AVOIDING OBSTACLES AND RESOLVING KINEMATIC REDUNDANCY*

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ABSTRACT

While obstacles in a robot workspace can effectively reduce the number of degrees of freedom, there need not be a corresponding loss of functionality for kinematically redundant mechanisms. In this paper, disk-like obstacles for a planar three-bar mechanism are classified. The extended Jacobian technique is used to implement an obstacle avoidance technique based on optimizing a distance criterion. The feasibility of applying multiple constraints or optimality criteria for resolving redundancy is investigated.

1. INTRODUCTION

Much has been written about the advantages of kinematically redundant robot manipulators for working in congested areas. (See, for example, Baillieul *et al.* 1984, Yoshikawa, 1984, Hollerbach, 1985, and Maciejewski and Klein, 1985.) To fully exploit these advantages, simple mathematical models of physical obstructions are needed which will facilitate the development of real-time joint space path-planning algorithms. While there exists a large body of literature on obstructions and path-planning, much of the work on topics such as the "movers' problem" and collision avoidance for mobile robots does not apply directly articulated manipulators for which not only the end effector but the entire mechanism must move so as to avoid obstructions in the environment. In the context of a nonredundant (PUMA) linkage, Brooks (1984) has developed an algorithm which finds collision- free paths taking both the end effector and links into account. Maciejewski and Klein (1985) have treated obstacle avoidance for kinematically redundant manipulators in a way which is fairly general and makes contact with previous work on the use of the pseudoinverse of the Jacobian for path generation in redundant manipulators. The algorithm requires knowing the position on the manipulator of an *obstacle avoidance point*, however, and in general it seems that tracking the motions of this point will be fairly complicated. Another general approach using the concept of "artificial potential field" has been proposed by Khatib (1985). An important feature of this work is the use of simple geometric "primitives" for representing obstacles. Work reported by Yoshikawa (1984b) focuses on the use of distance-like optimality criteria to simultaneously resolve redundancy and avoid obstacles.

The purpose of the present paper is threefold: First, we discuss the obstacle avoidance problem for redundant manipulators in which the redundancy is of the planar three-bar type. As in the work of Khatib cited above, simple geometric "primitives" (disks and cylinders) are used to represent obstacles. In terms of this representation, a simple classification of obstacles (based on how motions of the manipulator are impeded) is given. The second aim of our paper is to describe an algorithm which constitutes a very simple solution to the problem of avoiding one class of obstacles for such mechanisms and to show how this algorithm may be implemented in terms of the recently introduced *extended Jacobian technique* (Baillieul, 1985a). We recall that this technique may be viewed as a way of realizing functionally constrained motions of the joint variables. Any such kinematic constraint introduces *algorithmic singularities* which in turn may be characterized in terms of the extended Jacobian. The third purpose of this note is to discuss the resolution of redundancy by means of multiple constraints wherein the constraint which is active at any given time is determined by the position and orientation of the end effector in the workspace. Although multiple constraints may be useful for the practical resolu-

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tion of redundancy, it is shown that in general the possibility of encountering singular configurations is not (indeed cannot be) eliminated. A specific example is considered in detail for a planar three-bar mechanism.

2. AVOIDING OBSTACLES WITH PLANAR THREE-BAR MECHANISMS

Recently Hollerbach (1985) has shown that there are essentially 9 ways to add an extra sliding or rotary degree of freedom to a manipulator having the 6R geometry of a PUMA-without-offsets. One of these designs, schematically represented in *Figure 1*, is attractive for applications in which workspace obstacles are encountered which are of limited physical extent but which would always tend to lie in the plane of the links of a PUMA. Such obstacles can be formally modeled as solid cylinders in three space or disks in the plane. (We have chosen this formalism because the obstructions in which we are interested can be enveloped in a circular cylinder while at the same time it is particularly easy to describe

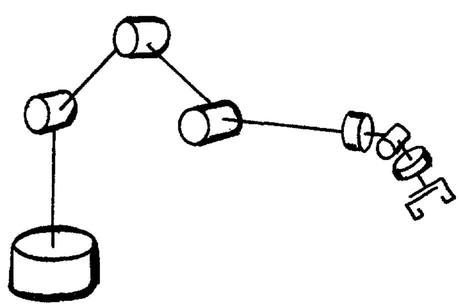


Figure 1: *A 7 d.o.f. manipulator with a planar-triple redundancy.*

the distance from such cylinders to the links of the manipulator.) *Figure 2* illustrates the reason that this manipulator design, which features the imbedded three bar mechanism, is well suited to working in and around such obstacles.

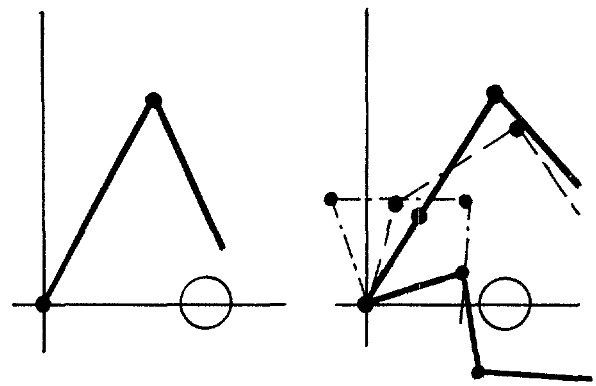
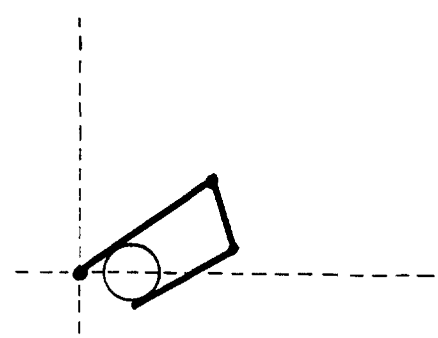
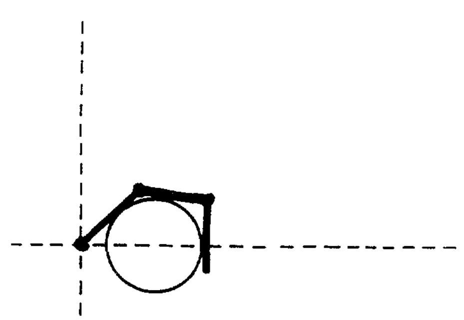


Figure 2: *A three-bar mechanism can get around obstacles in the plane of the links.*

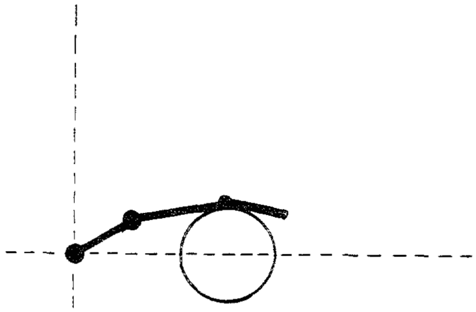
Workspace obstructions modeled as cylinders (or disks in 2-d) can be classified according to which links in a planar three-bar mechanism are potentially impeded by them. There are theoretically $2^3 - 1 = 7$ possible types of such obstructions, but because of practical design considerations, the number of classes actually reduces to four. (See Baillieul, 1985b.) These are depicted in *Figure 3*.



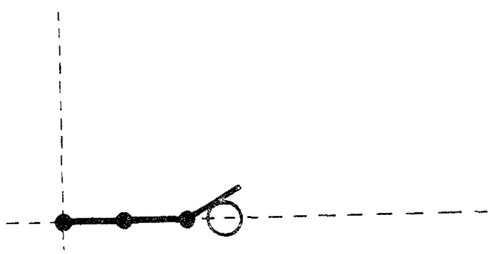
I: $x < r_1 - r$



II: $r_1 - r \leq x \leq r_1 - r$



III: $r_1 + r < x \leq r_1 + r_2 + r$



IV: $r_1 + r_2 + r < x$

Figure 3: *A classification of (positionally normalized) obstructions to the motions of a planar three-bar mechanism. Obstructions are circles of radius r centered on the x-axis x units from the origin. The link lengths are denoted by r_i.*

Classifying obstacles according to the scheme depicted in Figure 3 is useful for the purposes of designing motion planning algorithms. For positionally normalized Class IV obstructions, for instance, motions which maximize

$$g(\theta_1, \theta_2, \theta_3) = (sin(\theta_1 + \theta_2 + \theta_3)x - r_1 sin(\theta_2 + \theta_3) - r_2 sin\theta_3)^2$$

can be expected to avoid the obstacle by the widest possible margin. Using such optimality criteria, it is possible to simultaneously resolve kinematic redundancy and, to the greatest possible extent, avoid collisions with obstacles. In the next section, we recall the basic features of the *extended Jacobian technique* and illustrate its use in planning collision free motions for redundant manipulators.

3. OBSTACLE AVOIDANCE AND THE EXTENDED JACOBIAN TECHNIQUE

Suppose that the relationship between joint settings and end effector postion and orientation is described mathematically by an equation $f(\theta) = x$. (θ denotes joint variables, and x designates end effector position and orientation.) For kinematically redundant linkages, the dimension of the θ variables exceeds the dimension of x, and the Jacobian matrix $\frac{\partial f}{\partial \theta}$ will have more columns than rows. Although such nonsquare matrices are not invertible in the strictest sense, there is a very rich theory of generalized inverses (see e.g. Noble and Daniel, 1977) which have stimulated considerable interest for solving inverse kinematics problems for redundant robot linkages. Differentiating both sides of $x = f(\theta)$ with respect to t, we obtain the forward kinematics equation

$$\dot{x}(t) = \frac{\partial f}{\partial \theta}(\theta(t)) \cdot \dot{\theta}(t). \tag{3.1}$$

This may be solved by writing

$$\dot{\theta}_0(t) = \frac{\partial f}{\partial \theta(\theta(t))}^{+} \dot{x}(t), \tag{3.2}$$

where for any $m \times n$ matrix A of rank m ($m < n$) the Moore-Penrose (generalized) inverse is given by

$$A^+ = A^T(AA^T)^{-1}.$$

Recall that if $\dot{\theta}_0$ is the solution to (3.1) given by (3.2), then for any other solution $\dot{\theta}$, $\|\dot{\theta}_0\| < \|\dot{\theta}\|$. Because the Moore-Penrose inverse solution provides this instantaneous minimization of joint velocities, it was conjectured (and even claimed by some authors) that using this solution, kinematically singular configurations would automatically be avoided. Baillieul *et al.* (1984b) have shown that the Moore-Penrose solution to the inverse kinematics problem does not generate joint space trajectories which avoid singular configurations in any practical sense.

An alternative approach to the inverse kinematics problem called the *extended Jacobian technique* has been proposed by the author (Baillieul, 1985a). This may be described as follows. Observe that if I denotes the $n \times n$ identity matrix, $P(\theta) = I - \frac{\partial f}{\partial \theta}^+ \cdot \frac{\partial f}{\partial \theta}$ defines a projection of $\mathbf{R}^n$ onto the null space of $\frac{\partial f}{\partial \theta}$.

Theorem 3.1: (Baillieul, 1985a) *A necessary condition for θ_0 to maximize an objective function $g(\theta)$ subject to the positional constraint $f(\theta) = x$ is that*

$$P^T \cdot \frac{\partial g}{\partial \theta}(\theta_0) = 0. \tag{3.3}$$

Intuitively what this theorem states is that a necessary condition for θ_0 to define a configuration which optimizes g, there cannot be any self motion of the mechanism which would improve the value of g. Notice that if the Jacobian $\frac{\partial f}{\partial \theta}$ has rank m then $P(\theta)$ has rank $n-m$. Thus we find that (3.3) generically determines a set of $n-m$ independent constraints

$$G_1(\theta) = 0, \quad G_2(\theta) = 0, \quad \dots \quad G_{n-m}(\theta) = 0.$$

If the end effector traces a trajectory $x(t)$ along which the corresponding joint configuration $\theta(t)$ extremizes g at each point, then we have

$$\begin{pmatrix} f(\theta) \\ G_1(\theta) \\ \vdots \\ G_{n-m}(\theta) \end{pmatrix} = \begin{pmatrix} x(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Differentiating both sides provides a (forward) kinematic relationship between joint angle velocities and end effector velocity.

$$\begin{pmatrix} \frac{\partial f}{\partial \theta} \\ \dots \\ \frac{\partial G}{\partial \theta} \end{pmatrix} \cdot \dot{\theta}(t) = \begin{pmatrix} \dot{x}(t) \\ 0 \end{pmatrix}. \tag{3.4}$$

The coefficient matrix

$$J_e = \begin{pmatrix} \frac{\partial f}{\partial \theta} \\ \dots \\ \frac{\partial G}{\partial \theta} \end{pmatrix}$$

is square, and we shall call this the *extended Jacobian.*

Provided the extended Jacobian is nonsingular along a trajectory of interest, we may solve the inverse kinematics problem by writing

$$\dot{\theta}(t) = J_e^{-1} \begin{pmatrix} \dot{x}(t) \\ 0 \end{pmatrix}.$$

Obviously the extended Jacobian will be singular at any point where $\frac{\partial f}{\partial \theta}$ is singular. Other singular points of the extended Jacobian are characterized by the following.

Theorem 3.2: *Suppose θ_s is a joint angle configuration at which J_e is singular but where $\frac{\partial f}{\partial \theta}(\theta_s)$ has full rank (= m). Then (and only then) the following two conditions hold simultaneously:*

(i) The $n \times n$ matrix $P(\theta)$ has rank $n-m$, and

(ii) The $(n-m) \times n$ matrix $\frac{\partial G}{\partial \theta} \cdot P(\theta)$ has rank $< n-m$.

Proof: Suppose θ_s is a point where J_e is singular but $\frac{\partial f}{\partial \theta}(\theta_s)$ has full rank. The null space of $\frac{\partial f}{\partial \theta}(\theta_s)$ has dimension $n-m$, and since $P(\theta_s)$ is a projection onto this space, $P(\theta_s)$ has rank $n-m$. Since J_e is singular, there exists a vector v such that $J_e^T \cdot v = 0$, and because $\frac{\partial f}{\partial \theta}(\theta_s)$ has full rank, there is at least one $i > m$ such that the i-*th* entry $v_i \neq 0$. This means the row vector $\frac{\partial G}{\partial \theta}(\theta_s)$ is a linear combination of the rows of $\frac{\partial f}{\partial \theta}(\theta_s)$. Hence $\frac{\partial G_i}{\partial \theta} \cdot P$ is the zero vector, and the $(n-m) \times n$ matrix $\frac{\partial G}{\partial \theta} \cdot P$ has a zero row at this point, proving the theorem in one direction.

On the other hand, assuming the two conditions hold, the fact that $P(\theta_s)$ has rank $n-m$ implies that $\frac{\partial f}{\partial \theta}(\theta_s)$ has (full) rank m. $\frac{\partial f}{\partial \theta}^+$ thus has rank m, and there are $n-m$ columns of $P(\theta_s)$, labeled $p_1, p_2, \dots p_{n-m}$, spanning the column space of $P(\theta)$ and such that the matrix $M(\theta_s) = \left(\frac{\partial f}{\partial \theta}^+, \; p_1, \; \dots \; , p_{n-m} \right)$ has rank n. Now

$$J_e \cdot M(\theta_s) = \begin{pmatrix} I & 0 \\ A(\theta) & B(\theta) \end{pmatrix}$$

where I = the $m \times m$ identity matrix, $A(\theta) = \frac{\partial G}{\partial \theta} \cdot \frac{\partial f}{\partial \theta}^+$, and $B(\theta) = \frac{\partial G}{\partial \theta}(\theta) \cdot (p_1, \; \dots, \; p_{n-m})$. Now the second condition immediately implies that $rank\ B(\theta) < n-m$. Since $rank\ J_e = m + rank\ B(\theta)$, it follows $J_e(\theta_s)$ is singular, proving the theorem.

It was shown in Baillieul *et al.* (1984b) that any C^1 joint space trajectory not passing through a singularity of $\frac{\partial f}{\partial \theta}$ could be generated by an appropriate choice of $v(\cdot)$ in

$$\dot{\theta} = \frac{\partial f}{\partial \theta}^+ \dot{x} + Pv(t). \tag{3.5}$$

The problem of specifying the vector $v(\cdot)$ in some useful way can be approached via the *extended Jacobian technique.* It is not difficult to show that

$$J_e \cdot \left(\frac{\partial f}{\partial \theta}^+ \;\vdots\; P \right) = \begin{pmatrix} I & 0 \\ A(\theta) & B(\theta) \end{pmatrix} \tag{3.6}$$

where I = the $m \times m$ identity matrix, $A(\theta)$ is the $(n-m) \times m$ matrix $\frac{\partial G}{\partial \theta} \cdot \frac{\partial f}{\partial \theta}^+$, and $B(\theta)$ is the $(n-m) \times n$ matrix $\frac{\partial G}{\partial \theta} \cdot P$. Multiplying both sides of (3.6) by the composite vector $\begin{pmatrix} \dot{x} \\ v \end{pmatrix}$, we see from (3.4) that

$$0 = A(\theta)\dot{x} + B(\theta)v. \tag{3.7}$$

If J_e is invertible, $B(\theta)$ has rank $n-m$. This follows from Theorem 3.2. In this case v can always be found (using, say, the Moore-Penrose generalized inverse of B). If J_e is not invertible, it may still be possible to find a v which solves (3.7), but the conditions are more delicate. This motivates the following.

Definition 3.1: *Points θ in joint space where (3.7) cannot be solved for some value of $\dot{x}$ are called* algorithmic singularities *associated with the constraint(s)* $G_1(\theta) = 0, \ldots, G_{n-m}(\theta) = 0$.

It is important to emphasize that algorithmic singularities are not artifacts of the extended Jacobian technique. Unfortunately, they are inevitably introduced if functional constraints (or optimality criteria) are used to resolve the redundancy in almost any manipulator having revolute joints.

Example 3.1: One appealing criterion to optimize is the so-called *manipulatability index* (Yoshikawa, 1984a). In general, this is defined as $g(\theta) = det(\frac{\partial f}{\partial \theta}\frac{\partial f}{\partial \theta}^T)$. For a three-bar planar mechanism as depicted in *Figure 4*, maximizing this g is equivalent to maximizing $g(\theta) = \sin^2\theta_2 + \sin^2\theta_3$. As noted in Baillieul (1985a), the algorithmic singularitites for this criterion occur when the tip of the mechanism touches the base, forming a closed kinematic chain.

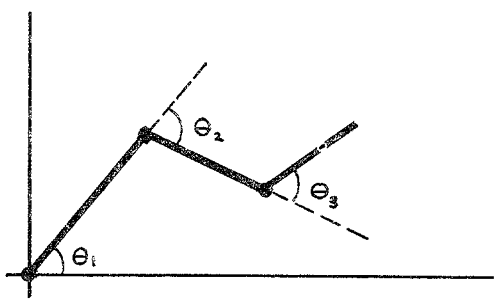


Figure 4: *A Planar three-bar mechanism.*

Example 3.2 (obstacle avoidance): For the same three-bar linkage, maximizing the criterion $g(\theta) = (\sin(\theta_1+\theta_2+\theta_3)\bar{x} - r_1\sin(\theta_2+\theta_3) - r_2\sin\theta_3)^2$ keeps the third link of the manipulator maximally distant from the point $(\bar{x}, 0)$ on the $x-axis$. This criterion appears appropriate for avoidance of *class 4* obstacles as defined in the preceding section. It can be shown that if this criterion is optimized, $(\bar{x}, 0)$ lies on a line passing through the tip of the mechanism and perpendicular to the third link. Moreover, the constraint $G(\theta_1,\theta_2,\theta_3) = \bar{x}\cos(\theta_1+\theta_2+\theta_3) - r_1\cos(\theta_2+\theta_3) - r_2\cos\theta_3 - r_3 = 0$ is satisfied. It is a straightforward calculation to show the algorithmic singularities associated with this constraint are the points where the equations $G(\theta_1,\theta_2,\theta_3) = 0$ and $\sin\theta_2 = 0$ hold simultaneously.

4. THE INTERFACE BETWEEN MULTIPLE CONSTRAINTS

We conclude with a remark on the incorporation of multiple constraints or optimality criteria into the control of kinematically redundant manipulators. It is reasonable to expect that if an obstruction lies in the workspace of a robot manipulator, some tasks will require more attention to obstacle avoidance than others. Thus, as the manipulator moves, it may be desirable to switch at certain points from a motion planning algorithm designed to avoid the obstacle to an algorithm which optimizes some other criterion. Some idea of the feasibility of doing this can be gleaned from the examples of the previous section.

When $r_1 = r_2 = r_3 = 1$, the constraint imposed in Example 3.1 is that $\theta_2 = \theta_3$, while the constraint of Example 3.2 is that $\bar{x}\cos(\theta_1+\theta_2+\theta_3) - \cos(\theta_2+\theta_3) - \cos\theta_3 - 1 = 0$. In either case, the set of points in joint space staisfying the criterion defines a multi-sheeted covering of the workspace. The number of preimages of each workspace point is either two or four (counting multiplicities). For example when the point $\begin{pmatrix} x \\ y \end{pmatrix}$ is near the base of the manipulator, the system of kinematic equations

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta_1 + \cos(\theta_1+\theta_2) + \cos(\theta_1+\theta_2+\theta_3) \\ \sin\theta_1 + \sin(\theta_1+\theta_2) + \sin(\theta_1+\theta_2+\theta_3) \end{pmatrix}$$

has four solutions satisfying the constraint $\theta_2 = \theta_3$. There is one solution for each of the ranges I, II, III, and IV as described in *Figure 5*. A similar statement holds for the other constraint where the four ranges for θ-values are A, B, C, and D as depicted in *Figure 5*. In general, these two constraints cannot be met simultaneously, but there is a one-parameter family of values of the joint variables (and a corresponding set of positions of the tip of the manipulator in the work space) where both criteria are satisfied simultaneously. The set of positions in the work space where both constraints are satisfied is the closed curve depicted in *Figure 6*.

I. $-2\pi/3 \geq \theta_2 \geq -\pi$ II. $0 \geq \theta_2 \geq -2\pi/3$ III. $0 \leq \theta_2 \leq 2\pi/3$ IV. $2\pi/3 \leq \theta_2 \leq \pi$	A. $\theta_2 \leq 0;\ \theta_3 \leq \theta_2$ B. $\theta_2 \leq 0;\ \theta_3 \geq \theta_2$ C. $\theta_2 \geq 0;\ \theta_3 \geq \theta_2$ D. $\theta_2 \geq 0;\ \theta_3 \leq \theta_2$
$\theta_2 = \theta_3$	$0 = \bar{x}\cos(\theta_1 + \theta_2 + \theta_3)$ $-\cos(\theta_2 + \theta_3) - \cos\theta_3 - 1$

Figure 5: *Each of the criteria in Examples 3.1 and 3.2 define multi-sheeted coverings of the workspace as indicated.*

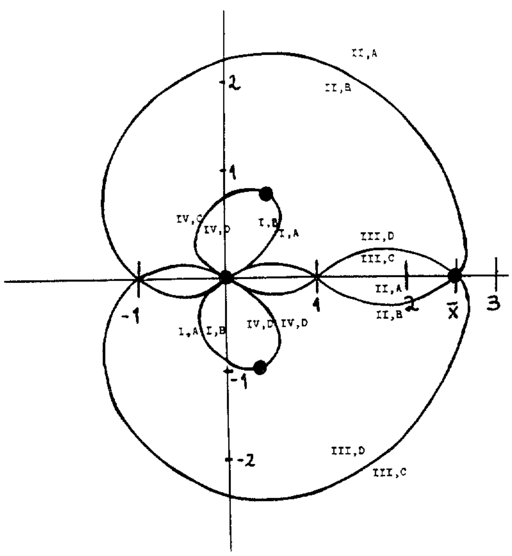


Figure 6: *The simultaneous zero locus of the constraint functions of Examples 3.1, 3.2 mapped by the kinematic equations to work space coordinates. The curve is subdivided into eight smooth segments (separated here by solid black circles) over each of which covering sheets of the two criteria (as labeled in* Fig. 5*) abut in joint space.*

Also indicated in *Figure 6* are some of the possible transitions between sheets of the constraint loci. Rules for making transitions between contraints along these curve segments may be generated to prescribe smooth motions which avoid any *class 4* obstacle in certain portions of the work space while maximizing *manipulatablilty* in other parts. Unfortunately, it is not possible to move from one constraint to another in such a way as to avoid the possibility of encountering an algorithmic or kinematic singularity. Indeed, it can be shown that switching between functional constraints will not in general eliminate the possibiltity of singular configurations being encountered in redundant manipulator motions. For this reason current research is focused on defining task oriented rules for switching between constraint surfaces so as to minimize the likelihood of encountering a singularity.

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REFERENCES

1. Baillieul, J., Brockett, R.W., and Hollerbach, J.M. (1984a) "Kinematically Redundant Manipulators—Final Report," AFWAL/MLTC Contract No. F33615-83-C-5115.

2. Baillieul, J., Hollerbach, J., and Brockett,R.W. (1984b) "Programming and Control of Kinematically Redundant Manipulators," **23-rd IEEE Conf. on Dec. and Control**, Las Vegas, December, pp. 768-774.

3. Baillieul, J. (1985a) "Kinematic Programming Alternatives for Redundant Manipulators," **IEEE Conf. on Robotics and Automation**, St. Louis, March 25–28, pp. 722-728.

4. Baillieul, J. (1985b) "Design of Kinematically Redundant Mechanisms," **24-th IEEE Conf. on Decision and Control**, Ft. Lauderdale, FL.

5. Brooks, R.A. (1984) "Planning Collision Free Motions for Pick and Place Operations," **Robotics Research: The First International Sysmposium**, Brady and Paul, Eds., MIT Press, Cambridge, MA, pp. 5-37.

6. Khatib, O. (1985) "Real-time Obstacle Avoidance for Manipulators and Mobile Robots," **IEEE Conf. on Robotics and Automation**, St. Louis, March 25–28, pp. 500-505.

7. Hollerbach, J.M. (1985) "Evaluation of Redundant Mnipulators Derived from the PUMA Geometry," **ASME Winter Annual Meeting**, Miami Beach, Nov. 1985.

8. Maciejewski, A.A. and Klein, C.A. (1985) "Obstacle Avoidance for Kinematically Redundant Manipulators in Dynamically Varying Environments," **The Int. J. Of Robotics Research**, v. 4, no. 3, pp. 109-117.

9. Noble, B. and Daniel, J. (1977) **Applied Linear Algebra**, (Second Edition), Prentice-Hall, Englewood Cliffs, NJ.

10. Yoshikawa, T. (1984a) "Manipulatability of Robotic Mechanisms," **Second Int. Symposium on Robotics Research**, Kyoto, Japan, August 20-23.

11. Yoshikawa, T. (1984b) "Analysis and Control of Robot Manipulators with Redundancy," **Robotics Research: The First International Sysmposium**, Brady and Paul, Eds., MIT Press, Cambridge, MA, pp. 735-747.