CLASSICAL RELATIONS AND FUZZY RELATIONS

CLASSICAL RELATIONS AND FUZZY RELATIONS

- This chapter introduces the notion of a relation as the basic idea behind numerous operations on sets such as Cartesian products, composition of relations, and equivalence properties.
- Like a set, a relation is of fundamental importance in all engineering, science, and mathematically based fields.

CLASSICAL RELATIONS AND FUZZY RELATIONS

- Relations are intimately involved in logic, approximate reasoning, rule based systems, nonlinear simulation, synthetic evaluation, classification, pattern recognition, and control.
- Relations represent mappings for sets just as mathematical functions do;
- Relations are also very useful in representing connectives in logic (see Chapter 5).

CARTESIAN PRODUCT

- An ordered sequence of *r* elements, written in the form $(a_1, a_2, a_3, \ldots, a_r)$, is called an ordered *r*-tuple.
- For crisp sets $A_1, A_2, \ldots, A_r$, the set of all *r*-tuples $(a_1, a_2, a_3, \ldots, a_r)$, where $a_1 \in A_1, a_2 \in A_2$, and $a_r \in A_r$, is called the *Cartesian product* of $A_1, A_2, \ldots, A_r$, and is denoted by

$$A_1 \times A_2 \times \cdots \times A_r .$$

(The Cartesian product of two or more sets is not the same thing as the arithmetic product of two or more sets.)

- **Example 3.1.** The elements in two sets A and B are given as A = {0, 1} and B = {*a*, *b*, *c*}.

  Various Cartesian products of these two sets can be written as shown:

$A \times B = \{(0, a), (0, b), (0, c), (1, a), (1, b), (1, c)\}$

$B \times A = \{(a, 0), (a, 1), (b, 0), (b, 1), (c, 0), (c, 1)\}$

$A \times A = A^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$

$B \times B = B^2 = \{(a, a), (a, b), (a, c), (b, a), (b, b),$
$(b, c), (c, a), (c, b), (c, c)\}$

Crisp Relations

- A subset of the Cartesian product $A_1 \times A_2 \times \cdot \cdot \cdot \times A_r$ is called an *r-ary relation* over $A_1, A_2, \ldots, A_r$.
- the most common case is for $r = 2$; the relation is a subset of the Cartesian product $A_1 \times A_2$ (i.e., a set of pairs, the first coordinate of which is from A_1 and the second from A_2). This subset of the full Cartesian product is called a *binary relation from* A_1 *into* A_2.
- If three, four, or five sets are involved in a subset of the full Cartesian product, the relations are called ternary, quaternary, and quinary.

Cartesian product

- The Cartesian product of two universes X and Y is determined as

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

which forms an ordered pair of every $x \in X$ with every $y \in Y$, forming *unconstrained* matches between X and Y.

That is, every element in universe X is related completely to every element in universe Y.

Strength

- The *strength* of this relationship between ordered pairs of elements in each universe is measured by the characteristic function, denoted χ, (where a value of unity is associated with *complete relationship* and a value of zero is associated with *no relationship*)i.e.,

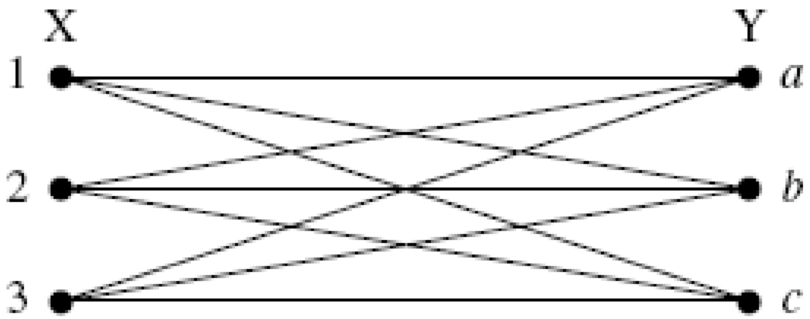
$$\chi_{X \times Y}(x, y) = \begin{cases} 1, & (x, y) \in X \times Y \\ 0, & (x, y) \notin X \times Y \end{cases}$$

Relation matrix

- One can think of this strength of relation as a mapping from ordered pairs of the universe, or ordered pairs of sets defined on the universes, to the characteristic function.
- When the universes, or sets, are **finite** the relation can be conveniently represented by a matrix, called a *relation matrix.* An *r*-ary relation can be represented by an *r*-dimensional relation matrix. Hence, binary relations can be represented by two-dimensional matrices.

Example: An example of the strength of relation for the unconstrained case

- X = {1, 2, 3} and Y = {*a*, *b*, *c*}.



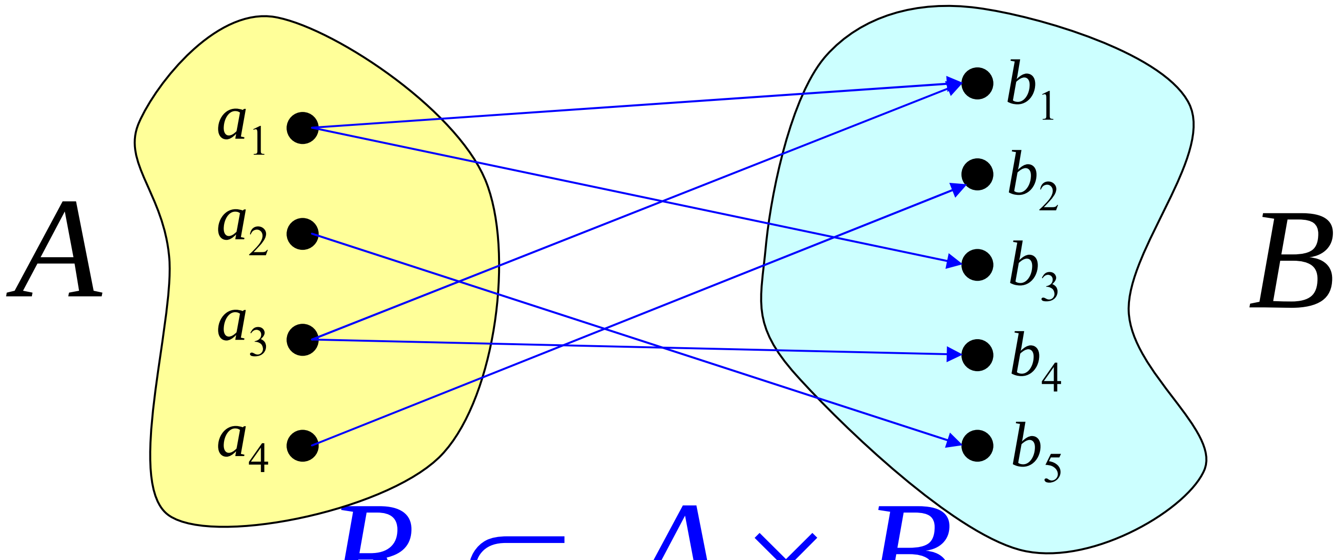
$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

Constrained Crisp Relations

- A more general crisp relation, R, exists when matches between elements in two universes are *constrained.*
- Again, the characteristic function is used to assign values of relationship in the mapping of the Cartesian space X $\times$ Y to the binary values of *(*0*,* 1*)*:

$$\chi_R(x, y) = \begin{cases} 1, & (x, y) \in R \\ 0, & (x, y) \notin R \end{cases}$$

Constrained Binary Relation (R)



$$R \subseteq A \times B$$

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$a_1 R b_1 \quad a_1 R b_3 \quad a_2 R b_5$

$$R = \left\{ \begin{matrix} (a_1, b_1), (a_1, b_3), (a_2, b_5) \\ (a_3, b_1), (a_3, b_4), (a_4, b_2) \end{matrix} \right\}$$

$a_3 R b_1 \quad a_3 R b_4 \quad a_4 R b_2$

Identity relation, Universal relation

- Special cases of the constrained and the unconstrained Cartesian product for sets where $r = 2$ (i.e., for A^2) are called the *identity relation* and the *universal relation.*
- For example, for $A = \{0, 1, 2\}$ the universal relation, denoted U_A, and the identity relation, denoted I_A, are found to be

$U_A = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\}$

$I_A = \{(0, 0), (1, 1), (2, 2)\}$

- **Example 3.3.** Relations can also be defined for continuous universes.

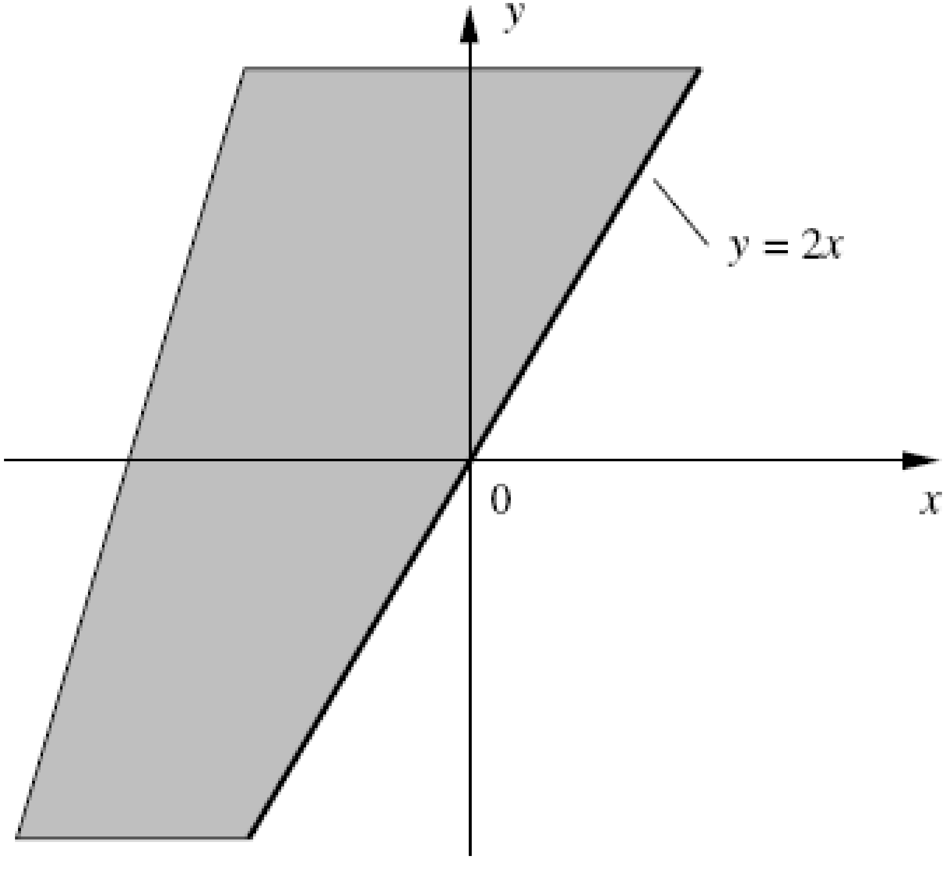
Consider, for example, the continuous relation defined by the following expression:

$$R = \{(x, y) \mid y \geqslant 2x, x \in X, y \in Y\}$$

which is also given in function-theoretic form using the characteristic function as

$$\chi_R(x, y) = \begin{cases} 1, & y \geq 2x \\ 0, & y < 2x \end{cases}$$

Graphically, this relation is equivalent to the shaded region shown



Cardinality of Crisp Relations

Suppose n elements of the universe X are related (paired) to m elements of the universe Y. If the cardinality of X is n_X and the cardinality of Y is n_Y, then the cardinality of the relation, R, between these two universes is $n_{X \times Y} = n_X * n_Y$. The cardinality of the power set describing this relation, P(X $\times$ Y), is then

$$n_{P(X \times Y)} = 2^{(n_x n_y)}$$

Operations on Crisp Relations

Define R and S as two separate relations on the Cartesian universe $X \times Y$, and define the null relation and the complete relation as the relation matrices $\underset{\sim}{\mathbf{O}}$ and $\underset{\sim}{\mathbf{E}}$, respectively. An example of a 4×4 form of the $\underset{\sim}{\mathbf{O}}$ and $\underset{\sim}{\mathbf{E}}$ matrices is given here:

$$\mathbf{O} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Operations on Crisp Relations

The following function-theoretic operations for the two crisp relations (R, S) can now be defined.

Union $\mathrm{R} \cup \mathrm{S} \longrightarrow \chi_{\mathrm{R}\cup\mathrm{S}}(x, y) : \chi_{\mathrm{R}\cup\mathrm{S}}(x, y) = \max[\chi_{\mathrm{R}}(x, y), \chi_{\mathrm{S}}(x, y)]$ (3.4)

Intersection $\mathrm{R} \cap \mathrm{S} \longrightarrow \chi_{\mathrm{R}\cap\mathrm{S}}(x, y) : \chi_{\mathrm{R}\cap\mathrm{S}}(x, y) = \min[\chi_{\mathrm{R}}(x, y), \chi_{\mathrm{S}}(x, y)]$ (3.5)

Complement $\overline{\mathrm{R}} \longrightarrow \chi_{\overline{\mathrm{R}}}(x, y) : \chi_{\overline{\mathrm{R}}}(x, y) = 1 - \chi_{\mathrm{R}}(x, y)$ (3.6)

Containment $\mathrm{R} \subset \mathrm{S} \longrightarrow \chi_{\mathrm{R}}(x, y) : \chi_{\mathrm{R}}(x, y) \leq \chi_{\mathrm{S}}(x, y)$ (3.7)

Identity $\emptyset \longrightarrow \mathbf{O}$ and $\mathrm{X} \longrightarrow \mathbf{E}$ (3.8)

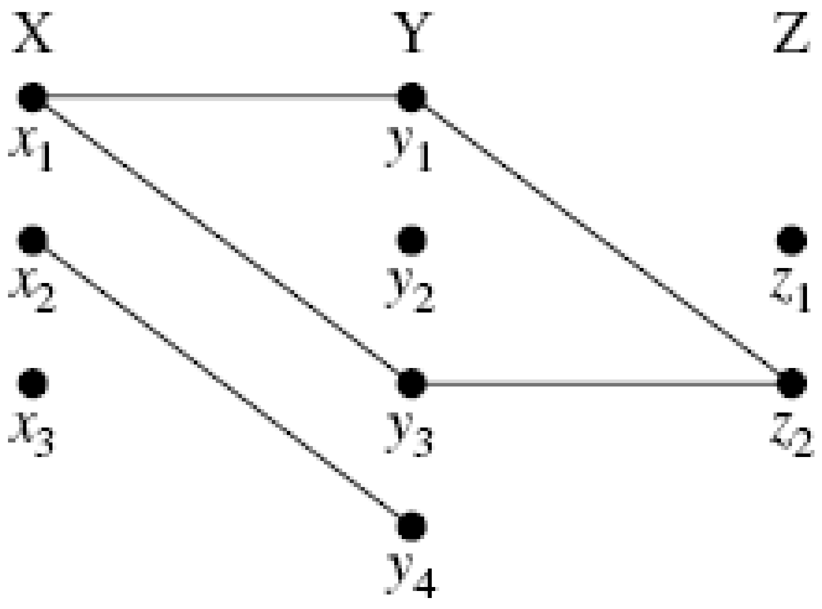
Properties of Crisp Relations

The properties of commutativity, associativity, distributivity, involution, and idempotency all hold for crisp relations just as they do for classical set operations.

Moreover, *De Morgan’sprinciples* and *the excluded middle axioms* also hold for crisp (classical) relations. The null relation, **O**, and the complete relation, **E**, are analogous to the null set, $\varnothing$, and the whole set, X, respectively, in the set-theoretic case.

Composition

- Let R be a relation that relates elements from universe X to universe Y, and S from universe Y to universe Z.
- $R = \{(x_1, y_1), (x_1, y_3), (x_2, y_4)\}$
- $S = \{(y_1, z_2), (y_3, z_2)\}$



•The max–min composition is defined by the set-theoretic and membership function-theoretic expressions

$$T = R \circ S$$

$$\chi_T(x, z) = \bigvee_{y \in Y} (\chi_R(x, y) \wedge \chi_S(y, z))$$

•max–product (sometimes called max–dot) composition

$$T = R \circ S$$

$$\chi_T(x, z) = \bigvee_{y \in Y} (\chi_R(x, y) \bullet \chi_S(y, z))$$

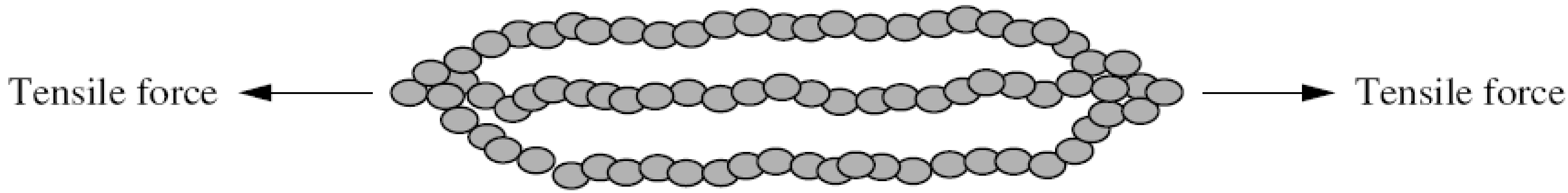


FIGURE 3.5
Chain strength analogy for max–min composition.

- The *minimum* ($\wedge$) strength of all the links in the chain governs the strength of the overall chain.
- the *maximum* ($\vee$) strength of all the chains in the chain system would govern the overall strength of the chain system.
- Each chain in the system is analogous to the min operation in the max–min composition, and the overall chain system strength is analogous to the max operation in the max–min composition.

Example 3.4. The matrix expression for the crisp relations shown in Fig. 3.4 can be found using the max–min composition operation. Relation matrices for R and S would be expressed as

$$\mathrm{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad \mathrm{S} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

The resulting relation T would then be determined by max–min composition, Eq. (3.9), or max–product composition, Eq. (3.10). (In the crisp case these forms of the composition operators produce identical results; other forms of this operator, such as those listed at the end of this chapter, will not produce identical results.) For example,

$$\mu_{\mathrm{T}}(x_1, z_1) = \max[\min(1, 0), \min(0, 0), \min(1, 0), \min(0, 0)] = 0$$

$$\mu_{\mathrm{T}}(x_1, z_2) = \max[\min(1, 1), \min(0, 0), \min(1, 1), \min(0, 0)] = 1$$

and for the rest,

$$\mathrm{T} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

The Real-Life Relation

- x is close to y
  - x and y are numbers
- x depends on y
  - x and y are events
- x and y look alike
  - x and y are persons or objects
- If x is large, then y is small
  - x is an observed reading and y is a corresponding action

Fuzzy Relations

- Fuzzy relations also map elements of one universe, say X, to those of another universe, say Y, through the Cartesian product of the two universes. However, the "strength" of the relation between ordered pairs of the two universes is not measured with the characteristic function, but rather with a membership function expressing various "degrees" of strength of the relation on the unit interval [0,1]
- Hence, a fuzzy relation $\underset{\sim}{R}$ is a mapping from the Cartesian space X $\times$ Y to the interval [0,1], where the strength of the mapping is expressed by the membership function of the relation for ordered pairs from the two universes, or $\mu_{\underset{\sim}{R}}(x, y)$.

Cardinality of Fuzzy Relations

- Since the cardinality of fuzzy sets on any universe is infinity, the cardinality of a fuzzy relation between two or more universes is also infinity.

Operations on Fuzzy Relations

Union	$\mu_{\underset{\sim}{R} \cup \underset{\sim}{S}}(x, y) = \max(\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y))$
Intersection	$\mu_{\underset{\sim}{R} \cap \underset{\sim}{S}}(x, y) = \min(\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y))$
Complement	$\mu_{\overline{\underset{\sim}{R}}}(x, y) = 1 - \mu_{\underset{\sim}{R}}(x, y)$
Containment	$\underset{\sim}{R} \subset \underset{\sim}{S} \Rightarrow \mu_{\underset{\sim}{R}}(x, y) \leq \mu_{\underset{\sim}{S}}(x, y)$

Properties of Fuzzy Relations

- commutativity, associativity, distributivity, involution, idempotency and De Morgan's principles all hold for fuzzy relations.
- the null relation, **O**, and the complete relation, **E**, are analogous to the null set and the whole set in set-theoretic form.
- excluded middle axioms doesn't hold.

$$\underset{\sim}{R} \cup \overline{\underset{\sim}{R}} \neq \mathbf{E}$$

$$\underset{\sim}{R} \cap \overline{\underset{\sim}{R}} \neq \mathbf{O}$$

Fuzzy Cartesian Product and composition

Let $\underset{\sim}{A}$ be a fuzzy set on universe X and $\underset{\sim}{B}$ be a

fuzzy set on universe Y; then the Cartesian product between fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ will result in a fuzzy relation $\underset{\sim}{R}$, which is contained within the full Cartesian product space, or

$$\underset{\sim}{A} \times \underset{\sim}{B} = \underset{\sim}{R} \subset X \times Y \tag{3.15}$$

where the fuzzy relation $\underset{\sim}{R}$ has membership function

$$\mu_{\underset{\sim}{R}}(x, y) = \mu_{\underset{\sim}{A} \times \underset{\sim}{B}}(x, y) = \min\left(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)\right)$$

Example 3.5. Suppose we have two fuzzy sets, $\underset{\sim}{A}$ defined on a universe of three discrete temperatures, $X = \{x_1, x_2, x_3\}$, and $\underset{\sim}{B}$ defined on a universe of two discrete pressures, $Y = \{y_1, y_2\}$, and we want to find the fuzzy Cartesian product between them. Fuzzy set $\underset{\sim}{A}$ could represent the "ambient" temperature and fuzzy set $\underset{\sim}{B}$ the "near optimum" pressure for a certain heat exchanger, and the Cartesian product might represent the conditions (temperature–pressure pairs) of the exchanger that are associated with "efficient" operations. For example, let

$$\underset{\sim}{A} = \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} \quad \text{and} \quad \underset{\sim}{B} = \frac{0.3}{y_1} + \frac{0.9}{y_2}$$

Note that $\underset{\sim}{A}$ can be represented as a column vector of size 3×1 and $\underset{\sim}{B}$ can be represented by a row vector of 1×2. Then the fuzzy Cartesian product, using Eq. (3.16), results in a fuzzy relation $\underset{\sim}{R}$ (of size 3×2) representing "efficient" conditions, or

$$\underset{\sim}{A} \times \underset{\sim}{B} = \underset{\sim}{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{bmatrix} \end{array}$$

Fuzzy composition

- fuzzy max–min composition

$$\underset{\sim}{\mathrm{T}} = \underset{\sim}{\mathrm{R}} \circ \underset{\sim}{\mathrm{S}}$$

$$\mu_{\underset{\sim}{\mathrm{T}}}(x, z) = \bigvee_{y \in \mathrm{Y}} (\mu_{\underset{\sim}{\mathrm{R}}}(x, y) \wedge \mu_{\underset{\sim}{\mathrm{S}}}(y, z))$$

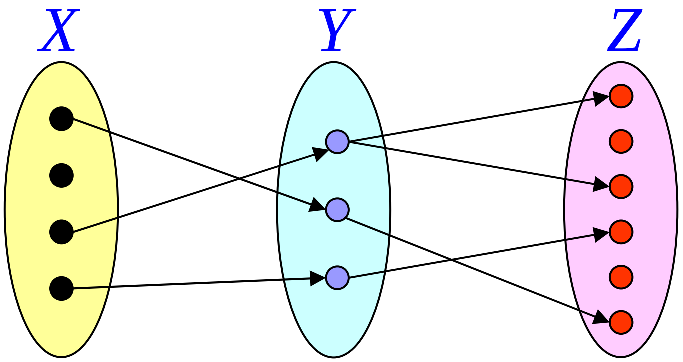
- fuzzy max–product composition

$$\mu_{\underset{\sim}{\mathrm{T}}}(x, z) = \bigvee_{y \in \mathrm{Y}} (\mu_{\underset{\sim}{\mathrm{R}}}(x, y) \bullet \mu_{\underset{\sim}{\mathrm{S}}}(y, z))$$

It should be pointed out that neither crisp nor fuzzy compositions are commutative in general; that is,

$$\underset{\sim}{\mathrm{R}} \circ \underset{\sim}{\mathrm{S}} \neq \underset{\sim}{\mathrm{S}} \circ \underset{\sim}{\mathrm{R}} \tag{3.18}$$

Max-Min Composition



R: fuzzy relation defined on X and Y.

S: fuzzy relation defined on Y and Z.

$\underbrace{R \circ S}_{}$: the composition of R and S.

A fuzzy relation defined on X an Z.

$$\mu_{R\circ S}(x,z) = \max_y \min\left(\mu_R(x,y), \mu_S(y,z)\right)$$

$$= \vee_y \left(\mu_R(x,y) \wedge \mu_S(y,z)\right)$$

Example

$$\mu_{S \circ R}(x, y) = \max_v \min\left(\mu_R(x,v), \mu_S(v,y)\right)$$

R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

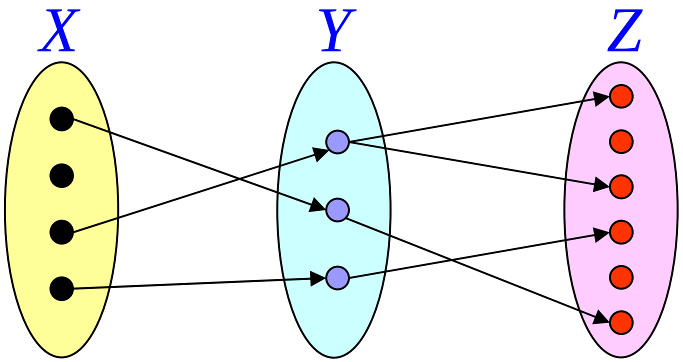
S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

	0.1	0.2	0.0	1.0
min	0.9	0.2	0.8	0.4
max	0.1	0.2	0.0	0.4

$R \circ S$	α	β	γ
1	$\boxed{0.4}$	0.2	0.3
2	0.3	0.3	0.3
3	0.8	0.9	0.8

Max-min composition is not mathematically tractable, therefore other compositions such as max-product composition have been suggested.

Max-Product Composition



R: fuzzy relation defined on X and Y.

S: fuzzy relation defined on Y and Z.

$\underbrace{R \circ S}$: the composition of R and S.

A fuzzy relation defined on X an Z.

$$\mu_{R\circ S}(x,y) = \max_v \left(\mu_R(x,v)\mu_S(v,y) \right)$$

Example 3.6. Let us extend the information contained in the Sagittal diagram shown in Fig. 3.4 to include fuzzy relationships for $X \times Y$ (denoted by the fuzzy relation $\underset{\sim}{R}$) and $Y \times Z$ (denoted by the fuzzy relation $\underset{\sim}{S}$). In this case we change the elements of the universes to,

$$X = \{x_1, x_2\}, \quad Y = \{y_1, y_2\}, \quad \text{and} \quad Z = \{z_1, z_2, z_3\}$$

Consider the following fuzzy relations:

$$\underset{\sim}{R} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \end{array} \quad \text{and} \quad \underset{\sim}{S} = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.1 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

Then the resulting relation, $\underset{\sim}{T}$, which relates elements of universe X to elements of universe Z, i.e., defined on Cartesian space $X \times Z$, can be found by max–min composition, Eq. (3.17*a*), tc be, for example,

$$\mu_{\underset{\sim}{T}}(x_1, z_1) = \max[\min(0.7, 0.9), \min(0.5, 0.1)] = 0.7$$

and the rest,

$$\underset{\sim}{T} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}$$

and by max–product composition, Eq. (3.17*b*), to be, for example,

$$\mu_{\underset{\sim}{T}}(x_2, z_2) = \max[(0.8 \cdot 0.6), (0.4 \cdot 0.7)] = 0.48$$

and the rest,

$$\underset{\sim}{T} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.63 & 0.42 & 0.25 \\ 0.72 & 0.48 & 0.20 \end{bmatrix} \end{array}$$

Example 3.7. A certain type of virus attacks cells of the human body. The infected cells can be visualized using a special microscope. The microscope generates digital images that medical doctors can analyze and identify the infected cells. The virus causes the infected cells to have a black spot, within a darker grey region (Fig. 3.6).

A digital image process can be applied to the image. This processing generates two variables: the first variable, P, is related to black spot quantity (black pixels), and the second variable, S, is related to the shape of the black spot, i.e., if they are circular or elliptic. In these images it is often difficult to actually count the number of black pixels, or to identify a perfect circular cluster of pixels; hence, both these variables must be estimated in a linguistic way.

Suppose that we have two fuzzy sets, $\underset{\sim}{P}$ which represents the number of black pixels (e.g., none with black pixels, C_1, a few with black pixels, C_2, and a lot of black pixels, C_3), and $\underset{\sim}{S}$ which represents the shape of the black pixel clusters, e.g., S_1 is an ellipse and S_2 is a circle. So we have

$$\underset{\sim}{P} = \left\{ \frac{0.1}{C_1} + \frac{0.5}{C_2} + \frac{1.0}{C_3} \right\} \quad \text{and} \quad \underset{\sim}{S} = \left\{ \frac{0.3}{S_1} + \frac{0.8}{S_2} \right\}$$

and we want to find the relationship between quantity of black pixels in the virus and the shape of the black pixel clusters. Using a Cartesian product between $\underset{\sim}{P}$ and $\underset{\sim}{S}$ gives

$$\underset{\sim}{R} = \underset{\sim}{P} \times \underset{\sim}{S} = \begin{array}{c} \\ C_1 \\ C_2 \end{array} \begin{array}{c} \begin{array}{cc} S_1 & S_2 \end{array} \\ \begin{bmatrix} 0.1 & 0.1 \\ 0.3 & 0.5 \end{bmatrix} \end{array}$$

Now, suppose another microscope image is taken and the number of black pixels is slightly different; let the new black pixel quantity be represented by a fuzzy set, $\underset{\sim}{P}'$:

$$\underset{\sim}{P}' = \left\{ \frac{0.4}{C_1} + \frac{0.7}{C_2} + \frac{1.0}{C_3} \right\}$$

Using max–min composition with the relation $\underset{\sim}{R}$ will yield a new value for the fuzzy set of pixel cluster shapes that are associated with the new black pixel quantity:

$$\underset{\sim}{S}' = \underset{\sim}{P}' \circ \underset{\sim}{R} = \begin{bmatrix} 0.4 & 0.7 & 1.0 \end{bmatrix} \circ \begin{bmatrix} 0.1 & 0.1 \\ 0.3 & 0.5 \\ 0.3 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.3 & 0.8 \end{bmatrix}$$

Tolerance and Equivalence Relations

relations can be used in graph theory

- When a relation is reflexive every vertex in the graph originates a single loop.
- If a relation is symmetric, then in the graph for every edge pointing (the arrows on the edge lines in Fig. 3.8*b*) from vertex i to vertex j ($i,j = 1, 2, 3$), there is an edge pointing in the opposite direction, i.e., from vertex j to vertex i.
- When a relation is transitive, then for every pair of edges in the graph, one pointing from vertex i to vertex j and the other from vertex j to vertex k ($i,j,k = 1, 2, 3$), there is an edge pointing from vertex i directly to vertex k, (e.g., an arrow from vertex 1 to vertex 2, an arrow from vertex 2 to vertex 3, and an arrow from vertex 1 to vertex 3).

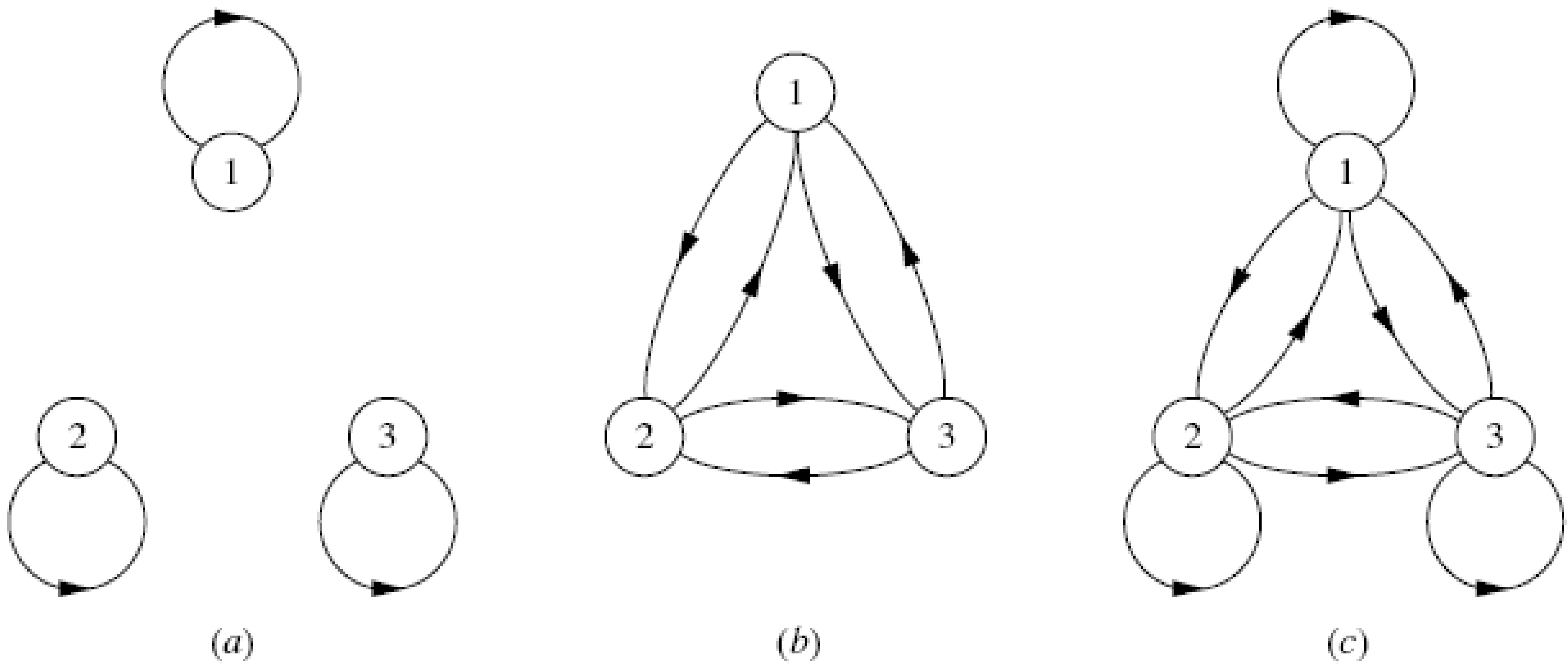


FIGURE 3.8
Three-vertex graphs for properties of (*a*) reflexivity, (*b*) symmetry, (*c*) transitivity [Gill, 1976].

Crisp Equivalence Relation

- A relation R on a universe X can also be thought of as a relation from X to X. The relation R is an equivalence relation if it has the following three properties: (1) reflexivity, (2) symmetry, and (3) transitivity.
- Example: 1.equality among elements of a set;

  2.the relation of parallelism among lines in plane geometry;

  3.the relation of similarity among triangles;

  4.the relation “works in the same building as” among workers of a given city.

Equivalence Relation Example

- for a matrix relation the following properties will hold:

Reflexivity $\quad (x_i, x_i) \in \mathrm{R}$ or $\chi_{\mathrm{R}}(x_i, x_i) = 1$

Symmetry $\quad (x_i, x_j) \in \mathrm{R} \longrightarrow (x_j, x_i) \in \mathrm{R}$

or $\quad \chi_{\mathrm{R}}(x_i, x_j) = \chi_{\mathrm{R}}(x_j, x_i)$

Transitivity $\quad (x_i, x_j) \in \mathrm{R}$ and $(x_j, x_k) \in \mathrm{R} \longrightarrow (x_i, x_k) \in \mathrm{R}$

or $\quad \chi_{\mathrm{R}}(x_i, x_j)$ and $\chi_{\mathrm{R}}(x_j, x_k) = 1 \longrightarrow \chi_{\mathrm{R}}(x_i, x_k) = 1$

Crisp Tolerance Relation

- A tolerance relation R (also called a *proximity* relation) on a universe X is a relation that exhibits only the properties of reflexivity and symmetry.
- A tolerance relation, R, can be reformed into an equivalence relation by at most *(n* − 1*)* compositions with itself, where *n* is the cardinal number of the set defining R, in this case X, i.e.,

$$R_1^{n-1} = R_1 \circ R_1 \circ \cdots \circ R_1 = R$$

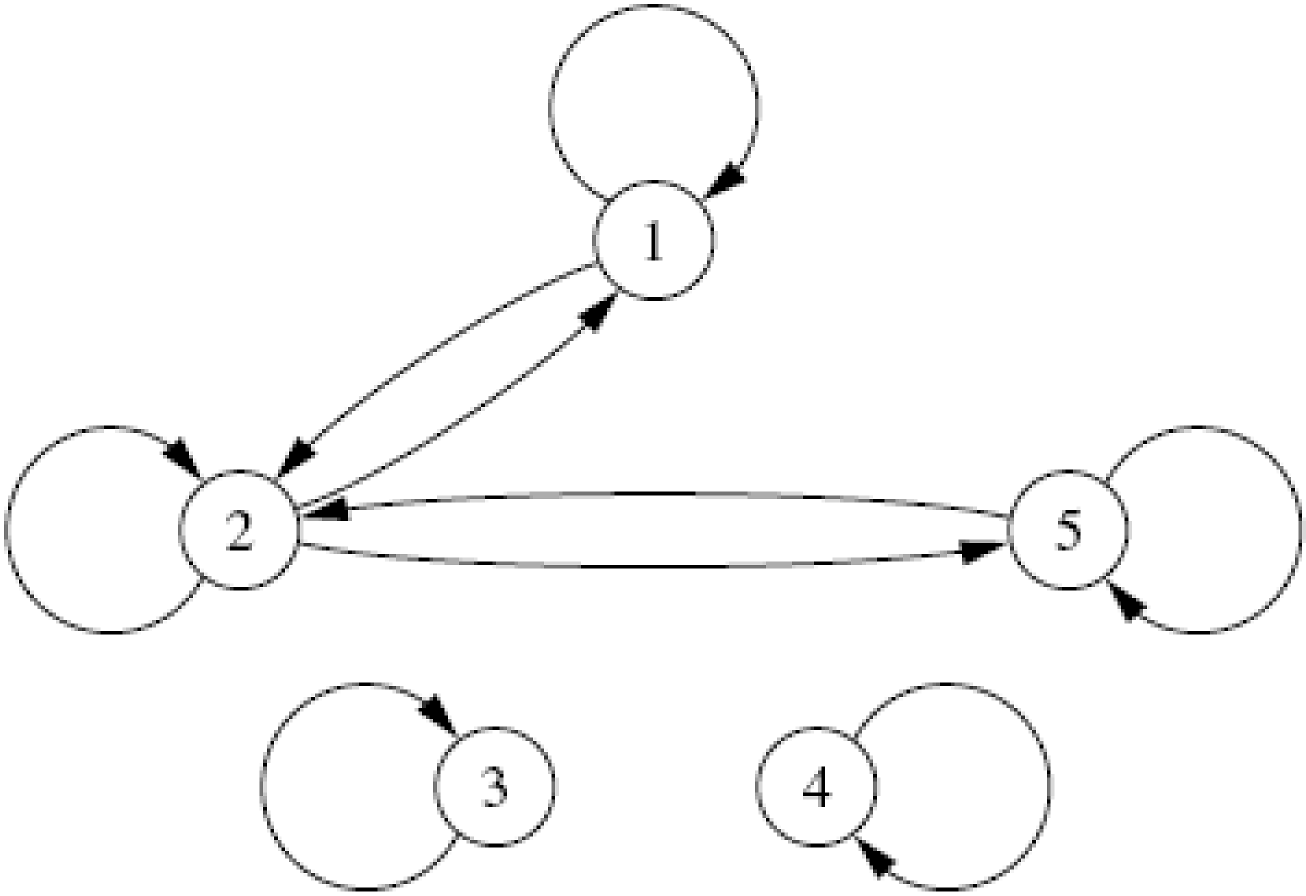
- **Example 3.10.** Suppose in an airline transportation system we have a universe composed of five elements: the cities Omaha, Chicago, Rome, London, and Detroit. The airline is studying locations of potential hubs in various countries and must consider air mileage between cities and takeoff and landing policies in the various countries. These cities can be enumerated as the elements of a set, i.e.,

$$X = \{x_1, x_2, x_3, x_4, x_5\}$$
$$= \{\text{Omaha, Chicago, Rome, London, Detroit}\}$$

Further, suppose we have a tolerance relation, R_1, that expresses relationships among these cities:

$$R_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

- This relation is reflexive and symmetric.
- but not transitivity



Crisp Tolerance Relation

- The property of reflexivity (diagonal elements equal unity) simply indicates that a city is totally related to itself.
- The property of symmetry might represent proximity: Omaha and Chicago (x_1 and x_2) are close (in a binary sense) geographically, and Chicago and Detroit (x_2 and x_5) are close geographically.
- This relation, R_1, does not have properties of transitivity, e.g.,$(x_1, x_2) \in R_1$ $(x_2, x_5) \in R_1$ but $(x_1, x_5) \notin R_1$

Crisp Tolerance Relation

R_1 can become an equivalence relation through one ($1 \leq n$, where $n = 5$) composition. Using Eq. (3.20), we get

$$R_1 \circ R_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} = R$$

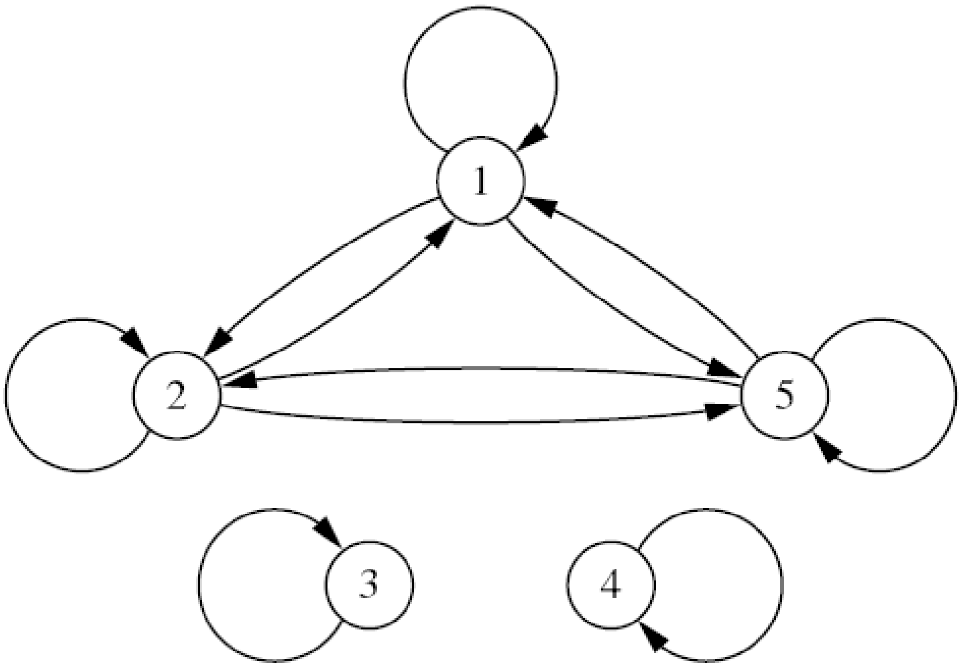


FIGURE 3.10
Five-vertex graph of equivalence relation (reflexive, symmetric, transitive) in Example 3.10.

Fuzzy tolerance and equivalence relations

- A fuzzy relation $\underset{\sim}{R}$, on a single universe X is also a relation from X to X.
- It is a fuzzy equivalence relation if the following properties for matrix relations define it:

Reflexivity $\quad \mu_{\underset{\sim}{R}}(x_i, x_i) = 1$

Symmetry $\quad \mu_{\underset{\sim}{R}}(x_i, x_j) = \mu_{\underset{\sim}{R}}(x_j, x_i)$

Transitivity $\quad \mu_{\underset{\sim}{R}}(x_i, x_j) = \lambda_1 \quad \text{and} \quad \mu_{\underset{\sim}{R}}(x_j, x_k) = \lambda_2 \longrightarrow \mu_{\underset{\sim}{R}}(x_i, x_k) = \lambda$

where $\lambda \geq \min[\lambda_1, \lambda_2]$.

It can be shown that any fuzzy tolerance relation, $\underset{\sim}{R}_1$, that has properties of reflexivity and symmetry can be reformed into a fuzzy equivalence relation by at most $(n-1)$ compositions, just as a crisp tolerance relation can be reformed into a crisp equivalence relation. That is,

$$\underset{\sim}{R}_1^{n-1} = \underset{\sim}{R}_1 \circ \underset{\sim}{R}_1 \circ \cdots \circ \underset{\sim}{R}_1 = \underset{\sim}{R} \tag{3.22}$$

- **Example 3.11.** Suppose, in a biotechnology experiment, five potentially new strains of bacteria have been detected in the area around an anaerobic corrosion pit on a new aluminum–lithium alloy used in the fuel tanks of a new experimental aircraft. In order to propose methods to eliminate the biocorrosion caused by these bacteria, the five strains must first be categorized. One way to categorize them is to compare them to one another. In a pairwise comparison, the following "similarity" relation, $\underset{\sim}{R}_1$ is developed.

For example, the first strain (column 1) has a strength of similarity to the second strain of 0.8.

$$\underset{\sim}{\mathrm{R}}_1 = \begin{bmatrix} 1 & 0.8 & 0 & 0.1 & 0.2 \\ 0.8 & 1 & 0.4 & 0 & 0.9 \\ 0 & 0.4 & 1 & 0 & 0 \\ 0.1 & 0 & 0 & 1 & 0.5 \\ 0.2 & 0.9 & 0 & 0.5 & 1 \end{bmatrix}$$

is reflexive and symmetric. However, it is not transitive, e.g.,

$$\mu_{\underset{\sim}{\mathrm{R}}}(x_1, x_2) = 0.8, \quad \mu_{\underset{\sim}{\mathrm{R}}}(x_2, x_5) = 0.9 \geq 0.8$$

but

$$\mu_{\underset{\sim}{\mathrm{R}}}(x_1, x_5) = 0.2 \leq \min(0.8, 0.9)$$

One composition results in the following relation:

$$\underset{\sim}{R}_1^2 = \underset{\sim}{R}_1 \circ \underset{\sim}{R}_1 = \begin{bmatrix} 1 & 0.8 & 0.4 & 0.2 & 0.8 \\ 0.8 & 1 & 0.4 & 0.5 & 0.9 \\ 0.4 & 0.4 & 1 & 0 & 0.4 \\ 0.2 & 0.5 & 0 & 1 & 0.5 \\ 0.8 & 0.9 & 0.4 & 0.5 & 1 \end{bmatrix}$$

where transitivity still does not result; for example,

$$\mu_{\underset{\sim}{R}^2}(x_1, x_2) = 0.8 \geq 0.5 \qquad \text{and} \qquad \mu_{\underset{\sim}{R}^2}(x_2, x_4) = 0.5$$

but

$$\mu_{R^2}(x_1, x_4) = 0.2 \leq \min(0.8, 0.5)$$

Finally, after one or two more compositions, transitivity results:

$$\underset{\sim}{R}_1^3 = \underset{\sim}{R}_1^4 = \underset{\sim}{R} = \begin{bmatrix} 1 & 0.8 & 0.4 & 0.5 & 0.8 \\ 0.8 & 1 & 0.4 & 0.5 & 0.9 \\ 0.4 & 0.4 & 1 & 0.4 & 0.4 \\ 0.5 & 0.5 & 0.4 & 1 & 0.5 \\ 0.8 & 0.9 & 0.4 & 0.5 & 1 \end{bmatrix}$$

$$\underset{\sim}{R}_1^3(x_1, x_2) = 0.8 \geq 0.5$$

$$\underset{\sim}{R}_1^3(x_2, x_4) = 0.5 \geq 0.5$$

$$\underset{\sim}{R}_1^3(x_1, x_4) = 0.5 \geq 0.5$$

VALUE ASSIGNMENTS

An appropriate question regarding relations is: Where do the membership values that are contained in a relation come from? At least seven different ways to develop the numerical values that characterize a relation:

- 1. Cartesian product
- 2. Closed-form expression
- 3. Lookup table
- 4. Linguistic rules of knowledge(expressed as if – then rules.)
- 5. Classification
- 6. Automated methods from input/output data
- 7. Similarity methods in data manipulation

7. Similarity methods

- It is one of the most prevalent forms of determining the values in relations, and which is simpler than the sixth method.
- The more robust a data set, the more accurate the relational entities are in establishing relationships among elements of two or more data sets.
- All of these methods attempt to determine some sort of similar pattern or structure in data through various metrics.

There are many of these methods available, but the two most prevalent will be discussed here.

- **Cosine Amplitude**
- **Max–Min Method**

Cosine Amplitude

- If these data samples are collected they form a data array, **X**,

$$\mathbf{X} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$$

Each of the elements, $\mathbf{x}_i$, in the data array **X** is itself a vector of length *m*, i.e.,

$$\mathbf{x}_i = \{x_{i1}, x_{i2}, \ldots, \mathbf{x}_{im}\}$$

Each element of a relation, r_{ij} ,
results from a pairwise comparison of two data samples, say $\mathbf{x}_i$ and $\mathbf{x}_j$, where the strength of the relationship between data sample $\mathbf{x}_i$ and data sample $\mathbf{x}_j$ is given by the membership value expressing that strength, i.e., $r_{ij} = \mu_R(x_i, y_j)$.

$$r_{ij} = \frac{\left|\sum_{k=1}^{m} x_{ik} x_{jk}\right|}{\sqrt{\left(\sum_{k=1}^{m} x_{ik}^2\right)\left(\sum_{k=1}^{m} x_{jk}^2\right)}}, \quad \text{where } i, j = 1, 2, \ldots, n$$

Example 3.12

Regions	x_1	x_2	x_3	x_4	x_5
x_{i1} –Ratio with no damage	0.3	0.2	0.1	0.7	0.4
x_{i2} –Ratio with medium damage	0.6	0.4	0.6	0.2	0.6
x_{i3} –Ratio with serious damage	0.1	0.4	0.3	0.1	0.0

$$r_{ij} = \frac{\left|\sum_{k=1}^{3} x_{ik} x_{jk}\right|}{\sqrt{\left(\sum_{k=1}^{3} x_{ik}^2\right)\left(\sum_{k=1}^{3} x_{jk}^2\right)}}$$

For example, for $i = 1$ and $j = 2$ we get

$$r_{12} = \frac{0.3 \times 0.2 + 0.6 \times 0.4 + 0.1 \times 0.4}{\left[\left(0.3^2 + 0.6^2 + 0.1^2\right)\left(0.2^2 + 0.4^2 + 0.4^2\right)\right]^{1/2}} = \frac{0.34}{[0.46 \times 0.36]^{1/2}} = 0.836$$

Computing the other elements of the relation results in the following tolerance relation:

$$\underset{\sim}{R}_1 = \begin{bmatrix} 1 & & & & \\ 0.836 & 1 & & \text{sym} & \\ 0.914 & 0.934 & 1 & & \\ 0.682 & 0.6 & 0.441 & 1 & \\ 0.982 & 0.74 & 0.818 & 0.774 & 1 \end{bmatrix}$$

and two compositions of $\underset{\sim}{R}_1$ produce the equivalence relation, $\underset{\sim}{R}$:

$$\underset{\sim}{R} = \underset{\sim}{R}_1^3 = \begin{bmatrix} 1 & & & & \\ 0.914 & 1 & & \text{sym} & \\ 0.914 & 0.934 & 1 & & \\ 0.774 & 0.774 & 0.774 & 1 & \\ 0.982 & 0.914 & 0.914 & 0.774 & 1 \end{bmatrix}$$

The tolerance relation, $\underset{\sim}{R}_1$, expressed the pairwise similarity of damage for each of the regions; the equivalence relation, $\underset{\sim}{R}$, also expresses this same information but additionally can be used to classify the regions into categories with *like properties* (see Chapter 11).

Max–Min Method

$$r_{ij} = \frac{\sum_{k=1}^{m} \min(x_{ik}, x_{jk})}{\sum_{k=1}^{m} \max(x_{ik}, x_{jk})}, \quad \text{where } i, j = 1, 2, \ldots, n$$

Example 3.13. If we reconsider Example 3.12, the min–max method will produce the following result for $i = 1$, $j = 2$:

$$r_{12} = \frac{\sum_{k=1}^{3}[\min(0.3, 0.2), \min(0.6, 0.4), \min(0.1, 0.4)]}{\sum_{k=1}^{3}[\max(0.3, 0.2), \max(0.6, 0.4), \max(0.1, 0.4)]} = \frac{0.2 + 0.4 + 0.1}{0.3 + 0.6 + 0.4} = 0.538$$

$$r_{ij} = \frac{\sum_{k=1}^{m}\min(x_{ik}, x_{jk})}{\sum_{k=1}^{m}\max(x_{ik}, x_{jk})}, \quad \text{where } i, j = 1, 2, \ldots, n$$

OTHER FORMS OF THE COMPOSITION OPERATION

$$\text{min–max} \quad \mu_{\underset{\sim}{B}}(y) = \min_{x \in X}\{\max[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{R}}(x, y)]\}$$

$$\text{max–max} \quad \mu_{\underset{\sim}{B}}(y) = \max_{x \in X}\{\max[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{R}}(x, y)]\}$$

$$\text{min–min} \quad \mu_{\underset{\sim}{B}}(y) = \min_{x \in X}\{\min[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{R}}(x, y)]\}$$

$$\text{max–average} \quad \mu_{\underset{\sim}{B}}(y) = \frac{1}{2}\max_{x \in X}[\mu_{\underset{\sim}{A}}(x) + \mu_{\underset{\sim}{R}}(x, y)]$$

$$\text{sum–product} \quad \mu_{\underset{\sim}{B}}(y) = f\left\{\sum_{x \in X}[\mu_{\underset{\sim}{A}}(x) \cdot \mu_{\underset{\sim}{R}}(x, y)]\right\}$$

SUMMARY

- properties and operations of crisp and fuzzy relations

- The idea of a relation is most powerful;dealing with such issues as logic, nonlinear simulation,classification, and control.
- Composition of a relation is similar to a method used to extend fuzziness into functions, called the *extension principle.*
- Tolerant and equivalent relations are used in similarity applications and classification applications, respectively.
- several similarity metrics were shown to be useful in developing the relational *strengths,* or distances, within fuzzy relations from data sets.