MEMBERSHIP FUNCTIONS, FUZZIFICATION, AND DEFUZZIFICATION

- It is one thing to compute, to reason, and to model with fuzzy information; it is another to apply the fuzzy results to the world around us.
- Despite the fact that the bulk of the information we assimilate every day is fuzzy, most of the actions or decisions implemented by humans or machines are crisp or binary.

(The decisions we make that require an action are binary, the hardware we use is binary, and certainly the computers we use are based on binary digital instructions.)

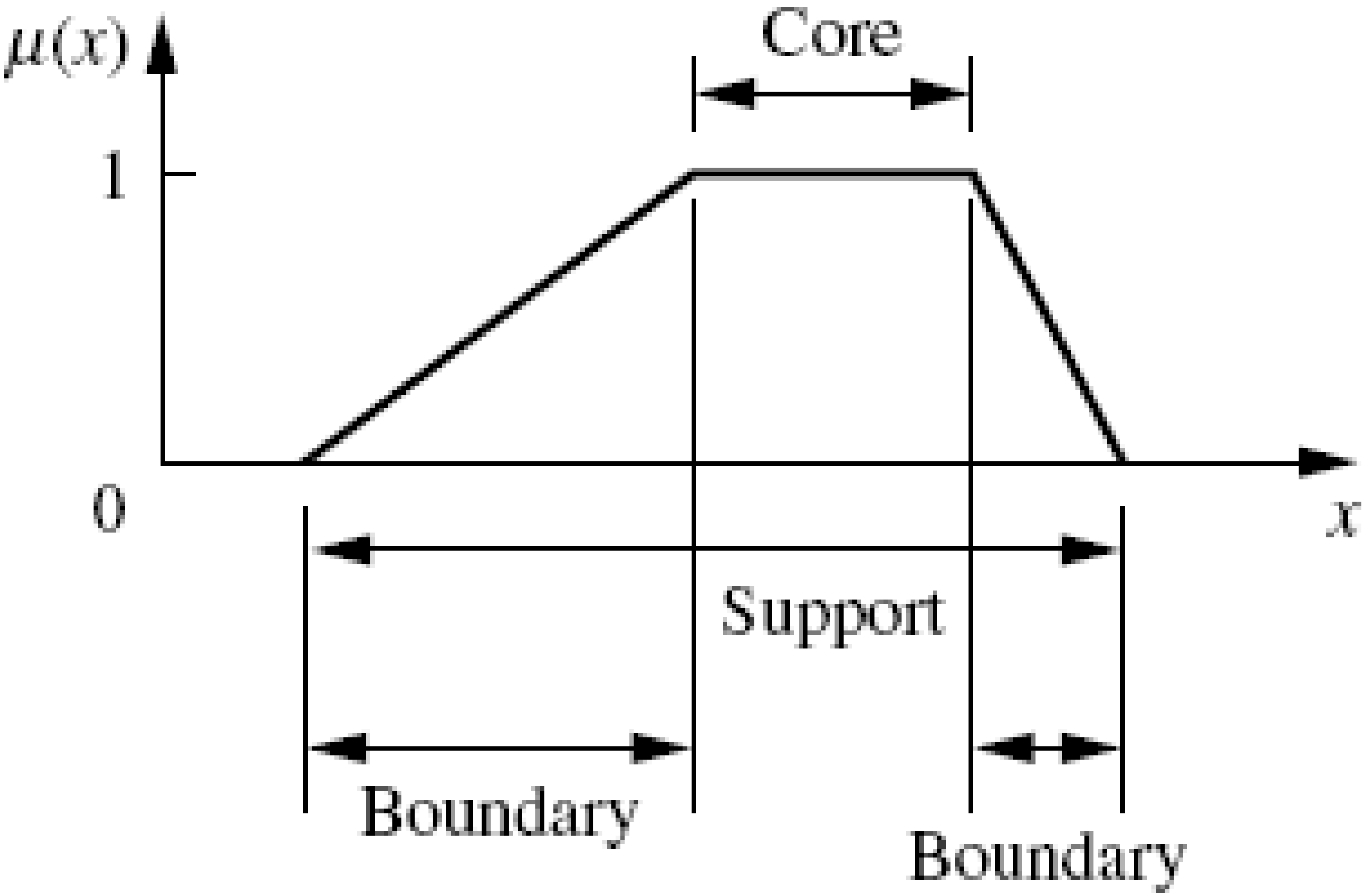
- The bulk of this textbook illustrates procedures to “fuzzify” the mathematical and engineering principles we have so long considered to be deterministic.
- But in various applications and engineering scenarios there will be a need to “defuzzify” the fuzzy results.

(In other words, we may eventually find a need to convert the fuzzy results to crisp results.)

Mathematically, the defuzzification of a fuzzy set is the process of “rounding it off” from its location in the unit hypercube to the nearest (in a geometric sense) vertex

Features of the membership function

- The *core* of a membership function for some fuzzy set $\underline{A}$ is defined as that region of the universe that is characterized by complete and full membership in the set *A*. That is, the core comprises those elements *x* of the universe such that $\mu_{\underline{A}}(x) = 1$.
- The *support* of a membership function for some fuzzy set *A* is defined as that region of the universe that is characterized by nonzero membership in the set $\underline{A}$. That is, the support comprises those elements *x* of the universe such that $\mu_{\underline{A}}(x) > 0$.
- The *boundaries* of a membership function for some fuzzy set $\underline{A}$ are defined as that region of the universe containing elements that have a nonzero membership but not complete membership.



$\mu(x)$
Core
1
0
x
Support
Boundary
Boundary

Features of the membership function

- A *normal* fuzzy set is one whose membership function has at least one element *x* in the universe whose membership value is unity.

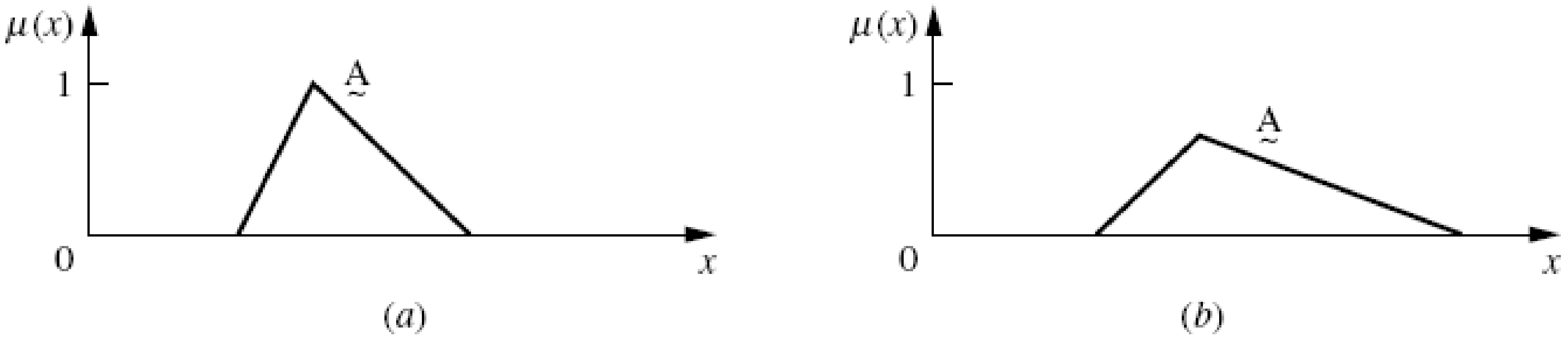


FIGURE 4.2
Fuzzy sets that are normal (*a*) and subnormal (*b*).

Convex fuzzy set

- A *convex* fuzzy set is described by a membership function whose membership values are strictly monotonically increasing, or strictly monotonically decreasing, or strictly monotonically increasing then strictly monotonically decreasing with increasing values for elements in the universe.

  (Said another way, if, for any elements *x*, *y*, and *z* in a fuzzy set $\underline{A}$, the relation $x < y < z$ implies that

$$\mu_{\underset{\sim}{A}}(y) \geq \min[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{A}}(z)]$$

  then $\underline{A}$ is said to be a convex fuzzy set.)

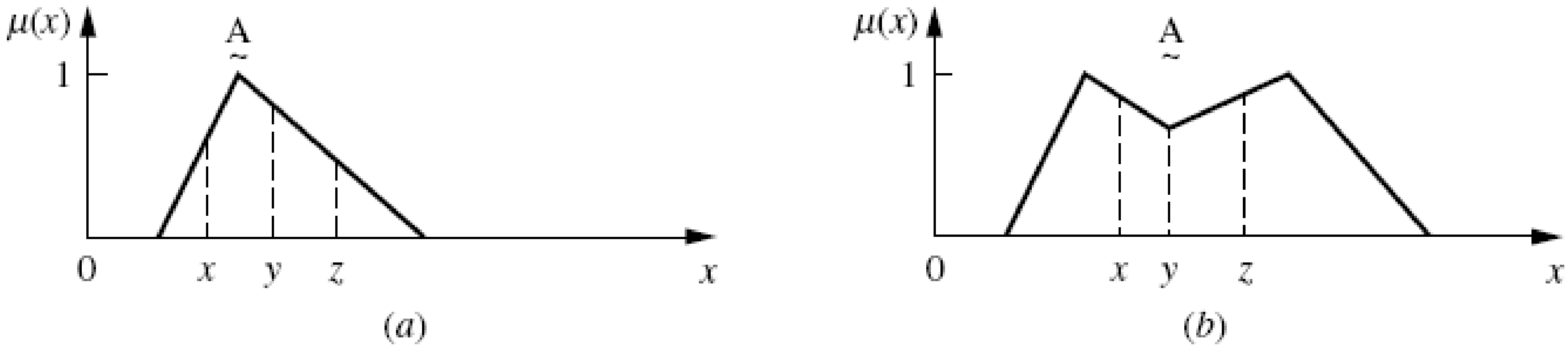


FIGURE 4.3
Convex, normal fuzzy set (a) and nonconvex, normal fuzzy set (b).

this definition of convexity is *different* from some definitions of the same term in mathematics. In some areas of mathematics, convexity of shape has to do with whether a straight line through any part of the shape goes outside the boundaries of that shape. This definition of convexity is *not* used here.

Convex fuzzy set

A special property of two convex fuzzy sets, say $\underset{\sim}{A}$ and $\underset{\sim}{B}$, is that the intersection of these two convex fuzzy sets is also a convex fuzzy set, as shown in Fig. 4.4. That is, for $\underset{\sim}{A}$ and $\underset{\sim}{B}$, which are both convex, $\underset{\sim}{A} \cap \underset{\sim}{B}$ is also convex.

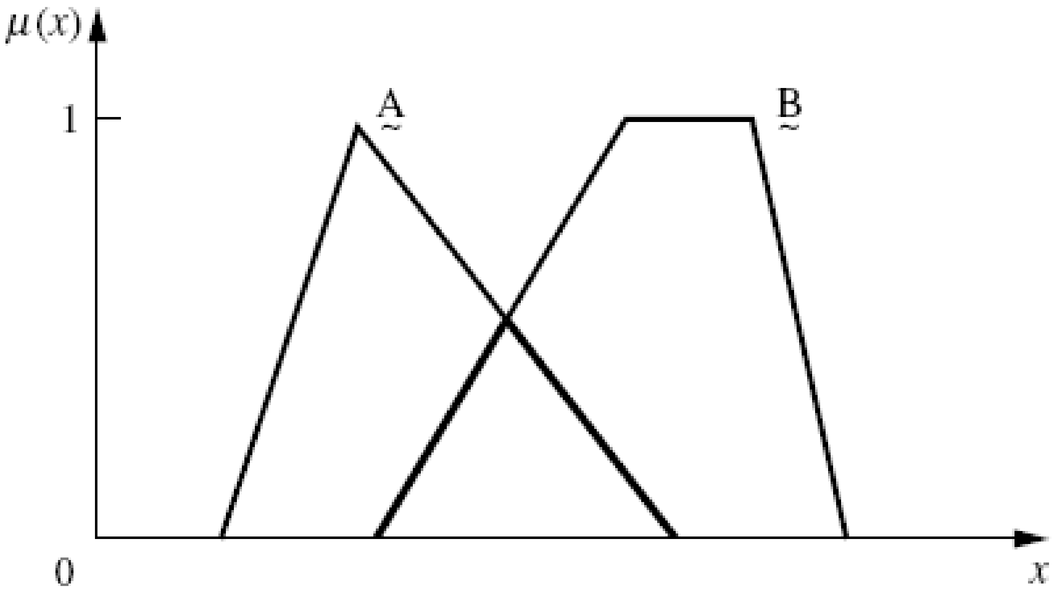


FIGURE 4.4
The intersection of two convex fuzzy sets produces a convex fuzzy set.

Convex fuzzy set

- The *crossover points* of a membership function are defined as the elements in the universe for which a particular fuzzy set *A* has values equal to 0.5, i.e., for which $\mu_A(x) = 0.5$.
- The *height* of a fuzzy set *A* is the maximum value of the membership function, i.e.,

$$\text{hgt}(\underset{\sim}{A}) = \max\{\mu_{\underset{\sim}{A}}(x)\}$$

- If the hgt$(\underline{A}) < 1$, the fuzzy set is said to be subnormal.
- If $\underline{A}$ is a convex single-point normal fuzzy set defined on the real line, then $\underline{A}$ is often termed a *fuzzy number*.

VARIOUS FORMS

- Membership functions can be symmetrical or asymmetrical. They are typically defined on one-dimensional universes, or on multidimensional universes.
- For example, the membership functions shown in this chapter are one-dimensional curves. In two dimensions these curves become surfaces and for three or more dimensions these surfaces become hypersurfaces.

VARIOUS FORMS

- *ordinary membership functions*.

(require that each element of the universe *x* on which the fuzzy set $\underset{\sim}{A}$ is defined be assigned a specific membership value, $\mu_{\underset{\sim}{A}}(x)$)

- *generalized membership functions*.
- *interval-valued membership function*
- (Suppose the level of information is not adequate to specify membership functions with this precision. For example, we may only know the upper and lower bounds of membership grades for each element of the universe for a fuzzy set. such as the one shown in Fig. 4.5.)

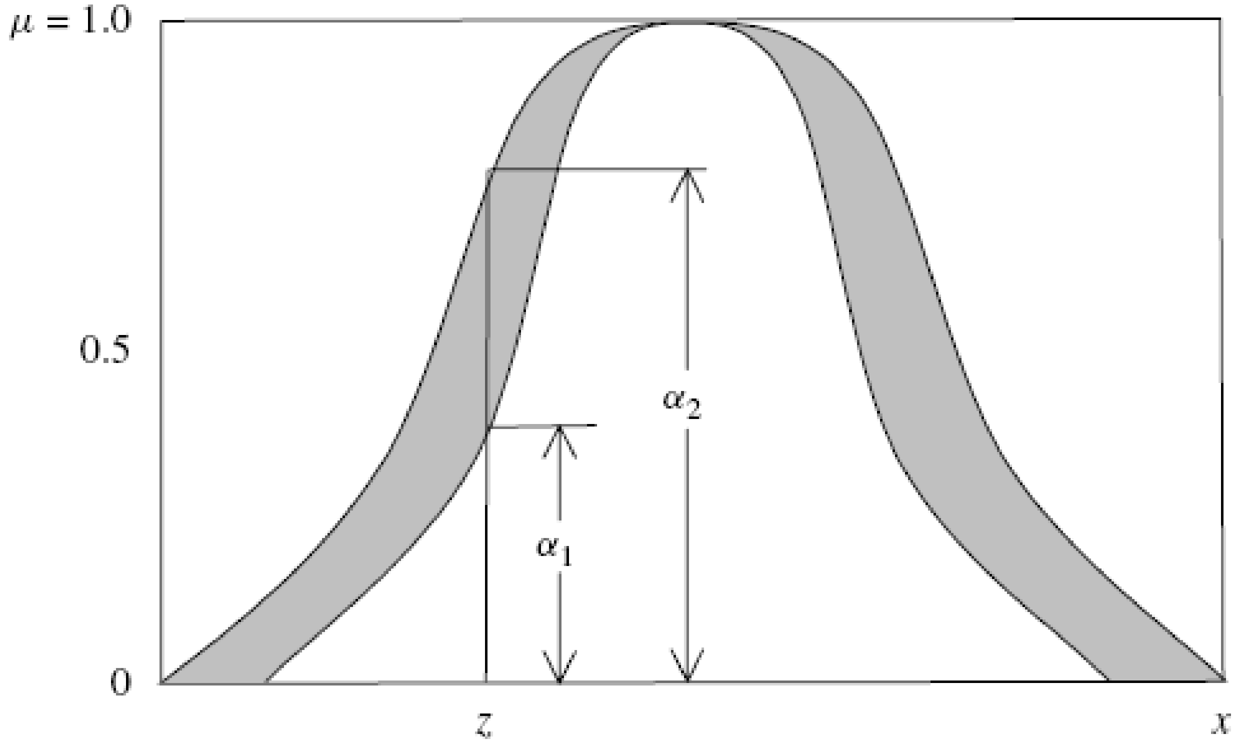


FIGURE 4.5
An interval-valued membership function.

In this figure, for a particular element, $x = z$, the membership in a fuzzy set A, i.e., $\mu_A(z)$, would be expressed by the membership interval $[\alpha_1, \alpha_2]$.

VARIOUS FORMS

- Interval-valued fuzzy sets can be generalized further by allowing their intervals to become fuzzy. Each membership interval then becomes an ordinary fuzzy set. This type of membership function is referred to in the literature as a *type-2 fuzzy set*.

FUZZIFICATION

- Fuzzification is the process of making a crisp quantity fuzzy.
- In the real world, hardware such as a digital voltmeter generates crisp data, but these data are subject to experimental error.

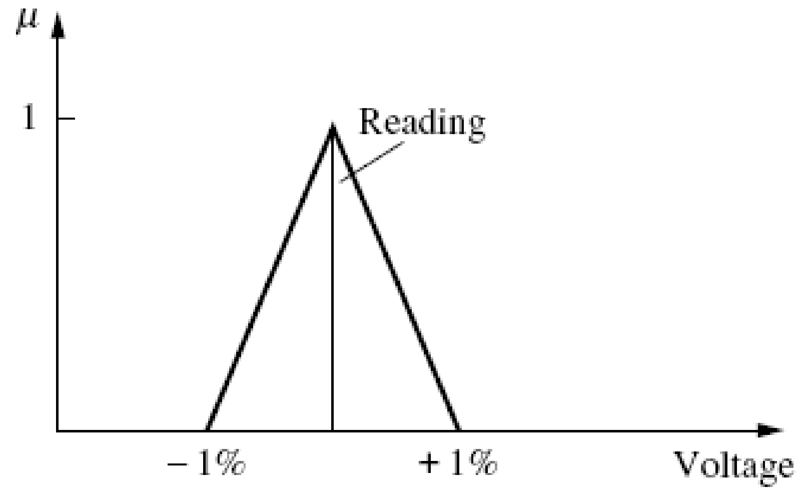


FIGURE 4.6
Membership function representing imprecision in "crisp voltage reading."

- The representation of imprecise data as fuzzy sets is a useful but not mandatory step when those data are used in fuzzy systems. This idea is shown in Fig. 4.7, where we consider the data as a crisp reading, Fig. 4.7*a*, or as a fuzzy reading, as shown in Fig. 4.7*b*.

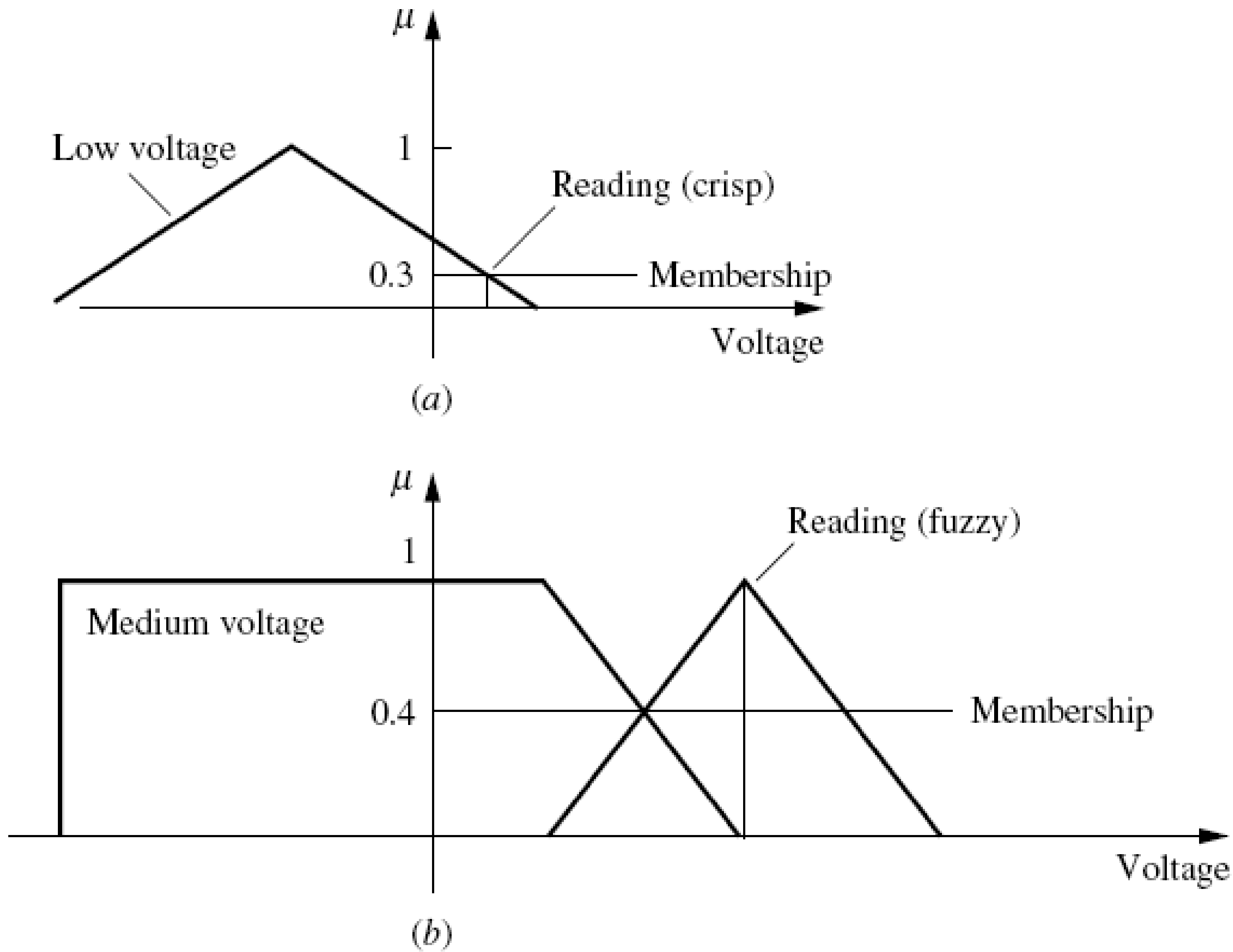


FIGURE 4.7
Comparisons of fuzzy sets and crisp or fuzzy readings: (*a*) fuzzy set and crisp reading; (*b*) fuzzy set and fuzzy reading.

Defuzzification to crisp sets

- $0 \leqslant \lambda \leqslant 1$. The set A_λ is a crisp set called the lambda (λ)-cut (or alpha-cut) set of the fuzzy set $\underline{A}$, where

$$A_\lambda = \{x \mid \mu_A(x) \geqslant \lambda\}.$$

- λ-cut set A_λ is a crisp set.
- Any element $x \in A_\lambda$ belongs to $\underline{A}$ with a grade of membership that is greater than or equal to the value λ.

Example 4.1. Let us consider the discrete fuzzy set, using Zadeh's notation, defined on universe $X = \{a, b, c, d, e, f\}$,

$$\underset{\sim}{A} = \left\{ \frac{1}{a} + \frac{0.9}{b} + \frac{0.6}{c} + \frac{0.3}{d} + \frac{0.01}{e} + \frac{0}{f} \right\}$$

This fuzzy set is shown schematically in Fig. 4.8. We can reduce this fuzzy set into several λ-cut sets, all of which are crisp. For example, we can define λ-cut sets for the values of $\lambda = 1$, 0.9, 0.6, 0.3, 0^+, and 0.

$$A_1 = \{a\}, \quad A_{0.9} = \{a, b\}$$
$$A_{0.6} = \{a, b, c\}, \quad A_{0.3} = \{a, b, c, d\}$$
$$A_{0^+} = \{a, b, c, d, e\}, \quad A_0 = X$$

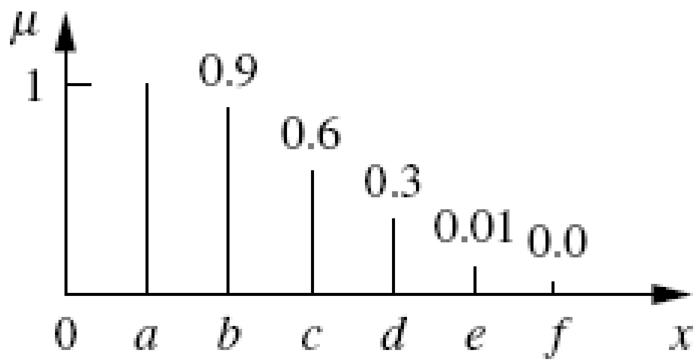


FIGURE 4.8
A discrete fuzzy set $\underset{\sim}{A}$.

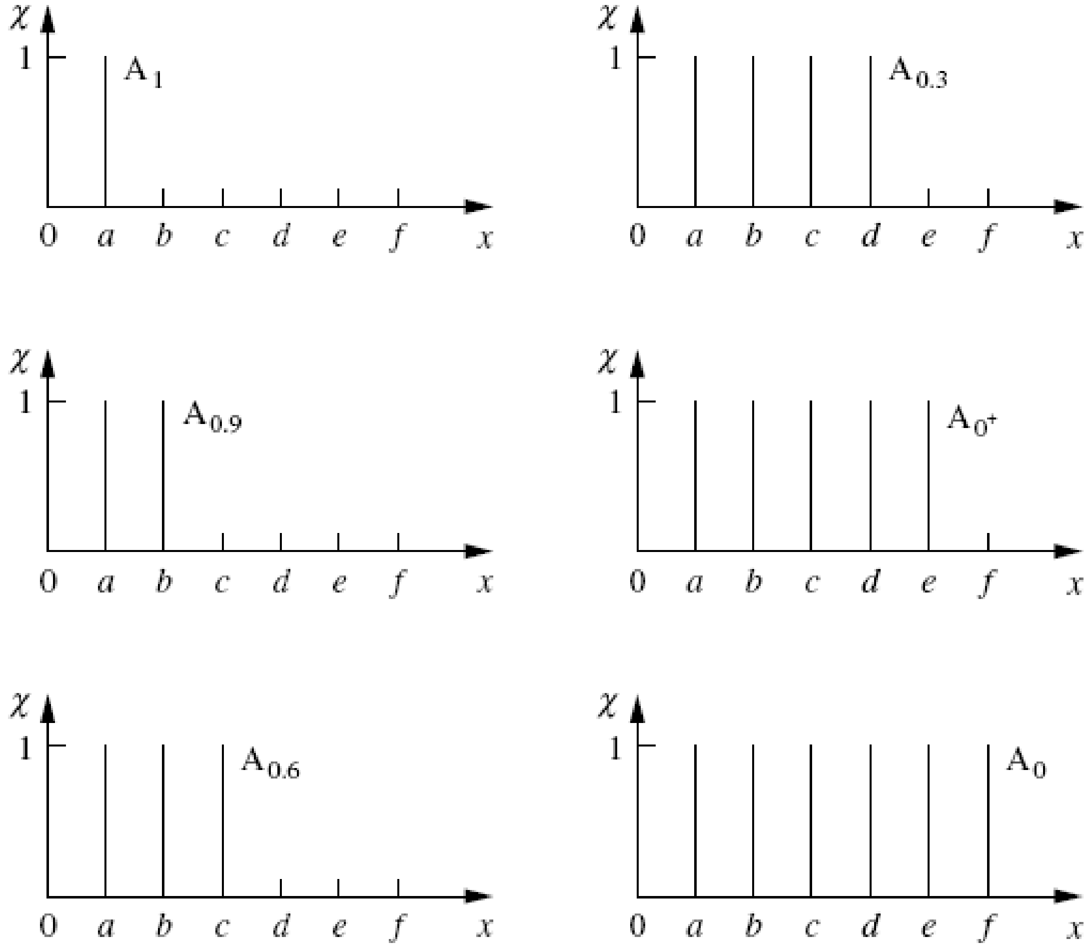


FIGURE 4.9
Lambda-cut sets for $\lambda = 1, 0.9, 0.6, 0.3, 0^+, 0$.

We can express λ-cut sets using Zadeh's notation. For the example, λ-cut sets for the values $\lambda = 0.9$ and 0.25 are given here:

$$A_{0.9} = \left\{ \frac{1}{a} + \frac{1}{b} + \frac{0}{c} + \frac{0}{d} + \frac{0}{e} + \frac{0}{f} \right\} \qquad A_{0.25} = \left\{ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{0}{e} + \frac{0}{f} \right\}$$

λ-cut sets obey the following four very special properties:

1. $(\underset{\sim}{A} \cup \underset{\sim}{B})_\lambda = A_\lambda \cup B_\lambda$ (4.1a)

2. $(\underset{\sim}{A} \cap \underset{\sim}{B})_\lambda = A_\lambda \cap B_\lambda$ (4.1b)

3. $(\overline{\underset{\sim}{A}})_\lambda \neq \overline{A}_\lambda$ except for a value of $\lambda = 0.5$ (4.1c)

4. For any $\lambda \leq \alpha$, where $0 \leq \alpha \leq 1$, it is true that $A_\alpha \subseteq A_\lambda$, where $A_0 = X$ (4.1d)

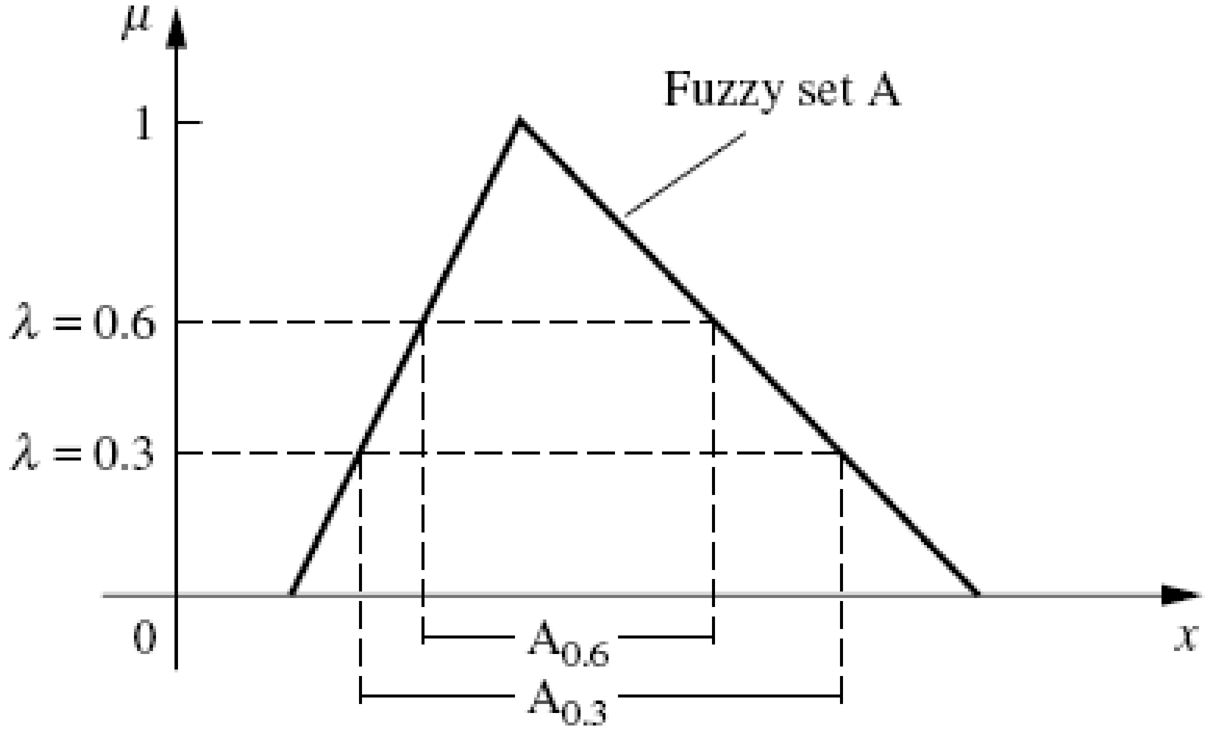


FIGURE 4.10
Two different λ-cut sets for a continuous-valued fuzzy set.

In this chapter, various definitions of a membership function are discussed and illustrated. Many of these same definitions arise through the use of λ-cut sets. As seen in Fig. 4.1, we can provide the following definitions for a convex fuzzy set $\underset{\sim}{A}$. The core of $\underset{\sim}{A}$ is the $\lambda = 1$ cut set, A_1. The support of $\underset{\sim}{A}$ is the λ-cut set A_{0+}, where $\lambda = 0^+$, or symbolically, $A_{0+} = \{x \mid \mu_{\underset{\sim}{A}(x)} > 0\}$. The intervals $[A_{0+}, A_1]$ form the boundaries of the fuzzy set $\underset{\sim}{A}$, i.e., those regions that have membership values between 0 and 1 (exclusive of 0 and 1): that is, for $0 < \lambda < 1$.

λ-cuts for fuzzy relations

- Define $R_\lambda = \{(x, y) \mid \mu_{\underset{\sim}{R}}(x, y) \geq \lambda\}$

as a λ-cut relation of the fuzzy relation $\underset{\sim}{R}$

Any pair *(x, y)* $\in$ R $_\lambda$ belongs to $\underset{\sim}{R}$ with a "strength" of relation greater than or equal to λ.

$$\underset{\sim}{R} = \begin{bmatrix} 1 & 0.8 & 0 & 0.1 & 0.2 \\ 0.8 & 1 & 0.4 & 0 & 0.9 \\ 0 & 0.4 & 1 & 0 & 0 \\ 0.1 & 0 & 0 & 1 & 0.5 \\ 0.2 & 0.9 & 0 & 0.5 & 1 \end{bmatrix}$$

Example 4.2. Suppose we take the fuzzy relation from the biotechnology example in Chapter 3 (Example 3.11), and perform λ-cut operations for the values of $\lambda = 1, 0.9, 0$. These crisp relations are given below:

$$\lambda = 1,\ R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\lambda = 0.9,\ R_{0.9} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$\lambda = 0,\ R_0 = \underset{\sim}{\mathbf{E}}$ (whole relation; see Chapter 3)

Properties of λ-cuts on fuzzy relations

λ-cuts on fuzzy relations obey certain properties, just as λ-cuts on fuzzy sets do (see Eqs. (4.1)), as given in Eqs. (4.2):

1. $(\underset{\sim}{R} \cup \underset{\sim}{S})_{\lambda} = R_{\lambda} \cup S_{\lambda}$ (4.2a)
2. $(\underset{\sim}{R} \cap \underset{\sim}{S})_{\lambda} = R_{\lambda} \cap S_{\lambda}$ (4.2b)
3. $(\overline{\underset{\sim}{R}})_{\lambda} \neq \overline{R}_{\lambda}$ (4.2c)
4. For any $\lambda \leq \alpha$, $0 \leq \alpha \leq 1$, then $R_{\alpha} \subseteq R_{\lambda}$ (4.2d)

Defuzzification to scalars

- Defuzzification is the conversion of a fuzzy quantity to a precise quantity, just as fuzzification is the conversion of a precise quantity to a fuzzy quantity.
- The output of a fuzzy process can be the logical union of two or more fuzzy membership functions defined on the universe of discourse of the output variable.

$$\underset{\sim}{C}_k = \bigcup_{i=1}^{k} \underset{\sim}{C}_i = \underset{\sim}{C}$$

For example, suppose a fuzzy output is comprised of two parts:

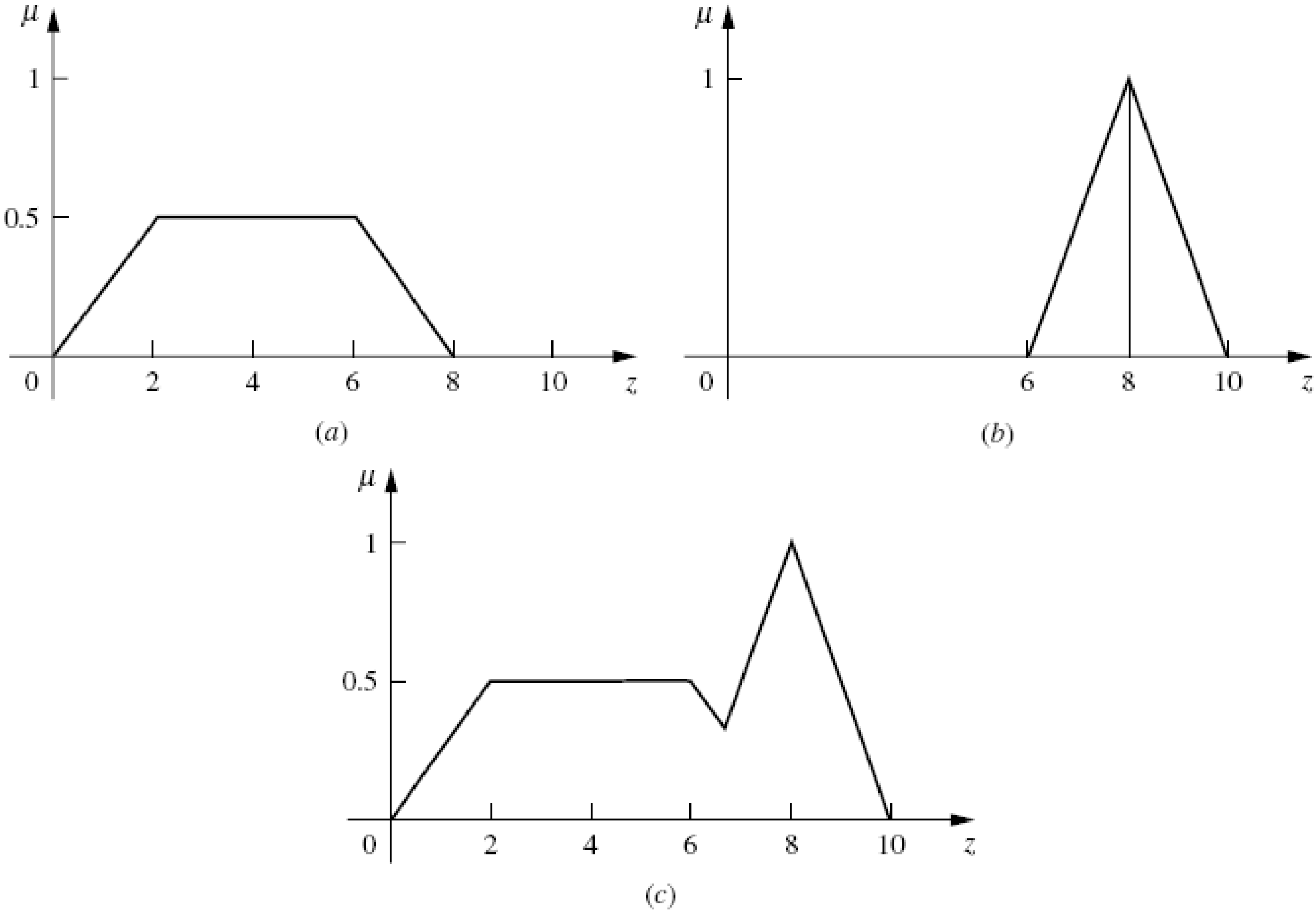


FIGURE 4.11
Typical fuzzy process output: (*a*) first part of fuzzy output; (*b*) second part of fuzzy output; (*c*) union of both parts.

Methods 1 for defuzzifying fuzzy output functions

- 1. *Max membership principle*: (Also known as the *height* method)

$$\mu_C(z^*) \geq \mu_C(z) \qquad \text{for all } z \in Z$$

where $z*$ is the defuzzified value.

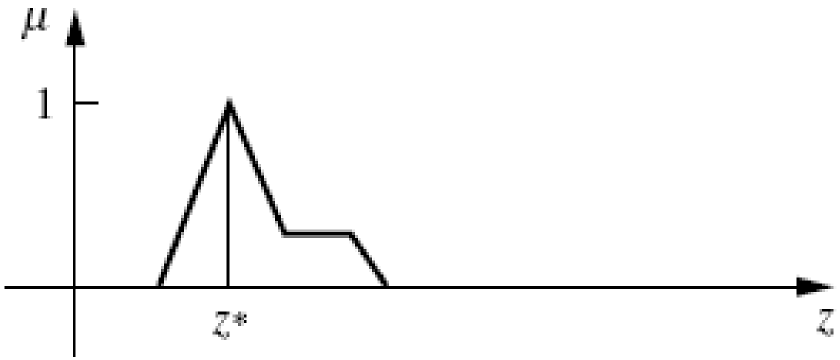


FIGURE 4.12
Max membership defuzzification method.

Methods 2 for defuzzifying fuzzy output functions

- 2. *Centroid method*:

  (also called center of area, center of gravity)

$$z^* = \frac{\int \mu_{\underset{\sim}{C}}(z) \cdot z \, dz}{\int \mu_{\underset{\sim}{C}}(z) \, dz}$$

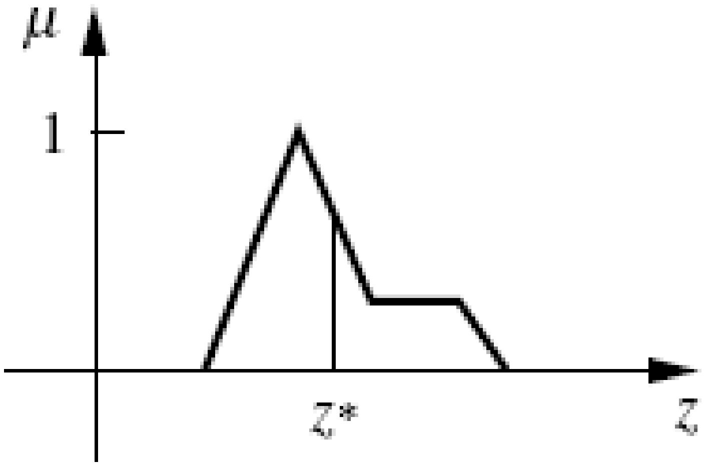


FIGURE 4.13
Centroid defuzzification method.

Methods 3 for defuzzifying fuzzy output functions

- 3. *Weighted average method*:

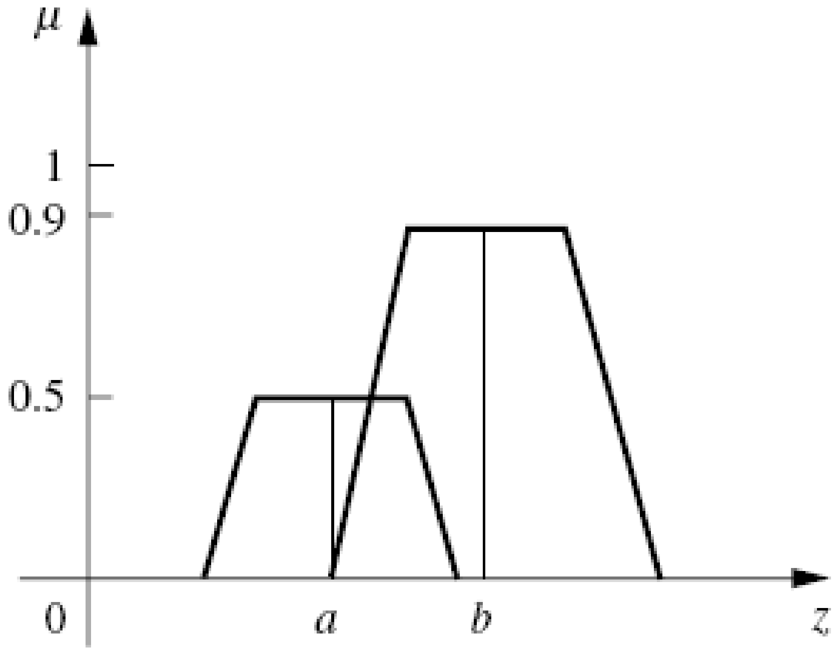
  (it is *usually* restricted to symmetrical output membership functions.)

$$z^* = \frac{\sum \mu_{\underset{\sim}{C}}(\bar{z}) \cdot \bar{z}}{\sum \mu_{\underset{\sim}{C}}(\bar{z})}$$

$\bar{z}$ is the centroid of each symmetric membership function.

Weighted average method

- As an example,



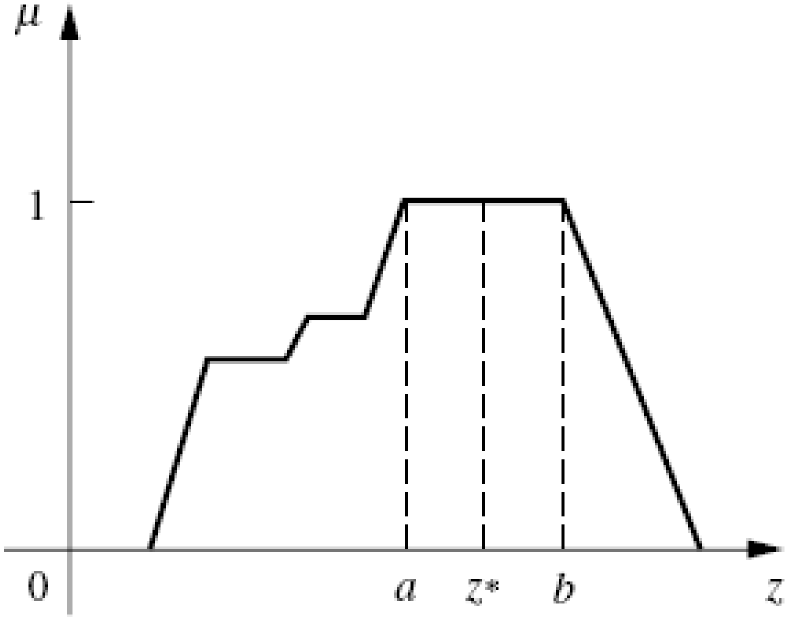
$$z^* = \frac{a(0.5) + b(0.9)}{0.5 + 0.9}$$

Methods 4 for defuzzifying fuzzy output functions

- 4. *Mean max membership*:

  (also called middle-of-maxima)

$$z^* = \frac{a+b}{2}$$



the maximum membership can be a plateau rather than a single point).

Example 4.3. A railroad company intends to lay a new rail line in a particular part of a county. The whole area through which the new line is passing must be purchased for right-of-way considerations. It is surveyed in three stretches, and the data are collected for analysis. The surveyed data for the road are given by the sets, $\underset{\sim}{B}_1$, $\underset{\sim}{B}_2$, and $\underset{\sim}{B}_3$, where the sets are defined on the universe of right-of-way widths, in meters. For the railroad to purchase the land, it must have an assessment of the amount of land to be bought. The three surveys on right-of-way width are ambiguous, however, because some of the land along the proposed railway route is already public domain and will not need to be purchased. Additionally, the original surveys are so old (*circa* 1860) that some ambiguity exists on boundaries and public right-of-way for old utility lines and old roads. The three fuzzy sets, $\underset{\sim}{B}_1$, $\underset{\sim}{B}_2$, and $\underset{\sim}{B}_3$, shown in Figs. 4.16, 4.17, and 4.18, respectively, represent the uncertainty in each survey as to the membership of right-of-way width, in meters, in privately owned land.

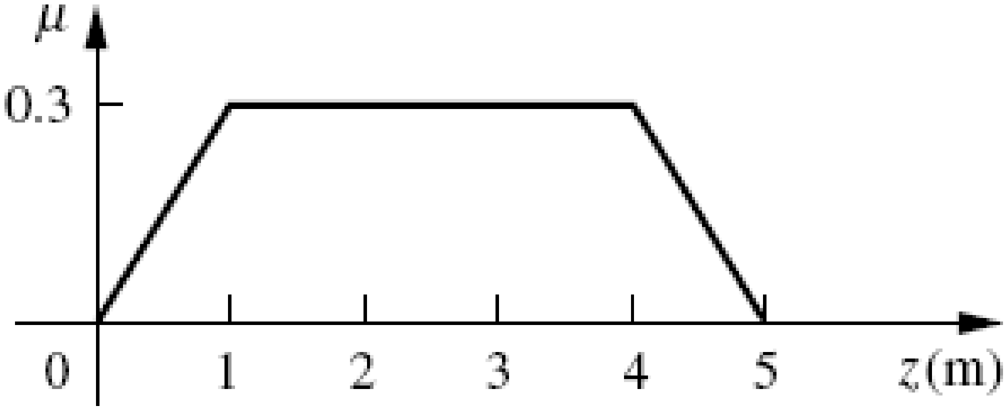


FIGURE 4.16
Fuzzy set $\underset{\sim}{B}_1$: public right-of-way width (z) for survey 1.

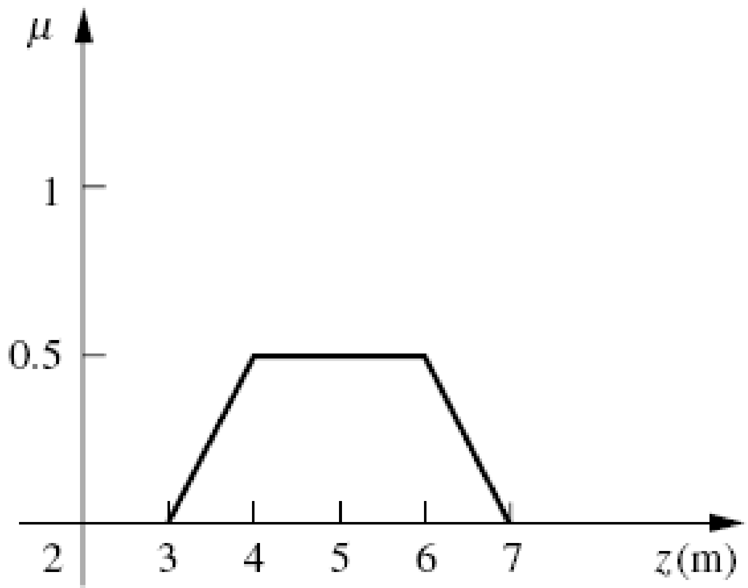


FIGURE 4.17
Fuzzy set $\underset{\sim}{B}_2$: public right-of-way width (z) for survey 2.

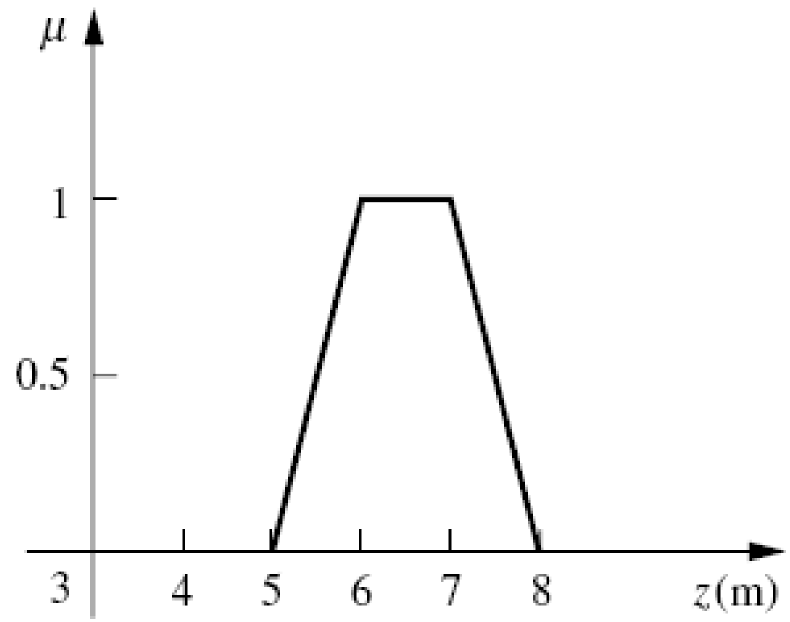


FIGURE 4.18
Fuzzy set $\underset{\sim}{B}_3$: public right-of-way width (z) for survey 3.

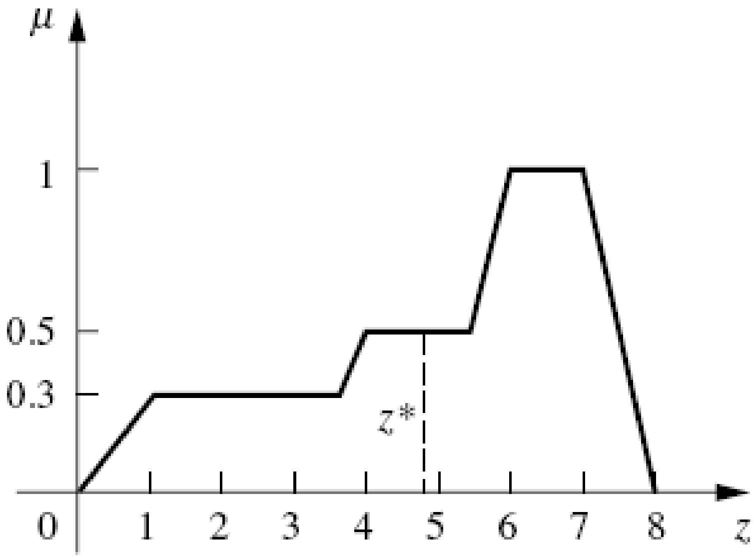


FIGURE 4.19
The centroid method for finding z^*.

We now want to aggregate these three survey results to find the single most nearly representative right-of-way width (z) to allow the railroad to make its initial estimate of the right-of-way purchasing cost. Using Eqs. (4.5)–(4.7) and the preceding three fuzzy sets, we want to find z^*.

According to the centroid method, Eq. (4.5), z^* can be found using

$$z^* = \frac{\int \mu_{\underset{\sim}{B}}(z) \cdot z \, dz}{\int \mu_{\underset{\sim}{B}}(z) \, dz}$$

$$= \left[\int_0^1 (0.3z) z \, dz + \int_1^{3.6} (0.3z) \, dz + \int_{3.6}^{4} \left(\frac{z - 3.6}{2} \right) z \, dz + \int_4^{5.5} (0.5) z \, dz \right.$$

$$\left. + \int_{5.5}^{6} (z - 5.5) z \, dz + \int_6^7 z \, dz + \int_7^8 \left(\frac{7 - z}{2} \right) z \, dz \right]$$

$$\div \left[\int_0^1 (0.3z) \, dz + \int_1^{3.6} (0.3) \, dz + \int_{3.6}^{4} \left(\frac{z - 3.6}{2} \right) dz + \int_4^{5.5} (0.5) \, dz \right.$$

$$\left. + \int_{5.5}^{6} \left(\frac{z - 5.5}{2} \right) dz + \int_6^7 dz + \int_7^8 \left(\frac{7 - z}{2} \right) dz \right]$$

$$= 4.9 \text{ meters}$$

According to the weighted average method, Eq. (4.6),

$$z^* = \frac{(0.3 \times 2.5) + (0.5 \times 5) + (1 \times 6.5)}{0.3 + 0.5 + 1} = 5.41 \text{ meters}$$

$$z^* = \frac{\sum \mu_{\underset{\sim}{C}}(\overline{z}) \cdot \overline{z}}{\sum \mu_{\underset{\sim}{C}}(\overline{z})}$$

$\overline{z}$ is the centroid of each symmetric membership function.

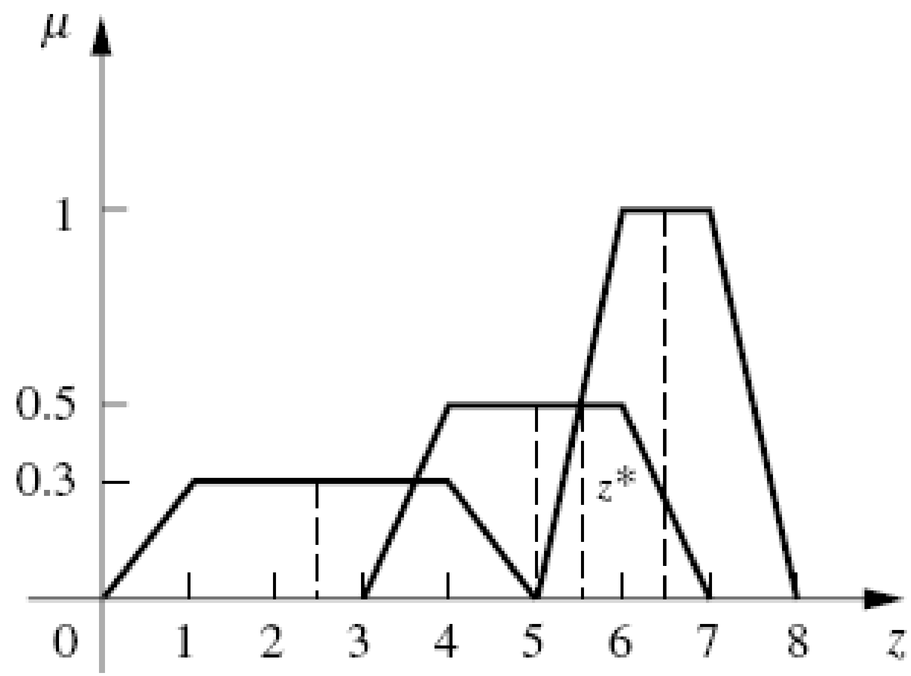


FIGURE 4.20
The weighted average method for finding z^*.

- According to the mean max membership method, Eq. (4.7), $z*$ is given by $(6 + 7)/2 = 6.5$ meters.

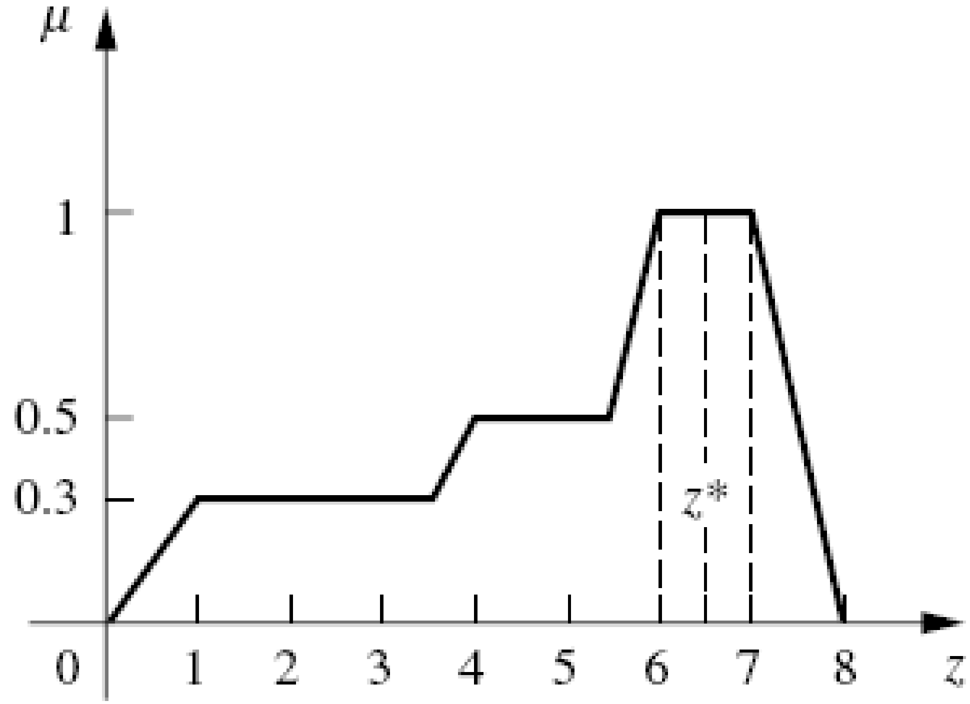


FIGURE 4.21
The mean max membership method for finding z^*.

Methods 5 for defuzzifying fuzzy output functions

- 5. *Center of sums*:
- This is faster than many defuzzification methods, the method is not restricted to symmetric membership functions.
- This process involves the algebraic sum of individual output fuzzy sets, instead of their union.
- Two drawbacks: the intersecting areas are added twice, and the method also involves finding the centroids of the individual membership functions.

$$z^* = \frac{\int_Z \bar{z} \sum_{k=1}^{n} \mu_{\underset{\sim}{C}_k}(z)\,dz}{\int_z \sum_{k=1}^{n} \mu_{\underset{\sim}{C}_k}(z)\,dz}$$

where the symbol $\bar{Z}$ is the distance to the centroid of each of the respective membership functions.

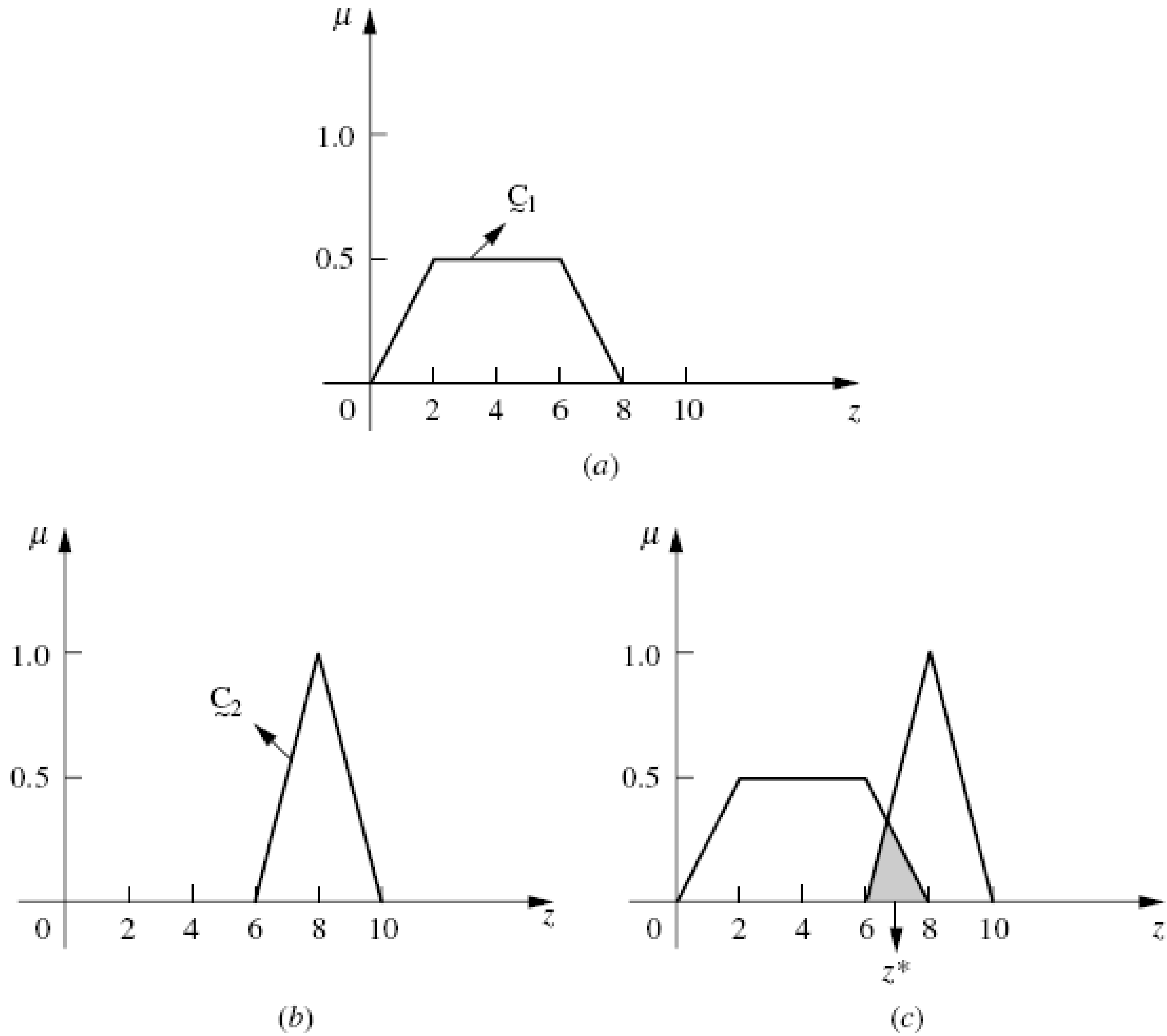


FIGURE 4.28
Center of sums method: (*a*) first membership function; (*b*) second membership function; and (*c*) defuzzification step.

Methods 6 for defuzzifying fuzzy output functions

6. *Center of largest area*:

If the output fuzzy set has at least two convex regions, then the center of gravity (i.e., z^* is calculated using the centroid method, Eq. (4.5)) of the convex fuzzy subregion with the largest area is used to obtain the defuzzified value z^* of the output.

$$z^* = \frac{\int \mu_{\underset{\sim}{C}_m}(z) z \, dz}{\int \mu_{\underset{\sim}{C}_m}(z) \, dz}$$

where $\underline{C}_m$ is the convex subregion that has the largest area making up $\underline{C}_k$

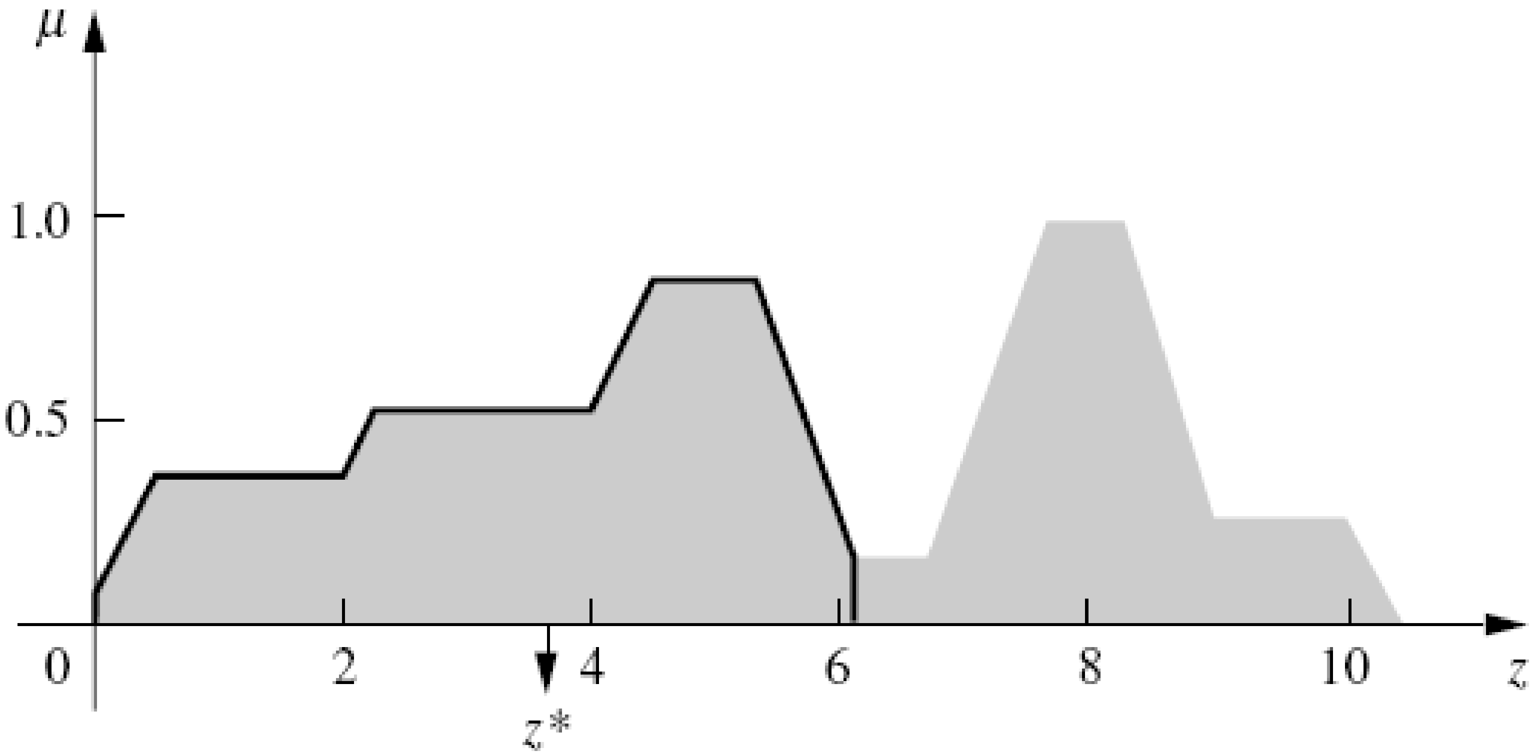


FIGURE 4.29
Center of largest area method (outlined with bold lines), shown for a nonconvex $\underset{\sim}{C}_k$.

Methods 7 for defuzzifying fuzzy output functions

- *First (or last) of maxima*:
- This method uses the overall output or union of all individual output fuzzy sets $\underline{C}_k$ to determine the smallest value of the domain with maximized membership degree in $\underline{C}_k$.

First (or last) of maxima

- First, the largest height in the union (denoted hgt($\underset{\sim}{C}_k$)) is determined,

$$\text{hgt}(\underset{\sim}{C}_k) = \sup_{z \in Z} \mu_{\underset{\sim}{C}_k}(z)$$

Then the first of the maxima is found,

$$z* = \inf_{z \in Z} \left\{ z \in Z \mid \mu_{\underset{\sim}{C}_k}(z) = \text{hgt}(\underset{\sim}{C}_k) \right\}$$

An alternative to this method is called the last of maxima

$$z^* = \sup_{z \in Z} \left\{ z \in Z \mid \mu_{\underset{\sim}{C}_k}(z) = \text{hgt}(\underset{\sim}{C}_k) \right\}$$

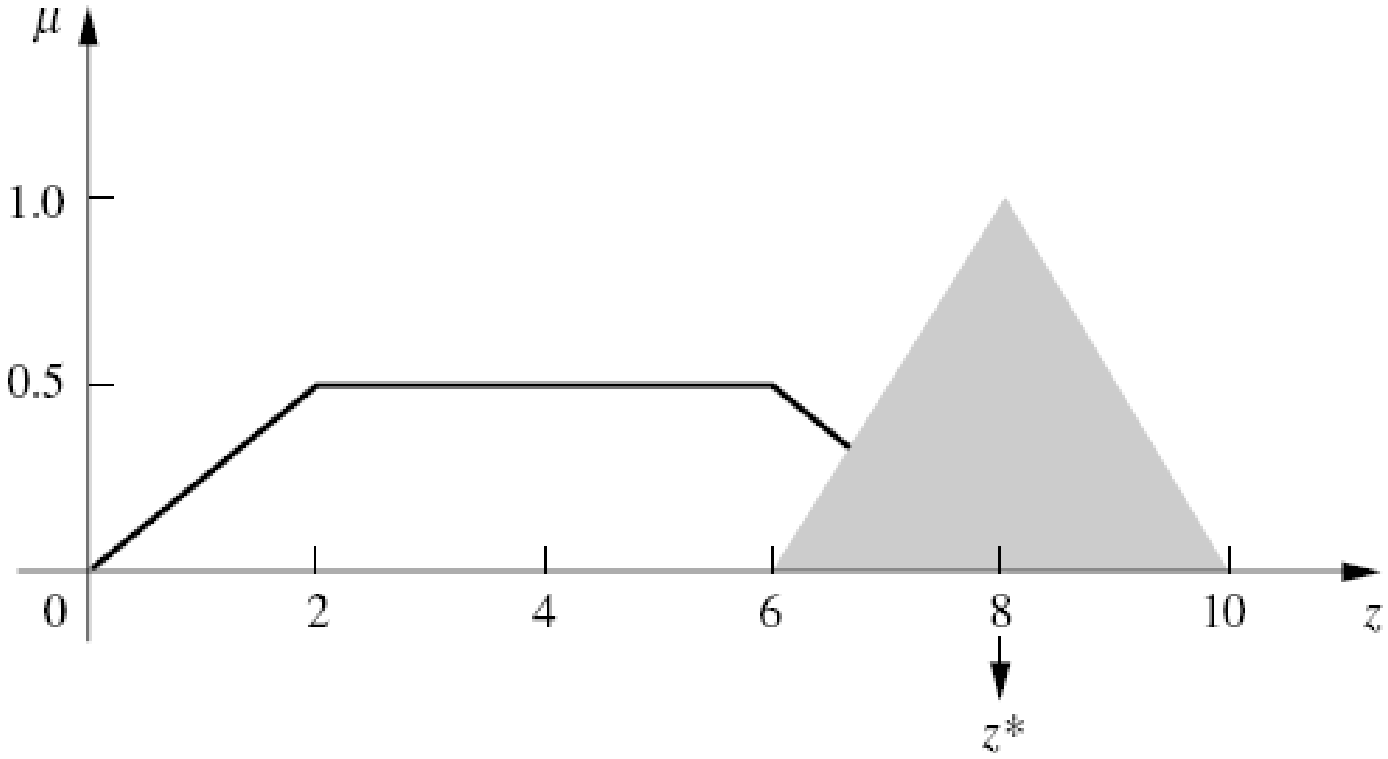


FIGURE 4.30
First of max (and last of max) method.

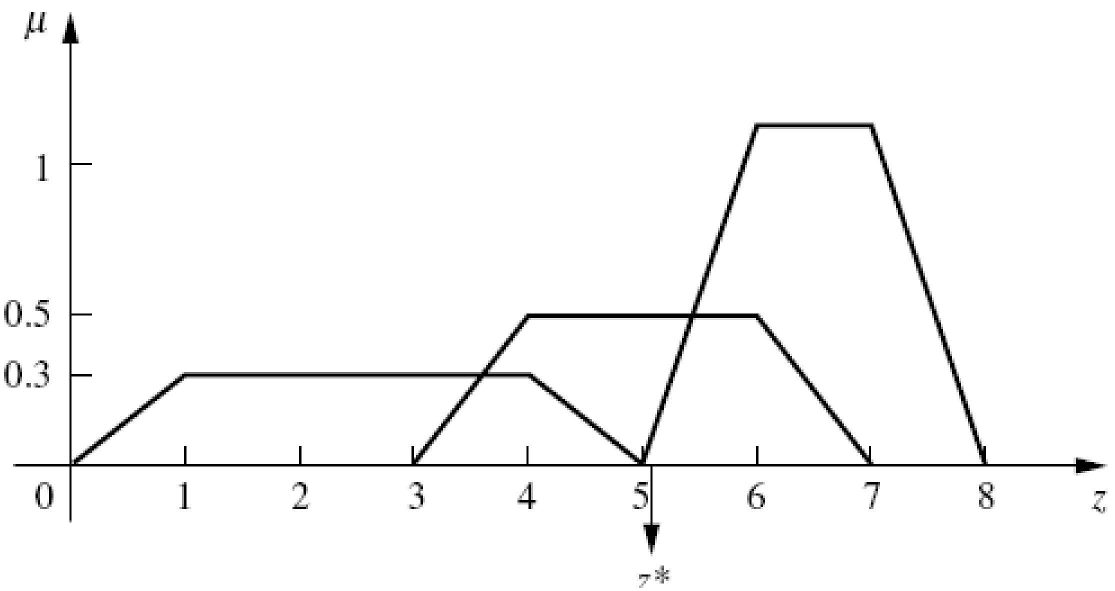
Hence, the methods presented in Eqs. (4.4) (max or height), (4.7) (mean max), (4.11) (first max), and (4.12) (last max) all provide the same defuzzified value, $z*$, for the particular situation illustrated in Fig. 4.30.

Example 4.5. Continuing with Example 4.3 on the railroad company planning to lay a new rail line, we will calculate the defuzzified values using the (1) center of sums method, (2) center of largest area, and (3) first maxima and last maxima.

According to the center of sums method, Eq. (4.8), z^* will be as follows:

$$z^* = \frac{[2.5 \times 0.5 \times 0.3(3+5) + 5 \times 0.5 \times 0.5(2+4) + 6.5 \times 0.5 \times 1(3+1)]}{[0.5 \times 0.3(3+5) + 0.5 \times 0.5(2+4) + 0.5 \times 1(3+1)]}$$

$$= 5.0 \text{ m}$$



$$z^* = \frac{\int_Z \bar{z} \sum_{k=1}^{n} \mu_{\underset{\sim}{C}_k}(z)\, dz}{\int_z \sum_{k=1}^{n} \mu_{\underset{\sim}{C}_k}(z)\, dz}$$

FIGURE 4.31
Center of sums result for Example 4.5.

The center of largest area method, Eq. (4.9), provides the same result (i.e., $z^* = 4.9$) as the centroid method, Eq. (4.5), because the complete output fuzzy set is convex, as seen in Fig. 4.32. According to the first of maxima and last of maxima methods, Eqs. (4.11) and (4.12), z^* is shown as z_1^* and z_2^*, respectively, in Fig. 4.33.

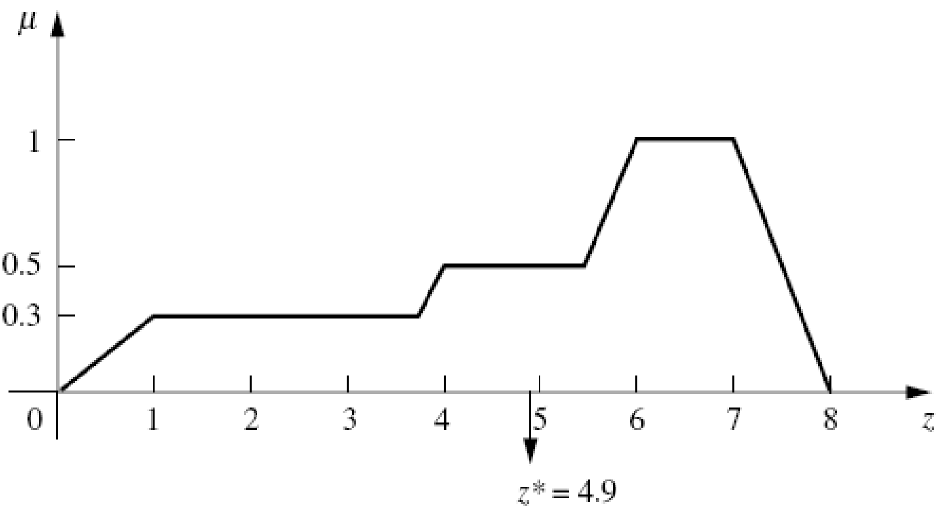


FIGURE 4.32
Output fuzzy set for Example 4.5 is convex.

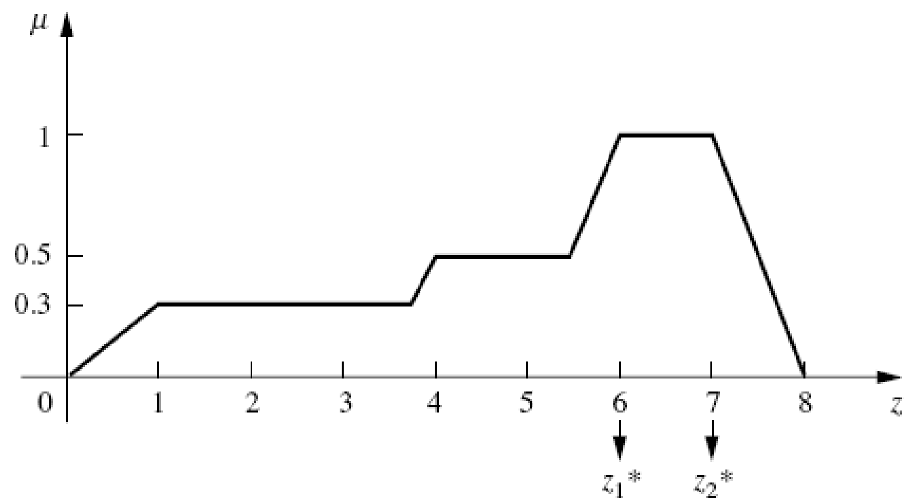


FIGURE 4.33
First of maxima solution ($z_1^* = 6$) and last of maxima solution ($z_2^* = 7$).

SUMMARY

- Of the seven defuzzification methods presented, which is the best?
- it is context- or problem-dependent.

SUMMARY

five criteria:

1)*continuity*.

A small change in the input of a fuzzy process should not produce a large change in the output.

2) *Disambiguity*

a defuzzification method should always result in a unique value for z^*, *i.e., no ambiguity in the defuzzified* value.

3) *plausibility*.

To be plausible, z^* should lie approximately in the middle of the support region of $\underset{\sim}{C}_k$ *and have a high degree* of membership in $\underset{\sim}{C}_k$

SUMMARY

five criteria:

4) *computational simplicity*,

the more time consuming a method is, the less value it should have in a computation system.

5) *weighting method.*

- which weights the output fuzzy sets.
- The problem with the fifth criterion is that it is problem-dependent, as there is little by which to judge the best weighting method; the weighted average method involves less computation than the center of sums, but that attribute falls under the fourth criterion, computational simplicity.