HQASTS402_1117

PERMUTATION AND COMBINATION

Fundamental principles of Counting

Addition Rule

If a work can be performed in m different ways and a second independent work can be performed in n different ways, either of the two works can be performed in (m + n) different ways.

Multiplication Rule

If a work can be performed in m different ways and following which a second work can be performed in n different ways, then the two works in succession can be performed in m × n different ways.

Application 1

How many numbers of four digits can be formed with the digits 1, 3, 5 and 7? (repetition of digits is not allowed)

Solution

Forming 4 digit numbers can be equated to the work of filling the boxes given below.

Here, first box can be filled in 4 ways (that is by filling any one of 1 or 3 or 5 or 7).

Second box can be filled in 3 ways (that is by filling any one of the remaining 3 numbers).

Third box can be filled in 2 ways (that is by filling any one of the remaining 2 numbers).

And the fourth box can be filled by the remaining 1 number i.e. 1 way.

So, by applying multiplication rule,

Number of 4 digit numbers = 4 × 3 × 2 × 1 = 24

Application 2

How many 5-digit secret codes can be constructed using the digits 0 to 9, if each number starts with 98 and no digit appears more than once?

Solution

Constructing 5-digit secret codes can be equated to work of filling the boxes given below.

9	8			

Now, the box next to 8 can be filled by 8 i.e., (10 – 2) ways.

Fourth box can filled by 7 i.e., (10 – 3) ways.

Fifth box can be filled by 6 i.e., (10 – 4) ways.

By applying multiplication rule,

Number of secret codes = 8 × 7 × 6 = 336

Application 3

In a team of data entry operators, there are 15 male members and 20 female members. To lead the team a leader or leaders has/have to be selected from the team.

(i) Find the number of ways a male and a female leader can be selected.

(ii) Find the number of ways a leader can be selected from male or female members.

Solution

(i) Selecting male and female leaders can be equated to filling the M and F boxes respectively.

M	F

Now, M box can be filled by 15 ways and F box can filled by 20 ways.

So, both the leaders can be selected in 15 × 20 = 300 ways.

(By multiplication principle of counting)

(ii) Selecting a male or a female leader can be done in 15 + 20 = 35 ways. (By addition principle of counting)

PERMUTATION

A permutation is an arrangement in a defined order of a number of objects taken some or all at a time.

Permutations when all the objects are distinct:

The number of permutations of n different objects taken r at a time, where $0 < r \le n$ is $n(n-1)(n-2)\ldots(n-r+1)$, which is denoted by ${}^{n}P_{r}$ can be written in factorial form as follows

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

Application 4

Compute: ${}^{50}P_{3}$

Solution

$${}^{50}P_{3} = \frac{50!}{(50-3)!} = 50 \times 49 \times 48 = 117600$$

Application 5

How many numbers can be formed between 400 and 500 with the digits 0, 1, 2, 3 and 4? (If all it is digits have to be distinct).

Solution

Hundred place can be filled by 1 way (by filling 4).

Remaining 2 places can be filled by ${}^{4}P_{2}$ ways.

So, required number = 1× ${}^{4}P_{2}$= 6

Application 6

If no two consonants are together then in how many ways can the word RAINBOW be arranged?

Solution

First arrange the 3 vowels which can be done in 3! ways.

Now there are 4 places created by 3 vowels which can be filled up by 4 consonants in $^4P_4$ ways as follows

C V C V C V C

Therefore, the total number of permutations is $3! \times {}^4P_4 = 6 \times 24 = 144$

Application 7

How many 5-digit numbers can be formed by using the digits 1 to 9 if repetition of digits is not allowed?

Solution

Here, we need to find the number of permutations of 9 numerals taken 5 at a time.

So, number of 5-digit numbers = ${}^9P_5 = 9 \times 8 \times 7 \times 6 \times 5 = 15120$

Application 8

Find the number of 6-letter words with or without meaning, that can be formed from the letters of the word 'COMBINED'.

Solution

No. of arrangement of 6-letter words from 8 letters

$$= {}^8P_6 = 8 \times 7 \times 6 \times 5 \times 4 \times 3 = 20160$$

Permutations of objects not all distinct

The number of mutually distinguishable permutations of n things, taken all at a time, of which p are alike of one kind, q alike of second such that $p + q = n$, is $\frac{n!}{p!q!}$.

Application 9

Find the number of different 6-letter arrangements that can be made from the letters of the word SIMPLE so that (i) all vowels occur together (ii) all vowels do not occur together

Solution

(i) To get the vowels occur together, let us consider them as a single unit (I and E).

In the 6 letter arrangements with two vowels and 4 consonants to be kept.

Number of required arrangements = $2 \times {}^5P_5 = 240$

(Two vowels can be arranged in 2 ways)

(ii) Number of required arrangements

= Total 6 letter arrangements – arrangements with vowels together

$= {}^6P_6 - 2 \times {}^5P_5 = 720 - 240 = 480$

Application 10

How many words can be formed using the letters of the word 'EXTRA' so that the vowels are never together?

Solution

The given word contains 5 different letters.

Total number of words formed using all the five letters $= 5!$ $= 5 \times 4 \times 3 \times 2 \times 1 = 120$

Suppose the vowels are together, EA can be treated as single unit and they can be arranged themselves in 2! ways.

Now the number of letters = 4

$\therefore$ No. of words each having vowels together $= 4! \times 2!$

$= 24 \times 2 = 48$

$\therefore$ No. of words each having vowels never together = 120 – 48

= 72

Application 11

How many words can be formed using the letters of the word ALLAHABAD?

Solution

The word ALLAHABAD contains 9 letters namely 4 A, 2 L, 1 H, 1 B and 1 D.

$\therefore$ Required number of words $= \frac{9!}{4!\,2!} = \frac{5 \times 6 \times 7 \times 8 \times 9}{2} = 7560$

Application 12

Find the total number of 9 digit numbers of different digits.

Solution

No. of ways of filling the first place from the left = 9

(0 is excluded)

The number of ways of filling the remaining 8 places by the remaining 9 digits (0 included) = ${}^9P_8$

$\therefore$ The required number of 9 digit numbers $= 9 \times {}^9P_8$

$= 9 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 3265920$

Application 13

Raju attempts a multiple choice question paper consisting of 10 questions and each question having 4 choices. Find the number of ways in which he can attempt the entire paper if he is marking the answers at random.

Solution

No. of ways of answering each question = 4

No. of questions = 10

$\therefore$ Required number of ways $= 4^{10}$

Application 14

In how many ways can 7 letters be posted into 3 letter boxes?

Solution

The first letter can be posted into any of the 3 boxes in 3 ways.

Similarly, each of the other letters can also be posted in 3 ways.

$\therefore$ The total number of ways $= 3\times3\times3\times3\times3\times3\times3 = 3^7$

$n!+1$ is not divisible by any number between 2 and n.

In any party where 'n' persons are present, the total number of handshakes $= {}^{n}C_2$.

CIRCULAR PERMUTATION

A circular permutation is one in which objects are arranged along a circle. It is also called as closed permutation.

The number of circular permutations of *n* distinct objects is (*n* – 1)!, if the clockwise and anticlockwise directions are different.

If there are *n* things and if the direction is not taken into consideration, the number of circular permutations is $\frac{(n-1)!}{2}$.

Application 15

There are 5 gentle men and 4 ladies to dine at a round table. In how many ways can they seat themselves so that no two ladies are together?

Solution

The number of ways in which 5 gentlemen may be arranged around a round table $= 4! = 24$

Since no two ladies are to be together, the ladies can sit one between 2 gentle men, there are 5 places for ladies and they can use any 4 and this can be done in ${}^{5}P_4 = 120$ ways

$\therefore$ The number of ways in which both the gentle men and the ladies arrange themselves $= 24\times120 = 2880$

Application 16

Rosy has 20 beads of different colours. Find the number of ways she can make chains of different arrangements.

Solution

Number of different arrangements = $\frac{19!}{2}$

COMBINATION

Each of the different groups of selection which can be made by taking some or all of a number of items is called a combination. For example, the combination of three items a, b, c taken two at a time are ab, bc, ca.

Here the order of the items is not important.

The number of combination of n things taken 'r' at a time is denoted by ${}^{n}C_r$, which can be written as ${}^{n}C_r = \frac{n!}{(n-r)!\,r!}$

Total number of combinations

Out of n given objects, the number of ways of selecting one or more objects is where we can select 1 or 2 or 3 .... or n objects at a time.

$\therefore$ The number of ways $= {}^{n}C_1 + {}^{n}C_2 + {}^{n}C_3 + ... + {}^{n}C_n$

This is called the total number of combinations and is equal to $2^n - 1$.

$\therefore {}^{n}C_1 + {}^{n}C_2 + {}^{n}C_3 + ... + {}^{n}C_n = 2^n - 1$

$\therefore$ The number of ways of selecting one or more objects from 'n' given objects is $2^n - 1$.

Application 17

Find the values of ${}^{10}C_4$.

Solution

$${}^{10}C_4 = \frac{10!}{6!\,4!} = \frac{7\times8\times9\times10}{1\times2\times3\times4} = 210$$

Out of 'n' non concurrent and non parallel lines the number of points of intersection is ${}^{n}C_2$.

The number of straight lines passing through 'n' points is ${}^{n}C_2$.

The number of straight lines passing through 'n' points out of which 'm' points are collinear is ${}^{n}C_2 - {}^{m}C_2 + 1$.

Application 18

A cultural committee of eight students is to be formed from 9 boys and 5 girls. In how many ways can it be done when the committee consists of (i) exactly 3 girls (ii) atleast 3 girls?

Solution

(i) We have to choose 3 girls out of 5 girls and 5 boys out of 9 boys.

No. of ways $= {}^5C_3 \times {}^9C_5 = \frac{5!}{2!3!} \times \frac{9!}{5!\,4!}$

$= 1260$ ways

(ii) The committee may take 3 or 4 or 5 girls

No. of ways of selecting 3 girls and 5 boys

$= {}^5C_3 \times {}^9C_5 = 1260$

No. of ways of selecting 4 girls and 4 boys

$= {}^5C_4 \times {}^9C_4 = 630$

No. of ways of selecting 5 girls and 3 boys $= {}^5C_5 \times {}^9C_3 = 84$

$\therefore$ Required number of ways = 1260 +630 + 84

= 1974 ways

In a polygon of n sides the total number of diagonals is $\frac{n(n-3)}{2}$.

Number of triangles formed by joining 'n' points is ${}^nC_3$.

Number of triangles formed by joining 'n' points out of which 'm' are collinear is ${}^nC_3 - {}^mC_3$.

Application 19

If there are 6 periods on each working day of a school, in how many ways can one arrange 5 subjects such that each subject is allowed atleast one period?

Solution

No. of ways of arranging 5 subjects in 6 periods $= {}^6P_5$

$= \frac{6!}{(6-5)!} = 720$ ways

The remaining period can be given to any of the 5 subjects in 5 ways.

$\therefore$ The required number of arrangements $= 5 \times 720 = 3600$

Application 20

How many different signals can be made by waving 5 different coloured flags one along the other when one or more of them may be waved at a time?

Solution

There could be 5, 4, 3, 2 or 1 flags.

If we consider a selection of 'r' flags we get ${}^5C_r$ such selections where $r = 5, 4, 3, 2, 1$

In each of these selections, we can again arrange them depending on their colours.

If the selection has 'r' flags, these r flags can be arranged in r! ways.

$\therefore$ Total number of signals

$= {}^5C_5 \times 5! + {}^5C_4 \times 4! + {}^5C_3 \times 3! + {}^5C_2 \times 2! + {}^5C_1 \times 1!$

$= 1 \times 120 + 5 \times 24 + 10 \times 6 + 10 \times 2 + 5$

$= 120 + 120 + 60 + 20 + 5 = 325$

Application 21

Out of 6 consonants and 3 vowels, how many words can be made so that each word contains 2 consonants and 3 vowels?

Solution

No. of ways of selecting 2 consonants from 6 consonants = ${}^6C_2$

No. of ways of selecting 3 vowels from 3 vowels $= {}^3C_3$

Each set selected can be arranged into words in 5! ways

$\therefore$ Required number of ways $= {}^6C_2 \times 3C_3 \times 5! = \frac{6 \times 5}{1 \times 2} \times 1 \times 120$

$= 1800$

Remember!

1. *Total number of selections of one or more things out of n different things* $= {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n = 2^n - 1$
2. *Number of ways of distributing n identical things among r persons when each person may get any number of things* $= {}^{(n+r-1)}C_{r-1}$
3. *Number of squares in a square of n*n side =* $1^2 + 2^2 + \ldots\ldots + n^2$
4. *Number of rectangles in a square of n*n side =* $1^3 + 2^3 + \ldots\ldots + n^3$
5. *A rectangle having m rows and n columns will have* $[mn + (m-1)(n-1) + \ldots\ldots + 1]$ *squares and* $[(1 + 2 + 3 + \ldots\ldots + m)(1 + 2 + \ldots\ldots + n)]$ *rectangles.*

SESSION – 1

1. Suppose an automobile number plate contains two letters followed by three digits with the first digit not zero. How many different number plates can be printed? (Assuming repetition allowed).
(a) 754320 (b) 608400 (c) 546754 (d) 875640

2. Three persons enter a railway carriage, where there are 5 vacant seats. In how many ways can they seat themselves?
(a) 60 (b) 70 (c) 80 (d) 90

3. How many numbers between 0 and one million can be formed using 0, 7 and 8?
(a) 486 (b) 1086 (c) 728 (d) None of these

4. A palindrome is a number that reads the same left to right as it does from right to left, such as 252. How many six-digit palindromes are there which are even?
(a) 900 (b) 500 (c) 9 × 105 (d) 400

5. Find the number of squares that we can find on a chess board.
(a) 64 (b) 160 (c) 224 (d) 204

6. In how many ways, we can choose a black and a white square on a chessboard such that the two are not in the same row or column?
(a) 432 (b) 768 (c) 869 (d) None of these

7. There are 6 boxes numbered 1, 2, …., 6. Each box is to be filled up either with a red or a green ball in such a way that at least 1 box contains a green ball and the boxes containing green balls are consecutively numbered. The total number of ways in which this can be done is
(a) 5 (b) 21 (c) 33 (d) 60

8. Of the 128 boxes of oranges, each box contains at least 120 and at most 144 oranges. The number of boxes containing the same number of oranges is at least
(a) 5 (b) 103 (c) 6 (d) 125

9. There are 5 letters and 5 directed envelopes. The number of ways in which all the letters can be put into wrong envelope is
(a) 119 (b) 44 (c) 59 (d) 40

10. A five-digit number is formed using digits 1, 3, 5, 7 and 9 without repetition. What is the sum of all such possible numbers?
(a) 6666600 (b) 6666660
(c) 6666666 (d) None of these

11. A party of 'n' persons sits at around table. Find the odds against two specified persons sitting next to each other.
(a) (n – 1)/2 (b) (n – 3)/2
(c) (n + 3)/2 (d) None of these

12. The number of non-negative integral solutions of the equation a + b + c + d = 20 will be
(a) 1208 (b) 4024 (c) 1140 (d) 1771

13. A, B, C and D are four towns, any three of which are non-collinear. Then the number of ways to construct three roads each joining a pair of towns so that the roads do not form triangle is
(a) 7 (b) 8 (c) 9 (d) more than 9

14. Find the number of ways in which five gentlemen and three ladies can be seated in a row so that no two ladies are together.
(a) 720 (b) 7,200 (c) 14,400 (d) 5,040

15. Six mobsters have arrived at the theater for the premiere of the film "Good buddies". One of the mobsters, Frankie, is an informer, and he's afraid that another member of his crew, Joey, is on to him. Frankie, wanting to keep Joey in his sights, insists upon standing behind Joey in line at the concession stand, though not necessarily right behind him. How many ways can the six arrange themselves inline such that Frankie's requirement is satisfied?
(a) 24 (b) 120 (c) 360 (d) 720

16. Serena and Venus were only two women participating in a chess tournament. Every participant played two games with every other participant. The number of games that men played between themselves proved to exceed by 66, compared to the number of games the men played with women. How many participants were there?
(a) 156 (b) 610 (c) 13 (d) 108

17. Find the number of whole numbers formed on the screen of a calculator which can be recognized as numbers with (unique) correct digits when they are read inverted. The greatest number that can be formed on the screen of the calculator is 999999.
(a) 98970 (b) 89912 (c) 110050 (d) 100843

18. A question paper has two parts - Part A and Part B. Part A contains 5 questions and part B has 4. Each question in part A has an alternative. A student has to attempt at least one question from each part. Find the number of ways in which the student can attempt the question paper
(a) 3360 (b) 1258 (c) 3850 (d) 3630

19. There are 5 different boxes and 7 different balls. All the 7 balls are to be distributed in the 5 boxes placed in a row so that any box can receive any number of balls. In how many ways can these balls be distributed into these boxes if ball 2 can be put into either box 2 or box 4?
(a) 12360 (b) 31250 (c) 13490 (d) 31526

20. Two packs of 52 playing cards are shuffled together. Find the number of ways in which a man can be dealt 26 cards so that he does not get two cards of the same suit and same denomination.
(a) $^{52}C_{26} * 2^{26}$ (b) 52!/(2!)26
(c) 2^{26} (d) None of these

SESSION – 2

1. In how many ways 3 boys and 3 girls can be seated in a row so that boys and girls are alternate?
(a) 6! (b) 3!
(c) $3! \times 3!$ (d) None of these

2. How many numbers between 2000 and 3000 can be formed with the digits 0, 1, 2, 3, 4, 5, 6, 7 ? (repetition of digits not allowed)
(a) 210 (b) 300 (c) 420 (d) 360

3. These are 3 different rings to be worn in 4 fingers with atmost one in each finger. Find the number of ways in which this is done.
(a) 6 (b) 12 (c) 24 (d) 36

4. In how many ways can the letters of the word 'HEXAGON' be arranged?
(a) 2520 (b) 5040 (c) 8020 (d) 900

5. How many different signals can be transmitted by arranging 3 red, 2 yellow and 2 green flags on a pole?
(a) 210 (b) 420 (c) 360 (d) 400

6. How many straight lines can be formed from 8 non-collinear points on the XY plane?
(a) 24 (b) 28 (c) 32 (d) 48

7. Find the number of ways in which 20 different pearls of 2 different colours can be set alternately on a necklace there being 10 pearls of each colour.
(a) $9!10!$ (b) $5(9!)^2$
(c) $(9!)^2$ (d) None of these

8. A box contains two red, three green and four blue balls. In how many ways can three balls be drawn from the box if atleast one green ball is to be included in the draw.
(a) 23 (b) 64 (c) 46 (d) None of these

9. How many 4 digit numbers divisible by 5 can be formed with the digits 0, 1, 2, 3, 4, 5, 6 and 6?
(a) 220 (b) 249 (c) 432 (d) 288

10. How many new words can be formed with the word 'MANAGEMENT' all ending in G?
(a) $\frac{10!}{(2!)^4-1}$ (b) $\frac{9!}{(2!)^4}$ (c) $\frac{10!}{(2!)^4}$ (d) $\frac{9!}{(2!)^4-1}$

11. How many different nine digit numbers can be formed from the number 223355888 by rearranging its digits so that the odd digits occupy even positions
(a) 60 (b) 75 (c) 88 (d) 77

12. Five balls of different colours are to be placed in three boxes of different sizes. Each box can hold all the five balls. In how many ways can we place the balls so that no box remains empty?
(a) $^5C_3$ (b) 5! (c) 150 (d) 5^3

13. There are 4 letters and 4 envelopes. In how many ways can wrong choices be made?
(a) 4^3 (b) $4!-1$ (c) 16 (d) 4^4-1

14. If a team of 4 persons have to be selected from 8 males and 8 females, find the number of ways in which the selection is made to include atleast 1 male.
(a) 3500 (b) 875 (c) 1200 (d) 1750

15. If we have to make 7 boys sit with 7 girls around a round table, then find the number of different relative arrangements of boys and girls that we can make so that not two boys are no two girls sitting next to each other.
(a) $2\times(7!)^2$ (b) $6!\times7!$
(c) $7!\times7!$ (d) $2\times6!\times7!$

16. In how many ways can one divide 12 books into 4 equal bundles?
(a) $\frac{12!}{4!(3!)^4}$ (b) $\frac{12!}{(3!)^2}$
(c) $\frac{12!}{(4!)^2(3!)^2}$ (d) None of these

17. Find the number of ways in which 21 balls can be distributed among 3 persons such that each person does not receive less than 5 balls.
(a) 28 (b) 14 (c) 21 (d) 7

18. In how many ways can the letters of the English alphabet be arranged so that there are 7 letters between A and B?
(a) $31!2!$ (b) $24\,P_7\times18!$
(c) $36\times24!$ (d) None of these

19. Find the number of circle that can be drawn out of 10 points of which 7 are collinear.
(a) 130 (b) 85 (c) 45 (d) 65

20. Find the number of ways of selecting the committee of 5 with a maximum of 2 women and having at the maximum one women holding one of the two posts of the committee. There are 4 men and 4 women.
(a) 16 (b) 512 (c) 608 (d) 324

21. There are 8 orators A, B, C, D, E, F, G and H. Find the number of ways in which the arrangement is made so that A always comes before B and B always comes before C
(a) $\frac{8!}{3!}$ (b) $\frac{8!}{6!}$ (c) $\frac{8!}{5!3!}$ (d) $5!3!$

22. In how many ways can 12 persons among whom are 2 brothers be arranged along a circle so that there is exactly one person between the 2 brothers?
(a) $9!2!$ (b) $11!$ (c) $10!2!$ (d) $10!$

SESSION – 3

Permutation and Combination – Case Study

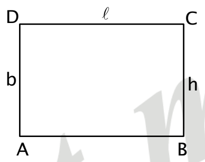
MENSURATION AND GEOMETRY

MENSURATION

Area of Geometrical Figures

1. **Rectangle**

ABCD is a rectangle with length AB = ℓ units and breadth BC = b units.



Area of the rectangle = length × breadth

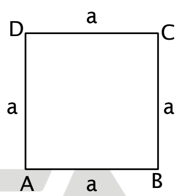
$= \ell b$ sq. units

$\therefore$ **Area of the rectangle with sides 'ℓ' and 'b' $= \ell$b sq. units**

2. **Square**

ABCD is a square with side 'a' units.

Since all the sides of a square are equal,



Area of the square = side × side $= a^2$ sq. units

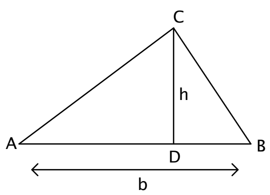
$\therefore$ **Area of a square with side 'a' $= a^2$ sq. units**

3. **Triangle**

ABC is a triangle with base 'b' units and height 'h' units.

Area of the triangle $= \frac{1}{2}$ base × height

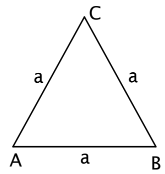
$= \frac{1}{2}$ bh sq. units



$\therefore$ **Area of the triangle with base 'b' and height 'h' $= \frac{1}{2}$ bh sq. units**

If the triangle is equilateral and the length of the sides is 'a' units,

Area of the triangle $= \frac{\sqrt{3}}{4} a^2$ sq. units

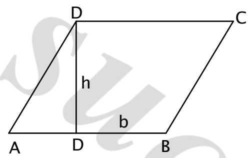


$\therefore$ **Area of the equilateral triangle with side 'a' $= \frac{\sqrt{3}}{4} a^2$ sq. units**

4. **Parallelogram**

ABCD is a parallelogram with base AB = 'b' and height 'h'.

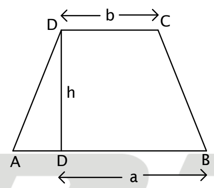
Area $= b \times h$ sq. units



$\therefore$ **Area of the parallelogram with the base 'b' and height 'h' = bh sq. units**

5. **Trapezium**

ABCD is a trapezium with parallel sides AB and CD with length 'a' units and 'b' units and the distance between them is 'h' units.



Area of the trapezium

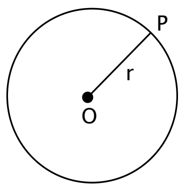
$= \frac{1}{2}(\text{sum of the} \parallel \text{sides}) \times \text{height}$

$= \frac{1}{2}(a+b)h$ sq. units

$\therefore$ **Area of the trapezium with parallel sides 'a' and 'b' and distance between them $h = \frac{1}{2}(a+b)\,h$ sq.units**

6. **Circle**

Let 'r' be the radius of the circle with Centre O.



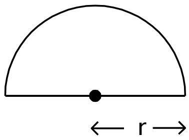
Area of the circle $= \pi r^2$ sq. units

Circumference of the circle $= 2\pi r$ units

7. **Semicircle**

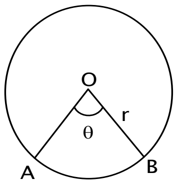
Area of a semicircle $= \frac{1}{2}\pi r^2$ **sq. units**

Perimeter $= \pi r + 2r$ **units**



8. **Sector of a circle**

Let Q be the angle subtended by an arc AB at the centre of a circle of radius 'r'.



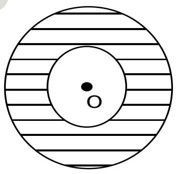
Length of the arc $AB = \frac{\theta}{360} \times 2\pi r$

Area of the sector $AOB = \frac{\theta}{360} \times \pi r^2$ **sq. units**

9. **Ring**

A ring is formed by two concentric circles.

Let the radii of the 2 circles be 'R' and 'r'.

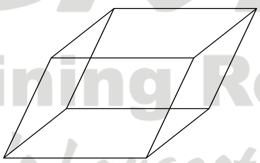


Area of the ring $= \pi\left(R^2 - r^2\right)$ **sq.units**

Area and Volume of Solids

1. **Parallelopiped**

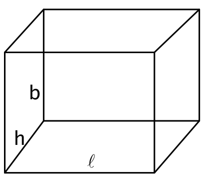
A solid bounded by three pairs of parallelograms is called a parallelopiped.



The opposite faces are congruent parallelograms.

2. **Cuboid**

Cuboid is a parallelopiped where all the faces are rectangles.



A cuboid has 12 edges and 8 vertices.

If the length, breadth and height of a cuboid are 'ℓ', 'b', 'h' units,

Volume of the cuboid = $\ell \times b \times h$ cubic units

Total surface area of the cuboid

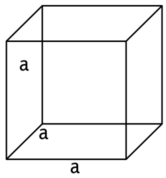
$$= 2(\ell \times b + b \times h + h \times \ell) \text{ sq. units}$$

Diagonal of the cuboid = $\sqrt{\ell^2 + b^2 + h^2}$ **units**

3. **Cube**

When all the edges of a cuboid are equal, it is called a cube.

Let 'a' be the side of the cube.



Volume of the cube

$= a^3$ **cubic units**

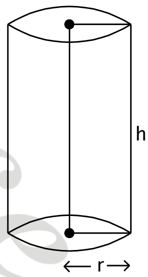
Total surface area of the cube $= 6a^2$ **sq. units**

Diagonal of the cube $= \sqrt{3}a$ **units**

4. **Cylinder**

A right circular cylinder is the solid generated by revolving a rectangle about one of its sides as axis. The cross section of a cylinder is a circle.

Let the cylinder be with base radius 'r' units and height 'h' units.



Volume of the cylinder

= Area of the base × height

$= \pi r^2 \times h \;\; = \pi r^2 \times h$ cubic units

∴ **Volume of the cylinder with base radius 'r' and height 'h'** $= \pi r^2 h$ **cubic units**

Curved surface area of the cylinder $= 2\pi rh$ **sq. units**

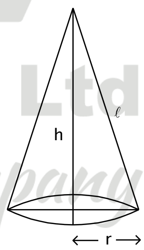
Total surface area of a solid cylinder

$= \left(2\pi rh + 2\pi r^2\right)$ **sq. units**

5. **Cone**

If a right angled triangle is rotated about one of its sides containing the right angle as axis, the solid generated is called a cone.

Let 'r' be the base radius and 'h' be the height of the cone.



'ℓ' is the slant height of the cone.

$\ell = \sqrt{h^2 + r^2}$ units

or The slant height of a cone $= \sqrt{h^2 + r^2}$ **units**

Volume of the cone $= \frac{1}{3}\pi r^2 h$ **cu. units**

Curved surface of the cone $= \pi r\ell$ **sq. units**

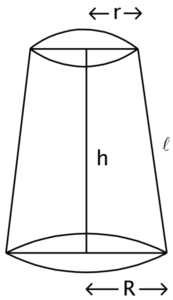
Total surface of the cone $= \pi r\ell + \pi r^2$

$= \pi r(\ell + r)$ **sq. units**

6. **Frustum of a cone**

If a cone is cut by a plane parallel to the base of the cone the portion between the plane and the base is called frustum.

If 'R' be the radius of the base of the cone and 'r' the radius of the top and 'h' the height of the frustum, then



Volume of the frustum of the cone

$=\frac{\pi h}{3}\left(R^2+r^2+Rr\right)$ **cu. units**

Lateral surface of the frustum $=\pi\ell(R+r)$ **where**

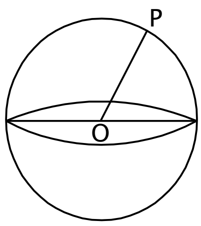
$\ell=\sqrt{h^2+(R-r)^2}$

Total surface area of the frustum $=\pi R^2+\pi r^2+\pi\ell(R+r)$ **sq. units.**

7. **Sphere**

A sphere is the solid generated by a circle while revolving about its diameter.

Let O be the centre of the sphere of radius 'r' units.



Volume of the sphere

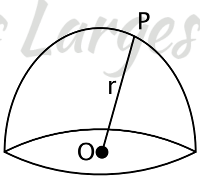
$=\frac{4}{3}\pi r^3$ **cu. units**

Surface area of the sphere $=4\pi r^2$ **sq. units**

Volume of a hollow sphere with external radius 'R' and internal radius 'r' $=\frac{4}{3}\pi\left(R^3-r^3\right)$ **cu. units**

8. **Hemisphere**

The diameter of a sphere divides it into two equal parts. Each of these parts is called a hemisphere.



Volume of the hemisphere of radius 'r'

$=\frac{2}{3}\pi r^3$ **cu. units**

Curved surface area of the hemisphere of radius 'r' $=2\pi r^2$ **sq. units**

Total surface area $=2\pi r^2+\pi r^2=3\pi r^2$ **sq. units**

GEOMETRY

The pre-requisite for solving problems in geometry is a firm grip on relevant theorems and results.

Important Theorems and Results

Straight Line

1. Two angles x and y are said to be <u>complementary</u> if $(x + y) = 90°$.
2. Two angles x and y are said to be <u>supplementary</u> if $(x + y) = 180°$.
3. When a straight line stands on another straight line, the two adjacent angles are supplementary. i.e. the two angles add up to 180°.
4. When two straight lines intersect, the vertically opposite angles are equal.
5. When a pair of parallel lines is intersected by a transversal,
   (i) the corresponding angles are equal
   (ii) the alternate angles are equal
   (iii) the interior angles are supplementary.
6. If a straight line L is parallel to another straight line L1, then it is parallel to all straight lines parallel to L1.

Triangle

1. The sum of all interior angles of a triangle is 180º.
2. Each exterior angle of a triangle is equal to the sum of the opposite interior angles.
3. The sum of any two sides of a triangle is always greater than the third side.
4. The line joining the midpoints of any two sides of a triangle is parallel to the third side and half its length. [This is known as *<u>midpoint theorem</u>*]
5. A triangle with any two sides equal is called an *<u>isosceles triangle</u>*.
6. A triangle with all three sides equal is called an *<u>equilateral triangle</u>*.
7. A triangle with one angle as 90º is called a *<u>right triangle</u>*.
8. Area of a triangle $=\left(1/2\right)$ Any side $\times$ the altitude on that side, $=\sqrt{\{s(s-a)(s-b)(s-c)\}}$, $[s=\frac{(a+b+c)}{2}$, a, b, c being the sides],
   $=\left(1/2\right)$ Product of any two sides $\times$ sin (included angle)
9. In a right triangle,
   (i) the side opposite to the right angle is called the hypotenuse,
   (ii) the square of the hypotenuse is equal to the sum of the squares of the other two sides. [This is the famous Pythagoras Theorem.]
10. In a 30° – 60° – 90° right triangle, the sides are in the ratio $1:\sqrt{3}:2$.

11. In a 45° – 45° – 90° right triangle, the sides are in the ratio $1:1:\sqrt{2}$.

12. In an equilateral triangle with edge a,
 (i) the altitude = $(\sqrt{3}/2)a$,
 (ii) the area = $(\sqrt{3})a^2/4$,
 (iii) in-radius = $(1/2)$ circum-radius,
 (iv) area of the circum-circle = 4 × area of the in-circle.

13. In an isosceles triangle, the angles opposite to the equal sides are also equal.

KEY POINTS

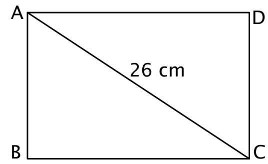
1. Area of a segment of a circle is the difference of areas of the corresponding sector and the triangle.
2. The perimeter of a semicircle is $\pi r + 2r$.
3. The formulae used for the polygons hold good only for regular polygons.
4. Area of an equilateral triangle of side 'a' is $\frac{\sqrt{3}}{4}a^2$ and Area of a regular hexagon $= 6 \times \frac{\sqrt{3}}{4}a^2$.
5. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{b}=\frac{c+d}{d}, \frac{a}{a+b}=\frac{c}{c+d}$ and $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.
6. If the sides of a triangle are a, b, c and $a+b+c=2s$, then area $=\sqrt{s(s-a)(s-b)(s-c)}$
7. If a chord is drawn through the point of contact of a tangent to a circle, the angle between the chord and the tangent is equal to the angle in the alternate segment.
8. If two circles touch each other, the point of contact lies on the line of centres.

GUARD AGAINST POPULAR ERRORS

1. A line segment is only a part of a line. In closed geometrical figures segment and line are one and the same.
2. All linear pairs have supplementary angles but not all supplementary angles form a linear pair.
3. When a number of lines intersect at a point, sum of all the angles formed at the point is 360°.
4. Isosceles triangles need not have only lateral sides equal. The base and one of the lateral sides also can be equal.
5. Exterior angles of a triangle are obtained by extending the sides of the triangle.
6. The sum of the length of any two sides of a triangle is greater than the third side and the difference of the length of any two sides is less than the third side.
7. In triangles if all the sides are equal, the angles are also equal. But in a polygon "all sides are equal" does not necessarily imply "all angles are equal".
8. In a circle all the angles subtended by a chord are equal only if all the angles lie in the same segment.

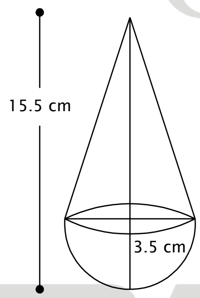
SESSION – 4

1. The diagonal of a rectangle is 26 cm long and its perimeter is 68 cm. Find the area of the rectangle.



(a) 240 sq.cm (b) 235 sq.cm
(c) 260 sq.cm (d) 225 sq.cm

2. A toy is in the form of a cone mounted by a hemisphere of radius 3.5 cm. The total length of the toy is 15.5 cm. Find the volume and the total surface area of the solid.

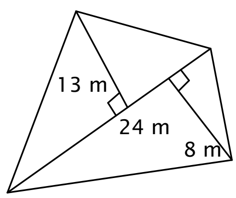


(a) $234.82, 205.5\,cm^2$ (b) $243.82\,cm^3, 214.5\,cm^2$
(c) $224.82, 220.5\,cm^2$ (d) $214.82, 234.5\,cm^2$

3. Length of the fence of the trapezium shaped field ABCD is 120 m. If BC = 48 m, CD = 17 m, AD = 40 m, find the area of the field. Side AB is perpendicular to the parallel sides AD and BC.

(a) $400\,m^2$ (b) $660\,m^2$ (c) $600\,m^2$ (d) $480\,m^2$

4. The diagonal of a quadrilateral shaped field is 24 m and the perpendiculars dropped on it from the remaining vertices are 8 m and 13 m. Find the area of the field.



(a) 250 sq. m. (b) 225 sq. m.
(c) 252 sq. m. (d) 196 sq. m.

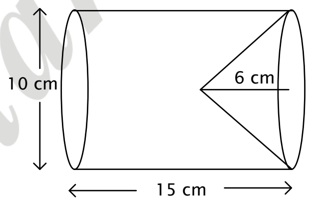
5. If the lateral surface area of a cube is increased by 44%, find the percentage change in the volume of the cube.
(a) 68% (b) 44% (c) 72.8% (d) 60%

6. A solid metallic spherical ball of diameter 6 cm is melted and recast into a cone with diameter of the base 12 cm. Find the height of the cone.
(a) 2 cm (b) 3 cm (c) 4 cm (d) 6 cm

7. The surface area of a cube is 1734 cm². Find its volume.
(a) $17\,cm^3$ (b) $289\,cm^3$
(c) $4913\,cm^3$ (d) None of these

8. Find the length of the canvas 1.25 m wide required to build a conical tent of base radius 7 m and height 24 m.
(a) 440 m (b) 550 m
(c) 44 m (d) None of these

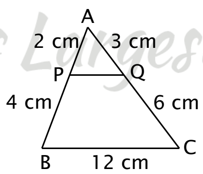
9. From a cylinder with a diameter of 10 cm and height 15 cm, a cone with base diameter 10 cm and height 6 cm is hollowed out. Find the volume of the remaining solid.



(a) $1021.45\,cm^3$ (b) $157.14\,cm^3$
(c) $707.15\,cm^3$ (d) None of these

10. The radius of a circle is 10 cm and the length of one of its chords is 12 cm. Find the distance of the chord from the centre.
(a) 64 cm (b) 4 cm (c) 16 cm (d) 8 cm

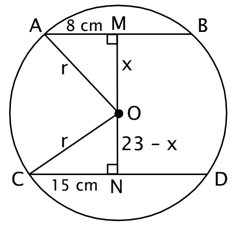
11. P and Q are points on the sides AB and AC of ΔABC, If AP = 2 cm, PB = 4 cm, AQ = 3 cm, QC = 6 cm and BC = 12 cm, find PQ.



(a) 4 cm (b) 6 cm (c) 8 cm (d) 10 cm

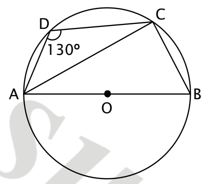
12. One angle of a pentagon is 140°. If the remaining angles are in the ratio 1:2:3:4, find the size of the greatest angle.
(a) 150° (b) 160° (c) 165° (d) 170°

13. AB and CD are two parallel chords of a circle such that AB = 16 cm and CD = 30 cm. If the chords are on the opposite sides of the centre and the distance between them is 23 cm, find the radius of the circle.



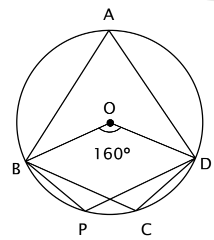
(a) 10 cm (b) 13 cm (c) 17 cm (d) 20 cm

14. In the adjoining figure ABCD is a cyclic quadrilateral whose side AB is a diameter of the circle through A, B, C, D. If $\angle ADC = 130°$, find $\angle BAC$.



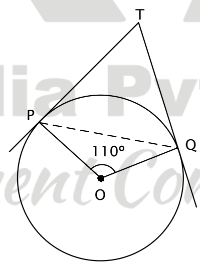
(a) 20° (b) 30° (c) 35° (d) 40°

15. ABCD is a cyclic quadrilateral. O is the centre of the circle. If $\angle BOD = 160°$, find $\angle BCD$. If P is any point on minor arc BC of the circle, find $\angle BPD$.



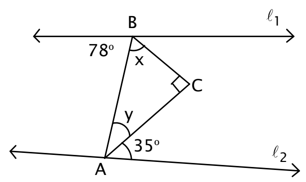
(a) 90° (b) 100° (c) 110° (d) 120°

16. In the given figure PT and TQ are 2 tangents to a circle with centre O. $\angle POQ = 110°$. Find $\angle PTQ$ and $\angle OPQ$.



(a) 50° (b) 60° (c) 70° (d) 80°

17. In the accompanying figure for which value of x is $\ell_1 || \ell_2$.



(a) 37° (b) 45° (c) 47° (d) 55°

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18. Two chords AB and CD of a circle intersect each other at a point E inside the circle. If AAE = 3 cm, CE = 5 cm, EB = 4 cm, find ED.
(a) 3.75 cm (b) 2.4 cm
(c) 4.8 cm (d) None of these

19. The diagonals AC and BD of a parallelogram ABCD intersect each other at the point O. If $\angle DAC = 32°$ and $\angle AOB = 70°$, find $\angle DBC$.
(a) 24° (b) 86° (c) 38° (d) 32°

20. The sum of the interior angles of a polygon is three times the sum of the exterior angles. Find the number of sides of the polygon.
(a) 5 (b) 6 (c) 7 (d) 8

SESSION – 5

1. Find the length of the wire required to fence 25 times around a circular garden of area 154 hectare.
(a) 110 km (b) 1100 km
(c) 11 km (d) None of these

2. A cone of height 9 cm with diameter of its base 18 cm is carved out from a wooden solid sphere of radius 9 cm. Find the percentage of the wood wasted.
(a) 25% (b) 25π% (c) 50% (d) 75%

3. There is a circular pond of diameter 2 m at the centre of the square plot of side 8 m and a cow is tethered at the centre of the pond with a rope of length 3 m. Find the area of the plot that the cow can cover.
(a) 10π sq. m. (b) 8π sq. m.
(c) 4π sq. m. (d) 6π sq. m.

4. A man walked diagonally across a square plot. Find the percent saved by not walking along the edges approximately.
(a) 20% (b) 24% (c) 30% (d) 33%

5. A sphere of maximum volume is cut out from a solid hemisphere of radius 'r'. The ratio of the volume of the hemisphere to that of the cutout sphere is
(a) 3:2 (b) 4:1 (c) 4:3 (d) 7:4

6. The circumference of the front wheel of a cart is 30 ft long and that of the back wheel is 36 ft long. Find the distance travelled by the cart when the front wheel has done five more revolutions than the back wheel.
(a) 20 ft (b) 25 ft (c) 750 ft (d) 900 ft

7. If the sides of a triangle measure 72 cm, 75 cm and 21 cm, find the measure of its in-radius.
(a) 37.5 cm (b) 24 cm (c) 9 cm (d) 15 cm

8. The height of a cylinder is to the diameter of the base as 1:2. Find the ratio of the area of its surface to the sum of the areas of its two ends.
(a) 1:1 (b) 1:2 (c) 2:1 (d) 1:3

9. A square is drawn by joining the mid points of the sides of a square of side 16 cm. Another square is drawn by joining the midpoints of the sides of the previous square. Another square is drawn by joining the mid points of the sides of the new square. The process is continued infinitely. Find the sum of the areas of all the squares.
(a) 256 sq. cm. (b) 128 sq. cm.
(c) 512 sq. cm. (d) 1024 sq. cm.

10. A solid cylinder of glass whose diameter is 1.5 m and height 1 m is melted and turned into a sphere. Find the diameter of the sphere.
(a) 1 m (b) 0.75 m
(c) 1.25 m (d) 1.5 m

11. The perimeter of a triangle is 28 cm and the inradius 2.5 cm. Find the area.
(a) $25\,cm^2$ (b) $42\,cm^2$
(c) $49\,cm^2$ (d) $35\,cm^2$

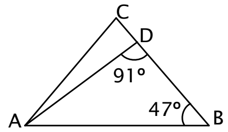
12. A cylindrical can of radius 14 cm and height 20 cm is completely filled with milk. A pipe is opened at the bottom of the can to fill the milk into identical bottles each which has a capacity of 770 ml. How many such bottles can be filled with the milk in the can?
(a) 4 (b) 12 (c) 8 (d) 16

13. Side AB of a triangle ABC is 80 cm long, whose perimeter is 170 cm. One of its angle = 60 degrees, the shortest side of triangle ABC measures (cm) _______
(a) 40 (b) 36 (c) 17 (d) 14

14. Δ ABC and Δ DBC are right triangle with common hypotenuse BC. The side AC and BD are extended to intersect at P, then AP × PC/DP × PB = ?
(a) 2 (b) 1/3
(c) 1 (d) None of these

15. In a Δ ABC, P, Q and R are the mid-points of sides BC, CA and AB respectively. If AC = 21 cm, BC = 29 cm and AB = 30 cm. The perimeter of the quadratic ARPQ is
(a) 91 cm (b) 60 cm (c) 51 cm (d) 70 cm

16. In the adjoining figure $AC = BC$ and $\underline{|ABC} = 47°$ $\underline{|ADB} = 91°$, find $\underline{|DAB}$. .



(a) 30° (b) 36° (c) 42° (d) 47°

17. If the circumference of a circle increases from 4π to 8π, find the change that occurs in its area.
(a) It is halved (b) It doubles
(c) It triples (d) It quadruples

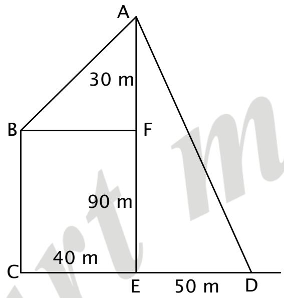
18. 27 identical spheres are filled into a cube such that no more such spheres can be placed. Find the part of the cube occupied by the spheres.

(a) $\frac{\pi}{6}$ (b) $\frac{3\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

19. The ratio of the measure of an angle of a regular octagon to the measure of its exterior angle is

(a) 1:2 (b) 3:1 (c) 2:3 (d) 3:4

20. In the adjoining figure, a plot ABCD is given. AF = 30 m, CE = 40 m, ED = 50 m, AE = 120 m. Find the area of the plot ABCD.



(a) 1800 sq. m. (b) 2400 sq. m.
(c) 3600 sq. m. (d) 7200 sq. m.

PROBABILITY

Probability is the measure of happening of a particular event. It is classically defined as, if a sample space (S) contains n outcomes and if m of them are favourable to an event A, then, we write n(S) = n and n(A) = m and the probability of the event A, $P(A) = \frac{n(A)}{n(S)} = \frac{m}{n}$

Some important classifications on events are as follows:

Equally likely events

Two or more events are said to be equally likely if each one of them has an equal chance of occurrence.

Mutually exclusive events

Two or more events are said to be mutually exclusive if the occurrence of one event prevents the occurrence of other events. That is, mutually exclusive events can't occur simultaneously. Thus, if A and B are two mutually exclusive events, then $A \cap B = \phi$

Complementary events

Let E be an event of a random experiment and S be its sample space. The set containing all the other outcomes which are not in E but in the sample space is called the complimentary event of E.

It is denoted by E'. Thus, E' = S – E.

Note that E and E' are mutually exclusive events.

Exhaustive events

Events $E_1, E_2, ..., E_n$ are exhaustive events if their union is the sample space S.

Sure event

The sample space of a random experiment is called sure or certain event as any one of its elements will surely occur in any trail of the experiment. So, probability of a sure event is 1.

Impossible event

An event which will not occur on any account is called an impossible event. It is denoted by ϕ. So, probability of an impossible event is 0.

Important results of probability

(1) $0 \le P(A) \le 1$

(2) P(S) = 1 (S – sample space)

Application 1

The probability of raining today is 0.54. What is the probability of not raining?

Solution

The events of raining and not raining or mutually exclusive events. So, the probability of not raining is (1 – 0.54) i.e. 0.46.

Application 2

A bag contains 10 yellow balls and some red balls. If the probability of drawing a yellow ball is twice that of drawing a red ball, then find the number of red balls.

Solution

Let the number of red balls be x.

Then the probability of drawing yellow and red balls are $\frac{10}{10+x}, \frac{x}{10+x}$ respectively.

By the given condition, $\frac{10}{10+x} = \frac{2x}{10+x}$

or $100 + 10x = 2x^2 + 20x$

or $2x^2 + 10x - 100 = 0$

$x^2 + 5x - 50 = 0$

$(x + 10)(x - 5) = 0$

So, number of red ball is 5 (as –10 is not possible).

Application 3

Find the probability that

(i) a leap year selected at random will have 53 Sundays.

(ii) a leap year selected at random will have 52 Mondays.

(iii) a non-leap year selected at random will have 53 Sundays.

Solution

Number of days in a leap year = 366 days (52 weeks and 2 days).

52 weeks contains 52 Sundays, remaining 2 days can be any one of the following: {Sunday and Monday, Monday and Tuesday, Tuesday and Wednesday, Wednesday and Thursday, Thursday and Friday, Friday and Saturday, Saturday and Sunday}

So, S = {Sun and Mon, Mon and Tue, Tue and Wed, Wed and Thu, Thu and Fri, Fri and Sat, Sat and Sun}

(i) Here, A = {Sun and Mon, Sat and Sun}

Therefore, P(A) = $\frac{2}{7}$

(ii) Here, A = {Tue and Wed, Wed and Thu, Thu and Fri, Fri and Sat, Sat and Sun}

Therefore, P(A) = $\frac{5}{7}$

(iii) Number of days in a non-leap year = 365 (52 weeks and a day)

So, sample space S = {Sun, Mon, Tue, Wed, Thu, Fri, Sat}

Here, A = {Sun}

Therefore, P(A) = $\frac{1}{7}$

A bag contains x red balls, y blue balls and z green balls. 'r' balls are drawn from the box at random. The probability that all the balls are of same colour is given by $\frac{{}^{x}C_r + {}^{y}C_r + {}^{z}C_r}{{}^{(x+y+z)}C_r}$

Application 4

Three covering letters are written to three different companies and addresses on 3 envelopes are also written. Without looking at the addresses, letters are kept inside the envelopes and sent to the companies. What is the probability that,

(i) right letters reach right companies.

(ii) none of the letters reach the right companies.

Solution

Let A, B and C the covering letters written to 3 different companies and EA, EB and EC be their envelopes respectively.

The different combinations of letters and envelopes can be showed as follows.

	C_1	C_2	C_3	C_4	C_5	C_6
EA	A	A	B	B	C	C
EB	B	C	A	C	A	B
EC	C	B	C	A	B	A

C_i's are different combinations of letters and envelopes.

(i) Then, the probability of right letters reaching right companies = $\frac{1}{6}$

(ii) And the probability that none of the letters reach the right companies = $\frac{2}{6} = \frac{1}{3}$

Aliter for (ii):

Number of ways none of the letters reach the right companies

$$= 3!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right) = 3! \times \frac{6-6+3-1}{6} = 2$$

So, probability that none of the letters reach the right companies

$= \frac{2}{6} = \frac{1}{3}$

Addition Theorem on Probability

If the subsets A and B are events of a random experiment and if the set S is the sample space of the experiment, then P(A ∪ B) = P(A) + P(B) – P(A ∩ B). This result is known as the addition theorem on probability.

Similarly, if A, B and C are events of a random experiment, then P(A ∪ B ∪ C) = P(A) + P(B) + P(C) – P(A ∩ B) – P(B ∩ C) – P(C ∩ A) + P(A ∩ B ∩ C)

Note: If A and B are mutually exclusive events, P(A ∪ B) = P(A) + P(B)

Application 5

One number is chosen randomly from the integers 23 to 61. Find the probability that it is divisible by 3 and 7.

Solution

Let A be the event of choosing a number divisible by 3 and B be the event of choosing a number divisible by 7. Then A = {24, 27, 30, 33, ... , 60}, B = {28, 35, 42, ... , 56} and A ∩ B = {42}

So, P(A ∪ B) = $\frac{13}{39} + \frac{5}{39} - \frac{1}{39} = \frac{13+5-1}{39} = \frac{17}{39}$

Application 6

A card is drawn at random from a well-shuffled deck of 52 cards. Find the probability that it will be a diamond or a queen.

Solution

Let D be the event of drawing diamond card and Q be the event of drawing queen card.

So, P(D) = $\frac{13}{52}$, P(Q) = $\frac{4}{52}$ and P(D ∩ Q) = $\frac{1}{52}$

Therefore, P(D ∪ Q) = $\frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{13+4-1}{52} = \frac{16}{52} = \frac{4}{13}$

Application 7

The probability that A, B and C can solve a problem are $\frac{1}{5}, \frac{1}{4}$ and $\frac{2}{5}$ respectively. The probability of the problem being solved by A and B is $\frac{1}{9}$, B and C is $\frac{1}{10}$, A and C is $\frac{2}{25}$ and the probability of the problem being solved by all the three is $\frac{1}{50}$.

Find the probability that the problem is not solved by any of the three.

Solution

From addition theorem of probability,

$$P(A \cup B \cup C) = \frac{1}{5}+\frac{1}{4}+\frac{2}{5}-\frac{1}{9}-\frac{1}{10}-\frac{2}{25}+\frac{1}{50}$$

$$= \frac{180+225+360-100-90-36+18}{900} = \frac{557}{900}$$

So, the probability that the problem is not solved by any of the three = $1-\frac{557}{900} = \frac{343}{900}$

Application 8

Out of 9 employees in a company there are 5 males and 4 females. A team of 3 is selected at random for a special training. Find the probability that there are atleast 2 female employees.

Solution

Total number of ways of selecting the team = ${}^9C_3$ = 84

Number of ways of selecting the team with atleast 2 female employees $= {}^4C_2 \times {}^5C_1 + {}^4C_3 \times {}^5C_0 = 6\times5+4 = 34$

Probability of selecting 2 female employees = $\frac{34}{84} = \frac{17}{42}$

Conditional Probability

The conditional probability of an event B, assuming that the event A has already happened is denoted by P (B/A).

And $P(B/A) = \frac{P(A \cap B)}{P(A)}$ provided $P(A) \neq 0$.

Multiplication Theorem on Probability

The probability of the simultaneous happening of two events A and B is given by $P(A \cap B) = P(A) . P(B/A)$

$$\text{or } P(A \cap B) = P(B) . P(A/B)$$

Independent Events

Events are said to be independent if the occurrence or non occurrence of any one of the event does not affect the probability of occurrence or non-occurrence of the other event.

Two events A and B are independent if $P(A \cap B) = P(A) . P(B)$

That is, $P(A/B) = P(A)$ and $P(B/A) = P(B)$

Application 9

From a well shuffled pack of 52 cards, two cards are drawn.
Find the probability that both are Jack.
(i) if the first card is replaced,
(ii) if the first card is not replaced.

Solution

Let A and B be the event of taking first and second card respectively.

(i) n(A) = n(B) = 4
The occurrence of A will not disturb the probability of B.
So, $P(A \cap B) = P(A) . P(B) = \frac{4}{52}\times\frac{4}{52} = \frac{1}{169}$

(ii) n(A) = 4 and n(B) = 3 (as the first card is not replaced)
Now, A and B are dependent events.
So, $P(A \cap B) = P(A). P(B/A) = \frac{4}{52}\times\frac{3}{51} = \frac{1}{221}$

Application 10

The probability of solving a problem by A, B and C is given as $\frac{1}{12}, \frac{3}{11}$ and $\frac{3}{10}$ respectively. What is the probability that the problem is solved?

Solution

Probability of A not solving = $1-\frac{1}{12} = \frac{11}{12}$

Probability of B not solving = $1-\frac{3}{11} = \frac{8}{11}$

Probability of C not solving = $1-\frac{3}{10} = \frac{7}{10}$

Probability that the problem is not solved by any one

$$= \frac{11}{12}\times\frac{8}{11}\times\frac{7}{10} = \frac{7}{15}$$

Therefore, the probability that the problem is solved

$$= 1-\frac{7}{15} = \frac{8}{15}$$

Total Probability

If $A_1, A_2 \ldots A_n$ are mutually exclusive and exhaustive events and B is any event in S then $P(B) = P(A_1) . P(B/A_1) + P(A_2) . P(B/A_2) \ldots + P(A_n). P(B/A_n)$

Here, P(B) is called the total probability of event B.

Application 11

A factory has two machines I and II. Machine I produces 40% of items of the output and Machine II produces 60% of the items. Further 2% of items produced by Machine I are defective and 1% of items produced by Machine II are defective. If an item is drawn at random, find the probability that it is a defective item.

Solution

Let A_1 and A_2 be the event of production by Machine I and Machine II respectively and B be the event of drawing a defective item.

$P(A_1) = \frac{40}{100}$ and $P(A_2) = \frac{60}{100}$

$P(B/A_1) = \frac{2}{100}$ and $P(B/A_2) = \frac{1}{100}$

$P(B) = P(A_1) . P(B/A_1) + P(A_2) . P(B/A_2)$

$$= \frac{1}{10000}(80+60) = \frac{140}{10000} = \frac{7}{50}$$

Bayes Theorem

$A_1, A_2 \dots A_n$ are mutually exclusive and exhaustive events and B is any event in S such that $P(A_i) > 0$ for i = 1, 2 … n.

Let B be any event with P(B) > 0 then

$$P(A_i/B) = \frac{P(A_i)P(B/A_i)}{P(A_1)P(B/A_1)+P(A_2)P(B/A_2)+\dots+P(A_n)P(B/A_n)}$$

Application 12

Urn I contains 4 yellow and 5 blue balls while urn II contains 3 yellow and 6 blue balls. One ball is drawn at random from one of the bags and it is found to be yellow. Find the probability that it is drawn from Urn I.

Solution

Let A_1 and A_2 be the events of choosing urn I and urn II respectively and B be the event of drawing yellow balls.

$P(A_1) = P(A_2) = \frac{1}{2}$

$P(B/A_1) = \frac{4}{9}$ and $P(B/A_2) = \frac{3}{9}$

$P(B) = P(A_1) . P(B/A_1) + P(A_2) . P(B/A_2) = \frac{1}{2}\left(\frac{4}{9}+\frac{3}{9}\right) = \frac{1}{2}\times\frac{7}{9}$

$$P(A_1/B) = \frac{P(A_1).P(B/A_1)}{P(A_1).P(B/A_1)+P(A_2).P(B/A_2)} = \frac{\frac{4}{9}}{\frac{7}{9}} = \frac{4}{7}$$

Application 13

The chances of A and B getting selected in a company are 2:3. The probabilities that they give treat are 0.3 and 0.2 respectively. If one of them gave treat, what is the probability that A is one who has got selected?

Solution

Let A and B are the events of them getting selected to the company respectively and T be the event of giving treat.

$P(A) = \frac{2}{5}$, $P(B) = \frac{3}{5}$

$P(T/A) = 0.3$ and $P(T/B) = 0.2$

$P(T) = P(A)P(T/A) + P(B)P(T/B) = \frac{2}{5}\times\frac{3}{10}+\frac{3}{5}\times\frac{2}{10} = \frac{6}{25}$

$$P(A/T) = \frac{P(A)\cdot P(T/A)}{P(A)\cdot P(T/A)+P(B)\cdot P(T/B)} = \frac{\frac{6}{50}}{\frac{6}{25}} = \frac{1}{2}$$

SESSION – 6

1. A bag contains 4 white balls and 3 black balls. Another bag contains 3 white balls and 4 black balls. A bag and a ball are picked at random. The probability that the ball will be white is

   (a) $\frac{1}{7}$ (b) $\frac{1}{5}$ (c) $\frac{1}{2}$ (d) 1

2. What is the probability that there are 53 Wednesdays and 52 Mondays in a leap year?

   (a) 0 (b) $\frac{1}{7}$ (c) $\frac{2}{7}$ (d) $\frac{5}{7}$

3. If events A and B are independent and P(A) = 0.15, P(AUB) = 0.45, then P(B) =

   (a) $\frac{6}{13}$ (b) $\frac{6}{17}$ (c) $\frac{6}{19}$ (d) $\frac{6}{23}$

4. Two cards are drawn at random from a pack of 52 cards. What is the probability that one is a spade and the other a king?

   (a) $\frac{1}{13}$ (b) $\frac{4}{13}$ (c) $\frac{1}{52}$ (d) $\frac{1}{26}$

5. A four-digit number is formed by using digits 2, 4, 6 and 8 without repetition. What is the probability that the number is divisible by 4?

   (a) $\frac{1}{5}$ (b) $\frac{5}{12}$ (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

6. Consider a circle and a point is chosen inside the circle. What is the probability that this point is closer to the centre rather than to the circumference?

   (a) $\frac{3}{4}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{2}{5}$

7. Four persons A, B, C and D are to speak at a function along with four others. If they all speak in random order, find the probability that A speaks before B, B speaks before C and C speaks before D.

   (a) $\frac{1}{4}$ (b) $\frac{1}{6}$ (c) $\frac{1}{24}$ (d) $\frac{1}{25}$

8. A cube has 3 blank but different coloured faces and the remaining faces numbered 1, 2 and 3. What is the probability of obtaining a total of exactly 10 in 4 throws?

   (a) $\frac{5}{648}$ (b) $\frac{5}{584}$ (c) $\frac{5}{512}$ (d) None of these

9. Bag A contains 6 red and 4 blue balls. Bag B contains 4 red and 6 blue balls. One ball is drawn at random from Bag A and placed in Bag B. Then one ball drawn at random from Bag B is placed in Bag A. If one ball is now drawn from Bag A, what is the probability that it is found to be red?

   (a) $\frac{31}{55}$ (b) $\frac{32}{55}$ (c) $\frac{33}{55}$ (d) $\frac{34}{55}$

10. Six white balls and four black balls are randomly placed in a row. The probability that no two black balls are placed adjacently is

    (a) $\frac{1}{4}$ (b) $\frac{1}{6}$ (c) $\frac{1}{8}$ (d) None of these

11. A bag contains ten balls numbered from 1 to 10. A person chose a ball at random and replaced it in the bag after noting its number. He repeated this process 3 more times. What is the probability that the ball chosen first is numbered higher than the ball chosen second and the second ball numbered higher than the third ball and the third ball numbered higher than the fourth ball?

    (a) $\frac{21}{10^4}$ (b) $\frac{21}{10^3}$ (c) $\frac{16}{10^4}$ (d) $\frac{16}{10^3}$

12. On a toss of two dice, A throws a total of 4. Then the probability that he would throw another 4 before he throws 6 is
(a) $\frac{1}{4}$ (b) $\frac{2}{5}$ (c) $\frac{3}{8}$ (d) None of these

13. If the integers x and y are chosen at random from integers 1 to 100 with replacement, what is the probability that a number of the form 3^x+3^y is divisible by 5?
(a) $\frac{1}{4}$ (b) $\frac{1}{7}$ (c) $\frac{1}{8}$ (d) $\frac{1}{49}$

14. There are 7 seats in a row. Three persons take seats at random. What is the probability that the middle seat is always occupied and no two persons are sitting on adjacent seats?
(a) $\frac{7}{70}$ (b) $\frac{14}{35}$ (c) $\frac{4}{35}$ (d) None of these

15. Three numbers are chosen at random from the numbers 10 to 99 with replacement. What is the probability that the product of the digits of the number is 12 at least once?
(a) $1-\left(\frac{43}{45}\right)^3$ (b) $\left(\frac{43}{45}\right)^3$
(c) $1-\left(\frac{43}{90}\right)^3$ (d) $\left(\frac{43}{90}\right)^3$

SESSION – 7

1. There are 10 pairs of socks in a cupboard from which 4 individual socks are picked at random. The probability that there is at least one pair is
(a) $\frac{195}{323}$ (b) $\frac{99}{323}$ (c) $\frac{198}{323}$ (d) $\frac{185}{323}$

2. A thought of a two-digit number and divided the number by the sum of the digits of the number. He found that the remainder is 3. B also thought of a two-digit number and divided the number by the sum of the digits of the number. He also found that the remainder is 3. Find the probability that the two digit numbers thought by A and B are same?
(a) $\frac{1}{12}$ (b) $\frac{1}{13}$ (c) $\frac{1}{14}$ (d) $\frac{1}{15}$

3. Five different balls numbered 1, 2, 3, 4, 5 are distributed at random in five boxes numbered 1, 2, 3, 4, 5. What is the probability that exactly one ball occupies the place corresponding to its number?
(a) $\frac{5}{120}$ (b) $\frac{3}{8}$ (c) $\frac{1}{2}$ (d) None of these

4. There are totally 7 persons in a room. What is the probability that exactly two of them were born on the same day of the week?
(a) $\frac{1080}{7^5}$ (b) $\frac{2160}{7^5}$ (c) $\frac{540}{7^4}$ (d) None of these

5. 100 identical coins each with probability p of showing up Heads are tossed once. If the probability of Heads showing on 50 coins is equal to that of Heads showing on 51 coins, then value of p is
(a) $\frac{1}{21}$ (b) $\frac{49}{101}$ (c) $\frac{50}{101}$ (d) $\frac{51}{101}$

6. A and B are throwing an unbiased die. If B throws 2, what is the probability that A will throw a higher number?
(a) $\frac{1}{4}$ (b) $\frac{2}{3}$ (c) $\frac{5}{6}$ (d) $\frac{1}{8}$

7. A five digit number is formed by using digits 1, 2, 3, 4 and 5 without repetition. What is the probability that the number is divisible by 3?
(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) 1

8. Three dice are thrown simultaneously. Find the probability that all of them show the same number.
(a) $\frac{1}{216}$ (b) $\frac{1}{36}$ (c) $\frac{4}{216}$ (d) $\frac{3}{216}$

9. If four coins are tossed at random, what is the chance that these will turn up head and tail alternately but not necessarily head in the first toss?
(a) $\frac{1}{8}$ (b) $\frac{1}{4}$ (c) $\frac{7}{8}$ (d) $\frac{3}{8}$

10. What is the probability that there are 52 Thursdays in a normal year?
(a) 0 (b) $\frac{1}{2}$ (c) $\frac{1}{7}$ (d) 1

11. A bag contains 7 objects - 3 cubes and 4 cuboids having been painted in three different colours. No two cubes have the same colour. An item is taken out from the box. What is the probability that this item is a cube having red colour?
(a) $\frac{1}{7}$ (b) $\frac{3}{7}$
(c) $\frac{1}{3}$ (d) Cannot be determined

12. In a km race where 10 runners are running, the probability that A will win is 0.25, that B will win is 0.1 and that C will win is 0.3. What is the probability that one of them will win if tie is impossible?
(a) 0.75 (b) 0.65 (c) 0.5 (d) 0.3

13. An urn contains 9 white and 11 black balls. Two balls are drawn in succession without replacement. What is the probability that first is black and second is white?
(a) $\frac{99}{380}$ (b) $\frac{43}{380}$ (c) $\frac{37}{380}$ (d) $\frac{23}{380}$

14. Three cards are drawn at random from a pack of 52 cards. What is the probability of getting all the three cards of the same suit?
(a) $\frac{11}{425}$ (b) $\frac{22}{425}$ (c) $\frac{11}{850}$ (d) None of these

15. An urn contains 3 red, 5 green and 2 blue balls. Two balls are drawn at random. What is the probability that none of the balls drawn is blue?
(a) $\frac{8}{10}$ (b) $\frac{13}{45}$ (c) $\frac{28}{45}$ (d) None of these

16. A box contains 20 electric bulbs, out of which 4 are defective. Two bulbs are chosen at random from this box. The probability that atleast one of these is defective is
(a) 7/19 (b) 32/95 (c) 3/95 (d) 7/95

17. A speaks truth in 75% of cases and B in 80% of cases. In what percentage of cases are they likely to contradict each other, narrating the same incident
(a) 25% (b) 50% (c) 35% (d) 45%

18. Tickets numbered 1 to 20 are mixed up and then a ticket is drawn at random. What is the probability that the ticket drawn has a number which is a multiple of 3 or 5?
(a) 6/20 (b) 9/20
(c) 10/20 (d) None of these

19. A box contains 5 green, 4 yellow and 3 white balls. Three balls are drawn at random. What is the probability that they are not of same color?
(a) 3/44 (b) 3/55 (c) 52/55 (d) 41/44

20. The odds in favor of standing first of three students Amit, Vikas and Vivek appearing at an examination are 1:2, 2:5 and 1:7 respectively. What is the probability that either of them will stand first (assume that a tie for the first place is not possible)?
(a) 168/178 (b) 122/168
(c) 5/168 (d) 125/168

21. A box contains 6 red balls, 7 green balls and 5 blue balls. Each ball is of a different size. The probability that the red ball selected is the smallest red ball is
(a) 1/18 (b) 1/3 (c) 1/6 (d) 2/3

22. From a group of 13 scientists which contain 5 mathematicians and 8 physicists, it is required to appoint a committee of two. If the selection is made without knowing the identity of the scientists. What is the probability that one will be mathematician and the other a physicist?
(a) 20/39 (b) 1/4
(c) 7/13 (d) None of these

23. Of a total of 600 bolts, 20% are too large and 10% are too small. The remaining are considered as suitable bolts. If a bolt is selected at random, what is the probability that it will be suitable?
(a) 7/10 (b) 3/5 (c) 2/7 (d) 3/10

24. A problem in mathematics is given to three students whose chances of solving it are 1/2, 1/3 and 1/4 respectively. What is the probability that the problem will be solved?
(a) 3/5 (b) 3/4 (c) 5/7 (d) 7/10

25. In a two-child family, one child is a boy. What is the probability that the other child is a girl?
(a) 2/3 (b) 2/5 (c) 5/3 (d) 3/2

SESSION – 8

Probability Case Study

LOGARITHM

Let a be a positive number other than 1 and let x be a real number.

If $a^x = b$, then this can be written as $\log_a b = x$.

From the definition of logarithm, we can deduce that, log1 = 0 and $\log_a a = 1$

Product rule

$\log_a mn = \log_a m + \log_a n$, where m and n are positive numbers and $a \neq 0$

Quotient rule

$\log_a \frac{m}{n} = \log_a m - \log_a n$, where m and n are positive numbers and $a \neq 0$

Power rule

$\log_a m^n = n\log_a m$, where m is a positive number and $a \neq 0$

Change of base rule

$\log_a m = \log_b m \times \log_a b$, where m, a and b are positive numbers and $a \neq 0$ and $b \neq 0$

Important results on logarithm

- $\log_a b = \frac{1}{\log_b a}$
- $a^{\log_a n} = n$
- $\log_{b^y} a^x = \frac{x}{y}\log_b a$
- $\log_a x = \frac{\log_b x}{\log_b a}$

Application 1

If $\log_{2\sqrt{2}}\left(\frac{1}{128}\right) = x$, find the value of x.

Solution

$\left(2\sqrt{2}\right)^x = 128^{-1} = 2^{-7}$

$2^{\frac{3}{2}x} = 2^{-7}$

So, $x = -\frac{14}{3}$

Aliter

$$x=\log_{2\sqrt{2}}\left(\frac{1}{128}\right)=\log_{2^{\frac{3}{2}}}2^{-7}=\frac{-7}{\frac{3}{2}}\times\log_2 2$$

So, $x=-\frac{14}{3}$

Application 2

Find the value of $4^{2+\log_4 8-\log_{16}2}$.

Solution

$$4^{2+\log_4 8-\log_{16}2}=4^2\times4^{\log_4 8}\times4^{-\log_{16}2}$$

$$=4^2\times4^{\log_4 8}\times4^{-\frac{1}{2}\log_4 2}$$

(as $\log_{16}2\times\log_4 16=\log_4 2$ and $\log_{16}2=\frac{1}{2}\log_4 2$)

$$=4^2\times8\times2^{-\frac{1}{2}}=64\sqrt{2}$$

SMART Tip

- *If $\log(x+y)=\log x+\log y$ then $x=\frac{y}{y-1}$.*
- $a^{\log_b c}=c^{\log_b a}$.

Application 3

Find the value of $\log_{16}64-\log_9 27$.

Solution

$$\log_{16}64-\log_9 27=\log_{4^2}4^3-\log_{3^2}3^3$$

$$=\frac{3}{2}-\frac{3}{2}=0$$

Application 4

If $\log_{125}x+\log_{25}x+\log_5 x=11$, find X.

Solution

Given equation can be written as,

$$\frac{1}{\log_x 125}+\frac{1}{\log_x 25}+\frac{1}{\log_x 5}=11$$

$$\frac{1}{3\log_x 5}+\frac{1}{2\log_x 5}+\frac{1}{\log_x 5}=11$$

$$\Rightarrow\frac{1}{\log_x 5}\left(\frac{1}{3}+\frac{1}{2}+1\right)=11$$

$$\frac{2+3+6}{6}=11\times\log_x 5 \text{ or } \log_x 5=\frac{1}{6}$$

$$\Rightarrow x^{\frac{1}{6}}=5 \text{ or } x=5^6$$

SMART Tip

- *Logarithm is not defined for negative value.*
- *$\log_a 0$ is not defined.*

Application 5

If $\log_{12}27=a$, then find $\log_6 8$.

Solution

Given, $\log_{12}27=\frac{\log 27}{\log 12}=a$

$$a=\frac{3\log 3}{\log 3+\log 4}$$

$$a(\log 3+2\log 2)=3\log 3$$

$(3-a)\log 3 = 2a\log 2$

$$\frac{\log 2}{\log 3}=\frac{3-a}{2a}$$

Now, $\log_6 8=\frac{\log 8}{\log 6}=\frac{3\log 2}{\log 3+\log 2}$

$$=\frac{3\frac{\log 2}{\log 3}}{1+\frac{\log 2}{\log 3}}=\frac{3\left(\frac{3-a}{2a}\right)}{1+\frac{3-a}{2a}}=\frac{3(3-a)}{3+a}$$

Application 6

Find the number of digits in 16^{60} (given that $\log_{10}2=0.3010$).

Solution

Number of digits in 16^{60} = Integral part of $\left(60\log 2^4\right)+1$

= Integral part of (240 × 0.3010) + 1

= Integral part of 73.24 = 73

KEY POINTS

1. $\log_b a=\frac{1}{\log_a b}$
2. $\log_{b^y}a^x=\frac{x}{y}\log_b a$

SESSION – 9

1. The value of $\log_2(\log_2(\log_3(\log_3 27^3)))$ is
   (a) 14 (b) 16 (c) 0 (d) 25
2. Simplify: $[1/\log_{xy}(xyz) + 1/\log_{yz}(xyz) + 1/\log_{xz}(xyz)]$
   (a) 4 (b) 5 (c) 2 (d) 0
3. Product of roots of $\log_5(x^2) = 6$
   (a) – 15625 (b) 15625 (c) 15265 (d) – 15265

4. $\log_2 3 . \log_3 4 . \log_4 5 \ldots \log_n (n + 1) = 10$ then the value of n is
   (a) 9 (b) 99 (c) 1023 (d) 999

5. $\dfrac{\log_3 \sqrt{243\sqrt{81\sqrt[3]{3}}}}{\log_2 \sqrt[4]{64} + \log_e e^{-10}} =$
   (a) – 43/102 (b) 64
   (c) 27 (d) None of these

6. If log 2 = 0.30103, the number of digits in 5^{20} is
   (a) 18 (b) 14 (c) 24 (d) 20

7. If $\log_{10}5 + \log_{10}(5x + 1) = \log_{10} (x + 5) + 1$, then x is equal to
   (a) 5 (b) 3 (c) 10 (d) 1

8. If $\log_{32} A = 0.8$, then A is equal to
   (a) 16 (b) 256 (c) 10 (d) 128

9. If $\log_{10}36 = y$, what does $\log_{10} 36000$ equal?
   (a) 1000 (b) y
   (c) y + 1000 (d) y + 3

10. $\log_8 2 + \log_8(1/2) = ?$
    (a) 0 (b) 1 (c) 2 (d) 8

11. $\log_{\sqrt{2}}(\frac{1}{8}) = ?$
    (a) 1 (b) – 2 (c) – 6 (d) – 8

12. What is the value of $\log_{0.2} 2$? (Take value of log 2 = 0.3)
    (a) 0.3 (b) – 3/7 (c) – 0.06 (d) – 1/10

13. Find $\log_x 2$, if $x^3 = 128$
    (a) 4/3 (b) 3/4 (c) 3/7 (d) 7/3

14. Find x if $\log_{(\frac{1}{x})} x^2 = -\sqrt{2}$ x.
    (a) 4 (b) 2 (c) $\sqrt{2}$ (d) $\sqrt[3]{2}$

15. Find x if $a^{(2\log_a x)} = 49$.
    (a) 7/2 (b) 49/2 (c) 7 (d) 49

16. Find $\log_2 100 + \log_{\sqrt{2}} 100 + \log_{\sqrt[3]{2}} 100 + \log_{\sqrt[4]{2}} 100 + \ldots\ldots\ldots$ upto 20 terms.
    (a) 40/log2 (b) 420/log2
    (c) 380/log2 (d) None of these

17. $\log_x a + \log_{x^2} a + \log_{x^4} a + \log_{x^8} a$ ..... upto ∝.
    (a) $\log_{x^2} a$ (b) $\log_{\sqrt{x}} a$
    (c) $\log_{\sqrt[4]{x}} a$ (d) $\log_{\sqrt[8]{x}} a$

18. Find the value of $\log_{\sqrt{2}} 32 + \log_5\left(\dfrac{1}{125}\right)$.
    (a) $\dfrac{11}{2}$ (b) 7 (c) 0 (d) $\dfrac{10}{3}$

19. If $\log_8 x + \log_8 \dfrac{1}{6} = \dfrac{1}{3}$, find the value of x.
    (a) 12 (b) 16 (c) 18 (d) 24

20. Simplify: $\dfrac{1}{\log_{p/q} x} + \dfrac{1}{\log_{q/r} x} + \dfrac{1}{\log_{r/p} x}$
    (a) 0 (b) 1 (c) 2 (d) 3

QUADRATIC EQUATIONS

An equation of form $ax^2 + bx + c = 0$ where a, b and c ∈ R and a ≠ 0 is called as the quadratic equation. It is a polynomial of degree 2.

Roots of the quadratic can be found using factorization method or the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

If α and β are the roots of the equation $ax^2 + bx + c = 0$, then the sum of the roots $\alpha + \beta = -\dfrac{b}{a}$ and the product of the roots $\alpha\beta = \dfrac{c}{a}$.

NATURE OF THE ROOTS

The nature of the roots is decided by the expression $b^2 - 4ac$, called the discriminant.

If the expression $b^2 - 4ac > 0$, then the roots are real and unequal.

If the expression $b^2 - 4ac = 0$, then the roots are real and equal.

If the expression $b^2 - 4ac < 0$, then the roots are not real but will be imaginary and conjugate to each other.

Application 1

Find the roots of the equation $x^2 - x - 12 = 0$.

Solution

$x^2 - x - 12 = x^2 - 4x + 3x - 12 = x(x - 4) + 3(x - 4)$

→ $(x - 4)(x + 3) = 0$

So, the roots of the given equation are 4 and –3.

Application 2

Find the roots of the equation $x^2 + 2x - 2 = 0$.

Solution

$$x = \frac{-2 \pm \sqrt{4+8}}{2} = \frac{-2 \pm 2\sqrt{3}}{2}$$

So, the roots of the given equation are $-1+\sqrt{3}$ and $-1-\sqrt{3}$

Application 3

If $\sqrt{2}$ – 2 is a root of the equation $x^2 + 4x + 2 = 0$, find its other root.

Solution

Let the other root be β.

Sum of the roots $= \frac{-b}{a} = -4$

$\Rightarrow \sqrt{2} - 2 + \beta = -4$ or $\beta = -2 - \sqrt{2}$

$\therefore$ Other root is $-2 - \sqrt{2}$.

Application 4

If α and β are the roots of the equation $6x^2 - x - 1 = 0$, find the equation whose roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

Solution

$\alpha+\beta=\frac{-b}{a}=\frac{1}{6}$ and $\alpha\beta=\frac{c}{a}=-\frac{1}{6}$

Sum of the roots of the required equation:

$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=\frac{\frac{1}{6}}{-\frac{1}{6}}=-1$

Product of the roots of the required equation: $\frac{1}{\alpha\beta}=-6$

So, the required equation is $x^2-\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)x+\frac{1}{\alpha\beta}=0$

$\Rightarrow x^2 + x - 6 = 0$

Application 5

If α and β are the roots of the equation $x^2 + 4x - 45 = 0$, find the value of $\alpha^4 + \beta^4$.

Solution

Here, $\alpha + \beta = -4$ and $\alpha\beta = -45$

$\alpha^4+\beta^4=\left(\alpha^2+\beta^2\right)^2-2\alpha^2\beta^2$

$=\left(\left(\alpha+\beta\right)^2-2\alpha\beta\right)^2-2\alpha^2\beta^2=(16 + 90)^2 - 4050$

$= 11236 - 4050 = 7186$

Application 6

A and B solves a quadratic equation. In solving, A commits a mistake in constant term and finds the roots as –2 and –8. In solving, B commits a mistake in the coefficient of x and finds the roots as 8 and 3. Find the correct roots of the equation.

Solution

Let the correct equation be $x^2 + ax + b = 0$

A's equation is $x^2 + 10x + 16 = 0$

As he commits mistake in constant term, correct value of a = 10

B's equation is $x^2 - 11x + 24 = 0$

As he commits mistake in the coefficient of x, correct value of b = 24

So, the given equation is $x^2 + 10x + 24 = 0$

Therefore, the correct roots of the given equation is –6 and –4

SMART Tip

In a quadratic equation $ax^2+bx+c=0$.

If b = 0, then roots are equal in magnitude but opposite in sign.

If c = 0, then one of it root is zero.

If a = c, then roots are reciprocal to each other.

Application 7

For what value of k, the equation $4x^2 - 4x + k = 0$ has equal roots?

Solution

Determinant = $b^2-4ac=16 - 16k = 0$ or $1 - k = 0$

So, for k = 1, the given equation has equal roots.

Application 8

Find the range of p, so that the equation $x^2 + px + 16 = 0$ has real roots.

Solution

For the solution to be real, determinant ≥ 0.

Determinant = $p^2 - 64 \geq 0$ or $(p + 8)(p - 8) \geq 0$

So, the required range is $p \geq 8$ and $p \leq -8$

Application 9

Solve: $\sqrt{x+3}=2x-9$

Solution

On squaring the given equation,

$x+3=4x^2-36x+81$

$4x^2 - 37x + 78 = 0$

$4x^2 - 24x - 13x + 78 = 0$

$4x(x - 6) - 13(x - 6) = 0$ or $(4x - 13)(x - 6) = 0$

Therefore, x = 6 (as $x = \frac{13}{4}$ does not satisfy the given equation)

SMART Tip

For $ax^2+bx+c=0$ with a > 0, the minimum value of the expression is $\frac{4ac-b^2}{4a}$ for $x=\frac{-b}{2a}$.

For $ax^2+bx+c=0$ with a < 0, the maximum value of the expression is $\frac{4ac-b^2}{4a}$ for $x=\frac{-b}{2a}$.

Application 10

The difference between a number and its reciprocal is $\frac{5}{6}$. Find the number.

Solution

Let the given number be x.

By the given data, $x-\frac{1}{x}=\frac{5}{6}$

On rewriting, $6(x^2 - 1) = 5x$ or $6x^2 - 5x - 6 = 0$

On factorizing, $(3x + 2)(2x - 3) = 0$

Therefore, the required number is $-\frac{2}{3}$ or $\frac{3}{2}$.

Application 11

The height of a triangle is 1.3 cm shorter than the base of it. If its area is 2.1 sq. cm, find its base.

Solution

If base of the triangle is x cm, its height is x – 1.3 cm.

Area $=\frac{1}{2}(\text{base}\times\text{height})=\frac{1}{2}\times x\times(x-1.3)=2.1$ or $x^2 - 1.3x = 4.2$

Or $10x^2 - 13x - 42 = 0$

On factorizing, $(2x + 3)(5x - 14) = 0$

$\Rightarrow x = \frac{14}{5}$ or $-\frac{3}{2}$ (negative value is neglected as x represent a length)

Therefore, the base is 2.8 cm.

SMART Tip

If a, b and c are all of same signs, both roots are negative.

If the sign of b is different from both a & c $\Rightarrow$ both roots are positive.

In a quadratic equation, if a + b + c = 0, then one root is 1 & the other root is $\frac{c}{a}$.

Application 12

A left an hour later than the scheduled time. In order to reach his native place 600 km away in time, he had to increase the speed by 20 km/h from his usual speed. Find his usual speed.

Solution

Let A's usual speed be x km/h.

By the given data, $\frac{600}{x}-\frac{600}{x+20}=1$

On simplifying, $x^2 + 20x - 12000 = 0$

On factorizing, $(x + 120)(x - 100) = 0$

$\Rightarrow x = 100$ (negative value is neglected as x represents speed)

So, his usual speed = 100 km/h

KEY POINTS

1. Quadratic equations with imaginary roots will not cut x-axis.
2. Graph of ax^2+bx+c will be in U form or inverted U form if a > 0 or a < 0 respectively.
3. A quadratic equation has either both roots real or both imaginary roots.

SESSION – 10

Find the correct answer.

1. If $7a + 8b = 53$ and $9a + 5b = 47$, find the values of a and b.
   (a) (4, 5) (b) (4, 3) (c) (3, 4) (d) (5, 4)
2. If I add 7 times my age seven years from now and 3 times my age three years ago, I get 12 times my current age. How old will I be 3 years from now?
   (a) 22 years (b) 23 years
   (c) 24 years (d) 25 years
3. A farmer has some chickens and feed for them. The feed lasts for exactly 30 days. He sold 10 of the chickens and purchased some feed such that the entire feed with him will last for 150 days. Now, he has thrice the initial feed. Find the initial number of chickens.
   (a) 25 (b) 30
   (c) 40 (d) Cannot be determined
4. Tree I grows at $\frac{3}{7}^{th}$ of tree II. If both the trees together grow 3 ft for every 3 years, find the time required by tree II to grow 7 ft.
   (a) 7 years (b) 10 years
   (c) 8 years (d) 12 years
5. Find the quadratic equation whose roots are the reciprocals of the roots of the equation $x^2-7x+12=0$.
   (a) $x^2-12x+7=0$ (b) $x^2+12x-7=0$
   (c) $12x^2+7x-1=0$ (d) $12x^2-7x+1=0$
6. The equation $\sqrt{4x+9}-\sqrt{11x+1}-\sqrt{7x+4}=0$ has
   (a) no solution
   (b) 1 solution
   (c) 2 solutions
   (d) more than 2 solutions
7. The equation $(\cos p-1)x^2+(\cos p)x+\sin p=0$ has real roots if p lies in the interval
   (a) $(0, 2\pi)$ (b) $(-\pi, 0)$
   (c) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ (d) $(0, \pi)$

8. If the sum of the roots of the equation $ax^2+bx+c=0$ is equal to the sum of the squares of their reciprocals, then
(a) ab^2, ca^2, bc^2 are in A.P.
(b) ab^2, bc^2, ca^2 are in A.P.
(c) ab^2, bc^2, ac^2 are in A.P.
(d) None of these

9. Let a, b, c be $\in R$ and $a \neq 0$. If α is a root of $a^2x^2+bx+c=0$, β is a root of $a^2x^2-bx-c=0$ and $0<\alpha<\beta$, then the equation $a^2x^2+2bx+2c=0$ has a root γ that always satisfies
(a) $\gamma-\frac{1}{2}(\alpha+\beta)$ (b) $\gamma=\alpha+\frac{\beta}{2}$
(c) $\gamma=\alpha+\beta$ (d) $\alpha<\gamma<\beta$

10. If for the quadratic equation $x^2-kx+1=0$, one of the roots is A such that $\tan A=2\sqrt{5}-1$, then the other root is
(a) greater than 1 (b) greater than 2
(c) less than 1 (d) None of these

11. If p and q are the roots of the equation $x^2+px+q=0$, then
(a) $p=1$ (b) $p=1$ or 0
(c) $p=-2$ (d) $p=-2$ or 0

12. If a, b, c are positive real numbers which are in G.P., then the equation $ax^2+2bx+c=0$ and $dx^2+2ex+f=0$ have a common root if $\frac{a}{d}, \frac{b}{e}, \frac{c}{f}$ are in
(a) A.P. (b) G.P.
(c) H.P. (d) None of these

13. If $f(x)=x^2+2x-5$ and $g(x)=5x+30$, find the roots of the quadratic equation $g[f(x)]=0$.
(a) $-1,-1$
(b) $2,-1$
(c) $-1+\sqrt{2}, -1-\sqrt{2}$
(d) 1, 2

14. The only value of x satisfying the equation $6\sqrt{\frac{x}{4+x}}-2\sqrt{\frac{4+x}{x}}=11$, $x\in R$ is
(a) $\frac{4}{35}$ (b) $\frac{16}{3}$ (c) $-\frac{4}{35}$ (d) $-\frac{16}{3}$

15. The number of real values of x satisfying the equation $2\left(x^2+\frac{1}{x^2}\right)-9\left(x+\frac{1}{x}\right)+14=0$ is
(a) 1 (b) 2 (c) 3 (d) 4

16. Let α, β be the roots of $x^2-x+p=0$ and γ, δ be the roots of $x^2-4x+q=0$. If $\alpha, \beta, \gamma, \delta$ are in G.P., find the values of p and q.
(a) $-2, -32$ (b) $-2, 3$
(c) $-6, 3$ (d) $-6, -32$

17. If one root of the quadratic equation $ax^2+bx+c=0$ is three times the other, find the relationship between a, b and c.
(a) $3b^2=16\,ac$ (b) $b^2=4\,ac$
(c) $(a+c)^2=4b$ (d) $\frac{a^2+c^2}{ac}=\frac{b}{2}$

18. If the roots of the equation $(a^2+b^2)x^2-2b(a+c)x+(b^2+c^2)=0$ are equal, then a, b, c are in
(a) A.P. (b) G.P.
(c) H.P. (d) Cannot be determined

19. If 2 quadratic equations $ax^2 + ax + b=0$ and $x^2 + x + b = 0$ have a common root, x = 1 find out which of the following statements is correct.
(a) a + b = -3.5 (b) ab = 3
(c) $\frac{a}{b}=\frac{3}{4}$ (d) a – b = – 0.5

20. Find the least value of $x^2 + 6x - 27$.
(a) – 63 (b) – 36 (c) 42 (d) 24

TRIGONOMETRY

Trigonometric Properties

Tangent and Cotangent Identities

1. tan θ = sin θ /cos θ
2. cot θ = cos θ /sin θ

Reciprocal Identities

3. sin θ = 1/ cosec θ
4. cosec θ = 1 /sin θ
5. cos θ = 1 /sec θ
6. sec θ = 1 /cos θ
7. tan θ = 1/ cot θ
8. cot θ = 1/ tan θ

Pythagorean Identities

10. $\sin^2\theta + \cos^2\theta = 1$
12. $\tan^2\theta + 1 = \sec^2\theta$
13. $1 + \cot^2\theta = \text{cosec}^2\theta$

Sum and Difference Formulae

14. sin(α ± β) = sin α cos β ± cos α sin β

15. cos(α ± β) = cos α cos β ∓ sin α sin β

16. tan(α ± β) = (tan α ± tan β)/(1 ∓ tan α tan β)

SESSION – 11

1. If tan A = 2, evaluate $\sec A.\sin A+\tan^2 A-\operatorname{cosec} A$

 (a) $\frac{12-\sqrt{5}}{2}$ (b) $\frac{\sqrt{3}+1}{2}$

 (c) $\frac{1-\sqrt{3}}{2\sqrt{2}}$ (d) None of these

2. If $\sin B=\frac{1}{2}$ find the value of $3\cos B-4\cos^3 B$

 (a) $\sqrt{2}$ (b) 5 (c) 0 (d) 1

3. Evaluate: $\tan 30^\circ \sec 45^\circ + \tan 60^\circ \sec 30^\circ$

 (a) $\left(2+\sqrt{3}\right)$ (b) $\left(\sqrt{3}-\sqrt{2}\right)$

 (c) $\frac{\sqrt{2}+2\sqrt{3}}{\sqrt{3}}$ (d) $\left(1+\sqrt{3}\right)$

4. Find the value of $\left\{\frac{\frac{4}{\tan^2 60^\circ}+\frac{1}{\cos^2 30^\circ}-2\sin^2 45^\circ}{\left(\sin^2 60^\circ+\cos^2 45^\circ\right)}\right\}$

 (a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{4}{3}$ (d) $\frac{1}{7}$

5. Find the value of
 $\cos^2 60^\circ \tan^2 30^\circ + \sin 30^\circ \cos 0^\circ \sin 60^\circ \tan 45^\circ$

 (a) $\frac{3\sqrt{3}}{14}$ (b) $\frac{1+3\sqrt{3}}{12}$

 (c) $\frac{1}{\sqrt{3}+12}$ (d) None of these

6. Simplify
 $\tan^2 30^\circ + \frac{1}{2}\sin^2 45^\circ + \frac{1}{3}\cos^2 30^\circ + \frac{1}{\tan^2 60^\circ}$

 (a) $1\frac{1}{7}$ (b) $1\frac{1}{9}$ (c) $1\frac{1}{6}$ (d) $2\frac{2}{7}$

7. A wheel makes 360 revolutions in one minute through how many radians does it turn in one second?
 (a) 10π radians (b) 8π radians
 (c) 12π radians (d) 16π radians

8. Find the value of
 $\sin 420^\circ \cos 390^\circ + \cos\left(-300^\circ\right)\sin\left(-330^\circ\right)$

 (a) 0 (b) 1 (c) $\sqrt{2}$ (d) $\sqrt{3}$

9. If A, B, C, D are angles of a cyclic quadrilateral find the value of $\cos A+\cos B+\cos C+\cos D$

 (a) 0 (b) 1 (c) $\sqrt{2}$ (d) $\sqrt{3}$

10. If $x=r\sin\theta.\cos\varphi$
 $y=r\sin\theta.\sin\varphi$
 $z=r\cos\varphi$, find the value of $x^2+y^2+z^2$

 (a) r (b) r^2 (c) r^3 (d) r^4

11. The difference between two complementary angles is 2 π/5 radians. Find the angles in degrees.
 (a) 30°, 60° (b) 9°, 81°
 (c) 45°, 45° (d) 25°, 65°

12. A moving horse, tied to a post by a rope along a circular path always keeping the rope tight and describes 88 metres when it traces 45° at the centre, find the length of the rope.
 (a) 125 m (b) 112 m
 (c) 108 m (d) 98 m

13. A person ties a balloon on 550 m of a string and flies at an angle of 45°. It is observed that the angle formed by the balloon on the ground is 75°. Find the distance of the balloon from the angular point it makes on the ground.
 (a) 401.6 m (b) 400 m
 (c) 402.9 m (d) 408.6 m

14. If 3tan θ = – 4 and θ is 90° ≤ θ ≤ 180°, find cosec θ.
 (a) 4/5 (b) 3 – √2
 (c) 3 + √2 (d) 5/4

15. The length of the arc of a semicircle is equal to the perimeter of a certain sector of a circle having the same radius. Find the angle of the sector.
 (a) 65.5 ° (b) 63.8° (c) 67° (d) 62°

16. An UFO is hovering 800 m above a river. It is observed by the passengers of a boat as they look upwards at an angle of 30°. Thirty seconds later, passengers look up at an angle of 60° to see the UFO. How far did the boat in 30 seconds?
 (a) 923.7 m (b) 925 m
 (c) 925.8 m (d) 929 m

17. In Δ ABC, b:c = √3:√2 and the angles are in A.P. Find ∠A.
 (a) 65° (b) 75° (c) 85° (d) 70°

18. The base of a tower erected vertically on a hill side makes an angle of 15° with the horizontal. From a point on the ground 50 m down the hill from the base of the tower, the angle of elevation of the top of the tower is 45°. Find the height of the tower.
 (a) 25 (b) 35 (c) 45 (d) 50

19. Find the value of θ if sin θ + cos θ = √2
 (a) 45° (b) 60° (c) 75° (d) 30°

20. Solve $\cot^2\pi/6$ + cosec 5π/6 + 3 $\tan^2\pi/6$
 (a) 4 (b) 5 (c) 6 (d) 7

FUNCTIONS

Properties of Functions:

Definition of a Function: A function is a rule or formula that associates each element in the set *X* (an input) to exactly one and only one element in the set *Y* (the output). Different elements in *X* can have the same output, and not every element in *Y* has to be an output.

Definition of the Domain of a Function: The set of all possible inputs of a function is defined as the **domain**. The domain of a real-valued function defined by a formula for y in terms of x will be the set of all x input-values that result in a real y output-value unless the domain of the function is further restricted.

Definition of the Range of a Function: The set of all possible outputs of a function is defined as the **range**. The range of a real-valued function defined by a formula for y in terms of x will be the set of all y output-values that result from the x input-values in the domain.

Function Notation: Given that f(x) is given by some formula containing x, f(B) will be the same formula with each x replaced by B.

Linear Function Definition: If a function may be written in the form f(x) = mx + b where x is the independent variable and m and b are constants, then f(x) represents a linear function. The constant m is defined as the slope and the point (0, b) represents the y-intercept. An equation in this form is known to be in *Slope-Intercept Form*.

Linear Function Slope Definition: Given that f(x) = mx + b, then m is defined as the slope where: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ for any two points (x_1, y_1) and (x_2, y_2) on the line. Graphically, the slope represents the change in y with respect to x on the graph of the line.

Linear Functions of Parallel Lines: If two linear functions are given by $f(x) = m_1x + b$ and $g(x) = m_2x + b_1$ and $m_1 = m_2$, then the graphs of f(x) and g(x) will consist of two lines that are parallel to each other.

Linear Functions of Perpendicular Lines: If two linear functions are given by $f(x) = m_1x + b$ and $g(x) = m_2x + b$, and $m_1 = -1/m_2$, then the graphs of f(x) and g(x) will consist of two lines that are perpendicular to each other.

Graphs of Even Functions: Given a function f(x), if f(*c*) = f(–*c*) for all *c* in the domain, then f(x) is an *even* function and its graph will have symmetry with respect to the y-axis.

Graphs of Odd Functions: Given a function f(x), if f(*c*) = – f(– *c*) for all *c* in the domain, then f(x) is called an *odd* function and its graph will have symmetry with respect to the origin. Symmetry with respect to the origin implies that a 180 degree rotation of the graph about (0,0) results in an identical graph.

Functions Shifted Left: Given a function f(x) and its graph and a value of c>0, the graph of f(x + c) will be a shift of the graph of f(x) left by "c" units. This is known as the *Left Shift Function Rule*.

Functions Shifted Right: Given a function f(x) and its graph and a value of c>0, the graph of f(x – c) will be a shift of the graph of f(x) right by "c" units. This is known as the *Right Shift Function Rule*.

Functions Shifted Up: Given a function f(x) and its graph and a value of c >0, the graph of f(x) + c will be a shift of the graph of f(x) up by "c" units. This is known as the *Vertical Shift up Function Rule*.

Functions Shifted Down: Given a function f(x) and its graph and a value of c > 0, the graph of f(x) – c will be a shift of the graph of f(x) down by "c" units. This is known as the *Vertical Shift down Function Rule*.

Function Reflected Across X-axis Given a function f(x) and its graph, the graph of g(x) = – f(x) will be a reflection of the graph of f(x) across the x-axis. This is known as the *X-axis Reflection Function Rule*.

Function Reflected Across Y-axis Given a function f(x) and its graph, the graph of g(x) = f(– x) will be a reflection of the graph of f(x) across the x-axis. This is known as the *Y-axis Reflection Function Rule*.

Function Vertically Stretched Or Shrunk Given a function f(x) and its graph and a value of *c* > 0, the graph of g(x) = *c* ●f(x) will be a vertical stretch of the graph of f(x). This means that all y-values of g(x) will be equal to *c* times the respective y-values of f(x). This is known as the *Vertical Stretch Function Rule*.

Definition of a Polynomial Function If f(x) may be written in the form $a_1x^n + a_2x^{n-1} + a_3x^{n-2} + \ldots + a_n$, then f(x) is a polynomial function of degree n where $a_1, a_2, \ldots a_n$ are real coefficients. Linear functions are 1st degree polynomials and quadratic functions are 2nd degree polynomials.

Graphs of Polynomials Given a function f(x) is a polynomial, it's x-intercepts will be located at the x-values x = c such that f(c) = 0. Other solution points on the graph will be located between each two x-intercepts.

Standard Form of Quadratic Functions: Quadratic functions of the form $f(x) = ax^2 + bx + c$ may always be rewritten in the form $y = a(x - h)^2 + k$. Function shift rules may then be applied to state that the graph will be a vertical stretch of $y = x^2$ and will be shifted right, left, up, or down according to the values of h and k.

Graphs of Quadratic Functions in Form $f(x) = ax^2 + bx + c$: Given $f(x) = ax^2 + bx + c$, the graph will be a shift of $g(x) = ax^2$ (meaning it has the same shape), and will have a vertex at $x = -b/2a$, $y = f(-b/2a)$.

Property of The Vertex of a Quadratic Function: The vertex of $f(x) = ax^2 + bx + c$ will be the lowest point of the graph if $a > 0$ and will be the highest point of the graph if $a < 0$. The vertex represents the minimum value of the function for $a > 0$ and represents the maximum value of the function if $a < 0$.

Function Operations: Given two functions f(x) and g(x), the operations (f + g)(x), (f – g)(x), (fg)(x), and (f/g)(x) are defined in the following way:

- $(f + g)(x) = f(x) + g(x)$ and is sometimes denoted $f + g$
- $(f - g)(x) = f(x) - g(x)$ and is sometimes denoted $f - g$
- $(fg)(x) = f(x) \bullet g(x)$ and is sometimes denoted fg
- $(f/g)(x) = f(x)/g(x)$ provided $g(x) \neq 0$. This is sometimes denoted f/g

Function Composition: Given two functions f(x) and g(x), the function composition (fog)(x), is defined in the following way:

$(f \, o \, g)(x) = f[g(x)]$ and is sometimes denoted as $f \, o \, g$

In essence, composition implies that you input the entire formula of the second function in for each x-value of the the formula in the first function, assuming x is the variable used.

Definition of Inverse Functions: Given two functions f(x) and g(x), if (f o g)(x) = x and (g o f)(x) = x, then f(x) is the inverse of g(x) and g(x) is the inverse of f(x). Each of these functions reverses the operations of the other function in reverse order. In that sense, the inverse of f(x) will consist of the identical formula with x and y interchanged - the solution for y results in "reversing" all operations on x and thus results in the formula for the inverse function.

We denote the inverse of f(x) as $f^{-1}(x)$ and we denote the inverse of g(x) as $g^{-1}(x)$.

Domain and Range of Functions That Are Inverses of Each Other: Given two functions f(x) and g(x) are inverses of each other, then

The domain of f(x) will consist of the same interval as the range of g(x).

The range of f(x) will consist of the same interval as the domain of g(x).

One-To-One Requirement For f(x) To Have an Inverse Function: Given a function f(x), it will only have an inverse if and only if each y-value in it's range corresponds to only 1 x-value in it's specified domain. When this is the case that each y is obtained from only 1 x-value, we say *f(x) is a one-to-one function.*

Note that a graphical way to determine that f(x) is not one-to-one is to show that a horizontal line passes through more than 1 point. This is often referred to as the **Horizontal Line Test**.

SESSION – 12

1. Let f(x) = max(2x + 1, 3 – 4x), where x is any real number. Then the minimum possible value of f (x) is:
   (a) 1/3 (b) 1/2 (c) 2/3 (d) 5/3

2. Let $f(x) = ax^2 - b|x|$, where a and b are constants. Then, at x = 0, f(x) is:
   (a) maximized whenever a > 0, b > 0
   (b) maximized whenever a > 0, b < 0
   (c) minimized whenever a > 0, b > 0
   (d) minimized whenever a > 0, b < 0

3. For the function f(x) = 2x – 1, g(x) = 5 – x, and $h(x) = x^2 + x + 1$, find range of x for which $\min\{f(x^2), h(x)\} < 3$.
   (a) $-2 < x < \sqrt{2}$ (b) $-\sqrt{2} < x < \sqrt{2}$
   (c) $-2 < x < 2$ (d) $-\sqrt{2} < x < 2$

4. The function f(x) = |x – 2| + |2.5 – x| + |3.6 – x|, where x is a real number, attains a minimum at:
   (a) x = 2.3 (b) x = 2.5
   (c) x = 2.7 (d) None of these

5. Find the minimum value of f(x) = |3x – 2| + |2x – 3|.
   (a) 5/6 (b) 5/3 (c) 5/2 (d) None of these

6. Find the minimum value of f(x) = max(k – x, |x| + k).
   (a) k – 1 (b) k (c) 2k (d) None of these

7. Let $f(x) = ax^2 + bx + c$, where a, b and c are certain constants and a ≠ 0. it is know that f(5) = – 3f(2) and that 3 is the root of f(x) = 0. What is the other root of f(x) = 0?
   (a) – 7 (b) – 4
   (c) 2 (d) Cannot be determined

8. If $f(x) = x^3 - 4x + p$ and if f(0) and f(1) are of opposite sign, then which of the following is necessarily true?
   (a) – 1 < p < 2 (b) 0 < p < 3
   (c) – 2 < p < 1 (d) – 3 < p < 0

9. The domain of $y = \frac{1}{\sqrt{|x| - x}}$ is
   (a) (0, ∝) (b) (∝, ∝) (c) (– ∝, 0) (d) (1, ∝)

10. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$, then
    (a) f(x) is even (b) $f(x_1).f(x_2) = f(x_1+x_2)$
    (c) $\frac{f(x_1)}{f(x_2)} = f(x_1 - x_2)$ (d) f(x) is odd

11. What is the minimum and maximum value of $\frac{2x}{x^2+1}$ respectively?
    (a) – 1, 1 (b) – 2, 1 (c) $-\frac{1}{3}$, 0 (d) None of these

12. Let f(x) = max (2x + 1, 3 – 4x), where x is any real number. Then, the minimum possible value of f(x) is:
(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{5}{3}$

13. Minimum value of f(x) = |3 – x| + |2 + x| + |5 – x|, will be:
(a) 0 (b) 7 (c) 8 (d) 10

14. A function f(x) is defined as follows:
(i) f(1) = 1
(ii) f(2x) = 4 f(x) + 6
(iii) f(x + 2) = f(x) + 12x + 12
then calculate f(6).
(a) 106 (b) 96 (c) 86 (d) 76

15. Let f(x) = |x – 2| + |2.5 – x| + |3.6 – x|, where x is a real number, attains a minimum at
(a) x = 2.3 (b) x = 2.5
(c) x = 2.7 (d) None of these

16. Find for what value of a is: f(n) = (a – 2)n + 3a – 4 an even function?
(a) – 2 (b) 2 (c) 3 (d) 4

17. Let g(x) = max (5 – x, x + 2). The smallest possible value of g(x) is?
(a) 4.0 (b) 4.5 (c) 1.5 (d) None of these

18. Find the maximum value of the functions $1/(x^2 – 3x + 2)$?
(a) 11/4 (b) 1/4 (c) 0 (d) None of these

19. Let g(x) be a function such that g(x + 1) + g(x – 1) = g(x) for every real x. Then, for what value of p is the relation g(x + p) = g(x) necessarily true for every real x ?
(a) 5 (b) 3 (c) 2 (d) 6

20. A function f(x) satisfies f(1) = 3600 and f(1) + f(2) +……f(n) = n^2f(n), for all positive integers n > 1. What is the value of f(9)?
(a) 200 (b) 100 (c) 120 (d) 80

SET THEORY

Set Theory - Properties

Notations used in set theory formulae:

$n(A)$ – Cardinal number of set A.

$n_o(A)$ – Cardinality of set A.

$\overline{A} = A^C$ – Complement of set A.

U – Universal set

$A \subset B$ – Set A is proper subset of set B.

$A \subseteq B$ – Set A is subset of set B.

ϕ – Null set.

$a \in A$ – Element "a" belongs to set A.

$A \cup B$ – Union of set A and set B.

$A \cap B$ – Intersection of set A and set B.

Formulae:

For a group of two sets:

1. If A and B are overlapping sets,
$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
2. If A and B are disjoint set, $n(A \cup B) = n(A) + n(B)$
3. $n(A) = n(A \cup B) + n(A \cap B) - n(B)$
4. $n(A \cap B) = n(A) + n(B) - n(A \cup B)$
5. $n(B) = n(A \cup B) + n(A \cap B) - n(A)$
6. $n(U) = n(A) + n(B) - n(A \cap B) + n((A \cup B)^c)$
7. $n((A \cup B)^C) = n(U) + n(A \cap B) - n(A) - n(B)$
8. $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$
9. $n(A - B) = n(A \cup B) - n(B)$
10. $n(A - B) = n(A) - n(A \cap B)$
11. $n(A^c) = n(U) - n(A)$

SESSION – 13

1. Let A and B be two finite sets such that n(A) = 20, n(B) = 28 and n(A ∪ B) = 36, find n(A ∩ B).
(a) 10 (b) 12 (c) 14 (d) 16

2. In a group of 60 people, 27 like cold drinks and 42 like hot drinks and each person likes at least one of the two drinks. How many like both coffee and tea?
(a) 6 (b) 7 (c) 8 (d) 9

Directions for Q3 and Q4: Consider the sets defined below and answer the questions that follow.

Universal set U = [positive real numbers less than 300]
A= {x € N}[x is divisible by 15]
B = {2, 4, 6, 8, ……… 198, 200}
C = the set of all natural numbers divisible by 5
D = {x € N}[x < 100]
E = {3, 6, 9, 12}

3. How many of the sets described above are finite sets?
(a) 2 (b) 3 (c) 4 (d) 5

4. Are sets A/B and B/A equal?
   (a) Yes (b) No
   (c) Data insufficient (d) None of these

Directions for Q5 to Q7: The Power of a set is defined as the number of distinct subsets of that set. Let A= {x < 100: x is a prime number}. Let A^P denote the power set of A

5. Is {2, 5, 3} a subset of $A^{P?}$
   (a) Yes (b) No
   (c) Can't be determined (d) None of these

6. Find the number of sets in A^P that have exactly 3 elements
   (a) 2300 (b) 13800 (c) 25800 (d) 2800

7. How many non-empty sets are elements of the power set of A^P?
   (a) (2 ^ 2 ^ 25) – 1 (b) 1
   (c) 2 ^ 25 – 1 (d) 2 ^ 2 ^ 25

8. In a group of 100 persons, 72 people can speak English and 43 can speak French. How many can speak English only? How many can speak French only and how many can speak both English and French?
   (a) 25 (b) 28 (c) 30 (d) 35

9. On its annual sports day, School awarded 35 medals in athletics, 15 Judo and 18 in swimming. If these medals goes to a total of 58 students and only three of them got medals in all the three sports. The number of students who received medals exactly two of the three sports are
   (a) 9 (b) 4 (c) 5 (d) 7

10. In a community it is found that 52% people like Idly and 73% like sambar. If p% like both idly and sambar then
   (a) $p \geq 25$ (b) $p \leq 52$
   (c) $25 \leq p \leq 52$ (d) All of these

11. In a committee, 50 people speak Hindi, 20 speak Bengali and 10 speak both Hindi and Bengali. How many speak at least one of these two languages?
   (a) 60 (b) 50 (c) 70 (d) None of these

12. In a certain locality of Delhi, there are 1000 families. A survey indicated that 300 subscribe to the Hindustan Times daily newspaper and 250 subscribe to the Hindu daily newspaper. Of these two categories, 100 subscribe to both. Find the number of families which do not subscribe to any of these newspapers.
   (a) 650 (b) 750 (c) 850 (d) 550

13. In a class of 25 students, 12 students have taken economics; 8 have taken economics but not politics. Find the number of students who have taken politics but not economics.
   (a) 30 (b) 20 (c) 24 (d) 13

14. In a town of 840 persons, 450 persons read Hindi, 300 read English and 200 read both. The number of persons who read neither is
   (a) 210 (b) 290 (c) 180 (d) 260

15. A dinner party is to be fixed for a group consisting of 100 persons. In this party, 50 persons do not prefer fish, 60 prefer chicken and 10 do not prefer either chicken or fish. The number of persons who prefer both fish and chicken is:
   (a) 10 (b) 30 (c) 40 (d) 20

16. A class of 60 students appeared for an examination of Mercantile law, Statistics and Accountancy. 25 students failed in Mercantile law, 24 failed in statistics, 32 failed in Accountancy, 9 failed in Mercantile law alone, 6 failed in Statistics alone; 5 failed in Accountancy and Statistics only and 3 failed in Mercantile law and Statistics only. Find how many failed in all three subjects.
   (a) 60 (b) 10 (c) 20 (d) 50

17. In a certain group of 100 people, 60 use product 'A', 75 pursue vocation 'B' and 80 like watching channel 'C'. What is the minimum possible number of people in this group who use product 'A' as well as pursue vocation 'B' and who also like watching channel 'C'?
   (a) 10 (b) 16 (c) 20 (d) 15

Directions for Q18 to Q20: Read the following passage and answer the questions given below.

A survey of a society whose residents are mainly businessmen sows that one-sixth of the society residents own bungalows, one-fourth own flats and one-eighth of those who own bungalows do not own flats.

18. What fraction of the society residents owns both a bungalow and a flat?
   (a) $\frac{7}{12}$ (b) $\frac{7}{48}$
   (c) $\frac{7}{16}$ (d) None of these

19. What fraction of the society residents owns either a bungalow or a flat or both?
   (a) $\frac{4}{5}$ (b) $\frac{1}{48}$ (c) $\frac{13}{48}$ (d) $\frac{15}{48}$

20. What fraction of the society residents owns neither a bungalow nor a flat?
   (a) $\frac{35}{48}$ (b) $\frac{15}{16}$
   (c) $\frac{30}{44}$ (d) None of these

HQASTS402_1117

SESSION – 14

CONSOLIDATED LEARNING

1. In how many ways can the letters of the word BEAUTY be rearranged such that the vowels always appear together?
(a) 4! * 3! (b) 5! * 6! (c) 6! (d) 3 * 4!

2. How many different words can be formed using all the letters of the word 'INSTITUTE'?
(a) $\frac{9!}{3!2!}$ (b) $\frac{9!}{3!}$ (c) 5! (d) $\frac{9!}{3!5!}$

3. From a set of 3 capital consonants, 5 small consonants and 4 small vowels how many words can be made each starting with a capital consonant and containing 3 small consonants and 2 small vowels?
(a) 3600 (b) 7200 (c) 21600 (d) 28800

4. A software engineer creates a LAN game where an 8 digit code made up of 1, 2, 3, 4, 5, 6, 7, 8 has to be decided on, as a universal code. There is a condition that each number has to be used and no number can be repeated. What is the probability that first 4 digits of the code are even numbers?
(a) 1/70 (b) 1/840 (c) 1/8 (d) 1/40320

5. Ritu visited a mall where tokens are given while submitting the belongings at the entrance. Tokens are lettered a, b, c, …, z. Guard gives the token at random. What is the probability that token given to Ritu is consonant?
(a) 5/21 (b) 21/26 (c) 5/26 (d) 26/21

6. A coin is tossed thrice. What is the probability that the first toss of coin lands head, second tail and third land tail as well?
(a) 1/16 (b) 3/8 (c) 1/8 (d) None of these

7. Sides are added to a convex polygon so that the sum of its interior angles measures is increased by 540°. How many sides are added to the polygon?
(a) 4 (b) 3 (c) 2 (d) None of these

8. Adjacent sides of a rectangle are in the ratio 5:12. If the perimeter of the rectangle is 34 cm, find the length of the diagonal.
(a) 12 cm (b) 11 cm (c) 13 cm (d) 10 cm

9. ABCDE is a regular pentagon. The bisector of $\angle A$ of the pentagon meets the side CD in M. Find the measure of $\angle AMC$.
(a) 54° (b) 45° (c) 90° (d) 100°

10. Simplify: $\tan 203^o + \tan 22^o + \tan 203^o \tan 22^o$
(a) -1 (b) 0 (c) 1 (d) 2

11. There are two buildings, one on each bank of the river, opposite to each other. From the top of one building, which is 60 m high, the angles of depression of the top and the foot of the other buildings are 30 and 60 respectively. What is the height of the other building?
(a) 30 m (b) 18 m (c) 40 m (d) 20 m

12. A car is being driven, in a straight line and at a uniform speed towards the base of a vertical tower. The top of the tower is observed from the car and in the process, it takes 10 minutes for the angle of elevation to change from 45 to 60 degrees. After how much more time will this car reach the base of the tower? (in minutes)
(a) $5(\sqrt{3} + 1)$ (b) $6(\sqrt{3} + \sqrt{2})$
(c) $7(\sqrt{3} - 1)$ (d) $8(\sqrt{3} - 2)$

13. If log2 = 0.3010 and log3 = 0.4771, find the value of log1.5
(a) 0.1761 (b) 0.7116 (c) 0.7161 (d) 0.7611

14. If log 2 = 0.3010 and log 3 = 0.4771, the value of $\log_5 512$ is
(a) 2.870 (b) 2.967 (c) 3.876 (d) 3.912

15. Function f is defined by $f(x) = 3x^2 - 7x - 5$. Find $f(x - 2)$
(a) $3x^2 - 7x - 7$ (b) $3x^2 - 19x + 21$
(c) $x^2 - 5x - 5$ (d) $3x^2 - 19x + 7$

16. Find the domain of the real valued function f defined by f(x) = SQRT(x + 11).
(a) $(-\infty, \infty)$ (b) $(0, \infty)$
(c) $[-11, \infty)$ (d) $(-11, \infty)$

17. The set of values of p for which the roots of the equation $5x^2 + 4x + p(p-2) = 0$ are of opposite sign is
(a) $(-\infty, 0)$ (b) $(0, 2)$
(c) $(0, \infty)$ (d) None of these

18. If α, β are the roots of the equation $(x-a)(x-b) = c$ with $c \neq 0$, find the roots of the equation $(x-\alpha)(x-\beta) + c = 0$.
(a) a, c (b) b, c
(c) a, b (d) a + c, b + c

19. A firm has 40 workers working in the factory premises, 30 working in its office and 20 working in both the factory and the office. How many workers are there in the firm if all worker work either in the factory or office?
(a) 50 (b) 60 (c) 70 (d) 80

20. If A and B are two sets such that neither A is a subset of B nor B is a subset of A, then which of the following statement is true?
(a) $A \cup B \subseteq A$ (b) $A \subseteq A \cap B$
(c) $A \subseteq A \cup B$ (d) $A \cup B \subseteq B$