LOGIC AND FUZZY SYSTEMS

PART II FUZZY SYSTEMS

NATURAL LANGUAGE

- Natural language is perhaps the most powerful form of conveying information that humans possess for any given problem or situation that requires solving or reasoning.
- This power has largely remained untapped in today's mathematical paradigms; not so anymore with the utility of fuzzy logic.

NATURAL LANGUAGE

- Our natural language consists of fundamental terms characterized as atoms [□ætəm] in the literature.
- A collection of these atoms will form the molecules [□mɔlikju:l], or phrases, of our natural language.
- The fundamental terms can be called *atomic* terms.

Examples: *slow*, *medium*, *young*, *beautiful*, etc.

- A collection of atomic terms is called a composite, or simply a set of terms.

Examples: *very slow horse*, *medium-weight female*, *young tree*, *fairly beautiful painting*, etc.

NATURAL LANGUAGE

- Leibniz, claimed, “If we could find characters or signs appropriate for expressing all our thoughts as definitely and as exactly as arithmetic expresses numbers or geometric analysis expresses lines, we could in all subjects, in so far as they are amenable to reasoning, accomplish what is done in arithmetic and geometry.”
- Fuzzy sets are a relatively new quantitative method to accomplish just what Leibniz had suggested.

NATURAL LANGUAGE

- natural language can be expressed as a mapping $\underset{\sim}{M}$ from a set of atomic terms in X to a corresponding set of interpretations defined on universe Y.

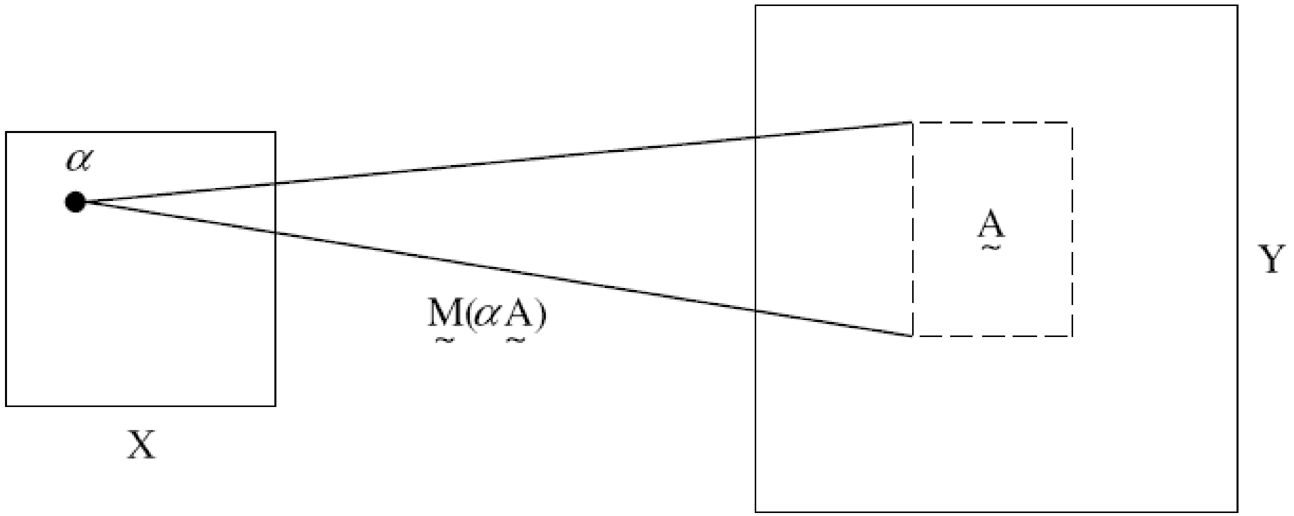


FIGURE 5.6
Mapping of a linguistic atom, α, to a cognitive interpretation, $\underset{\sim}{A}$.

The fuzzy set $\underset{\sim}{A}$ represents the fuzziness in the mapping between an atomic term and its interpretation, and can be denoted by the membership function $\mu_{\underset{\sim}{M}}(\alpha, y)$, or more simply by

$$\mu_{\underset{\sim}{M}}(\alpha, y) = \mu_{\underset{\sim}{A}}(y) \tag{5.24}$$

Example

- We have the atomic term "young" (α) and we want to interpret this linguistic atom in terms of age, *y*, by a membership function that expresses the term "young" .
- $$\underset{\sim}{A} = \text{``young''} = \int_0^{25} \frac{1}{y} + \int_{25}^{100} \frac{1}{y}\left[1+\left(\frac{y-25}{5}\right)^2\right]^{-1}$$
- or alternatively,

$$\mu_{\underset{\sim}{M}}(\text{young}, y) = \begin{cases} \left[1+\left(\dfrac{y-25}{5}\right)^2\right]^{-1} & y > 25 \text{ years} \\ 1 & y \leq 25 \text{ years} \end{cases}$$

- On the basis of the foregoing, we can call α a natural language variable whose "value" is defined by the fuzzy set $\mu_{\alpha}(y)$. Hereinafter, the "value" of a linguistic variable will be synonymous with its *interpretation*.
- A composite is a collection, or set, of atomic terms combined by various linguistic connectives such as *and*, *or*, and *not*.

$$\alpha \text{ or } \beta : \mu_{\alpha \text{ or } \beta}(y) = \max(\mu_\alpha(y), \mu_\beta(y))$$

$$\alpha \text{ and } \beta : \mu_{\alpha \text{ and } \beta}(y) = \min(\mu_\alpha(y), \mu_\beta(y))$$

$$\text{Not } \alpha = \overline{\alpha} : \mu_{\overline{\alpha}}(y) = 1 - \mu_\alpha(y)$$

LINGUISTIC HEDGES

- In linguistics, fundamental atomic terms are often modified with adjectives (nouns) or adverbs (verbs) like *very, low, slight, more or less, fairly, slightly, almost, barely, mostly, roughly, approximately*. We will call these modifiers "linguistic hedges".

Example:

Define $\alpha = \int_Y \mu_\alpha(y)/y$; then

$$\left.\begin{aligned}
&\text{``Very''}\ \alpha = \alpha^2 = \int_Y \frac{[\mu_\alpha(y)]^2}{y} \\
&\text{``Very, very''}\ \alpha = \alpha^4 \\
&\text{``Plus''}\ \alpha = \alpha^{1.25}
\end{aligned}\right\}\ \textit{concentrations}$$

$$\left.\begin{aligned}
&\text{``Slightly''}\ \alpha = \sqrt{\alpha} = \int_Y \frac{[\mu_\alpha(y)]^{0.5}}{y} \\
&\text{``Minus''}\ \alpha = \alpha^{0.75}
\end{aligned}\right\}\ \begin{array}{l}\textit{dilations} \\ \textit{(or dilutions)}\end{array}$$

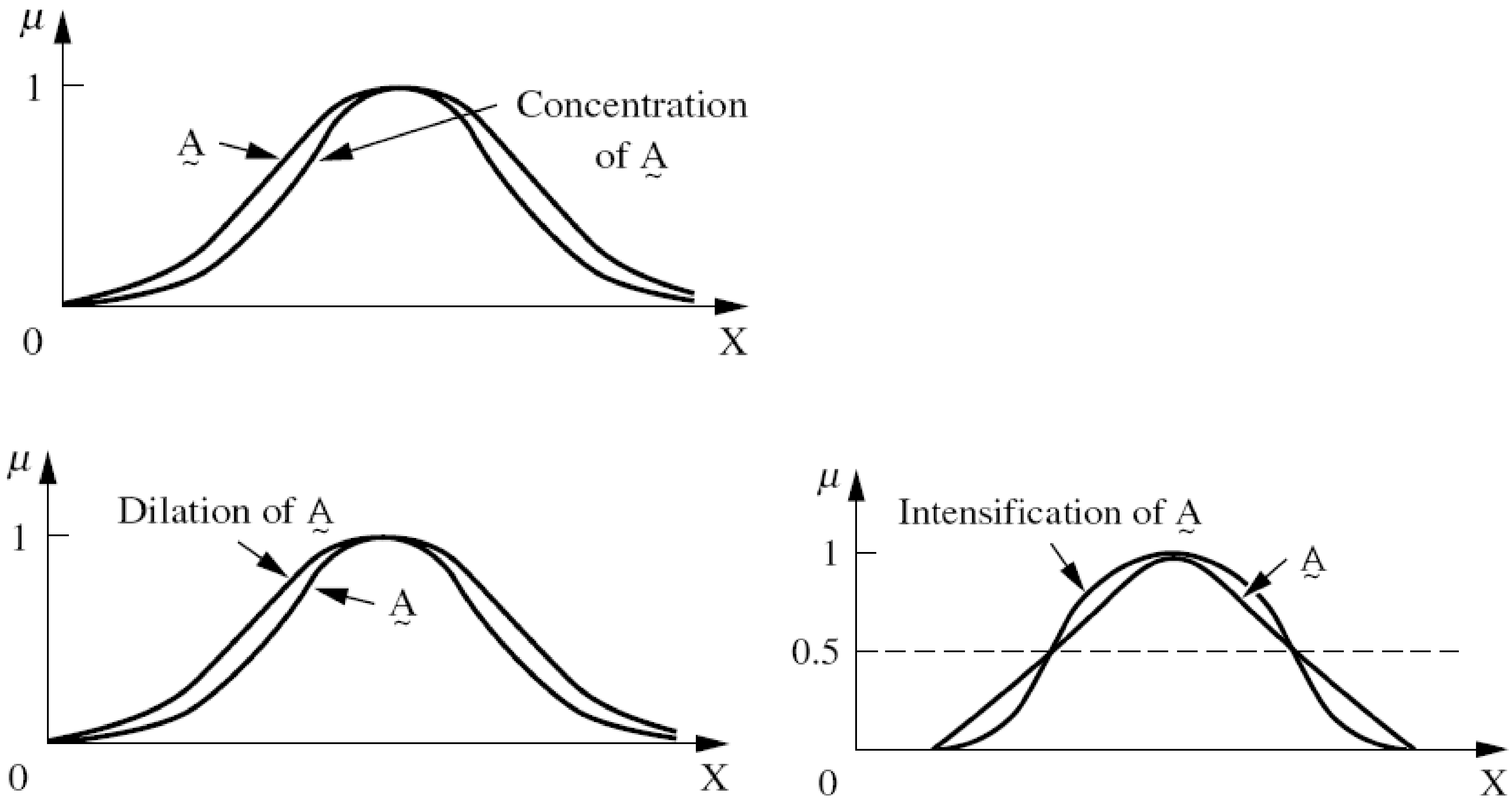
Concentrations and Dilations

- Concentrations tend to concentrate the elements of a fuzzy set by reducing the degree of membership of all elements that are only “partly” in the set.
- Dilations stretch or dilate a fuzzy set by increasing the membership of elements that are “partly” in the set.

Intensification

- Another operation on linguistic fuzzy sets is known as *intensification*. This operation acts in a combination of concentration and dilation. This also has the effect of making the boundaries of the membership function steeper.
- For example:

$$\text{``intensify''}\ \alpha = \begin{cases} 2{\mu_\alpha}^2(y) & \text{for } 0 \le \mu_\alpha(y) \le 0.5 \\ 1 - 2\left[1 - \mu_\alpha(y)\right]^2 & \text{for } 0.5 \le \mu_\alpha(y) \le 1 \end{cases}$$



μ
1
0
X
Concentration
of A
A
μ
1
0
X
Dilation of A
A
μ
1
0.5
0
X
Intensification of A
A

Composite terms

- Composite terms can be formed from one or more combinations of atomic terms, logical connectives, and linguistic hedges.

- For example, suppose we have two atomic terms "small" and "red" , and their associated membership functions, and we pose the following linguistic expression:

  a "not small" and "very red" fruit.

  Which of the operations, i.e., not, and, very, would we perform first, which would we perform second, and so on?

TABLE 5.7
Precedence for linguistic hedges and logical operations

Precedence	Operation
First	Hedge. not
Second	And
Third	Or

Source: Zadeh [1973]

Parentheses may be used to change the precedence order and ambiguities may be resolved by the use of association-to-the-right. For example:

plus (very (minus (very (small))))

Example 5.14.

- Suppose we have a universe of integers, Y = {1, 2, 3, 4, 5}. We define the following linguistic terms as a mapping onto Y:

$$\text{``Small''} = \left\{ \frac{1}{1} + \frac{0.8}{2} + \frac{0.6}{3} + \frac{0.4}{4} + \frac{0.2}{5} \right\}$$

$$\text{``Large''} = \left\{ \frac{0.2}{1} + \frac{0.4}{2} + \frac{0.6}{3} + \frac{0.8}{4} + \frac{1}{5} \right\}$$

- Now we modify these two linguistic terms with hedges,

$$\text{``Very small''} = \text{``small''}^2 \text{ (Eq. (5.26))} = \left\{\frac{1}{1} + \frac{0.64}{2} + \frac{0.36}{3} + \frac{0.16}{4} + \frac{0.04}{5}\right\}$$

$$\text{``Not very small''} = 1 - \text{``very small''} = \left\{\frac{0}{1} + \frac{0.36}{2} + \frac{0.64}{3} + \frac{0.84}{4} + \frac{0.96}{5}\right\}$$

- Then we construct a phrase, or a composite term:

α = "not very zvsmall and not very, very large"

which involves the following set-theoretic operations:

$$\alpha = \left(\frac{0.36}{2} + \frac{0.64}{3} + \frac{0.84}{4} + \frac{0.96}{5}\right) \cap \left(\frac{1}{1} + \frac{1}{2} + \frac{0.9}{3} + \frac{0.6}{4}\right) =$$

$$\left(\frac{0.36}{2} + \frac{0.64}{3} + \frac{0.6}{4}\right)$$

- Suppose we want to construct a linguistic variable "intensely small" (extremely small); we will make use of Eq. (5.31) to modify "small" as follows:

$$\text{“Intensely small”} = \left\{ \frac{1-2[1-1]^2}{1} + \frac{1-2[1-0.8]^2}{2} + \frac{1-2[1-0.6]^2}{3} + \frac{2[0.4]^2}{4} + \frac{2[0.2]^2}{5} \right\}$$

$$= \left\{ \frac{1}{1} + \frac{0.92}{2} + \frac{0.68}{3} + \frac{0.32}{4} + \frac{0.08}{5} \right\}$$

$$\text{“intensify” } \alpha = \begin{cases} 2\mu_\alpha{}^2(y) & \text{for } 0 \le \mu_\alpha(y) \le 0.5 \\ 1-2\,[1-\mu_\alpha(y)]^2 & \text{for } 0.5 \le \mu_\alpha(y) \le 1 \end{cases}$$

Eq. (5.31)

FUZZY (RULE-BASED) SYSTEMS

- In the field of artificial intelligence (machine intelligence) the most common way to represent human knowledge is to form it into natural language expressions :

IF premise (antecedent), THEN conclusion (consequent)

The form referred to as the *IF–THEN rule-based* form; this form generally is referred to as the *deductive* form.

Fuzzy rule-based system

The fuzzy rule-based system is most useful in modeling some complex systems that can be observed by humans because it makes use of linguistic variables as its antecedents and consequents; as described here these linguistic variables can be naturally represented by fuzzy sets and logical connectives of these sets.

- By using the basic properties and operations defined for fuzzy sets ,any compound rule structure may be decomposed and reduced to a number of simple canonical rules as given here:

TABLE 5.8
The canonical form for a fuzzy rule-based system

Rule 1:	IF condition C^1, THEN restriction R^1
Rule 2:	IF condition C^2, THEN restriction R^2
⋮	
Rule r:	IF condition C^r, THEN restriction R^r

Multiple conjunctive antecedents

IF x is $\underset{\sim}{A}^1$ and $\underset{\sim}{A}^2 \ldots$ and $\underset{\sim}{A}^L$ THEN y is $\underset{\sim}{B}^s$

Assuming a new fuzzy subset A^s as

$$\underset{\sim}{A}^s = \underset{\sim}{A}^1 \cap \underset{\sim}{A}^2 \cap \cdots \cap \underset{\sim}{A}^L$$

expressed by means of membership function

$$\mu_{\underset{\sim}{A}^s}(x) = \min[\mu_{\underset{\sim}{A}^1}(x), \mu_{\underset{\sim}{A}^2}(x), \ldots, \mu_{\underset{\sim}{A}^L}(x)]$$

The compound rule may be rewritten as

$$\text{IF } \underset{\sim}{A}^s \text{ THEN } \underset{\sim}{B}^s$$

Multiple disjunctive antecedents

$$\text{IF } x \text{ is } \underset{\sim}{A}^1 \text{ OR } x \text{ is } \underset{\sim}{A}^2 \ldots \text{OR } x \text{ is } \underset{\sim}{A}^L \text{ THEN } y \text{ is } \underset{\sim}{B}^s$$

could be rewritten as

$$\text{IF } x \text{ is } \underset{\sim}{A}^s \text{ THEN } y \text{ is } \underset{\sim}{B}^s$$

where the fuzzy set $\underset{\sim}{A}^s$ is defined as

$$\underset{\sim}{A}^s = \underset{\sim}{A}^1 \cup \underset{\sim}{A}^2 \cup \cdots \cup \underset{\sim}{A}^L$$

$$\mu_{\underset{\sim}{A}^s}(x) = \max\left[\mu_{\underset{\sim}{A}^1}(x), \mu_{\underset{\sim}{A}^2}(x), \ldots, \mu_{\underset{\sim}{A}^L}(x)\right]$$

which is based on the definition of the standard fuzzy union operation.

Aggregation of fuzzy rules

- 1. *Conjunctive system of rules*.

In the case of a system of rules that must be jointly satisfied, the rules are connected by "and" connectives. In this case the aggregated output(consequent), *y*, is found by the fuzzy intersection of all individual rule consequents, y_i , where $i = 1, 2, \ldots r$ as

$$y = y^1 \text{ and } y^2 \text{ and } \ldots \text{ and } y^r$$

or

$$y = y^1 \cap y^2 \cap \cdots \cap y^r$$

$$\mu_y(y) = \min(\mu_{y^1}(y),\ \mu_{y^2}(y),\ \ldots,\ \mu_{y^r}(y)) \text{ for } y \in Y$$

Aggregation of fuzzy rules

- 2. *Disjunctive system of rules*.

  The aggregated output is found by the fuzzy union of all individual rule contributions, as

$$y = y^1 \text{ or } y^2 \text{ or } \ldots \text{ or } y^r$$

or

$$y = y^1 \cup y^2 \cup \cdots \cup y^r$$

$$\mu_y(y) = \max(\mu_{y^1}(y), \mu_{y^2}(y), \ldots, \mu_{y^r}(y)) \text{ for } y \in Y$$

Graphical techniques of inference

- This section will describe three common methods of deductive inference for fuzzy systems based on linguistic rules:
- (1) Mamdani systems,
- (2) Sugeno models,
- (3) Tsukamoto models.

(1) Mamdani systems

we consider a simple two-rule system where each rule comprises two antecedents and one consequent. This is analogous to a dual-input and single-output fuzzy system.

$$\text{IF } x_1 \text{ is } \underset{\sim}{A}_1^k \text{ and } x_2 \text{ is } \underset{\sim}{A}_2^k \text{ THEN } y^k \text{ is } \underset{\sim}{B}^k \qquad \text{for } k = 1,\ 2,\ \ldots,\ r$$

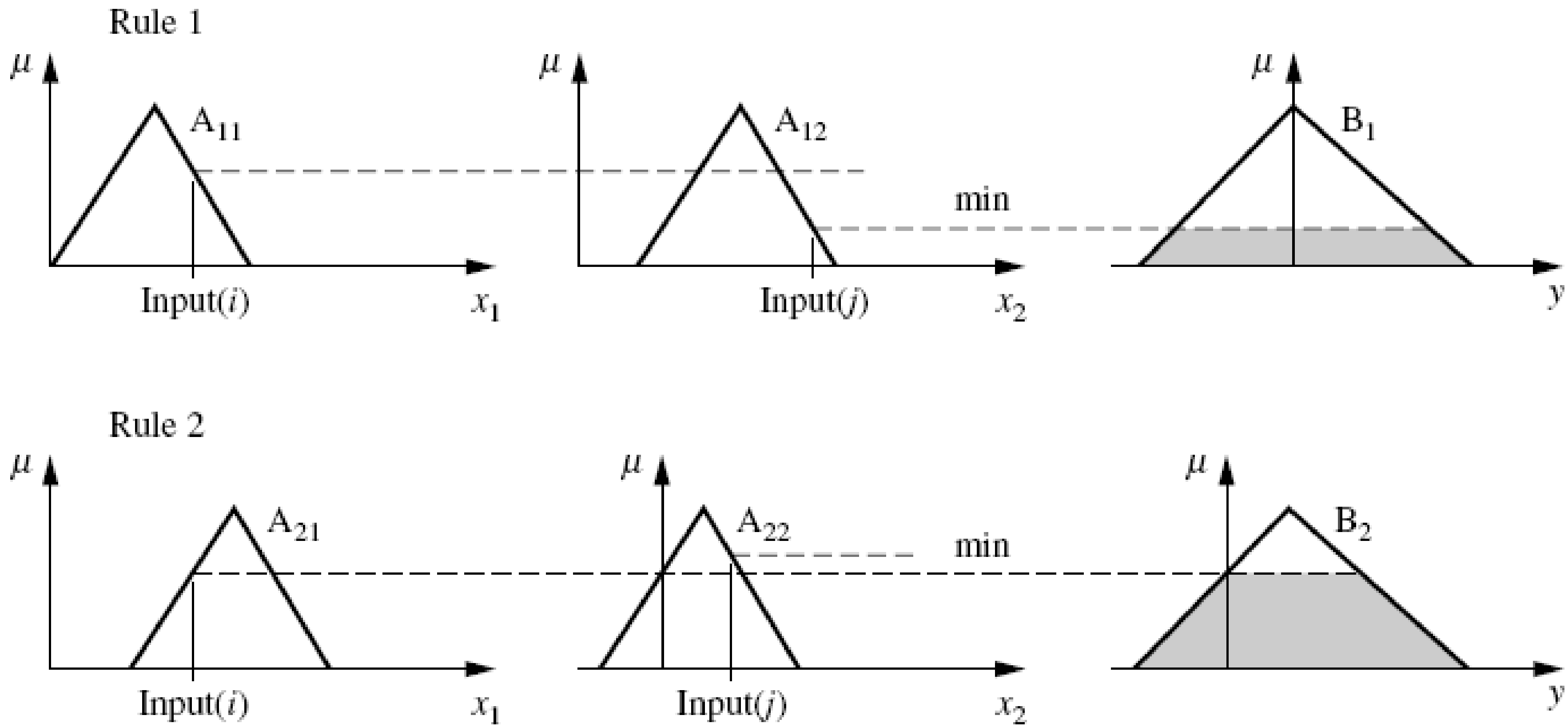
Case 1: The inputs are scalar values, and we use a max–min inference method.

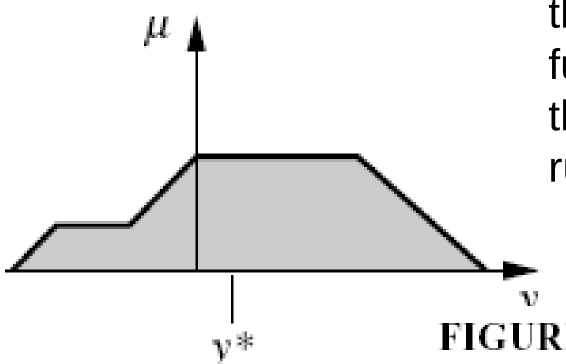
$$\mu(x_1) = \delta(x_1 - \text{input }(i)) = \begin{cases} 1, & x_1 = \text{input}(i) \\ 0, & \text{otherwise} \end{cases}$$

$$\mu(x_2) = \delta(x_2 - \text{input }(j)) = \begin{cases} 1, & x_2 = \text{input}(j) \\ 0, & \text{otherwise} \end{cases}$$

$$\mu_{\underset{\sim}{\mathrm{B}}^k}(y) = \max_k[\min[\mu_{\underset{\sim}{\mathrm{A}}_1^k}(\text{input}(i)), \mu_{\underset{\sim}{\mathrm{A}}_2^k}(\text{input}(j))]] \quad k = 1,\ 2, \ldots, r$$

$$\text{IF } x_1 \text{ is } \underset{\sim}{\mathrm{A}}_1^k \text{ and } x_2 \text{ is } \underset{\sim}{\mathrm{A}}_2^k \text{ THEN } y^k \text{ is } \underset{\sim}{\mathrm{B}}^k \qquad \text{for } k = 1,\ 2,\ \ldots,\ r$$





the symbols A11 and A12 refer to the first and second fuzzy antecedents of the first rule, respectively,
the symbol B1 refers to the fuzzy consequent of the first rule;

FIGURE 5.10
Graphical Mamdani (max–min) inference method with crisp inputs.

Case 2: the inputs to the system are scalar values, and we use a max–product inference method.

$$\mu_{\underset{\sim}{B}^k}(y) = \max_k[\mu_{\underset{\sim}{A}_1^k}(\text{input}(i)) \cdot \mu_{\underset{\sim}{A}_2^k}(\text{input}(j))] \quad k = 1,\ 2, \ldots, r$$

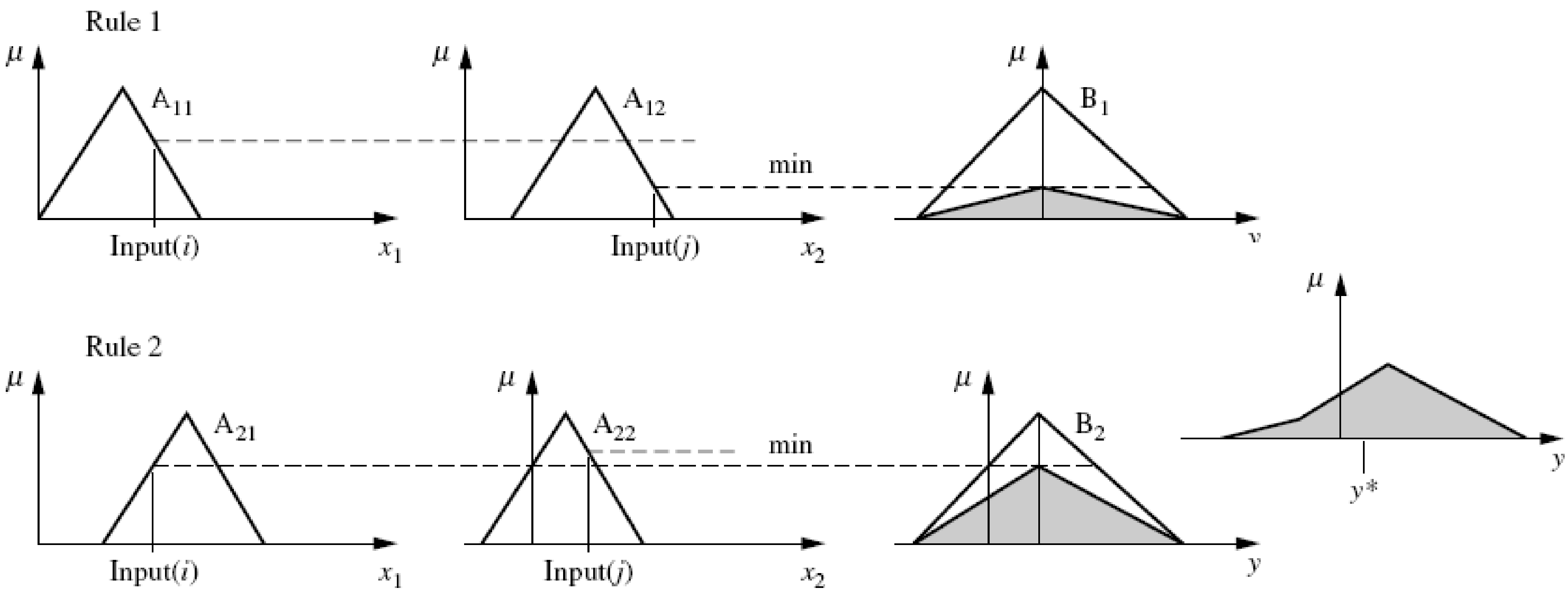


FIGURE 5.11
Graphical Mamdani (max–product) implication method with crisp inputs.

Example 5.15.

- In mechanics, the energy of a moving body is called kinetic energy. If an object of mass *m* (kilograms) is moving with a velocity *v* (meters per second), then the kinetic energy *k* (in joules) is given by the equation $k = (1/2)mv^2$. Suppose we model the mass and velocity as inputs to a system (moving body) and the energy as output, then observe the system for a while and deduce the following two disjunctive rules of inference based on our observations:

Rule 1 : IF x_1 is $\underset{\sim}{A}_1^1$ (small mass) *and* x_2 is $\underset{\sim}{A}_2^1$ (high velocity),

THEN y is $\underset{\sim}{B}^1$ (medium energy).

Rule 2 : IF x_1 is $\underset{\sim}{A}_1^2$ (large mass) *or* x_2 is $\underset{\sim}{A}_2^2$ (medium velocity),

THEN y is $\underset{\sim}{B}^2$ (high energy).

- For example, let input*(i)* = 0.35 kg (mass) and input*(j)* = 55 m/s (velocity).

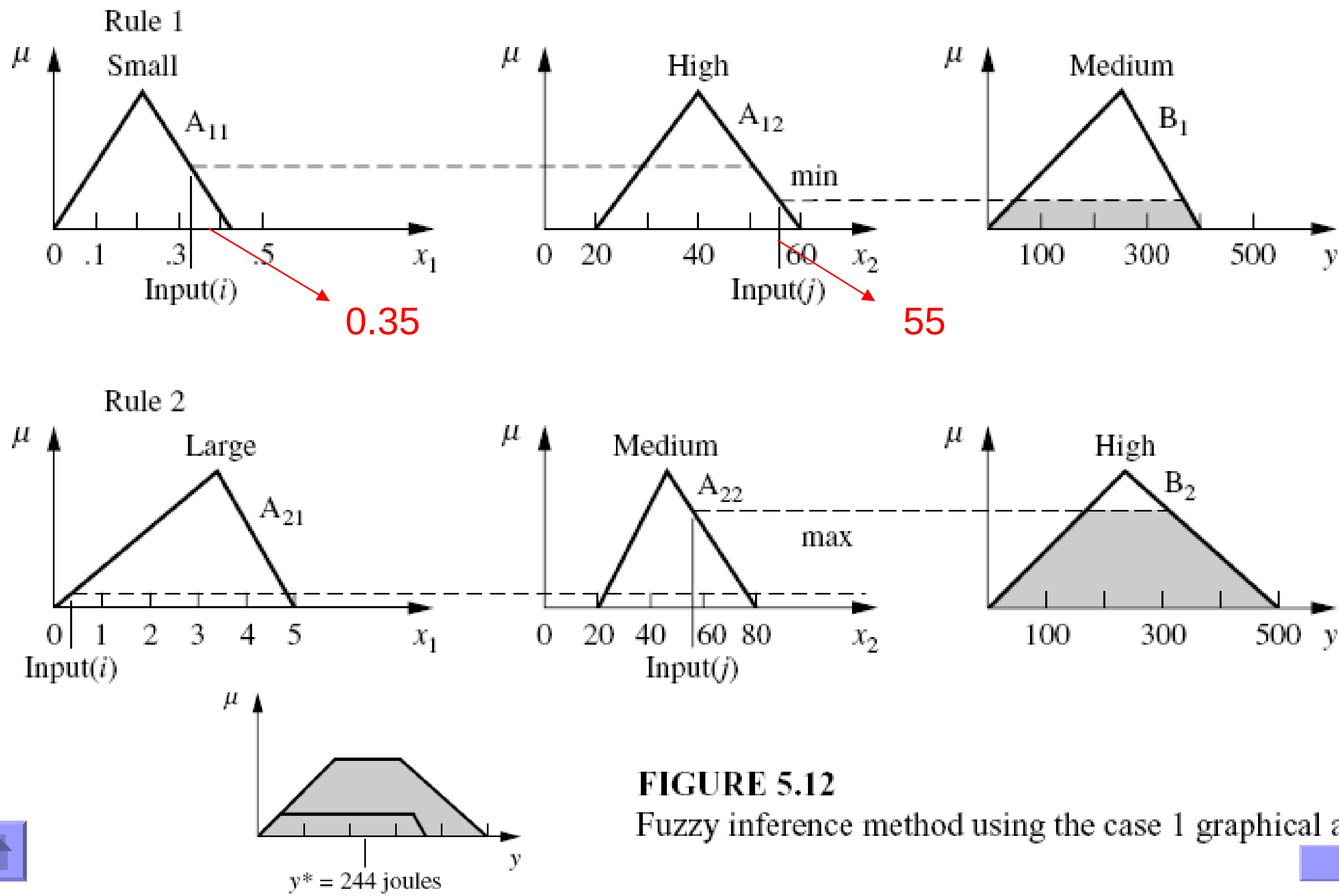


FIGURE 5.12
Fuzzy inference method using the case 1 graphical approach.

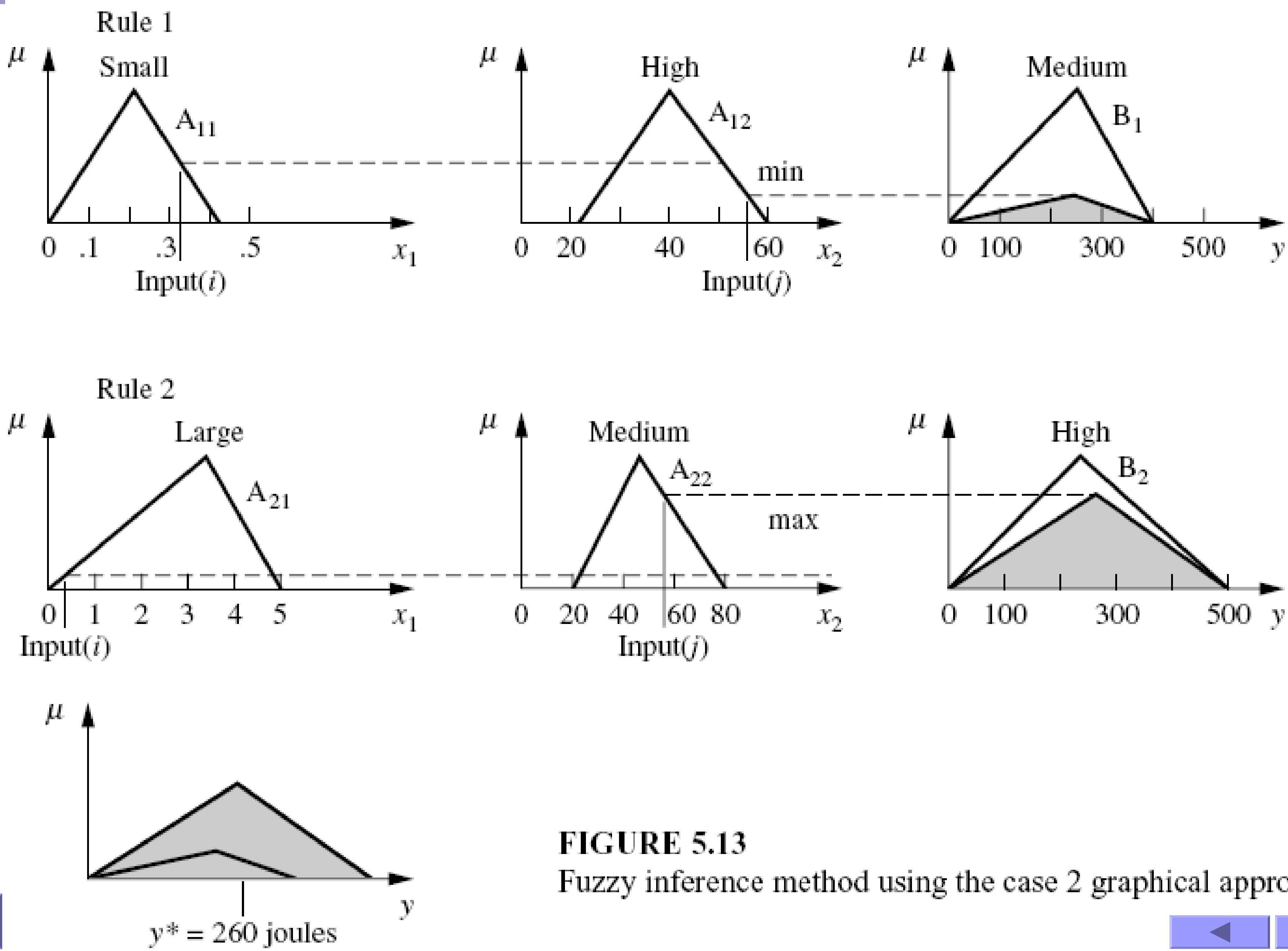


FIGURE 5.13
Fuzzy inference method using the case 2 graphical approach.

Variations of the Mamdani method

- The Mamdani method has several variations.
- There are different t-norms to use for the connectives of the antecedents, different aggregation operators for the rules, and numerous defuzzification methods that could be used.
- The power of fuzzy rule-based systems is their ability to yield “good” results with reasonably simple mathematical operations.

(2) Sugeno models (TSK method)

- It is a systematic approach to generating fuzzy rules from a given input–output data set.

$$\text{IF } x \text{ is } \underset{\sim}{\mathrm{A}} \text{ } and \text{ } y \text{ is } \underset{\sim}{\mathrm{B}}, \text{ THEN } z \text{ is } z = f(x, y)$$

where $z = f(x, y)$ is a crisp function in the consequent.

Usually $f(x, y)$ is a polynomial function in the inputs x and y, but it can be any general function .

(2) Sugeno models (TSK method)

- When $f(x, y)$ is a constant the inference system is called a *zero-order Sugeno Model*.
- When $f(x, y)$ is a linear function of x and y, the inference system is called a *first-order Sugeno model*.

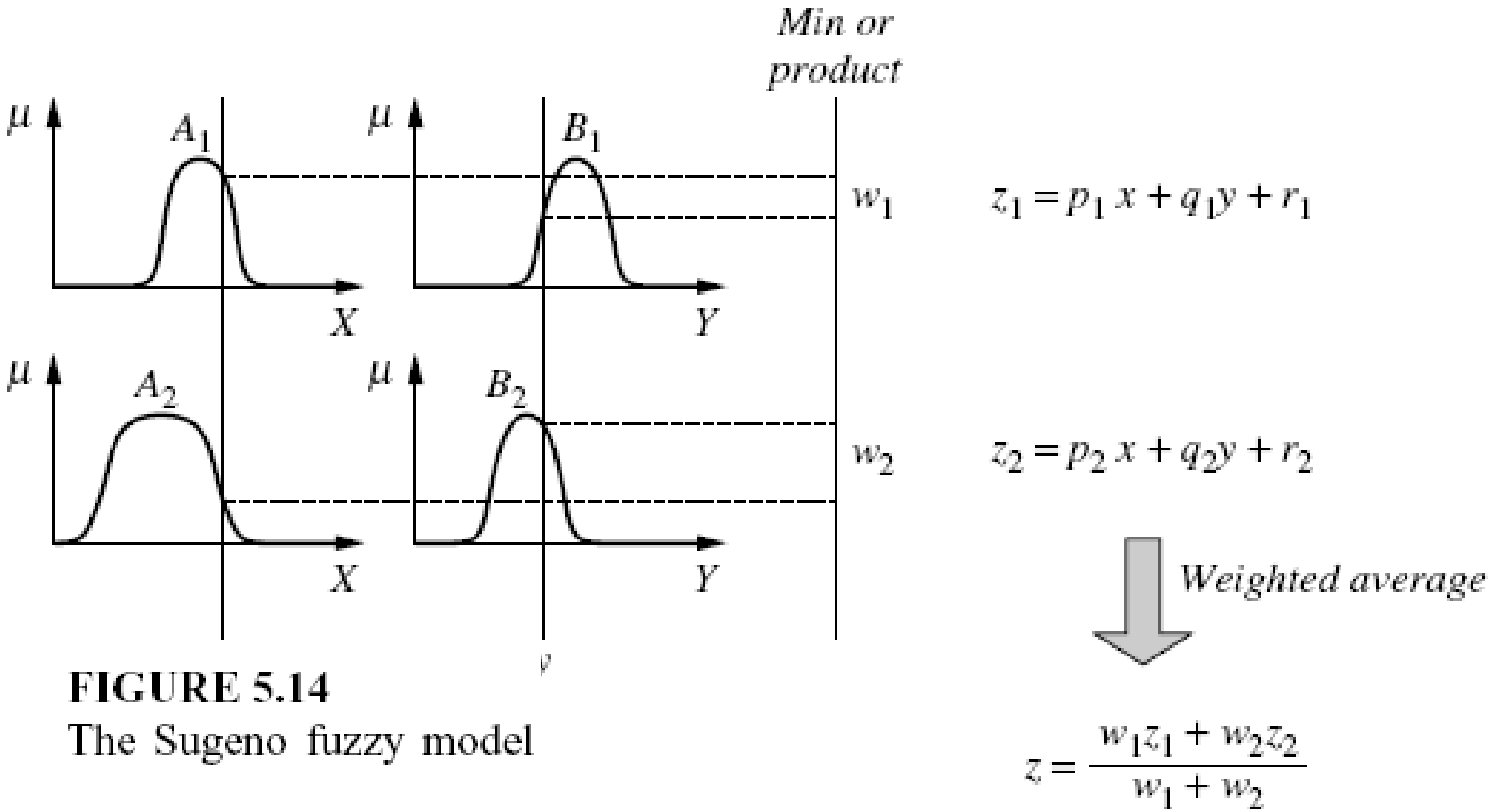


FIGURE 5.14
The Sugeno fuzzy model

In a Sugeno model each rule has a crisp output, given by a function; because of this the overall output is obtained via a weighted average defuzzification (Eq. (4.6)), as shown in Fig. 5.14. This process avoids the time-consuming methods of defuzzification necessary in the Mamdani model.

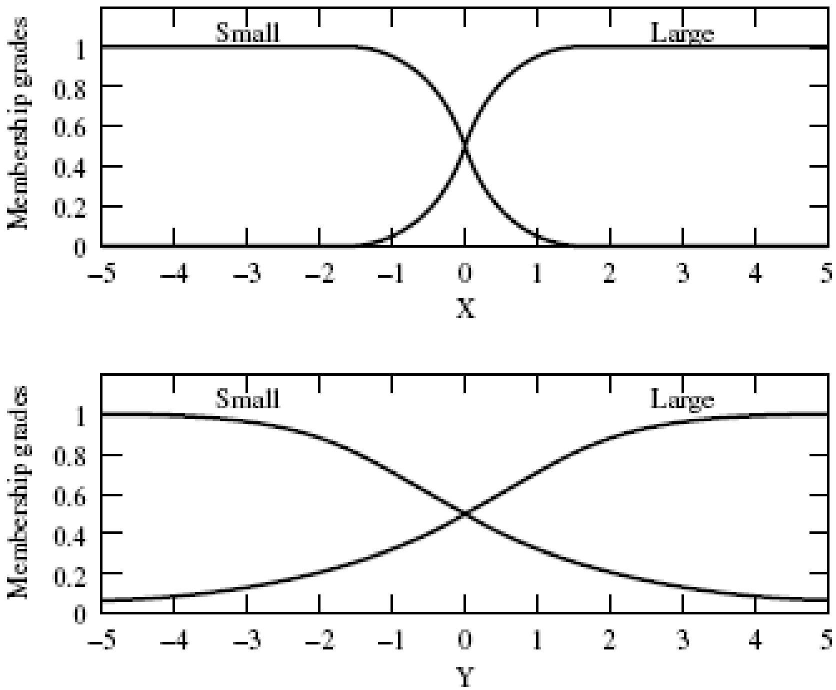
Example 5.16. An example of a two-input single-output Sugeno model with four rules is repeated from Jang et al. [1997]:

IF X is small and Y is small, THEN $z = -x + y + 1$

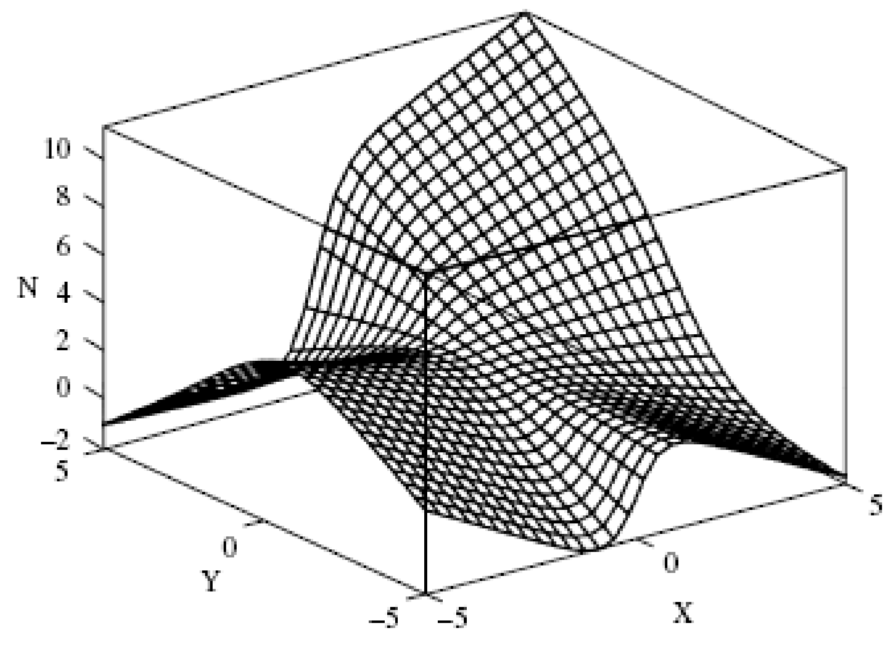
IF X is small and Y is large, THEN $z = -y + 3$

IF X is large and Y is small, THEN $z = -x + 3$

IF X is large and Y is large, THEN $z = x + y + 2$



(a)



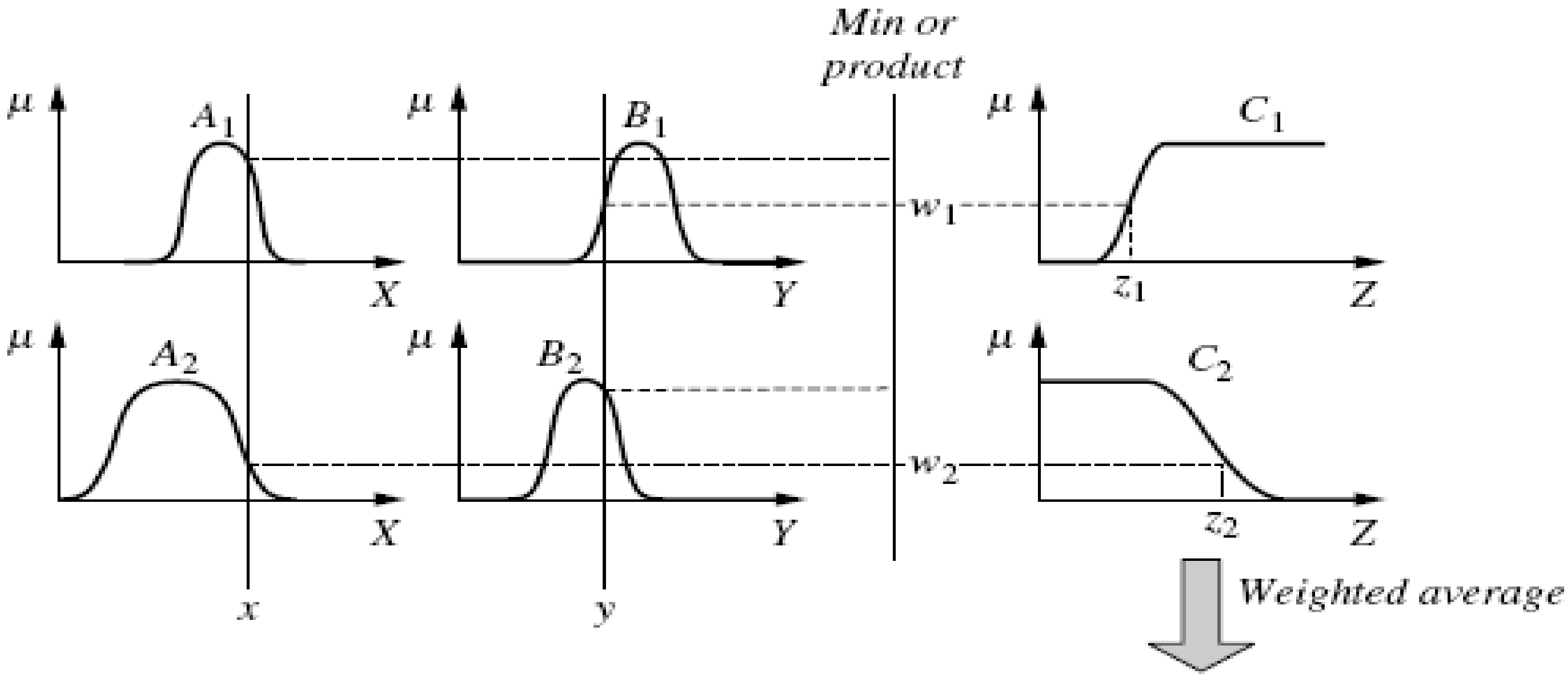
(b)

- Figure *a plots the membership function of inputs X and Y, and Fig. b is the* resulting input – output surface of the system.
- The surface is complex, but it is still obvious that the surface is comprised of four planes, each of which is specified by the output function of each of the four rules.
- Figure *b shows that there is a smooth transition between the four* output planes.

- Without the mathematically difficult process of a defuzzification operation, the Sugeno model is a very popular method for sample-based fuzzy systems modeling.

(3) Tsukamoto models

- In this method, the consequent of each fuzzy rule is represented by a fuzzy set with a monotonic membership function.
- In a monotonic membership function, sometimes called a *shoulder function*, the inferred output of each rule is defined as a crisp value induced by the membership value coming from the antecedent clause of the rule.
- The overall output is calculated by the weighted average of each rule's output.



$$z = \frac{w_1 z_1 + w_2 z_2}{w_1 + w_2}$$

Since each rule infers a crisp output, the Tsukamoto model's aggregation of the overall output also avoids the time-consuming process of defuzzification.

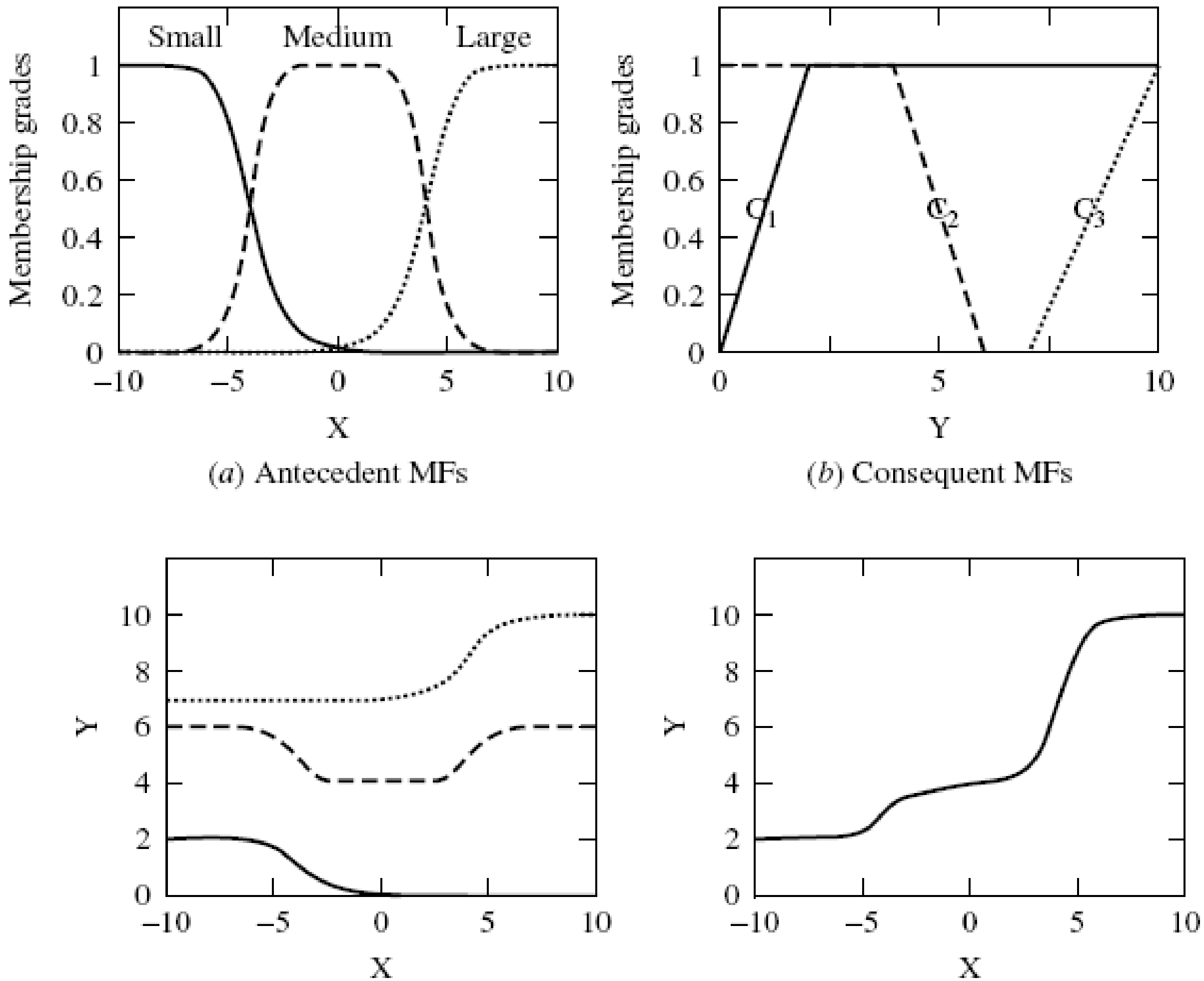
Because of the special nature of the output membership functions required by the method, it is not as useful as a general approach, and must be employed in specific situations.

Example 5.17. An example of a single input, single-output Tsukamoto fuzzy model is given by the following rules:

IF X is small, THEN Y is C_1

IF X is medium, THEN Y is C_2

IF X is large, THEN Y is C_3



(*a*) Antecedent MFs

(*b*) Consequent MFs

(*c*) Each rule's output

(*d*) Overall input-output curve

- Since the reasoning mechanism of the Tsukamoto fuzzy model does not strictly follow a composition operation in its inference it always generates a crisp output even when the input and output membership functions are fuzzy membership functions.

SUMMARY

- This chapter has presented the basic axioms, operations, and properties of binary logic and fuzzy logic.

(the only significant difference between a

binary logic and a fuzzy logic stems from the logical equivalent of the excluded middle axioms.)

- This chapter has also summarized the seminal works of Zadeh [1972, 1973, 1975*a,b]* in the area of linguistic modeling.