EE 779 Advanced Topics in Signal Processing
Assignment 2
Assigned: 29/08/13, Due: 06/09/13
Indian Institute of Technology Bombay

Note

- Submit the starred ($*$) problems and **all** simulations.
- For the simulation problems, you can use the Matlab functions provided by S & M [1] or Hayes [2]. A copy of the files is in moodle.

Problems, Due: 06/09/13

1. [$*$]The estimated autocorrelation sequence of a random process $x(n)$ is

$$r_x(k) = 2, 1, 1, 0.5, 0; \text{ for lags } k = 0, 1, 2, 3, 4;$$

   Estimate the power spectrum of $x(n)$ for each of the following cases.

   (a) $x(n)$ is an AR(2) process.

   (b) $x(n)$ is an MA(2) process.

   (c) $x(n)$ is an ARMA(1,1) process.

2. Given the autocorrelation sequence

$$r_x(k) = 1, 0.8, 0.5, 0.1; \text{ for lags } k = 0, 1, 2, 3;$$

   find the reflection coefficients, Γ_j, the model parameters, $a_j(k)$ and the modeling errors, ϵ_j, for $j = 1, 2, 3$. Use Levinson Durbin's algorithm.

3. [$*$]Determine whether the following statements are True or False. Justify.

   (a) If $r_x(k)$ is an autocorrelation sequence with $r_x(k) = 0$ for $|k| > p$, then $\Gamma_k = 0$ for $|k| > p$.

   (b) Given an autocorrelation sequence, $r_x(k)$ for $k = 0, \ldots, p$, if the $(p+1) \times (p+1)$ Toeplitz matrix

$$\mathbf{R}_p = Toep\{r_x(0), r_x(1), \ldots, r_p(p)\}$$

   is positive definite, then

$$\mathbf{r}_x = [r_x(0), r_x(1), \ldots, r_x(p), 0, 0, \ldots]^T$$

   will always be a valid autocorrelation sequence, i.e., extending $r_x(k)$ with zeros is a valid autocorrelation extension. Note: A Toeplitz matrix is described by one of its column or row.

   (c) If $r_x(k)$ is periodic, then Γ_j will be periodic with the same period.

4. A random process may be classified in terms of the properties of the prediction error sequence ϵ_k that is produced when fitting an all–pole model to the process. Listed below are different classifications for the error sequence:

   (a) $\epsilon_k = c > 0$ for all $k \geq 0$.

   (b) $\epsilon_k = c > 0$ for all $k \geq k_0$ for some $k_0 > 0$.

   (c) $\epsilon_k \to c$ as $k \to \infty$ where $c > 0$.

   (d) $\epsilon_k \to 0$ as $k \to \infty$

   (e) $\epsilon = 0$ for all $k \geq k_0$ for some $k_0 > 0$.

For each of these classifications, describe as completely as possible the characteristics that may be attributed to the process and its power spectrum.

5. Show that the optimal one-step linear predictor for an AR(p) process based on the infinite past is the same as one based on the previous p samples. *Hint:* Show that the solution of the Yule-Walker equation yields $a(k) = 0$ for $k > p$ for an AR(p) process.

6. [∗]Estimates are made of the correlation function of a particular signal and the values obtained are: $r_x(0) = 7.24, r_x(1) = 3.6$. Determine the parameters of the MA(1) model:

$$H(z) = b_0 + b_1 z^{-1},$$

which matches these correlation values using:

   (a) Direct solution of the Yule-Walker MA equations.

   (b) By spectral factorization.

Sketch the power spectral estimate obtained using this MA model. Fit a AR(1) model for the given correlation data and sketch the resulting spectral estimate. Is this estimate better than that obtained using the MA model ?

7. In the MUSIC algorithm, finding the peaks of the frequency estimation function

$$\mathrm{P}_{\mathrm{MUSIC}} = \frac{1}{\sum\limits_{i=P+1}^{M} |\mathbf{e}^H \mathbf{v}_i|^2}$$

is equivalent to finding the minima of the denominator. Here M is the number of samples, $\mathbf{e}$ represents a complex exponential, P is the number of complex exponentials, and $\mathbf{v}_i$ are the normalized eigenvectors of the input correlation matrix. Show that the minima of the denominator is equivalent to the maxima of

$$\sum_{i=1}^{P} |\mathbf{e}^H \mathbf{v}_i|^2.$$

8. [∗]The 3×3 autocorrelation matrix of a harmonic process is

$$\mathbf{R}_x = \begin{bmatrix} 3 & -j & -1 \\ j & 3 & -j \\ -1 & j & 3 \end{bmatrix}.$$

   (a) Using Pisarenko harmonic decomposition, find the complex exponential frequencies and the variance of the white noise.

   (b) Repeat (a) using MUSIC algorithm.

   (c) Repeat (a) using Min-norm method.

9. In ESPRIT technique, the signal space eigenvector matrices $\mathbf{V}_{S1}$ and $\mathbf{V}_{S2}$ are obtained as

$$\mathbf{V}_{S1} = [\mathbf{I}_{\tilde{M}} \quad \mathbf{0}]\mathbf{V}_S \text{ and } \mathbf{V}_{S2} = [\mathbf{0} \quad \mathbf{I}_{\tilde{M}}]\mathbf{V}_S.$$

The translation matrix $\mathbf{\Phi}$, whose eigenvalues will be used to estimate the complex frequencies, can be obtained by solving $\widehat{\mathbf{V}}_{S2} \simeq \widehat{\mathbf{V}}_{S1}\mathbf{\Phi}$. Obtain the least-squares solution for $\mathbf{\Phi}$.

Simulations, Due: 23/09/13

1. In this simulation you will use the same data set you used in your assignment 1. You will first estimate the AR parameters for the data and use these parameters to estimate the power spectrum.
   (a) Use the autocorrelation method to estimate a 3×3 Toeplitz correlation matrix for the signal data. Show this matrix in your report.
   (b) Solve the Yule-Walker equations corresponding to the matrix generated in part (a) to obtain the second-order linear prediction filter parameters and the prediction error variance.
   (c) Apply the filter to the original data set and generate the prediction error signal. Plot this signal and compute its variance. Does it compare well with the theoretical prediction error variance you obtained in part (b).
   (d) Take the upper $2 times 2$ block of the correlation matrix you generated in part (a) and solve for the coefficients and prediction error variance of a first-order linear predictive filter. How does this first-order prediction filter compare to the second-order filter you generated in part (b).
   (e) Compute and plot AR power spectral estimates for the data sets using the first-order model parameters.
   (f) Compute and plot AR power spectral estimates for the data sets using the second-order model parameters.
   (g) Compare the power spectrum obtained using the periodogram method (best case) with that obtained using the AR models.

2. The data sets $R01$, $R10$, $R40$ and $I01$, $I10$, $I40$ contain 32 samples of the real and imaginary parts respectively of the following complex signal in white noise:

$$x[n] = s[n] + Kw[n]$$

   for $K = 0.01$, $K = 0.10$, and $K = 0.40$. The signal $s[n]$ is defined by

$$s[n] = e^{\frac{j3\pi n}{8}} + e^{\frac{j5\pi n}{8}} + e^{\frac{j\pi n}{2}}$$

   and $w[n]$ is a zero-mean, unit-variance white noise sequence.
   (a) Compute the periodogram for $K = 0.01$ and plot it in the range $0 \leq \omega \leq \pi$. This will be the reference plot.
   (b) Plot and compare the spectral estimates for each noise value using the following methods.
      i. An AR model using the autocorrelation method. You can use $p = 7$.
      ii. An AR model using the covariance method. You can use $p = 7$.
      iii. MUSIC method. You can use a 8×8 autocorrelation matrix. Use the covariance method to estimate the correlation matrix.
      iv. Minimum-norm method. You can use a 8×8 autocorrelation matrix. Use the covariance method to estimate the correlation matrix.

      Place the plots close by for easy comparison. Also have a plot where all four methods can be overlaid on a relative dB scale.

Reference

1. Petre Stoica and Randolph Moses, "Spectral analysis of signals", Prentice Hall, 2005. (Indian edition available)
2. Monson H. Hayes, "Statistical signal processing and modeling", Wiley India Pvt. Ltd., 2002. (Indian edition available)
3. Charles W. Therrien, "Discrete random signals and statistical signal processing", Charles W. Therrien, 2004.