations of $G_{\mathbb{Q}_p}$ and $GL_2(\mathbb{Q}_p)$ representations. Astérisque (2010), vol. 330, pp. 511–528.
10. G. Böckle, Deformation rings for some mod 3 Galois representations of the absolute Galois group of $\mathbb{Q}_3$. Astérisque (2010), vol. 330, pp. 529–542.
11. G. Chenevier, Sur la variété des caractères p-adique du groupe de galois absolu de $\mathbb{Q}_p$ (2010). http://gaetan.chenevier.perso.math.cnrs.fr/articles/lieugalois.pdf.
12. P. Colmez, G. Dospinescu, V. Paškūnas, Irreducible components of deformation spaces: Wild 2-adic exercises. *Int. Math. Res. Not. (IMRN)* **14**, 5333–5356 (2015).
13. G. Böckle, A.-K. Juschka, Irreducibility of versal deformation rings in the (p, p)-case for 2-dimensional representations. *J. Algebra* **444**, 81–123 (2015).
14. B. Conrad, Lifting global Galois representations with local properties. Unpublished note (2011). http://math.stanford.edu/~conrad/papers/locchar.pdf.
15. J. P. Serre, *Local Fields, Graduate Texts in Mathematics* (Springer, Berlin, 1979), vol. 67, pp. viii+241. Translated from the French by Marvin Jay Greenberg.
16. J. P. Serre, "Local class field theory" in *Algebraic Number Theory (Proceedings of Instructional Conference, Brighton, 1965)* (Thompson, Washington, D.C., 1967), pp. 128–161.
17. T. Gee, F. Herzig, D. Savitt, General Serre weight conjectures. *J. Eur. Math. Soc. (JEMS)* **20**, 2859–2949 (2018).
18. B. Conrad, Irreducible components of rigid spaces. *Ann. Inst. Fourier (Grenoble)* **49**, 473–541 (1999).
19. J. Bellaïche, G. Chenevier, Families of Galois representations and Selmer groups. Astérisque (2009), vol. 324, pp. xii+314.
20. G. Chenevier, "Une application des variétés de Hecke des groupes unitaires" in *Shimura Varieties, London Mathematical Society. Lecture Note Series* (Cambridge University Press, Cambridge, 2020), vol. 457, pp. 266–296.
21. M. Kisin, Potentially semi-stable deformation rings. *J. Am. Math. Soc.* **21**, 513–546 (2008).
22. A. J. de Jong, Crystalline Dieudonné module theory via formal and rigid geometry. *Inst. Hautes Études Sci. Publ. Math.* **82**, 5–96 (1995).