CLASSICAL SETS AND FUZZY SETS

Classical sets and fuzzy sets

- A classical set is defined by *crisp* boundaries, i.e., there is no uncertainty in the prescription or location of the boundaries of the set.
- A fuzzy set, on the other hand, is prescribed by vague or ambiguous properties; hence its boundaries are ambiguously specified.

(the notation used throughout this text for a fuzzy set is a set symbol with a tilde underscore, say, $\underset{\sim}{A}$, where the functional mapping is given by

$$\mu_{\underset{\sim}{A}}(x) \in [0, 1]$$

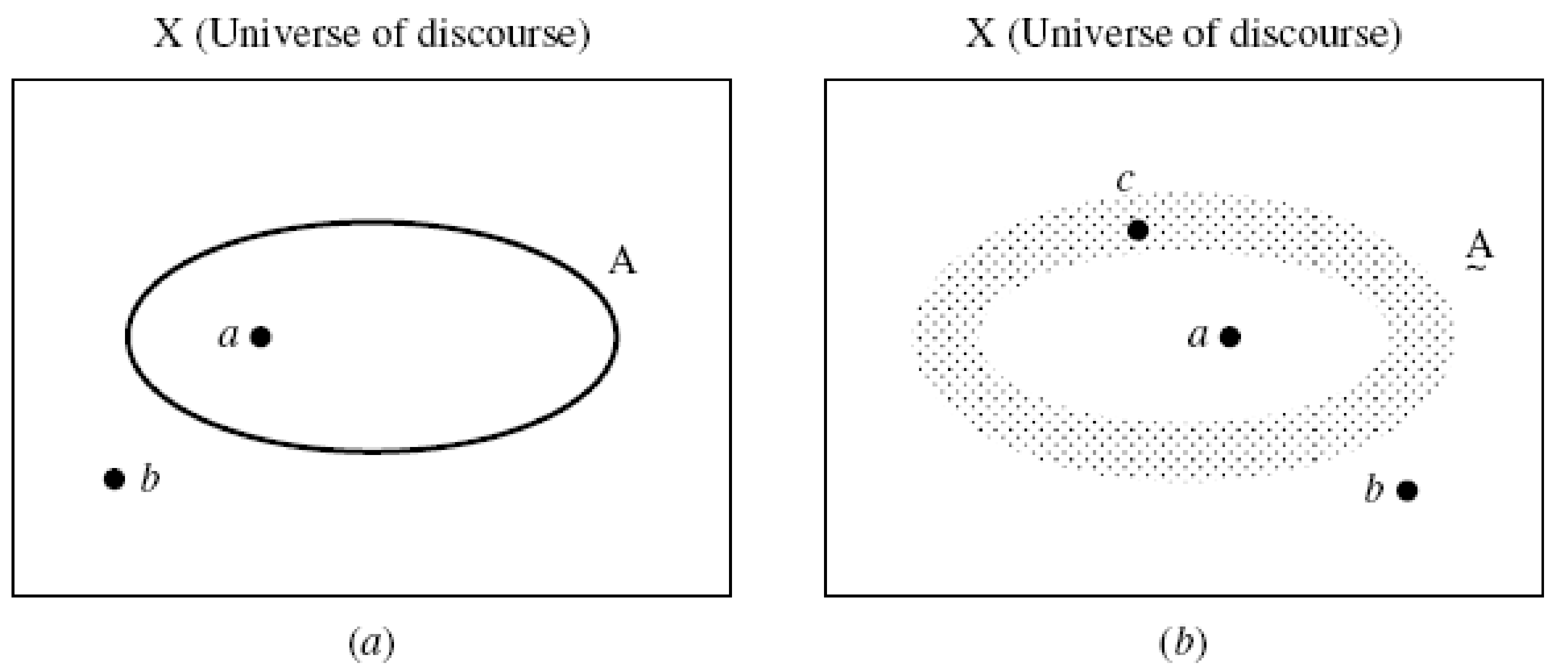


FIGURE 2.1
Diagrams for (*a*) crisp set boundary and (*b*) fuzzy set boundary.

Universe of discourse

- Define a universe of discourse, X, as a collection of objects all having the same characteristics.
- Examples:

The clock speeds of computer CPUs

The operating currents of an electronic motor

The operating temperature of a heat pump (in degrees Celsius)

The Richter magnitudes of an earthquake

The integers 1 to 10

- Most real-world engineering processes contain elements that are real and non-negative.
- For purposes of modeling, most engineering problems are simplified to consider only integer values of the elements in a universe of discourse.
- Further, most engineering processes are simplified to consider only finite-sized universes.
- upper bound

Classical sets

- The total number of elements in a universe X is called its cardinal number, denoted n_x, where *x* is a label for individual elements in the universe.

  (Discrete universes that are composed of a countably finite collection of elements will have a finite cardinal number;

continuous universes comprised of an infinite collection of elements will have an infinite cardinality.)

Classical sets

- Collections of elements within a universe are called sets, and collections of elements within sets are called subsets.
- We define the null set, $\varnothing$, as the set containing no elements, and the whole set, X, as the set of all elements in the universe.

Classical sets

For crisp sets A and B consisting of collections of some elements in X, the following notation is defined:

$x \in X$	$\Rightarrow$	x belongs to X
$x \in A$	$\Rightarrow$	x belongs to A
$x \notin A$	$\Rightarrow$	x does not belong to A

For sets A and B on X, we also have

$A \subset B$	$\Rightarrow$	A is fully contained in B (if $x \in A$, then $x \in B$)
$A \subseteq B$	$\Rightarrow$	A is contained in or is equivalent to B
$(A \leftrightarrow B)$	$\Rightarrow$	$A \subseteq B$ and $B \subseteq A$ (A is equivalent to B)

Classical sets

- All possible sets of X constitute a special set called the power set, P(X).
- **Example 2.1.** We have a universe comprised of three elements, X = {a, b, c}, so the cardinal number is $n_x = 3$. The power set is

P(X) = {∅, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}}

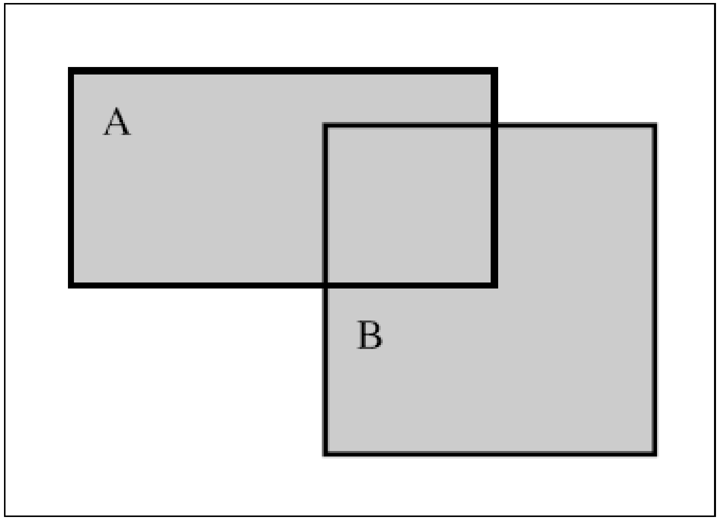
The cardinality of the power set, denoted $n_{P(X)}$, is found as

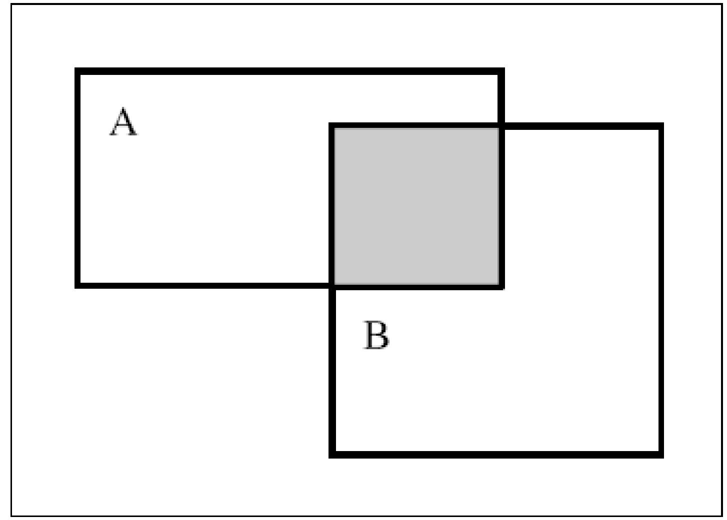
- $$n_{P(X)} = 2^{n_X} = 2^3 = 8$$

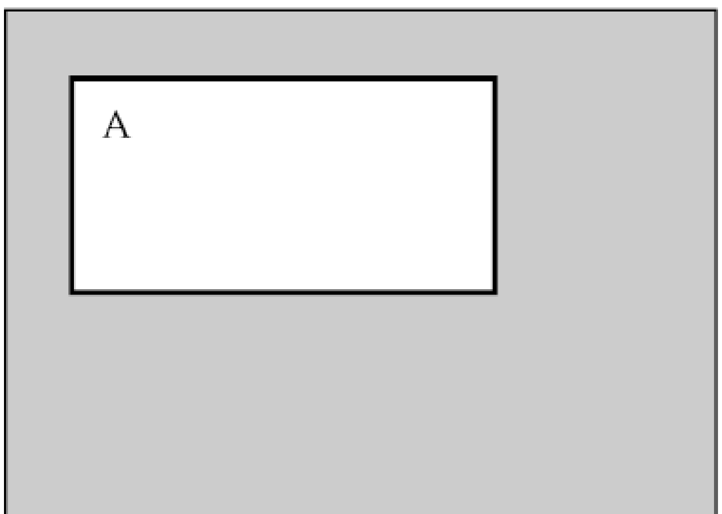
Operations on Classical Sets

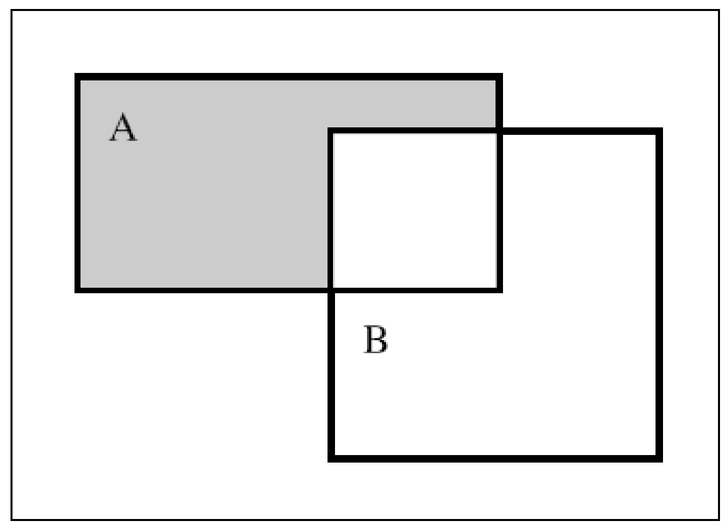
Union	$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
Intersection	$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
Complement	$\overline{A} = \{x \mid x \notin A, x \in X\}$
Difference	$A \mid B = \{x \mid x \in A \text{ and } x \notin B\}$

These four operations are shown in terms of Venn diagrams as following:


A
B


A
B


A


A
B

Properties of classical(Crisp)Sets

Commutativity	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
Associativity	$A \cup (B \cup C) = (A \cup B) \cup C$
	$A \cap (B \cap C) = (A \cap B) \cap C$
Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Idempotency	$A \cup A = A$
	$A \cap A = A$
Identity	$A \cup \emptyset = A$
	$A \cap X = A$
	$A \cap \emptyset = \emptyset$
	$A \cup X = X$
Transitivity	If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$
Involution	$\overline{\overline{A}} = A$

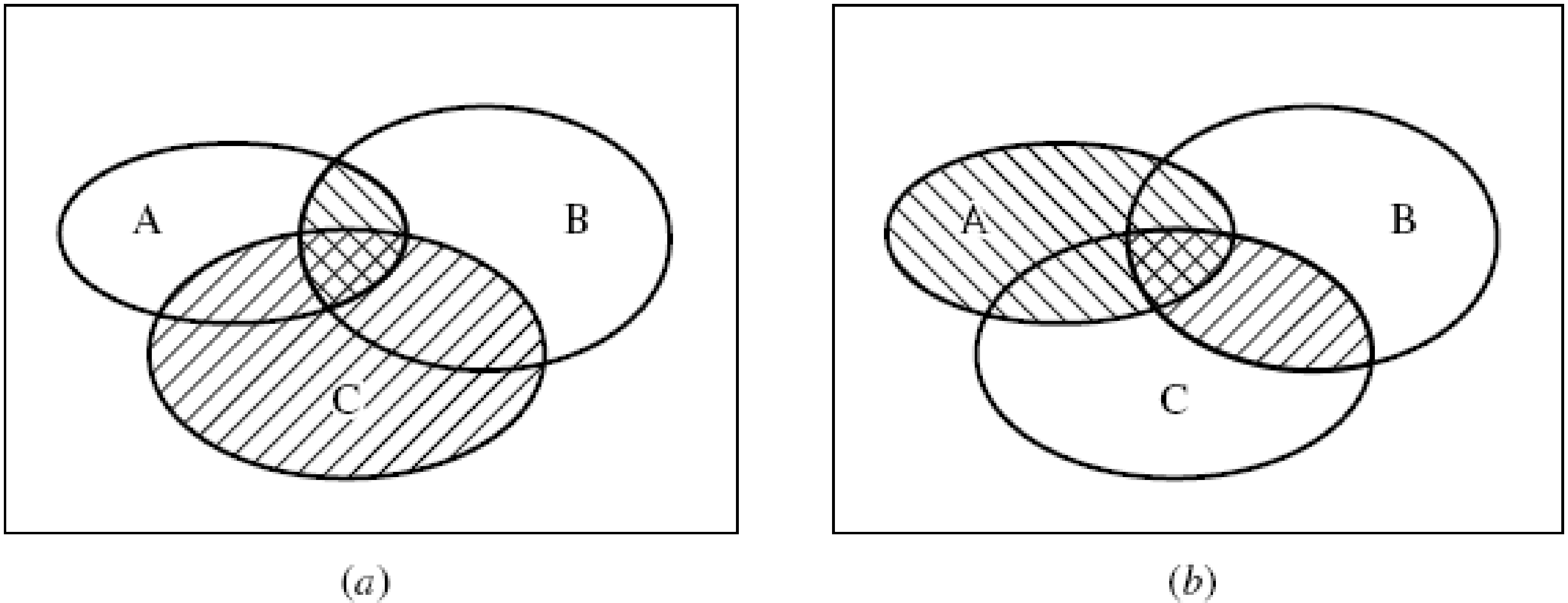


FIGURE 2.6
Venn diagrams for (*a*) $(A \cap B) \cap C$ and (*b*) $A \cap (B \cap C)$.

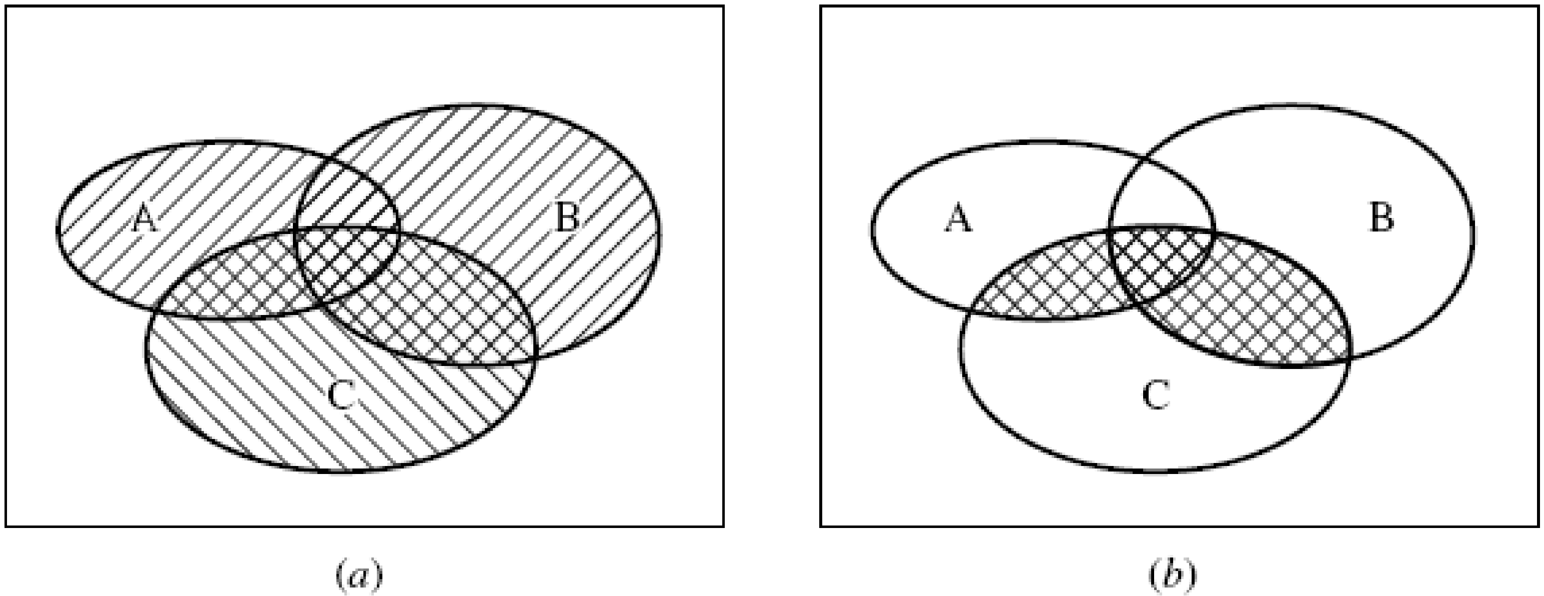


FIGURE 2.7
Venn diagrams for (*a*) $(A \cup B) \cap C$ and (*b*) $(A \cap C) \cup (B \cap C)$.

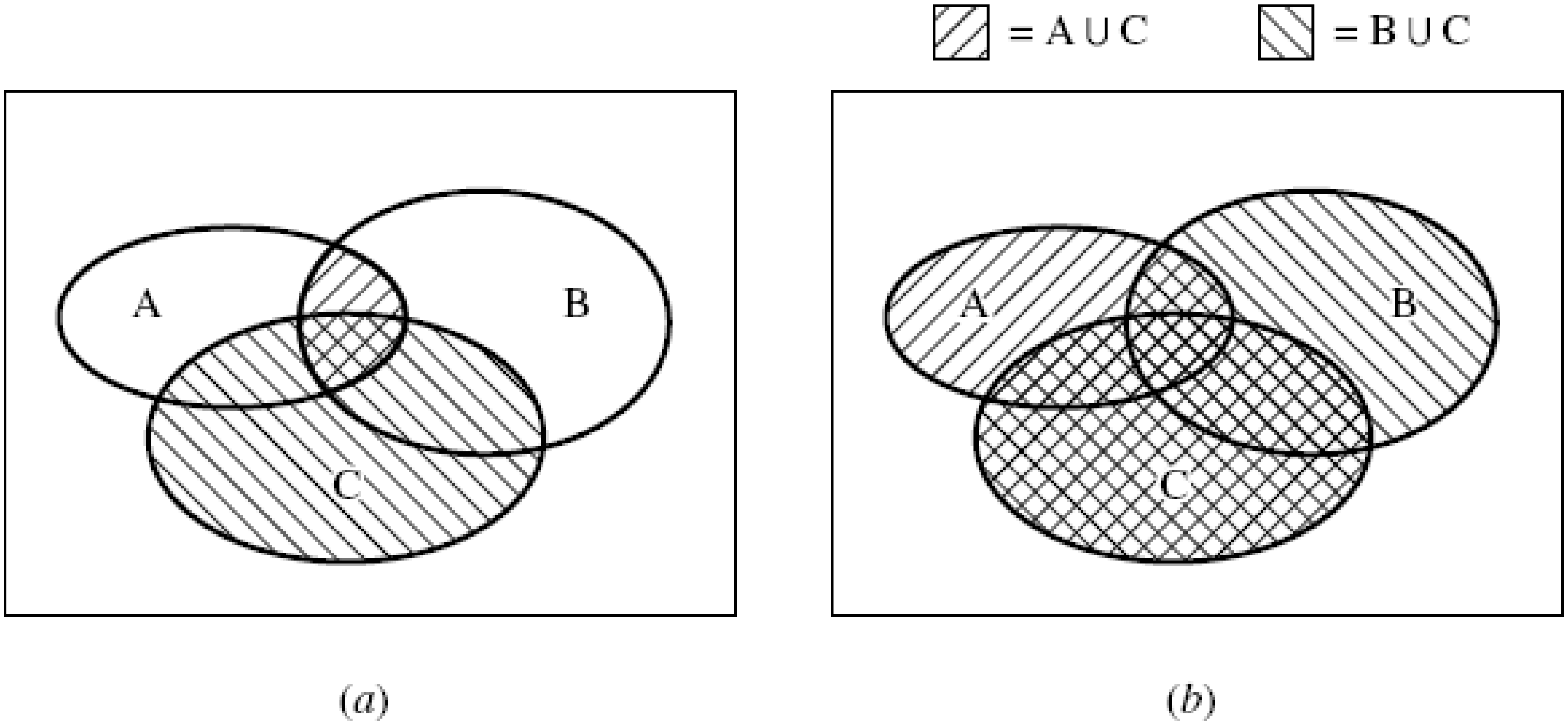


FIGURE 2.8
Venn diagrams for (*a*) $(A \cap B) \cup C$ and (*b*) $(A \cup C) \cap (B \cup C)$.

Properties of Crisp Sets

The excluded middle axioms are very important because these are the only set operations described here that are not valid for both classical sets and fuzzy sets.

Axiom of the excluded middle $A \cup \overline{A} = X$

Axiom of the contradiction $A \cap \overline{A} = \emptyset$

Properties of Crisp Sets

- *De Morgan's principles* are important because of their usefulness in proving tautologies and contradictions in logic, as well as in a host of other set operations and proofs.

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

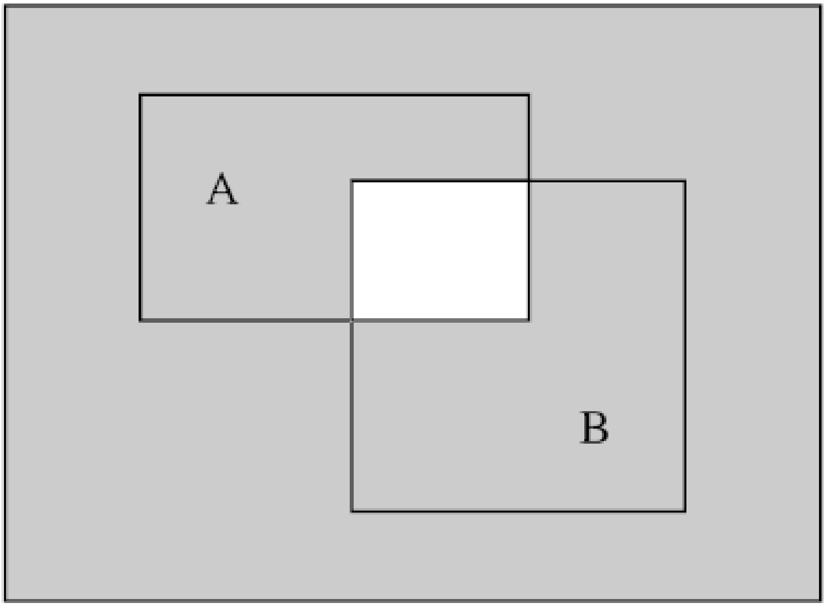


FIGURE 2.9
De Morgan's principle ($\overline{A \cap B}$).

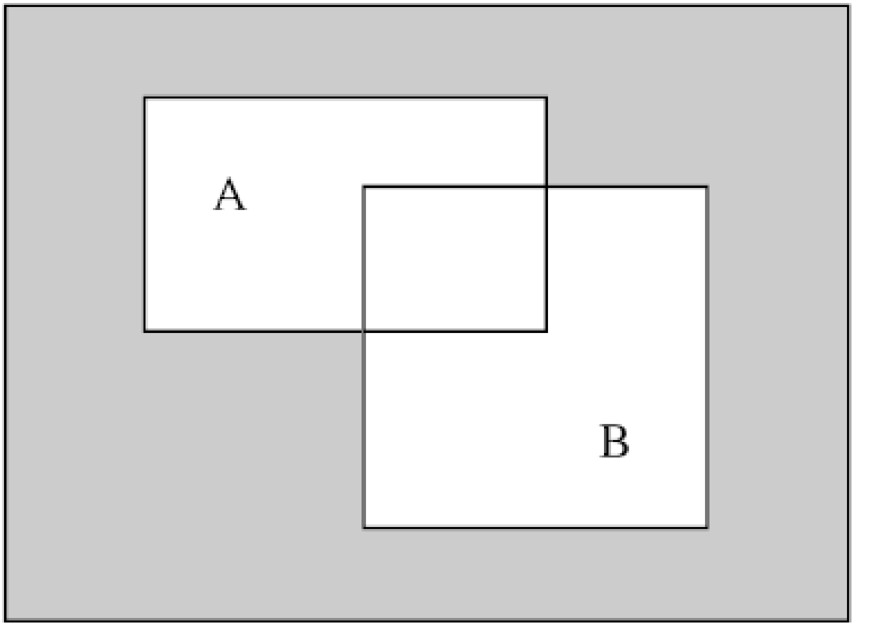


FIGURE 2.10
De Morgan's principle ($\overline{A \cup B}$).

Properties of Crisp Sets

$$\overline{E_1 \cup E_2 \cup \cdots \cup E_n} = \overline{E_1} \cap \overline{E_2} \cap \cdots \cap \overline{E_n}$$

$$\overline{E_1 \cap E_2 \cap \cdots \cap E_n} = \overline{E_1} \cup \overline{E_2} \cup \cdots \cup \overline{E_n}$$

De Morgan’s principles can be stated for *n* sets

Mapping of Crisp Sets to Functions

- If an element *x* is contained in X and corresponds to an element *y* contained in Y, it is generally termed a mapping from X to Y, or $f : X \rightarrow Y$.
- As a mapping, the characteristic (indicator) function χ_A is defined by

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

where χ_A expresses "membership" in set A for the element *x* in the universe.

Mapping of Crisp Sets to Functions

- For any set A defined on the universe X, there exists a function-theoretic set, called a value set, denoted V(A), under the mapping of the characteristic function, χ.
- By convention, the null set $\varnothing$ is assigned the membership value 0 and the whole set X is assigned the membership value 1.

- **Example 2.4.** For a universe with three elements, $X = \{a, b, c\}$, we desire to map the elements of the power set of X, i.e., P(X), to a universe, Y, consisting of only two elements (the characteristic function), $Y = \{0, 1\}$

the power set are

$P(X) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

Thus, the elements in the value set V(A) as determined from the mapping are

$V\{P(X)\} = \{\{0, 0, 0\}, \{1, 0, 0\}, \{0, 1, 0\}, \{0, 0, 1\}, \{1, 1, 0\}, \{0, 1, 1\}, \{1, 0, 1\}, \{1, 1, 1\}\}$

Define two sets, A and B, on the universe X. (the symbol $\vee$ is the maximum operator and $\wedge$ is the minimum operator):

Union $$\mathrm{A} \cup \mathrm{B} \longrightarrow \chi_{\mathrm{A} \cup \mathrm{B}}(x) = \chi_{\mathrm{A}}(x) \vee \chi_{\mathrm{B}}(x) = \max(\chi_{\mathrm{A}}(x), \chi_{\mathrm{B}}(x)) \quad (2.16)$$

The intersection of these two sets in function-theoretic terms is given by

Intersection $$\mathrm{A} \cap \mathrm{B} \longrightarrow \chi_{\mathrm{A} \cap \mathrm{B}}(x) = \chi_{\mathrm{A}}(x) \wedge \chi_{\mathrm{B}}(x) = \min(\chi_{\mathrm{A}}(x), \chi_{\mathrm{B}}(x)) \quad (2.17)$$

The complement of a single set on universe X, say A, is given by

Complement $$\overline{\mathrm{A}} \longrightarrow \chi_{\overline{\mathrm{A}}}(x) = 1 - \chi_{\mathrm{A}}(x) \quad (2.18)$$

For two sets on the same universe, say A and B, if one set (A) is contained in another set (B), then

Containment $$\mathrm{A} \subseteq \mathrm{B} \longrightarrow \chi_{\mathrm{A}}(x) \leq \chi_{\mathrm{B}}(x) \quad (2.19)$$

Fuzzy sets

- A fuzzy set, then, is a set containing elements that have varying degrees of membership in the set.
- A notation convention for fuzzy sets when the universe of discourse, X, is discrete and finite, is as follows for a fuzzy set $\underset{\sim}{A}$

$$\underset{\sim}{A} = \left\{ \frac{\mu_{\underset{\sim}{A}}(x_1)}{x_1} + \frac{\mu_{\underset{\sim}{A}}(x_2)}{x_2} + \cdots \right\} = \left\{ \sum_i \frac{\mu_{\underset{\sim}{A}}(x_i)}{x_i} \right\}$$

When the universe, X, is continuous and infinite, the fuzzy set $\underset{\sim}{A}$ is denoted by

$$\underset{\sim}{A} = \left\{ \int \frac{\mu_{\underset{\sim}{A}}(x)}{x} \right\}$$

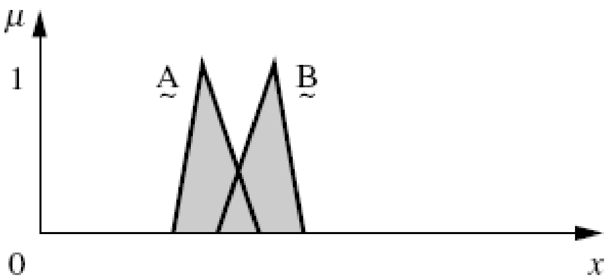
- The horizontal bar is not a quotient but rather a delimiter.
- The summation symbol is not for algebraic summation but
- rather denotes the collection or aggregation of each element;
- the "+"signs in the first notation are not the algebraic "add"
- the integral sign is not an algebraic integral but a continuous
- function-theoretic aggregation operator for continuous
- variables.

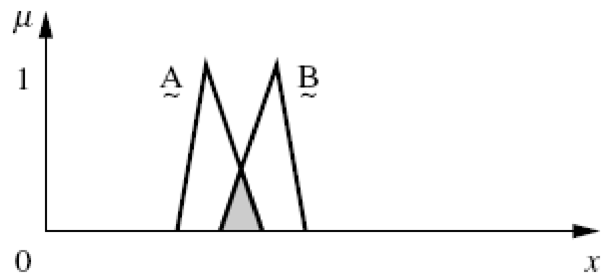
Fuzzy Set Operations

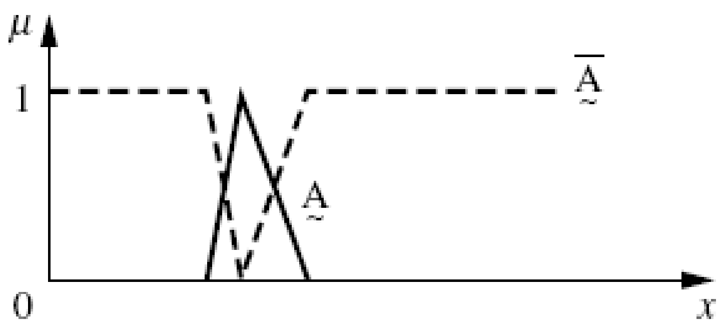
Union $$\mu_{\underset{\sim}{A}\cup\underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \vee \mu_{\underset{\sim}{B}}(x)$$

Intersection $$\mu_{\underset{\sim}{A}\cap\underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \wedge \mu_{\underset{\sim}{B}}(x)$$

Complement $$\mu_{\overline{\underset{\sim}{A}}}(x) = 1 - \mu_{\underset{\sim}{A}}(x)$$







- all other operations on classical sets also hold for fuzzy sets, except for the excluded middle axioms.
- Since fuzzy sets can overlap, a set and its complement can also overlap.
- The *excluded middle axioms,* extended for fuzzy sets, are expressed by

$$\underset{\sim}{A} \cup \overline{\underset{\sim}{A}} \neq X$$

$$\underset{\sim}{A} \cap \overline{\underset{\sim}{A}} \neq \emptyset$$

Properties of Fuzzy Sets

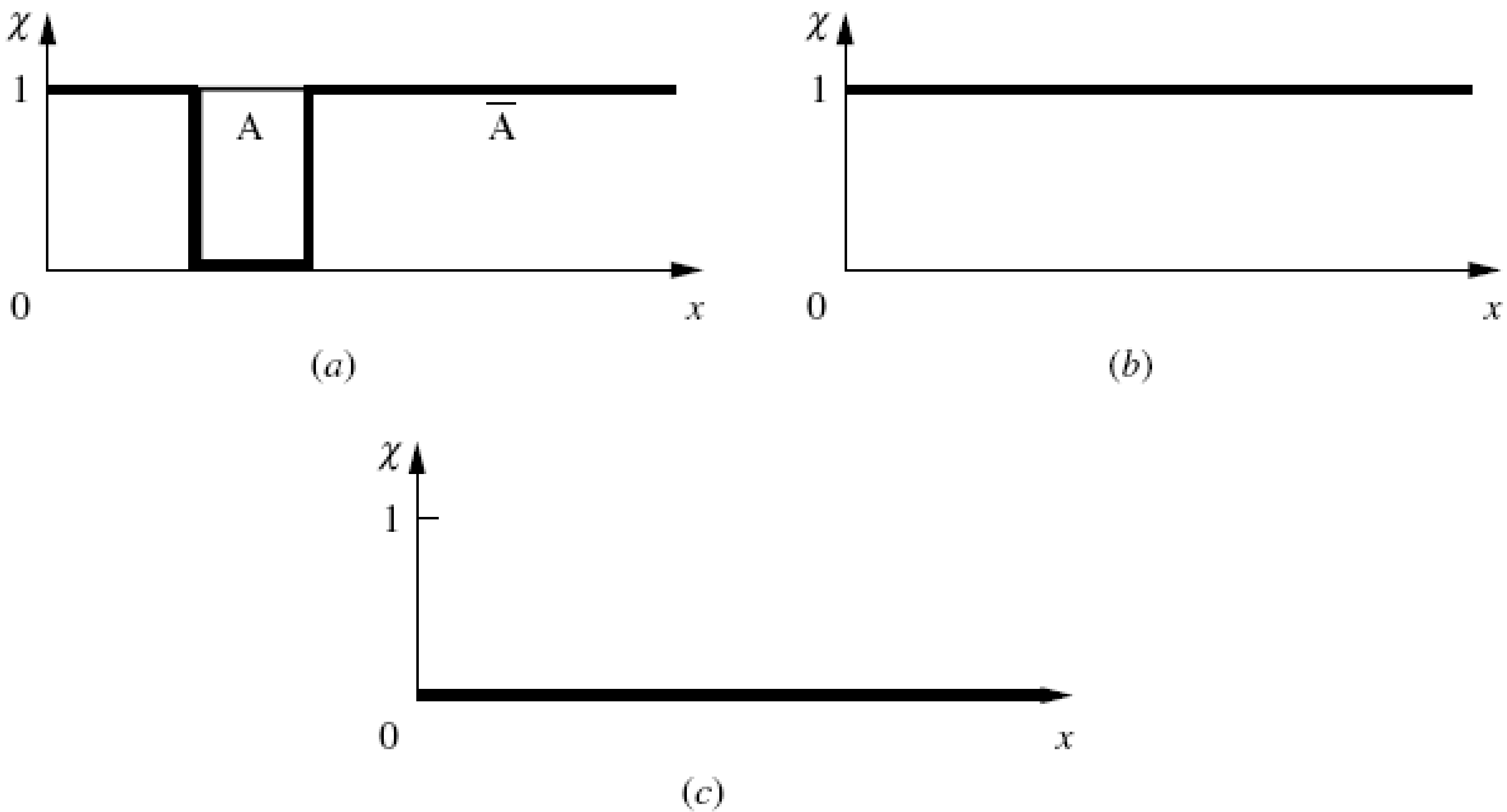


FIGURE 2.18
Excluded middle axioms for crisp sets. (*a*) Crisp set A and its complement; (*b*) crisp $A \cup \overline{A} = X$ (axiom of excluded middle); (*c*) crisp $A \cap \overline{A} = \emptyset$ (axiom of contradiction).

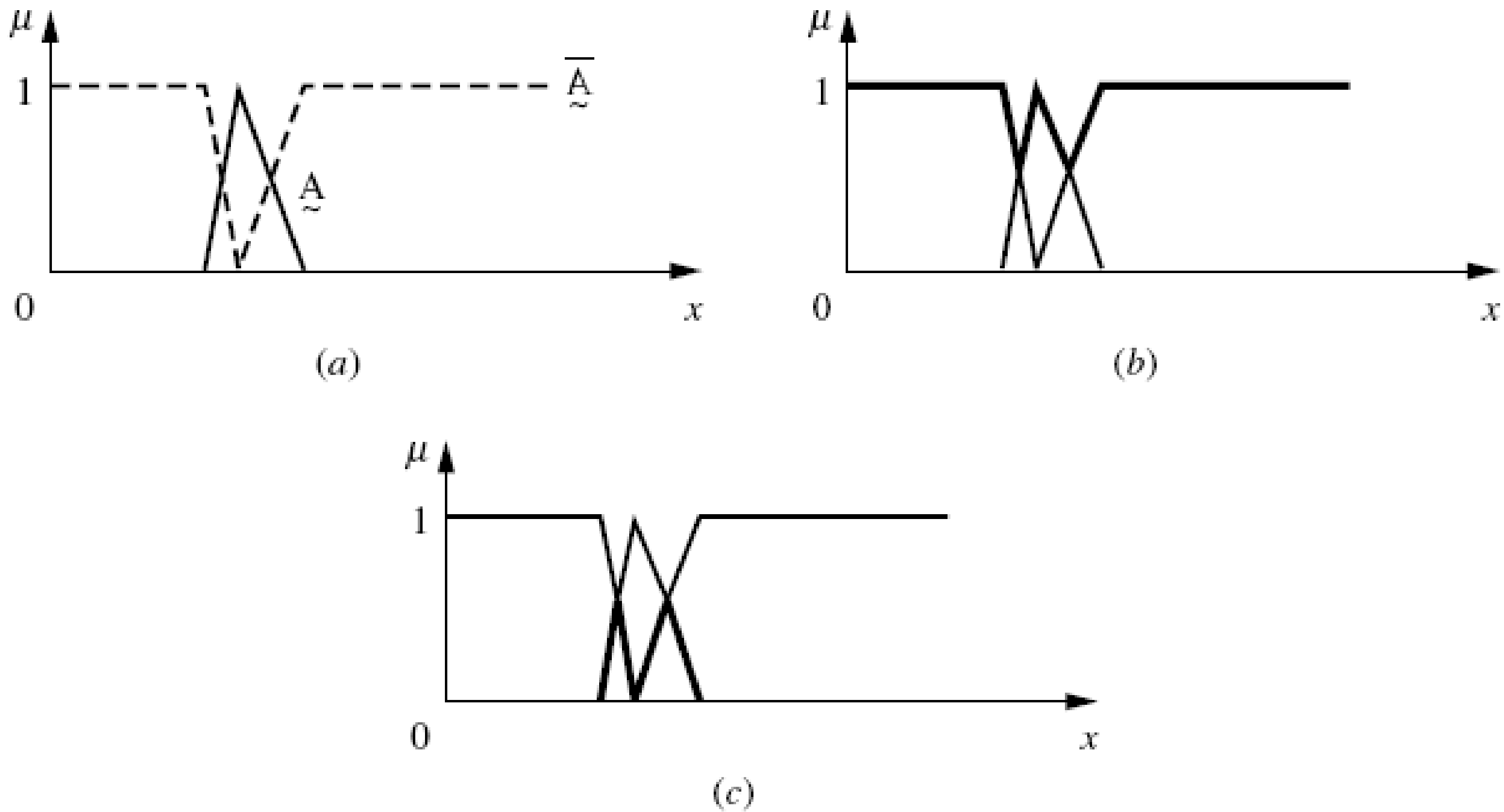


FIGURE 2.19
Excluded middle axioms for fuzzy sets. (*a*) Fuzzy set $\underset{\sim}{A}$ and its complement; (*b*) fuzzy $\underset{\sim}{A} \cup \overline{\underset{\sim}{A}} \neq X$ (axiom of excluded middle); (*c*) fuzzy $A \cap \overline{A} \neq \emptyset$ (axiom of contradiction).

Properties of Fuzzy Sets

Commutativity $$\underset{\sim}{A} \cup \underset{\sim}{B} = \underset{\sim}{B} \cup \underset{\sim}{A}$$

$$\underset{\sim}{A} \cap \underset{\sim}{B} = \underset{\sim}{B} \cap \underset{\sim}{A}$$

Associativity $$\underset{\sim}{A} \cup \left(\underset{\sim}{B} \cup \underset{\sim}{C}\right) = (\underset{\sim}{A} \cup \underset{\sim}{B}) \cup \underset{\sim}{C}$$

$$\underset{\sim}{A} \cap \left(\underset{\sim}{B} \cap \underset{\sim}{C}\right) = (\underset{\sim}{A} \cap \underset{\sim}{B}) \cap \underset{\sim}{C}$$

Distributivity $$\underset{\sim}{A} \cup \left(\underset{\sim}{B} \cap \underset{\sim}{C}\right) = (\underset{\sim}{A} \cup \underset{\sim}{B}) \cap \left(\underset{\sim}{A} \cup \underset{\sim}{C}\right)$$

$$\underset{\sim}{A} \cap \left(\underset{\sim}{B} \cup \underset{\sim}{C}\right) = (\underset{\sim}{A} \cap \underset{\sim}{B}) \cup \left(\underset{\sim}{A} \cap \underset{\sim}{C}\right)$$

Idempotency $$\underset{\sim}{A} \cup \underset{\sim}{A} = \underset{\sim}{A} \quad \text{and} \quad \underset{\sim}{A} \cap \underset{\sim}{A} = \underset{\sim}{A}$$

Identity $$\underset{\sim}{A} \cup \emptyset = \underset{\sim}{A} \quad \text{and} \quad \underset{\sim}{A} \cap X = \underset{\sim}{A}$$

$$\underset{\sim}{A} \cap \emptyset = \emptyset \quad \text{and} \quad \underset{\sim}{A} \cup X = X$$

Transitivity $$\text{If } \underset{\sim}{A} \subseteq \underset{\sim}{B} \text{ and } \underset{\sim}{B} \subseteq \underset{\sim}{C}, \text{ then } \underset{\sim}{A} \subseteq \underset{\sim}{C}$$

Involution $$\overline{\overline{\underset{\sim}{A}}} = \underset{\sim}{A}$$

Example: we have two discrete fuzzy sets

$$\underset{\sim}{A} = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\} \quad \text{and} \quad \underset{\sim}{B} = \left\{ \frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\}$$

(membership for element 1 in both sets is implicitly)

Complement
$$\overline{\underset{\sim}{A}} = \left\{ \frac{1}{1} + \frac{0}{2} + \frac{0.5}{3} + \frac{0.7}{4} + \frac{0.8}{5} \right\}$$

$$\overline{\underset{\sim}{B}} = \left\{ \frac{1}{1} + \frac{0.5}{2} + \frac{0.3}{3} + \frac{0.8}{4} + \frac{0.6}{5} \right\}$$

Union
$$\underset{\sim}{A} \cup \underset{\sim}{B} = \left\{ \frac{1}{2} + \frac{0.7}{3} + \frac{0.3}{4} + \frac{0.4}{5} \right\}$$

Intersection
$$\underset{\sim}{A} \cap \underset{\sim}{B} = \left\{ \frac{0.5}{2} + \frac{0.5}{3} + \frac{0.2}{4} + \frac{0.2}{5} \right\}$$

Difference
$$\underset{\sim}{A} \mid \underset{\sim}{B} = \underset{\sim}{A} \cap \overline{\underset{\sim}{B}} = \left\{ \frac{0.5}{2} + \frac{0.3}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\}$$

$$\underset{\sim}{B} \mid \underset{\sim}{A} = \underset{\sim}{B} \cap \overline{\underset{\sim}{A}} = \left\{ \frac{0}{2} + \frac{0.5}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\}$$

De Morgan's principles

$$\overline{\underset{\sim}{A} \cup \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \cap \overline{\underset{\sim}{B}} = \left\{\frac{1}{1} + \frac{0}{2} + \frac{0.3}{3} + \frac{0.7}{4} + \frac{0.6}{5}\right\}$$

$$\overline{\underset{\sim}{A} \cap \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \cup \overline{\underset{\sim}{B}} = \left\{\frac{1}{1} + \frac{0.5}{2} + \frac{0.5}{3} + \frac{0.8}{4} + \frac{0.8}{5}\right\}$$

$$\underset{\sim}{A} = \left\{\frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5}\right\} \quad \text{and} \quad \underset{\sim}{B} = \left\{\frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5}\right\}$$