Kinematic and Dynamic Analysis of a Surgical Robot Positioning Arm

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This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,
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Abstract

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Over time, operating rooms have become increasingly crowded. The introduction of surgical robots into the operating field has only exacerbated the problem. In an effort to reduce crowding, the BioRobotics Lab at the University of Washington (Seattle, WA) developed the RAVEN, a small form-factor surgical robot. To increase the RAVEN's workspace and decrease setup time, the C-Arm was also developed to accurately support and position the RAVEN during surgery. The macro-micro manipulator system created by the C-Arm and the RAVEN occupies less space than a single manipulator with a similar workspace, can be used to automate setup procedures, and also increases patient safety.

This thesis presents an analysis and computational implementation of the C-Arm's forward kinematics and dynamics, along with a fast numerical solution for the inverse kinematics. The inverse kinematics solution can be generalized to provide a solution for any six degree-of-freedom manipulator, assuming the forward kinematics are known and that it is possible to solve for the remaining joint angles if one joint angle's value is known. With a fast numerical method and the current levels of computing power, designing a manipulator with closed-form inverse kinematics is no longer necessary. When designing the C-Arm, we therefore chose to weigh other factors, such as actuator size and patient safety, more heavily than the ability to find a closed-form inverse kinematics solution. The equations presented here will be used when controlling the C-Arm either alone or with the RAVEN.

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GLOSSARY

CLOSED-FORM SOLUTION: From [17], "a solution method based on analytic expressions or on the solution of a polynomial of degree 4 or less, such that noniterative calculations suffice to arrive at a solution."

DEGREES OF FREEDOM (DOFS): From [17], "the number of independent position variables that would have to be specified in order to locate all parts of the mechanism." For the mechanism being considered here, the number of joints is equal to the number of DOFs.

DYNAMIC EQUATIONS: The amount of torque applied to each joint based on the angle, angular velocity, and angular acceleration of each joint. Also referred to as "torque equations."

FORWARD KINEMATICS: The position and orientation of the end effector given the angle (or position) of each joint.

GOLD C-ARM: When standing at the foot of the operating table looking towards the patient's head, the GOLD C-Arm is located to the left of the table.

GREEN C-ARM: When standing at the foot of the operating table looking towards the patient's head, the GREEN C-Arm is located to the right of the table.

HYPER-REDUNDANT MANIPULATOR: From [13], hyper-redundant mechanisms "have a very large or infinite degree of kinematic redundancy." In practice, three-dimensional manipulators with thirty DOFs or more are considered hyper-redundant.

INVERSE KINEMATICS: The combination of joint angles needed to place the end effector in a desired position and orientation.

ITERATIVE SOLUTION: A solution method where an initial guess x_0 is chosen, and then calculations are performed to obtain better and better approximations $x_1, x_2, \ldots$ until a solution with the desired accuracy is obtained.

JACOBIAN MATRIX: Also known as the manipulator Jacobian, or simply as the Jacobian. From [17], "the Jacobian specifies a mapping from velocities in joint space to velocities in Cartesian space."

MACRO/MICRO MANIPULATOR PAIR: Macro/micro manipulator pairs traditionally consist of a large (macro) manipulator with a large workspace that positions a small (micro) dexterous manipulator. The manipulators work together to accomplish a given task.

MINIMALLY INVASIVE SURGERY (MIS): Also known as laparoscopic surgery. Procedures are performed through small incisions in the body, rather than through a large opening.

POSE: Position and orientation. In three-dimensional space, three positions and three orientations are required to define a pose. In two-dimensional space, two positions and one orientation are required.

SINGULAR POSE: Also known as a singularity. A manipulator is in a singular pose when its end effector is unable to move in one or more directions in Cartesian space.

REDUNDANT MANIPULATOR: A manipulator with more DOFs than necessary to obtain a desired pose. 3 DOFs are necessary for two-dimensional systems, and 6 DOFs are necessary for three-dimensional systems.

Chapter 1

INTRODUCTION

Since the mid-1980s, the use of robots in surgery has slowly gained popularity. With systems present in hospitals around the world, the FDA-approved da Vinci, from Intuitive Surgical, Inc. (Sunnyvale, CA), is arguably the most well-known surgical robot [26]. Some aspects of the da Vinci system could be improved, however. Two of the most prevalent complaints pertain to the size of the system and to the amount of time required to set up the system for a procedure. The da Vinci occupies a large footprint on the operating floor, most of one side of the operating table, and a large portion of the area above the patient. Several research groups are working to develop smaller systems. The BioRobotics Lab's RAVEN at the University of Washington, Seattle [41] is one such system. The RAVEN was designed primarily for minimally invasive surgery (MIS), in which long, narrow tools are introduced into the insufflated abdominal cavity through ports in the abdominal wall, as shown in Figure 1.1. The RAVEN was designed with a spherical mechanism, allowing up to six degrees of freedom of manipulator motion plus grasping without displacing the tool shaft where it enters the abdominal wall (the insertion point).

One drawback to small systems is that their workspace is generally reduced along with their size. The RAVEN was designed to reach a patient's entire abdomen if the system is positioned precisely [39]. In practice, the system's performance is optimal over only a subset of the abdomen, and decreases at the edge of the workspace. In addition, the ports for MIS are not always placed in the same location for every surgery. Depending on the port placement, the system may not be able to reach the entire abdomen without repositioning its base. Rather than increasing its workspace, and therefore the amount of space it occupies over the patient, the BioRobotics Lab opted to design a second arm,

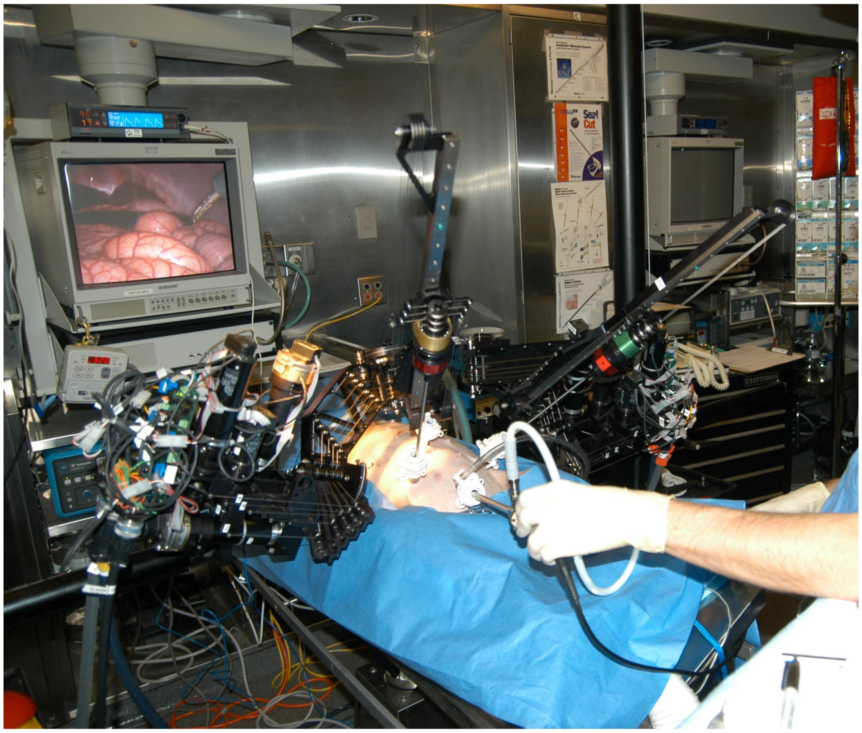

Figure 1.1: RAVEN Surgical Robot, shown operating on a porcine model. A human assistant positions the camera for the surgeon.

attached to the operating table, to support and position the RAVEN over the patient. The support arm, currently referred to as the "C-Arm," can travel the length of the table and extend the insertion point of the RAVEN to any point on the side or upper surface of the abdomen. The C-Arm can also extend beyond the patient to reach tools or supplies well outside the immediate vicinity of the operating table. When combined, the C-Arm/RAVEN system has more degrees of freedom (DOFs) than necessary to obtain a given position and orientation of the surgical tool in three-dimensional space. Many methods have been proposed for dealing with these extra, or redundant, DOFs.

This chapter contains a brief history of surgical robotics, followed by an overview of the methods used to control redundant manipulators, hyper-redundant manipulators, and macro/micro manipulators. Finally, a discussion of my research goals, including the goals of this thesis, is presented.

1.1 Surgical Robotics

In April 1985, Kwoh and colleagues used a Unimation Puma 200 robot to orient a biopsy needle for neurosurgery, marking the first use of robotics in surgery [37]. The latter half of the 1980s also saw the development of the system that would later become ROBODOC, which was used for precision bone machining in orthopedic surgeries such as cementless total hip replacements first in canines and then in humans [63, 64], as well as the use of a robot to perform a transurethral resection of the prostate, first with a Puma 560 and later with the specially designed Probot [18, 19, 27]. The use of robotics in surgery increased in popularity in the 1990s, with devices such as the SRI telepresence system [29], the IBM Research Center/Johns Hopkins University surgical robot [62], the system designed at the Politecnico di Milano in Italy [58], and the Black Falcon from Massachusetts Institute of Technology [43].

The Automated Endoscopic System for Optimal Positioning (AESOP) was the first robot approved for use in surgery by the US Food and Drug Administration (FDA). After its approval in 1994, the system assisted surgeons by supporting an endoscope and repositioning according to the surgeons' instructions [7, 31, 59]. Developed by Computer Motion, Inc. (Goleta, CA), the AESOP was later expanded into the Zeus robotic surgery system [22], which received FDA approval in October 2001 [4, 28]. The Zeus was used in the first (and only) transatlantic telesurgery, performed between Manhattan, New York, USA and Strasbourg, France in September 2001 [22, 46]. The Zeus's major competitor was the da Vinci surgical robot, produced by Intuitive Surgical, Inc. (Mountain View, CA) and FDA approved in July 2000 [26, 28, 61]. In June 2003, the companies merged under the name Intuitive Surgical, Inc. and production of the Zeus and AESOP systems ceased [28, 61]. Other commercially available systems include the NeuroMate (which, along with ROBODOC, was produced by Integrated Surgical Systems, Inc. in Davis, CA) [4, 16, 38] and the Naviot laparoscope manipulator [35].

Several research-level surgical robotic systems are currently in development around the world. The system designed at the University of Tokyo [49] has performed telesurgical experiments throughout Asia. The NeuRobot [30] has been used in clinical applications.

Other systems include the Berkeley/UCSF laparoscopic telesurgical workstation [8], the Light Endoscopic Robot [5], the MC^2E [74], and the University of Washington's RAVEN.

1.2 Redundant, Hyper-Redundant, and Macro/Micro Manipulators

Assuming a manipulator is operating in standard three-dimensional space, 6 DOFs are needed for the manipulator to obtain any position and orientation in its workspace. (In two-dimensional space, 3 DOFs are needed.) Many techniques have been proposed for controlling "redundant" manipulators with one, or perhaps two, additional DOFs. Other techniques are used to control "hyper-redundant" manipulators with thirty DOFs or more. The intermediate range is occupied in part by macro/micro manipulator pairs, which, as a group, are less widely studied than redundant manipulators.

A variety of methods have been proposed for controlling manipulators with one or two redundant DOFs. Some methods seek to avoid poor dynamic poses or singularities [69, 70, 71]. Others seek to avoid joint limits or high joint torques [9, 12]. Still others employ obstacle avoidance [3, 15, 33]. Some seek to control multiple redundant DOFs by avoiding several of these conditions at once [48, 53]. Depending on the function of the manipulator, some methods are more appropriate than others. In most cases, a function is minimized or maximized in order to avoid the undesirable condition. Such an approach works well for one or two redundant DOFs, but becomes tedious for systems with more redundant DOFs.

At the other end of the spectrum is the hyper-redundant mechanism. The theoretical hyper-redundant mechanism is an infinite-DOF system. In reality, the mechanisms tend to have between thirty and fifty DOFs. Hyper-redundant mechanisms are generally used in very constrained workspaces. As a result, their controllers are frequently focused on obstacle avoidance. The most common control method computes a curve through the field of obstacles and then uses curve-fitting techniques to deploy the mechanism along the curve [13, 14, 42]. Another control method is referred to as "follow the leader" [55, 67]. The end effector moves through the obstacles and the rest of the manipulator maneuvers to follow its path through the field. Both methods work well for hyper-redundant mechanisms, but would work poorly for manipulators with fewer DOFs.

Macro/micro manipulator pairs occupy a portion of the range between redundant and hyper-redundant mechanisms. Generally, three-dimensional macro/micro manipulator pairs have between eight and twelve DOFs in total and two-dimensional pairs have five or six DOFs. Manipulators in this class traditionally consist of a macro manipulator with a micro manipulator attached to its distal end. As the names suggest, the macro manipulator is usually quite large compared to the micro manipulator. Macro/micro pairs are very commonly used in space applications [20, 68, 73]. In some cases, a large manipulator arm will be treated as a macro manipulator and a smaller, faster manipulator at its distal end will be treated as a micro manipulator. In other cases, the spaceship itself acts as the macro manipulator and the large manipulator arm is treated as the micro manipulator. Underwater vehicles with attached arms [21], precision machining tasks [54], precision positioning stages [23, 32], and bridge or underground tank inspection [47] also make use of macro/micro manipulator pairs. In general, the macro manipulator provides a large workspace while the micro manipulator allows for fast precision control. The C-Arm/RAVEN system can easily be considered a macro/micro system.

1.3 Research Goals

My long-term research goal is to develop a controller for a macro/micro system that can then be implemented on the combined C-Arm/RAVEN system. I intend to promote patient safety by moving all joints, and most particularly the large, powerful joints of the C-Arm, as little as possible. More details about my plans for such a controller can be found in Chapter 5 of this thesis.

The work presented in this thesis represents the first step towards my long-term goal. Once development of the C-Arms is completed, we will need to control them. Specifically, we will need to know:

- The position and orientation of the end effector given the given the angle (or position) of each joint,

- The amount of torque applied to each joint based on the angle, angular velocity, and angular acceleration of each joint, and

- The combination of joint angles needed to place the end effector in a desired position and orientation.

Respectively, these objectives are referred to as the forward kinematics, the dynamics, and the inverse kinematics of the system. One goal of this thesis is to develop equations or models that achieve each of these objectives.

As I will discuss in more detail in Section 2.1, the design chosen for the C-Arms does not have a closed-form inverse kinematics solution. Until now, the lack of a closed-form solution would likely have sent us back to the drawing board to come up with a design that had closed-form inverse kinematics. Few groups over the last 20 years have chosen to continue with such a design [1, 6, 10, 66, 75]. If we assume that Moore's Law holds, and that transistor density will double every 1.5-2 years (thereby doubling processing speed every 1.5-2 years), then we can now perform calculations between 3 and 4 orders of magnitude faster than we could 20 years ago. With such an increase in our computing abilities, we felt that it might no longer be necessary to constrain ourselves to designs with closed-form inverse kinematics. It is therefore a significant goal of this thesis to determine whether numerical inverse kinematics solutions can be computed sufficiently quickly to be acceptable.

Chapter 2

C-ARM DESIGN

The C-Arms were developed as part of Phase I of the Trauma Pod Project[1]. Trauma Pod was led by SRI International with contributions from General Dynamics, General Electric, Intuitive Surgical Inc., Multi-Dimensional Imaging Inc., Oak Ridge National Laboratory, Robotic Surgical Technology, the University of Maryland, the University of Texas at Austin, and the University of Washington at Seattle. Figure 2.1 shows a SolidWorks model of an operating room consisting of an operating table, an insufflated patient, two C-Arms, and two RAVEN surgical robots. The GOLD C-Arm (as seen in Figure 2.2) and GREEN C-Arm are nearing completion and are expected to be fully operational shortly.

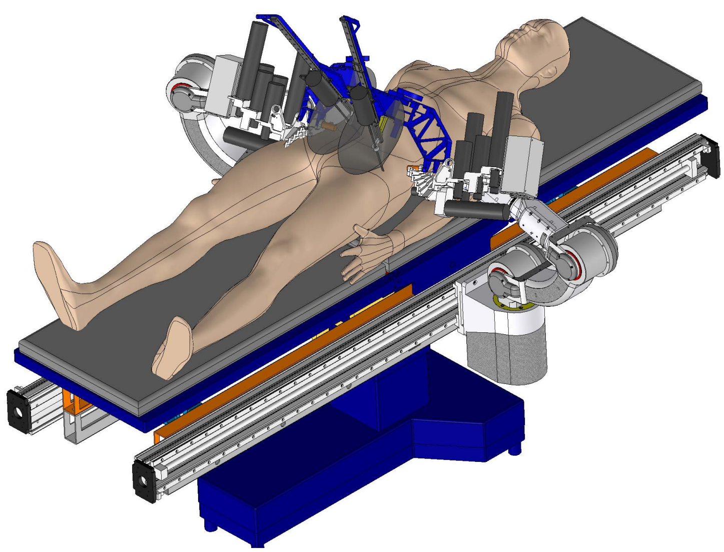

Figure 2.1: CAD rendering of the C-Arms supporting RAVENs over an insufflated patient. The gray cones located around the tools on each RAVEN illustrate the workspace in which the RAVENs can operate most effectively.

[1] `http://www.traumapod.org`

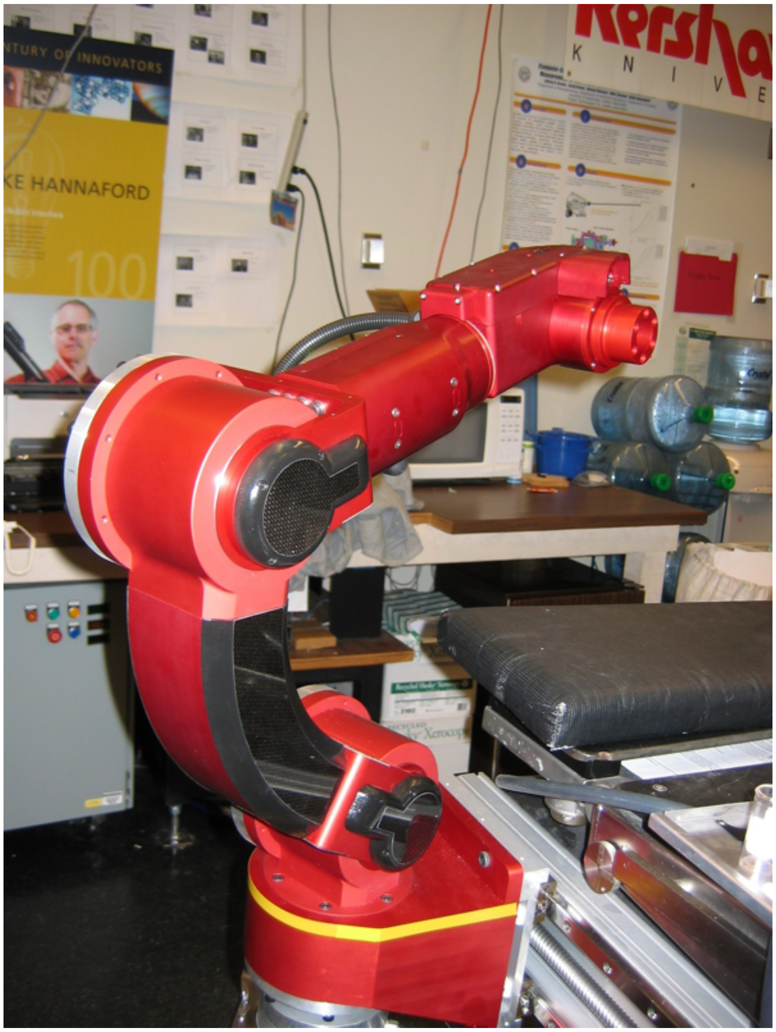

Figure 2.2: The C-Arm, currently under development in the BioRobotics Lab. Joint 1 slides along the linear rail seen in the lower right corner. The arm pictured is the GOLD Arm.

Design and control of the C-Arms has been a collaborative effort within the BioRobotics Lab. Mitch Lum, Tim Ramsey, and Jacob Rosen contributed to the conceptual design phase of the project. Joel Perry, Rainer Leuschke, and Shane Draney completed the majority of the final design and developed the SolidWorks model from which many properties used in this thesis were obtained. The electronics and control software was developed by a large team led by Tim Kowalewski, Radivoje "Ogi" Jovanovic, and Jesse Dosher.

The C-Arms were designed to support and position a variety of end effectors for use in robotic surgery. Potential end effectors range from rigid probes to the RAVEN surgical robot. Regardless of the end effector, a model of the C-Arm's kinematics, inverse kinematics, and dynamics is needed to accurately control the arm. This Chapter will discuss the goals and tradeoffs considered when designing the C-Arms and present the kinematic and dynamic model of the C-Arm. The inverse kinematics solution will be presented in Chapter 3.

2.1 Design Goals

The C-Arm's primary function is to support and move an arbitrary end effector during robotic surgery and conveniently position it as necessary over the patient. Up to four C-Arms may eventually be used, with the arms attached either to the side of the operating table or to a ceiling-mounted rail. The 12kg RAVEN is among the largest, heaviest end effector currently considered for use. The C-Arms must be able to easily manipulate a payload of that magnitude.

The function of the C-Arm varies based on the end effector. When the RAVEN is used, the C-Arm must provide a rigid support for the RAVEN's base. When a rigid probe is used, the C-Arm must accurately position the tool dynamically. The C-Arm must also be able to reach across an insufflated patient to allow C-Arms mounted on either side of an operating table to access MIS ports (or the operating site, in open surgery) located on one side of the patient. Other design goals include providing a high ratio of workspace to total volume and minimizing the size, weight, and power consumption of joint actuators. Given these requirements, it was decided that a better design could be achieved by sacrificing a traditional manipulator design goal: that the inverse kinematics have a closed form solution.

A well known result by Pieper and Roth [56, 57] is that a closed form inverse kinematics solution exists for any manipulator with three consecutive intersecting rotary axes. Another of Pieper's results is that a solution exists for manipulators with three pairs of intersecting rotary axes, as well as for manipulators with three rotary and three prismatic joints. It has been standard practice for many years to design manipulator arms so that a closed form solution exists. In our design process however, we felt that substantial increases in computing power available today reduce the penalty on numerical solutions and thus free the designer to match the architecture more closely to the task requirements.

For any system that will operate in close proximity to a human, safety is a primary concern. To reduce the risk of injury to the patient, we endeavored to minimize size and actuator strength of the entire C-Arm, especially in the vicinity of the patient. As we were constrained by the need to span a large workspace while supporting a 12kg payload, we searched for a design that would allow us to use the smallest actuators possible. Our analysis showed that exchanging the order of the last two joints from the design that was ultimately chosen would provide us with a design containing three consecutive intersecting rotary axes (Joints 3 through 5, defined in Section 2.2.1, are parallel and meet at a point at infinity), thereby allowing us to find a closed-form inverse kinematics solution. Exchanging the order of those joints, however, would increase the torques exerted on those joints and require the use of larger actuators. Using larger actuators on the most distal joints would increase the payload supported by more proximal joints, and would require those actuators to be enlarged as well. Other designs considered resulted in similar enlargements.

Increasing the size of both the actuators and the manipulator itself increases the risk to patient safety. Larger manipulators increase the risk of patient-manipulator collisions, and larger actuators increase the amount of damage that a patient could suffer in the case of catastrophic failure. In comparing potential designs, we determined that a slight increase in computing time due to choosing a design with no closed-form inverse kinematic solution was well worth the corresponding increase in patient safety.

An interesting side note is that our chosen design does contain a point where the first three joint axes intersect. Since one of those joints is prismatic, however, the conditions set forth by Pieper are not met, a fact that was not immediately clear from the description of Pieper's solution presented in [17].

2.2 System Model

Properties used to develop a system model of the C-Arm were derived from the SolidWorks model. Unfortunately, the properties of the physical C-Arm will not exactly match those of the SolidWorks model. The dimensions of components and their locations relative to a base frame may vary slightly due to machining tolerances and plating or coating thicknesses. These variations will affect not only the length of each link, but also the mass and distribution of mass within each link.

Another factor affecting the mass and its distribution within each link is the slight simplification of the SolidWorks model. When all nuts, screws, washers, and other fasteners are included, the SolidWorks model becomes unwieldy. When many of the fasteners were excluded, the model was more reliable and could be manipulated more easily.

Simplified models of circuit boards and off-the-shelf components were also used. Circuit board models were used to prevent interference with other C-Arm parts during assembly, rather than providing a detailed layout of the circuit board components. It was therefore more important to model the envelope of the board rather than the individual components. The resulting model allows for proper C-Arm assembly, but does not provide accurate mass properties of the circuit board. The same is true for models of many off-the-shelf components, where the dimensions and total mass are known, but not the distribution of the mass throughout the component.

Finally, all wires running within the C-Arm are not modeled. As the SolidWorks model was not intended to serve as a wiring guide, models of the wires would not have been instructive. The only information needed was the number and size of the wires passing through each opening to ensure appropriate clearance and strain relief.

It is assumed that the simplifications described here will not significantly affect the system model derived in this thesis. This assumption will need to be evaluated when the model is implemented on the physical system, and modifications will be made if necessary.

2.2.1 C-Arm Joints

Each C-Arm is a 6-DOF system with one linear joint and five rotational joints. The joints are numbered 1 through 6, with Joint 1 being the joint most proximal to the rigid base (consisting of a linear rail mounted to either the operating table or the ceiling) and Joint 6 being the most distal. Joint axes for the GOLD C-Arm are defined by the blue lines in Figure 2.3, with the arrowhead indicating either the direction of positive linear motion (for

Joint 1) or the direction of positive rotation using a right-handed coordinate system (for Joints 2-6). Figure 2.3 also indicates relevant lengths (in red, defined in Table 2.2 in Section 2.2.2) and the directions of the X- and Z-axes for each link frame (in green) as defined using the Denavit-Hartenberg (D-H) notation convention [17]. The D-H parameters defined in Table 2.1 take the pose in Figure 2.3 as the zero pose.

The GREEN C-Arm is a mirror image of the GOLD C-Arm through a plane perpendicular to the X_B axis. When defining positive directions for joint rotations and frame axes, the following statements were adhered to:

- Frame $\{B\}$ is the stationary base frame, attached to a corner at the foot of the operating table. For Frame $\{B\}$, the X axis points toward the center of the patient and the Z axis points toward the patient's head. The Z_B axes of the GOLD and GREEN C-Arms therefore point in the same direction, and the X_B axes point in opposite directions.

- The axes of Frame $\{1\}$ point in the same direction as the corresponding axes of Frame $\{B\}$ and the direction of positive motion for Joint 1 is in the direction indicated by the Z axis of each arm. The joint axes and Z_1 axes of the GOLD and GREEN C-Arms therefore point in the same direction (when the C-Arms are attached to opposite sides of an operating table), and the X_1 axes point in opposite directions.

- The positive direction of motion for Joint 2 is defined to be up. In the zero pose, the X_2 axis points in the same direction as the X_B and X_1 axes. In the zero pose, the joint axes and Z_2 axes of the GOLD and GREEN C-Arms therefore point in the same direction, and the X_2 axes point in opposite directions.

- For Joints 3 and 4, the X axes point up when the C-Arms are in the zero pose. The positive direction of joint rotation is defined to curl the arm in towards the patient for positive joint motion. In the zero pose, the joint axes and Z axes of the GOLD and GREEN C-Arms therefore point in opposite directions, and the X axes point in the same direction.

- The X axes of Joint 5 point up when the C-Arms are in the zero pose. The positive direction of joint rotation points to the patient's left when the C-Arms are in the zero pose. In the zero pose, the joint axes, X_5 axes, and Z_5 axes of the GOLD and GREEN C-Arms all point in the same direction.

- For Joint 6, the X axes point up when the C-Arms are in the zero pose. The positive direction of joint rotation is in the same direction as positive joint rotation for Joints 3 and 4 when the C-Arms are in the zero pose. In the zero pose, the joint axes and Z_6 axes of the GOLD and GREEN C-Arms therefore point in opposite directions, and the X_6 axes point in the same direction.

The D-H parameters for the GREEN C-Arm are also defined in Table 2.1. When either $\pm$ or $\mp$ is indicated, the top sign refers to the GOLD C-Arm and the bottom sign refers to the GREEN C-Arm.

Table 2.1: D-H Parameters of the C-Arms. When either $\pm$ or $\mp$ is indicated, the top sign refers to the GOLD C-Arm and the bottom sign refers to the GREEN C-Arm.

$\boldsymbol{i}$	$\boldsymbol{a_{i-1}}$	$\boldsymbol{\alpha_{i-1}}$	$\boldsymbol{d_i}$	$\boldsymbol{\theta_i}$
1	0	0	$d_1(t)$	0
2	0	$\pm\frac{\pi}{2}$	0	$\theta_2(t)$
3	0	$-\frac{\pi}{2}$	0	$-\frac{\pi}{2} + \theta_3(t)$
4	L_1	0	0	$\theta_4(t)$
5	0	$\mp\frac{\pi}{2}$	$\pm L_2$	$\theta_5(t)$
6	0	$\pm\frac{\pi}{2}$	0	$\theta_6(t)$

Joint Limits

As Joint 1 is linear, its limits depend on the length of rail to which it is attached. In our current lab setup, the linear rail attached to an operating table allows for 2000mm of travel.

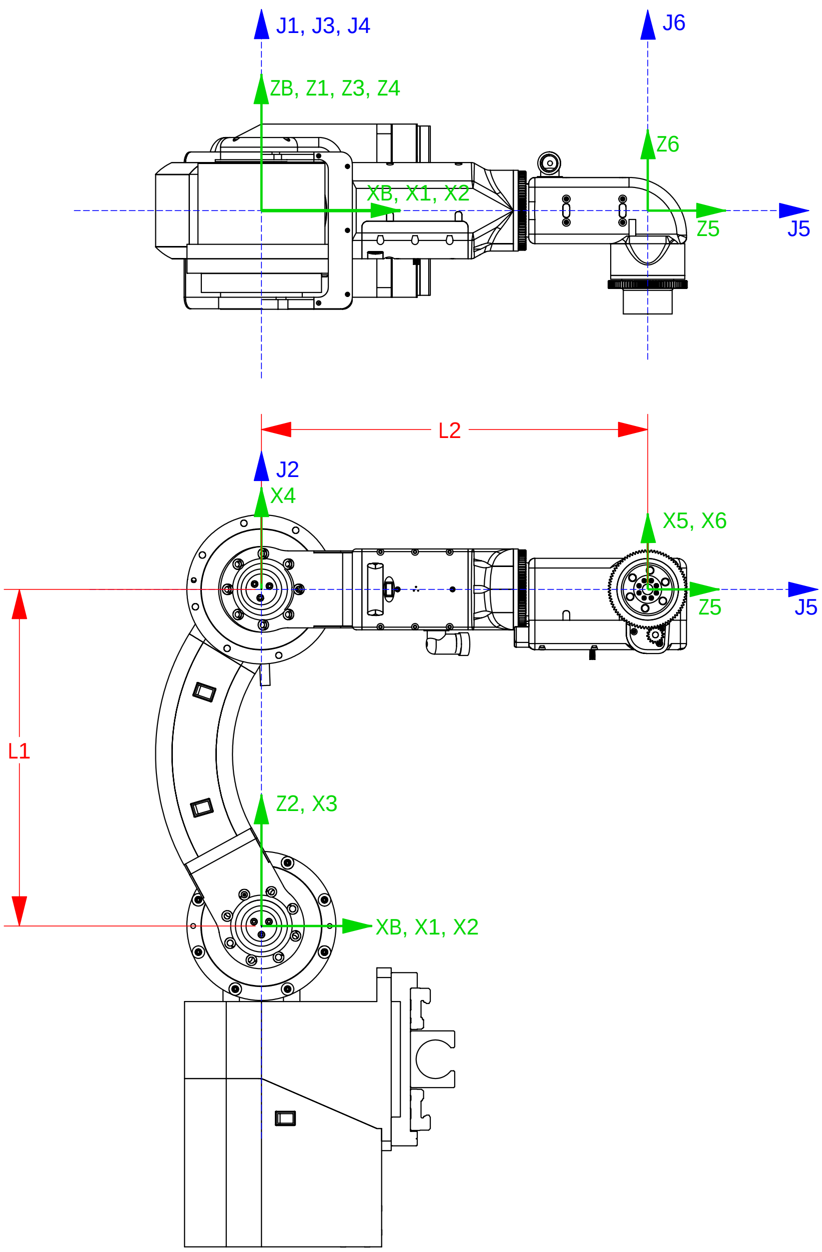

Figure 2.3: CAD rendering of the GOLD C-Arm showing joint axes, lengths, and coordinate systems used to determine the DH parameters. X_B and Z_B indicate the base frame, attached to the linear rail. Coordinate frames 1 through 6 are attached to Links 1 through 6, where Link i is defined as the portion of the C-Arm that falls between Joints i and $i+1$.

For both the GOLD and GREEN C-Arms, the zero point is defined as the end of the rail nearest the patient's feet with the positive direction of travel pointing toward the head.

Joints 2, 5, and 6 can theoretically perform one full revolution, with joint limits defined at $-180°$ and $180°$. For all three joints, these limits will likely be reduced somewhat as construction of the C-Arms nears completion, due to lengths of wires, the addition of safety stops, and other considerations. Safe limits may also vary in a clinical setting based on the relative positions of the patient and other medical equipment to the C-Arms.

Limits for Joints 3 and 4 are more difficult to define. In both cases, the joints are limited due to self-collisions as well as by external factors. As a result of the C-Arm's design, limits defined by self-collisions will vary based on manipulator pose. Joint 3, for instance, reaches its joint limits when Links 1 and 3 collide. (Note that Link i is defined as the portion of the C-Arm that falls between Joints i and $i + 1$.) For $\theta_2 = 0°$, this limit is reached when $\theta_3 = -84°$ or $\theta_3 = +116°$. Depending on the pose, Joint 4 may cause a collision between one of the links proximal to Joint 4 and one of the links distal to the joint. If Joints 2 and 3 are both in the zero pose, a collision will occur soonest between Links 3 and 4 when $\theta_4 < 0°$ and between Links 1 and 5 when $\theta_4 > 0°$. If θ_3 is set to a negative angle, θ_4 can move further in the positive direction, and a collision will occur between Links 2 and 5 instead. When θ_3 is set to a slight negative angle, the limits reached by Joint 4 are $\theta_4 = -178°$ and $\theta_4 = +66°$.

The range of motion achieved by each link may be reduced by wiring or the addition of safety stops, just as for Joints 2, 5, and 6. A much greater limit will be imposed on the motion of Joints 3 and 4, however, by the addition of the patient, operating table, other medical devices in the field, and an end effector. Before the C-Arms can be used in a clinical setting, further work will need to be completed to fully map regions where self-collision is a concern and to include obstacle avoidance in the control software.

2.2.2 C-Arm Forward Kinematics

Transformation matrices relate two frames and can be used to transform positions and orientations from one frame to the other. In robotics, a transformation matrix can be used to calculate the position and orientation of a manipulator's wrist or end effector given the values of each of the manipulator's joints. These specific transformation matrices are referred to as the manipulator's forward kinematics, or simply the manipulator kinematics.

The transformation from one joint to the next can be computed using the D-H parameters [17]. Note that c and s in Equation (2.1) refer to the cosine or sine of the angle that follows (either θ_i or α_{i-1}).

$$ {}_{i-1}^{\;\;i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}d_1 \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & -c\alpha_{i-1} & -c\alpha_{i-1}d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.1} $$

When the transformation matrices between each pair of neighboring frames are known, a transformation matrix between the base and the wrist or end effector can be calculated:

$$ {}_W^BT = {}_1^BT\, {}_2^1T\, {}_3^2T\, {}_4^3T\, {}_5^4T\, {}_6^5T\, {}_W^6T \tag{2.2} $$

where B refers to the base frame and W refers to the wrist frame. Note that for the C-Arm, the sixth link frame and the wrist frame are the same, so ${}_W^6T = I_{4\times4}$. When an end effector is attached to the C-Arm, an additional transform is included.

For clarity, and for use in Chapter 3, the transformation matrix is reported here in two parts. As before, when either $\pm$ or $\mp$ is indicated, the top sign refers to the GOLD C-Arm and the bottom sign refers to the GREEN C-Arm. For the rest of this thesis, C_2 is shorthand for $\cos(\theta_2)$, S_{34} is shorthand for $\sin(\theta_3+\theta_4)$, and so on. Additional examples are listed in Appendix A.

$$ {}^{B}_{W}T = {}^{B}_{2}T \, {}^{2}_{W}T \tag{2.3a} $$

$$ {}^{B}_{2}T = \begin{bmatrix} C_2 & -S_2 & 0 & 0 \\ 0 & 0 & \mp 1 & 0 \\ \pm S_2 & \pm C_2 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.3b} $$

$$ {}^{2}_{W}T = \begin{bmatrix} S_{34}C_5C_6 + C_{34}S_6 & C_{34}C_6 - S_{34}C_5S_6 & \pm S_{34}S_5 & S_3L_1 + C_{34}L_2 \\ \mp S_5C_6 & \pm S_5S_6 & C_5 & 0 \\ C_{34}C_5C_6 - S_{34}S_6 & -S_{34}C_6 - C_{34}C_5S_6 & \pm C_{34}S_5 & C_3L_1 - S_{34}L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.3c} $$

Another useful transformation matrix is ${}^{W}_{P}T$, the transformation from the wrist frame to the base frame of the RAVEN Surgical Robot. The origin of the RAVEN's base frame is the pivot point that marks the location of the MIS port. The RAVEN's base frame is referred to as Frame $\{P\}$. The transformation matrix is:

$$ {}^{W}_{P}T = \begin{bmatrix} 0 & 1 & 0 & -L_3 \\ 0 & 0 & -1 & L_5 \\ -1 & 0 & 0 & \mp L_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.4} $$

From Equations (2.3) and (2.4), the transformation matrix from the C-Arm's base to the MIS port can be calculated:

$$ {}^{B}_{P}T = {}^{B}_{2}T \; {}^{2}_{W}T \; {}^{W}_{P}T \tag{2.5} $$

2.2.3 Wrist or End Effector Pose

Transformation matrices contain sufficient information to calculate the wrist or end effector pose, consisting of three position terms and three orientation terms. Using the notation $T_{i,j}$ to refer to the $\{i, j\}^{\text{th}}$ element of the transformation matrix T, the position and orientation

Table 2.2: Lengths used to determine the D-H Parameters and Forward Kinematics of the C-Arms and RAVEN manipulator arms. Lengths were measured using the SolidWorks model.

Length	**Value (mm)**
L_1	350
L_2	402
L_3	15
L_4	186
L_5	253

terms $\{X, Y, Z, \alpha, \beta, \gamma\}$ are calculated as follows:

$$X = T_{1,4} \tag{2.6a}$$

$$Y = T_{2,4} \tag{2.6b}$$

$$Z = T_{3,4} \tag{2.6c}$$

$$\alpha = \operatorname{atan2}\left(T_{2,3}, T_{1,3}\right) \tag{2.6d}$$

$$\beta = \operatorname{atan2}\left(\sqrt{T_{1,3}^2 + T_{2,3}^2}, T_{3,3}\right) \tag{2.6e}$$

$$\gamma = \operatorname{atan2}\left(T_{3,2}, -T_{3,1}\right) \tag{2.6f}$$

If $T_{1,3}^2 + T_{2,3}^2 = 0$, an alternate equation for γ is used:

$$\gamma = \operatorname{atan2}\left(T_{2,1}, T_{2,2}\right) - \alpha T_{3,3} \tag{2.6g}$$

These equations produce orientation values α, β, and γ using the ZYZ Euler angle convention [17]. To reproduce this orientation, the following rotations are performed:

- From Frame $\{B\}$, rotate about the frame axis $\hat{Z}_B$ by an angle α into Frame $\{B'\}$.
- Rotate about the frame axis $\hat{Y}_{B'}$ by an angle β into Frame $\{B''\}$.
- Rotate about the frame axis $\hat{Z}_{B''}$ by an angle γ into Frame $\{W\}$.

2.2.4 C-Arm Jacobian Matrix

The Jacobian matrix, or simply the Jacobian, of a manipulator is used to find the angular and linear velocities of the wrist or end effector given the velocity of each joint. The Jacobian is calculated based on the D-H parameters via the method outlined in [17].

First, the angular and linear velocities of each link are calculated. For each Link $(i+1)$, the angular and linear velocities are calculated with respect to the base frame and expressed in terms of Frame $\{i+1\}$. The angular velocity ${}^{i+1}\omega_{i+1}$ of Link $(i+1)$ when Joint $(i+1)$ is rotational is:

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R \, {}^{i}\omega_i + \dot{\theta}_{i+1} \, {}^{i+1}\hat{Z}_{i+1} \tag{2.7} $$

Note that

$$ \dot{\theta}_{i+1} \, {}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} \tag{2.8} $$

If Joint $(i+1)$ is prismatic, the angular velocity ${}^{i+1}\omega_{i+1}$ of Link $(i+1)$ is:

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R \, {}^{i}\omega_i \tag{2.9} $$

The linear velocity ${}^{i+1}v_{i+1}$ of the origin of Frame $\{i+1\}$ when Joint $(i+1)$ is rotational is:

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R \left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1} \right) \tag{2.10} $$

where ${}^{i}P_{i+1}$ is the first three elements of the fourth column of ${}^{i}_{i+1}T$. If Joint $(i+1)$ is prismatic, the linear velocity ${}^{i+1}v_{i+1}$ of the origin of Frame $\{i+1\}$ is:

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R \left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1} \right) + \dot{d}_{i+1} \, {}^{i+1}\hat{Z}_{i+1} \tag{2.11} $$

where

$$ \dot{d}_{i+1} \, {}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix} \tag{2.12} $$

The velocity vectors ${}^{W}v_W \left(= {}^{6}v_6\right)$ and ${}^{W}\omega_W \left(= {}^{6}\omega_6\right)$ can be concatenated into a vector ${}^{W}\nu$:

$$ {}^{W}\nu = \begin{bmatrix} {}^{W}v_W \\ {}^{W}\omega_W \end{bmatrix} \tag{2.13} $$

The concatenated vector ${}^{W}\nu$ is used, along with the vector of joint angles:

$$\Theta = \begin{bmatrix} d_1 & \theta_2 & \theta_3 & \theta_4 & \theta_5 & \theta_6 \end{bmatrix}^T \tag{2.14}$$

to compute ${}^{W}J(\Theta)$, the Jacobian written in the wrist frame.

$${}^{W}J(\Theta) = \frac{\partial\, {}^{W}\nu}{\partial \Theta} \tag{2.15}$$

The Jacobian is generally more useful when written in the base frame. The base frame Jacobian is calculated in the same manner, except that the velocity vectors ${}^{B}v_W$ and ${}^{B}\omega_W$, representing the linear and angular velocities of the wrist expressed in the base frame, are used to form a concatenated vector ${}^{B}\nu$.

$${}^{B}\nu = \begin{bmatrix} {}^{B}v_W \\ {}^{B}\omega_W \end{bmatrix} \tag{2.16}$$

where

$${}^{B}v_W = {}^{B}_{W}R\, {}^{W}v_W \tag{2.17}$$

$${}^{B}\omega_W = {}^{B}_{W}R\, {}^{W}\omega_W \tag{2.18}$$

The Jacobian written in the base frame is:

$${}^{B}J(\Theta) = \begin{bmatrix} 0 & -S_2(S_3L_1 + C_{34}L_2) & C_2(C_3L_1 - S_{34}L_2) & -C_2S_{34}L_2 & 0 & 0 \\ 0 & 0 & \pm S_3L_1 \pm C_{34}L_2 & \pm C_{34}L_2 & 0 & 0 \\ 1 & \pm C_2(S_3L_1 + C_{34}L_2) & \pm S_2(C_3L_1 - S_{34}L_2) & \mp S_2S_{34}L_2 & 0 & 0 \\ 0 & 0 & -S_2 & -S_2 & \pm C_2C_{34} & \pm C_2S_{34}S_5 - S_2C_5 \\ 0 & \mp 1 & 0 & 0 & S_{34} & -C_{34}S_5 \\ 0 & 0 & \pm C_2 & \pm C_2 & S_2C_{34} & \pm C_2C_5 + S_2S_{34}S_5 \end{bmatrix} \tag{2.19}$$

As before, when either $\pm$ or $\mp$ is indicated, the top sign refers to the GOLD C-Arm and the bottom sign refers to the GREEN C-Arm.

The relationship between the Jacobian, the angular and linear velocities of the wrist or end effector, and the velocity of each joint is as follows:

$${}^{B}\nu = {}^{B}J(\Theta)\,\dot{\Theta} \tag{2.20}$$

2.2.5 Singular Poses

A manipulator is in a singular pose, or singularity, when its end effector is unable to move independently in one or more directions in Cartesian space. In terms of the Jacobian, the manipulator is in a singular pose when linear combinations of the Jacobian's columns fail to produce a desired set of end effector linear and angular velocities. When this occurs, the Jacobian is said to have lost rank. The determinant of a matrix is zero if and only if the matrix has lost rank [65]. Singular poses can therefore be identified using the determinant of the Jacobian:

$$det\left\{{}^{B}J\left(\Theta\right)\right\} = L_1\left(\pm S_2 S_3^2 S_5 L_1 + C_{34}\left(C_2 C_4 C_5 \pm S_2 S_3 S_5\right) L_2\right) \tag{2.21}$$

The determinant of the Jacobian is zero, and the C-Arm is in a singular pose, when any one of the following conditions occurs:

- $C_2 = 0$ and $S_3 = 0$
- $S_2 = 0$ and $C_4 = 0$
- $S_2 = 0$ and $C_{34} = 0$
- $S_2 = 0$ and $C_5 = 0$
- $C_2 = 0$ and $S_5 = 0$
- $S_3 = 0$ and $C_4 = 0$
- $S_3 = 0$ and $C_5 = 0$
- $C_4 = 0$ and $S_5 = 0$
- $C_{34} = 0$ and $S_5 = 0$
- $C_2 = 0$, $C_3 = 0$, and $S_4 = \dfrac{L_1}{L_2}$
- $C_3 = 0$, $S_4 = \dfrac{L_1}{L_2}$, and $C_5 = 0$
- $S_2 = 0$, $C_3 = 0$, and $S_4 = 0$

- $C_3 = 0$, $S_4 = 0$, and $S_5 = 0$
- $C_2 = 0$, $C_4 = 1$, and $(S_3L_1 + C_3L_2) = 0$
- $C_4 = 1$, $C_5 = 0$, and $(S_3L_1 + C_3L_2) = 0$
- $C_2 = 0$, $C_4 = -1$, and $(S_3L_1 - C_3L_2) = 0$
- $C_4 = -1$, $C_5 = 0$, and $(S_3L_1 - C_3L_2) = 0$

The equation $(S_3L_1 \pm C_3L_2) = 0$ has two solutions: $\theta_3 = \operatorname{atan2}(\pm L_2, -L_1)$ and $\theta_3 = \operatorname{atan2}(\mp L_2, L_1)$.

Several methods exist to deal with singular poses. In many cases, it may be feasible to simply avoid the singular poses altogether. When singularity avoidance is not feasible, approaches such as the damped least-squares inverse method [52], or the method presented by Maneewarn [44] can be used. Maneewarn's method determines the directions around a singularity in which the manipulator cannot move, and then restricts motion in those directions as the manipulator passes through the singularity.

2.2.6 C-Arm Dynamics

The dynamics of the C-Arms were modeled via the Lagrangian formulation as presented by Yoshikawa for a general n-link manipulator [72]. The mass properties of the C-Arms are presented, followed by an illustration of the method used to determine the dynamic model.

C-Arm Mass Properties

Mass properties for the C-Arms were measured from the SolidWorks model using the Mass Properties tool. Relevant mass properties of each link i include the mass (m_i, in kg), the location of the center of mass relative to the link frame (COM_i, in mm), and the Mass Moment of Inertia matrix centered at the origin of the link frame and aligned with the link frame axes (I_i, in kg·mm^2). Table 2.3 contains the measured mass properties.

Table 2.3: Mass Properties for the C-Arm, as measured from the SolidWorks model. When either $\pm$ or $\mp$ is indicated, the top sign refers to the GOLD C-Arm and the bottom sign refers to the GREEN C-Arm.

Link	Mass (kg)	Center of Mass (mm)	Mass Moment of Inertia Matrix (kg mm^2)
Link 1	$m_1 = 11.96$	$COM_1 = \begin{pmatrix} \mp 48.9 \\ 168.7 \\ -0.35 \end{pmatrix}$	$I_1 = \begin{pmatrix} 416488 & \pm 87026 & \mp 99 \\ \pm 87026 & 118753 & -1029 \\ \mp 99 & -1029 & 477372 \end{pmatrix}$
Link 2	$m_2 = 6.74$	$COM_2 = \begin{pmatrix} 0 \\ -24.4 \\ -18.5 \end{pmatrix}$	$I_2 = \begin{pmatrix} 32534 & \mp 0.4 & \pm 2.8 \\ \mp 0.4 & 27790 & -124 \\ \pm 2.8 & -124 & 21409 \end{pmatrix}$
Link 3	$m_3 = 6.92$	$COM_3 = \begin{pmatrix} \pm 309.1 \\ -9.9 \\ -24.2 \end{pmatrix}$	$I_3 = \begin{pmatrix} 25647 & \mp 12214 & \mp 57919 \\ \mp 12214 & 739345 & 201 \\ \mp 57919 & 201 & 737133 \end{pmatrix}$
Link 4	$m_4 = 2.32$	$COM_4 = \begin{pmatrix} \mp 0.96 \\ 153.1 \\ 0.97 \end{pmatrix}$	$I_4 = \begin{pmatrix} 70283 & \mp 409 & \mp 11.8 \\ \mp 409 & 4644 & 153 \\ \mp 11.8 & 153 & 68649 \end{pmatrix}$
Link 5	$m_5 = 2.34$	$COM_5 = \begin{pmatrix} \mp 10.2 \\ 10.6 \\ -51.0 \end{pmatrix}$	$I_5 = \begin{pmatrix} 15021 & \mp 222 & \pm 1081 \\ \mp 222 & 15032 & 187 \\ \pm 1081 & 187 & 3922 \end{pmatrix}$
Link 6	$m_6 = 0.34$	$COM_6 = \begin{pmatrix} 0 \\ 77.4 \\ 0 \end{pmatrix}$	$I_6 = \begin{pmatrix} 2203 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 2203 \end{pmatrix}$

When used in equations, elements of the (symmetric) mass moment of inertia matrices will be referred to using the following convention:

$$I_i = \begin{bmatrix} I_{ixx} & I_{ixy} & I_{ixz} \\ I_{ixy} & I_{iyy} & I_{iyz} \\ I_{ixz} & I_{iyz} & I_{izz} \end{bmatrix} \tag{2.22}$$

Elements of the center of mass vectors will be referred to as follows:

$$COM_i = \begin{bmatrix} COM_{ix} \\ COM_{iy} \\ COM_{iz} \end{bmatrix} \tag{2.23}$$

Torque Equations

After the mass properties are known, the pseudo inertia matrices $\hat{H}_i$ can be calculated for each link.

$$\hat{H}_i = \begin{bmatrix} \frac{-I_{ixx}+I_{iyy}+I_{izz}}{2} & I_{ixy} & I_{ixz} & m_i COM_{ix} \\ I_{ixy} & \frac{I_{ixx}-I_{iyy}+I_{izz}}{2} & I_{iyz} & m_i COM_{iy} \\ I_{ixz} & I_{iyz} & \frac{I_{ixx}+I_{iyy}-I_{izz}}{2} & m_i COM_{iz} \\ m_i COM_{ix} & m_i COM_{iy} & m_i COM_{iz} & m_i \end{bmatrix} \tag{2.24}$$

Elements of the symmetric (6×6) inertia matrix $M(\Theta)$ can then be calculated:

$$M_{ij} = M_{ji} = \sum_{k=max(i,j)}^{n} tr\left(\frac{\partial\, {}^0_k T}{\partial\, \Theta_j} \hat{H}_k \frac{\partial \left({}^0_k T \right)^T}{\partial\, \Theta_i} \right) \tag{2.25}$$

Note that Θ_i refers to the i^{th} element of the vector Θ as defined in Equation (2.14) and that $tr()$ refers to the trace of a matrix.

$V(\Theta, \dot{\Theta})$ is a 6×1 vector of centrifugal and Coriolis terms, with the i^{th} element given by:

$$V_i = \sum_{j=1}^{n} \sum_{m=1}^{n} \sum_{k=max(i,j,m)}^{n} tr\left(\frac{\partial^2\, {}^0_k T}{\partial\, \Theta_j \partial\, \Theta_m} \hat{H}_k \frac{\partial \left({}^0_k T \right)^T}{\partial\, \Theta_i} \right) \dot{\Theta}_j \dot{\Theta}_m \tag{2.26}$$

Finally, $G(\Theta)$ is a 6×1 vector of gravity terms, with the i^{th} element given by:

$$G_i = -\sum_{j=1}^{n} m_j \tilde{g}^T \frac{\partial\, {}^0_j T}{\partial\, \Theta_i} COM_j \tag{2.27}$$

where $\tilde{g}$ is the gravitational acceleration vector. For the C-Arms, $\tilde{g}$ is defined (in mm/s^2) as:

$$\tilde{g} = \begin{bmatrix} 0 & 9810 & 0 & 0 \end{bmatrix}^T \tag{2.28}$$

The torque equation for the C-Arms can then be calculated:

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta) \tag{2.29}$$

The final calculated values of $M(\Theta)$, $V(\Theta, \dot{\Theta})$, $G(\Theta)$, and τ can be found in Appendix A.

Chapter 3

INVERSE KINEMATICS

While the chosen design for the C-Arm was well suited to our needs, it does not have a closed-form inverse kinematics solution. I therefore began to consider using one of the numerical inverse kinematics solutions that others have presented. By far the most prominent category of solutions relies on the use of the Newton-Raphson method [10], the Newton-Gauss method [2], or some other iterative method [6, 25, 75], along with some method of approximating the inverse of the Jacobian, such as the Moore-Penrose generalized pseudoinverse. Other solutions avoided use of the Jacobian, as its presence tended to result in instability, but still relied on iterative methods [1, 24, 66].

Two drawbacks are common to all of these methods. First, these methods only find one solution when many are possible. Second, these methods iterate over all of the manipulator DOFs, so convergence can be slow. Rather than using one of these methods, I chose to develop a method that required iteration over only one DOF. My numerical solution is accomplished as follows:

1. Choose a set of possible positions along the linear rail.

2. Calculate the angles of the remaining joints for each possible position.

3. Choose the solution(s) that most closely match the desired position.

4. Choose a new set of possible positions that are close to the solution(s) from Step 3.

5. Repeat Steps 2-4 until a solution with the desired accuracy is obtained.

This method is similar to the one presented by Manseur and Doty [45]. Unlike their method, which assumes a rotational Joint 1, my method works for any 6-DOF manipulator with known forward kinematics for which it is possible to solve for the remaining joint angles if one joint angle's value is known. For clarity, I first present the equations that provide solutions for the remaining joints assuming the value of $d_1(t)$ is known, and then discuss my method for determining the value of $d_1(t)$.

Several factors influenced my decision to iterate over the values of $d_1(t)$ and use those values to solve for the remaining joints. As a prismatic joint, $d_1(t)$ will only affect the position of the end effector, which will limit its presence to at most three terms of Equation (3.2). Because of the distribution of transformation matrices in Equation (3.2), $d_1(t)$ only appears in two terms. As one of those terms contained only one other joint variable, a solution for that joint variable was easy to obtain. The solutions for other joint variables followed easily. In addition, the geometry of the mechanism could easily be used to eliminate regions of the joint's operational space, as explained more fully in the second half of this section. Finally, divisions of a prismatic joint are more intuitive than divisions of a rotational joint. While not a deciding factor, ease of understanding was an important consideration.

3.1 *Determination of Joint Angles* $\theta_2(t)$ *through* $\theta_6(t)$

Using the homogeneous transforms between frames, as determined from the D-H parameters, we can solve for a desired pose as follows:

$$T_D = {}^{B}_{W}T \tag{3.1}$$

$$\left({}^{B}_{2}T\right)^{-1} T_D = {}^{2}_{W}T \tag{3.2}$$

where ${}^{B}_{2}T$ and ${}^{2}_{W}T$ are as defined in Equation (2.3) and T_D is the desired pose, defined as:

$$T_D = \begin{bmatrix} R_{11} & R_{12} & R_{13} & P_x \\ R_{21} & R_{22} & R_{23} & P_y \\ R_{31} & R_{32} & R_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.3}$$

Multiplying out the matrices in Equation (3.2) results in twelve non-trivial equations:

$$C_2 R_{11} \pm S_2 R_{31} = S_{34} C_5 C_6 + C_{34} S_6 \tag{3.4a}$$

$$C_2 R_{12} \pm S_2 R_{32} = -S_{34} C_5 S_6 + C_{34} C_6 \tag{3.4b}$$

$$C_2 R_{13} \pm S_2 R_{33} = \pm S_{34} S_5 \tag{3.4c}$$

$$C_2 P_x \pm S_2 \left(P_z - d_1\right) = S_3 L_1 + C_{34} L_2 \tag{3.4d}$$

$$-S_2 R_{11} \pm C_2 R_{31} = \mp S_5 C_6 \tag{3.4e}$$

$$-S_2 R_{12} \pm C_2 R_{32} = \pm S_5 S_6 \tag{3.4f}$$

$$-S_2 R_{13} \pm C_2 R_{33} = C_5 \tag{3.4g}$$

$$-S_2 P_x \pm C_2 \left(P_z - d_1\right) = 0 \tag{3.4h}$$

$$\mp R_{21} = C_{34} C_5 C_6 - S_{34} S_6 \tag{3.4i}$$

$$\mp R_{22} = -C_{34} C_5 S_6 - S_{34} C_6 \tag{3.4j}$$

$$\mp R_{23} = \pm C_{34} S_5 \tag{3.4k}$$

$$\mp P_y = C_3 L_1 - S_{34} L_2 \tag{3.4l}$$

If we know $d_1(t)$, we can use a subset of the Equations (3.4) to solve for the other five joint angles. Figure 3.1 shows the calculations used to solve for the joint angles. Using Equation (3.4h), we can solve for $\theta_2(t)$. Two solutions are obtained, and are referred to as θ_{2a} and θ_{2b}. As before, C_2 is shorthand for $\cos(\theta_2)$, S_{34} is shorthand for $\sin(\theta_3 + \theta_4)$, and so on.

$$\theta_{2a} = \operatorname{atan2}\left(\mp P_z \pm d_1, -P_x\right) \tag{3.5a}$$

$$\theta_{2b} = \operatorname{atan2}\left(\pm P_z \mp d_1, P_x\right) \tag{3.5b}$$

Once θ_2 is known, we can use Equation (3.4g) to solve for C_5. The value of C_5 is useful because it tells us something about the value of S_5. Specifically, we are interested in determining whether $S_5 = 0$. This corresponds to $C_5 = \pm 1$. One set of equations, presented in Section 3.1.1, can be used to solve for the remaining joints if S_5 is nonzero. An alternate set of equations is presented in Section 3.1.2 for the case where $S_5 = 0$.

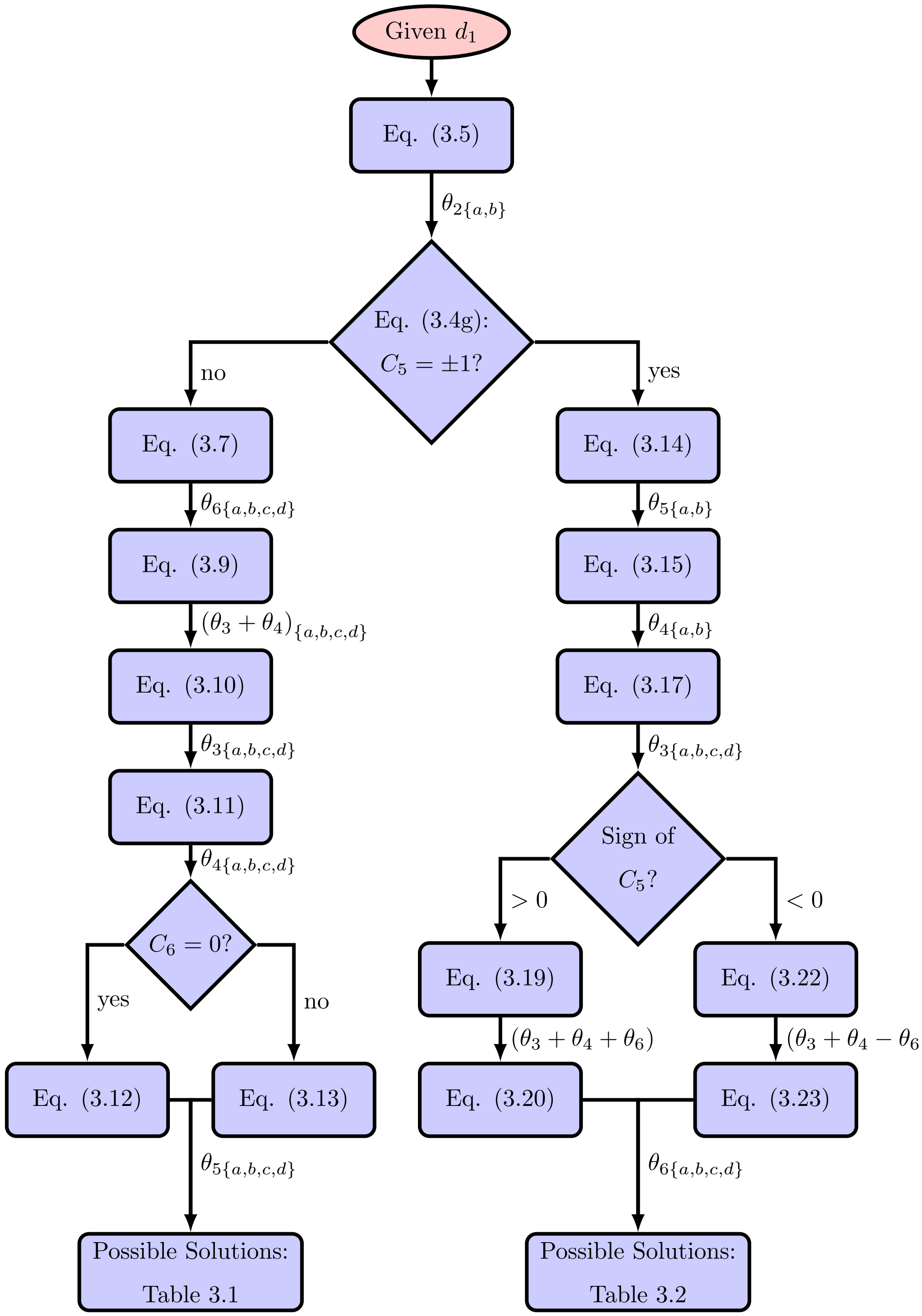

Figure 3.1: Flowchart of the calculations performed to solve for joint angles θ_2 through θ_6 given the value of d_1.

3.1.1 *Determination of Joint Angles $\theta_3(t)$ through $\theta_6(t)$ if $S_5 \neq 0$*

Once we know that $S_5 \neq 0$, Equations (3.4e) and (3.4f) can be used to solve for θ_6. Solving the two equations for either the sine or cosine of θ_6 gives:

$$C_6 = \frac{1}{S_5}\left(\pm S_2 R_{11} - C_2 R_{31}\right) \tag{3.6a}$$

$$S_6 = \frac{1}{S_5}\left(\mp S_2 R_{12} + C_2 R_{32}\right) \tag{3.6b}$$

These two equations can be used to solve for θ_6 using the atan2 function. We know that $S_5 \neq 0$, so replacing the $\frac{1}{S_5}$ term from each equation with $sign\left(S_5\right)$ is allowed. As we do not yet know the sign of S_5, we will solve for θ_6 using both signs. Later, the erroneous solutions will be eliminated due to either violations of the joint angle constraints or large values of error. The two equations for θ_6, designated θ_{6p} for positive values of S_5 and θ_{6n} for negative values of S_5, are as follows:

$$\theta_{6p} = \operatorname{atan2}\left(\mp S_2 R_{12} + C_2 R_{32}, \pm S_2 R_{11} - C_2 R_{31}\right) \tag{3.7a}$$

$$\theta_{6n} = \operatorname{atan2}\left(\pm S_2 R_{12} - C_2 R_{32}, \mp S_2 R_{11} + C_2 R_{31}\right) \tag{3.7b}$$

Note that we have a total of four values for θ_6:

- $\theta_{6a} = \theta_{6p}\left(\theta_{2a}\right)$,
- $\theta_{6b} = \theta_{6p}\left(\theta_{2b}\right)$,
- $\theta_{6c} = \theta_{6n}\left(\theta_{2a}\right)$, and
- $\theta_{6d} = \theta_{6n}\left(\theta_{2b}\right)$

The summation $(\theta_3 + \theta_4)$ can be calculated from Equations (3.4c) and (3.4k). Solving the two equations for either the sine or cosine of $(\theta_3 + \theta_4)$ gives:

$$C_{34} = \frac{1}{S_5}\left(-R_{23}\right) \tag{3.8a}$$

$$S_{34} = \frac{1}{S_5}\left(\pm C_2 R_{13} + S_2 R_{33}\right) \tag{3.8b}$$

As with θ_6, these two equations can be used to solve for the summation $(\theta_3 + \theta_4)$ using the atan2 function. Since we know that $S_5 \neq 0$ but do not yet its sign, we will solve for $(\theta_3 + \theta_4)$ using both signs, and later eliminate the erroneous solutions. The two equations for $(\theta_3 + \theta_4)$, designated θ_{34p} for positive values of S_5 and θ_{34n} for negative values of S_5, are as follows:

$$\theta_{34p} = \text{atan2}\left(\pm C_2 R_{13} + S_2 R_{33}, -R_{23}\right) \tag{3.9a}$$

$$\theta_{34n} = \text{atan2}\left(\mp C_2 R_{13} - S_2 R_{33}, R_{23}\right) \tag{3.9b}$$

Note that we have a total of four values for θ_6:

- $\theta_{34a} = \theta_{34p}\left(\theta_{2a}\right)$,
- $\theta_{34b} = \theta_{34p}\left(\theta_{2b}\right)$,
- $\theta_{34c} = \theta_{34n}\left(\theta_{2a}\right)$, and
- $\theta_{34d} = \theta_{34n}\left(\theta_{2b}\right)$

Using Equations (3.4d) and (3.4l), we can then solve for θ_3. As before, we solve the equations for S_3 and C_3 respectively, and then use the atan2 function to solve for θ_3:

$$\theta_3 = \text{atan2}\left(C_2 P_x \pm S_2\left(P_z - d_1\right) - C_{34} L_2, S_{34} L_2 \mp P_y\right) \tag{3.10}$$

Note that we have a total of four values for θ_3:

- $\theta_{3a} = \theta_3\left(\theta_{2a}, \theta_{34a}\right)$,
- $\theta_{3b} = \theta_3\left(\theta_{2b}, \theta_{34b}\right)$,
- $\theta_{3c} = \theta_3\left(\theta_{2a}, \theta_{34c}\right)$, and
- $\theta_{3d} = \theta_3\left(\theta_{2b}, \theta_{34d}\right)$

The value of θ_4 is then the difference between corresponding values of $(\theta_3 + \theta_4)$ and θ_3:

$$\theta_{4i} = \theta_{34i} - \theta_{3i} \quad \text{for} \quad i = \{a, b, c, d\} \tag{3.11}$$

The final joint angle we need to find is θ_5. From Equation (3.4g), we know C_5, and we can find S_5 using either Equation (3.4e) or (3.4f). We choose between the two equations for S_5 based on the value of C_6. We use Equation (3.4f) if $C_6 = 0$, since we know $S_6 \neq 0$. The equation for θ_5 derived from Equations (3.4f) and (3.4g) is:

$$\theta_5 = \text{atan2}\left(\frac{1}{S_6}\left(\mp S_2 R_{12} + C_2 R_{32}\right), -S_2 R_{13} \pm C_2 R_{33}\right) \tag{3.12}$$

If $C_6 \neq 0$, we use Equation (3.4e). The equation for θ_5 derived from Equations (3.4e) and (3.4g) is:

$$\theta_5 = \text{atan2}\left(\frac{1}{C_6}\left(\pm S_2 R_{11} - C_2 R_{31}\right), -S_2 R_{13} \pm C_2 R_{33}\right) \tag{3.13}$$

Note that we have a total of four values for θ_5:

- $\theta_{5a} = \theta_5\left(\theta_{2a}, \theta_{6a}\right)$,
- $\theta_{5b} = \theta_5\left(\theta_{2b}, \theta_{6b}\right)$,
- $\theta_{5c} = \theta_5\left(\theta_{2a}, \theta_{6c}\right)$, and
- $\theta_{5d} = \theta_5\left(\theta_{2b}, \theta_{6d}\right)$

The four possible solutions for joint angles θ_2 through θ_6 are shown in Table 3.1.

Table 3.1: Possible solutions for joint angles θ_2 through θ_6 if $S_5 \neq 0$.

Solution Number	$\boldsymbol{\theta_2}$	$\boldsymbol{\theta_3}$	$\boldsymbol{\theta_4}$	$\boldsymbol{\theta_5}$	$\boldsymbol{\theta_6}$
1	θ_{2a}	θ_{3a}	θ_{4a}	θ_{5a}	θ_{6a}
2	θ_{2b}	θ_{3b}	θ_{4b}	θ_{5b}	θ_{6b}
3	θ_{2a}	θ_{3c}	θ_{4c}	θ_{5c}	θ_{6c}
4	θ_{2b}	θ_{3d}	θ_{4d}	θ_{5d}	θ_{6d}

3.1.2 Determination of Joint Angles $\theta_3(t)$ through $\theta_6(t)$ if $S_5 = 0$

When $S_5 = 0$, Equation (3.4g) can be used to solve for θ_5:

$$\theta_5 = \arccos\left(-S_2 R_{13} \pm C_2 R_{33}\right) \tag{3.14}$$

A value of either 0 or π radians is possible for each of the two values of θ_5:

- $\theta_{5a} = \theta_5\left(\theta_{2a}\right)$ and
- $\theta_{5b} = \theta_5\left(\theta_{2b}\right)$

When $S_5 = 0$, the joint axes for J3, J4, and J6 are parallel (or anti-parallel, if $\theta_5 = \pi$). In such a case, it becomes necessary to determine the value of one joint angle geometrically. Figure 3.2 shows a top view of one such pose for the GOLD C-Arm (with an operating table included for reference). The C-Arm is in the same pose in Figure 3.3, but viewed from an angle such that the axes for Joints 3, 4, and 6 are pointing into the page. Note that in these Figures, $\theta_5 = 0$. If θ_5 were set to π, the Joint 6 axis would point out of the page.

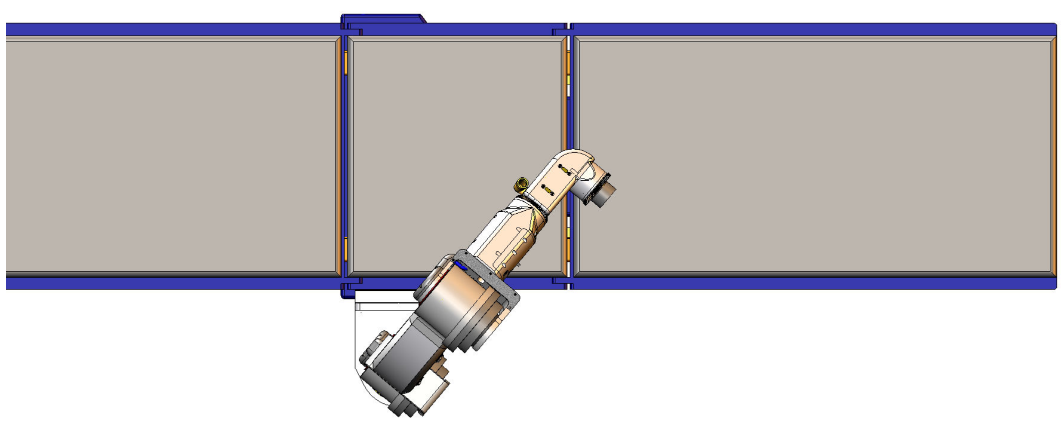

Figure 3.2: CAD rendering of the GOLD C-Arm over an operating table, with $S_5 = 0$.

Figure 3.4 shows the triangle formed by the C-Arm. The lower left corner is coincident with the origin of Frame $\{3\}$, and is located at the position $(0, 0, d_1)$ in base frame coordinates. The lower right corner is coincident with the Wrist Frame's origin, and is located

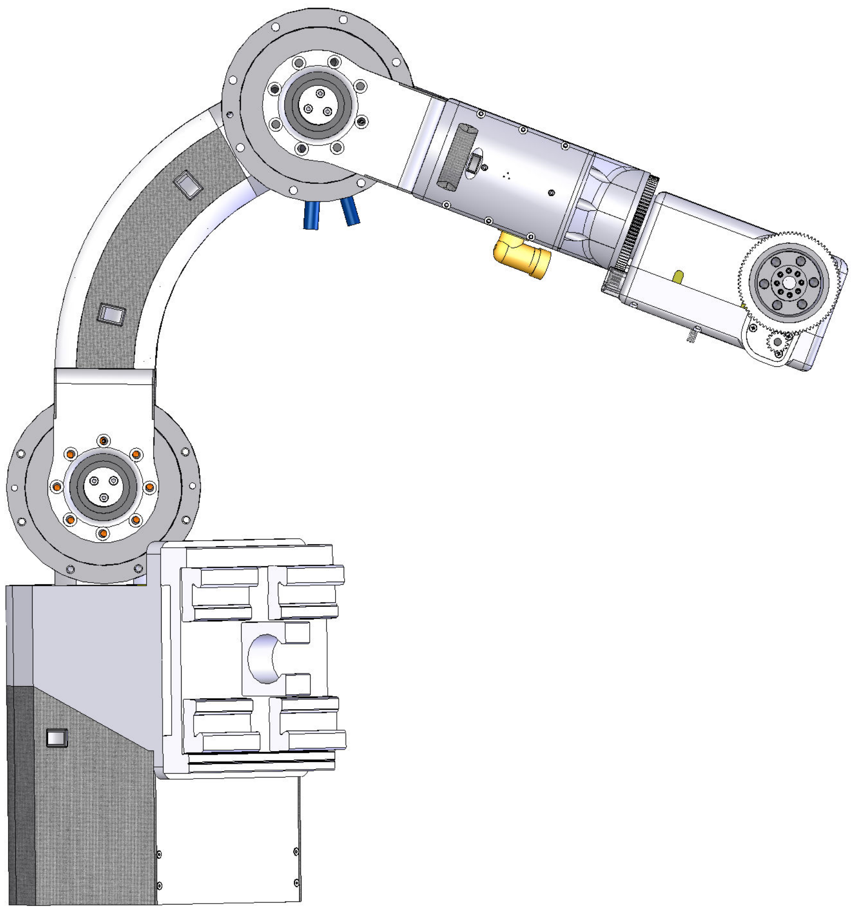

Figure 3.3: CAD rendering of the GOLD C-Arm with $S_5 = 0$, viewed along the J3, J4, and J6 axes.

at the position (P_x, P_y, P_z) in base frame coordinates. The top point of the triangle is coincident with the origin of Frame $\{4\}$. While the coordinates of Frame $\{4\}$'s origin are unknown, we do know the lengths of the C-Arm links. Based on the defined zero position of the arm and the direction of positive rotation for θ_4, we also know the measurement of one angle in terms of θ_4.

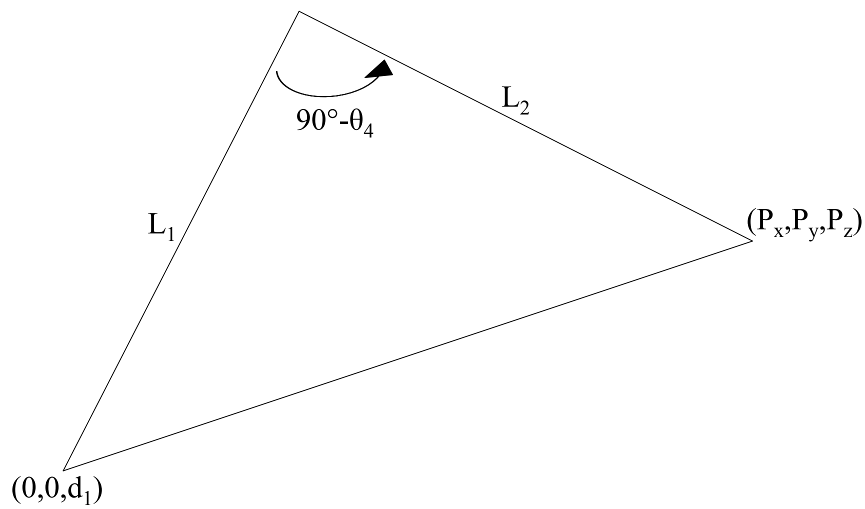

Figure 3.4: Triangle formed by the origins of frames $\{3\}$, $\{4\}$, and $\{6\}$.

Using the Law of Cosines, we can find an equation for $\cos(90° - \theta_4)\,(= S_4)$ in terms of known quantities. Two solutions must be included, since the point representing the axis of Joint 4 may be on either side of the line connecting the origins of Frames $\{3\}$ and $\{6\}$.

$$\theta_{4a} = \arcsin\left(\frac{L_1^2 + L_2^2 - \left|P_x^2 + P_y^2 + (P_z - d_1)^2\right|}{2L_1L_2}\right) \tag{3.15a}$$

$$\theta_{4b} = \pi - \arcsin\left(\frac{L_1^2 + L_2^2 - \left|P_x^2 + P_y^2 + (P_z - d_1)^2\right|}{2L_1L_2}\right) \tag{3.15b}$$

Once θ_4 is known, Equations (3.4d) and (3.4l) can be used to solve for θ_3. Equations (3.4d) and (3.4l) are equivalent to:

$$(L_1 - S_4 L_2) C_3 - (C_4 L_2) S_3 = \mp P_y \tag{3.16a}$$

$$(L_1 - S_4 L_2) C_3 + (C_4 L_2) S_3 = C_2 P_x \pm S_2 (P_z - d_1) \tag{3.16b}$$

The atan2 function is then used to find θ_3:

$$\begin{aligned}\theta_3 = \operatorname{atan2}\big(&(L_1 - S_4 L_2)(C_2 P_x \pm S_2 (P_z - d_1)) \pm C_4 L_2 P_y,\\ &\mp (L_1 - S_4 L_2) P_y + C_4 L_2 (C_2 P_x \pm S_2 (P_z - d_1))\big)\end{aligned} \tag{3.17}$$

Note that we have a total of four values for θ_3:

- $\theta_{3a} = \theta_3 (\theta_{2a}, \theta_{4a})$,
- $\theta_{3b} = \theta_3 (\theta_{2b}, \theta_{4a})$,
- $\theta_{3c} = \theta_3 (\theta_{2a}, \theta_{4b})$, and
- $\theta_{3d} = \theta_3 (\theta_{2b}, \theta_{4b})$

Finally, we can solve for θ_6 using Equations (3.4i) and (3.4j). If $C_5 = +1$, Equations (3.4i) and (3.4j) simplify to:

$$\mp R_{21} = \cos (\theta_3 + \theta_4 + \theta_6) \tag{3.18a}$$

$$\mp R_{22} = -\sin (\theta_3 + \theta_4 + \theta_6) \tag{3.18b}$$

The atan2 function is then used to find $(\theta_3 + \theta_4 + \theta_6)$:

$$(\theta_3 + \theta_4 + \theta_6) = \operatorname{atan2} (\pm R_{22}, \mp R_{21}) \tag{3.19}$$

Note that one unique value is obtained for the summation of the three joint values. The value of θ_6 is then obtained by subtracting the values of θ_3 and θ_4 from the summation:

$$\theta_6 = (\theta_3 + \theta_4 + \theta_6) - \theta_3 - \theta_4 \tag{3.20}$$

Note that we have a total of four values for θ_6:

- $\theta_{6a} = \theta_6\left(\theta_{3a}, \theta_{4a}, (\theta_3 + \theta_4 + \theta_6)\right)$,
- $\theta_{6b} = \theta_6\left(\theta_{3b}, \theta_{4a}, (\theta_3 + \theta_4 + \theta_6)\right)$,
- $\theta_{6c} = \theta_6\left(\theta_{3c}, \theta_{4b}, (\theta_3 + \theta_4 + \theta_6)\right)$, and
- $\theta_{6d} = \theta_6\left(\theta_{3d}, \theta_{4b}, (\theta_3 + \theta_4 + \theta_6)\right)$

If $C_5 = -1$, Equations (3.4i) and (3.4j) simplify to:

$$\mp R_{21} = -\cos\left(\theta_3 + \theta_4 - \theta_6\right) \tag{3.21a}$$

$$\mp R_{22} = -\sin\left(\theta_3 + \theta_4 - \theta_6\right) \tag{3.21b}$$

The atan2 function is then used to find $(\theta_3 + \theta_4 - \theta_6)$:

$$(\theta_3 + \theta_4 - \theta_6) = \text{atan2}\left(\pm R_{22}, \pm R_{21}\right) \tag{3.22}$$

Again, one unique value is obtained. The value of θ_6 is then obtained by combining Equations (3.15), (3.17), and (3.22) appropriately:

$$\theta_6 = \theta_3 + \theta_4 - (\theta_3 + \theta_4 - \theta_6) \tag{3.23}$$

As before, we have a total of four values for θ_6:

- $\theta_{6a} = \theta_6\left(\theta_{3a}, \theta_{4a}, (\theta_3 + \theta_4 - \theta_6)\right)$,
- $\theta_{6b} = \theta_6\left(\theta_{3b}, \theta_{4a}, (\theta_3 + \theta_4 - \theta_6)\right)$,
- $\theta_{6c} = \theta_6\left(\theta_{3c}, \theta_{4b}, (\theta_3 + \theta_4 - \theta_6)\right)$, and
- $\theta_{6d} = \theta_6\left(\theta_{3d}, \theta_{4b}, (\theta_3 + \theta_4 - \theta_6)\right)$

The four possible solutions for joint angles θ_2 through θ_6 are shown in Table 3.2.

Table 3.2: Possible solutions for joint angles θ_2 through θ_6 if $S_5 = 0$.

Solution Number	$\boldsymbol{\theta_2}$	$\boldsymbol{\theta_3}$	$\boldsymbol{\theta_4}$	$\boldsymbol{\theta_5}$	$\boldsymbol{\theta_6}$
1	θ_{2a}	θ_{3a}	θ_{4a}	θ_{5a}	θ_{6a}
2	θ_{2b}	θ_{3b}	θ_{4a}	θ_{5b}	θ_{6b}
3	θ_{2a}	θ_{3c}	θ_{4b}	θ_{5a}	θ_{6c}
4	θ_{2b}	θ_{3d}	θ_{4b}	θ_{5b}	θ_{6d}

3.1.3 Final Solution(s) for Joint Angles θ_2 Through θ_6

For any given desired pose and value of d_1, four possible solutions are found for the remaining five joint angles. The four solutions are not equally good, however. It is likely that at least one of the possible solutions violates the C-Arm's joint limit restrictions. Additionally, some of the solutions may do a poor job of achieving the desired pose. Solutions that perform poorly or are impossible to achieve need to be eliminated from consideration. Figure 3.5 outlines the steps for choosing amongst the solutions.

Joint Limits

Joint limits for the C-Arms are discussed in Section 2.2.1. If a solution requires one or more joints to position itself outside of these limits, the solution is eliminated from consideration. The forward kinematics, presented in Section 2.2.2, and the wrist pose calculations, presented in Section 2.2.3, are then used to find the C-Arm's wrist pose resulting from each remaining solution.

Calculate Error

Each resulting wrist pose is then compared to the desired wrist pose. Due to the nature of angle measurements, it is occasionally possible to report errors in orientation that are inaccurately large. For instance, if $\alpha_D = -3.124$ rad $(= -179°)$ and $\alpha = +3.124$ rad $(= +179°)$, the absolute error $|\alpha - \alpha_D|$ would be reported as 6.248 rad $(358°)$. In reality, the error

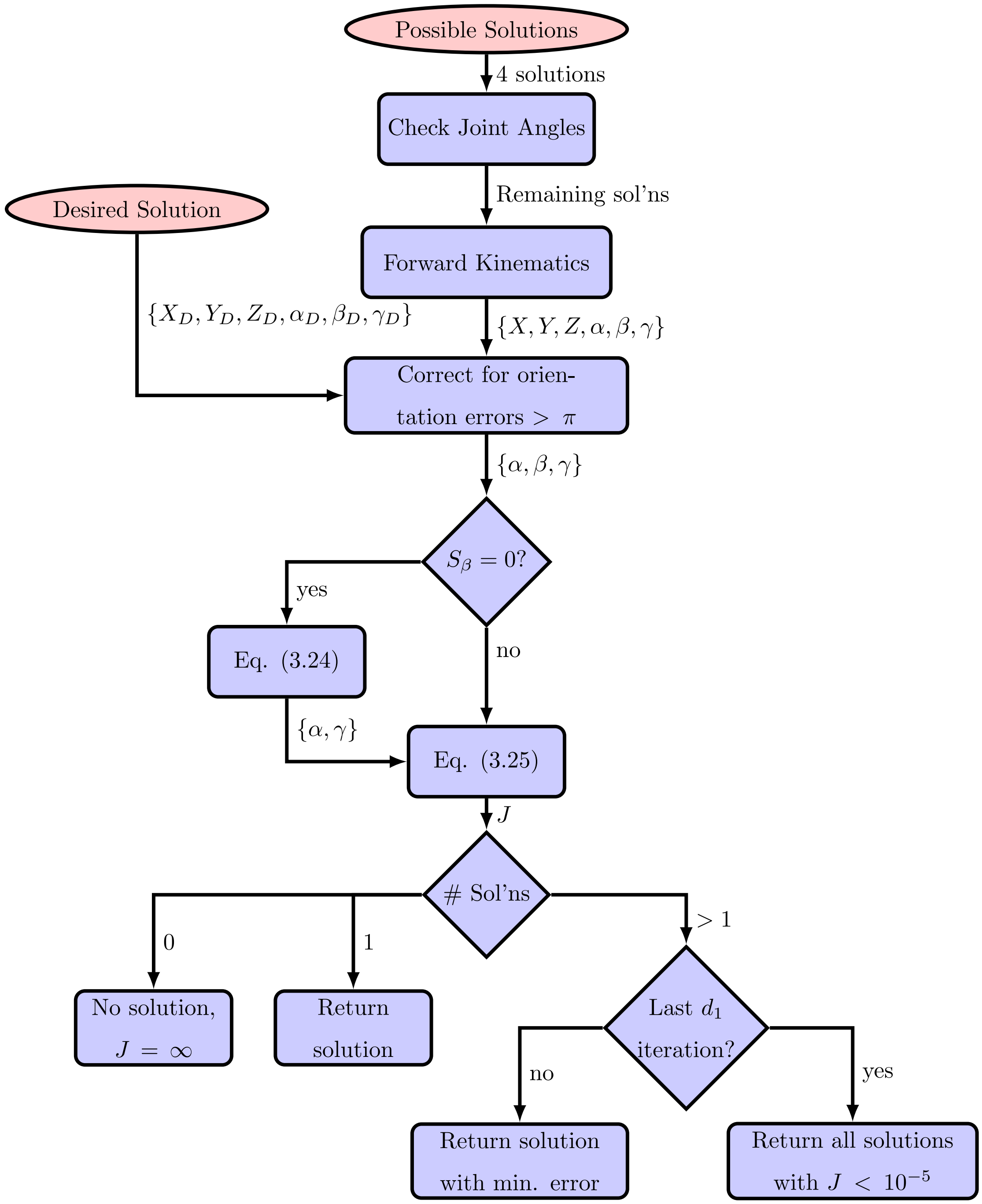

Figure 3.5: Flowchart showing the steps performed to choose between possible solutions for joint angles θ_2 through θ_6 given the value of d_1.

should be reported as 0.035 rad (2°). To correct for this error, the calculated wrist orientation values are compared to the desired wrist orientation values. If the absolute error between any of the corresponding values is greater than π rad, the calculated orientation values are offset by 2π in the appropriate direction.

A second type of inaccurate orientation error is also possible. If $\sin(\beta) = 0$, the axes about which rotations α and γ occur are either parallel (if $\beta = 0$) or antiparallel (if $\beta = \pi$). In this case, we are actually concerned with the error in either the sum or difference of the two angles. For our error calculations, we use α_{new} and γ_{new} in place of α and γ.

$$\alpha_{new} = \alpha_D \tag{3.24a}$$

$$\gamma_{new} = \gamma - \text{sign}\left(C_\beta\right)\left(\alpha - \alpha_D\right) \tag{3.24b}$$

Once all forms of inaccurate orientation error are eliminated, the solution error can be calculated. The total solution error J consists of two components: J_p, the position error; and J_o, the orientation error.

$$J_p = \sqrt{(X - X_D)^2 + (Y - Y_D)^2 + (Z - Z_D)^2} \tag{3.25a}$$

$$J_o = \sqrt{(\alpha - \alpha_D)^2 + (\beta - \beta_D)^2 + (\gamma - \gamma_D)^2} \tag{3.25b}$$

$$J = \sqrt{J_p^2 + 10^4 J_o^2} \tag{3.25c}$$

The total solution error J is calculated using a weighted least squares approach to minimize the error in position and orientation. Error in orientation (in radians) is weighted 10^4 times greater than error in position to obtain error values of similar magnitude. A weight of $10^4\left(= 100^2\right)$ makes physical sense for this problem if you consider angles and lengths to be related by a radius of 100mm.

Choose Best Solution(s)

After eliminating solutions that violate the joint limit constraints, between zero and four possible solutions remain. If no solutions remain, an error of $J = \infty$ is returned. If one solution remains, that solution and its corresponding error are returned. If more than one solution remains, a subset of the solutions is returned.

For all but the last iteration of d_1 values, the solution with the lowest total error is reported. If multiple solutions have the same minimum error, any one of those solutions may be reported. For the last iteration of d_1 values, all solutions with an error below some threshhold are reported. In practice, a threshhold of $J < 10^{-5}$ (equivalent to a position error of 10nm) was reasonable.

3.2 Numerical Solution for $d_1(t)$

The linear rails along which the C-Arms slide are 2000mm long with a positioning accuracy of 0.01mm. Rather than checking 200,000 possible joint positions, we opted to search along the entire rail with a 10mm accuracy, then search around the best points with a 1mm accuracy, and so on until we obtain the best result with a 0.01mm accuracy. Figure 3.6 outlines the steps for iterating through values of d_1 to find the inverse kinematics solution.

For any desired position and orientation, we have found that there exist at most four values of d_1 that could be considered "best." Our method finds all solutions, and we choose the most appropriate solution for the given situation. For trajectory generation, for instance, this would be the solution with joint angles most similar to the current joint angles. Assuming four local minima are identified, and searching ten points on either side of the "best" point for each iteration, we need to find solutions for only 441 points instead of the initial 200,000. We can reduce the number of points further with a simple geometric constraint. Specifically, we limit the values of d_1 considered to the range bounded by the points where the fully extended manipulator can reach the desired position.

$$(d_1)_{min,max} = P_z \pm \sqrt{(L_1 + L_2)^2 - P_x^2 - P_y^2} \tag{3.26}$$

In practice, we found that the best results were achieved when we computed values of d_1 down to a precision 0.001 mm (one-tenth the precision of our linear rails) and then rounded to the actual joint precision. This change added at most 80 more points to our calculation for a maximum of 521 points.

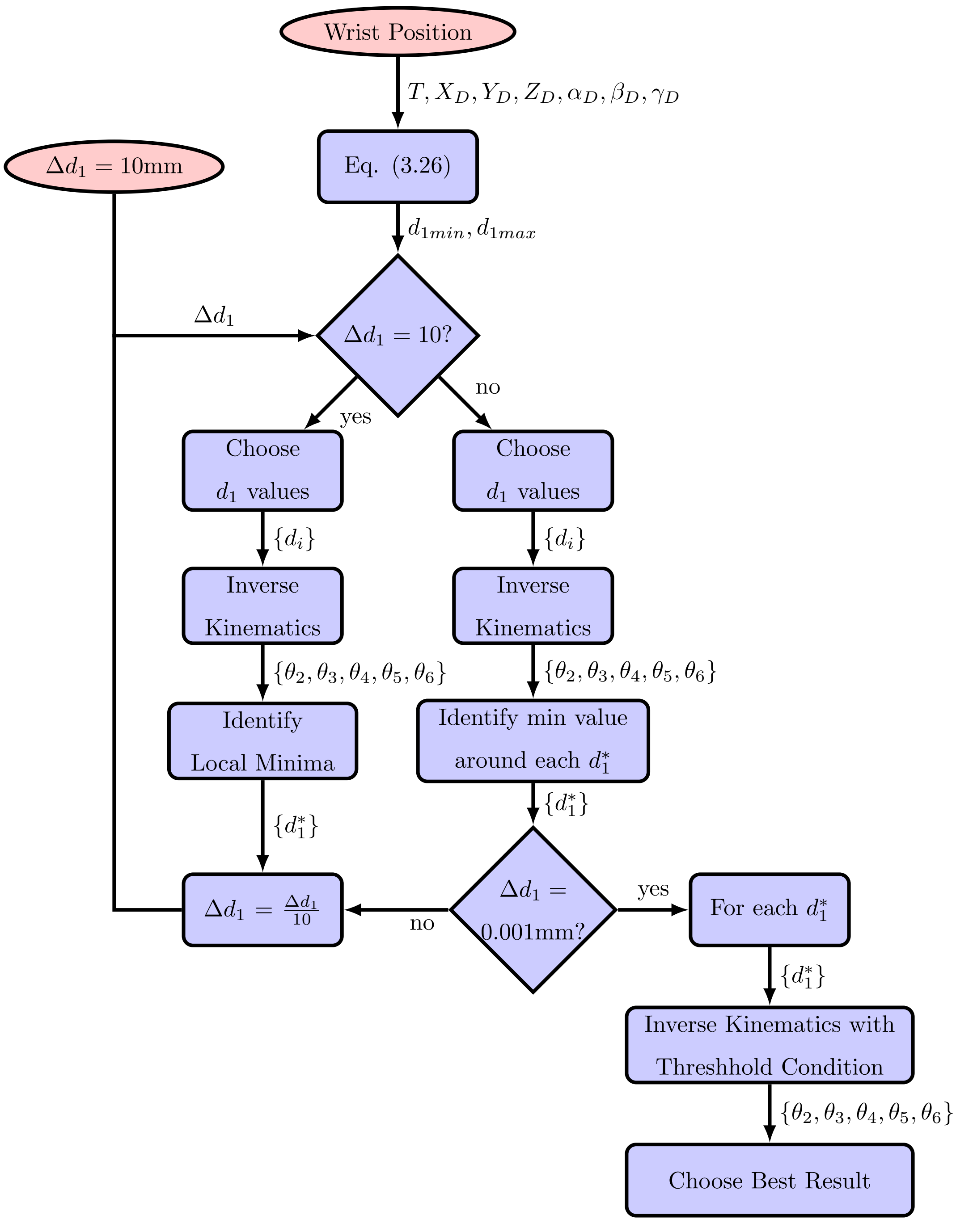

Figure 3.6: Flowchart showing the steps performed to iterate through values of d_1 to find the inverse kinematics solution.

In its current form, finding the inverse kinematic solution for one point takes between 0.06 and 0.1 seconds, depending on other processor demands and the number of final solutions. If necessary, calculation time can be decreased in several ways, some of which are easier to implement than others. Some examples are:

- Use C-code to find a solution rather than Matlab code.
- Implement a more efficient iterative method, such as Newton-Raphson, to find each local minima after the initial sweep along the linear rail.
- Identify the local minima closest to the current position after the initial sweep along the linear rail, and then iterate to the desired position at that local minima alone.
- Adapt the code to run on several parallel processors simultaneously.

In most cases, implementing at most one or two of these changes should be sufficient to obtain the desired speed-up.

3.3 Validation

When validating the inverse kinematics solution, equations for the GOLD C-Arm were used. The inverse kinematics solution was validated in two ways. First, a series of points within the C-Arm's workspace were chosen. For each point, the joint angles were calculated as described, and the error (calculated via Equation (3.25)) was noted. The results of interest were the maximum error and mean error, as well as the mean runtime for the inverse kinematics solution. Next, we chose a trajectory in Cartesian coordinates and calculated the inverse kinematics at points along the trajectory. As with the validation points, we are interested in maximum and mean error and mean runtime.

3.3.1 Validation Points

Validation points were chosen using joint coordinates, and then were converted to wrist coordinates using the forward kinematics and the wrist pose calculations (Sections 2.2.2

and 2.2.3 respectively). Choosing the validation points using joint coordinates ensured that the inverse kinematics solutions were valid over a large portion of the C-Arm's motion range. It also provided a known desired position against which to compare our results.

Between 3 and 11 possible values were chosen for each joint. The possible values are listed in Table 3.3. The inverse kinematics solution was then calculated for all combinations of the possible joint values except those that produced a singular pose where finding an acceptable solution was trivial. For example, an infinite number of solutions exist when $C_2 = 0$ and $S_5 = 0$, so any d_1 value within range produced a valid solution. Out of 16,500 possible combination, 600 were eliminated.

Table 3.3: Possible values for joints d_1 through θ_6 used to validate inverse kinematics solution.

Joint	**Possible Values**
d_1	500mm, 1000mm, 1500mm
θ_2	$-150°$, $-120°$, $-90°$, $-60°$, $-30°$, $0°$, $30°$, $60°$, $90°$, $120°$, $150°$
θ_3	$-60°$, $-30°$, $0°$, $30°$, $60°$
θ_4	$-60°$, $-30°$, $0°$, $30°$
θ_5	$-120°$, $-60°$, $0°$, $60°$, $120°$
θ_6	$-60°$, $-30°$, $0°$, $30°$, $60°$

For each remaining test point, the inverse kinematics solution was calculated. Between one and five solutions were found for each point (up to four values of d_1 were possible, and some values of d_1 had multiple solutions). In this instance, we chose the solution that most closely matched the original joint values. Figure 3.7 outlines the steps used to choose between solutions. Out of 15,900 points, 10,178 returned multiple solutions. 3,795 solutions were chosen based on their value for θ_2 and the remaining 6,383 solutions were chosen based on their value for θ_3.

Figure 3.8 illustrates the multiplicity of solutions. All but θ_2 was held constant ($d_1 =$ 500mm, $\theta_3 = 60°$, $\theta_4 = -60°$, $\theta_5 = 60°$, $\theta_6 = 0°$) and θ_2 was allowed to take each of the 11

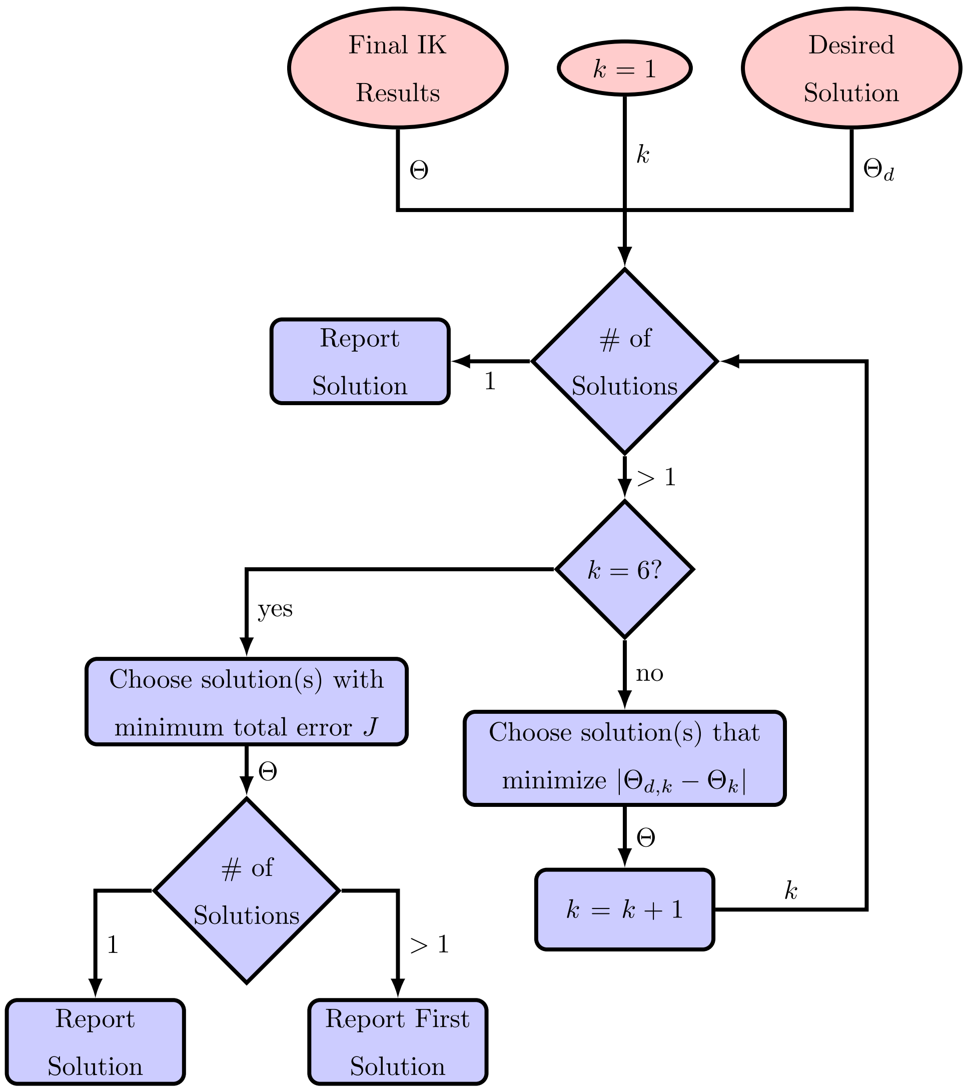

Figure 3.7: Flowchart showing the steps used to choose the inverse kinematics solution that most closely matched the validation point. Note that $\Theta = \{d_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6\}$ and Θ_k refers to the k^{th} element of Θ.

possible values listed in Table 3.3. The values of θ_2 that correspond to the desired solution are indicated by blue $\circ$'s. The value of θ_2 for all possible solutions are indicated by red $\times$'s. If more than one solution for a validation point shared the value of θ_2, the number of solutions sharing that value is indicated by a number in parentheses next to the $\times$. The desired solution was always found, as indicated by an $\times$ inside a circle. It is also very common to have an alternate solution for θ_2 that is 180° off from the desired solution.

A histogram of the position error J_p, the orientation error J_o, and the total error J calculated for all 15,900 points is shown in Figure 3.9. The error is divided into ten bins, with 15,840 of the 15,900 points having total error $0 \leq J \leq 3.49 \times 10^{-7}$. Of the remaining 60 points, 15 have total error $1.40 \times 10^{-6} \leq J \leq 1.75 \times 10^{-6}$, 25 have total error $2.09 \times 10^{-6} \leq J \leq 2.44 \times 10^{-6}$, and 20 have total error $3.14 \times 10^{-6} \leq J \leq 3.49 \times 10^{-6}$.

The maximum error and mean error are displayed in Table 3.4, along with the threshhold value under which 99.6% of the error values fall. Table 3.4 also includes length and angle measurements that correspond to position-only and orientation-only error. When the inverse kinematics equations are used to position the physical C-Arms, the error due to limitations in joint angle precision will far outweigh the algorithmic error shown here. On average, the inverse kinematics calculations took 0.098 seconds per point.

Table 3.4: Maximum error, mean error, and 99.6% error threshhold for 15,900 validation points.

Error Measurement	$\boldsymbol{J}$	**Equivalent Position-Only Error**	**Equivalent Orientation-Only Error**
J_{max}	3.49×10^{-6}	3.49nm	0.02 deg
J_{mean}	9.51×10^{-9}	9.51pm	5.45×10^{-5} deg
$J_{99.6\%}$	3.44×10^{-9}	3.44pm	1.97×10^{-5} deg

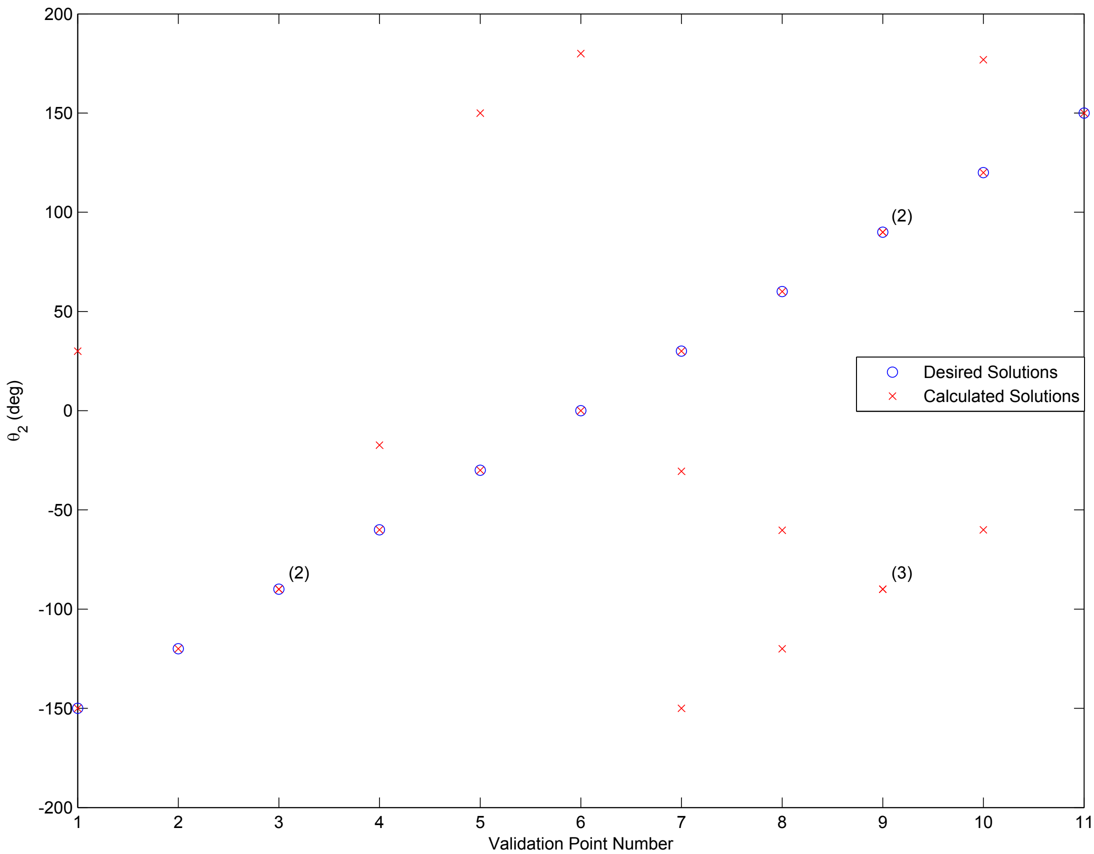

Figure 3.8: θ_2 solutions for a subset of the validation points. Validation points chosen have $d_1 = 500\text{mm}$, $\theta_3 = 60°$, $\theta_4 = -60°$, $\theta_5 = 60°$, $\theta_6 = 0°$. θ_2 was allowed to take each of the 11 possible values listed in Table 3.3. The desired value of θ_2 is marked by a blue $\circ$ for each validation point. The value of θ_2 for each possible solution is marked by a red $\times$. If multiple solutions exist with the same θ_2 value for any validation point, the multiplicity of that solution is indicated in parentheses next to the point.

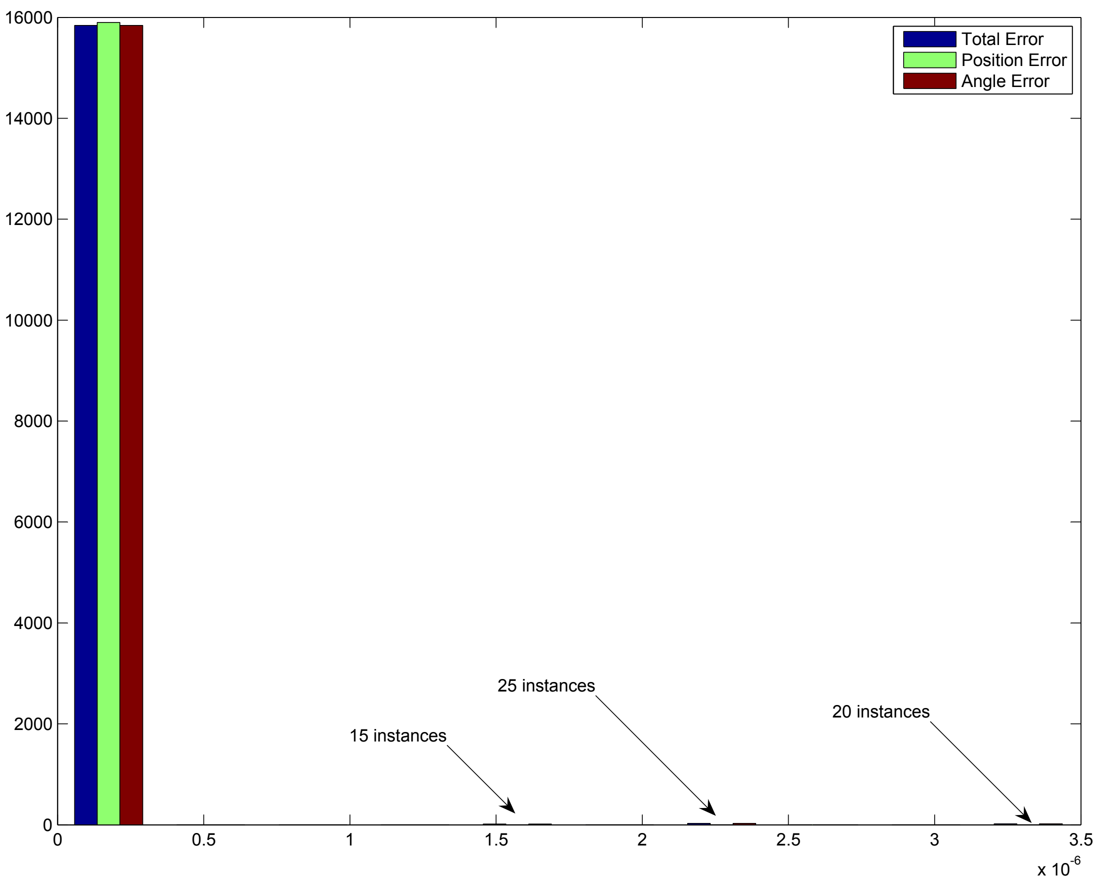

Figure 3.9: Histogram with 10 bins showing the error in inverse kinematics calculation of 15,900 validation points. 99.6% of points have a total error $J \leq 3.49 \times 10^{-7}$ and 100% of points have a total error $J \leq 3.49 \times 10^{-6}$. Orientation error is scaled by 10^4 so its values are comparable to position error and total error.

3.3.2 Trajectory Validation

After testing the validation points, a rectangle within the C-Arm's workspace was defined. A trajectory was chosen that moved between corners of the rectangle and ten points were calculated between subsequent corner points. The orientation was kept constant and was chosen to be the same as the orientation when the C-Arm is in the zero pose. Figure 3.10 shows the rectangle (red), the trajectory (blue), and the calculated points along the trajectory (black $\times$'s). Table 3.5 lists the eight points that define the corners of the rectangle and lists the order in which the points are visited in the trajectory.

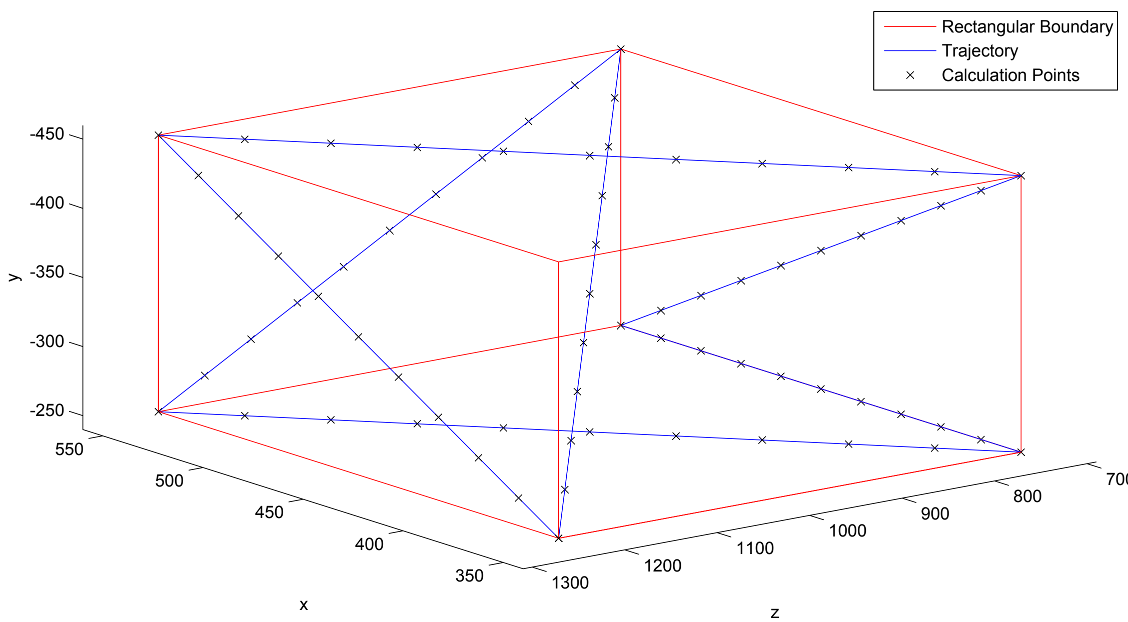

Figure 3.10: Validation trajectory, defined in wrist space and holding orientation constant. The blue line shows the trajectory of the arm starting and ending at $(350, -250, 1250)$. The black $\times$'s show the points calculated along the trajectory. The red rectangle is included for reference.

A histogram of the position error J_p, the orientation error J_o, and the total error J is shown in Figure 3.11. As in Figure 3.9, the error is divided into ten bins. The maximum error and mean error are displayed in Table 3.6, along with length and angle measurements that correspond to position-only and orientation-only error. As before, when the inverse

Table 3.5: Rectangle used to define a trajectory for validation. Coordinates are defined relative to the C-Arm's base frame.

X Coordinate (mm)	**Y Coordinate (mm)**	**Z Coordinate (mm)**	**Order in Trajectory**
350	-450	750	3
350	-450	1250	
350	-250	750	5
350	-250	1250	1, 8
550	-450	750	7
550	-450	1250	2
550	-250	750	4
550	-250	1250	6

kinematics equations are used to position the physical C-Arms, the error due to limitations in joint angle precision will far outweigh the algorithmic error shown here.

Figure 3.12 shows the calculated joint angles at each point along the trajectory. As expected, d_1 has a value of either 750mm or 1250mm at the corners of the rectangle, and changes at a constant rate when moving between values. As the orientation at all points is the same as the C-Arm's zero pose orientation, both θ_2 and θ_5 remain at 0° for the entire trajectory. The remaining joints stay well within their defined joint limits.

Table 3.6: Maximum and mean errors for the wrist space validation trajectory.

Error Measurement	**J**	**Equivalent Position-Only Error**	**Equivalent Orientation-Only Error**
J_{max}	1.27×10^{-13}	1.27×10^{-4}pm	7.28×10^{-10} deg
J_{mean}	3.21×10^{-14}	3.21×10^{-5}pm	1.84×10^{-10} deg

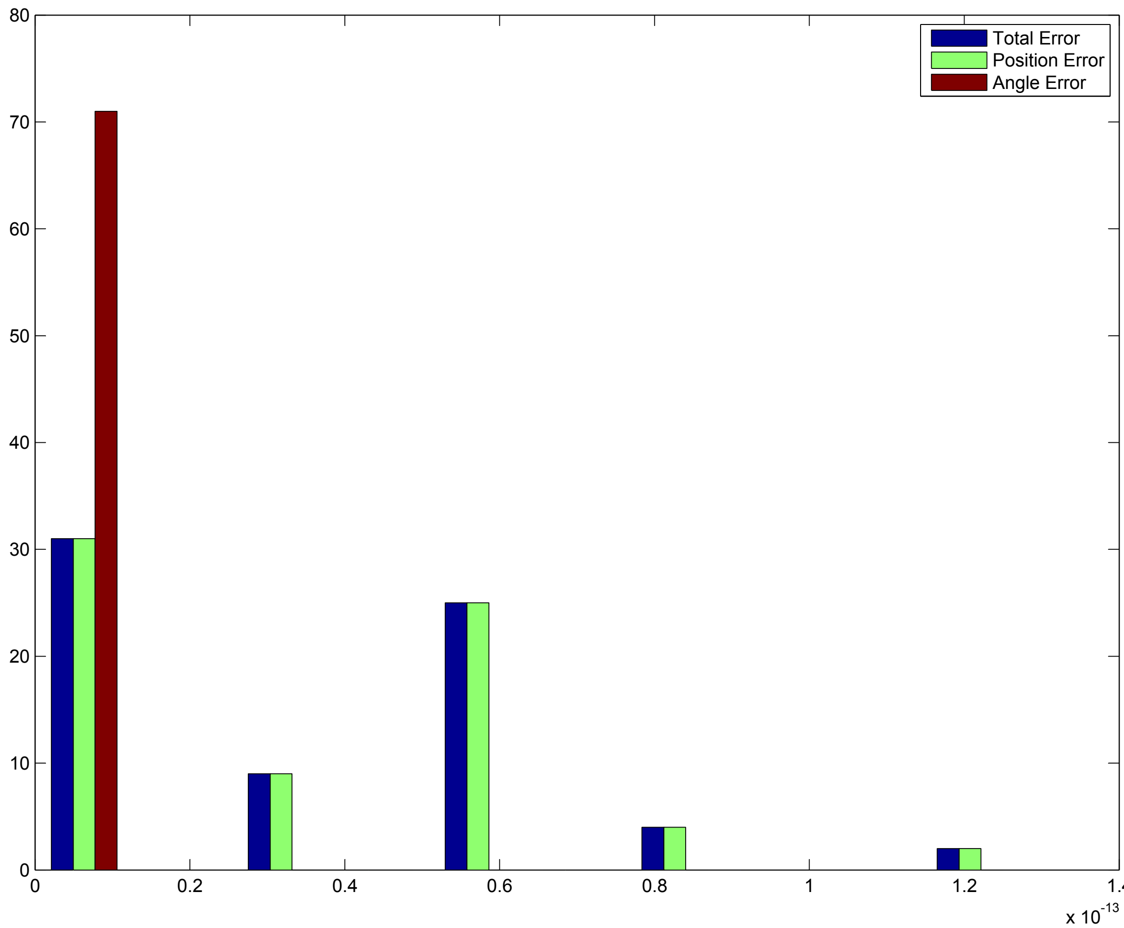

Figure 3.11: Histogram with 10 bins showing the error in inverse kinematics calculation of the wrist space validation trajectory. Maximum total error is $J = 1.27 \times 10^{-13}$.

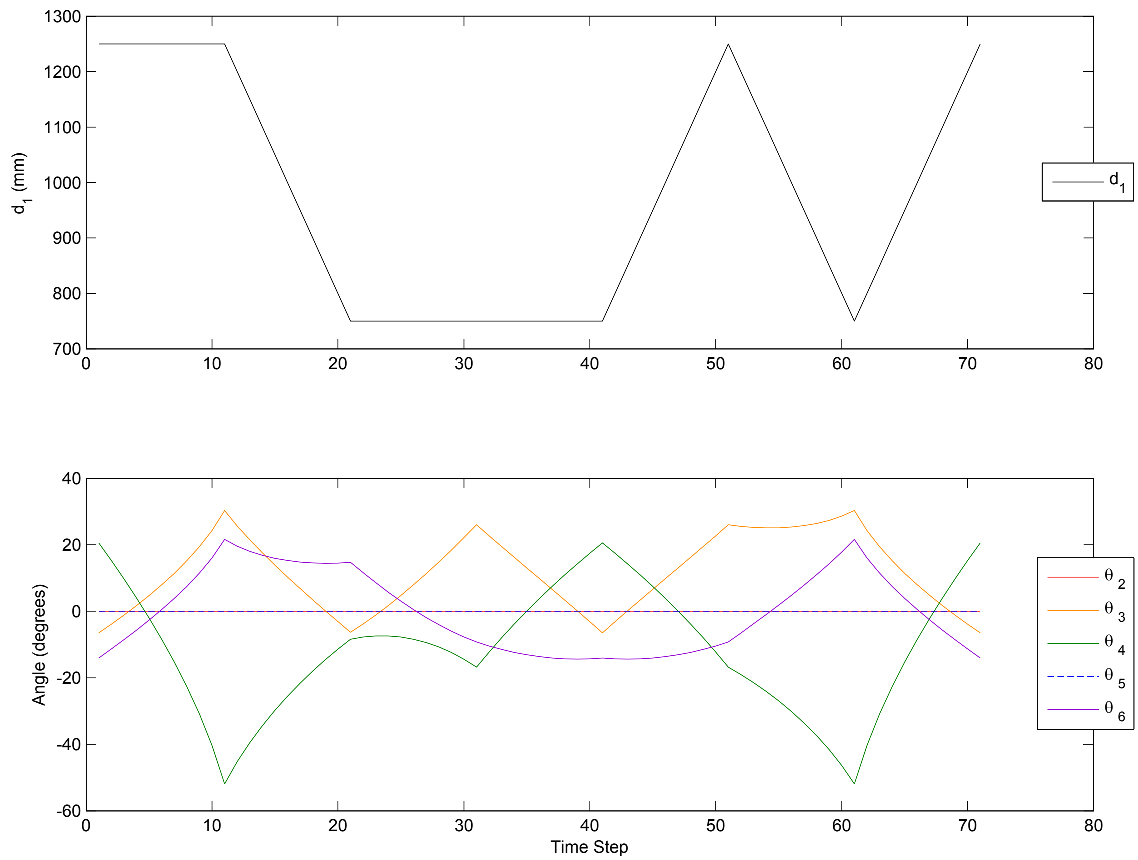

Figure 3.12: Plot of the joint angles needed to produce the trajectory in Figure 3.10. The numbers along the x-axis correspond to the $\times$'s in Figure 3.10. Note that Joints 2 and 5 remain at 0° throughout the entire trajectory.

Chapter 4
CONCLUSIONS

This thesis presents a detailed analysis of a robotic C-Arm. Joint frames and rotation directions are defined, along with relevant link lengths, according to the Denavit-Hartenberg notation convention [17]. Joint limits are presented and equations for the system's forward kinematics are calculated. The Jacobian matrix and singular poses are then calculated from the forward kinematics. Equations for the C-Arm's dynamics are also obtained based on mass properties measured from a SolidWorks model of the system.

A large portion of this thesis is dedicated to the development and verification of a numerical inverse kinematics solution. The solution is shown to produce accurate results over a large range of poses, and is also shown to be effective for trajectory generation. The solution differs from previous solutions in that it:

- Finds all possible cominations of joint angles that produce a desired pose, rather than just one combination,

- Shrinks the search space from six dimensions to one, thereby allowing for faster convergence, and

- Does not require iterations to be performed over the first joint, or require that iterations be performed over a rotational joint.

These differences represent a significant improvement over previous methods. Our method can be applied to more systems, can provide a solution in less time, and is far more likely to return the appropriate solution than previous methods.

Until now, having a closed-form solution for a mechanism's inverse kinematics was considered so important (largely because the time needed to compute a solution is substantially less for a closed-form solution than for a numerical solution) that manipulators were rarely built that did not have a closed-form solution. Without this constraint, many systems could likely have been built using a design that would better achieve their objectives. Previous numerical solutions, however, have generally been too slow to allow for the elimination of this constraint. The numerical inverse kinematics method presented in this thesis, coupled with the substantial increase in computing power that we have experienced in the last 20 years, means that it is no longer necessary to constrain robotic system designs to those with closed-form inverse kinematic solutions.

Chapter 5

FUTURE WORK

Once development of the C-Arm is completed, the equations presented in this thesis can be used to control the system. Implementing this analysis on the system will require conversion into C-code. The C-code will need to be tested thoroughly to ensure constants measured from the SolidWorks model are acceptable approximations of the true system parameters, and will be modified if necessary. In addition, equations or look-up tables will need to be developed to handle changing joint limits.

I also plan to develop a controller that minimizes the motions made by a macro/micro system, such as the C-Arm/RAVEN system. Control methods such as the one proposed by Chen and Halang [11], or the one proposed by Narikiyo et al. [54], use the macro manipulator for gross positioning and the micro manipulator for accurate tracking. As a result, they require both the macro manipulator and the micro manipulator to move continuously. I plan to develop a controller that only requires motion on the part of the macro manipulator if the required motions of the system are difficult for the micro manipulator to achieve alone.

Khatib [34] proposed a "dexterous dynamic coordination" strategy that accomplished a similar goal. His strategy minimized the instantaneous kinetic energy of the system, and used the macro manipulator to reduce joint torques on the micro manipulator. He treats the macro/micro system as a redundant manipulator and calculates the effective mass and effective inertia of the system based on the "pseudo kinetic energy matrix." I would like to develop a method that takes advantage of the physical separation of the macro and micro manipulators and seeks to minimize criteria other than the kinetic energy.

My proposed control algorithm is divided into three main parts. First, the recent movements of the micro manipulator are analyzed. If those movements are outside a predefined workspace, a new pose is determined that would locate the manipulator's most recent motions closer to the center of that workspace. The second component of the controller repositions the macro manipulator such that the micro manipulator is in the new pose. The third component of the controller determines a method to move the micro manipulator in time with the macro manipulator such that the position and orientation of the end effector do not change.

Once a basic macro/micro system controller is developed, I hope to improve the controller's usefulness. One goal is to include obstacle avoidance to the controller. Optimally, I would eventually like to avoid both static and dynamic obstacles. Without some sort of 3D detection, however, I may only be able to avoid static obstacles. I would also like to explore path planning techniques to deal with singular poses and to perform basic automated tasks. For surgical robotics, automated tasks might include changing tools or moving the system out of the way so a nurse can have direct access to the patient.

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Appendix A

ELEMENTS OF THE C-ARM'S DYNAMIC EQUATIONS

This Appendix contains the final calculated values of $M(\Theta)$, $V(\Theta, \dot{\Theta})$, $G(\Theta)$, and τ as defined in Section 2.2.6 of this thesis. For clarity, only equations for the GOLD C-Arm are presented. Also note that sines and cosines are abbreviated as follows:

- $C_2 = \cos(\theta_2)$
- $S_{34} = \sin(\theta_3 + \theta_4)$
- $C_{66} = \cos(2\theta_6)$
- $S_5^2 = \sin^2(\theta_5)$
- $C_{4-5} = \cos(\theta_4 - \theta_5)$
- $C_{5566} = \cos(2(\theta_5 + \theta_6))$
- $S_{55-66} = \sin(2(\theta_5 - \theta_6))$
- $C_{3344-566} = \cos(2\theta_3 + 2\theta_4 - \theta_5 - 2\theta_6)$
- etc.

The equations are presented in terms of the lengths L_1 and L_2 as presented in Table 2.2 and the mass properties as presented in Table 2.3. When referring to the mass properties, m_i refers to the mass of Link i and elements of the center of mass vectors and mass moment of intertia matrices follow the convention of Equations (2.22) and (2.23).

A.1 *Inertia Matrix* $M(\Theta)$

The inertia matrix $M(\Theta)$ is a 6×6 symmetric matrix. Its elements are defined as follows:

$$M(\Theta) = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} & M_{16} \\ M_{12} & M_{22} & M_{23} & M_{24} & M_{25} & M_{26} \\ M_{13} & M_{23} & M_{33} & M_{34} & M_{35} & M_{36} \\ M_{14} & M_{24} & M_{34} & M_{44} & M_{45} & M_{46} \\ M_{15} & M_{25} & M_{35} & M_{45} & M_{55} & M_{56} \\ M_{16} & M_{26} & M_{36} & M_{46} & M_{56} & M_{66} \end{bmatrix} \tag{A.1}$$

The following formulas, found using Equation (2.25), define each of the elements of $M(\Theta)$.

$$M_{11} = m_1 + m_2 + m_3 + m_4 + m_5 + m_6 \tag{A.2a}$$

$$\begin{aligned} M_{12} = {} & (\mathrm{COM}_{2x} C_2 - \mathrm{COM}_{2y} S_2)\, m_2 + (\mathrm{COM}_{3x} C_2 S_3 + \mathrm{COM}_{3y} C_2 C_3 - \mathrm{COM}_{3z} S_2)\, m_3 \\ & + (\mathrm{COM}_{4x} C_2 S_{34} + \mathrm{COM}_{4y} C_2 C_{34} m_4 - \mathrm{COM}_{4z} S_2 + C_2 S_3 L_1)\, m_4 + (\mathrm{COM}_{5x} (C_2 C_5 S_{34} \\ & + S_2 S_5) + \mathrm{COM}_{5y} (C_5 S_2 - C_2 S_{34} S_5) + \mathrm{COM}_{5z} C_2 C_{34} + C_2 (S_3 L_1 + C_{34} L_2))\, m_5 \\ & + (\mathrm{COM}_{6x} (C_6 S_2 S_5 + C_2 (C_4 (C_5 C_6 S_3 + C_3 S_6) + S_4 (C_3 C_5 C_6 - S_3 S_6))) \\ & + \mathrm{COM}_{6y} (C_2 (C_3 (C_4 C_6 - C_5 S_4 S_6) - S_3 (C_6 S_4 + C_4 C_5 S_6)) - S_2 S_5 S_6) \\ & + \mathrm{COM}_{6z} (C_2 S_{34} S_5 - C_5 S_2) + C_2 (S_3 L_1 + C_{34} L_2))\, m_6 \end{aligned} \tag{A.2b}$$

$$\begin{aligned} M_{13} = {} & S_2 ((\mathrm{COM}_{3x} C_3 - \mathrm{COM}_{3y} S_3)\, m_3 + (\mathrm{COM}_{4x} C_{34} - \mathrm{COM}_{4y} S_{34} + C_3 L_1)\, m_4 \\ & + (\mathrm{COM}_{5x} C_{34} C_5 - \mathrm{COM}_{5y} C_{34} S_5 - \mathrm{COM}_{5z} S_{34} + C_3 L_1 m_5 - S_{34} L_2)\, m_5 \\ & + (\mathrm{COM}_{6x} (C_3 C_4 C_5 C_6 - S_3 S_4 C_5 C_6 - S_3 C_4 S_6 - C_3 S_4 S_6) + \mathrm{COM}_{6y} (S_3 S_4 C_5 S_6 \\ & - C_3 C_4 C_5 S_6 - S_3 C_4 C_6 - C_3 S_4 C_6) + \mathrm{COM}_{6z} C_{34} S_5 + C_3 L_1 - S_{34} L_2))\, m_6 \end{aligned} \tag{A.2c}$$

$$\begin{aligned} M_{14} = {} & S_2 ((\mathrm{COM}_{4x} C_{34} - \mathrm{COM}_{4y} S_{34})\, m_4 + (\mathrm{COM}_{5x} C_{34} C_5 - \mathrm{COM}_{5y} C_{34} S_5 - \mathrm{COM}_{5z} S_{34} \\ & - S_{34} L_2)\, m_5 + (\mathrm{COM}_{6x} (C_3 (C_4 C_5 C_6 - S_4 S_6) - S_3 (C_5 C_6 S_4 + C_4 S_6)) \\ & + \mathrm{COM}_{6y} (S_4 (C_5 S_3 S_6 - C_3 C_6) - C_4 (C_6 S_3 + C_3 C_5 S_6)) + \mathrm{COM}_{6z} C_{34} S_5 - S_{34} L_2))\, m_6 \end{aligned} \tag{A.2d}$$

$$
\begin{aligned}
M_{15} = {} & \left(-\mathrm{COM}_{5x}\left(S_2S_{34}S_5 + C_2C_5\right) + \mathrm{COM}_{5y}\left(C_2S_5 - S_2S_{34}C - 5\right)\right) m_5 \\
& + \left(-\mathrm{COM}_{6x}\left(S_2S_{34}S_5C_6 + C_2C_5C_6\right) + \mathrm{COM}_{6y}\left(S_2S_{34}S_5S_6 + C_2C_5S_6\right)\right. \\
& \left.+\mathrm{COM}_{6z}\left(S_2S_{34}C_5 - C_2S_5\right)\right) m_6
\end{aligned} \tag{A.2e}
$$

$$
\begin{aligned}
M_{16} = {} & \left(\mathrm{COM}_{6x}\left(C_2S_5S_6 - S_2S_3S_4C_6 - S_2S_3C_4C_5S_6 + S_2C_3C_4C_6 - S_2C_3S_4C_5C_6\right)\right. \\
& \left.+\mathrm{COM}_{6y}\left(S_2S_3S_4S_6 + C_2S_5C_6 - S_2S_3C_4C_5C_6 - S_2C_3C_4S_6 - S_2C_3S_4C_5C_6\right)\right) m_6
\end{aligned} \tag{A.2f}
$$

$$
\begin{aligned}
M_{22} = {} & \left(2\left(\mathrm{COM}_{4x}S_{34} + \mathrm{COM}_{4y}C_{34}\right) + S_3L_1\right) S_3L_1m_4 + \left(\left(2S_3S_{34}L_1 + S_{3344}L_2\right)\left(\mathrm{COM}_{5x}C_5\right.\right. \\
& \left.\left. -\mathrm{COM}_{5y}S_5\right) + 2\mathrm{COM}_{5z}C_{34}\left(S_3L_1 + 2C_{34}L_2\right) + \left(S_3L_1 + 2C_{34}L_2\right)^2\right) m_5 \\
& + \Big(\mathrm{COM}_{6x}\Big(\Big(S_{33}S_4C_5C_6 - 2S_3^2S_4S_6 + 2S_3^2C_4C_5C_6 + S_{33}C_4S_6\Big) L_1 + 2\left(S_3C_{34}\left(C_4C_5C_6\right.\right. \\
& \left.\left. -S_4S_6\right) + C_3C_{34}\left(C_4S_6 + S_4C_5C_6\right)\right) L_2\Big) - \mathrm{COM}_{6y}\Big(\Big(S_{33}S_4C_5S_6 - 2S_3^2S_4C_6 \\
& +2S_3^2C_4C_5S_6 - S_{33}C_4C_6\Big) L_1 + 2\left(S_3C_{34}\left(C_4C_5S_6 + S_4S_6\right) + C_3C_{34}\left(S_4C_5S_6\right.\right. \\
& \left.\left. -C_4C_6\right)\right) L_2\Big) + 2\mathrm{COM}_{6z}\left(S_3S_{34}S_5L_1 + S_3C_{34}C_4S_5L_2 + C_3C_{34}S_4S_5L_2\right) + \left(S_3L_1\right. \\
& \left. +2C_{34}L_2\right)^2\Big) m_6 + \mathrm{I}_{2zz} + \frac{1}{2}\left(\mathrm{I}_{3xx}\left(1 + C_{33}\right) + 2\mathrm{I}_{3xy}S_{33} + \mathrm{I}_{3yy}\left(1 - C_{33}\right)\right) \\
& + \frac{1}{2}\left(\mathrm{I}_{4xx}\left(1 + C_{3344}\right) + 2\mathrm{I}_{4xy}S_{3344} + \mathrm{I}_{4yy}\left(1 - C_{3344}\right)\right) + \frac{1}{8}\left(\mathrm{I}_{5xx}\left(2 + 2C_{3344} + C_{334455}\right.\right. \\
& \left. +C_{3344-55} + 2C_{55}\right) + 2\mathrm{I}_{5xy}\left(S_{334455} - S_{3344-55} + 2S_{55}\right) + \mathrm{I}_{5yy}\left(2 + 2C_{3344} - C_{334455}\right. \\
& \left. -C_{3344-55} - 2C_{55}\right) + 4\mathrm{I}_{5xz}\left(S_{33445} + S_{3344-5}\right) + 4\mathrm{I}_{5yz}\left(C_{33445} - C_{3344-5}\right) \\
& \left. +4\mathrm{I}_{5zz}\left(1 - C_{3344}\right)\right) + \frac{1}{32}\left(\left(\mathrm{I}_{6xx} + \mathrm{I}_{6yy}\right)\left(12 - 4C_{3344} + 2C_{334455} + 2C_{3344-55} + 4C_{55}\right)\right. \\
& + \left(\mathrm{I}_{6xx} - \mathrm{I}_{6yy}\right)\left(6C_{334466} + 4C_{3344566} + C_{33445566} + 4C_{334466-5} + C_{334466-55}\right. \\
& -4C_{3344-566} + S_{3344-5566} + 6C_{3344-66} - 4C_{33445-66} + C_{334455-66} + 2C_{5566} + 2C_{55-66} \\
& \left. -4C_{66}\right) + \mathrm{I}_{6xy}\left(12S_{334466} + 8S_{3344566} + 2S_{33445566} + 8S_{334466-5} + 2S_{334466-55}\right. \\
& +8S_{3344-566} - 2S_{3344-5566} - 12S_{3344-66} + 8S_{33445-66} - 2S_{334455-66} + 4S_{5566} - 4S_{55-66} \\
& \left. -8S_{66}\right) - \mathrm{I}_{6xz}\left(8S_{334456} + 4S_{3344556} - 8S_{33446-5} - 4S_{33446-55} + 8S_{3344-56} - 4S_{3344-556}\right. \\
& \left. -8S_{33445-6} + 4S_{334455-6} + 8S_{556} + 8S_{55-6}\right) - \mathrm{I}_{6yz}\left(8C_{334456} + 4C_{3344556} - 8C_{33446-5}\right. \\
& \left.\left. -4C_{33446-55} - 8C_{3344-56} + 4C_{3344-556} + 8C_{33445-6} - 4C_{334455-6} + 8C_{556} - 8C_{55-6}\right)\right)
\end{aligned} \tag{A.2g}
$$

$$M_{23} = \frac{1}{4}\big(4\mathrm{I}_{5xy}C_3C_4C_5^2 - 4\mathrm{I}_{6xz}C_3C_4C_6C_5^2 - 4\mathrm{I}_{5xy}S_3S_4C_5^2 + 4\mathrm{I}_{6xz}C_6S_3S_4C_5^2 + 4\mathrm{I}_{6yz}C_3C_4S_6C_5^2$$
$$-4\mathrm{I}_{6yz}S_3S_4S_6C_5^2 - 4\mathrm{I}_{5yz}C_4S_3C_5 + 4\mathrm{I}_{6yz}C_4C_6S_3C_5 - 4\mathrm{I}_{5yz}C_3S_4C_5 + 4\mathrm{I}_{6yz}C_3C_6S_4C_5$$
$$+4\mathrm{I}_{6xz}C_4S_3S_6C_5 + 4\mathrm{I}_{6xz}C_3S_4S_6C_5 - 4\mathrm{I}_{5xy}C_3C_4S_5^2 + 4\mathrm{I}_{6xz}C_3C_4C_6S_5^2 + 4\mathrm{I}_{5xy}S_3S_4S_5^2$$
$$-4\mathrm{I}_{6xz}C_6S_3S_4S_5^2 + 4\mathrm{I}_{6xy}C_4S_3S_5S_6^2 + 4\mathrm{I}_{6xy}C_3S_4S_5S_6^2 + \mathrm{I}_{6xx}C_3C_4S_{55}S_6^2$$
$$-\mathrm{I}_{6yy}C_3C_4S_{55}S_6^2 - \mathrm{I}_{6xx}S_3S_4S_{55}S_6^2 + \mathrm{I}_{6yy}S_3S_4S_{55}S_6^2 - 4\mathrm{I}_{3xz}C_3 - 4\mathrm{I}_{4xz}C_3C_4 + 4\mathrm{I}_{3yz}S_3$$
$$+4\mathrm{I}_{4yz}C_4S_3 + 4\mathrm{I}_{4yz}C_3S_4 + 4\mathrm{I}_{4xz}S_3S_4 - 4\mathrm{I}_{6xy}C_4C_6^2S_3S_5 - 4\mathrm{I}_{5xz}C_4S_3S_5$$
$$-4\mathrm{I}_{6xy}C_3C_6^2S_4S_5 - 4\mathrm{I}_{5xz}C_3S_4S_5 - \mathrm{I}_{6xx}C_3C_4C_6^2S_{55} + \mathrm{I}_{6yy}C_3C_4C_6^2S_{55} - 2\mathrm{I}_{5xx}C_3C_4S_{55}$$
$$+2\mathrm{I}_{5yy}C_3C_4S_{55} - \mathrm{I}_{6xx}C_3C_4S_{55} - \mathrm{I}_{6yy}C_3C_4S_{55} + 2\mathrm{I}_{6zz}C_3C_4S_{55} + \mathrm{I}_{6xx}C_6^2S_3S_4S_{55}$$
$$-\mathrm{I}_{6yy}C_6^2S_3S_4S_{55} + 2\mathrm{I}_{5xx}S_3S_4S_{55} - 2\mathrm{I}_{5yy}S_3S_4S_{55} + \mathrm{I}_{6xx}S_3S_4S_{55} + \mathrm{I}_{6yy}S_3S_4S_{55}$$
$$-2\mathrm{I}_{6zz}S_3S_4S_{55} - 4\mathrm{I}_{6yz}C_3C_4S_5^2S_6 + 4\mathrm{I}_{6yz}S_3S_4S_5^2S_6 + 2\mathrm{I}_{6xx}C_4S_3S_5S_{66} - 2\mathrm{I}_{6yy}C_4S_3S_5S_{66}$$
$$+2\mathrm{I}_{6xx}C_3S_4S_5S_{66} - 2\mathrm{I}_{6yy}C_3S_4S_5S_{66} - 2\mathrm{I}_{6xy}C_3C_4S_{55}S_{66} + 2\mathrm{I}_{6xy}S_3S_4S_{55}S_{66}$$
$$-4S_{34}L_2\left(\left(\mathrm{COM}_{5y}C_5 + \mathrm{COM}_{5x}S_5\right)m_5 - \left(\mathrm{COM}_{6z}C_5 + S_5\left(\mathrm{COM}_{6y}S_6 - \mathrm{COM}_{6x}C_6\right)\right)m_6\right)$$
$$-4C_3L_1\left(\mathrm{COM}_{4z}m_4 - \left(\mathrm{COM}_{5y}C_5 + \mathrm{COM}_{5x}S_5\right)m_5 + \left(\mathrm{COM}_{6z}C_5 + S_5\left(\mathrm{COM}_{6y}S_6\right.\right.\right.$$
$$\left.\left.\left.-\mathrm{COM}_{6x}C_6\right)\right)m_6\right)\big) \tag{A.2h}$$

$$M_{24} = \frac{1}{4}\Big(S_3\Big(S_4\Big(-4\left(\mathrm{I}_{5xy} - \mathrm{I}_{6xz}C_6 + \mathrm{I}_{6yz}S_6\right)C_5^2 - \mathrm{I}_{6xx}S_{55}S_6^2 + \mathrm{I}_{6yy}S_{55}S_6^2 + 4\mathrm{I}_{4xz} + \mathrm{I}_{6xx}C_6^2S_{55}$$
$$-\mathrm{I}_{6yy}C_6^2S_{55} + 2\mathrm{I}_{5xx}S_{55} - 2\mathrm{I}_{5yy}S_{55} + \mathrm{I}_{6xx}S_{55} + \mathrm{I}_{6yy}S_{55} - 2\mathrm{I}_{6zz}S_{55} + 4S_5^2\left(\mathrm{I}_{5xy} - \mathrm{I}_{6xz}C_6\right.$$
$$\left.+\mathrm{I}_{6yz}S_6\right) + 2\mathrm{I}_{6xy}S_{55}S_{66}\Big) + 2C_4\left(2\mathrm{I}_{4yz} + C_5\left(-2\mathrm{I}_{5yz} + 2\mathrm{I}_{6yz}C_6 + 2\mathrm{I}_{6xz}S_6\right) - S_5\left(2\mathrm{I}_{5xz}\right.\right.$$
$$\left.\left.+2\mathrm{I}_{6xy}C_{66} + (\mathrm{I}_{6yy} - \mathrm{I}_{6xx})S_{66}\right)\right)\Big) - C_3\Big(C_4\Big(-4\left(\mathrm{I}_{5xy} - \mathrm{I}_{6xz}C_6 + \mathrm{I}_{6yz}S_6\right)C_5^2 - \mathrm{I}_{6xx}S_{55}S_6^2$$
$$+\mathrm{I}_{6yy}S_{55}S_6^2 + 4\mathrm{I}_{4xz} + \mathrm{I}_{6xx}C_6^2S_{55} - \mathrm{I}_{6yy}C_6^2S_{55} + 2\mathrm{I}_{5xx}S_{55} - 2\mathrm{I}_{5yy}S_{55} + \mathrm{I}_{6xx}S_{55} + \mathrm{I}_{6yy}S_{55}$$
$$-2\mathrm{I}_{6zz}S_{55} + 4S_5^2\left(\mathrm{I}_{5xy} - \mathrm{I}_{6xz}C_6 + \mathrm{I}_{6yz}S_6\right) + 2\mathrm{I}_{6xy}S_{55}S_{66}\Big) + 2S_4\left(-2\mathrm{I}_{4yz} + 2C_5\left(\mathrm{I}_{5yz}\right.\right.$$
$$\left.\left.-\mathrm{I}_{6yz}C_6 - \mathrm{I}_{6xz}S_6\right) + S_5\left(2\mathrm{I}_{5xz} + 2\mathrm{I}_{6xy}C_{66} + (\mathrm{I}_{6yy} - \mathrm{I}_{6xx})S_{66}\right)\right)\Big) - 4S_{34}L_2\left(\left(\mathrm{COM}_{5y}C_5\right.\right.$$
$$\left.\left.+\mathrm{COM}_{5x}S_5\right)m_5 - \left(\mathrm{COM}_{6z}C_5 + S_5\left(\mathrm{COM}_{6y}S_6 - \mathrm{COM}_{6x}C_6\right)\right)m_6\right)\Big) \tag{A.2i}$$

$$M_{25} = \frac{1}{2}\left(-2\mathrm{I}_{6xy}C_3C_4C_5C_6^2 + \mathrm{I}_{6xx}C_4S_3C_6^2 - \mathrm{I}_{6yy}C_4S_3C_6^2 + \mathrm{I}_{6xx}C_3S_4C_6^2 - \mathrm{I}_{6yy}C_3S_4C_6^2\right.$$
$$+2\mathrm{I}_{6xy}C_5S_3S_4C_6^2 - 2\mathrm{I}_{6yz}C_3C_4S_5C_6 + 2\mathrm{I}_{6yz}S_3S_4S_5C_6 + 2\mathrm{I}_{6xy}C_3C_4C_5S_6^2 - \mathrm{I}_{6xx}C_4S_3S_6^2$$
$$+\mathrm{I}_{6yy}C_4S_3S_6^2 - \mathrm{I}_{6xx}C_3S_4S_6^2 + \mathrm{I}_{6yy}C_3S_4S_6^2 - 2\mathrm{I}_{6xy}C_5S_3S_4S_6^2 - 2\mathrm{I}_{5xz}C_3C_4C_5$$
$$-2\mathrm{I}_{5zz}C_4S_3 - \mathrm{I}_{6xx}C_4S_3 - \mathrm{I}_{6yy}C_4S_3 - 2\mathrm{I}_{5zz}C_3S_4 - \mathrm{I}_{6xx}C_3S_4 - \mathrm{I}_{6yy}C_3S_4 + 2\mathrm{I}_{5xz}C_5S_3S_4$$
$$+2\mathrm{I}_{5yz}C_3C_4S_5 - 2\mathrm{I}_{5yz}S_3S_4S_5 - 2\mathrm{I}_{6xz}C_3C_4S_5S_6 + 2\mathrm{I}_{6xz}S_3S_4S_5S_6 + \mathrm{I}_{6xx}C_3C_4C_5S_{66}$$
$$-\mathrm{I}_{6yy}C_3C_4C_5S_{66} + 2\mathrm{I}_{6xy}C_4S_3S_{66} + 2\mathrm{I}_{6xy}C_3S_4S_{66} - \mathrm{I}_{6xx}C_5S_3S_4S_{66} + \mathrm{I}_{6yy}C_5S_3S_4S_{66}$$
$$-2\left(S_3L_1 + C_{34}L_2\right)\left(\left(\mathrm{COM}_{5x}C_5 - \mathrm{COM}_{5y}S_5\right)m_5 + \left(\mathrm{COM}_{6z}S_5 + C_5\left(\mathrm{COM}_{6x}C_6\right.\right.\right.$$
$$\left.\left.\left.\left.-\mathrm{COM}_{6y}S_6\right)\right)m_6\right)\right) \tag{A.2j}$$

$$M_{26} = -\mathrm{I}_{6xz}C_3C_4C_5C_6 + \mathrm{I}_{6yz}C_4S_3C_6 + \mathrm{I}_{6yz}C_3S_4C_6 + \mathrm{I}_{6xz}C_5S_3S_4C_6 + \mathrm{I}_{6zz}C_3C_4S_5 - \mathrm{I}_{6zz}S_3S_4S_5$$
$$+\mathrm{I}_{6yz}C_3C_4C_5S_6 + \mathrm{I}_{6xz}C_4S_3S_6 + \mathrm{I}_{6xz}C_3S_4S_6 - \mathrm{I}_{6yz}C_5S_3S_4S_6 + S_3S_5\left(\mathrm{COM}_{6y}C_6\right.$$
$$\left.+\mathrm{COM}_{6x}S_6\right)L_1m_6 + C_{34}S_5\left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right)L_2m_6 \tag{A.2k}$$

$$M_{33} = \frac{1}{4}\left(4\left(m_4 + m_5 + m_6\right)L_1^2 + 8\left(\left(\mathrm{COM}_{4x}C_4 - \mathrm{COM}_{4y}S_4\right)m_4 + \left(-\mathrm{COM}_{5z}S_4 - L_2S_4\right.\right.\right.$$
$$\left.+C_4\left(\mathrm{COM}_{5x}C_5 - \mathrm{COM}_{5y}S_5\right)\right)m_5 + \left(-S_4\left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right) + C_4\left(\mathrm{COM}_{6z}S_5\right.\right.$$
$$\left.\left.\left.+C_5\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right)\right) - S_4L_2\right)m_6\right)L_1 + \frac{1}{2}\left(8\mathrm{I}_{3zz} + 8\mathrm{I}_{4zz} + 4\mathrm{I}_{5xx} + 4\mathrm{I}_{5yy}\right.$$
$$+2\mathrm{I}_{6xx} + 2\mathrm{I}_{6yy} + 4\mathrm{I}_{6zz} - 2(2\mathrm{I}_{5xx} - 2\mathrm{I}_{5yy} + \mathrm{I}_{6xx} + \mathrm{I}_{6yy} - 2\mathrm{I}_{6zz})C_{55} + (\mathrm{I}_{6yy} - \mathrm{I}_{6xx})C_{55-66}$$
$$-4\mathrm{I}_{6yz}C_{55-6} + 2\mathrm{I}_{6xx}C_{66} - 2\mathrm{I}_{6yy}C_{66} - \mathrm{I}_{6xx}C_{5566} + \mathrm{I}_{6yy}C_{5566} + 4\mathrm{I}_{6yz}C_{556} - 8\mathrm{I}_{5xy}S_{55}$$
$$\left.+2\mathrm{I}_{6xy}S_{55-66} + 4\mathrm{I}_{6xz}S_{55-6} + 4\mathrm{I}_{6xy}S_{66} - 2\mathrm{I}_{6xy}S_{5566} + 4\mathrm{I}_{6xz}S_{556}\right) + 4L_2^2\left(m_5 + m_6\right)$$
$$\left.+8L_2\left(\mathrm{COM}_{5z}m_5 + \left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right)m_6\right)\right) \tag{A.2l}$$

$$M_{34} = \frac{1}{4}\Bigg(4\left(m_5+m_6\right)L_2^2 + 8\left(\mathrm{COM}_{5z}m_5 + \left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right)m_6\right)L_2 + \frac{1}{2}\left(8\mathrm{I}_{4zz} + 4\mathrm{I}_{5xx}\right.$$
$$+4\mathrm{I}_{5yy} + 2\mathrm{I}_{6xx} + 2\mathrm{I}_{6yy} + 4\mathrm{I}_{6zz} - 2(2\mathrm{I}_{5xx} - 2\mathrm{I}_{5yy} + \mathrm{I}_{6xx} + \mathrm{I}_{6yy} - 2\mathrm{I}_{6zz})C_{55} + (\mathrm{I}_{6yy}$$
$$-\mathrm{I}_{6xx})C_{55-66} - 4\mathrm{I}_{6yz}C_{55-6} + 2\mathrm{I}_{6xx}C_{66} - 2\mathrm{I}_{6yy}C_{66} - \mathrm{I}_{6xx}C_{5566} + \mathrm{I}_{6yy}C_{5566} + 4\mathrm{I}_{6yz}C_{556}$$
$$\left.-8\mathrm{I}_{5xy}S_{55} + 2\mathrm{I}_{6xy}S_{55-66} + 4\mathrm{I}_{6xz}S_{55-6} + 4\mathrm{I}_{6xy}S_{66} - 2\mathrm{I}_{6xy}S_{5566} + 4\mathrm{I}_{6xz}S_{556}\right)$$
$$+4L_1\left(\left(\mathrm{COM}_{4x}C_4 - \mathrm{COM}_{4y}S_4\right)m_4 + \left(-\mathrm{COM}_{5z}S_4 - L_2S_4 + C_4\left(\mathrm{COM}_{5x}C_5\right.\right.\right.$$
$$\left.\left.-\mathrm{COM}_{5y}S_5\right)\right)m_5 + \left(-S_4\left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right) + C_4\left(\mathrm{COM}_{6z}S_5 + C_5\left(\mathrm{COM}_{6x}C_6\right.\right.\right.$$
$$\left.\left.\left.\left.-\mathrm{COM}_{6y}S_6\right)\right) - S_4L_2\right)m_6\right)\Bigg) \tag{A.2m}$$

$$M_{35} = \mathrm{I}_{6xy}S_5\left(C_6^2 - S_6^2\right) - \mathrm{I}_{6yz}C_5C_6 + \mathrm{I}_{6yy}S_5S_6C_6 + \mathrm{I}_{5yz}C_5 + \mathrm{I}_{5xz}S_5 - \mathrm{I}_{6xz}C_5S_6 - \frac{1}{2}\mathrm{I}_{6xx}S_5S_{66}$$
$$- S_4L_1\left(\left(\mathrm{COM}_{5y}C_5 + \mathrm{COM}_{5x}S_5\right)m_5 - \left(\mathrm{COM}_{6z}C_5 + S_5\left(\mathrm{COM}_{6y}S_6 - \mathrm{COM}_{6x}C_6\right)\right)m_6\right)$$
$$+ L_2\left(\left(\mathrm{COM}_{5y}C_5 + \mathrm{COM}_{5x}S_5\right)m_5 - \left(\mathrm{COM}_{6z}C_5 + S_5\left(\mathrm{COM}_{6y}S_6 - \mathrm{COM}_{6x}C_6\right)\right)m_6\right) \tag{A.2n}$$

$$M_{36} = \mathrm{I}_{6zz}C_5 + \left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right)L_2m_6C_5 + \mathrm{I}_{6xz}C_6S_5 - \mathrm{I}_{6yz}S_5S_6 + \left(C_4\left(\mathrm{COM}_{6x}C_6\right.\right.$$
$$\left.\left.-\mathrm{COM}_{6y}S_6\right) - C_5S_4\left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right)\right)L_1m_6 \tag{A.2o}$$

$$M_{44} = \frac{1}{8}\left(8\left(m_5+m_6\right)L_2^2 + 16\left(\mathrm{COM}_{5z}m_5 + \left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right)m_6\right)L_2 + 8\mathrm{I}_{4zz} + 4\mathrm{I}_{5xx}\right.$$
$$+4\mathrm{I}_{5yy} + 2\mathrm{I}_{6xx} + 2\mathrm{I}_{6yy} + 4\mathrm{I}_{6zz} - 2(2\mathrm{I}_{5xx} - 2\mathrm{I}_{5yy} + \mathrm{I}_{6xx} + \mathrm{I}_{6yy} - 2\mathrm{I}_{6zz})C_{55} - \mathrm{I}_{6xx}C_{55-66}$$
$$+\mathrm{I}_{6yy}C_{55-66} - 4\mathrm{I}_{6yz}C_{55-6} + 2\mathrm{I}_{6xx}C_{66} - 2\mathrm{I}_{6yy}C_{66} - \mathrm{I}_{6xx}C_{5566} + \mathrm{I}_{6yy}C_{5566} + 4\mathrm{I}_{6yz}C_{556}$$
$$\left.-8\mathrm{I}_{5xy}S_{55} + 2\mathrm{I}_{6xy}S_{55-66} + 4\mathrm{I}_{6xz}S_{55-6} + 4\mathrm{I}_{6xy}S_{66} - 2\mathrm{I}_{6xy}S_{5566} + 4\mathrm{I}_{6xz}S_{556}\right) \tag{A.2p}$$

$$M_{45} = C_5\left(\mathrm{I}_{5yz} - \mathrm{I}_{6yz}C_6 - \mathrm{I}_{6xz}S_6\right) + \frac{1}{2}S_5\left(2\mathrm{I}_{5xz} + 2\mathrm{I}_{6xy}C_{66} + (\mathrm{I}_{6yy} - \mathrm{I}_{6xx})S_{66}\right) + L_2\left(\left(\mathrm{COM}_{5y}C_5\right.\right.$$
$$\left.\left.+\mathrm{COM}_{5x}S_5\right)m_5 - \left(\mathrm{COM}_{6z}C_5 + S_5\left(\mathrm{COM}_{6y}S_6 - \mathrm{COM}_{6x}C_6\right)\right)m_6\right) \tag{A.2q}$$

$$M_{46} = \mathrm{I}_{6zz}C_5 + \left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right)L_2m_6C_5 + S_5\left(\mathrm{I}_{6xz}C_6 - \mathrm{I}_{6yz}S_6\right) \tag{A.2r}$$

$$M_{55} = \frac{1}{2}\left(2\mathrm{I}_{5zz} + \mathrm{I}_{6xx} + \mathrm{I}_{6yy} + (\mathrm{I}_{6yy} - \mathrm{I}_{6xx})C_{66} - 2\mathrm{I}_{6xy}S_{66}\right) \tag{A.2s}$$

$$M_{56} = -\mathrm{I}_{6yz}C_6 - \mathrm{I}_{6xz}S_6 \tag{A.2t}$$

$$M_{66} = \mathrm{I}_{6zz} \tag{A.2u}$$

A.2 Centrifugal and Coriolis Terms $V(\Theta, \dot{\Theta})$

The centrifugal and Coriolis terms $V(\Theta, \dot{\Theta})$ form a 6×1 vector. The term V_i indicates the i^{th} term of V. The following formulas, found using Equation (2.26), define each of the elements of $V(\Theta, \dot{\Theta})$.

$$\begin{aligned}
V_1 = &-\left(\mathrm{COM}_{2y}C_2 + \mathrm{COM}_{2x}S_2\right) m_2\dot{\theta}_2{}^2 - \left(\mathrm{COM}_{3z}C_2 + S_2\left(\mathrm{COM}_{3y}C_3 + \mathrm{COM}_{3x}S_3\right)\right) m_3\dot{\theta}_2{}^2 - \left(\mathrm{COM}_{4z}C_2\right.\\
&+S_2\left(\mathrm{COM}_{4y}C_{34} + \mathrm{COM}_{4x}S_{34}\right) + S_2S_3L_1\big) m_4\dot{\theta}_2{}^2 + \left(\mathrm{COM}_{5y}C_2C_5 - \mathrm{COM}_{5x}S_2S_{34}C_5\right.\\
&-\mathrm{COM}_{5z}C_{34}S_2 + \mathrm{COM}_{5x}C_2S_5 + \mathrm{COM}_{5y}S_2S_{34}S_5 - S_2S_3L_1 - C_{34}S_2L_2\big) m_5\dot{\theta}_2{}^2 + \left(-\mathrm{COM}_{6z}\left(C_2C_5\right.\right.\\
&+S_2S_{34}S_5\big) + \mathrm{COM}_{6x}\left(C_2C_6S_5 - S_2\left(C_4\left(C_5C_6S_3 + C_3S_6\right) + S_4\left(C_3C_5C_6 - S_3S_6\right)\right)\right)\\
&+\mathrm{COM}_{6y}\left(-C_2S_5S_6 - S_2\left(C_3\left(C_4C_6 - C_5S_4S_6\right) - S_3\left(C_6S_4 + C_4C_5S_6\right)\right)\right) - S_2\left(S_3L_1\right.\\
&+C_{34}L_2\big)\big) m_6\dot{\theta}_2{}^2 + 2C_2\left(\mathrm{COM}_{3x}C_3 - \mathrm{COM}_{3y}S_3\right) m_3\dot{\theta}_3\dot{\theta}_2 + 2C_2\left(\mathrm{COM}_{4x}C_{34} - \mathrm{COM}_{4y}S_{34}\right.\\
&+C_3L_1\big) m_4\dot{\theta}_3\dot{\theta}_2 + 2C_2\left(\mathrm{COM}_{5x}C_{34}C_5 - \mathrm{COM}_{5z}S_{34} - \mathrm{COM}_{5y}C_{34}S_5 + C_3L_1 - S_{34}L_2\right) m_5\dot{\theta}_3\dot{\theta}_2\\
&+2C_2\left(\mathrm{COM}_{6z}C_{34}S_5 + \mathrm{COM}_{6y}\left(S_4\left(C_5S_3S_6 - C_3C_6\right) - C_4\left(C_6S_3 + C_3C_5S_6\right)\right)\right.\\
&+\mathrm{COM}_{6x}\left(C_3\left(C_4C_5C_6 - S_4S_6\right) - S_3\left(C_5C_6S_4 + C_4S_6\right)\right) + C_3L_1 - S_{34}L_2\big) m_6\dot{\theta}_3\dot{\theta}_2\\
&+2C_2\left(\mathrm{COM}_{4x}C_{34} - \mathrm{COM}_{4y}S_{34}\right) m_4\dot{\theta}_4\dot{\theta}_2 + 2C_2\left(-\mathrm{COM}_{5z}S_{34} - L_2S_{34} + C_{34}\left(\mathrm{COM}_{5x}C_5\right.\right.\\
&-\mathrm{COM}_{5y}S_5\big)\big) m_5\dot{\theta}_4\dot{\theta}_2 + 2C_2\left(-\mathrm{COM}_{6y}C_4C_6S_3 - \mathrm{COM}_{6x}C_5C_6S_4S_3 - \mathrm{COM}_{6x}C_4S_6S_3\right.\\
&+\mathrm{COM}_{6y}C_5S_4S_6S_3 + \mathrm{COM}_{6z}C_{34}S_5 + C_3\left(C_4C_5\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right) - S_4\left(\mathrm{COM}_{6y}C_6\right.\right.\\
&+\mathrm{COM}_{6x}S_6\big)\big) - S_{34}L_2\big) m_6\dot{\theta}_4\dot{\theta}_2 - 2\left(C_5\left(\mathrm{COM}_{5y}C_2S_{34} - \mathrm{COM}_{5x}S_2\right) + \left(\mathrm{COM}_{5y}S_2\right.\right.\\
&+\mathrm{COM}_{5x}C_2S_{34}\big) S_5\big) m_5\dot{\theta}_5\dot{\theta}_2 + 2\left(C_5\left(\mathrm{COM}_{6x}C_6S_2 - \mathrm{COM}_{6y}S_6S_2 + \mathrm{COM}_{6z}C_2S_{34}\right) + S_5\left(\mathrm{COM}_{6z}S_2\right.\right.\\
&+C_2S_{34}\left(\mathrm{COM}_{6y}S_6 - \mathrm{COM}_{6x}C_6\right)\big)\big) m_6\dot{\theta}_5\dot{\theta}_2 + 2\left(C_2\left(C_3\left(C_4\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right)\right.\right.\right.\\
&-C_5S_4\left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right)\big) - S_3\left(C_4C_5\left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right) + S_4\left(\mathrm{COM}_{6x}C_6\right.\right.\\
&-\mathrm{COM}_{6y}S_6\big)\big)\big) - S_2S_5\left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right)\big) m_6\dot{\theta}_6\dot{\theta}_2 - S_2\left(\mathrm{COM}_{3y}C_3 + \mathrm{COM}_{3x}S_3\right) m_3\dot{\theta}_3{}^2 \ldots
\end{aligned}$$

$$
\begin{aligned}
& - S_2 \left(\mathrm{COM}_{4y} C_{34} + \mathrm{COM}_{4x} S_{34} + S_3 L_1\right) m_4 \dot{\theta}_3{}^2 - S_2 \left(\mathrm{COM}_{5z} C_{34} + L_2 C_{34} + \mathrm{COM}_{5x} C_5 S_{34}\right. \\
& -\mathrm{COM}_{5y} S_{34} S_5 + S_3 L_1\big) m_5 \dot{\theta}_3{}^2 + S_2 \left(-\mathrm{COM}_{6z} S_{34} S_5 + \mathrm{COM}_{6x} \left(S_4 \left(S_3 S_6 - C_3 C_5 C_6\right) - C_4 \left(C_5 C_6 S_3\right.\right.\right. \\
& \left.\left. +C_3 S_6\right)\right) + \mathrm{COM}_{6y} \left(S_3 \left(C_6 S_4 + C_4 C_5 S_6\right) + C_3 \left(C_5 S_4 S_6 - C_4 C_6\right)\right) - S_3 L_1 - C_{34} L_2\big) m_6 \dot{\theta}_3{}^2 \\
& - S_2 \left(\mathrm{COM}_{4y} C_{34} + \mathrm{COM}_{4x} S_{34}\right) m_4 \dot{\theta}_4{}^2 - S_2 \left(\mathrm{COM}_{5z} C_{34} + L_2 C_{34} + S_{34} \left(\mathrm{COM}_{5x} C_5\right.\right. \\
& \left.\left. -\mathrm{COM}_{5y} S_5\right)\right) m_5 \dot{\theta}_4{}^2 + S_2 \left(-\mathrm{COM}_{6x} C_4 C_5 C_6 S_3 + \mathrm{COM}_{6y} C_6 S_4 S_3 + \mathrm{COM}_{6y} C_4 C_5 S_6 S_3\right. \\
& +\mathrm{COM}_{6x} S_4 S_6 S_3 - \mathrm{COM}_{6z} S_{34} S_5 - C_3 \left(C_4 \left(\mathrm{COM}_{6y} C_6 + \mathrm{COM}_{6x} S_6\right) + C_5 S_4 \left(\mathrm{COM}_{6x} C_6\right.\right. \\
& \left.\left. -\mathrm{COM}_{6y} S_6\right)\right) - C_{34} L_2\big) m_6 \dot{\theta}_4{}^2 + \left(C_2 \left(\mathrm{COM}_{5y} C_5 + \mathrm{COM}_{5x} S_5\right) + S_2 S_{34} \left(\mathrm{COM}_{5y} S_5\right.\right. \\
& \left.\left. -\mathrm{COM}_{5x} C_5\right)\right) m_5 \dot{\theta}_5{}^2 + \left(-S_2 S_{34} \left(\mathrm{COM}_{6z} S_5 + C_5 \left(\mathrm{COM}_{6x} C_6 - \mathrm{COM}_{6y} S_6\right)\right) - C_2 \left(\mathrm{COM}_{6z} C_5\right.\right. \\
& \left.\left. +S_5 \left(\mathrm{COM}_{6y} S_6 - \mathrm{COM}_{6x} C_6\right)\right)\right) m_6 \dot{\theta}_5{}^2 + \left(\mathrm{COM}_{6y} C_6 S_2 S_3 S_4 + \mathrm{COM}_{6x} S_2 S_3 S_6 S_4 + \mathrm{COM}_{6x} C_2 C_6 S_5\right. \\
& -\mathrm{COM}_{6y} C_2 S_5 S_6 - C_4 C_5 S_2 S_3 \left(\mathrm{COM}_{6x} C_6 - \mathrm{COM}_{6y} S_6\right) - C_3 S_2 \left(C_4 \left(\mathrm{COM}_{6y} C_6 + \mathrm{COM}_{6x} S_6\right)\right. \\
& \left.\left. +C_5 S_4 \left(\mathrm{COM}_{6x} C_6 - \mathrm{COM}_{6y} S_6\right)\right)\right) m_6 \dot{\theta}_6{}^2 - 2 S_2 \left(\mathrm{COM}_{4y} C_{34} + \mathrm{COM}_{4x} S_{34}\right) m_4 \dot{\theta}_3 \dot{\theta}_4 \\
& - 2 S_2 \left(\mathrm{COM}_{5z} C_{34} + L_2 C_{34} + S_{34} \left(\mathrm{COM}_{5x} C_5 - \mathrm{COM}_{5y} S_5\right)\right) m_5 \dot{\theta}_3 \dot{\theta}_4 + 2 S_2 \left(-\mathrm{COM}_{6x} C_4 C_5 C_6 S_3\right. \\
& +\mathrm{COM}_{6y} C_6 S_4 S_3 + \mathrm{COM}_{6y} C_4 C_5 S_6 S_3 + \mathrm{COM}_{6x} S_4 S_6 S_3 - \mathrm{COM}_{6z} S_{34} S_5 - C_3 \left(C_4 \left(\mathrm{COM}_{6y} C_6\right.\right. \\
& \left.\left. +\mathrm{COM}_{6x} S_6\right) + C_5 S_4 \left(\mathrm{COM}_{6x} C_6 - \mathrm{COM}_{6y} S_6\right)\right) - C_{34} L_2\big) m_6 \dot{\theta}_3 \dot{\theta}_4 - 2 C_{34} S_2 \left(\mathrm{COM}_{5y} C_5\right. \\
& \left. +\mathrm{COM}_{5x} S_5\right) m_5 \dot{\theta}_3 \dot{\theta}_5 + 2 C_{34} S_2 \left(\mathrm{COM}_{6z} C_5 + S_5 \left(\mathrm{COM}_{6y} S_6 - \mathrm{COM}_{6x} C_6\right)\right) m_6 \dot{\theta}_3 \dot{\theta}_5 \\
& - 2 C_{34} S_2 \left(\mathrm{COM}_{5y} C_5 + \mathrm{COM}_{5x} S_5\right) m_5 \dot{\theta}_4 \dot{\theta}_5 + 2 C_{34} S_2 \left(\mathrm{COM}_{6z} C_5 + S_5 \left(\mathrm{COM}_{6y} S_6\right.\right. \\
& \left.\left. -\mathrm{COM}_{6x} C_6\right)\right) m_6 \dot{\theta}_4 \dot{\theta}_5 + 2 S_2 \left(S_3 \left(C_5 S_4 \left(\mathrm{COM}_{6y} C_6 + \mathrm{COM}_{6x} S_6\right) + C_4 \left(\mathrm{COM}_{6y} S_6 - \mathrm{COM}_{6x} C_6\right)\right)\right. \\
& \left. -C_3 \left(C_4 C_5 \left(\mathrm{COM}_{6y} C_6 + \mathrm{COM}_{6x} S_6\right) + S_4 \left(\mathrm{COM}_{6x} C_6 - \mathrm{COM}_{6y} S_6\right)\right)\right) m_6 \dot{\theta}_3 \dot{\theta}_6 \\
& + 2 S_2 \left(S_3 \left(C_5 S_4 \left(\mathrm{COM}_{6y} C_6 + \mathrm{COM}_{6x} S_6\right) + C_4 \left(\mathrm{COM}_{6y} S_6 - \mathrm{COM}_{6x} C_6\right)\right)\right. \\
& \left. -C_3 \left(C_4 C_5 \left(\mathrm{COM}_{6y} C_6 + \mathrm{COM}_{6x} S_6\right) + S_4 \left(\mathrm{COM}_{6x} C_6 - \mathrm{COM}_{6y} S_6\right)\right)\right) m_6 \dot{\theta}_4 \dot{\theta}_6 + 2 \left(C_2 C_5\right. \\
& \left. +S_2 S_{34} S_5\right) \left(\mathrm{COM}_{6y} C_6 + \mathrm{COM}_{6x} S_6\right) m_6 \dot{\theta}_5 \dot{\theta}_6
\end{aligned} \tag{A.3a}
$$

$$
\begin{aligned}
V_2 = \frac{1}{16} \Big(\dot{\theta}_2 \big(\big(16 S_{33} \left(m_4 + m_5 + m_6\right) L_1^2 + 8 \big(4 \left(\mathrm{COM}_{4y} C_{334} + \mathrm{COM}_{4x} S_{334}\right) m_4 \\
+4 \left(\mathrm{COM}_{5z} C_{334} + L_2 C_{334} + S_{334} \left(\mathrm{COM}_{5x} C_5 - \mathrm{COM}_{5y} S_5\right)\right) m_5 + \big(2 \mathrm{COM}_{6z} C_{334-5} \\
-2 \mathrm{COM}_{6z} C_{3345} + 2 \mathrm{COM}_{6y} C_{334-6} - \mathrm{COM}_{6y} C_{334-56} - \mathrm{COM}_{6y} C_{3345-6} + 2 \mathrm{COM}_{6y} C_{3346} \\
+\mathrm{COM}_{6y} C_{3346-5} + \mathrm{COM}_{6y} C_{33456} - 2 \mathrm{COM}_{6x} S_{334-6} + \mathrm{COM}_{6x} S_{334-56} + \mathrm{COM}_{6x} S_{3345-6} \\
+2 \mathrm{COM}_{6x} S_{3346} + \mathrm{COM}_{6x} S_{3346-5} + \mathrm{COM}_{6x} S_{33456} + 4 C_{334} L_2\big) m_6\big) L_1 + 32 \mathrm{I}_{3xy} C_{33} \\
+32 \mathrm{I}_{4xy} C_{3344} - 8 \mathrm{I}_{5xy} C_{3344-55} + 16 \mathrm{I}_{5xz} C_{3344-5} + 8 \mathrm{I}_{5xy} C_{334455} + 16 \mathrm{I}_{5xz} C_{33445} \\
+8 \mathrm{I}_{6xy} C_{3344-566} + 8 \mathrm{I}_{6xy} C_{33445-66} - 12 \mathrm{I}_{6xy} C_{3344-66} + 4 \mathrm{I}_{6xz} C_{3344-556} - 2 \mathrm{I}_{6xy} C_{3344-5566} \\
-8 \mathrm{I}_{6xz} C_{3344-56} - 2 \mathrm{I}_{6xy} C_{334455-66} + 8 \mathrm{I}_{6xz} C_{33445-6} - 4 \mathrm{I}_{6xz} C_{334455-6} + 12 \mathrm{I}_{6xy} C_{334466} \ \ldots
\end{aligned}
$$

$$+4\mathrm{I}_{6xz}C_{334466-55}+2\mathrm{I}_{6xy}C_{334466-55}+8\mathrm{I}_{6xz}C_{33446-5}+2\mathrm{I}_{6xy}C_{33445566}-8\mathrm{I}_{6xz}C_{334456}$$
$$-4\mathrm{I}_{6xz}C_{3344556}+8\mathrm{I}_{6xy}C_{334466-5}+8\mathrm{I}_{6xy}C_{3344566}-16\mathrm{I}_{3xx}S_{33}+16\mathrm{I}_{3yy}S_{33}-16\mathrm{I}_{4xx}S_{3344}$$
$$+16\mathrm{I}_{4yy}S_{3344}-8\mathrm{I}_{5xx}S_{3344}-8\mathrm{I}_{5yy}S_{3344}+16\mathrm{I}_{5zz}S_{3344}+4\mathrm{I}_{6xx}S_{3344}+4\mathrm{I}_{6yy}S_{3344}-8\mathrm{I}_{6zz}S_{3344}$$
$$-4\mathrm{I}_{5xx}S_{3344-55}+4\mathrm{I}_{5yy}S_{3344-55}-2\mathrm{I}_{6xx}S_{3344-55}-2\mathrm{I}_{6yy}S_{3344-55}+4\mathrm{I}_{6zz}S_{3344-55}+16\mathrm{I}_{5yz}S_{3344-5}$$
$$-4\mathrm{I}_{5xx}S_{334455}+4\mathrm{I}_{5yy}S_{334455}-2\mathrm{I}_{6xx}S_{334455}-2\mathrm{I}_{6yy}S_{334455}+4\mathrm{I}_{6zz}S_{334455}-16\mathrm{I}_{5yz}S_{33445}$$
$$+4\mathrm{I}_{6xx}S_{3344-566}-4\mathrm{I}_{6yy}S_{3344-566}+4\mathrm{I}_{6xx}S_{33445-66}-4\mathrm{I}_{6yy}S_{33445-66}-6\mathrm{I}_{6xx}S_{3344-66}$$
$$+6\mathrm{I}_{6yy}S_{3344-66}+4\mathrm{I}_{6yz}S_{3344-556}-\mathrm{I}_{6xx}S_{3344-5566}+\mathrm{I}_{6yy}S_{3344-5566}-8\mathrm{I}_{6yz}S_{3344-56}$$
$$-\mathrm{I}_{6xx}S_{334455-66}+\mathrm{I}_{6yy}S_{334455-66}+8\mathrm{I}_{6yz}S_{33445-6}-4\mathrm{I}_{6yz}S_{334455-6}-6\mathrm{I}_{6xx}S_{334466}+6\mathrm{I}_{6yy}S_{334466}$$
$$-4\mathrm{I}_{6yz}S_{334466-55}-\mathrm{I}_{6xx}S_{334466-55}+\mathrm{I}_{6yy}S_{334466-55}-8\mathrm{I}_{6yz}S_{33446-5}-\mathrm{I}_{6xx}S_{33445566}+\mathrm{I}_{6yy}S_{33445566}$$
$$+8\mathrm{I}_{6yz}S_{334456}+4\mathrm{I}_{6yz}S_{3344556}-4\mathrm{I}_{6xx}S_{334466-5}+4\mathrm{I}_{6yy}S_{334466-5}-4\mathrm{I}_{6xx}S_{3344566}+4\mathrm{I}_{6yy}S_{3344566}$$
$$-16S_{3344}L_2^2\,(m_5+m_6)+8L_2\,(2\,(\mathrm{COM}_{5x}C_{3344-5}+\mathrm{COM}_{5x}C_{33445}-2\mathrm{COM}_{5z}S_{3344}$$
$$+\mathrm{COM}_{5y}S_{3344-5}-\mathrm{COM}_{5y}S_{33445})\,m_5+(-2\mathrm{COM}_{6x}C_{3344-6}+\mathrm{COM}_{6x}C_{3344-56}+\mathrm{COM}_{6x}C_{33445-6}$$
$$+2\mathrm{COM}_{6x}C_{33446}+\mathrm{COM}_{6x}C_{33446-5}+\mathrm{COM}_{6x}C_{334456}-2\mathrm{COM}_{6z}S_{3344-5}+2\mathrm{COM}_{6z}S_{33445}$$
$$-2\mathrm{COM}_{6y}S_{3344-6}+\mathrm{COM}_{6y}S_{3344-56}+\mathrm{COM}_{6y}S_{33445-6}-2\mathrm{COM}_{6y}S_{33446}-\mathrm{COM}_{6y}S_{33446-5}$$
$$-\mathrm{COM}_{6y}S_{334456})\,m_6))\,\dot{\theta}_3+\left(-16S_{3344}\,(m_5+m_6)\,L_2^2+8\,(2\,(\mathrm{COM}_{5x}C_{3344-5}+\mathrm{COM}_{5x}C_{33445}\right.$$
$$-2\mathrm{COM}_{5z}S_{3344}+\mathrm{COM}_{5y}S_{3344-5}-\mathrm{COM}_{5y}S_{33445})\,m_5+(-2\mathrm{COM}_{6x}C_{3344-6}+\mathrm{COM}_{6x}C_{3344-56}$$
$$+\mathrm{COM}_{6x}C_{33445-6}+2\mathrm{COM}_{6x}C_{33446}+\mathrm{COM}_{6x}C_{33446-5}+\mathrm{COM}_{6x}C_{334456}-2\mathrm{COM}_{6z}S_{3344-5}$$
$$+2\mathrm{COM}_{6z}S_{33445}-2\mathrm{COM}_{6y}S_{3344-6}+\mathrm{COM}_{6y}S_{3344-56}+\mathrm{COM}_{6y}S_{33445-6}-2\mathrm{COM}_{6y}S_{33446}$$
$$-\mathrm{COM}_{6y}S_{33446-5}-\mathrm{COM}_{6y}S_{334456})\,m_6)\,L_2+32\mathrm{I}_{4xy}C_{3344}-8\mathrm{I}_{5xy}C_{3344-55}+16\mathrm{I}_{5xz}C_{3344-5}$$
$$+8\mathrm{I}_{5xy}C_{334455}+16\mathrm{I}_{5xz}C_{33445}+8\mathrm{I}_{6xy}C_{3344-566}+8\mathrm{I}_{6xy}C_{33445-66}-12\mathrm{I}_{6xy}C_{3344-66}$$
$$+4\mathrm{I}_{6xz}C_{3344-556}-2\mathrm{I}_{6xy}C_{3344-5566}-8\mathrm{I}_{6xz}C_{3344-56}-2\mathrm{I}_{6xy}C_{334455-66}+8\mathrm{I}_{6xz}C_{33445-6}$$
$$-4\mathrm{I}_{6xz}C_{334455-6}+12\mathrm{I}_{6xy}C_{334466}+4\mathrm{I}_{6xz}C_{334466-55}+2\mathrm{I}_{6xy}C_{334466-55}+8\mathrm{I}_{6xz}C_{33446-5}$$
$$+2\mathrm{I}_{6xy}C_{33445566}-8\mathrm{I}_{6xz}C_{334456}-4\mathrm{I}_{6xz}C_{3344556}+8\mathrm{I}_{6xy}C_{334466-5}+8\mathrm{I}_{6xy}C_{3344566}-16\mathrm{I}_{4xx}S_{3344}$$
$$+16\mathrm{I}_{4yy}S_{3344}-8\mathrm{I}_{5xx}S_{3344}-8\mathrm{I}_{5yy}S_{3344}+16\mathrm{I}_{5zz}S_{3344}+4\mathrm{I}_{6xx}S_{3344}+4\mathrm{I}_{6yy}S_{3344}-8\mathrm{I}_{6zz}S_{3344}$$
$$-4\mathrm{I}_{5xx}S_{3344-55}+4\mathrm{I}_{5yy}S_{3344-55}-2\mathrm{I}_{6xx}S_{3344-55}-2\mathrm{I}_{6yy}S_{3344-55}+4\mathrm{I}_{6zz}S_{3344-55}+16\mathrm{I}_{5yz}S_{3344-5}$$
$$-4\mathrm{I}_{5xx}S_{334455}+4\mathrm{I}_{5yy}S_{334455}-2\mathrm{I}_{6xx}S_{334455}-2\mathrm{I}_{6yy}S_{334455}+4\mathrm{I}_{6zz}S_{334455}-16\mathrm{I}_{5yz}S_{33445}$$
$$+4\mathrm{I}_{6xx}S_{3344-566}-4\mathrm{I}_{6yy}S_{3344-566}+4\mathrm{I}_{6xx}S_{33445-66}-4\mathrm{I}_{6yy}S_{33445-66}-6\mathrm{I}_{6xx}S_{3344-66}$$
$$+6\mathrm{I}_{6yy}S_{3344-66}+4\mathrm{I}_{6yz}S_{3344-556}-\mathrm{I}_{6xx}S_{3344-5566}+\mathrm{I}_{6yy}S_{3344-5566}-8\mathrm{I}_{6yz}S_{3344-56}$$
$$-\mathrm{I}_{6xx}S_{334455-66}+\mathrm{I}_{6yy}S_{334455-66}+8\mathrm{I}_{6yz}S_{33445-6}-4\mathrm{I}_{6yz}S_{334455-6}-6\mathrm{I}_{6xx}S_{334466}+6\mathrm{I}_{6yy}S_{334466}$$
$$-4\mathrm{I}_{6yz}S_{334466-55}-\mathrm{I}_{6xx}S_{334466-55}+\mathrm{I}_{6yy}S_{334466-55}-8\mathrm{I}_{6yz}S_{33446-5}-\mathrm{I}_{6xx}S_{33445566}\ \ldots$$

$+\mathrm{I}_{6yy}S_{33445566}+8\mathrm{I}_{6yz}S_{334456}+4\mathrm{I}_{6yz}S_{3344556}-4\mathrm{I}_{6xx}S_{334466-5}+4\mathrm{I}_{6yy}S_{334466-5}-4\mathrm{I}_{6xx}S_{3344566}$

$+4\mathrm{I}_{6yy}S_{3344566}-16S_3L_1\left(\left(2\mathrm{COM}_{4y}S_{34}-2\mathrm{COM}_{4x}C_{34}\right)m_4-\left(\mathrm{COM}_{5x}C_{34-5}+\mathrm{COM}_{5x}C_{345}\right.\right.$

$\left.-2\mathrm{COM}_{5z}S_{34}+\mathrm{COM}_{5y}S_{34-5}-\mathrm{COM}_{5y}S_{345}-2S_{34}L_2\right)m_5+2\left(C_3\left(S_4\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)\right.\right.$

$\left.\left.-C_4\left(\mathrm{COM}_{6z}S_5+C_5\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right)\right)+S_3\left(C_4\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)\right.$

$\left.\left.\left.+S_4\left(\mathrm{COM}_{6z}S_5+C_5\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right)\right)+S_{34}L_2\right)m_6\right)\right)\dot{\theta}_4+8\mathrm{I}_{5xy}C_{3344-55}\dot{\theta}_5$

$-8\mathrm{I}_{5xz}C_{3344-5}\dot{\theta}_5+16\mathrm{I}_{5xy}C_{55}\dot{\theta}_5+8\mathrm{I}_{5xy}C_{334455}\dot{\theta}_5+8\mathrm{I}_{5xz}C_{33445}\dot{\theta}_5-4\mathrm{I}_{6xy}C_{3344-566}\dot{\theta}_5$

$+4\mathrm{I}_{6xy}C_{33445-66}\dot{\theta}_5-4\mathrm{I}_{6xz}C_{3344-556}\dot{\theta}_5+2\mathrm{I}_{6xy}C_{3344-5566}\dot{\theta}_5+4\mathrm{I}_{6xz}C_{3344-56}\dot{\theta}_5-4\mathrm{I}_{6xy}C_{55-66}\dot{\theta}_5$

$-2\mathrm{I}_{6xy}C_{334455-66}\dot{\theta}_5+4\mathrm{I}_{6xz}C_{33445-6}\dot{\theta}_5-8\mathrm{I}_{6xz}C_{55-6}\dot{\theta}_5-4\mathrm{I}_{6xz}C_{334455-6}\dot{\theta}_5-4\mathrm{I}_{6xz}C_{334466-55}\dot{\theta}_5$

$-2\mathrm{I}_{6xy}C_{334466-55}\dot{\theta}_5-4\mathrm{I}_{6xz}C_{33446-5}\dot{\theta}_5+4\mathrm{I}_{6xy}C_{5566}\dot{\theta}_5+2\mathrm{I}_{6xy}C_{33445566}\dot{\theta}_5-4\mathrm{I}_{6xz}C_{334456}\dot{\theta}_5$

$-8\mathrm{I}_{6xz}C_{556}\dot{\theta}_5-4\mathrm{I}_{6xz}C_{3344556}\dot{\theta}_5-4\mathrm{I}_{6xy}C_{334466-5}\dot{\theta}_5+4\mathrm{I}_{6xy}C_{3344566}\dot{\theta}_5+4\mathrm{I}_{5xx}S_{3344-55}\dot{\theta}_5$

$-4\mathrm{I}_{5yy}S_{3344-55}\dot{\theta}_5+2\mathrm{I}_{6xx}S_{3344-55}\dot{\theta}_5+2\mathrm{I}_{6yy}S_{3344-55}\dot{\theta}_5-4\mathrm{I}_{6zz}S_{3344-55}\dot{\theta}_5-8\mathrm{I}_{5yz}S_{3344-5}\dot{\theta}_5$

$-8\mathrm{I}_{5xx}S_{55}\dot{\theta}_5+8\mathrm{I}_{5yy}S_{55}\dot{\theta}_5-4\mathrm{I}_{6xx}S_{55}\dot{\theta}_5-4\mathrm{I}_{6yy}S_{55}\dot{\theta}_5+8\mathrm{I}_{6zz}S_{55}\dot{\theta}_5-4\mathrm{I}_{5xx}S_{334455}\dot{\theta}_5+4\mathrm{I}_{5yy}S_{334455}\dot{\theta}_5$

$-2\mathrm{I}_{6xx}S_{334455}\dot{\theta}_5-2\mathrm{I}_{6yy}S_{334455}\dot{\theta}_5+4\mathrm{I}_{6zz}S_{334455}\dot{\theta}_5-8\mathrm{I}_{5yz}S_{33445}\dot{\theta}_5-2\mathrm{I}_{6xx}S_{3344-566}\dot{\theta}_5\dot{\theta}_5$

$+2\mathrm{I}_{6yy}S_{3344-566}+2\mathrm{I}_{6xx}S_{33445-66}\dot{\theta}_5-2\mathrm{I}_{6yy}S_{33445-66}\dot{\theta}_5-4\mathrm{I}_{6yz}S_{3344-556}\dot{\theta}_5+\mathrm{I}_{6xx}S_{3344-5566}\dot{\theta}_5$

$-\mathrm{I}_{6yy}S_{3344-5566}\dot{\theta}_5+4\mathrm{I}_{6yz}S_{3344-56}\dot{\theta}_5-2\mathrm{I}_{6xx}S_{55-66}\dot{\theta}_5+2\mathrm{I}_{6yy}S_{55-66}\dot{\theta}_5-\mathrm{I}_{6xx}S_{334455-66}\dot{\theta}_5$

$+\mathrm{I}_{6yy}S_{334455-66}\dot{\theta}_5+4\mathrm{I}_{6yz}S_{33445-6}\dot{\theta}_5-8\mathrm{I}_{6yz}S_{55-6}\dot{\theta}_5-4\mathrm{I}_{6yz}S_{334455-6}\dot{\theta}_5+4\mathrm{I}_{6yz}S_{334466-55}\dot{\theta}_5$

$+\mathrm{I}_{6xx}S_{334466-55}\dot{\theta}_5-\mathrm{I}_{6yy}S_{334466-55}\dot{\theta}_5+4\mathrm{I}_{6yz}S_{33446-5}\dot{\theta}_5-2\mathrm{I}_{6xx}S_{5566}\dot{\theta}_5+2\mathrm{I}_{6yy}S_{5566}\dot{\theta}_5$

$-\mathrm{I}_{6xx}S_{33445566}\dot{\theta}_5+\mathrm{I}_{6yy}S_{33445566}\dot{\theta}_5+4\mathrm{I}_{6yz}S_{334456}\dot{\theta}_5+8\mathrm{I}_{6yz}S_{556}\dot{\theta}_5+4\mathrm{I}_{6yz}S_{3344556}\dot{\theta}_5$

$+2\mathrm{I}_{6xx}S_{334466-5}\dot{\theta}_5-2\mathrm{I}_{6yy}S_{334466-5}\dot{\theta}_5-2\mathrm{I}_{6xx}S_{3344566}\dot{\theta}_5+2\mathrm{I}_{6yy}S_{3344566}\dot{\theta}_5-8\mathrm{COM}_{5y}C_{4-5}L_1m_5\dot{\theta}_5$

$+8\mathrm{COM}_{5y}C_{334-5}L_1m_5\dot{\theta}_5-8\mathrm{COM}_{5y}C_{45}L_1m_5\dot{\theta}_5+8\mathrm{COM}_{5y}C_{3345}L_1m_5\dot{\theta}_5+8\mathrm{COM}_{5x}S_{4-5}L_1m_5\dot{\theta}_5$

$-8\mathrm{COM}_{5x}S_{334-5}L_1m_5\dot{\theta}_5-8\mathrm{COM}_{5x}S_{45}L_1m_5\dot{\theta}_5+8\mathrm{COM}_{5x}S_{3345}L_1m_5\dot{\theta}_5$

$-8\mathrm{COM}_{5x}C_{3344-5}L_2m_5\dot{\theta}_5+8\mathrm{COM}_{5x}C_{33445}L_2m_5\dot{\theta}_5-8\mathrm{COM}_{5y}S_{3344-5}L_2m_5\dot{\theta}_5$

$-8\mathrm{COM}_{5y}S_{33445}L_2m_5\dot{\theta}_5+8\mathrm{COM}_{6z}C_{4-5}L_1m_6\dot{\theta}_5-8\mathrm{COM}_{6z}C_{334-5}L_1m_6\dot{\theta}_5$

$+8\mathrm{COM}_{6z}C_{45}L_1m_6\dot{\theta}_5-8\mathrm{COM}_{6z}C_{3345}L_1m_6\dot{\theta}_5-4\mathrm{COM}_{6y}C_{4-56}L_1m_6\dot{\theta}_5$

$+4\mathrm{COM}_{6y}C_{334-56}L_1m_6\dot{\theta}_5+4\mathrm{COM}_{6y}C_{45-6}L_1m_6\dot{\theta}_5-4\mathrm{COM}_{6y}C_{3345-6}L_1m_6\dot{\theta}_5$

$+4\mathrm{COM}_{6y}C_{46-5}L_1m_6\dot{\theta}_5-4\mathrm{COM}_{6y}C_{3346-5}L_1m_6\dot{\theta}_5-4\mathrm{COM}_{6y}C_{456}L_1m_6\dot{\theta}_5$

$+4\mathrm{COM}_{6y}C_{33456}L_1m_6\dot{\theta}_5+4\mathrm{COM}_{6x}S_{4-56}L_1m_6\dot{\theta}_5-4\mathrm{COM}_{6x}S_{334-56}L_1m_6\dot{\theta}_5$

$-4\mathrm{COM}_{6x}S_{45-6}L_1m_6\dot{\theta}_5+4\mathrm{COM}_{6x}S_{3345-6}L_1m_6\dot{\theta}_5+4\mathrm{COM}_{6x}S_{46-5}L_1m_6\dot{\theta}_5$

$-4\mathrm{COM}_{6x}S_{3346-5}L_1m_6\dot{\theta}_5-4\mathrm{COM}_{6x}S_{456}L_1m_6\dot{\theta}_5+4\mathrm{COM}_{6x}S_{33456}L_1m_6\dot{\theta}_5$

$-4\mathrm{COM}_{6x}C_{3344-56}L_2m_6\dot{\theta}_5+4\mathrm{COM}_{6x}C_{33445-6}L_2m_6\dot{\theta}_5-4\mathrm{COM}_{6x}C_{33446-5}L_2m_6\dot{\theta}_5\ \ldots$

$$+4\mathrm{COM}_{6x}C_{334456}L_2m_6\dot{\theta}_5+8\mathrm{COM}_{6z}S_{3344-5}L_2m_6\dot{\theta}_5+8\mathrm{COM}_{6z}S_{33445}L_2m_6\dot{\theta}_5$$
$$-4\mathrm{COM}_{6y}S_{3344-56}L_2m_6\dot{\theta}_5+4\mathrm{COM}_{6y}S_{33445-6}L_2m_6\dot{\theta}_5+4\mathrm{COM}_{6y}S_{33446-5}L_2m_6\dot{\theta}_5$$
$$-4\mathrm{COM}_{6y}S_{334456}L_2m_6\dot{\theta}_5-8\mathrm{I}_{6xy}C_{3344-566}\dot{\theta}_6-8\mathrm{I}_{6xy}C_{33445-66}\dot{\theta}_6+12\mathrm{I}_{6xy}C_{3344-66}\dot{\theta}_6$$
$$-2\mathrm{I}_{6xz}C_{3344-556}\dot{\theta}_6+2\mathrm{I}_{6xy}C_{3344-5566}\dot{\theta}_6+4\mathrm{I}_{6xz}C_{3344-56}\dot{\theta}_6+4\mathrm{I}_{6xy}C_{55-66}\dot{\theta}_6+2\mathrm{I}_{6xy}C_{334455-66}\dot{\theta}_6$$
$$-4\mathrm{I}_{6xz}C_{33445-6}\dot{\theta}_6+4\mathrm{I}_{6xz}C_{55-6}\dot{\theta}_6+2\mathrm{I}_{6xz}C_{334455-6}\dot{\theta}_6-8\mathrm{I}_{6xy}C_{66}\dot{\theta}_6+12\mathrm{I}_{6xy}C_{334466}\dot{\theta}_6$$
$$+2\mathrm{I}_{6xz}C_{334466-55}\dot{\theta}_6+2\mathrm{I}_{6xy}C_{334466-55}\dot{\theta}_6+4\mathrm{I}_{6xz}C_{33446-5}\dot{\theta}_6+4\mathrm{I}_{6xy}C_{5566}\dot{\theta}_6+2\mathrm{I}_{6xy}C_{33445566}\dot{\theta}_6$$
$$-4\mathrm{I}_{6xz}C_{334456}\dot{\theta}_6-4\mathrm{I}_{6xz}C_{556}\dot{\theta}_6-2\mathrm{I}_{6xz}C_{3344556}\dot{\theta}_6+8\mathrm{I}_{6xy}C_{334466-5}\dot{\theta}_6+8\mathrm{I}_{6xy}C_{3344566}\dot{\theta}_6$$
$$-4\mathrm{I}_{6xx}S_{3344-566}\dot{\theta}_6+4\mathrm{I}_{6yy}S_{3344-566}\dot{\theta}_6-4\mathrm{I}_{6xx}S_{33445-66}\dot{\theta}_6+4\mathrm{I}_{6yy}S_{33445-66}\dot{\theta}_6+6\mathrm{I}_{6xx}S_{3344-66}\dot{\theta}_6$$
$$-6\mathrm{I}_{6yy}S_{3344-66}\dot{\theta}_6-2\mathrm{I}_{6yz}S_{3344-556}\dot{\theta}_6+\mathrm{I}_{6xx}S_{3344-5566}\dot{\theta}_6-\mathrm{I}_{6yy}S_{3344-5566}\dot{\theta}_6+4\mathrm{I}_{6yz}S_{3344-56}\dot{\theta}_6$$
$$+2\mathrm{I}_{6xx}S_{55-66}\dot{\theta}_6-2\mathrm{I}_{6yy}S_{55-66}\dot{\theta}_6+\mathrm{I}_{6xx}S_{334455-66}\dot{\theta}_6-\mathrm{I}_{6yy}S_{334455-66}\dot{\theta}_6-4\mathrm{I}_{6yz}S_{33445-6}\dot{\theta}_6$$
$$+4\mathrm{I}_{6yz}S_{55-6}\dot{\theta}_6+2\mathrm{I}_{6yz}S_{334455-6}\dot{\theta}_6+4\mathrm{I}_{6xx}S_{66}\dot{\theta}_6-4\mathrm{I}_{6yy}S_{66}\dot{\theta}_6-6\mathrm{I}_{6xx}S_{334466}\dot{\theta}_6+6\mathrm{I}_{6yy}S_{334466}\dot{\theta}_6$$
$$-2\mathrm{I}_{6yz}S_{334466-55}\dot{\theta}_6-\mathrm{I}_{6xx}S_{334466-55}\dot{\theta}_6+\mathrm{I}_{6yy}S_{334466-55}\dot{\theta}_6-4\mathrm{I}_{6yz}S_{33446-5}\dot{\theta}_6-2\mathrm{I}_{6xx}S_{5566}\dot{\theta}_6$$
$$+2\mathrm{I}_{6yy}S_{5566}\dot{\theta}_6-\mathrm{I}_{6xx}S_{33445566}\dot{\theta}_6+\mathrm{I}_{6yy}S_{33445566}\dot{\theta}_6+4\mathrm{I}_{6yz}S_{334456}\dot{\theta}_6+4\mathrm{I}_{6yz}S_{556}\dot{\theta}_6+2\mathrm{I}_{6yz}S_{3344556}\dot{\theta}_6$$
$$-4\mathrm{I}_{6xx}S_{334466-5}\dot{\theta}_6+4\mathrm{I}_{6yy}S_{334466-5}\dot{\theta}_6-4\mathrm{I}_{6xx}S_{3344566}\dot{\theta}_6+4\mathrm{I}_{6yy}S_{3344566}\dot{\theta}_6+8\mathrm{COM}_{6y}C_{4-6}L_1m_6\dot{\theta}_6$$
$$-8\mathrm{COM}_{6y}C_{334-6}L_1m_6\dot{\theta}_6-4\mathrm{COM}_{6y}C_{4-56}L_1m_6\dot{\theta}_6+4\mathrm{COM}_{6y}C_{334-56}L_1m_6\dot{\theta}_6$$
$$-4\mathrm{COM}_{6y}C_{45-6}L_1m_6\dot{\theta}_6+4\mathrm{COM}_{6y}C_{3345-6}L_1m_6\dot{\theta}_6-8\mathrm{COM}_{6y}C_{46}L_1m_6\dot{\theta}_6+8\mathrm{COM}_{6y}C_{3346}L_1m_6\dot{\theta}_6$$
$$-4\mathrm{COM}_{6y}C_{46-5}L_1m_6\dot{\theta}_6+4\mathrm{COM}_{6y}C_{3346-5}L_1m_6\dot{\theta}_6-4\mathrm{COM}_{6y}C_{456}L_1m_6\dot{\theta}_6$$
$$+4\mathrm{COM}_{6y}C_{33456}L_1m_6\dot{\theta}_6-8\mathrm{COM}_{6x}S_{4-6}L_1m_6\dot{\theta}_6+8\mathrm{COM}_{6x}S_{334-6}L_1m_6\dot{\theta}_6$$
$$+4\mathrm{COM}_{6x}S_{4-56}L_1m_6\dot{\theta}_6-4\mathrm{COM}_{6x}S_{334-56}L_1m_6\dot{\theta}_6+4\mathrm{COM}_{6x}S_{45-6}L_1m_6\dot{\theta}_6$$
$$-4\mathrm{COM}_{6x}S_{3345-6}L_1m_6\dot{\theta}_6-8\mathrm{COM}_{6x}S_{46}L_1m_6\dot{\theta}_6+8\mathrm{COM}_{6x}S_{3346}L_1m_6\dot{\theta}_6-4\mathrm{COM}_{6x}S_{46-5}L_1m_6\dot{\theta}_6$$
$$+4\mathrm{COM}_{6x}S_{3346-5}L_1m_6\dot{\theta}_6-4\mathrm{COM}_{6x}S_{456}L_1m_6\dot{\theta}_6+4\mathrm{COM}_{6x}S_{33456}L_1m_6\dot{\theta}_6$$
$$+8\mathrm{COM}_{6x}C_{3344-6}L_2m_6\dot{\theta}_6-4\mathrm{COM}_{6x}C_{3344-56}L_2m_6\dot{\theta}_6-4\mathrm{COM}_{6x}C_{33445-6}L_2m_6\dot{\theta}_6$$
$$+16\mathrm{COM}_{6x}C_{6}L_2m_6\dot{\theta}_6+8\mathrm{COM}_{6x}C_{33446}L_2m_6\dot{\theta}_6+4\mathrm{COM}_{6x}C_{33446-5}L_2m_6\dot{\theta}_6$$
$$+4\mathrm{COM}_{6x}C_{334456}L_2m_6\dot{\theta}_6+8\mathrm{COM}_{6y}S_{3344-6}L_2m_6\dot{\theta}_6-4\mathrm{COM}_{6y}S_{3344-56}L_2m_6\dot{\theta}_6$$
$$-4\mathrm{COM}_{6y}S_{33445-6}L_2m_6\dot{\theta}_6-16\mathrm{COM}_{6y}S_{6}L_2m_6\dot{\theta}_6-8\mathrm{COM}_{6y}S_{33446}L_2m_6\dot{\theta}_6$$
$$-4\mathrm{COM}_{6y}S_{33446-5}L_2m_6\dot{\theta}_6-4\mathrm{COM}_{6y}S_{334456}L_2m_6\dot{\theta}_6\Big)+4\Big(\Big(4\mathrm{I}_{6xz}C_4C_6S_3C_5^2+4\mathrm{I}_{6xz}C_3C_6S_4C_5^2$$
$$-4\mathrm{I}_{5xy}S_{34}C_5^2-4\mathrm{I}_{6yz}C_4S_3S_6C_5^2-4\mathrm{I}_{6yz}C_3S_4S_6C_5^2-4\mathrm{I}_{5yz}C_{34}C_5+4\mathrm{I}_{6yz}C_3C_4C_6C_5$$
$$-4\mathrm{I}_{6yz}C_6S_3S_4C_5+4\mathrm{I}_{6xz}C_3C_4S_6C_5-4\mathrm{I}_{6xz}S_3S_4S_6C_5+4\mathrm{I}_{5xy}S_{34}S_5^2-4\mathrm{I}_{6xz}C_6S_{34}S_5^2$$
$$+4\mathrm{I}_{6xy}C_3C_4S_5S_6^2-4\mathrm{I}_{6xy}S_3S_4S_5S_6^2-\mathrm{I}_{6xx}C_4S_3S_{55}S_6^2+\mathrm{I}_{6yy}C_4S_3S_{55}S_6^2-\mathrm{I}_{6zz}C_4S_3S_{55}S_6^2$$
$$-\mathrm{I}_{6xx}C_3S_4S_{55}S_6^2+\mathrm{I}_{6yy}C_3S_4S_{55}S_6^2-\mathrm{I}_{6zz}C_3S_4S_{55}S_6^2+4\mathrm{I}_{3yz}C_3+4\mathrm{I}_{4yz}C_{34}+4\mathrm{I}_{3xz}S_3+4\mathrm{I}_{4xz}S_{34}\;\ldots$$

$$-4\mathrm{I}_{6xy}C_3C_4C_6^2S_5-4\mathrm{I}_{5xz}C_{34}S_5+4\mathrm{I}_{6xy}C_6^2S_3S_4S_5+\mathrm{I}_{6xx}C_4C_6^2S_3S_{55}-\mathrm{I}_{6yy}C_4C_6^2S_3S_{55}$$
$$-\mathrm{I}_{6zz}C_4C_6^2S_3S_{55}+\mathrm{I}_{6xx}C_3C_6^2S_4S_{55}-\mathrm{I}_{6yy}C_3C_6^2S_4S_{55}-\mathrm{I}_{6zz}C_3C_6^2S_4S_{55}+2\mathrm{I}_{5xx}S_{34}S_{55}$$
$$-2\mathrm{I}_{5yy}S_{34}S_{55}+\mathrm{I}_{6xx}S_{34}S_{55}+\mathrm{I}_{6yy}S_{34}S_{55}-\mathrm{I}_{6zz}S_{34}S_{55}+4\mathrm{I}_{6yz}S_{34}S_5^2S_6+2\mathrm{I}_{6xx}C_3C_4S_5S_{66}$$
$$-2\mathrm{I}_{6yy}C_3C_4S_5S_{66}-2\mathrm{I}_{6xx}S_3S_4S_5S_{66}+2\mathrm{I}_{6yy}S_3S_4S_5S_{66}+2\mathrm{I}_{6xy}C_4S_3S_{55}S_{66}+2\mathrm{I}_{6xy}C_3S_4S_{55}S_{66}$$
$$-4C_{34}L_2\left(\left(\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5\right)m_5-\left(\mathrm{COM}_{6z}C_5+S_5\left(\mathrm{COM}_{6y}S_6-\mathrm{COM}_{6x}C_6\right)\right)m_6\right)$$
$$+4S_3L_1\left(\mathrm{COM}_{4z}m_4-\left(\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5\right)m_5+\left(\mathrm{COM}_{6z}C_5+S_5\left(\mathrm{COM}_{6y}S_6\right.\right.\right.$$
$$\left.\left.\left.-\mathrm{COM}_{6x}C_6\right)\right)m_6\right)\right)\dot\theta_3{}^2+\left(2\left(4\mathrm{I}_{6xz}C_4C_6S_3C_5^2+4\mathrm{I}_{6xz}C_3C_6S_4C_5^2-4\mathrm{I}_{5xy}S_{34}C_5^2-4\mathrm{I}_{6yz}C_4S_3S_6C_5^2\right.\right.$$
$$-4\mathrm{I}_{6yz}C_3S_4S_6C_5^2+4\mathrm{I}_{6yz}C_3C_4C_6C_5-4\mathrm{I}_{6yz}C_6S_3S_4C_5+4\mathrm{I}_{6xz}C_3C_4S_6C_5-4\mathrm{I}_{6xz}S_3S_4S_6C_5$$
$$+4\mathrm{I}_{5xy}S_{34}S_5^2-4\mathrm{I}_{6xz}C_6S_{34}S_5^2+4\mathrm{I}_{6xy}C_3C_4S_5S_6^2-4\mathrm{I}_{6xy}S_3S_4S_5S_6^2-\mathrm{I}_{6xx}C_4S_3S_{55}S_6^2$$
$$+\mathrm{I}_{6yy}C_4S_3S_{55}S_6^2-\mathrm{I}_{6zz}C_4S_3S_{55}S_6^2-\mathrm{I}_{6xx}C_3S_4S_{55}S_6^2+\mathrm{I}_{6yy}C_3S_4S_{55}S_6^2-\mathrm{I}_{6zz}C_3S_4S_{55}S_6^2$$
$$+4\mathrm{I}_{4xz}S_{34}-4\mathrm{I}_{6xy}C_3C_4C_6^2S_5+4\mathrm{I}_{6xy}C_6^2S_3S_4S_5+4C_{34}\left(\mathrm{I}_{4yz}-\mathrm{I}_{5yz}C_5-\mathrm{I}_{5xz}S_5\right)+\mathrm{I}_{6xx}C_4C_6^2S_3S_{55}$$
$$-\mathrm{I}_{6yy}C_4C_6^2S_3S_{55}-\mathrm{I}_{6zz}C_4C_6^2S_3S_{55}+\mathrm{I}_{6xx}C_3C_6^2S_4S_{55}-\mathrm{I}_{6yy}C_3C_6^2S_4S_{55}-\mathrm{I}_{6zz}C_3C_6^2S_4S_{55}$$
$$+2\mathrm{I}_{5xx}S_{34}S_{55}-2\mathrm{I}_{5yy}S_{34}S_{55}+\mathrm{I}_{6xx}S_{34}S_{55}+\mathrm{I}_{6yy}S_{34}S_{55}-\mathrm{I}_{6zz}S_{34}S_{55}+4\mathrm{I}_{6yz}S_{34}S_5^2S_6$$
$$+2\mathrm{I}_{6xx}C_3C_4S_5S_{66}-2\mathrm{I}_{6yy}C_3C_4S_5S_{66}-2\mathrm{I}_{6xx}S_3S_4S_5S_{66}+2\mathrm{I}_{6yy}S_3S_4S_5S_{66}+2\mathrm{I}_{6xy}C_4S_3S_{55}S_{66}$$
$$+2\mathrm{I}_{6xy}C_3S_4S_{55}S_{66}-4C_{34}L_2\left(\left(\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5\right)m_5-\left(\mathrm{COM}_{6z}C_5+S_5\left(\mathrm{COM}_{6y}S_6\right.\right.\right.$$
$$\left.\left.\left.-\mathrm{COM}_{6x}C_6\right)\right)m_6\right)\right)\dot\theta_4-C_{34}\left(4\mathrm{I}_{5zz}+2\mathrm{I}_{6xx}+2\mathrm{I}_{6yy}+2(2\mathrm{I}_{5xx}-2\mathrm{I}_{5yy}+\mathrm{I}_{6xx}+\mathrm{I}_{6yy}-2\mathrm{I}_{6zz})C_{55}\right.$$
$$+(\mathrm{I}_{6xx}-\mathrm{I}_{6yy})C_{55-66}+4\mathrm{I}_{6yz}C_{55-6}-2\mathrm{I}_{6xx}C_{66}+2\mathrm{I}_{6yy}C_{66}+\mathrm{I}_{6xx}C_{5566}-\mathrm{I}_{6yy}C_{5566}-4\mathrm{I}_{6yz}C_{556}$$
$$\left.+8\mathrm{I}_{5xy}S_{55}-2\mathrm{I}_{6xy}S_{55-66}-4\mathrm{I}_{6xz}S_{55-6}-4\mathrm{I}_{6xy}S_{66}+2\mathrm{I}_{6xy}S_{5566}-4\mathrm{I}_{6xz}S_{556}\right)\dot\theta_5$$
$$+2\left(C_3\left(2S_4\left(2C_5\left(\mathrm{I}_{6xz}C_6-\mathrm{I}_{6yz}S_6\right)-S_5\left(\mathrm{I}_{6zz}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})C_{66}-2\mathrm{I}_{6xy}S_{66}\right)\right)+C_4\left(4\left(\mathrm{I}_{6yz}C_6\right.\right.\right.\right.$$
$$\left.\left.\left.+\mathrm{I}_{6xz}S_6\right)C_5^2+S_{55}\left((\mathrm{I}_{6xx}-\mathrm{I}_{6yy})S_{66}-2\mathrm{I}_{6xy}C_{66}\right)\right)\right)+S_3\left(2C_4\left(2C_5\left(\mathrm{I}_{6xz}C_6-\mathrm{I}_{6yz}S_6\right)-S_5\left(\mathrm{I}_{6zz}\right.\right.\right.$$
$$\left.\left.+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})C_{66}-2\mathrm{I}_{6xy}S_{66}\right)\right)+S_4\left(S_{55}\left(2\mathrm{I}_{6xy}C_{66}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})S_{66}\right)-4C_5^2\left(\mathrm{I}_{6yz}C_6\right.\right.$$
$$\left.\left.\left.\left.+\mathrm{I}_{6xz}S_6\right)\right)\right)\right)\dot\theta_6\Big)\dot\theta_3+\left(4\mathrm{I}_{6xz}C_4C_6S_3C_5^2+4\mathrm{I}_{6xz}C_3C_6S_4C_5^2-4\mathrm{I}_{5xy}S_{34}C_5^2-4\mathrm{I}_{6yz}C_4S_3S_6C_5^2\right.$$
$$-4\mathrm{I}_{6yz}C_3S_4S_6C_5^2+4\mathrm{I}_{6yz}C_3C_4C_6C_5-4\mathrm{I}_{6yz}C_6S_3S_4C_5+4\mathrm{I}_{6xz}C_3C_4S_6C_5-4\mathrm{I}_{6xz}S_3S_4S_6C_5$$
$$+4\mathrm{I}_{5xy}S_{34}S_5^2-4\mathrm{I}_{6xz}C_6S_{34}S_5^2+4\mathrm{I}_{6xy}C_3C_4S_5S_6^2-4\mathrm{I}_{6xy}S_3S_4S_5S_6^2-\mathrm{I}_{6xx}C_4S_3S_{55}S_6^2$$
$$+\mathrm{I}_{6yy}C_4S_3S_{55}S_6^2-\mathrm{I}_{6zz}C_4S_3S_{55}S_6^2-\mathrm{I}_{6xx}C_3S_4S_{55}S_6^2+\mathrm{I}_{6yy}C_3S_4S_{55}S_6^2-\mathrm{I}_{6zz}C_3S_4S_{55}S_6^2$$
$$+4\mathrm{I}_{4xz}S_{34}-4\mathrm{I}_{6xy}C_3C_4C_6^2S_5+4\mathrm{I}_{6xy}C_6^2S_3S_4S_5+4C_{34}\left(\mathrm{I}_{4yz}-\mathrm{I}_{5yz}C_5-\mathrm{I}_{5xz}S_5\right)+\mathrm{I}_{6xx}C_4C_6^2S_3S_{55}$$
$$-\mathrm{I}_{6yy}C_4C_6^2S_3S_{55}-\mathrm{I}_{6zz}C_4C_6^2S_3S_{55}+\mathrm{I}_{6xx}C_3C_6^2S_4S_{55}-\mathrm{I}_{6yy}C_3C_6^2S_4S_{55}-\mathrm{I}_{6zz}C_3C_6^2S_4S_{55}$$
$$+2\mathrm{I}_{5xx}S_{34}S_{55}-2\mathrm{I}_{5yy}S_{34}S_{55}+\mathrm{I}_{6xx}S_{34}S_{55}+\mathrm{I}_{6yy}S_{34}S_{55}-\mathrm{I}_{6zz}S_{34}S_{55}+4\mathrm{I}_{6yz}S_{34}S_5^2S_6$$
$$+2\mathrm{I}_{6xx}C_3C_4S_5S_{66}-2\mathrm{I}_{6yy}C_3C_4S_5S_{66}-2\mathrm{I}_{6xx}S_3S_4S_5S_{66}+2\mathrm{I}_{6yy}S_3S_4S_5S_{66}+2\mathrm{I}_{6xy}C_4S_3S_{55}S_{66}$$
$$+2\mathrm{I}_{6xy}C_3S_4S_{55}S_{66}-4C_{34}L_2\left(\left(\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5\right)m_5-\left(\mathrm{COM}_{6z}C_5+S_5\left(\mathrm{COM}_{6y}S_6\quad\ldots\right.\right.\right.$$

$$
\begin{aligned}
&-\mathrm{COM}_{6x}C_6))\,m_6))\,\dot\theta_4{}^2-2\dot\theta_4\left(\frac{1}{2}C_{34}\,(4\mathrm{I}_{5zz}+2\mathrm{I}_{6xx}+2\mathrm{I}_{6yy}+2(2\mathrm{I}_{5xx}-2\mathrm{I}_{5yy}+\mathrm{I}_{6xx}+\mathrm{I}_{6yy}\right.\\
&-2\mathrm{I}_{6zz})C_{55}+(\mathrm{I}_{6xx}-\mathrm{I}_{6yy})C_{55-66}+4\mathrm{I}_{6yz}C_{55-6}-2\mathrm{I}_{6xx}C_{66}+2\mathrm{I}_{6yy}C_{66}+\mathrm{I}_{6xx}C_{5566}-\mathrm{I}_{6yy}C_{5566}\\
&-4\mathrm{I}_{6yz}C_{556}+8\mathrm{I}_{5xy}S_{55}-2\mathrm{I}_{6xy}S_{55-66}-4\mathrm{I}_{6xz}S_{55-6}-4\mathrm{I}_{6xy}S_{66}+2\mathrm{I}_{6xy}S_{5566}-4\mathrm{I}_{6xz}S_{556})\,\dot\theta_5\\
&+(S_3\,(C_4\,(C_5\,(4\mathrm{I}_{6yz}S_6-4\mathrm{I}_{6xz}C_6)+2S_5\,(\mathrm{I}_{6zz}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})C_{66}-2\mathrm{I}_{6xy}S_{66}))+S_4\,(4\,(\mathrm{I}_{6yz}C_6\\
&+\mathrm{I}_{6xz}S_6)\,C_5^2+S_{55}\,((\mathrm{I}_{6xx}-\mathrm{I}_{6yy})S_{66}-2\mathrm{I}_{6xy}C_{66})))+C_3\,(2S_4\,(C_5\,(2\mathrm{I}_{6yz}S_6-2\mathrm{I}_{6xz}C_6)\\
&+S_5\,(\mathrm{I}_{6zz}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})C_{66}-2\mathrm{I}_{6xy}S_{66}))+C_4\,(S_{55}\,(2\mathrm{I}_{6xy}C_{66}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})S_{66})\\
&\left.-4C_5^2\,(\mathrm{I}_{6yz}C_6+\mathrm{I}_{6xz}S_6))))\,\dot\theta_6\right)+4\left(\frac{1}{2}\,(2S_3L_1\,((\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5)\,m_5-(\mathrm{COM}_{6z}C_5\right.\\
&+S_5\,(\mathrm{COM}_{6y}S_6-\mathrm{COM}_{6x}C_6))\,m_6)+C_{34}\,(2C_5\,(\mathrm{I}_{5yz}-\mathrm{I}_{6yz}C_6-\mathrm{I}_{6xz}S_6)+S_5\,(2\mathrm{I}_{5xz}+2\mathrm{I}_{6xy}C_{66}\\
&+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})S_{66})+2L_2\,((\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5)\,m_5-(\mathrm{COM}_{6z}C_5+S_5\,(\mathrm{COM}_{6y}S_6\\
&-\mathrm{COM}_{6x}C_6))\,m_6)))\,\dot\theta_5{}^2+\left(\mathrm{I}_{6xx}C_3C_4C_5C_6^2-\mathrm{I}_{6yy}C_3C_4C_5C_6^2+2\mathrm{I}_{6xy}C_4S_3C_6^2+2\mathrm{I}_{6xy}C_3S_4C_6^2\right.\\
&-\mathrm{I}_{6xx}C_5S_3S_4C_6^2+\mathrm{I}_{6yy}C_5S_3S_4C_6^2-\mathrm{I}_{6xx}C_3C_4C_5S_6^2+\mathrm{I}_{6yy}C_3C_4C_5S_6^2-2\mathrm{I}_{6xy}C_4S_3S_6^2\\
&-2\mathrm{I}_{6xy}C_3S_4S_6^2+\mathrm{I}_{6xx}C_5S_3S_4S_6^2-\mathrm{I}_{6yy}C_5S_3S_4S_6^2+\mathrm{I}_{6zz}C_3C_4C_5-\mathrm{I}_{6zz}C_5S_3S_4+2\mathrm{I}_{6xy}C_3C_4C_5S_{66}\\
&-\mathrm{I}_{6xx}C_4S_3S_{66}+\mathrm{I}_{6yy}C_4S_3S_{66}-\mathrm{I}_{6xx}C_3S_4S_{66}+\mathrm{I}_{6yy}C_3S_4S_{66}-2\mathrm{I}_{6xy}C_5S_3S_4S_{66}\\
&+2C_5S_3\,(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6)\,L_1m_6+2C_{34}C_5\,(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6)\,L_2m_6)\,\dot\theta_6\dot\theta_5\\
&+(\mathrm{I}_{6yz}C_3C_4C_5C_6+\mathrm{I}_{6xz}C_4S_3C_6+\mathrm{I}_{6xz}C_3S_4C_6-\mathrm{I}_{6yz}C_5S_3S_4C_6+\mathrm{I}_{6xz}C_3C_4C_5S_6-\mathrm{I}_{6yz}C_4S_3S_6\\
&-\mathrm{I}_{6yz}C_3S_4S_6-\mathrm{I}_{6xz}C_5S_3S_4S_6+S_3S_5\,(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6)\,L_1m_6+C_{34}S_5\,(\mathrm{COM}_{6x}C_6\\
&\left.\left.\left.-\mathrm{COM}_{6y}S_6)\,L_2m_6)\,\dot\theta_6{}^2\right)\right)\right)
\end{aligned}
\tag{A.3b}
$$

$$
\begin{aligned}
V_3=\frac{1}{32}\,\big(-\big(&16S_{33}\,(m_4+m_5+m_6)\,L_1^2+8\,(4\,(\mathrm{COM}_{4y}C_{334}+\mathrm{COM}_{4x}S_{334})\,m_4+4\,(\mathrm{COM}_{5z}C_{334}+L_2C_{334}\\
&+S_{334}\,(\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5))\,m_5+(2\mathrm{COM}_{6z}C_{334-5}-2\mathrm{COM}_{6z}C_{3345}+2\mathrm{COM}_{6y}C_{334-6}\\
&-\mathrm{COM}_{6y}C_{334-56}-\mathrm{COM}_{6y}C_{3345-6}+2\mathrm{COM}_{6y}C_{3346}+\mathrm{COM}_{6y}C_{3346-5}+\mathrm{COM}_{6y}C_{33456}\\
&-2\mathrm{COM}_{6x}S_{334-6}+\mathrm{COM}_{6x}S_{334-56}+\mathrm{COM}_{6x}S_{3345-6}+2\mathrm{COM}_{6x}S_{3346}+\mathrm{COM}_{6x}S_{3346-5}\\
&+\mathrm{COM}_{6x}S_{33456}+4C_{334}L_2)\,m_6)\,L_1+32\mathrm{I}_{3xy}C_{33}+32\mathrm{I}_{4xy}C_{3344}-8\mathrm{I}_{5xy}C_{3344-55}+16\mathrm{I}_{5xz}C_{3344-5}\\
&+8\mathrm{I}_{5xy}C_{334455}+16\mathrm{I}_{5xz}C_{33445}+8\mathrm{I}_{6xy}C_{3344-566}+8\mathrm{I}_{6xy}C_{33445-66}-12\mathrm{I}_{6xy}C_{3344-66}+4\mathrm{I}_{6xz}C_{3344-556}\\
&-2\mathrm{I}_{6xy}C_{3344-5566}-8\mathrm{I}_{6xz}C_{3344-56}-2\mathrm{I}_{6xy}C_{334455-66}+8\mathrm{I}_{6xz}C_{33445-6}-4\mathrm{I}_{6xz}C_{334455-6}\\
&+12\mathrm{I}_{6xy}C_{334466}+4\mathrm{I}_{6xz}C_{334466-55}+2\mathrm{I}_{6xy}C_{334466-55}+8\mathrm{I}_{6xz}C_{33446-5}+2\mathrm{I}_{6xy}C_{33445566}\\
&-8\mathrm{I}_{6xz}C_{334456}-4\mathrm{I}_{6xz}C_{3344556}+8\mathrm{I}_{6xy}C_{334466-5}+8\mathrm{I}_{6xy}C_{3344566}-16\mathrm{I}_{3xx}S_{33}+16\mathrm{I}_{3yy}S_{33}\\
&-16\mathrm{I}_{4xx}S_{3344}+16\mathrm{I}_{4yy}S_{3344}-8\mathrm{I}_{5xx}S_{3344}-8\mathrm{I}_{5yy}S_{3344}+16\mathrm{I}_{5zz}S_{3344}+4\mathrm{I}_{6xx}S_{3344}+4\mathrm{I}_{6yy}S_{3344}\\
&-8\mathrm{I}_{6zz}S_{3344}-4\mathrm{I}_{5xx}S_{3344-55}+4\mathrm{I}_{5yy}S_{3344-55}-2\mathrm{I}_{6xx}S_{3344-55}-2\mathrm{I}_{6yy}S_{3344-55}+4\mathrm{I}_{6zz}S_{3344-55}\\
&+16\mathrm{I}_{5yz}S_{3344-5}-4\mathrm{I}_{5xx}S_{334455}+4\mathrm{I}_{5yy}S_{334455}-2\mathrm{I}_{6xx}S_{334455}-2\mathrm{I}_{6yy}S_{334455}+4\mathrm{I}_{6zz}S_{334455}\ \ldots
\end{aligned}
$$

$$-16\mathrm{I}_{5yz}S_{33445}+4\mathrm{I}_{6xx}S_{3344-566}-4\mathrm{I}_{6yy}S_{3344-566}+4\mathrm{I}_{6xx}S_{33445-66}-4\mathrm{I}_{6yy}S_{33445-66}-6\mathrm{I}_{6xx}S_{3344-66}$$
$$+6\mathrm{I}_{6yy}S_{3344-66}+4\mathrm{I}_{6yz}S_{3344-556}-\mathrm{I}_{6xx}S_{3344-5566}+\mathrm{I}_{6yy}S_{3344-5566}-8\mathrm{I}_{6yz}S_{3344-56}-\mathrm{I}_{6xx}S_{334455-66}$$
$$+\mathrm{I}_{6yy}S_{334455-66}+8\mathrm{I}_{6yz}S_{33445-6}-4\mathrm{I}_{6yz}S_{334455-6}-6\mathrm{I}_{6xx}S_{334466}+6\mathrm{I}_{6yy}S_{334466}-4\mathrm{I}_{6yz}S_{334466-55}$$
$$-\mathrm{I}_{6xx}S_{334466-55}+\mathrm{I}_{6yy}S_{334466-55}-8\mathrm{I}_{6yz}S_{33446-5}-\mathrm{I}_{6xx}S_{33445566}+\mathrm{I}_{6yy}S_{33445566}+8\mathrm{I}_{6yz}S_{334456}$$
$$+4\mathrm{I}_{6yz}S_{3344556}-4\mathrm{I}_{6xx}S_{334466-5}+4\mathrm{I}_{6yy}S_{334466-5}-4\mathrm{I}_{6xx}S_{3344566}+4\mathrm{I}_{6yy}S_{3344566}-16S_{3344}L_2^2\,(m_5$$
$$+m_6)+8L_2\,(2\,(\mathrm{COM}_{5x}C_{3344-5}+\mathrm{COM}_{5x}C_{33445}-2\mathrm{COM}_{5z}S_{3344}+\mathrm{COM}_{5y}S_{3344-5}$$
$$-\mathrm{COM}_{5y}S_{33445})\,m_5+(-2\mathrm{COM}_{6x}C_{3344-6}+\mathrm{COM}_{6x}C_{3344-56}+\mathrm{COM}_{6x}C_{33445-6}+2\mathrm{COM}_{6x}C_{33446}$$
$$+\mathrm{COM}_{6x}C_{33446-5}+\mathrm{COM}_{6x}C_{334456}-2\mathrm{COM}_{6z}S_{3344-5}+2\mathrm{COM}_{6z}S_{33445}-2\mathrm{COM}_{6y}S_{3344-6}$$
$$+\mathrm{COM}_{6y}S_{3344-56}+\mathrm{COM}_{6y}S_{33445-6}-2\mathrm{COM}_{6y}S_{33446}-\mathrm{COM}_{6y}S_{33446-5}$$
$$-\mathrm{COM}_{6y}S_{334456})\,m_6))\,\dot{\theta}_2{}^2+4\,((8\mathrm{I}_{5zz}C_{34}+4\mathrm{I}_{6xx}C_{34}+4\mathrm{I}_{6yy}C_{34}-4\mathrm{I}_{5xx}C_{34-55}+4\mathrm{I}_{5yy}C_{34-55}$$
$$-2\mathrm{I}_{6xx}C_{34-55}-2\mathrm{I}_{6yy}C_{34-55}+4\mathrm{I}_{6zz}C_{34-55}+8\mathrm{I}_{5yz}C_{34-5}-8\mathrm{I}_{5yz}C_{345}-4\mathrm{I}_{5xx}C_{3455}+4\mathrm{I}_{5yy}C_{3455}$$
$$-2\mathrm{I}_{6xx}C_{3455}-2\mathrm{I}_{6yy}C_{3455}+4\mathrm{I}_{6zz}C_{3455}-2\mathrm{I}_{6xx}C_{34-66}+2\mathrm{I}_{6yy}C_{34-66}+2\mathrm{I}_{6xx}C_{34-566}-2\mathrm{I}_{6yy}C_{34-566}$$
$$+2\mathrm{I}_{6xx}C_{345-66}-2\mathrm{I}_{6yy}C_{345-66}-\mathrm{I}_{6xx}C_{3455-66}+\mathrm{I}_{6yy}C_{3455-66}+4\mathrm{I}_{6yz}C_{34-556}-4\mathrm{I}_{6yz}C_{34-56}$$
$$+4\mathrm{I}_{6yz}C_{345-6}-4\mathrm{I}_{6yz}C_{3455-6}-4\mathrm{I}_{6yz}C_{346-55}-4\mathrm{I}_{6yz}C_{346-5}+4\mathrm{I}_{6yz}C_{3456}+4\mathrm{I}_{6yz}C_{34556}$$
$$-2\mathrm{I}_{6xx}C_{3466}+2\mathrm{I}_{6yy}C_{3466}-\mathrm{I}_{6xx}S_{3466-55}+\mathrm{I}_{6yy}S_{3466-55}-2\mathrm{I}_{6xx}C_{3466-5}+2\mathrm{I}_{6yy}C_{3466-5}$$
$$-2\mathrm{I}_{6xx}C_{34566}+2\mathrm{I}_{6yy}C_{34566}-\mathrm{I}_{6xx}C_{34-5566}+\mathrm{I}_{6yy}C_{34-5566}-\mathrm{I}_{6xx}C_{345566}+\mathrm{I}_{6yy}C_{345566}$$
$$+8\mathrm{I}_{5xy}S_{34-55}-8\mathrm{I}_{5xz}S_{34-5}-8\mathrm{I}_{5xz}S_{345}-8\mathrm{I}_{5xy}S_{3455}+4\mathrm{I}_{6xy}S_{34-66}-4\mathrm{I}_{6xy}S_{34-566}-4\mathrm{I}_{6xy}S_{345-66}$$
$$+2\mathrm{I}_{6xy}S_{3455-66}-4\mathrm{I}_{6xz}S_{34-556}+4\mathrm{I}_{6xz}S_{34-56}-4\mathrm{I}_{6xz}S_{345-6}+4\mathrm{I}_{6xz}S_{3455-6}-4\mathrm{I}_{6xz}S_{346-55}$$
$$-4\mathrm{I}_{6xz}S_{346-5}+4\mathrm{I}_{6xz}S_{3456}+4\mathrm{I}_{6xz}S_{34556}-4\mathrm{I}_{6xy}S_{3466}-2\mathrm{I}_{6xy}S_{3466-55}-4\mathrm{I}_{6xy}S_{3466-5}-4\mathrm{I}_{6xy}S_{34566}$$
$$+2\mathrm{I}_{6xy}S_{34-5566}-2\mathrm{I}_{6xy}S_{345566}+16C_3L_1\,((\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5)\,m_5+(\mathrm{COM}_{6z}S_5$$
$$+C_5\,(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6))\,m_6)-16S_{34}L_2\,((\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5)\,m_5+(\mathrm{COM}_{6z}S_5$$
$$+C_5\,(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6))\,m_6))\,\dot{\theta}_5-8S_5\,\big(2\mathrm{I}_{6xy}C_3C_4C_5C_6^2-\mathrm{I}_{6xx}C_4S_3C_6^2+\mathrm{I}_{6yy}C_4S_3C_6^2$$
$$-\mathrm{I}_{6xx}C_3S_4C_6^2+\mathrm{I}_{6yy}C_3S_4C_6^2-2\mathrm{I}_{6xy}C_5S_3S_4C_6^2+2\mathrm{I}_{6yz}C_3C_4S_5C_6-2\mathrm{I}_{6yz}S_3S_4S_5C_6$$
$$-2\mathrm{I}_{6xy}C_3C_4C_5S_6^2+\mathrm{I}_{6xx}C_4S_3S_6^2-\mathrm{I}_{6yy}C_4S_3S_6^2+\mathrm{I}_{6xx}C_3S_4S_6^2-\mathrm{I}_{6yy}C_3S_4S_6^2+2\mathrm{I}_{6xy}C_5S_3S_4S_6^2$$
$$-\mathrm{I}_{6zz}C_4S_3-\mathrm{I}_{6zz}C_3S_4+2\mathrm{I}_{6xz}C_3C_4S_5S_6-2\mathrm{I}_{6xz}S_3S_4S_5S_6-\mathrm{I}_{6xx}C_3C_4C_5S_{66}+\mathrm{I}_{6yy}C_3C_4C_5S_{66}$$
$$-2\mathrm{I}_{6xy}C_4S_3S_{66}-2\mathrm{I}_{6xy}C_3S_4S_{66}+\mathrm{I}_{6xx}C_5S_3S_4S_{66}-\mathrm{I}_{6yy}C_5S_3S_4S_{66}+2C_3\,(\mathrm{COM}_{6y}C_6$$
$$+\mathrm{COM}_{6x}S_6)\,L_1m_6-2S_{34}\,(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6)\,L_2m_6)\,\dot{\theta}_6\big)\,\dot{\theta}_2-16\,\Big(4\mathrm{I}_{5xy}\dot{\theta}_3\dot{\theta}_5C_5^2$$
$$-4\mathrm{I}_{6xz}C_6\dot{\theta}_3\dot{\theta}_5C_5^2+4\mathrm{I}_{6yz}S_6\dot{\theta}_3\dot{\theta}_5C_5^2+4\mathrm{I}_{5xy}\dot{\theta}_4\dot{\theta}_5C_5^2-4\mathrm{I}_{6xz}C_6\dot{\theta}_4\dot{\theta}_5C_5^2+4\mathrm{I}_{6yz}S_6\dot{\theta}_4\dot{\theta}_5C_5^2$$
$$+2\mathrm{I}_{6xy}C_6^2\dot{\theta}_3\dot{\theta}_6C_5^2-2\mathrm{I}_{6xy}S_6^2\dot{\theta}_3\dot{\theta}_6C_5^2-\mathrm{I}_{6xx}S_{66}\dot{\theta}_3\dot{\theta}_6C_5^2+\mathrm{I}_{6yy}S_{66}\dot{\theta}_3\dot{\theta}_6C_5^2+2\mathrm{I}_{6xy}C_6^2\dot{\theta}_4\dot{\theta}_6C_5^2$$
$$-2\mathrm{I}_{6xy}S_6^2\dot{\theta}_4\dot{\theta}_6C_5^2-\mathrm{I}_{6xx}S_{66}\dot{\theta}_4\dot{\theta}_6C_5^2+\mathrm{I}_{6yy}S_{66}\dot{\theta}_4\dot{\theta}_6C_5^2-2\mathrm{I}_{6xy}C_6^2\dot{\theta}_5{}^2C_5+2\mathrm{I}_{6xy}S_6^2\dot{\theta}_5{}^2C_5\ \ldots$$

$$
\begin{aligned}
&-2\mathrm{I}_{5xz}\dot{\theta}_5{}^2 C_5 + \mathrm{I}_{6xx} S_{66}\dot{\theta}_5{}^2 C_5 - \mathrm{I}_{6yy} S_{66}\dot{\theta}_5{}^2 C_5 - 2\mathrm{COM}_{5x} L_2 m_5 \dot{\theta}_5{}^2 C_5 - 2\mathrm{COM}_{6x} C_6 L_2 m_6 \dot{\theta}_5{}^2 C_5 \\
&+2\mathrm{COM}_{6y} S_6 L_2 m_6 \dot{\theta}_5{}^2 C_5 - 2\mathrm{COM}_{6x} C_6 L_2 m_6 \dot{\theta}_6{}^2 C_5 + 2\mathrm{COM}_{6y} S_6 L_2 m_6 \dot{\theta}_6{}^2 C_5 + 2\mathrm{I}_{5yz} S_5 \dot{\theta}_5{}^2 \\
&-2\mathrm{I}_{6yz} C_6 S_5 \dot{\theta}_5{}^2 - 2\mathrm{I}_{6xz} S_5 S_6 \dot{\theta}_5{}^2 + 2\mathrm{COM}_{5y} S_5 L_2 m_5 \dot{\theta}_5{}^2 - 2\mathrm{COM}_{6z} S_5 L_2 m_6 \dot{\theta}_5{}^2 + 2\mathrm{I}_{6yz} C_6 S_5 \dot{\theta}_6{}^2 \\
&+2\mathrm{I}_{6xz} S_5 S_6 \dot{\theta}_6{}^2 - 4\mathrm{I}_{5xy} S_5^2 \dot{\theta}_3 \dot{\theta}_5 + 4\mathrm{I}_{6xz} C_6 S_5^2 \dot{\theta}_3 \dot{\theta}_5 + \mathrm{I}_{6xx} S_{55} S_6^2 \dot{\theta}_3 \dot{\theta}_5 - \mathrm{I}_{6yy} S_{55} S_6^2 \dot{\theta}_3 \dot{\theta}_5 - \mathrm{I}_{6xx} C_6^2 S_{55} \dot{\theta}_3 \dot{\theta}_5 \\
&+\mathrm{I}_{6yy} C_6^2 S_{55} \dot{\theta}_3 \dot{\theta}_5 - 2\mathrm{I}_{5xx} S_{55} \dot{\theta}_3 \dot{\theta}_5 + 2\mathrm{I}_{5yy} S_{55} \dot{\theta}_3 \dot{\theta}_5 - \mathrm{I}_{6xx} S_{55} \dot{\theta}_3 \dot{\theta}_5 - \mathrm{I}_{6yy} S_{55} \dot{\theta}_3 \dot{\theta}_5 + 2\mathrm{I}_{6zz} S_{55} \dot{\theta}_3 \dot{\theta}_5 \\
&-4\mathrm{I}_{6yz} S_5^2 S_6 \dot{\theta}_3 \dot{\theta}_5 - 2\mathrm{I}_{6xy} S_{55} S_{66} \dot{\theta}_3 \dot{\theta}_5 - 4\mathrm{I}_{5xy} S_5^2 \dot{\theta}_4 \dot{\theta}_5 + 4\mathrm{I}_{6xz} C_6 S_5^2 \dot{\theta}_4 \dot{\theta}_5 + \mathrm{I}_{6xx} S_{55} S_6^2 \dot{\theta}_4 \dot{\theta}_5 \\
&-\mathrm{I}_{6yy} S_{55} S_6^2 \dot{\theta}_4 \dot{\theta}_5 - \mathrm{I}_{6xx} C_6^2 S_{55} \dot{\theta}_4 \dot{\theta}_5 + \mathrm{I}_{6yy} C_6^2 S_{55} \dot{\theta}_4 \dot{\theta}_5 - 2\mathrm{I}_{5xx} S_{55} \dot{\theta}_4 \dot{\theta}_5 + 2\mathrm{I}_{5yy} S_{55} \dot{\theta}_4 \dot{\theta}_5 - \mathrm{I}_{6xx} S_{55} \dot{\theta}_4 \dot{\theta}_5 \\
&-\mathrm{I}_{6yy} S_{55} \dot{\theta}_4 \dot{\theta}_5 + 2\mathrm{I}_{6zz} S_{55} \dot{\theta}_4 \dot{\theta}_5 - 4\mathrm{I}_{6yz} S_5^2 S_6 \dot{\theta}_4 \dot{\theta}_5 - 2\mathrm{I}_{6xy} S_{55} S_{66} \dot{\theta}_4 \dot{\theta}_5 - 2\mathrm{I}_{6xy} C_6^2 \dot{\theta}_3 \dot{\theta}_6 - 2\mathrm{I}_{6xy} C_6^2 S_5^2 \dot{\theta}_3 \dot{\theta}_6 \\
&+2\mathrm{I}_{6xy} S_5^2 S_6^2 \dot{\theta}_3 \dot{\theta}_6 + 2\mathrm{I}_{6xy} S_6^2 \dot{\theta}_3 \dot{\theta}_6 + 2\mathrm{I}_{6yz} C_6 S_{55} \dot{\theta}_3 \dot{\theta}_6 + 2\mathrm{I}_{6xz} S_{55} S_6 \dot{\theta}_3 \dot{\theta}_6 + \mathrm{I}_{6xx} S_5^2 S_{66} \dot{\theta}_3 \dot{\theta}_6 \\
&-\mathrm{I}_{6yy} S_5^2 S_{66} \dot{\theta}_3 \dot{\theta}_6 + \mathrm{I}_{6xx} S_{66} \dot{\theta}_3 \dot{\theta}_6 - \mathrm{I}_{6yy} S_{66} \dot{\theta}_3 \dot{\theta}_6 - 4\mathrm{COM}_{6x} C_6 L_2 m_6 \dot{\theta}_3 \dot{\theta}_6 + 4\mathrm{COM}_{6y} S_6 L_2 m_6 \dot{\theta}_3 \dot{\theta}_6 \\
&-2\mathrm{I}_{6xy} C_6^2 \dot{\theta}_4 \dot{\theta}_6 - 2\mathrm{I}_{6xy} C_6^2 S_5^2 \dot{\theta}_4 \dot{\theta}_6 + 2\mathrm{I}_{6xy} S_5^2 S_6^2 \dot{\theta}_4 \dot{\theta}_6 + 2\mathrm{I}_{6xy} S_6^2 \dot{\theta}_4 \dot{\theta}_6 + 2\mathrm{I}_{6yz} C_6 S_{55} \dot{\theta}_4 \dot{\theta}_6 + 2\mathrm{I}_{6xz} S_{55} S_6 \dot{\theta}_4 \dot{\theta}_6 \\
&+\mathrm{I}_{6xx} S_5^2 S_{66} \dot{\theta}_4 \dot{\theta}_6 - \mathrm{I}_{6yy} S_5^2 S_{66} \dot{\theta}_4 \dot{\theta}_6 + \mathrm{I}_{6xx} S_{66} \dot{\theta}_4 \dot{\theta}_6 - \mathrm{I}_{6yy} S_{66} \dot{\theta}_4 \dot{\theta}_6 - 4\mathrm{COM}_{6x} C_6 L_2 m_6 \dot{\theta}_4 \dot{\theta}_6 \\
&+4\mathrm{COM}_{6y} S_6 L_2 m_6 \dot{\theta}_4 \dot{\theta}_6 - 2\mathrm{I}_{6xx} S_5 S_6^2 \dot{\theta}_5 \dot{\theta}_6 + 2\mathrm{I}_{6yy} S_5 S_6^2 \dot{\theta}_5 \dot{\theta}_6 + 2\mathrm{I}_{6xx} C_6^2 S_5 \dot{\theta}_5 \dot{\theta}_6 - 2\mathrm{I}_{6yy} C_6^2 S_5 \dot{\theta}_5 \dot{\theta}_6 \\
&+2\mathrm{I}_{6zz} S_5 \dot{\theta}_5 \dot{\theta}_6 + 4\mathrm{I}_{6xy} S_5 S_{66} \dot{\theta}_5 \dot{\theta}_6 + 4\mathrm{COM}_{6y} C_6 S_5 L_2 m_6 \dot{\theta}_5 \dot{\theta}_6 + 4\mathrm{COM}_{6x} S_5 S_6 L_2 m_6 \dot{\theta}_5 \dot{\theta}_6 \\
&+2L_1 \Big(\left(\mathrm{COM}_{4y} C_4 + \mathrm{COM}_{4x} S_4 \right) m_4 \dot{\theta}_4 \Big(2\dot{\theta}_3 + \dot{\theta}_4 \Big) + m_5 \left(\left(\mathrm{COM}_{5z} C_4 + L_2 C_4 + S_4 \left(\mathrm{COM}_{5x} C_5 \right.\right.\right. \\
&\quad \left.\left. - \mathrm{COM}_{5y} S_5 \right) \right) \dot{\theta}_4{}^2 + 2C_4 \left(\mathrm{COM}_{5y} C_5 + \mathrm{COM}_{5x} S_5 \right) \dot{\theta}_5 \dot{\theta}_4 + S_4 \left(\mathrm{COM}_{5x} C_5 - \mathrm{COM}_{5y} S_5 \right) \dot{\theta}_5{}^2 \\
&\quad +2\dot{\theta}_3 \Big(\left(\mathrm{COM}_{5z} C_4 + L_2 C_4 + S_4 \left(\mathrm{COM}_{5x} C_5 - \mathrm{COM}_{5y} S_5 \right) \right) \dot{\theta}_4 + C_4 \left(\mathrm{COM}_{5y} C_5 + \mathrm{COM}_{5x} S_5 \right) \dot{\theta}_5 \Big) \Big) \\
&+m_6 \Big(\left(C_4 \left(\mathrm{COM}_{6y} C_6 + \mathrm{COM}_{6x} S_6 \right) + S_4 \left(\mathrm{COM}_{6z} S_5 + C_5 \left(\mathrm{COM}_{6x} C_6 - \mathrm{COM}_{6y} S_6 \right) \right) + C_4 L_2 \right) \dot{\theta}_4{}^2 \\
&-2 \Big(C_4 \left(\mathrm{COM}_{6z} C_5 + S_5 \left(\mathrm{COM}_{6y} S_6 - \mathrm{COM}_{6x} C_6 \right) \right) \dot{\theta}_5 - \left(C_4 C_5 \left(\mathrm{COM}_{6y} C_6 + \mathrm{COM}_{6x} S_6 \right) \right. \\
&\quad \left. + S_4 \left(\mathrm{COM}_{6x} C_6 - \mathrm{COM}_{6y} S_6 \right) \right) \dot{\theta}_6 \Big) \dot{\theta}_4 + \mathrm{COM}_{6x} C_5 C_6 S_4 \dot{\theta}_5{}^2 + \mathrm{COM}_{6z} S_4 S_5 \dot{\theta}_5{}^2 - \mathrm{COM}_{6y} C_5 S_4 S_6 \dot{\theta}_5{}^2 \\
&\quad +\mathrm{COM}_{6y} C_4 C_6 \dot{\theta}_6{}^2 + \mathrm{COM}_{6x} C_5 C_6 S_4 \dot{\theta}_6{}^2 + \mathrm{COM}_{6x} C_4 S_6 \dot{\theta}_6{}^2 - \mathrm{COM}_{6y} C_5 S_4 S_6 \dot{\theta}_6{}^2 \\
&-2\mathrm{COM}_{6y} C_6 S_4 S_5 \dot{\theta}_5 \dot{\theta}_6 - 2\mathrm{COM}_{6x} S_4 S_5 S_6 \dot{\theta}_5 \dot{\theta}_6 + 2\dot{\theta}_3 \left(\left(C_4 \left(\mathrm{COM}_{6y} C_6 + \mathrm{COM}_{6x} S_6 \right) + S_4 \left(\mathrm{COM}_{6z} S_5 \right.\right.\right. \\
&\quad \left.\left. + C_5 \left(\mathrm{COM}_{6x} C_6 - \mathrm{COM}_{6y} S_6 \right) \right) + C_4 L_2 \right) \dot{\theta}_4 - C_4 \left(\mathrm{COM}_{6z} C_5 + S_5 \left(\mathrm{COM}_{6y} S_6 - \mathrm{COM}_{6x} C_6 \right) \right) \dot{\theta}_5 \\
&\quad \left. + \left(C_4 C_5 \left(\mathrm{COM}_{6y} C_6 + \mathrm{COM}_{6x} S_6 \right) + S_4 \left(\mathrm{COM}_{6x} C_6 - \mathrm{COM}_{6y} S_6 \right) \right) \dot{\theta}_6 \right) \Big) \Big) \Big) \qquad \text{(A.3c)}
\end{aligned}
$$

$$V_4 = \frac{1}{32}\left(\left(16S_{3344}\left(m_5+m_6\right)L_2^2 - 8\left(2\left(\mathrm{COM}_{5x}C_{3344-5} + \mathrm{COM}_{5x}C_{33445} - 2\mathrm{COM}_{5z}S_{3344} + \mathrm{COM}_{5y}S_{3344-5}\right.\right.\right.\right.$$
$$-\mathrm{COM}_{5y}S_{33445}\right)m_5 + \left(-2\mathrm{COM}_{6x}C_{3344-6} + \mathrm{COM}_{6x}C_{3344-56} + \mathrm{COM}_{6x}C_{33445-6} + 2\mathrm{COM}_{6x}C_{33446}$$
$$+\mathrm{COM}_{6x}C_{33446-5} + \mathrm{COM}_{6x}C_{334456} - 2\mathrm{COM}_{6z}S_{3344-5} + 2\mathrm{COM}_{6z}S_{33445} - 2\mathrm{COM}_{6y}S_{3344-6}$$
$$+\mathrm{COM}_{6y}S_{3344-56} + \mathrm{COM}_{6y}S_{33445-6} - 2\mathrm{COM}_{6y}S_{33446} - \mathrm{COM}_{6y}S_{33446-5} - \mathrm{COM}_{6y}S_{334456}\right)m_6\right)L_2$$
$$-32\mathrm{I}_{4xy}C_{3344} + 8\mathrm{I}_{5xy}C_{3344-55} - 16\mathrm{I}_{5xz}C_{3344-5} - 8\mathrm{I}_{5xy}C_{334455} - 16\mathrm{I}_{5xz}C_{33445} - 8\mathrm{I}_{6xy}C_{3344-566}$$
$$-8\mathrm{I}_{6xy}C_{33445-66} + 12\mathrm{I}_{6xy}C_{3344-66} - 4\mathrm{I}_{6xz}C_{3344-556} + 2\mathrm{I}_{6xy}C_{3344-5566} + 8\mathrm{I}_{6xz}C_{3344-56}$$
$$+2\mathrm{I}_{6xy}C_{334455-66} - 8\mathrm{I}_{6xz}C_{33445-6} + 4\mathrm{I}_{6xz}C_{334455-6} - 12\mathrm{I}_{6xy}C_{334466} - 4\mathrm{I}_{6xz}C_{334466-55}$$
$$-2\mathrm{I}_{6xy}C_{334466-55} - 8\mathrm{I}_{6xz}C_{33446-5} - 2\mathrm{I}_{6xy}C_{33445566} + 8\mathrm{I}_{6xz}C_{334456} + 4\mathrm{I}_{6xz}C_{3344556}$$
$$-8\mathrm{I}_{6xy}C_{334466-5} - 8\mathrm{I}_{6xy}C_{3344566} + 16\mathrm{I}_{4xx}S_{3344} - 16\mathrm{I}_{4yy}S_{3344} + 8\mathrm{I}_{5xx}S_{3344} + 8\mathrm{I}_{5yy}S_{3344}$$
$$-16\mathrm{I}_{5zz}S_{3344} - 4\mathrm{I}_{6xx}S_{3344} - 4\mathrm{I}_{6yy}S_{3344} + 8\mathrm{I}_{6zz}S_{3344} + 4\mathrm{I}_{5xx}S_{3344-55} - 4\mathrm{I}_{5yy}S_{3344-55}$$
$$+2\mathrm{I}_{6xx}S_{3344-55} + 2\mathrm{I}_{6yy}S_{3344-55} - 4\mathrm{I}_{6zz}S_{3344-55} - 16\mathrm{I}_{5yz}S_{3344-5} + 4\mathrm{I}_{5xx}S_{334455} - 4\mathrm{I}_{5yy}S_{334455}$$
$$+2\mathrm{I}_{6xx}S_{334455} + 2\mathrm{I}_{6yy}S_{334455} - 4\mathrm{I}_{6zz}S_{334455} + 16\mathrm{I}_{5yz}S_{33445} - 4\mathrm{I}_{6xx}S_{3344-566} + 4\mathrm{I}_{6yy}S_{3344-566}$$
$$-4\mathrm{I}_{6xx}S_{33445-66} + 4\mathrm{I}_{6yy}S_{33445-66} + 6\mathrm{I}_{6xx}S_{3344-66} - 6\mathrm{I}_{6yy}S_{3344-66} - 4\mathrm{I}_{6yz}S_{3344-556}$$
$$+\mathrm{I}_{6xx}S_{3344-5566} - \mathrm{I}_{6yy}S_{3344-5566} + 8\mathrm{I}_{6yz}S_{3344-56} + \mathrm{I}_{6xx}S_{334455-66} - \mathrm{I}_{6yy}S_{334455-66} - 8\mathrm{I}_{6yz}S_{33445-6}$$
$$+4\mathrm{I}_{6yz}S_{334455-6} + 6\mathrm{I}_{6xx}S_{334466} - 6\mathrm{I}_{6yy}S_{334466} + 4\mathrm{I}_{6yz}S_{334466-55} + \mathrm{I}_{6xx}S_{334466-55} - \mathrm{I}_{6yy}S_{334466-55}$$
$$+8\mathrm{I}_{6yz}S_{33446-5} + \mathrm{I}_{6xx}S_{33445566} - \mathrm{I}_{6yy}S_{33445566} - 8\mathrm{I}_{6yz}S_{334456} - 4\mathrm{I}_{6yz}S_{3344556} + 4\mathrm{I}_{6xx}S_{334466-5}$$
$$-4\mathrm{I}_{6yy}S_{334466-5} + 4\mathrm{I}_{6xx}S_{3344566} - 4\mathrm{I}_{6yy}S_{3344566} - 16L_1\left(2S_3\left(\mathrm{COM}_{4x}C_{34} - \mathrm{COM}_{4y}S_{34}\right)m_4\right.$$
$$+S_3\left(\mathrm{COM}_{5x}C_{34-5} + \mathrm{COM}_{5x}C_{345} - 2\mathrm{COM}_{5z}S_{34} + \mathrm{COM}_{5y}S_{34-5} - \mathrm{COM}_{5y}S_{345} - 2S_{34}L_2\right)m_5$$
$$-\left(S_4\left(2\mathrm{COM}_{6z}S_5S_3^2 + 2C_5\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right)S_3^2 + \mathrm{COM}_{6y}C_6S_{33} + \mathrm{COM}_{6x}S_{33}S_6\right)\right.$$
$$+C_4\left(2\mathrm{COM}_{6x}S_6S_3^2 + C_6\left(2\mathrm{COM}_{6y}S_3^2 - \mathrm{COM}_{6x}C_5S_{33}\right) + S_{33}\left(\mathrm{COM}_{6y}C_5S_6 - \mathrm{COM}_{6z}S_5\right)\right)$$
$$\left.\left.+2S_3S_{34}L_2\right)m_6\right)\right)\dot{\theta}_2{}^2 + 4\left(\left(4(2\mathrm{I}_{5zz} + \mathrm{I}_{6xx} + \mathrm{I}_{6yy})C_{34} - 4\mathrm{I}_{5xx}C_{34-55} + 4\mathrm{I}_{5yy}C_{34-55} - 2\mathrm{I}_{6xx}C_{34-55}\right.\right.$$
$$-2\mathrm{I}_{6yy}C_{34-55} + 4\mathrm{I}_{6zz}C_{34-55} + 8\mathrm{I}_{5yz}C_{34-5} - 8\mathrm{I}_{5yz}C_{345} - 4\mathrm{I}_{5xx}C_{3455} + 4\mathrm{I}_{5yy}C_{3455} - 2\mathrm{I}_{6xx}C_{3455}$$
$$-2\mathrm{I}_{6yy}C_{3455} + 4\mathrm{I}_{6zz}C_{3455} - 2\mathrm{I}_{6xx}C_{34-66} + 2\mathrm{I}_{6yy}C_{34-66} + 2\mathrm{I}_{6xx}C_{34-566} - 2\mathrm{I}_{6yy}C_{34-566}$$
$$+2\mathrm{I}_{6xx}C_{345-66} - 2\mathrm{I}_{6yy}C_{345-66} - \mathrm{I}_{6xx}C_{3455-66} + \mathrm{I}_{6yy}C_{3455-66} + 4\mathrm{I}_{6yz}C_{34-556} - 4\mathrm{I}_{6yz}C_{34-56}$$
$$+4\mathrm{I}_{6yz}C_{345-6} - 4\mathrm{I}_{6yz}C_{3455-6} - 4\mathrm{I}_{6yz}C_{346-55} - 4\mathrm{I}_{6yz}C_{346-5} + 4\mathrm{I}_{6yz}C_{3456} + 4\mathrm{I}_{6yz}C_{34556}$$
$$-2\mathrm{I}_{6xx}C_{3466} + 2\mathrm{I}_{6yy}C_{3466} - \mathrm{I}_{6xx}S_{3466-55} + \mathrm{I}_{6yy}S_{3466-55} - 2\mathrm{I}_{6xx}C_{3466-5} + 2\mathrm{I}_{6yy}C_{3466-5}$$
$$-2\mathrm{I}_{6xx}C_{34566} + 2\mathrm{I}_{6yy}C_{34566} - \mathrm{I}_{6xx}C_{34-5566} + \mathrm{I}_{6yy}C_{34-5566} - \mathrm{I}_{6xx}C_{345566} + \mathrm{I}_{6yy}C_{345566}$$
$$+8\mathrm{I}_{5xy}S_{34-55} - 8\mathrm{I}_{5xz}S_{34-5} - 8\mathrm{I}_{5xz}S_{345} - 8\mathrm{I}_{5xy}S_{3455} + 4\mathrm{I}_{6xy}S_{34-66} - 4\mathrm{I}_{6xy}S_{34-566} - 4\mathrm{I}_{6xy}S_{345-66}$$
$$+2\mathrm{I}_{6xy}S_{3455-66} - 4\mathrm{I}_{6xz}S_{34-556} + 4\mathrm{I}_{6xz}S_{34-56} - 4\mathrm{I}_{6xz}S_{345-6} + 4\mathrm{I}_{6xz}S_{3455-6} - 4\mathrm{I}_{6xz}S_{346-55}$$
$$-4\mathrm{I}_{6xz}S_{346-5} + 4\mathrm{I}_{6xz}S_{3456} + 4\mathrm{I}_{6xz}S_{34556} - 4\mathrm{I}_{6xy}S_{3466} - 2\mathrm{I}_{6xy}S_{3466-55} - 4\mathrm{I}_{6xy}S_{3466-5}\ \ldots$$

$$
\begin{aligned}
&-4\mathrm{I}_{6xy}S_{34566}+2\mathrm{I}_{6xy}S_{34-5566}-2\mathrm{I}_{6xy}S_{345566}-16S_{34}L_2\left(\left(\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5\right)m_5+\left(\mathrm{COM}_{6z}S_5\right.\right.\\
&\left.\left.+C_5\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right)m_6\right)\big)\dot{\theta}_5+4\left(C_4\left(2S_3S_5\left(\mathrm{I}_{6zz}+(\mathrm{I}_{6xx}-\mathrm{I}_{6yy})C_{66}+2\mathrm{I}_{6xy}S_{66}\right)\right.\right.\\
&-C_3\left(4\mathrm{I}_{6yz}C_6S_5^2+4\mathrm{I}_{6xz}S_6S_5^2+2\mathrm{I}_{6xy}C_{66}S_{55}-\mathrm{I}_{6xx}S_{55}S_{66}+\mathrm{I}_{6yy}S_{55}S_{66}\right)\big)+S_4\left(2C_3S_5\left(\mathrm{I}_{6zz}\right.\right.\\
&\left.+(\mathrm{I}_{6xx}-\mathrm{I}_{6yy})C_{66}+2\mathrm{I}_{6xy}S_{66}\right)+S_3\left(4\mathrm{I}_{6yz}C_6S_5^2+4\mathrm{I}_{6xz}S_6S_5^2+2\mathrm{I}_{6xy}C_{66}S_{55}-\mathrm{I}_{6xx}S_{55}S_{66}\right.\\
&\left.\left.+\mathrm{I}_{6yy}S_{55}S_{66}\right)\right)+4S_{34}S_5\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)L_2m_6\big)\dot{\theta}_6\Big)\dot{\theta}_2+8\left(4L_1\left(\left(\mathrm{COM}_{4y}C_4\right.\right.\right.\\
&\left.+\mathrm{COM}_{4x}S_4\right)m_4+\left(\mathrm{COM}_{5z}C_4+L_2C_4+S_4\left(\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5\right)\right)m_5+\left(C_4\left(\mathrm{COM}_{6y}C_6\right.\right.\\
&\left.\left.+\mathrm{COM}_{6x}S_6\right)+S_4\left(\mathrm{COM}_{6z}S_5+C_5\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right)+C_4L_2\right)m_6\big)\dot{\theta}_3{}^2+\left(\left(-8\mathrm{I}_{5xy}C_{55}\right.\right.\\
&+2\mathrm{I}_{6xy}C_{55-66}+4\mathrm{I}_{6xz}C_{55-6}-2\mathrm{I}_{6xy}C_{5566}+4\mathrm{I}_{6xz}C_{556}+4\mathrm{I}_{5xx}S_{55}-4\mathrm{I}_{5yy}S_{55}+2\mathrm{I}_{6xx}S_{55}+2\mathrm{I}_{6yy}S_{55}\\
&-4\mathrm{I}_{6zz}S_{55}+\mathrm{I}_{6xx}S_{55-66}-\mathrm{I}_{6yy}S_{55-66}+4\mathrm{I}_{6yz}S_{55-6}+\mathrm{I}_{6xx}S_{5566}-\mathrm{I}_{6yy}S_{5566}-4\mathrm{I}_{6yz}S_{556}\big)\dot{\theta}_5\\
&+4\left(2\mathrm{I}_{6xy}C_{66}S_5^2-\mathrm{I}_{6xx}S_{66}S_5^2+\mathrm{I}_{6yy}S_{66}S_5^2-\mathrm{I}_{6yz}C_6S_{55}-\mathrm{I}_{6xz}S_{55}S_6+2\left(\mathrm{COM}_{6x}C_6\right.\right.\\
&\left.\left.-\mathrm{COM}_{6y}S_6\right)L_2m_6\right)\dot{\theta}_6\Big)\dot{\theta}_3+2\left(\left(2S_5\left(-\mathrm{I}_{5yz}+\mathrm{I}_{6yz}C_6+\mathrm{I}_{6xz}S_6\right)+C_5\left(2\mathrm{I}_{5xz}+2\mathrm{I}_{6xy}C_{66}+(\mathrm{I}_{6yy}\right.\right.\right.\\
&\left.-\mathrm{I}_{6xx})S_{66}\right)+2L_2\left(\left(\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5\right)m_5+\left(\mathrm{COM}_{6z}S_5+C_5\left(\mathrm{COM}_{6x}C_6\right.\right.\right.\\
&\left.\left.\left.-\mathrm{COM}_{6y}S_6\right)\right)m_6\right)\big)\dot{\theta}_5{}^2-2S_5\left(\mathrm{I}_{6zz}+(\mathrm{I}_{6xx}-\mathrm{I}_{6yy})C_{66}+2\mathrm{I}_{6xy}S_{66}+2\left(\mathrm{COM}_{6y}C_6\right.\right.\\
&\left.\left.+\mathrm{COM}_{6x}S_6\right)L_2m_6\right)\dot{\theta}_6\dot{\theta}_5-2\left(S_5\left(\mathrm{I}_{6yz}C_6+\mathrm{I}_{6xz}S_6\right)-C_5\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)L_2m_6\right)\dot{\theta}_6{}^2\\
&+\dot{\theta}_4\left(\left(-4\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)C_5^2+4S_5^2\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)+S_{55}\left(2\mathrm{I}_{5xx}-2\mathrm{I}_{5yy}+\mathrm{I}_{6xx}\right.\right.\right.\\
&\left.\left.+\mathrm{I}_{6yy}-2\mathrm{I}_{6zz}+(\mathrm{I}_{6xx}-\mathrm{I}_{6yy})C_{66}+2\mathrm{I}_{6xy}S_{66}\right)\right)\dot{\theta}_5+2\left(2\mathrm{I}_{6xy}C_{66}S_5^2-\mathrm{I}_{6xx}S_{66}S_5^2+\mathrm{I}_{6yy}S_{66}S_5^2\right.\\
&\left.\left.\left.-\mathrm{I}_{6yz}C_6S_{55}-\mathrm{I}_{6xz}S_{55}S_6+2\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)L_2m_6\right)\dot{\theta}_6\right)\right)\Big)\Big)
\end{aligned} \tag{A.3d}
$$

$$
\begin{aligned}
V_5=\frac{1}{32}&\left(\left(32S_3S_{34}L_1\left(\left(\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5\right)m_5-\left(\mathrm{COM}_{6z}C_5+S_5\left(\mathrm{COM}_{6y}S_6-\mathrm{COM}_{6x}C_6\right)\right)m_6\right)\right.\right.\\
&-8C_{34}\left(S_3\left(S_4\left(-4\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)C_5^2+4S_5^2\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)+S_{55}\left(2\mathrm{I}_{5xx}-2\mathrm{I}_{5yy}\right.\right.\right.\right.\\
&\left.\left.+\mathrm{I}_{6xx}+\mathrm{I}_{6yy}-2\mathrm{I}_{6zz}+(\mathrm{I}_{6xx}-\mathrm{I}_{6yy})C_{66}+2\mathrm{I}_{6xy}S_{66}\right)\right)+C_4\left(C_5\left(-4\mathrm{I}_{5yz}+4\mathrm{I}_{6yz}C_6+4\mathrm{I}_{6xz}S_6\right)\right.\\
&\left.\left.-2S_5\left(2\mathrm{I}_{5xz}+2\mathrm{I}_{6xy}C_{66}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})S_{66}\right)\right)\right)+C_3\left(C_4\left(4\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)C_5^2\right.\right.\\
&-4S_5^2\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)-S_{55}\left(2\mathrm{I}_{5xx}-2\mathrm{I}_{5yy}+\mathrm{I}_{6xx}+\mathrm{I}_{6yy}-2\mathrm{I}_{6zz}+(\mathrm{I}_{6xx}-\mathrm{I}_{6yy})C_{66}\right.\\
&\left.\left.+2\mathrm{I}_{6xy}S_{66}\right)\right)+2S_4\left(C_5\left(-2\mathrm{I}_{5yz}+2\mathrm{I}_{6yz}C_6+2\mathrm{I}_{6xz}S_6\right)-S_5\left(2\mathrm{I}_{5xz}+2\mathrm{I}_{6xy}C_{66}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})S_{66}\right)\right)\big)\\
&-4S_{34}L_2\left(\left(\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5\right)m_5-\left(\mathrm{COM}_{6z}C_5+S_5\left(\mathrm{COM}_{6y}S_6-\mathrm{COM}_{6x}C_6\right)\right)m_6\right)\big)\big)\dot{\theta}_2{}^2\\
&-4\left(\left(8\mathrm{I}_{5zz}C_{34}+4\mathrm{I}_{6xx}C_{34}+4\mathrm{I}_{6yy}C_{34}-4\mathrm{I}_{5xx}C_{34-55}+4\mathrm{I}_{5yy}C_{34-55}-2\mathrm{I}_{6xx}C_{34-55}-2\mathrm{I}_{6yy}C_{34-55}\right.\right.\\
&+4\mathrm{I}_{6zz}C_{34-55}+8\mathrm{I}_{5yz}C_{34-5}-8\mathrm{I}_{5yz}C_{345}-4\mathrm{I}_{5xx}C_{3455}+4\mathrm{I}_{5yy}C_{3455}-2\mathrm{I}_{6xx}C_{3455}-2\mathrm{I}_{6yy}C_{3455}\\
&+4\mathrm{I}_{6zz}C_{3455}-2\mathrm{I}_{6xx}C_{34-66}+2\mathrm{I}_{6yy}C_{34-66}+2\mathrm{I}_{6xx}C_{34-566}-2\mathrm{I}_{6yy}C_{34-566}+2\mathrm{I}_{6xx}C_{345-66}\\
&-2\mathrm{I}_{6yy}C_{345-66}-\mathrm{I}_{6xx}C_{3455-66}+\mathrm{I}_{6yy}C_{3455-66}+4\mathrm{I}_{6yz}C_{34-556}-4\mathrm{I}_{6yz}C_{34-56}+4\mathrm{I}_{6yz}C_{345-6}\\
&-4\mathrm{I}_{6yz}C_{3455-6}-4\mathrm{I}_{6yz}C_{346-55}-4\mathrm{I}_{6yz}C_{346-5}+4\mathrm{I}_{6yz}C_{3456}+4\mathrm{I}_{6yz}C_{34556}-2\mathrm{I}_{6xx}C_{3466}\ \ldots
\end{aligned}
$$

$$
\begin{aligned}
&+2\mathrm{I}_{6yy}C_{3466}-\mathrm{I}_{6xx}S_{3466-55}+\mathrm{I}_{6yy}S_{3466-55}-2\mathrm{I}_{6xx}C_{3466-5}+2\mathrm{I}_{6yy}C_{3466-5}-2\mathrm{I}_{6xx}C_{34566}\\
&+2\mathrm{I}_{6yy}C_{34566}-\mathrm{I}_{6xx}C_{34-5566}+\mathrm{I}_{6yy}C_{34-5566}-\mathrm{I}_{6xx}C_{345566}+\mathrm{I}_{6yy}C_{345566}+8\mathrm{I}_{5xy}S_{34-55}\\
&-8\mathrm{I}_{5xz}S_{34-5}-8\mathrm{I}_{5xz}S_{345}-8\mathrm{I}_{5xy}S_{3455}+4\mathrm{I}_{6xy}S_{34-66}-4\mathrm{I}_{6xy}S_{34-566}-4\mathrm{I}_{6xy}S_{345-66}+2\mathrm{I}_{6xy}S_{3455-66}\\
&-4\mathrm{I}_{6xz}S_{34-556}+4\mathrm{I}_{6xz}S_{34-56}-4\mathrm{I}_{6xz}S_{345-6}+4\mathrm{I}_{6xz}S_{3455-6}-4\mathrm{I}_{6xz}S_{346-55}-4\mathrm{I}_{6xz}S_{346-5}\\
&+4\mathrm{I}_{6xz}S_{3456}+4\mathrm{I}_{6xz}S_{34556}-4\mathrm{I}_{6xy}S_{3466}-2\mathrm{I}_{6xy}S_{3466-55}-4\mathrm{I}_{6xy}S_{3466-5}-4\mathrm{I}_{6xy}S_{34566}\\
&+2\mathrm{I}_{6xy}S_{34-5566}-2\mathrm{I}_{6xy}S_{345566}+16C_3L_1\left(\left(\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5\right)m_5+\left(\mathrm{COM}_{6z}S_5\right.\right.\\
&\left.\left.+C_5\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right)m_6\right)-16S_{34}L_2\left(\left(\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5\right)m_5+\left(\mathrm{COM}_{6z}S_5\right.\right.\\
&\left.\left.+C_5\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right)m_6\right))\,\dot{\theta}_3+(4(2\mathrm{I}_{5zz}+\mathrm{I}_{6xx}+\mathrm{I}_{6yy})C_{34}-4\mathrm{I}_{5xx}C_{34-55}+4\mathrm{I}_{5yy}C_{34-55}\\
&-2\mathrm{I}_{6xx}C_{34-55}-2\mathrm{I}_{6yy}C_{34-55}+4\mathrm{I}_{6zz}C_{34-55}+8\mathrm{I}_{5yz}C_{34-5}-8\mathrm{I}_{5yz}C_{345}-4\mathrm{I}_{5xx}C_{3455}+4\mathrm{I}_{5yy}C_{3455}\\
&-2\mathrm{I}_{6xx}C_{3455}-2\mathrm{I}_{6yy}C_{3455}+4\mathrm{I}_{6zz}C_{3455}-2\mathrm{I}_{6xx}C_{34-66}+2\mathrm{I}_{6yy}C_{34-66}+2\mathrm{I}_{6xx}C_{34-566}-2\mathrm{I}_{6yy}C_{34-566}\\
&+2\mathrm{I}_{6xx}C_{345-66}-2\mathrm{I}_{6yy}C_{345-66}-\mathrm{I}_{6xx}C_{3455-66}+\mathrm{I}_{6yy}C_{3455-66}+4\mathrm{I}_{6yz}C_{34-556}-4\mathrm{I}_{6yz}C_{34-56}\\
&+4\mathrm{I}_{6yz}C_{345-6}-4\mathrm{I}_{6yz}C_{3455-6}-4\mathrm{I}_{6yz}C_{346-55}-4\mathrm{I}_{6yz}C_{346-5}+4\mathrm{I}_{6yz}C_{3456}+4\mathrm{I}_{6yz}C_{34556}\\
&-2\mathrm{I}_{6xx}C_{3466}+2\mathrm{I}_{6yy}C_{3466}-\mathrm{I}_{6xx}S_{3466-55}+\mathrm{I}_{6yy}S_{3466-55}-2\mathrm{I}_{6xx}C_{3466-5}+2\mathrm{I}_{6yy}C_{3466-5}\\
&-2\mathrm{I}_{6xx}C_{34566}+2\mathrm{I}_{6yy}C_{34566}-\mathrm{I}_{6xx}C_{34-5566}+\mathrm{I}_{6yy}C_{34-5566}-\mathrm{I}_{6xx}C_{345566}+\mathrm{I}_{6yy}C_{345566}\\
&+8\mathrm{I}_{5xy}S_{34-55}-8\mathrm{I}_{5xz}S_{34-5}-8\mathrm{I}_{5xz}S_{345}-8\mathrm{I}_{5xy}S_{3455}+4\mathrm{I}_{6xy}S_{34-66}-4\mathrm{I}_{6xy}S_{34-566}\\
&-4\mathrm{I}_{6xy}S_{345-66}+2\mathrm{I}_{6xy}S_{3455-66}-4\mathrm{I}_{6xz}S_{34-556}+4\mathrm{I}_{6xz}S_{34-56}-4\mathrm{I}_{6xz}S_{345-6}+4\mathrm{I}_{6xz}S_{3455-6}\\
&-4\mathrm{I}_{6xz}S_{346-55}-4\mathrm{I}_{6xz}S_{346-5}+4\mathrm{I}_{6xz}S_{3456}+4\mathrm{I}_{6xz}S_{34556}-4\mathrm{I}_{6xy}S_{3466}-2\mathrm{I}_{6xy}S_{3466-55}\\
&-4\mathrm{I}_{6xy}S_{3466-5}-4\mathrm{I}_{6xy}S_{34566}+2\mathrm{I}_{6xy}S_{34-5566}-2\mathrm{I}_{6xy}S_{345566}-16S_{34}L_2\left(\left(\mathrm{COM}_{5x}C_5\right.\right.\\
&\left.\left.-\mathrm{COM}_{5y}S_5\right)m_5+\left(\mathrm{COM}_{6z}S_5+C_5\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right)m_6\right))\,\dot{\theta}_4+8\left(C_3\left(S_4\left((\mathrm{I}_{6xx}-\mathrm{I}_{6yy})S_{66}\right.\right.\right.\\
&\left.-2\mathrm{I}_{6xy}C_{66}\right)+C_4\left(2S_5\left(\mathrm{I}_{6xz}C_6-\mathrm{I}_{6yz}S_6\right)+C_5\left(\mathrm{I}_{6zz}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})C_{66}-2\mathrm{I}_{6xy}S_{66}\right)\right))\\
&-S_3\left(C_4\left(2\mathrm{I}_{6xy}C_{66}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})S_{66}\right)+S_4\left(2S_5\left(\mathrm{I}_{6xz}C_6-\mathrm{I}_{6yz}S_6\right)+C_5\left(\mathrm{I}_{6zz}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})C_{66}\right.\right.\right.\\
&\left.\left.\left.\left.-2\mathrm{I}_{6xy}S_{66}\right)\right)\right)\right)\dot{\theta}_6\Big)\,\dot{\theta}_2+8\left(\left(4\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)C_5^2-4\mathrm{I}_{5xy}S_5^2+4\mathrm{I}_{6xz}C_6S_5^2+\mathrm{I}_{6xx}S_{55}S_6^2\right.\right.\\
&-\mathrm{I}_{6yy}S_{55}S_6^2-\mathrm{I}_{6xx}C_6^2S_{55}+\mathrm{I}_{6yy}C_6^2S_{55}-2\mathrm{I}_{5xx}S_{55}+2\mathrm{I}_{5yy}S_{55}-\mathrm{I}_{6xx}S_{55}-\mathrm{I}_{6yy}S_{55}+2\mathrm{I}_{6zz}S_{55}\\
&-4\mathrm{I}_{6yz}S_5^2S_6-2\mathrm{I}_{6xy}S_{55}S_{66}+4C_4L_1\left(\left(\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5\right)m_5-\left(\mathrm{COM}_{6z}C_5+S_5\left(\mathrm{COM}_{6y}S_6\right.\right.\right.\\
&\left.\left.\left.\left.-\mathrm{COM}_{6x}C_6\right)\right)m_6\right))\,{\dot{\theta}_3}^2+2\left(\left(4\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)C_5^2-4S_5^2\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)\right.\right.\\
&\left.-S_{55}\left(2\mathrm{I}_{5xx}-2\mathrm{I}_{5yy}+\mathrm{I}_{6xx}+\mathrm{I}_{6yy}-2\mathrm{I}_{6zz}+(\mathrm{I}_{6xx}-\mathrm{I}_{6yy})C_{66}+2\mathrm{I}_{6xy}S_{66}\right)\right)\dot{\theta}_4+2\left(C_5\left(2\mathrm{I}_{6yz}S_6\right.\right.\\
&\left.\left.-2\mathrm{I}_{6xz}C_6\right)+S_5\left(\mathrm{I}_{6zz}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})C_{66}-2\mathrm{I}_{6xy}S_{66}\right)\right)\dot{\theta}_6\Big)\,\dot{\theta}_3+\left(4\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)C_5^2\right.\\
&-4S_5^2\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)-S_{55}\left(2\mathrm{I}_{5xx}-2\mathrm{I}_{5yy}+\mathrm{I}_{6xx}+\mathrm{I}_{6yy}-2\mathrm{I}_{6zz}+(\mathrm{I}_{6xx}-\mathrm{I}_{6yy})C_{66}\right.\\
&\left.\left.+2\mathrm{I}_{6xy}S_{66}\right)\right){\dot{\theta}_4}^2-4\left(2C_5\left(\mathrm{I}_{6xz}C_6-\mathrm{I}_{6yz}S_6\right)-S_5\left(\mathrm{I}_{6zz}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})C_{66}-2\mathrm{I}_{6xy}S_{66}\right)\right)\dot{\theta}_4\dot{\theta}_6\\
&-4\dot{\theta}_6\Big(\left(2\mathrm{I}_{6xy}C_{66}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})S_{66}\right)\dot{\theta}_5+\left(\mathrm{I}_{6xz}C_6-\mathrm{I}_{6yz}S_6\right)\dot{\theta}_6\Big)\Big)\Big)
\end{aligned}
\tag{A.3e}
$$

$$
\begin{aligned}
V_6 = \frac{1}{32}\Big(&(8\mathrm{I}_{6xy}C_{3344-566} + 8\mathrm{I}_{6xy}C_{33445-66} - 12\mathrm{I}_{6xy}C_{3344-66} + 2\mathrm{I}_{6xz}C_{3344-556} - 2\mathrm{I}_{6xy}C_{3344-5566} \\
&-4\mathrm{I}_{6xz}C_{3344-56} - 4\mathrm{I}_{6xy}C_{55-66} - 2\mathrm{I}_{6xy}C_{334455-66} + 4\mathrm{I}_{6xz}C_{33445-6} - 4\mathrm{I}_{6xz}C_{55-6} - 2\mathrm{I}_{6xz}C_{334455-6} \\
&+8\mathrm{I}_{6xy}C_{66} - 12\mathrm{I}_{6xy}C_{334466} - 2\mathrm{I}_{6xz}C_{334466-55} - 2\mathrm{I}_{6xy}C_{334466-55} - 4\mathrm{I}_{6xz}C_{33446-5} - 4\mathrm{I}_{6xy}C_{5566} \\
&-2\mathrm{I}_{6xy}C_{33445566} + 4\mathrm{I}_{6xz}C_{334456} + 4\mathrm{I}_{6xz}C_{556} + 2\mathrm{I}_{6xz}C_{3344556} - 8\mathrm{I}_{6xy}C_{334466-5} - 8\mathrm{I}_{6xy}C_{3344566} \\
&+4\mathrm{I}_{6xx}S_{3344-566} - 4\mathrm{I}_{6yy}S_{3344-566} + 4\mathrm{I}_{6xx}S_{33445-66} - 4\mathrm{I}_{6yy}S_{33445-66} - 6\mathrm{I}_{6xx}S_{3344-66} \\
&+6\mathrm{I}_{6yy}S_{3344-66} + 2\mathrm{I}_{6yz}S_{3344-556} - \mathrm{I}_{6xx}S_{3344-5566} + \mathrm{I}_{6yy}S_{3344-5566} - 4\mathrm{I}_{6yz}S_{3344-56} - 2\mathrm{I}_{6xx}S_{55-66} \\
&+2\mathrm{I}_{6yy}S_{55-66} - \mathrm{I}_{6xx}S_{334455-66} + \mathrm{I}_{6yy}S_{334455-66} + 4\mathrm{I}_{6yz}S_{33445-6} - 4\mathrm{I}_{6yz}S_{55-6} - 2\mathrm{I}_{6yz}S_{334455-6} \\
&-4\mathrm{I}_{6xx}S_{66} + 4\mathrm{I}_{6yy}S_{66} + 6\mathrm{I}_{6xx}S_{334466} - 6\mathrm{I}_{6yy}S_{334466} + 2\mathrm{I}_{6yz}S_{334466-55} + \mathrm{I}_{6xx}S_{334466-55} \\
&-\mathrm{I}_{6yy}S_{334466-55} + 4\mathrm{I}_{6yz}S_{33446-5} + 2\mathrm{I}_{6xx}S_{5566} - 2\mathrm{I}_{6yy}S_{5566} + \mathrm{I}_{6xx}S_{33445566} - \mathrm{I}_{6yy}S_{33445566} \\
&-4\mathrm{I}_{6yz}S_{334456} - 4\mathrm{I}_{6yz}S_{556} - 2\mathrm{I}_{6yz}S_{3344556} + 4\mathrm{I}_{6xx}S_{334466-5} - 4\mathrm{I}_{6yy}S_{334466-5} + 4\mathrm{I}_{6xx}S_{3344566} \\
&-4\mathrm{I}_{6yy}S_{3344566} + 32S_3\left(S_3\left(C_4C_5\left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right) + S_4\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right)\right)\right. \\
&\left.+C_3\left(C_5S_4\left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right) + C_4\left(\mathrm{COM}_{6y}S_6 - \mathrm{COM}_{6x}C_6\right)\right)\right)L_1m_6 \\
&-32C_{34}\left(C_3\left(C_4\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right) - C_5S_4\left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right)\right) - S_3\left(C_4C_5\left(\mathrm{COM}_{6y}C_6\right.\right.\right. \\
&\left.\left.\left.+\mathrm{COM}_{6x}S_6\right) + S_4\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right)\right)\right)L_2m_6)\,\dot{\theta}_2{}^2 - 32\left(S_5\left(-2\mathrm{I}_{6xy}C_3C_4C_5C_6^2 + \mathrm{I}_{6xx}C_4S_3C_6^2\right.\right. \\
&-\mathrm{I}_{6yy}C_4S_3C_6^2 + \mathrm{I}_{6xx}C_3S_4C_6^2 - \mathrm{I}_{6yy}C_3S_4C_6^2 + 2\mathrm{I}_{6xy}C_5S_3S_4C_6^2 - 2\mathrm{I}_{6yz}C_3C_4S_5C_6 + 2\mathrm{I}_{6yz}S_3S_4S_5C_6 \\
&+2\mathrm{I}_{6xy}C_3C_4C_5S_6^2 - \mathrm{I}_{6xx}C_4S_3S_6^2 + \mathrm{I}_{6yy}C_4S_3S_6^2 - \mathrm{I}_{6xx}C_3S_4S_6^2 + \mathrm{I}_{6yy}C_3S_4S_6^2 - 2\mathrm{I}_{6xy}C_5S_3S_4S_6^2 \\
&+\mathrm{I}_{6zz}C_4S_3 + \mathrm{I}_{6zz}C_3S_4 - 2\mathrm{I}_{6xz}C_3C_4S_5S_6 + 2\mathrm{I}_{6xz}S_3S_4S_5S_6 + \mathrm{I}_{6xx}C_3C_4C_5S_{66} - \mathrm{I}_{6yy}C_3C_4C_5S_{66} \\
&+2\mathrm{I}_{6xy}C_4S_3S_{66} + 2\mathrm{I}_{6xy}C_3S_4S_{66} - \mathrm{I}_{6xx}C_5S_3S_4S_{66} + \mathrm{I}_{6yy}C_5S_3S_4S_{66} - 2C_3\left(\mathrm{COM}_{6y}C_6\right. \\
&\left.+\mathrm{COM}_{6x}S_6\right)L_1m_6 + 2S_{34}\left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right)L_2m_6)\,\dot{\theta}_3 - S_5\left(C_3\left(C_4\left(2S_5\left(\mathrm{I}_{6yz}C_6 + \mathrm{I}_{6xz}S_6\right)\right.\right.\right. \\
&\left.+C_5\left(2\mathrm{I}_{6xy}C_{66} + (\mathrm{I}_{6yy} - \mathrm{I}_{6xx})S_{66}\right)\right) - S_4\left(\mathrm{I}_{6zz} + (\mathrm{I}_{6xx} - \mathrm{I}_{6yy})C_{66} + 2\mathrm{I}_{6xy}S_{66}\right)) - S_3\left(C_4\left(\mathrm{I}_{6zz}\right.\right. \\
&\left.+(\mathrm{I}_{6xx} - \mathrm{I}_{6yy})C_{66} + 2\mathrm{I}_{6xy}S_{66}\right) + S_4\left(2S_5\left(\mathrm{I}_{6yz}C_6 + \mathrm{I}_{6xz}S_6\right) + C_5\left(2\mathrm{I}_{6xy}C_{66} + (\mathrm{I}_{6yy} - \mathrm{I}_{6xx})S_{66}\right)\right))) \\
&-2S_{34}\left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right)L_2m_6)\,\dot{\theta}_4 + \left(C_3\left(S_4\left(2\mathrm{I}_{6xy}C_{66} + (\mathrm{I}_{6yy} - \mathrm{I}_{6xx})S_{66}\right) + C_4\left(2S_5\left(\mathrm{I}_{6yz}S_6\right.\right.\right.\right. \\
&\left.\left.-\mathrm{I}_{6xz}C_6\right) - C_5\left(\mathrm{I}_{6zz} + (\mathrm{I}_{6yy} - \mathrm{I}_{6xx})C_{66} - 2\mathrm{I}_{6xy}S_{66}\right)\right)) + S_3\left(C_4\left(2\mathrm{I}_{6xy}C_{66} + (\mathrm{I}_{6yy} - \mathrm{I}_{6xx})S_{66}\right)\right. \\
&\left.\left.+S_4\left(2S_5\left(\mathrm{I}_{6xz}C_6 - \mathrm{I}_{6yz}S_6\right) + C_5\left(\mathrm{I}_{6zz} + (\mathrm{I}_{6yy} - \mathrm{I}_{6xx})C_{66} - 2\mathrm{I}_{6xy}S_{66}\right)\right)\right))\,\dot{\theta}_5\Big)\,\dot{\theta}_2 + 16\left(\left(-2\mathrm{I}_{6xy}C_{66}S_5^2\right.\right. \\
&+\mathrm{I}_{6xx}S_{66}S_5^2 - \mathrm{I}_{6yy}S_{66}S_5^2 + \mathrm{I}_{6yz}C_6S_{55} + \mathrm{I}_{6xz}S_{55}S_6 + 2\left(C_4C_5\left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right)\right. \\
&\left.+S_4\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right)\right)L_1m_6 - 2\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right)L_2m_6)\,\dot{\theta}_3{}^2 + 2\left(\left(-2\mathrm{I}_{6xy}C_{66}S_5^2\right.\right. \\
&\left.+\mathrm{I}_{6xx}S_{66}S_5^2 - \mathrm{I}_{6yy}S_{66}S_5^2 + \mathrm{I}_{6yz}C_6S_{55} + \mathrm{I}_{6xz}S_{55}S_6 - 2\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right)L_2m_6\right)\dot{\theta}_4 \\
&\left.+\left(2C_5\left(\mathrm{I}_{6xz}C_6 - \mathrm{I}_{6yz}S_6\right) - S_5\left(\mathrm{I}_{6zz} + (\mathrm{I}_{6yy} - \mathrm{I}_{6xx})C_{66} - 2\mathrm{I}_{6xy}S_{66}\right)\right)\dot{\theta}_5\right)\dot{\theta}_3\ \ldots
\end{aligned}
$$

$$+\frac{1}{4}\left(4S_5\left(2C_5\left(\mathrm{I}_{6yz}C_6+\mathrm{I}_{6xz}S_6\right)+S_5\left((\mathrm{I}_{6xx}-\mathrm{I}_{6yy})S_{66}-2\mathrm{I}_{6xy}C_{66}\right)\right)-8\left(\mathrm{COM}_{6x}C_6\right.\right.$$
$$\left.-\mathrm{COM}_{6y}S_6\right)L_2m_6)\,\dot{\theta}_4{}^2+\left(2\mathrm{I}_{6xy}C_{66}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})S_{66}\right)\dot{\theta}_5{}^2+2\left(2C_5\left(\mathrm{I}_{6xz}C_6-\mathrm{I}_{6yz}S_6\right)-S_5\left(\mathrm{I}_{6zz}\right.\right.$$
$$\left.\left.\left.+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})C_{66}-2\mathrm{I}_{6xy}S_{66}\right)\right)\dot{\theta}_4\dot{\theta}_5\right)\Big) \tag{A.3f}$$

A.3 Gravitational Terms $G(\Theta)$

The gravitational terms $G(\Theta)$ form a 6×1 vector. The term G_i indicates the i^{th} term of G. The following formulas, found using Equation (2.27), define each of the elements of $G(\Theta)$.

$$G_1 = 0 \tag{A.4a}$$

$$G_2 = 0 \tag{A.4b}$$

$$G_3 = -9810\left(\mathrm{COM}_{3y}C_3+\mathrm{COM}_{3x}S_3\right)m_3-9810\left(\left(\mathrm{COM}_{4y}C_{34}+\mathrm{COM}_{4x}S_{34}\right)m_4\right.$$
$$+\left(\mathrm{COM}_{5z}C_{34}+S_{34}\left(\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5\right)\right)m_5+\left(C_{34}\left(\mathrm{COM}_{6y}C_6\right.\right.$$
$$\left.\left.\left.+\mathrm{COM}_{6x}S_6\right)+S_{34}\left(\mathrm{COM}_{6z}S_5+C_5\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right)\right)m_6\right) \tag{A.4c}$$

$$G_4 = -9810\left(\left(\mathrm{COM}_{4y}C_{34}+\mathrm{COM}_{4x}S_{34}\right)m_4+\left(\mathrm{COM}_{5z}C_{34}+S_{34}\left(\mathrm{COM}_{5x}C_5\right.\right.\right.$$
$$\left.\left.-\mathrm{COM}_{5y}S_5\right)\right)m_5+\left(C_{34}\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)+S_{34}\left(\mathrm{COM}_{6z}S_5\right.\right.$$
$$\left.\left.\left.+C_5\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right)\right)m_6\right) \tag{A.4d}$$

$$G_5 = -9810C_{34}\left(\left(\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5\right)m_5-\left(\mathrm{COM}_{6z}C_5+S_5\left(\mathrm{COM}_{6y}S_6\right.\right.\right.$$
$$\left.\left.\left.-\mathrm{COM}_{6x}C_6\right)\right)m_6\right) \tag{A.4e}$$

$$G_6 = -9810\left(C_{34}C_5\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)+S_{34}\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right)m_6 \tag{A.4f}$$

A.4 Joint Torque Equations τ

The 6×1 vector τ represents the torque applied to each joint. The formulas found using Equation (2.29) produce results with units $\dfrac{kg \cdot mm^2}{s^2}$. The formulas shown here have been multiplied by 10^{-6}, and therefore have units $N \cdot m$.

$$
\begin{aligned}
\tau_1 = 10^{-6} \Big(& -(\mathrm{COM}_{2y} C_2 + \mathrm{COM}_{2x} S_2)\, m_2 \dot{\theta}_2{}^2 - (\mathrm{COM}_{3z} C_2 + S_2 (\mathrm{COM}_{3y} C_3 + \mathrm{COM}_{3x} S_3))\, m_3 \dot{\theta}_2{}^2 \\
& - (\mathrm{COM}_{4z} C_2 + S_2 (\mathrm{COM}_{4y} C_{34} + \mathrm{COM}_{4x} S_{34}) + S_2 S_3 L_1)\, m_4 \dot{\theta}_2{}^2 + (\mathrm{COM}_{5y} C_2 C_5 - \mathrm{COM}_{5x} S_2 S_{34} C_5 \\
& - \mathrm{COM}_{5z} C_{34} S_2 + \mathrm{COM}_{5x} C_2 S_5 + \mathrm{COM}_{5y} S_2 S_{34} S_5 - S_2 S_3 L_1 - C_{34} S_2 L_2)\, m_5 \dot{\theta}_2{}^2 + (-\mathrm{COM}_{6z} (C_2 C_5 \\
& + S_2 S_{34} S_5) + \mathrm{COM}_{6x} (C_2 C_6 S_5 - S_2 (C_4 (C_5 C_6 S_3 + C_3 S_6) + S_4 (C_3 C_5 C_6 - S_3 S_6))) \\
& + \mathrm{COM}_{6y} (-C_2 S_5 S_6 - S_2 (C_3 (C_4 C_6 - C_5 S_4 S_6) - S_3 (C_6 S_4 + C_4 C_5 S_6))) - S_2 (S_3 L_1 \\
& + C_{34} L_2))\, m_6 \dot{\theta}_2{}^2 + 2 C_2 (\mathrm{COM}_{3x} C_3 - \mathrm{COM}_{3y} S_3)\, m_3 \dot{\theta}_3 \dot{\theta}_2 + 2 C_2 (\mathrm{COM}_{4x} C_{34} - \mathrm{COM}_{4y} S_{34} \\
& + C_3 L_1)\, m_4 \dot{\theta}_3 \dot{\theta}_2 + 2 C_2 (\mathrm{COM}_{5x} C_{34} C_5 - \mathrm{COM}_{5z} S_{34} - \mathrm{COM}_{5y} C_{34} S_5 + C_3 L_1 - S_{34} L_2)\, m_5 \dot{\theta}_3 \dot{\theta}_2 \\
& + 2 C_2 (\mathrm{COM}_{6z} C_{34} S_5 + \mathrm{COM}_{6y} (S_4 (C_5 S_3 S_6 - C_3 C_6) - C_4 (C_6 S_3 + C_3 C_5 S_6)) \\
& + \mathrm{COM}_{6x} (C_3 (C_4 C_5 C_6 - S_4 S_6) - S_3 (C_5 C_6 S_4 + C_4 S_6)) + C_3 L_1 - S_{34} L_2)\, m_6 \dot{\theta}_3 \dot{\theta}_2 \\
& + 2 C_2 (\mathrm{COM}_{4x} C_{34} - \mathrm{COM}_{4y} S_{34})\, m_4 \dot{\theta}_4 \dot{\theta}_2 + 2 C_2 (-\mathrm{COM}_{5z} S_{34} - L_2 S_{34} + C_{34} (\mathrm{COM}_{5x} C_5 \\
& - \mathrm{COM}_{5y} S_5))\, m_5 \dot{\theta}_4 \dot{\theta}_2 + 2 C_2 (-\mathrm{COM}_{6y} C_4 C_6 S_3 - \mathrm{COM}_{6x} C_5 C_6 S_4 S_3 - \mathrm{COM}_{6x} C_4 S_6 S_3 \\
& + \mathrm{COM}_{6y} C_5 S_4 S_6 S_3 + \mathrm{COM}_{6z} C_{34} S_5 + C_3 (C_4 C_5 (\mathrm{COM}_{6x} C_6 - \mathrm{COM}_{6y} S_6) - S_4 (\mathrm{COM}_{6y} C_6 \\
& + \mathrm{COM}_{6x} S_6)) - S_{34} L_2)\, m_6 \dot{\theta}_4 \dot{\theta}_2 - 2 (C_5 (\mathrm{COM}_{5y} C_2 S_{34} - \mathrm{COM}_{5x} S_2) + (\mathrm{COM}_{5y} S_2 \\
& + \mathrm{COM}_{5x} C_2 S_{34}) S_5)\, m_5 \dot{\theta}_5 \dot{\theta}_2 + 2 (C_5 (\mathrm{COM}_{6x} C_6 S_2 - \mathrm{COM}_{6y} S_6 S_2 + \mathrm{COM}_{6z} C_2 S_{34}) + S_5 (\mathrm{COM}_{6z} S_2 \\
& + C_2 S_{34} (\mathrm{COM}_{6y} S_6 - \mathrm{COM}_{6x} C_6)))\, m_6 \dot{\theta}_5 \dot{\theta}_2 + 2 (C_2 (C_3 (C_4 (\mathrm{COM}_{6x} C_6 - \mathrm{COM}_{6y} S_6) \\
& - C_5 S_4 (\mathrm{COM}_{6y} C_6 + \mathrm{COM}_{6x} S_6)) - S_3 (C_4 C_5 (\mathrm{COM}_{6y} C_6 + \mathrm{COM}_{6x} S_6) + S_4 (\mathrm{COM}_{6x} C_6 \\
& - \mathrm{COM}_{6y} S_6))) - S_2 S_5 (\mathrm{COM}_{6y} C_6 + \mathrm{COM}_{6x} S_6))\, m_6 \dot{\theta}_6 \dot{\theta}_2 - S_2 (\mathrm{COM}_{3y} C_3 + \mathrm{COM}_{3x} S_3)\, m_3 \dot{\theta}_3{}^2 \\
& - S_2 (\mathrm{COM}_{4y} C_{34} + \mathrm{COM}_{4x} S_{34} + S_3 L_1)\, m_4 \dot{\theta}_3{}^2 - S_2 (\mathrm{COM}_{5z} C_{34} + L_2 C_{34} + \mathrm{COM}_{5x} C_5 S_{34} \\
& - \mathrm{COM}_{5y} S_{34} S_5 + S_3 L_1)\, m_5 \dot{\theta}_3{}^2 + S_2 (-\mathrm{COM}_{6z} S_{34} S_5 + \mathrm{COM}_{6x} (S_4 (S_3 S_6 - C_3 C_5 C_6) - C_4 (C_5 C_6 S_3 \\
& + C_3 S_6)) + \mathrm{COM}_{6y} (S_3 (C_6 S_4 + C_4 C_5 S_6) + C_3 (C_5 S_4 S_6 - C_4 C_6)) - S_3 L_1 - C_{34} L_2)\, m_6 \dot{\theta}_3{}^2 \\
& - S_2 (\mathrm{COM}_{4y} C_{34} + \mathrm{COM}_{4x} S_{34})\, m_4 \dot{\theta}_4{}^2 - S_2 (\mathrm{COM}_{5z} C_{34} + L_2 C_{34} + S_{34} (\mathrm{COM}_{5x} C_5 \\
& - \mathrm{COM}_{5y} S_5))\, m_5 \dot{\theta}_4{}^2 + S_2 (-\mathrm{COM}_{6x} C_4 C_5 C_6 S_3 + \mathrm{COM}_{6y} C_6 S_4 S_3 + \mathrm{COM}_{6y} C_4 C_5 S_6 S_3 \\
& + \mathrm{COM}_{6x} S_4 S_6 S_3 - \mathrm{COM}_{6z} S_{34} S_5 - C_3 (C_4 (\mathrm{COM}_{6y} C_6 + \mathrm{COM}_{6x} S_6) + C_5 S_4 (\mathrm{COM}_{6x} C_6 \\
& - \mathrm{COM}_{6y} S_6)) - C_{34} L_2)\, m_6 \dot{\theta}_4{}^2 + (C_2 (\mathrm{COM}_{5y} C_5 + \mathrm{COM}_{5x} S_5) + S_2 S_{34} (\mathrm{COM}_{5y} S_5 \\
& - \mathrm{COM}_{5x} C_5))\, m_5 \dot{\theta}_5{}^2 + (-S_2 S_{34} (\mathrm{COM}_{6z} S_5 + C_5 (\mathrm{COM}_{6x} C_6 - \mathrm{COM}_{6y} S_6)) - C_2 (\mathrm{COM}_{6z} C_5 \;\ldots
\end{aligned}
$$

$$
\begin{aligned}
&+S_5\left(\mathrm{COM}_{6y}S_6-\mathrm{COM}_{6x}C_6\right)))\,m_6\dot{\theta}_5{}^2+(\mathrm{COM}_{6y}C_6S_2S_3S_4+\mathrm{COM}_{6x}S_2S_3S_6S_4+\mathrm{COM}_{6x}C_2C_6S_5\\
&-\mathrm{COM}_{6y}C_2S_5S_6-C_4C_5S_2S_3\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)-C_3S_2\left(C_4\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)\right.\\
&+C_5S_4\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)))\,m_6\dot{\theta}_6{}^2-2S_2\left(\mathrm{COM}_{4y}C_{34}+\mathrm{COM}_{4x}S_{34}\right)m_4\dot{\theta}_3\dot{\theta}_4\\
&-2S_2\left(\mathrm{COM}_{5z}C_{34}+L_2C_{34}+S_{34}\left(\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5\right)\right)m_5\dot{\theta}_3\dot{\theta}_4+2S_2\left(-\mathrm{COM}_{6x}C_4C_5C_6S_3\right.\\
&+\mathrm{COM}_{6y}C_6S_4S_3+\mathrm{COM}_{6y}C_4C_5S_6S_3+\mathrm{COM}_{6x}S_4S_6S_3-\mathrm{COM}_{6z}S_{34}S_5-C_3\left(C_4\left(\mathrm{COM}_{6y}C_6\right.\right.\\
&+\mathrm{COM}_{6x}S_6)+C_5S_4\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right))-C_{34}L_2)\,m_6\dot{\theta}_3\dot{\theta}_4-2C_{34}S_2\left(\mathrm{COM}_{5y}C_5\right.\\
&+\mathrm{COM}_{5x}S_5)\,m_5\dot{\theta}_3\dot{\theta}_5+2C_{34}S_2\left(\mathrm{COM}_{6z}C_5+S_5\left(\mathrm{COM}_{6y}S_6-\mathrm{COM}_{6x}C_6\right)\right)m_6\dot{\theta}_3\dot{\theta}_5\\
&-2C_{34}S_2\left(\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5\right)m_5\dot{\theta}_4\dot{\theta}_5+2C_{34}S_2\left(\mathrm{COM}_{6z}C_5+S_5\left(\mathrm{COM}_{6y}S_6\right.\right.\\
&-\mathrm{COM}_{6x}C_6))\,m_6\dot{\theta}_4\dot{\theta}_5+2S_2\left(S_3\left(C_5S_4\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)+C_4\left(\mathrm{COM}_{6y}S_6-\mathrm{COM}_{6x}C_6\right)\right)\right.\\
&-C_3\left(C_4C_5\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)+S_4\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right))\,m_6\dot{\theta}_3\dot{\theta}_6\\
&+2S_2\left(S_3\left(C_5S_4\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)+C_4\left(\mathrm{COM}_{6y}S_6-\mathrm{COM}_{6x}C_6\right)\right)-C_3\left(C_4C_5\left(\mathrm{COM}_{6y}C_6\right.\right.\right.\\
&+\mathrm{COM}_{6x}S_6)+S_4\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)))\,m_6\dot{\theta}_4\dot{\theta}_6+2\left(C_2C_5+S_2S_{34}S_5\right)\left(\mathrm{COM}_{6y}C_6\right.\\
&+\mathrm{COM}_{6x}S_6)\,m_6\dot{\theta}_5\dot{\theta}_6+\left(m_1+m_2+m_3+m_4+m_5+m_6\right)\ddot{d}_1+(\mathrm{COM}_{2x}C_2m_2-\mathrm{COM}_{2y}S_2m_2\\
&+\mathrm{COM}_{3y}C_2C_3m_3-\mathrm{COM}_{3z}S_2m_3+\mathrm{COM}_{3x}C_2S_3m_3+\mathrm{COM}_{4y}C_2C_{34}m_4-\mathrm{COM}_{4z}S_2m_4\\
&+\mathrm{COM}_{4x}C_2S_{34}m_4+C_2S_3L_1m_4+\mathrm{COM}_{5z}C_2C_{34}m_5+\mathrm{COM}_{5x}\left(C_2C_5S_{34}+S_2S_5\right)m_5\\
&+\mathrm{COM}_{5y}\left(C_5S_2-C_2S_{34}S_5\right)m_5+C_2\left(S_3L_1+C_{34}L_2\right)m_5+\mathrm{COM}_{6z}\left(C_2S_{34}S_5-C_5S_2\right)m_6\\
&+\mathrm{COM}_{6x}\left(C_6S_2S_5+C_2\left(C_4\left(C_5C_6S_3+C_3S_6\right)+S_4\left(C_3C_5C_6-S_3S_6\right)\right)\right)m_6+\mathrm{COM}_{6y}\left(C_2\left(C_3\left(C_4C_6\right.\right.\right.\\
&-C_5S_4S_6)-S_3\left(C_6S_4+C_4C_5S_6\right))-S_2S_5S_6)\,m_6+C_2\left(S_3L_1+C_{34}L_2\right)m_6)\,\ddot{\theta}_2+S_2\left(\left(\mathrm{COM}_{3x}C_3\right.\right.\\
&-\mathrm{COM}_{3y}S_3)\,m_3+\left(\mathrm{COM}_{4x}C_{34}-\mathrm{COM}_{4y}S_{34}+C_3L_1\right)m_4+\mathrm{COM}_{5x}C_{34}C_5m_5-\mathrm{COM}_{5z}S_{34}m_5\\
&-\mathrm{COM}_{5y}C_{34}S_5m_5+C_3L_1m_5-S_{34}L_2m_5+\mathrm{COM}_{6x}C_3C_4C_5C_6m_6-\mathrm{COM}_{6y}C_4C_6S_3m_6\\
&-\mathrm{COM}_{6y}C_3C_6S_4m_6-\mathrm{COM}_{6x}C_5C_6S_3S_4m_6+\mathrm{COM}_{6z}C_{34}S_5m_6-\mathrm{COM}_{6y}C_3C_4C_5S_6m_6\\
&-\mathrm{COM}_{6x}C_4S_3S_6m_6-\mathrm{COM}_{6x}C_3S_4S_6m_6+\mathrm{COM}_{6y}C_5S_3S_4S_6m_6+C_3L_1m_6-S_{34}L_2m_6)\,\ddot{\theta}_3\\
&+S_2(\mathrm{COM}_{4x}C_{34}m_4-\mathrm{COM}_{4y}S_{34}m_4+\mathrm{COM}_{5x}C_{34}C_5m_5-\mathrm{COM}_{5z}S_{34}m_5-\mathrm{COM}_{5y}C_{34}S_5m_5\\
&-S_{34}L_2m_5+\mathrm{COM}_{6z}C_{34}S_5m_6+\mathrm{COM}_{6y}\left(S_4\left(C_5S_3S_6-C_3C_6\right)-C_4\left(C_6S_3+C_3C_5S_6\right)\right)m_6\\
&+\mathrm{COM}_{6x}\left(C_3\left(C_4C_5C_6-S_4S_6\right)-S_3\left(C_5C_6S_4+C_4S_6\right)\right)m_6-S_{34}L_2m_6)\,\ddot{\theta}_4+((S_2S_{34}\left(\mathrm{COM}_{6z}C_5\right.\\
&+S_5\left(\mathrm{COM}_{6y}S_6-\mathrm{COM}_{6x}C_6\right))-C_2\left(\mathrm{COM}_{6z}S_5+C_5\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right))\,m_6\\
&-\left(S_2S_{34}\left(\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5\right)+C_2\left(\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5\right)\right)m_5)\,\ddot{\theta}_5+(-\mathrm{COM}_{6x}C_6S_2S_3S_4\\
&+\mathrm{COM}_{6y}S_2S_3S_6S_4+\mathrm{COM}_{6y}C_2C_6S_5+\mathrm{COM}_{6x}C_2S_5S_6-C_4C_5S_2S_3\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)\\
&+C_3S_2\left(C_4\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)-C_5S_4\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)\right))\,m_6\ddot{\theta}_6\Big)
\end{aligned}
\tag{A.5a}
$$

$$
\begin{aligned}
\tau_2 = 10^{-6} \Bigg(\frac{1}{16} \Big(\dot{\theta}_2 \big(\big(16 S_{33} \left(m_4 + m_5 + m_6 \right) L_1^2 + 8 \left(4 \left(\mathrm{COM}_{4y} C_{334} + \mathrm{COM}_{4x} S_{334} \right) m_4 + 4 \left(\mathrm{COM}_{5z} C_{334} \right.\right. \\
& + L_2 C_{334} + S_{334} \left(\mathrm{COM}_{5x} C_5 - \mathrm{COM}_{5y} S_5 \right) \big) m_5 + \left(2\mathrm{COM}_{6z} C_{334-5} - 2\mathrm{COM}_{6z} C_{3345} \right. \\
& + 2\mathrm{COM}_{6y} C_{334-6} - \mathrm{COM}_{6y} C_{334-56} - \mathrm{COM}_{6y} C_{3345-6} + 2\mathrm{COM}_{6y} C_{3346} + \mathrm{COM}_{6y} C_{3346-5} \\
& + \mathrm{COM}_{6y} C_{33456} - 2\mathrm{COM}_{6x} S_{334-6} + \mathrm{COM}_{6x} S_{334-56} + \mathrm{COM}_{6x} S_{3345-6} + 2\mathrm{COM}_{6x} S_{3346} \\
& + \mathrm{COM}_{6x} S_{3346-5} + \mathrm{COM}_{6x} S_{33456} + 4 C_{334} L_2 \big) m_6 \big) L_1 + 32 \mathrm{I}_{3xy} C_{33} + 32 \mathrm{I}_{4xy} C_{3344} - 8 \mathrm{I}_{5xy} C_{3344-55} \\
& + 16 \mathrm{I}_{5xz} C_{3344-5} + 8 \mathrm{I}_{5xy} C_{334455} + 16 \mathrm{I}_{5xz} C_{33445} + 8 \mathrm{I}_{6xy} C_{3344-566} + 8 \mathrm{I}_{6xy} C_{33445-66} - 12 \mathrm{I}_{6xy} C_{3344-66} \\
& + 4 \mathrm{I}_{6xz} C_{3344-556} - 2 \mathrm{I}_{6xy} C_{3344-5566} - 8 \mathrm{I}_{6xz} C_{3344-56} - 2 \mathrm{I}_{6xy} C_{334455-66} + 8 \mathrm{I}_{6xz} C_{33445-6} \\
& - 4 \mathrm{I}_{6xz} C_{334455-6} + 12 \mathrm{I}_{6xy} C_{334466} + 4 \mathrm{I}_{6xz} C_{334466-55} + 2 \mathrm{I}_{6xy} C_{334466-55} + 8 \mathrm{I}_{6xz} C_{33446-5} \\
& + 2 \mathrm{I}_{6xy} C_{33445566} - 8 \mathrm{I}_{6xz} C_{334456} - 4 \mathrm{I}_{6xz} C_{3344556} + 8 \mathrm{I}_{6xy} C_{334466-5} + 8 \mathrm{I}_{6xy} C_{3344566} - 16 \mathrm{I}_{3xx} S_{33} \\
& + 16 \mathrm{I}_{3yy} S_{33} - 16 \mathrm{I}_{4xx} S_{3344} + 16 \mathrm{I}_{4yy} S_{3344} - 8 \mathrm{I}_{5xx} S_{3344} - 8 \mathrm{I}_{5yy} S_{3344} + 16 \mathrm{I}_{5zz} S_{3344} + 4 \mathrm{I}_{6xx} S_{3344} \\
& + 4 \mathrm{I}_{6yy} S_{3344} - 8 \mathrm{I}_{6zz} S_{3344} - 4 \mathrm{I}_{5xx} S_{3344-55} + 4 \mathrm{I}_{5yy} S_{3344-55} - 2 \mathrm{I}_{6xx} S_{3344-55} - 2 \mathrm{I}_{6yy} S_{3344-55} \\
& + 4 \mathrm{I}_{6zz} S_{3344-55} + 16 \mathrm{I}_{5yz} S_{3344-5} - 4 \mathrm{I}_{5xx} S_{334455} + 4 \mathrm{I}_{5yy} S_{334455} - 2 \mathrm{I}_{6xx} S_{334455} - 2 \mathrm{I}_{6yy} S_{334455} \\
& + 4 \mathrm{I}_{6zz} S_{334455} - 16 \mathrm{I}_{5yz} S_{33445} + 4 \mathrm{I}_{6xx} S_{3344-566} - 4 \mathrm{I}_{6yy} S_{3344-566} + 4 \mathrm{I}_{6xx} S_{33445-66} - 4 \mathrm{I}_{6yy} S_{33445-66} \\
& - 6 \mathrm{I}_{6xx} S_{3344-66} + 6 \mathrm{I}_{6yy} S_{3344-66} + 4 \mathrm{I}_{6yz} S_{3344-556} - \mathrm{I}_{6xx} S_{3344-5566} + \mathrm{I}_{6yy} S_{3344-5566} - 8 \mathrm{I}_{6yz} S_{3344-56} \\
& - \mathrm{I}_{6xx} S_{334455-66} + \mathrm{I}_{6yy} S_{334455-66} + 8 \mathrm{I}_{6yz} S_{33445-6} - 4 \mathrm{I}_{6yz} S_{334455-6} - 6 \mathrm{I}_{6xx} S_{334466} \\
& + 6 \mathrm{I}_{6yy} S_{334466} - 4 \mathrm{I}_{6yz} S_{334466-55} - \mathrm{I}_{6xx} S_{334466-55} + \mathrm{I}_{6yy} S_{334466-55} - 8 \mathrm{I}_{6yz} S_{33446-5} - \mathrm{I}_{6xx} S_{33445566} \\
& + \mathrm{I}_{6yy} S_{33445566} + 8 \mathrm{I}_{6yz} S_{334456} + 4 \mathrm{I}_{6yz} S_{3344556} - 4 \mathrm{I}_{6xx} S_{334466-5} + 4 \mathrm{I}_{6yy} S_{334466-5} - 4 \mathrm{I}_{6xx} S_{3344566} \\
& + 4 \mathrm{I}_{6yy} S_{3344566} - 16 S_{3344} L_2^2 \left(m_5 + m_6 \right) + 8 L_2 \big(2 \left(\mathrm{COM}_{5x} C_{3344-5} + \mathrm{COM}_{5x} C_{33445} - 2\mathrm{COM}_{5z} S_{3344} \right. \\
& + \mathrm{COM}_{5y} S_{3344-5} - \mathrm{COM}_{5y} S_{33445} \big) m_5 + \big(-2\mathrm{COM}_{6x} C_{3344-6} + \mathrm{COM}_{6x} C_{3344-56} + \mathrm{COM}_{6x} C_{33445-6} \\
& + 2\mathrm{COM}_{6x} C_{33446} + \mathrm{COM}_{6x} C_{33446-5} + \mathrm{COM}_{6x} C_{334456} - 2\mathrm{COM}_{6z} S_{3344-5} + 2\mathrm{COM}_{6z} S_{33445} \\
& - 2\mathrm{COM}_{6y} S_{3344-6} + \mathrm{COM}_{6y} S_{3344-56} + \mathrm{COM}_{6y} S_{33445-6} - 2\mathrm{COM}_{6y} S_{33446} - \mathrm{COM}_{6y} S_{33446-5} \\
& - \mathrm{COM}_{6y} S_{334456} \big) m_6 \big) \big) \dot{\theta}_3 + \big(-16 S_{3344} \left(m_5 + m_6 \right) L_2^2 + 8 \big(2 \big(\mathrm{COM}_{5x} C_{3344-5} + \mathrm{COM}_{5x} C_{33445} \\
& - 2\mathrm{COM}_{5z} S_{3344} + \mathrm{COM}_{5y} S_{3344-5} - \mathrm{COM}_{5y} S_{33445} \big) m_5 + \big(-2\mathrm{COM}_{6x} C_{3344-6} + \mathrm{COM}_{6x} C_{3344-56} \\
& + \mathrm{COM}_{6x} C_{33445-6} + 2\mathrm{COM}_{6x} C_{33446} + \mathrm{COM}_{6x} C_{33446-5} + \mathrm{COM}_{6x} C_{334456} - 2\mathrm{COM}_{6z} S_{3344-5} \\
& + 2\mathrm{COM}_{6z} S_{33445} - 2\mathrm{COM}_{6y} S_{3344-6} + \mathrm{COM}_{6y} S_{3344-56} + \mathrm{COM}_{6y} S_{33445-6} - 2\mathrm{COM}_{6y} S_{33446} \\
& - \mathrm{COM}_{6y} S_{33446-5} - \mathrm{COM}_{6y} S_{334456} \big) m_6 \big) L_2 + 32 \mathrm{I}_{4xy} C_{3344} - 8 \mathrm{I}_{5xy} C_{3344-55} + 16 \mathrm{I}_{5xz} C_{3344-5} \\
& + 8 \mathrm{I}_{5xy} C_{334455} + 16 \mathrm{I}_{5xz} C_{33445} + 8 \mathrm{I}_{6xy} C_{3344-566} + 8 \mathrm{I}_{6xy} C_{33445-66} - 12 \mathrm{I}_{6xy} C_{3344-66} \\
& + 4 \mathrm{I}_{6xz} C_{3344-556} - 2 \mathrm{I}_{6xy} C_{3344-5566} - 8 \mathrm{I}_{6xz} C_{3344-56} - 2 \mathrm{I}_{6xy} C_{334455-66} + 8 \mathrm{I}_{6xz} C_{33445-6} \\
& - 4 \mathrm{I}_{6xz} C_{334455-6} + 12 \mathrm{I}_{6xy} C_{334466} + 4 \mathrm{I}_{6xz} C_{334466-55} + 2 \mathrm{I}_{6xy} C_{334466-55} + 8 \mathrm{I}_{6xz} C_{33446-5} \\
& + 2 \mathrm{I}_{6xy} C_{33445566} - 8 \mathrm{I}_{6xz} C_{334456} - 4 \mathrm{I}_{6xz} C_{3344556} + 8 \mathrm{I}_{6xy} C_{334466-5} + 8 \mathrm{I}_{6xy} C_{3344566} - 16 \mathrm{I}_{4xx} S_{3344} \quad \ldots
\end{aligned}
$$

$$+16\mathrm{I}_{4yy}S_{3344} - 8\mathrm{I}_{5xx}S_{3344} - 8\mathrm{I}_{5yy}S_{3344} + 16\mathrm{I}_{5zz}S_{3344} + 4\mathrm{I}_{6xx}S_{3344} + 4\mathrm{I}_{6yy}S_{3344} - 8\mathrm{I}_{6zz}S_{3344}$$
$$-4\mathrm{I}_{5xx}S_{3344-55} + 4\mathrm{I}_{5yy}S_{3344-55} - 2\mathrm{I}_{6xx}S_{3344-55} - 2\mathrm{I}_{6yy}S_{3344-55} + 4\mathrm{I}_{6zz}S_{3344-55} + 16\mathrm{I}_{5yz}S_{3344-5}$$
$$-4\mathrm{I}_{5xx}S_{334455} + 4\mathrm{I}_{5yy}S_{334455} - 2\mathrm{I}_{6xx}S_{334455} - 2\mathrm{I}_{6yy}S_{334455} + 4\mathrm{I}_{6zz}S_{334455} - 16\mathrm{I}_{5yz}S_{33445}$$
$$+4\mathrm{I}_{6xx}S_{3344-566} - 4\mathrm{I}_{6yy}S_{3344-566} + 4\mathrm{I}_{6xx}S_{33445-66} - 4\mathrm{I}_{6yy}S_{33445-66} - 6\mathrm{I}_{6xx}S_{3344-66}$$
$$+6\mathrm{I}_{6yy}S_{3344-66} + 4\mathrm{I}_{6yz}S_{3344-556} - \mathrm{I}_{6xx}S_{3344-5566} + \mathrm{I}_{6yy}S_{3344-5566} - 8\mathrm{I}_{6yz}S_{3344-56}$$
$$-\mathrm{I}_{6xx}S_{334455-66} + \mathrm{I}_{6yy}S_{334455-66} + 8\mathrm{I}_{6yz}S_{33445-6} - 4\mathrm{I}_{6yz}S_{334455-6} - 6\mathrm{I}_{6xx}S_{334466}$$
$$+6\mathrm{I}_{6yy}S_{334466} - 4\mathrm{I}_{6yz}S_{334466-55} - \mathrm{I}_{6xx}S_{334466-55} + \mathrm{I}_{6yy}S_{334466-55} - 8\mathrm{I}_{6yz}S_{33446-5} - \mathrm{I}_{6xx}S_{33445566}$$
$$+\mathrm{I}_{6yy}S_{33445566} + 8\mathrm{I}_{6yz}S_{334456} + 4\mathrm{I}_{6yz}S_{3344556} - 4\mathrm{I}_{6xx}S_{334466-5} + 4\mathrm{I}_{6yy}S_{334466-5} - 4\mathrm{I}_{6xx}S_{3344566}$$
$$+4\mathrm{I}_{6yy}S_{3344566} - 16S_3L_1\left(\left(2\mathrm{COM}_{4y}S_{34} - 2\mathrm{COM}_{4x}C_{34}\right)m_4 - \left(\mathrm{COM}_{5x}C_{34-5} + \mathrm{COM}_{5x}C_{345}\right.\right.$$
$$-2\mathrm{COM}_{5z}S_{34} + \mathrm{COM}_{5y}S_{34-5} - \mathrm{COM}_{5y}S_{345} - 2S_{34}L_2\big)m_5 + 2\left(C_3\left(S_4\left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right)\right.\right.$$
$$-C_4\left(\mathrm{COM}_{6z}S_5 + C_5\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right)\right)\big) + S_3\left(C_4\left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right)\right.$$
$$\left.\left.\left.+S_4\left(\mathrm{COM}_{6z}S_5 + C_5\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right)\right)\right) + S_{34}L_2\right)m_6\big)\big)\dot{\theta}_4 + 8\mathrm{I}_{5xy}C_{3344-55}\dot{\theta}_5$$
$$-8\mathrm{I}_{5xz}C_{3344-5}\dot{\theta}_5 + 16\mathrm{I}_{5xy}C_{55}\dot{\theta}_5 + 8\mathrm{I}_{5xy}C_{334455}\dot{\theta}_5 + 8\mathrm{I}_{5xz}C_{33445}\dot{\theta}_5 - 4\mathrm{I}_{6xy}C_{3344-566}\dot{\theta}_5$$
$$+4\mathrm{I}_{6xy}C_{33445-66}\dot{\theta}_5 - 4\mathrm{I}_{6xz}C_{3344-556}\dot{\theta}_5 + 2\mathrm{I}_{6xy}C_{3344-5566}\dot{\theta}_5 + 4\mathrm{I}_{6xz}C_{3344-56}\dot{\theta}_5 - 4\mathrm{I}_{6xy}C_{55-66}\dot{\theta}_5$$
$$-2\mathrm{I}_{6xy}C_{334455-66}\dot{\theta}_5 + 4\mathrm{I}_{6xz}C_{33445-6}\dot{\theta}_5 - 8\mathrm{I}_{6xz}C_{55-6}\dot{\theta}_5 - 4\mathrm{I}_{6xz}C_{334455-6}\dot{\theta}_5 - 4\mathrm{I}_{6xz}C_{334466-55}\dot{\theta}_5$$
$$-2\mathrm{I}_{6xy}C_{334466-55}\dot{\theta}_5 - 4\mathrm{I}_{6xz}C_{33446-5}\dot{\theta}_5 + 4\mathrm{I}_{6xy}C_{5566}\dot{\theta}_5 + 2\mathrm{I}_{6xy}C_{33445566}\dot{\theta}_5 - 4\mathrm{I}_{6xz}C_{334456}\dot{\theta}_5$$
$$-8\mathrm{I}_{6xz}C_{556}\dot{\theta}_5 - 4\mathrm{I}_{6xz}C_{3344556}\dot{\theta}_5 - 4\mathrm{I}_{6xy}C_{334466-5}\dot{\theta}_5 + 4\mathrm{I}_{6xy}C_{3344566}\dot{\theta}_5 + 4\mathrm{I}_{5xx}S_{3344-55}\dot{\theta}_5$$
$$-4\mathrm{I}_{5yy}S_{3344-55}\dot{\theta}_5 + 2\mathrm{I}_{6xx}S_{3344-55}\dot{\theta}_5 + 2\mathrm{I}_{6yy}S_{3344-55}\dot{\theta}_5 - 4\mathrm{I}_{6zz}S_{3344-55}\dot{\theta}_5 - 8\mathrm{I}_{5yz}S_{3344-5}\dot{\theta}_5$$
$$-8\mathrm{I}_{5xx}S_{55}\dot{\theta}_5 + 8\mathrm{I}_{5yy}S_{55}\dot{\theta}_5 - 4\mathrm{I}_{6xx}S_{55}\dot{\theta}_5 - 4\mathrm{I}_{6yy}S_{55}\dot{\theta}_5 + 8\mathrm{I}_{6zz}S_{55}\dot{\theta}_5 - 4\mathrm{I}_{5xx}S_{334455}\dot{\theta}_5 + 4\mathrm{I}_{5yy}S_{334455}\dot{\theta}_5$$
$$-2\mathrm{I}_{6xx}S_{334455}\dot{\theta}_5 - 2\mathrm{I}_{6yy}S_{334455}\dot{\theta}_5 + 4\mathrm{I}_{6zz}S_{334455}\dot{\theta}_5 - 8\mathrm{I}_{5yz}S_{33445}\dot{\theta}_5 - 2\mathrm{I}_{6xx}S_{3344-566}\dot{\theta}_5$$
$$+2\mathrm{I}_{6yy}S_{3344-566}\dot{\theta}_5 + 2\mathrm{I}_{6xx}S_{33445-66}\dot{\theta}_5 - 2\mathrm{I}_{6yy}S_{33445-66}\dot{\theta}_5 - 4\mathrm{I}_{6yz}S_{3344-556}\dot{\theta}_5 + \mathrm{I}_{6xx}S_{3344-5566}\dot{\theta}_5$$
$$-\mathrm{I}_{6yy}S_{3344-5566}\dot{\theta}_5 + 4\mathrm{I}_{6yz}S_{3344-56}\dot{\theta}_5 - 2\mathrm{I}_{6xx}S_{55-66}\dot{\theta}_5 + 2\mathrm{I}_{6yy}S_{55-66}\dot{\theta}_5 - \mathrm{I}_{6xx}S_{334455-66}\dot{\theta}_5$$
$$+\mathrm{I}_{6yy}S_{334455-66}\dot{\theta}_5 + 4\mathrm{I}_{6yz}S_{33445-6}\dot{\theta}_5 - 8\mathrm{I}_{6yz}S_{55-6}\dot{\theta}_5 - 4\mathrm{I}_{6yz}S_{334455-6}\dot{\theta}_5 + 4\mathrm{I}_{6yz}S_{334466-55}\dot{\theta}_5$$
$$+\mathrm{I}_{6xx}S_{334466-55}\dot{\theta}_5 - \mathrm{I}_{6yy}S_{334466-55}\dot{\theta}_5 + 4\mathrm{I}_{6yz}S_{33446-5}\dot{\theta}_5 - 2\mathrm{I}_{6xx}S_{5566}\dot{\theta}_5 + 2\mathrm{I}_{6yy}S_{5566}\dot{\theta}_5$$
$$-\mathrm{I}_{6xx}S_{33445566}\dot{\theta}_5 + \mathrm{I}_{6yy}S_{33445566}\dot{\theta}_5 + 4\mathrm{I}_{6yz}S_{334456}\dot{\theta}_5 + 8\mathrm{I}_{6yz}S_{556}\dot{\theta}_5 + 4\mathrm{I}_{6yz}S_{3344556}\dot{\theta}_5$$
$$+2\mathrm{I}_{6xx}S_{334466-5}\dot{\theta}_5 - 2\mathrm{I}_{6yy}S_{334466-5}\dot{\theta}_5 - 2\mathrm{I}_{6xx}S_{3344566}\dot{\theta}_5 + 2\mathrm{I}_{6yy}S_{3344566}\dot{\theta}_5 - 8\mathrm{COM}_{5y}C_{4-5}L_1m_5\dot{\theta}_5$$
$$+8\mathrm{COM}_{5y}C_{334-5}L_1m_5\dot{\theta}_5 - 8\mathrm{COM}_{5y}C_{45}L_1m_5\dot{\theta}_5 + 8\mathrm{COM}_{5y}C_{3345}L_1m_5\dot{\theta}_5 + 8\mathrm{COM}_{5x}S_{4-5}L_1m_5\dot{\theta}_5$$
$$-8\mathrm{COM}_{5x}S_{334-5}L_1m_5\dot{\theta}_5 - 8\mathrm{COM}_{5x}S_{45}L_1m_5\dot{\theta}_5 + 8\mathrm{COM}_{5x}S_{3345}L_1m_5\dot{\theta}_5 - 8\mathrm{COM}_{5x}C_{3344-5}L_2m_5\dot{\theta}_5$$
$$+8\mathrm{COM}_{5x}C_{33445}L_2m_5\dot{\theta}_5 - 8\mathrm{COM}_{5y}S_{3344-5}L_2m_5\dot{\theta}_5 - 8\mathrm{COM}_{5y}S_{33445}L_2m_5\dot{\theta}_5$$
$$+8\mathrm{COM}_{6z}C_{4-5}L_1m_6\dot{\theta}_5 - 8\mathrm{COM}_{6z}C_{334-5}L_1m_6\dot{\theta}_5 + 8\mathrm{COM}_{6z}C_{45}L_1m_6\dot{\theta}_5 \ \ldots$$

$$-8\mathrm{COM}_{6z}C_{3345}L_1m_6\dot{\theta}_5 - 4\mathrm{COM}_{6y}C_{4-56}L_1m_6\dot{\theta}_5 + 4\mathrm{COM}_{6y}C_{334-56}L_1m_6\dot{\theta}_5$$
$$+4\mathrm{COM}_{6y}C_{45-6}L_1m_6\dot{\theta}_5 - 4\mathrm{COM}_{6y}C_{3345-6}L_1m_6\dot{\theta}_5 + 4\mathrm{COM}_{6y}C_{46-5}L_1m_6\dot{\theta}_5$$
$$-4\mathrm{COM}_{6y}C_{3346-5}L_1m_6\dot{\theta}_5 - 4\mathrm{COM}_{6y}C_{456}L_1m_6\dot{\theta}_5 + 4\mathrm{COM}_{6y}C_{33456}L_1m_6\dot{\theta}_5$$
$$+4\mathrm{COM}_{6x}S_{4-56}L_1m_6\dot{\theta}_5 - 4\mathrm{COM}_{6x}S_{334-56}L_1m_6\dot{\theta}_5 - 4\mathrm{COM}_{6x}S_{45-6}L_1m_6\dot{\theta}_5$$
$$+4\mathrm{COM}_{6x}S_{3345-6}L_1m_6\dot{\theta}_5 + 4\mathrm{COM}_{6x}S_{46-5}L_1m_6\dot{\theta}_5 - 4\mathrm{COM}_{6x}S_{3346-5}L_1m_6\dot{\theta}_5$$
$$-4\mathrm{COM}_{6x}S_{456}L_1m_6\dot{\theta}_5 + 4\mathrm{COM}_{6x}S_{33456}L_1m_6\dot{\theta}_5 - 4\mathrm{COM}_{6x}C_{3344-56}L_2m_6\dot{\theta}_5$$
$$+4\mathrm{COM}_{6x}C_{33445-6}L_2m_6\dot{\theta}_5 - 4\mathrm{COM}_{6x}C_{33446-5}L_2m_6\dot{\theta}_5 + 4\mathrm{COM}_{6x}C_{334456}L_2m_6\dot{\theta}_5$$
$$+8\mathrm{COM}_{6z}S_{3344-5}L_2m_6\dot{\theta}_5 + 8\mathrm{COM}_{6z}S_{33445}L_2m_6\dot{\theta}_5 - 4\mathrm{COM}_{6y}S_{3344-56}L_2m_6\dot{\theta}_5$$
$$+4\mathrm{COM}_{6y}S_{33445-6}L_2m_6\dot{\theta}_5 + 4\mathrm{COM}_{6y}S_{33446-5}L_2m_6\dot{\theta}_5 - 4\mathrm{COM}_{6y}S_{334456}L_2m_6\dot{\theta}_5$$
$$-8\mathrm{I}_{6xy}C_{3344-566}\dot{\theta}_6 - 8\mathrm{I}_{6xy}C_{33445-66}\dot{\theta}_6 + 12\mathrm{I}_{6xy}C_{3344-66}\dot{\theta}_6 - 2\mathrm{I}_{6xz}C_{3344-556}\dot{\theta}_6 + 2\mathrm{I}_{6xy}C_{3344-5566}\dot{\theta}_6$$
$$+4\mathrm{I}_{6xz}C_{3344-56}\dot{\theta}_6 + 4\mathrm{I}_{6xy}C_{55-66}\dot{\theta}_6 + 2\mathrm{I}_{6xy}C_{334455-66}\dot{\theta}_6 - 4\mathrm{I}_{6xz}C_{33445-6}\dot{\theta}_6 + 4\mathrm{I}_{6xz}C_{55-6}\dot{\theta}_6$$
$$+2\mathrm{I}_{6xz}C_{334455-6}\dot{\theta}_6 - 8\mathrm{I}_{6xy}C_{66}\dot{\theta}_6 + 12\mathrm{I}_{6xy}C_{334466}\dot{\theta}_6 + 2\mathrm{I}_{6xz}C_{334466-55}\dot{\theta}_6 + 2\mathrm{I}_{6xy}C_{334466-55}\dot{\theta}_6$$
$$+4\mathrm{I}_{6xz}C_{33446-5}\dot{\theta}_6 + 4\mathrm{I}_{6xy}C_{5566}\dot{\theta}_6 + 2\mathrm{I}_{6xy}C_{33445566}\dot{\theta}_6 - 4\mathrm{I}_{6xz}C_{334456}\dot{\theta}_6 - 4\mathrm{I}_{6xz}C_{556}\dot{\theta}_6$$
$$-2\mathrm{I}_{6xz}C_{3344556}\dot{\theta}_6 + 8\mathrm{I}_{6xy}C_{334466-5}\dot{\theta}_6 + 8\mathrm{I}_{6xy}C_{3344566}\dot{\theta}_6 - 4\mathrm{I}_{6xx}S_{3344-566}\dot{\theta}_6 + 4\mathrm{I}_{6yy}S_{3344-566}\dot{\theta}_6$$
$$-4\mathrm{I}_{6xx}S_{33445-66}\dot{\theta}_6 + 4\mathrm{I}_{6yy}S_{33445-66}\dot{\theta}_6 + 6\mathrm{I}_{6xx}S_{3344-66}\dot{\theta}_6 - 6\mathrm{I}_{6yy}S_{3344-66}\dot{\theta}_6 - 2\mathrm{I}_{6yz}S_{3344-556}\dot{\theta}_6$$
$$+\mathrm{I}_{6xx}S_{3344-5566}\dot{\theta}_6 - \mathrm{I}_{6yy}S_{3344-5566}\dot{\theta}_6 + 4\mathrm{I}_{6yz}S_{3344-56}\dot{\theta}_6 + 2\mathrm{I}_{6xx}S_{55-66}\dot{\theta}_6 - 2\mathrm{I}_{6yy}S_{55-66}\dot{\theta}_6$$
$$+\mathrm{I}_{6xx}S_{334455-66}\dot{\theta}_6 - \mathrm{I}_{6yy}S_{334455-66}\dot{\theta}_6 - 4\mathrm{I}_{6yz}S_{33445-6}\dot{\theta}_6 + 4\mathrm{I}_{6yz}S_{55-6}\dot{\theta}_6 + 2\mathrm{I}_{6yz}S_{334455-6}\dot{\theta}_6$$
$$+4\mathrm{I}_{6xx}S_{66}\dot{\theta}_6 - 4\mathrm{I}_{6yy}S_{66}\dot{\theta}_6 - 6\mathrm{I}_{6xx}S_{334466}\dot{\theta}_6 + 6\mathrm{I}_{6yy}S_{334466}\dot{\theta}_6 - 2\mathrm{I}_{6yz}S_{334466-55}\dot{\theta}_6 - \mathrm{I}_{6xx}S_{334466-55}\dot{\theta}_6$$
$$+\mathrm{I}_{6yy}S_{334466-55}\dot{\theta}_6 - 4\mathrm{I}_{6yz}S_{33446-5}\dot{\theta}_6 - 2\mathrm{I}_{6xx}S_{5566}\dot{\theta}_6 + 2\mathrm{I}_{6yy}S_{5566}\dot{\theta}_6 - \mathrm{I}_{6xx}S_{33445566}\dot{\theta}_6$$
$$+\mathrm{I}_{6yy}S_{33445566}\dot{\theta}_6 + 4\mathrm{I}_{6yz}S_{334456}\dot{\theta}_6 + 4\mathrm{I}_{6yz}S_{556}\dot{\theta}_6 + 2\mathrm{I}_{6yz}S_{3344556}\dot{\theta}_6 - 4\mathrm{I}_{6xx}S_{334466-5}\dot{\theta}_6$$
$$+4\mathrm{I}_{6yy}S_{334466-5}\dot{\theta}_6 - 4\mathrm{I}_{6xx}S_{3344566}\dot{\theta}_6 + 4\mathrm{I}_{6yy}S_{3344566}\dot{\theta}_6 + 8\mathrm{COM}_{6y}C_{4-6}L_1m_6\dot{\theta}_6$$
$$-8\mathrm{COM}_{6y}C_{334-6}L_1m_6\dot{\theta}_6 - 4\mathrm{COM}_{6y}C_{4-56}L_1m_6\dot{\theta}_6 + 4\mathrm{COM}_{6y}C_{334-56}L_1m_6\dot{\theta}_6$$
$$-4\mathrm{COM}_{6y}C_{45-6}L_1m_6\dot{\theta}_6 + 4\mathrm{COM}_{6y}C_{3345-6}L_1m_6\dot{\theta}_6 - 8\mathrm{COM}_{6y}C_{46}L_1m_6\dot{\theta}_6 + 8\mathrm{COM}_{6y}C_{3346}L_1m_6\dot{\theta}_6$$
$$-4\mathrm{COM}_{6y}C_{46-5}L_1m_6\dot{\theta}_6 + 4\mathrm{COM}_{6y}C_{3346-5}L_1m_6\dot{\theta}_6 - 4\mathrm{COM}_{6y}C_{456}L_1m_6\dot{\theta}_6$$
$$+4\mathrm{COM}_{6y}C_{33456}L_1m_6\dot{\theta}_6 - 8\mathrm{COM}_{6x}S_{4-6}L_1m_6\dot{\theta}_6 + 8\mathrm{COM}_{6x}S_{334-6}L_1m_6\dot{\theta}_6 + 4\mathrm{COM}_{6x}S_{4-56}L_1m_6\dot{\theta}_6$$
$$-4\mathrm{COM}_{6x}S_{334-56}L_1m_6\dot{\theta}_6 + 4\mathrm{COM}_{6x}S_{45-6}L_1m_6\dot{\theta}_6 - 4\mathrm{COM}_{6x}S_{3345-6}L_1m_6\dot{\theta}_6$$
$$-8\mathrm{COM}_{6x}S_{46}L_1m_6\dot{\theta}_6 + 8\mathrm{COM}_{6x}S_{3346}L_1m_6\dot{\theta}_6 - 4\mathrm{COM}_{6x}S_{46-5}L_1m_6\dot{\theta}_6 + 4\mathrm{COM}_{6x}S_{3346-5}L_1m_6\dot{\theta}_6$$
$$-4\mathrm{COM}_{6x}S_{456}L_1m_6\dot{\theta}_6 + 4\mathrm{COM}_{6x}S_{33456}L_1m_6\dot{\theta}_6 + 8\mathrm{COM}_{6x}C_{3344-6}L_2m_6\dot{\theta}_6$$
$$-4\mathrm{COM}_{6x}C_{3344-56}L_2m_6\dot{\theta}_6 - 4\mathrm{COM}_{6x}C_{33445-6}L_2m_6\dot{\theta}_6 + 16\mathrm{COM}_{6x}C_{6}L_2m_6\dot{\theta}_6$$
$$+8\mathrm{COM}_{6x}C_{33446}L_2m_6\dot{\theta}_6 + 4\mathrm{COM}_{6x}C_{33446-5}L_2m_6\dot{\theta}_6 + 4\mathrm{COM}_{6x}C_{334456}L_2m_6\dot{\theta}_6 \ \ldots$$

$$+8\mathrm{COM}_{6y}S_{3344-6}L_2m_6\dot{\theta}_6 - 4\mathrm{COM}_{6y}S_{3344-56}L_2m_6\dot{\theta}_6 - 4\mathrm{COM}_{6y}S_{33445-6}L_2m_6\dot{\theta}_6$$
$$-16\mathrm{COM}_{6y}S_6L_2m_6\dot{\theta}_6 - 8\mathrm{COM}_{6y}S_{33446}L_2m_6\dot{\theta}_6 - 4\mathrm{COM}_{6y}S_{33446-5}L_2m_6\dot{\theta}_6$$
$$-4\mathrm{COM}_{6y}S_{334456}L_2m_6\dot{\theta}_6\Big) + 4\left(\left(4\mathrm{I}_{6xz}C_4C_6S_3C_5^2 + 4\mathrm{I}_{6xz}C_3C_6S_4C_5^2 - 4\mathrm{I}_{5xy}S_{34}C_5^2 - 4\mathrm{I}_{6yz}C_4S_3S_6C_5^2\right.\right.$$
$$-4\mathrm{I}_{6yz}C_3S_4S_6C_5^2 - 4\mathrm{I}_{5yz}C_{34}C_5 + 4\mathrm{I}_{6yz}C_3C_4C_6C_5 - 4\mathrm{I}_{6yz}C_6S_3S_4C_5 + 4\mathrm{I}_{6xz}C_3C_4S_6C_5$$
$$-4\mathrm{I}_{6xz}S_3S_4S_6C_5 + 4\mathrm{I}_{5xy}S_{34}S_5^2 - 4\mathrm{I}_{6xz}C_6S_{34}S_5^2 + 4\mathrm{I}_{6xy}C_3C_4S_5S_6^2 - 4\mathrm{I}_{6xy}S_3S_4S_5S_6^2$$
$$-\mathrm{I}_{6xx}C_4S_3S_{55}S_6^2 + \mathrm{I}_{6yy}C_4S_3S_{55}S_6^2 - \mathrm{I}_{6zz}C_4S_3S_{55}S_6^2 - \mathrm{I}_{6xx}C_3S_4S_{55}S_6^2 + \mathrm{I}_{6yy}C_3S_4S_{55}S_6^2$$
$$-\mathrm{I}_{6zz}C_3S_4S_{55}S_6^2 + 4\mathrm{I}_{3yz}C_3 + 4\mathrm{I}_{4yz}C_{34} + 4\mathrm{I}_{3xz}S_3 + 4\mathrm{I}_{4xz}S_{34} - 4\mathrm{I}_{6xy}C_3C_4C_6^2S_5 - 4\mathrm{I}_{5xz}C_{34}S_5$$
$$+4\mathrm{I}_{6xy}C_6^2S_3S_4S_5 + \mathrm{I}_{6xx}C_4C_6^2S_3S_{55} - \mathrm{I}_{6yy}C_4C_6^2S_3S_{55} - \mathrm{I}_{6zz}C_4C_6^2S_3S_{55} + \mathrm{I}_{6xx}C_3C_6^2S_4S_{55}$$
$$-\mathrm{I}_{6yy}C_3C_6^2S_4S_{55} - \mathrm{I}_{6zz}C_3C_6^2S_4S_{55} + 2\mathrm{I}_{5xx}S_{34}S_{55} - 2\mathrm{I}_{5yy}S_{34}S_{55} + \mathrm{I}_{6xx}S_{34}S_{55} + \mathrm{I}_{6yy}S_{34}S_{55}$$
$$-\mathrm{I}_{6zz}S_{34}S_{55} + 4\mathrm{I}_{6yz}S_{34}S_5^2S_6 + 2\mathrm{I}_{6xx}C_3C_4S_5S_{66} - 2\mathrm{I}_{6yy}C_3C_4S_5S_{66} - 2\mathrm{I}_{6xx}S_3S_4S_5S_{66}$$
$$+2\mathrm{I}_{6yy}S_3S_4S_5S_{66} + 2\mathrm{I}_{6xy}C_4S_3S_{55}S_{66} + 2\mathrm{I}_{6xy}C_3S_4S_{55}S_{66} - 4C_{34}L_2\left(\left(\mathrm{COM}_{5y}C_5 + \mathrm{COM}_{5x}S_5\right)m_5\right.$$
$$\left.-\left(\mathrm{COM}_{6z}C_5 + S_5\left(\mathrm{COM}_{6y}S_6 - \mathrm{COM}_{6x}C_6\right)\right)m_6\right) + 4S_3L_1\left(\mathrm{COM}_{4z}m_4 - \left(\mathrm{COM}_{5y}C_5\right.\right.$$
$$\left.\left.\left.+\mathrm{COM}_{5x}S_5\right)m_5 + \left(\mathrm{COM}_{6z}C_5 + S_5\left(\mathrm{COM}_{6y}S_6 - \mathrm{COM}_{6x}C_6\right)\right)m_6\right)\right)\dot{\theta}_3{}^2 + \left(2\left(4\mathrm{I}_{6xz}C_4C_6S_3C_5^2\right.\right.$$
$$+4\mathrm{I}_{6xz}C_3C_6S_4C_5^2 - 4\mathrm{I}_{5xy}S_{34}C_5^2 - 4\mathrm{I}_{6yz}C_4S_3S_6C_5^2 - 4\mathrm{I}_{6yz}C_3S_4S_6C_5^2 + 4\mathrm{I}_{6yz}C_3C_4C_6C_5$$
$$-4\mathrm{I}_{6yz}C_6S_3S_4C_5 + 4\mathrm{I}_{6xz}C_3C_4S_6C_5 - 4\mathrm{I}_{6xz}S_3S_4S_6C_5 + 4\mathrm{I}_{5xy}S_{34}S_5^2 - 4\mathrm{I}_{6xz}C_6S_{34}S_5^2$$
$$+4\mathrm{I}_{6xy}C_3C_4S_5S_6^2 - 4\mathrm{I}_{6xy}S_3S_4S_5S_6^2 - \mathrm{I}_{6xx}C_4S_3S_{55}S_6^2 + \mathrm{I}_{6yy}C_4S_3S_{55}S_6^2 - \mathrm{I}_{6zz}C_4S_3S_{55}S_6^2$$
$$-\mathrm{I}_{6xx}C_3S_4S_{55}S_6^2 + \mathrm{I}_{6yy}C_3S_4S_{55}S_6^2 - \mathrm{I}_{6zz}C_3S_4S_{55}S_6^2 + 4\mathrm{I}_{4xz}S_{34} - 4\mathrm{I}_{6xy}C_3C_4C_6^2S_5$$
$$+4\mathrm{I}_{6xy}C_6^2S_3S_4S_5 + 4C_{34}\left(\mathrm{I}_{4yz} - \mathrm{I}_{5yz}C_5 - \mathrm{I}_{5xz}S_5\right) + \mathrm{I}_{6xx}C_4C_6^2S_3S_{55} - \mathrm{I}_{6yy}C_4C_6^2S_3S_{55}$$
$$-\mathrm{I}_{6zz}C_4C_6^2S_3S_{55} + \mathrm{I}_{6xx}C_3C_6^2S_4S_{55} - \mathrm{I}_{6yy}C_3C_6^2S_4S_{55} - \mathrm{I}_{6zz}C_3C_6^2S_4S_{55} + 2\mathrm{I}_{5xx}S_{34}S_{55}$$
$$-2\mathrm{I}_{5yy}S_{34}S_{55} + \mathrm{I}_{6xx}S_{34}S_{55} + \mathrm{I}_{6yy}S_{34}S_{55} - \mathrm{I}_{6zz}S_{34}S_{55} + 4\mathrm{I}_{6yz}S_{34}S_5^2S_6$$
$$+2\mathrm{I}_{6xx}C_3C_4S_5S_{66} - 2\mathrm{I}_{6yy}C_3C_4S_5S_{66} - 2\mathrm{I}_{6xx}S_3S_4S_5S_{66} + 2\mathrm{I}_{6yy}S_3S_4S_5S_{66} + 2\mathrm{I}_{6xy}C_4S_3S_{55}S_{66}$$
$$+2\mathrm{I}_{6xy}C_3S_4S_{55}S_{66} - 4C_{34}L_2\left(\left(\mathrm{COM}_{5y}C_5 + \mathrm{COM}_{5x}S_5\right)m_5 - \left(\mathrm{COM}_{6z}C_5 + S_5\left(\mathrm{COM}_{6y}S_6\right.\right.\right.$$
$$\left.\left.\left.-\mathrm{COM}_{6x}C_6\right)\right)m_6\right)\right)\dot{\theta}_4 - C_{34}\left(4\mathrm{I}_{5zz} + 2\mathrm{I}_{6xx} + 2\mathrm{I}_{6yy} + 2(2\mathrm{I}_{5xx} - 2\mathrm{I}_{5yy} + \mathrm{I}_{6xx} + \mathrm{I}_{6yy} - 2\mathrm{I}_{6zz})C_{55}\right.$$
$$+(\mathrm{I}_{6xx} - \mathrm{I}_{6yy})C_{55-66} + 4\mathrm{I}_{6yz}C_{55-6} - 2\mathrm{I}_{6xx}C_{66} + 2\mathrm{I}_{6yy}C_{66} + \mathrm{I}_{6xx}C_{5566} - \mathrm{I}_{6yy}C_{5566} - 4\mathrm{I}_{6yz}C_{556}$$
$$\left.+8\mathrm{I}_{5xy}S_{55} - 2\mathrm{I}_{6xy}S_{55-66} - 4\mathrm{I}_{6xz}S_{55-6} - 4\mathrm{I}_{6xy}S_{66} + 2\mathrm{I}_{6xy}S_{5566} - 4\mathrm{I}_{6xz}S_{556}\right)\dot{\theta}_5$$
$$+2\left(C_3\left(2S_4\left(2C_5\left(\mathrm{I}_{6xz}C_6 - \mathrm{I}_{6yz}S_6\right) - S_5\left(\mathrm{I}_{6zz} + (\mathrm{I}_{6yy} - \mathrm{I}_{6xx})C_{66} - 2\mathrm{I}_{6xy}S_{66}\right)\right) + C_4\left(4\left(\mathrm{I}_{6yz}C_6\right.\right.\right.\right.$$
$$\left.\left.\left.\left.+\mathrm{I}_{6xz}S_6\right)C_5^2 + S_{55}\left((\mathrm{I}_{6xx} - \mathrm{I}_{6yy})S_{66} - 2\mathrm{I}_{6xy}C_{66}\right)\right)\right) + S_3\left(2C_4\left(2C_5\left(\mathrm{I}_{6xz}C_6 - \mathrm{I}_{6yz}S_6\right) - S_5\left(\mathrm{I}_{6zz}\right.\right.\right.$$
$$\left.\left.+(\mathrm{I}_{6yy} - \mathrm{I}_{6xx})C_{66} - 2\mathrm{I}_{6xy}S_{66}\right)\right) + S_4\left(S_{55}\left(2\mathrm{I}_{6xy}C_{66} + (\mathrm{I}_{6yy} - \mathrm{I}_{6xx})S_{66}\right) - 4C_5^2\left(\mathrm{I}_{6yz}C_6\right.\right.$$
$$\left.\left.\left.\left.+\mathrm{I}_{6xz}S_6\right)\right)\right)\dot{\theta}_6\Big)\dot{\theta}_3 + \left(4\mathrm{I}_{6xz}C_4C_6S_3C_5^2 + 4\mathrm{I}_{6xz}C_3C_6S_4C_5^2 - 4\mathrm{I}_{5xy}S_{34}C_5^2 - 4\mathrm{I}_{6yz}C_4S_3S_6C_5^2\right.$$
$$-4\mathrm{I}_{6yz}C_3S_4S_6C_5^2 + 4\mathrm{I}_{6yz}C_3C_4C_6C_5 - 4\mathrm{I}_{6yz}C_6S_3S_4C_5 + 4\mathrm{I}_{6xz}C_3C_4S_6C_5 - 4\mathrm{I}_{6xz}S_3S_4S_6C_5 \quad \ldots$$

$$+4\mathrm{I}_{5xy}S_{34}S_5^2-4\mathrm{I}_{6xz}C_6S_{34}S_5^2+4\mathrm{I}_{6xy}C_3C_4S_5S_6^2-4\mathrm{I}_{6xy}S_3S_4S_5S_6^2-\mathrm{I}_{6xx}C_4S_3S_{55}S_6^2$$
$$+\mathrm{I}_{6yy}C_4S_3S_{55}S_6^2-\mathrm{I}_{6zz}C_4S_3S_{55}S_6^2-\mathrm{I}_{6xx}C_3S_4S_{55}S_6^2+\mathrm{I}_{6yy}C_3S_4S_{55}S_6^2-\mathrm{I}_{6zz}C_3S_4S_{55}S_6^2$$
$$+4\mathrm{I}_{4xz}S_{34}-4\mathrm{I}_{6xy}C_3C_4C_6^2S_5+4\mathrm{I}_{6xy}C_6^2S_3S_4S_5+4C_{34}\left(\mathrm{I}_{4yz}-\mathrm{I}_{5yz}C_5-\mathrm{I}_{5xz}S_5\right)+\mathrm{I}_{6xx}C_4C_6^2S_3S_{55}$$
$$-\mathrm{I}_{6yy}C_4C_6^2S_3S_{55}-\mathrm{I}_{6zz}C_4C_6^2S_3S_{55}+\mathrm{I}_{6xx}C_3C_6^2S_4S_{55}-\mathrm{I}_{6yy}C_3C_6^2S_4S_{55}-\mathrm{I}_{6zz}C_3C_6^2S_4S_{55}$$
$$+2\mathrm{I}_{5xx}S_{34}S_{55}-2\mathrm{I}_{5yy}S_{34}S_{55}+\mathrm{I}_{6xx}S_{34}S_{55}+\mathrm{I}_{6yy}S_{34}S_{55}-\mathrm{I}_{6zz}S_{34}S_{55}+4\mathrm{I}_{6yz}S_{34}S_5^2S_6$$
$$+2\mathrm{I}_{6xx}C_3C_4S_5S_{66}-2\mathrm{I}_{6yy}C_3C_4S_5S_{66}-2\mathrm{I}_{6xx}S_3S_4S_5S_{66}+2\mathrm{I}_{6yy}S_3S_4S_5S_{66}+2\mathrm{I}_{6xy}C_4S_3S_{55}S_{66}$$
$$+2\mathrm{I}_{6xy}C_3S_4S_{55}S_{66}-4C_{34}L_2\left(\left(\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5\right)m_5-\left(\mathrm{COM}_{6z}C_5+S_5\left(\mathrm{COM}_{6y}S_6\right.\right.\right.$$
$$\left.\left.\left.-\mathrm{COM}_{6x}C_6\right)\right)m_6\right)\right)\dot{\theta}_4{}^2-2\dot{\theta}_4\left(\frac{1}{2}C_{34}\left(4\mathrm{I}_{5zz}+2\mathrm{I}_{6xx}+2\mathrm{I}_{6yy}+2(2\mathrm{I}_{5xx}-2\mathrm{I}_{5yy}+\mathrm{I}_{6xx}+\mathrm{I}_{6yy}\right.\right.$$
$$-2\mathrm{I}_{6zz})C_{55}+(\mathrm{I}_{6xx}-\mathrm{I}_{6yy})C_{55-66}+4\mathrm{I}_{6yz}C_{55-6}-2\mathrm{I}_{6xx}C_{66}+2\mathrm{I}_{6yy}C_{66}+\mathrm{I}_{6xx}C_{5566}-\mathrm{I}_{6yy}C_{5566}$$
$$\left.-4\mathrm{I}_{6yz}C_{556}+8\mathrm{I}_{5xy}S_{55}-2\mathrm{I}_{6xy}S_{55-66}-4\mathrm{I}_{6xz}S_{55-6}-4\mathrm{I}_{6xy}S_{66}+2\mathrm{I}_{6xy}S_{5566}-4\mathrm{I}_{6xz}S_{556}\right)\dot{\theta}_5$$
$$+\left(S_3\left(C_4\left(C_5\left(4\mathrm{I}_{6yz}S_6-4\mathrm{I}_{6xz}C_6\right)+2S_5\left(\mathrm{I}_{6zz}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})C_{66}-2\mathrm{I}_{6xy}S_{66}\right)\right)+S_4\left(4\left(\mathrm{I}_{6yz}C_6\right.\right.\right.\right.$$
$$\left.\left.\left.+\mathrm{I}_{6xz}S_6\right)C_5^2+S_{55}\left((\mathrm{I}_{6xx}-\mathrm{I}_{6yy})S_{66}-2\mathrm{I}_{6xy}C_{66}\right)\right)\right)+C_3\left(2S_4\left(C_5\left(2\mathrm{I}_{6yz}S_6-2\mathrm{I}_{6xz}C_6\right)+S_5\left(\mathrm{I}_{6zz}\right.\right.\right.$$
$$\left.\left.+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})C_{66}-2\mathrm{I}_{6xy}S_{66}\right)\right)+C_4\left(S_{55}\left(2\mathrm{I}_{6xy}C_{66}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})S_{66}\right)-4C_5^2\left(\mathrm{I}_{6yz}C_6\right.\right.$$
$$\left.\left.\left.\left.+\mathrm{I}_{6xz}S_6\right)\right)\right)\right)\dot{\theta}_6\Big)+4\left(\frac{1}{2}\left(2S_3L_1\left(\left(\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5\right)m_5-\left(\mathrm{COM}_{6z}C_5+S_5\left(\mathrm{COM}_{6y}S_6\right.\right.\right.\right.\right.$$
$$\left.\left.-\mathrm{COM}_{6x}C_6\right)\right)m_6\right)+C_{34}\left(2C_5\left(\mathrm{I}_{5yz}-\mathrm{I}_{6yz}C_6-\mathrm{I}_{6xz}S_6\right)+S_5\left(2\mathrm{I}_{5xz}+2\mathrm{I}_{6xy}C_{66}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})S_{66}\right)\right.$$
$$\left.\left.+2L_2\left(\left(\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5\right)m_5-\left(\mathrm{COM}_{6z}C_5+S_5\left(\mathrm{COM}_{6y}S_6-\mathrm{COM}_{6x}C_6\right)\right)m_6\right)\right)\right)\dot{\theta}_5{}^2$$
$$+\left(\mathrm{I}_{6xx}C_3C_4C_5C_6^2-\mathrm{I}_{6yy}C_3C_4C_5C_6^2+2\mathrm{I}_{6xy}C_4S_3C_6^2+2\mathrm{I}_{6xy}C_3S_4C_6^2-\mathrm{I}_{6xx}C_5S_3S_4C_6^2\right.$$
$$+\mathrm{I}_{6yy}C_5S_3S_4C_6^2-\mathrm{I}_{6xx}C_3C_4C_5S_6^2+\mathrm{I}_{6yy}C_3C_4C_5S_6^2-2\mathrm{I}_{6xy}C_4S_3S_6^2-2\mathrm{I}_{6xy}C_3S_4S_6^2+\mathrm{I}_{6xx}C_5S_3S_4S_6^2$$
$$-\mathrm{I}_{6yy}C_5S_3S_4S_6^2+\mathrm{I}_{6zz}C_3C_4C_5-\mathrm{I}_{6zz}C_5S_3S_4+2\mathrm{I}_{6xy}C_3C_4C_5S_{66}-\mathrm{I}_{6xx}C_4S_3S_{66}+\mathrm{I}_{6yy}C_4S_3S_{66}$$
$$-\mathrm{I}_{6xx}C_3S_4S_{66}+\mathrm{I}_{6yy}C_3S_4S_{66}-2\mathrm{I}_{6xy}C_5S_3S_4S_{66}+2C_5S_3\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)L_1m_6$$
$$\left.+2C_{34}C_5\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)L_2m_6\right)\dot{\theta}_6\dot{\theta}_5+\left(\mathrm{I}_{6yz}C_3C_4C_5C_6+\mathrm{I}_{6xz}C_4S_3C_6+\mathrm{I}_{6xz}C_3S_4C_6\right.$$
$$-\mathrm{I}_{6yz}C_5S_3S_4C_6+\mathrm{I}_{6xz}C_3C_4C_5S_6-\mathrm{I}_{6yz}C_4S_3S_6-\mathrm{I}_{6yz}C_3S_4S_6-\mathrm{I}_{6xz}C_5S_3S_4S_6+S_3S_5\left(\mathrm{COM}_{6x}C_6\right.$$
$$\left.\left.\left.-\mathrm{COM}_{6y}S_6\right)L_1m_6+C_{34}S_5\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)L_2m_6\right)\dot{\theta}_6{}^2\right)\Big)+16\left(\mathrm{COM}_{2x}C_2m_2\right.$$
$$-\mathrm{COM}_{2y}S_2m_2+\mathrm{COM}_{3y}C_2C_3m_3-\mathrm{COM}_{3z}S_2m_3+\mathrm{COM}_{3x}C_2S_3m_3+\mathrm{COM}_{4y}C_2C_{34}m_4$$
$$-\mathrm{COM}_{4z}S_2m_4+\mathrm{COM}_{4x}C_2S_{34}m_4+C_2S_3L_1m_4+\mathrm{COM}_{5z}C_2C_{34}m_5+\mathrm{COM}_{5x}\left(C_2C_5S_{34}\right.$$
$$\left.+S_2S_5\right)m_5+\mathrm{COM}_{5y}\left(C_5S_2-C_2S_{34}S_5\right)m_5+C_2\left(S_3L_1+C_{34}L_2\right)m_5+\mathrm{COM}_{6z}\left(C_2S_{34}S_5\right.$$
$$\left.-C_5S_2\right)m_6+\mathrm{COM}_{6x}\left(C_6S_2S_5+C_2\left(C_4\left(C_5C_6S_3+C_3S_6\right)+S_4\left(C_3C_5C_6-S_3S_6\right)\right)\right)m_6$$
$$+\mathrm{COM}_{6y}\left(C_2\left(C_3\left(C_4C_6-C_5S_4S_6\right)-S_3\left(C_6S_4+C_4C_5S_6\right)\right)-S_2S_5S_6\right)m_6+C_2\left(S_3L_1\right.$$
$$\left.\left.+C_{34}L_2\right)m_6\right)\ddot{d}_1+8\left(2S_3^2\left(m_4+m_5+m_6\right)L_1^2+2\left(2S_3\left(\mathrm{COM}_{4y}C_{34}+\mathrm{COM}_{4x}S_{34}\right)m_4\right.\right.$$
$$+2S_3\left(\mathrm{COM}_{5z}C_{34}+L_2C_{34}+S_{34}\left(\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5\right)\right)m_5\quad\ldots$$

$$+ (-2\mathrm{COM}_{6y}C_6S_4S_3^2 - 2\mathrm{COM}_{6x}S_4S_6S_3^2 + 2\mathrm{COM}_{6z}S_{34}S_5S_3 + 2C_{34}L_2S_3 + \mathrm{COM}_{6x}C_5C_6S_{33}S_4$$
$$-\mathrm{COM}_{6y}C_5S_{33}S_4S_6 + C_4 (2C_5 (\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6) S_3^2 + S_{33} (\mathrm{COM}_{6y}C_6$$
$$+\mathrm{COM}_{6x}S_6))) m_6) L_1 + \frac{1}{16} (32C_{34}^2 (m_5 + m_6) L_2^2 + 32 ((2\mathrm{COM}_{5z}C_{34}^2 + S_{3344} (\mathrm{COM}_{5x}C_5$$
$$-\mathrm{COM}_{5y}S_5)) m_5 + 2C_{34} (S_3 (C_4 (\mathrm{COM}_{6z}S_5 + C_5 (\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6)) - S_4 (\mathrm{COM}_{6y}C_6$$
$$+\mathrm{COM}_{6x}S_6)) + C_3 (C_4 (\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6) + S_4 (\mathrm{COM}_{6z}S_5 + C_5 (\mathrm{COM}_{6x}C_6$$
$$-\mathrm{COM}_{6y}S_6)))) m_6) L_2 + 32\mathrm{I}_{2zz} + 16\mathrm{I}_{3xx} + 16\mathrm{I}_{3yy} + 16\mathrm{I}_{4xx} + 16\mathrm{I}_{4yy} + 8\mathrm{I}_{5xx} + 8\mathrm{I}_{5yy} + 16\mathrm{I}_{5zz}$$
$$+12\mathrm{I}_{6xx} + 12\mathrm{I}_{6yy} + 8\mathrm{I}_{6zz} + 16(\mathrm{I}_{3xx} - \mathrm{I}_{3yy})C_{33} + 16\mathrm{I}_{4xx}C_{3344} - 16\mathrm{I}_{4yy}C_{3344} + 8\mathrm{I}_{5xx}C_{3344}$$
$$+8\mathrm{I}_{5yy}C_{3344} - 16\mathrm{I}_{5zz}C_{3344} - 4\mathrm{I}_{6xx}C_{3344} - 4\mathrm{I}_{6yy}C_{3344} + 8\mathrm{I}_{6zz}C_{3344} + 4\mathrm{I}_{5xx}C_{3344-55} - 4\mathrm{I}_{5yy}C_{3344-55}$$
$$+2\mathrm{I}_{6xx}C_{3344-55} + 2\mathrm{I}_{6yy}C_{3344-55} - 4\mathrm{I}_{6zz}C_{3344-55} - 16\mathrm{I}_{5yz}C_{3344-5} + 8\mathrm{I}_{5xx}C_{55} - 8\mathrm{I}_{5yy}C_{55} + 4\mathrm{I}_{6xx}C_{55}$$
$$+4\mathrm{I}_{6yy}C_{55} - 8\mathrm{I}_{6zz}C_{55} + 4\mathrm{I}_{5xx}C_{334455} - 4\mathrm{I}_{5yy}C_{334455} + 2\mathrm{I}_{6xx}C_{334455} + 2\mathrm{I}_{6yy}C_{334455} - 4\mathrm{I}_{6zz}C_{334455}$$
$$+16\mathrm{I}_{5yz}C_{33445} - 4\mathrm{I}_{6xx}C_{3344-566} + 4\mathrm{I}_{6yy}C_{3344-566} - 4\mathrm{I}_{6xx}C_{33445-66} + 4\mathrm{I}_{6yy}C_{33445-66}$$
$$+6\mathrm{I}_{6xx}C_{3344-66} - 6\mathrm{I}_{6yy}C_{3344-66} - 4\mathrm{I}_{6yz}C_{3344-556} + \mathrm{I}_{6xx}C_{3344-5566} - \mathrm{I}_{6yy}C_{3344-5566}$$
$$+8\mathrm{I}_{6yz}C_{3344-56} + 2\mathrm{I}_{6xx}C_{55-66} - 2\mathrm{I}_{6yy}C_{55-66} + \mathrm{I}_{6xx}C_{334455-66} - \mathrm{I}_{6yy}C_{334455-66} - 8\mathrm{I}_{6yz}C_{33445-6}$$
$$+8\mathrm{I}_{6yz}C_{55-6} + 4\mathrm{I}_{6yz}C_{334455-6} - 4\mathrm{I}_{6xx}C_{66} + 4\mathrm{I}_{6yy}C_{66} + 6\mathrm{I}_{6xx}C_{334466} - 6\mathrm{I}_{6yy}C_{334466}$$
$$+4\mathrm{I}_{6yz}C_{334466-55} + \mathrm{I}_{6xx}C_{334466-55} - \mathrm{I}_{6yy}C_{334466-55} + 8\mathrm{I}_{6yz}C_{33446-5} + 2\mathrm{I}_{6xx}C_{5566} - 2\mathrm{I}_{6yy}C_{5566}$$
$$+\mathrm{I}_{6xx}C_{33445566} - \mathrm{I}_{6yy}C_{33445566} - 8\mathrm{I}_{6yz}C_{334456} - 8\mathrm{I}_{6yz}C_{556} - 4\mathrm{I}_{6yz}C_{3344556} + 4\mathrm{I}_{6xx}C_{334466-5}$$
$$-4\mathrm{I}_{6yy}C_{334466-5} + 4\mathrm{I}_{6xx}C_{3344566} - 4\mathrm{I}_{6yy}C_{3344566} + 32\mathrm{I}_{3xy}S_{33} + 32\mathrm{I}_{4xy}S_{3344} - 8\mathrm{I}_{5xy}S_{3344-55}$$
$$+16\mathrm{I}_{5xz}S_{3344-5} + 16\mathrm{I}_{5xy}S_{55} + 8\mathrm{I}_{5xy}S_{334455} + 16\mathrm{I}_{5xz}S_{33445} + 8\mathrm{I}_{6xy}S_{3344-566} + 8\mathrm{I}_{6xy}S_{33445-66}$$
$$-12\mathrm{I}_{6xy}S_{3344-66} + 4\mathrm{I}_{6xz}S_{3344-556} - 2\mathrm{I}_{6xy}S_{3344-5566} - 8\mathrm{I}_{6xz}S_{3344-56} - 4\mathrm{I}_{6xy}S_{55-66}$$
$$-2\mathrm{I}_{6xy}S_{334455-66} + 8\mathrm{I}_{6xz}S_{33445-6} - 8\mathrm{I}_{6xz}S_{55-6} - 4\mathrm{I}_{6xz}S_{334455-6} - 8\mathrm{I}_{6xy}S_{66} + 12\mathrm{I}_{6xy}S_{334466}$$
$$+4\mathrm{I}_{6xz}S_{334466-55} + 2\mathrm{I}_{6xy}S_{334466-55} + 8\mathrm{I}_{6xz}S_{33446-5} + 4\mathrm{I}_{6xy}S_{5566} + 2\mathrm{I}_{6xy}S_{33445566} - 8\mathrm{I}_{6xz}S_{334456}$$
$$-8\mathrm{I}_{6xz}S_{556} - 4\mathrm{I}_{6xz}S_{3344556} + 8\mathrm{I}_{6xy}S_{334466-5} + 8\mathrm{I}_{6xy}S_{3344566})) \ddot{\theta}_2 + 4 (4\mathrm{I}_{5xy}C_3C_4C_5^2$$
$$-4\mathrm{I}_{6xz}C_3C_4C_6C_5^2 - 4\mathrm{I}_{5xy}S_3S_4C_5^2 + 4\mathrm{I}_{6xz}C_6S_3S_4C_5^2 + 4\mathrm{I}_{6yz}C_3C_4S_6C_5^2 - 4\mathrm{I}_{6yz}S_3S_4S_6C_5^2$$
$$-4\mathrm{I}_{5yz}C_4S_3C_5 + 4\mathrm{I}_{6yz}C_4C_6S_3C_5 - 4\mathrm{I}_{5yz}C_3S_4C_5 + 4\mathrm{I}_{6yz}C_3C_6S_4C_5 + 4\mathrm{I}_{6xz}C_4S_3S_6C_5$$
$$+4\mathrm{I}_{6xz}C_3S_4S_6C_5 - 4\mathrm{I}_{5xy}C_3C_4S_5^2 + 4\mathrm{I}_{6xz}C_3C_4C_6S_5^2 + 4\mathrm{I}_{5xy}S_3S_4S_5^2 - 4\mathrm{I}_{6xz}C_6S_3S_4S_5^2$$
$$+4\mathrm{I}_{6xy}C_4S_3S_5S_6^2 + 4\mathrm{I}_{6xy}C_3S_4S_5S_6^2 + \mathrm{I}_{6xx}C_3C_4S_{55}S_6^2 - \mathrm{I}_{6yy}C_3C_4S_{55}S_6^2 - \mathrm{I}_{6xx}S_3S_4S_{55}S_6^2$$
$$+\mathrm{I}_{6yy}S_3S_4S_{55}S_6^2 - 4\mathrm{I}_{3xz}C_3 - 4\mathrm{I}_{4xz}C_3C_4 + 4\mathrm{I}_{3yz}S_3 + 4\mathrm{I}_{4yz}C_4S_3 + 4\mathrm{I}_{4yz}C_3S_4 + 4\mathrm{I}_{4xz}S_3S_4$$
$$-4\mathrm{I}_{6xy}C_4C_6^2S_3S_5 - 4\mathrm{I}_{5xz}C_4S_3S_5 - 4\mathrm{I}_{6xy}C_3C_6^2S_4S_5 - 4\mathrm{I}_{5xz}C_3S_4S_5 - \mathrm{I}_{6xx}C_3C_4C_6^2S_{55}$$
$$+\mathrm{I}_{6yy}C_3C_4C_6^2S_{55} - 2\mathrm{I}_{5xx}C_3C_4S_{55} + 2\mathrm{I}_{5yy}C_3C_4S_{55} - \mathrm{I}_{6xx}C_3C_4S_{55} - \mathrm{I}_{6yy}C_3C_4S_{55} + 2\mathrm{I}_{6zz}C_3C_4S_{55}$$
$$+\mathrm{I}_{6xx}C_6^2S_3S_4S_{55} - \mathrm{I}_{6yy}C_6^2S_3S_4S_{55} + 2\mathrm{I}_{5xx}S_3S_4S_{55} - 2\mathrm{I}_{5yy}S_3S_4S_{55} + \mathrm{I}_{6xx}S_3S_4S_{55} + \mathrm{I}_{6yy}S_3S_4S_{55}$$
$$-2\mathrm{I}_{6zz}S_3S_4S_{55} - 4\mathrm{I}_{6yz}C_3C_4S_5^2S_6 + 4\mathrm{I}_{6yz}S_3S_4S_5^2S_6 + 2\mathrm{I}_{6xx}C_4S_3S_5S_{66} - 2\mathrm{I}_{6yy}C_4S_3S_5S_{66} \ \ldots$$

$$+2\mathrm{I}_{6xx}C_3S_4S_5S_{66} - 2\mathrm{I}_{6yy}C_3S_4S_5S_{66} - 2\mathrm{I}_{6xy}C_3C_4S_{55}S_{66} + 2\mathrm{I}_{6xy}S_3S_4S_{55}S_{66} - 4S_{34}L_2\left(\left(\mathrm{COM}_{5y}C_5\right.\right.$$
$$\left.+\mathrm{COM}_{5x}S_5\right)m_5 - \left(\mathrm{COM}_{6z}C_5 + S_5\left(\mathrm{COM}_{6y}S_6 - \mathrm{COM}_{6x}C_6\right)\right)m_6\right) - 4C_3L_1\left(\mathrm{COM}_{4z}m_4\right.$$
$$\left.-\left(\mathrm{COM}_{5y}C_5 + \mathrm{COM}_{5x}S_5\right)m_5 + \left(\mathrm{COM}_{6z}C_5 + S_5\left(\mathrm{COM}_{6y}S_6 - \mathrm{COM}_{6x}C_6\right)\right)m_6\right)\right)\ddot{\theta}_3$$
$$+4\left(S_3\left(S_4\left(-4\left(\mathrm{I}_{5xy} - \mathrm{I}_{6xz}C_6 + \mathrm{I}_{6yz}S_6\right)C_5^2 - \mathrm{I}_{6xx}S_{55}S_6^2 + \mathrm{I}_{6yy}S_{55}S_6^2 + 4\mathrm{I}_{4xz} + \mathrm{I}_{6xx}C_6^2S_{55}\right.\right.\right.$$
$$-\mathrm{I}_{6yy}C_6^2S_{55} + 2\mathrm{I}_{5xx}S_{55} - 2\mathrm{I}_{5yy}S_{55} + \mathrm{I}_{6xx}S_{55} + \mathrm{I}_{6yy}S_{55} - 2\mathrm{I}_{6zz}S_{55} + 4S_5^2\left(\mathrm{I}_{5xy} - \mathrm{I}_{6xz}C_6 + \mathrm{I}_{6yz}S_6\right)$$
$$\left.+2\mathrm{I}_{6xy}S_{55}S_{66}\right) + 2C_4\left(2\mathrm{I}_{4yz} + C_5\left(-2\mathrm{I}_{5yz} + 2\mathrm{I}_{6yz}C_6 + 2\mathrm{I}_{6xz}S_6\right) - S_5\left(2\mathrm{I}_{5xz} + 2\mathrm{I}_{6xy}C_{66} + \left(\mathrm{I}_{6yy}\right.\right.\right.$$
$$\left.\left.\left.\left.-\mathrm{I}_{6xx}\right)S_{66}\right)\right)\right) - C_3\left(C_4\left(-4\left(\mathrm{I}_{5xy} - \mathrm{I}_{6xz}C_6 + \mathrm{I}_{6yz}S_6\right)C_5^2 - \mathrm{I}_{6xx}S_{55}S_6^2 + \mathrm{I}_{6yy}S_{55}S_6^2 + 4\mathrm{I}_{4xz}\right.\right.$$
$$+\mathrm{I}_{6xx}C_6^2S_{55} - \mathrm{I}_{6yy}C_6^2S_{55} + 2\mathrm{I}_{5xx}S_{55} - 2\mathrm{I}_{5yy}S_{55} + \mathrm{I}_{6xx}S_{55} + \mathrm{I}_{6yy}S_{55} - 2\mathrm{I}_{6zz}S_{55} + 4S_5^2\left(\mathrm{I}_{5xy}\right.$$
$$\left.\left.-\mathrm{I}_{6xz}C_6 + \mathrm{I}_{6yz}S_6\right) + 2\mathrm{I}_{6xy}S_{55}S_{66}\right) + 2S_4\left(-2\mathrm{I}_{4yz} + 2C_5\left(\mathrm{I}_{5yz} - \mathrm{I}_{6yz}C_6 - \mathrm{I}_{6xz}S_6\right) + S_5\left(2\mathrm{I}_{5xz}\right.\right.$$
$$\left.\left.\left.+2\mathrm{I}_{6xy}C_{66} + \left(\mathrm{I}_{6yy} - \mathrm{I}_{6xx}\right)S_{66}\right)\right)\right) - 4S_{34}L_2\left(\left(\mathrm{COM}_{5y}C_5 + \mathrm{COM}_{5x}S_5\right)m_5 - \left(\mathrm{COM}_{6z}C_5\right.\right.$$
$$\left.\left.\left.+S_5\left(\mathrm{COM}_{6y}S_6 - \mathrm{COM}_{6x}C_6\right)\right)m_6\right)\right)\ddot{\theta}_4 + 8\left(-2\mathrm{I}_{6xy}C_3C_4C_5C_6^2 + \mathrm{I}_{6xx}C_4S_3C_6^2 - \mathrm{I}_{6yy}C_4S_3C_6^2\right.$$
$$+\mathrm{I}_{6xx}C_3S_4C_6^2 - \mathrm{I}_{6yy}C_3S_4C_6^2 + 2\mathrm{I}_{6xy}C_5S_3S_4C_6^2 - 2\mathrm{I}_{6yz}C_3C_4S_5C_6 + 2\mathrm{I}_{6yz}S_3S_4S_5C_6$$
$$+2\mathrm{I}_{6xy}C_3C_4C_5S_6^2 - \mathrm{I}_{6xx}C_4S_3S_6^2 + \mathrm{I}_{6yy}C_4S_3S_6^2 - \mathrm{I}_{6xx}C_3S_4S_6^2 + \mathrm{I}_{6yy}C_3S_4S_6^2 - 2\mathrm{I}_{6xy}C_5S_3S_4S_6^2$$
$$-2\mathrm{I}_{5xz}C_3C_4C_5 - 2\mathrm{I}_{5zz}C_4S_3 - \mathrm{I}_{6xx}C_4S_3 - \mathrm{I}_{6yy}C_4S_3 - 2\mathrm{I}_{5zz}C_3S_4 - \mathrm{I}_{6xx}C_3S_4 - \mathrm{I}_{6yy}C_3S_4$$
$$+2\mathrm{I}_{5xz}C_5S_3S_4 + 2\mathrm{I}_{5yz}C_3C_4S_5 - 2\mathrm{I}_{5yz}S_3S_4S_5 - 2\mathrm{I}_{6xz}C_3C_4S_5S_6 + 2\mathrm{I}_{6xz}S_3S_4S_5S_6 + \mathrm{I}_{6xx}C_3C_4C_5S_{66}$$
$$-\mathrm{I}_{6yy}C_3C_4C_5S_{66} + 2\mathrm{I}_{6xy}C_4S_3S_{66} + 2\mathrm{I}_{6xy}C_3S_4S_{66} - \mathrm{I}_{6xx}C_5S_3S_4S_{66} + \mathrm{I}_{6yy}C_5S_3S_4S_{66}$$
$$-2S_3L_1\left(\left(\mathrm{COM}_{5x}C_5 - \mathrm{COM}_{5y}S_5\right)m_5 + \left(\mathrm{COM}_{6z}S_5 + C_5\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right)\right)m_6\right)$$
$$\left.-2C_{34}L_2\left(\left(\mathrm{COM}_{5x}C_5 - \mathrm{COM}_{5y}S_5\right)m_5 + \left(\mathrm{COM}_{6z}S_5 + C_5\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right)\right)m_6\right)\right)\ddot{\theta}_5$$
$$+16\left(-\mathrm{I}_{6xz}C_3C_4C_5C_6 + \mathrm{I}_{6yz}C_4S_3C_6 + \mathrm{I}_{6yz}C_3S_4C_6 + \mathrm{I}_{6xz}C_5S_3S_4C_6 + \mathrm{I}_{6zz}C_3C_4S_5 - \mathrm{I}_{6zz}S_3S_4S_5\right.$$
$$+\mathrm{I}_{6yz}C_3C_4C_5S_6 + \mathrm{I}_{6xz}C_4S_3S_6 + \mathrm{I}_{6xz}C_3S_4S_6 - \mathrm{I}_{6yz}C_5S_3S_4S_6 + S_3S_5\left(\mathrm{COM}_{6y}C_6\right.$$
$$\left.\left.\left.+\mathrm{COM}_{6x}S_6\right)L_1m_6 + C_{34}S_5\left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right)L_2m_6\right)\ddot{\theta}_6\right)\right) \tag{A.5b}$$

$$\tau_3 = 10^{-6}\left(-9810\left(\mathrm{COM}_{3y}C_3 + \mathrm{COM}_{3x}S_3\right)m_3 - 9810\left(\left(\mathrm{COM}_{4y}C_{34} + \mathrm{COM}_{4x}S_{34}\right)m_4 + \left(\mathrm{COM}_{5z}C_{34}\right.\right.\right.$$
$$\left.+S_{34}\left(\mathrm{COM}_{5x}C_5 - \mathrm{COM}_{5y}S_5\right)\right)m_5 + \left(C_{34}\left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right) + S_{34}\left(\mathrm{COM}_{6z}S_5\right.\right.$$
$$\left.\left.\left.+C_5\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right)\right)\right)m_6\right) + \frac{1}{32}\left(-\left(16S_{33}\left(m_4 + m_5 + m_6\right)L_1^2 + 8\left(4\left(\mathrm{COM}_{4y}C_{334}\right.\right.\right.\right.$$
$$\left.+\mathrm{COM}_{4x}S_{334}\right)m_4 + 4\left(\mathrm{COM}_{5z}C_{334} + L_2C_{334} + S_{334}\left(\mathrm{COM}_{5x}C_5 - \mathrm{COM}_{5y}S_5\right)\right)m_5$$
$$+\left(2\mathrm{COM}_{6z}C_{334-5} - 2\mathrm{COM}_{6z}C_{3345} + 2\mathrm{COM}_{6y}C_{334-6} - \mathrm{COM}_{6y}C_{334-56} - \mathrm{COM}_{6y}C_{3345-6}\right.$$
$$+2\mathrm{COM}_{6y}C_{3346} + \mathrm{COM}_{6y}C_{3346-5} + \mathrm{COM}_{6y}C_{33456} - 2\mathrm{COM}_{6x}S_{334-6} + \mathrm{COM}_{6x}S_{334-56}$$
$$\left.\left.+\mathrm{COM}_{6x}S_{3345-6} + 2\mathrm{COM}_{6x}S_{3346} + \mathrm{COM}_{6x}S_{3346-5} + \mathrm{COM}_{6x}S_{33456} + 4C_{334}L_2\right)m_6\right)L_1$$
$$+32\mathrm{I}_{3xy}C_{33} + 32\mathrm{I}_{4xy}C_{3344} - 8\mathrm{I}_{5xy}C_{3344-55} + 16\mathrm{I}_{5xz}C_{3344-5} + 8\mathrm{I}_{5xy}C_{334455} + 16\mathrm{I}_{5xz}C_{33445}\ \ldots$$

$$+8\mathrm{I}_{6xy}C_{3344-566}+8\mathrm{I}_{6xy}C_{33445-66}-12\mathrm{I}_{6xy}C_{3344-66}+4\mathrm{I}_{6xz}C_{3344-556}-2\mathrm{I}_{6xy}C_{3344-5566}$$
$$-8\mathrm{I}_{6xz}C_{3344-56}-2\mathrm{I}_{6xy}C_{334455-66}+8\mathrm{I}_{6xz}C_{33445-6}-4\mathrm{I}_{6xz}C_{334455-6}+12\mathrm{I}_{6xy}C_{334466}$$
$$+4\mathrm{I}_{6xz}C_{334466-55}+2\mathrm{I}_{6xy}C_{334466-55}+8\mathrm{I}_{6xz}C_{33446-5}+2\mathrm{I}_{6xy}C_{33445566}-8\mathrm{I}_{6xz}C_{334456}$$
$$-4\mathrm{I}_{6xz}C_{3344556}+8\mathrm{I}_{6xy}C_{334466-5}+8\mathrm{I}_{6xy}C_{3344566}-16\mathrm{I}_{3xx}S_{33}+16\mathrm{I}_{3yy}S_{33}-16\mathrm{I}_{4xx}S_{3344}$$
$$+16\mathrm{I}_{4yy}S_{3344}-8\mathrm{I}_{5xx}S_{3344}-8\mathrm{I}_{5yy}S_{3344}+16\mathrm{I}_{5zz}S_{3344}+4\mathrm{I}_{6xx}S_{3344}+4\mathrm{I}_{6yy}S_{3344}-8\mathrm{I}_{6zz}S_{3344}$$
$$-4\mathrm{I}_{5xx}S_{3344-55}+4\mathrm{I}_{5yy}S_{3344-55}-2\mathrm{I}_{6xx}S_{3344-55}-2\mathrm{I}_{6yy}S_{3344-55}+4\mathrm{I}_{6zz}S_{3344-55}+16\mathrm{I}_{5yz}S_{3344-5}$$
$$-4\mathrm{I}_{5xx}S_{334455}+4\mathrm{I}_{5yy}S_{334455}-2\mathrm{I}_{6xx}S_{334455}-2\mathrm{I}_{6yy}S_{334455}+4\mathrm{I}_{6zz}S_{334455}-16\mathrm{I}_{5yz}S_{33445}$$
$$+4\mathrm{I}_{6xx}S_{3344-566}-4\mathrm{I}_{6yy}S_{3344-566}+4\mathrm{I}_{6xx}S_{33445-66}-4\mathrm{I}_{6yy}S_{33445-66}-6\mathrm{I}_{6xx}S_{3344-66}$$
$$+6\mathrm{I}_{6yy}S_{3344-66}+4\mathrm{I}_{6yz}S_{3344-556}-\mathrm{I}_{6xx}S_{3344-5566}+\mathrm{I}_{6yy}S_{3344-5566}-8\mathrm{I}_{6yz}S_{3344-56}-\mathrm{I}_{6xx}S_{334455-66}$$
$$+\mathrm{I}_{6yy}S_{334455-66}+8\mathrm{I}_{6yz}S_{33445-6}-4\mathrm{I}_{6yz}S_{334455-6}-6\mathrm{I}_{6xx}S_{334466}+6\mathrm{I}_{6yy}S_{334466}-4\mathrm{I}_{6yz}S_{334466-55}$$
$$-\mathrm{I}_{6xx}S_{334466-55}+\mathrm{I}_{6yy}S_{334466-55}-8\mathrm{I}_{6yz}S_{33446-5}-\mathrm{I}_{6xx}S_{33445566}+\mathrm{I}_{6yy}S_{33445566}+8\mathrm{I}_{6yz}S_{334456}$$
$$+4\mathrm{I}_{6yz}S_{3344556}-4\mathrm{I}_{6xx}S_{334466-5}+4\mathrm{I}_{6yy}S_{334466-5}-4\mathrm{I}_{6xx}S_{3344566}+4\mathrm{I}_{6yy}S_{3344566}-16S_{3344}L_2^2\,(m_5$$
$$+m_6)+8L_2\,(2\,(\mathrm{COM}_{5x}C_{3344-5}+\mathrm{COM}_{5x}C_{33445}-2\mathrm{COM}_{5z}S_{3344}+\mathrm{COM}_{5y}S_{3344-5}$$
$$-\mathrm{COM}_{5y}S_{33445})\,m_5+(-2\mathrm{COM}_{6x}C_{3344-6}+\mathrm{COM}_{6x}C_{3344-56}+\mathrm{COM}_{6x}C_{33445-6}+2\mathrm{COM}_{6x}C_{33446}$$
$$+\mathrm{COM}_{6x}C_{33446-5}+\mathrm{COM}_{6x}C_{334456}-2\mathrm{COM}_{6z}S_{3344-5}+2\mathrm{COM}_{6z}S_{33445}-2\mathrm{COM}_{6y}S_{3344-6}$$
$$+\mathrm{COM}_{6y}S_{3344-56}+\mathrm{COM}_{6y}S_{33445-6}-2\mathrm{COM}_{6y}S_{33446}-\mathrm{COM}_{6y}S_{33446-5}$$
$$-\mathrm{COM}_{6y}S_{334456})\,m_6))\,\dot{\theta}_2{}^2+4\,((8\mathrm{I}_{5zz}C_{34}+4\mathrm{I}_{6xx}C_{34}+4\mathrm{I}_{6yy}C_{34}-4\mathrm{I}_{5xx}C_{34-55}+4\mathrm{I}_{5yy}C_{34-55}$$
$$-2\mathrm{I}_{6xx}C_{34-55}-2\mathrm{I}_{6yy}C_{34-55}+4\mathrm{I}_{6zz}C_{34-55}+8\mathrm{I}_{5yz}C_{34-5}-8\mathrm{I}_{5yz}C_{345}-4\mathrm{I}_{5xx}C_{3455}+4\mathrm{I}_{5yy}C_{3455}$$
$$-2\mathrm{I}_{6xx}C_{3455}-2\mathrm{I}_{6yy}C_{3455}+4\mathrm{I}_{6zz}C_{3455}-2\mathrm{I}_{6xx}C_{34-66}+2\mathrm{I}_{6yy}C_{34-66}+2\mathrm{I}_{6xx}C_{34-566}-2\mathrm{I}_{6yy}C_{34-566}$$
$$+2\mathrm{I}_{6xx}C_{345-66}-2\mathrm{I}_{6yy}C_{345-66}-\mathrm{I}_{6xx}C_{3455-66}+\mathrm{I}_{6yy}C_{3455-66}+4\mathrm{I}_{6yz}C_{34-556}-4\mathrm{I}_{6yz}C_{34-56}$$
$$+4\mathrm{I}_{6yz}C_{345-6}-4\mathrm{I}_{6yz}C_{3455-6}-4\mathrm{I}_{6yz}C_{346-55}-4\mathrm{I}_{6yz}C_{346-5}+4\mathrm{I}_{6yz}C_{3456}+4\mathrm{I}_{6yz}C_{34556}$$
$$-2\mathrm{I}_{6xx}C_{3466}+2\mathrm{I}_{6yy}C_{3466}-\mathrm{I}_{6xx}S_{3466-55}+\mathrm{I}_{6yy}S_{3466-55}-2\mathrm{I}_{6xx}C_{3466-5}+2\mathrm{I}_{6yy}C_{3466-5}$$
$$-2\mathrm{I}_{6xx}C_{34566}+2\mathrm{I}_{6yy}C_{34566}-\mathrm{I}_{6xx}C_{34-5566}+\mathrm{I}_{6yy}C_{34-5566}-\mathrm{I}_{6xx}C_{345566}+\mathrm{I}_{6yy}C_{345566}$$
$$+8\mathrm{I}_{5xy}S_{34-55}-8\mathrm{I}_{5xz}S_{34-5}-8\mathrm{I}_{5xz}S_{345}-8\mathrm{I}_{5xy}S_{3455}+4\mathrm{I}_{6xy}S_{34-66}-4\mathrm{I}_{6xy}S_{34-566}-4\mathrm{I}_{6xy}S_{345-66}$$
$$+2\mathrm{I}_{6xy}S_{3455-66}-4\mathrm{I}_{6xz}S_{34-556}+4\mathrm{I}_{6xz}S_{34-56}-4\mathrm{I}_{6xz}S_{345-6}+4\mathrm{I}_{6xz}S_{3455-6}-4\mathrm{I}_{6xz}S_{346-55}$$
$$-4\mathrm{I}_{6xz}S_{346-5}+4\mathrm{I}_{6xz}S_{3456}+4\mathrm{I}_{6xz}S_{34556}-4\mathrm{I}_{6xy}S_{3466}-2\mathrm{I}_{6xy}S_{3466-55}-4\mathrm{I}_{6xy}S_{3466-5}-4\mathrm{I}_{6xy}S_{34566}$$
$$+2\mathrm{I}_{6xy}S_{34-5566}-2\mathrm{I}_{6xy}S_{345566}+16C_3L_1\,((\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5)\,m_5+(\mathrm{COM}_{6z}S_5$$
$$+C_5\,(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6))\,m_6)-16S_{34}L_2\,((\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5)\,m_5+(\mathrm{COM}_{6z}S_5$$
$$+C_5\,(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6))\,m_6))\,\dot{\theta}_5-8S_5\,(2\mathrm{I}_{6xy}C_3C_4C_5C_6^2-\mathrm{I}_{6xx}C_4S_3C_6^2+\mathrm{I}_{6yy}C_4S_3C_6^2$$
$$-\mathrm{I}_{6xx}C_3S_4C_6^2+\mathrm{I}_{6yy}C_3S_4C_6^2-2\mathrm{I}_{6xy}C_5S_3S_4C_6^2+2\mathrm{I}_{6yz}C_3C_4S_5C_6-2\mathrm{I}_{6yz}S_3S_4S_5C_6\ \ \ldots$$

$$-2\mathrm{I}_{6xy}C_3C_4C_5S_6^2+\mathrm{I}_{6xx}C_4S_3S_6^2-\mathrm{I}_{6yy}C_4S_3S_6^2+\mathrm{I}_{6xx}C_3S_4S_6^2-\mathrm{I}_{6yy}C_3S_4S_6^2+2\mathrm{I}_{6xy}C_5S_3S_4S_6^2$$

$$-\mathrm{I}_{6zz}C_4S_3-\mathrm{I}_{6zz}C_3S_4+2\mathrm{I}_{6xz}C_3C_4S_5S_6-2\mathrm{I}_{6xz}S_3S_4S_5S_6-\mathrm{I}_{6xx}C_3C_4C_5S_{66}+\mathrm{I}_{6yy}C_3C_4C_5S_{66}$$

$$-2\mathrm{I}_{6xy}C_4S_3S_{66}-2\mathrm{I}_{6xy}C_3S_4S_{66}+\mathrm{I}_{6xx}C_5S_3S_4S_{66}-\mathrm{I}_{6yy}C_5S_3S_4S_{66}+2C_3\left(\mathrm{COM}_{6y}C_6\right.$$

$$\left.+\mathrm{COM}_{6x}S_6\right)L_1m_6-2S_{34}\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)L_2m_6\big)\dot\theta_6\Big)\dot\theta_2-16\Big(4\mathrm{I}_{5xy}\dot\theta_3\dot\theta_5C_5^2$$

$$-4\mathrm{I}_{6xz}C_6\dot\theta_3\dot\theta_5C_5^2+4\mathrm{I}_{6yz}S_6\dot\theta_3\dot\theta_5C_5^2+4\mathrm{I}_{5xy}\dot\theta_4\dot\theta_5C_5^2-4\mathrm{I}_{6xz}C_6\dot\theta_4\dot\theta_5C_5^2+4\mathrm{I}_{6yz}S_6\dot\theta_4\dot\theta_5C_5^2$$

$$+2\mathrm{I}_{6xy}C_6^2\dot\theta_3\dot\theta_6C_5^2-2\mathrm{I}_{6xy}S_6^2\dot\theta_3\dot\theta_6C_5^2-\mathrm{I}_{6xx}S_{66}\dot\theta_3\dot\theta_6C_5^2+\mathrm{I}_{6yy}S_{66}\dot\theta_3\dot\theta_6C_5^2+2\mathrm{I}_{6xy}C_6^2\dot\theta_4\dot\theta_6C_5^2$$

$$-2\mathrm{I}_{6xy}S_6^2\dot\theta_4\dot\theta_6C_5^2-\mathrm{I}_{6xx}S_{66}\dot\theta_4\dot\theta_6C_5^2+\mathrm{I}_{6yy}S_{66}\dot\theta_4\dot\theta_6C_5^2-2\mathrm{I}_{6xy}C_6^2{\dot\theta_5}^2C_5+2\mathrm{I}_{6xy}S_6^2{\dot\theta_5}^2C_5-2\mathrm{I}_{5xz}{\dot\theta_5}^2C_5$$

$$+\mathrm{I}_{6xx}S_{66}{\dot\theta_5}^2C_5-\mathrm{I}_{6yy}S_{66}{\dot\theta_5}^2C_5-2\mathrm{COM}_{5x}L_2m_5{\dot\theta_5}^2C_5-2\mathrm{COM}_{6x}C_6L_2m_6{\dot\theta_5}^2C_5$$

$$+2\mathrm{COM}_{6y}S_6L_2m_6{\dot\theta_5}^2C_5-2\mathrm{COM}_{6x}C_6L_2m_6{\dot\theta_6}^2C_5+2\mathrm{COM}_{6y}S_6L_2m_6{\dot\theta_6}^2C_5+2\mathrm{I}_{5yz}S_5{\dot\theta_5}^2$$

$$-2\mathrm{I}_{6yz}C_6S_5{\dot\theta_5}^2-2\mathrm{I}_{6xz}S_5S_6{\dot\theta_5}^2+2\mathrm{COM}_{5y}S_5L_2m_5{\dot\theta_5}^2-2\mathrm{COM}_{6z}S_5L_2m_6{\dot\theta_5}^2+2\mathrm{I}_{6yz}C_6S_5{\dot\theta_6}^2$$

$$+2\mathrm{I}_{6xz}S_5S_6{\dot\theta_6}^2-4\mathrm{I}_{5xy}S_5^2\dot\theta_3\dot\theta_5+4\mathrm{I}_{6xz}C_6S_5^2\dot\theta_3\dot\theta_5+\mathrm{I}_{6xx}S_{55}S_6^2\dot\theta_3\dot\theta_5-\mathrm{I}_{6yy}S_{55}S_6^2\dot\theta_3\dot\theta_5-\mathrm{I}_{6xx}C_6^2S_{55}\dot\theta_3\dot\theta_5$$

$$+\mathrm{I}_{6yy}C_6^2S_{55}\dot\theta_3\dot\theta_5-2\mathrm{I}_{5xx}S_{55}\dot\theta_3\dot\theta_5+2\mathrm{I}_{5yy}S_{55}\dot\theta_3\dot\theta_5-\mathrm{I}_{6xx}S_{55}\dot\theta_3\dot\theta_5-\mathrm{I}_{6yy}S_{55}\dot\theta_3\dot\theta_5+2\mathrm{I}_{6zz}S_{55}\dot\theta_3\dot\theta_5$$

$$-4\mathrm{I}_{6yz}S_5^2S_6\dot\theta_3\dot\theta_5-2\mathrm{I}_{6xy}S_{55}S_{66}\dot\theta_3\dot\theta_5-4\mathrm{I}_{5xy}S_5^2\dot\theta_4\dot\theta_5+4\mathrm{I}_{6xz}C_6S_5^2\dot\theta_4\dot\theta_5+\mathrm{I}_{6xx}S_{55}S_6^2\dot\theta_4\dot\theta_5$$

$$-\mathrm{I}_{6yy}S_{55}S_6^2\dot\theta_4\dot\theta_5-\mathrm{I}_{6xx}C_6^2S_{55}\dot\theta_4\dot\theta_5+\mathrm{I}_{6yy}C_6^2S_{55}\dot\theta_4\dot\theta_5-2\mathrm{I}_{5xx}S_{55}\dot\theta_4\dot\theta_5+2\mathrm{I}_{5yy}S_{55}\dot\theta_4\dot\theta_5-\mathrm{I}_{6xx}S_{55}\dot\theta_4\dot\theta_5$$

$$-\mathrm{I}_{6yy}S_{55}\dot\theta_4\dot\theta_5+2\mathrm{I}_{6zz}S_{55}\dot\theta_4\dot\theta_5-4\mathrm{I}_{6yz}S_5^2S_6\dot\theta_4\dot\theta_5-2\mathrm{I}_{6xy}S_{55}S_{66}\dot\theta_4\dot\theta_5-2\mathrm{I}_{6xy}C_6^2\dot\theta_3\dot\theta_6-2\mathrm{I}_{6xy}C_6^2S_5^2\dot\theta_3\dot\theta_6$$

$$+2\mathrm{I}_{6xy}S_5^2S_6^2\dot\theta_3\dot\theta_6+2\mathrm{I}_{6xy}S_6^2\dot\theta_3\dot\theta_6+2\mathrm{I}_{6yz}C_6S_{55}\dot\theta_3\dot\theta_6+2\mathrm{I}_{6xz}S_{55}S_6\dot\theta_3\dot\theta_6+\mathrm{I}_{6xx}S_5^2S_{66}\dot\theta_3\dot\theta_6$$

$$-\mathrm{I}_{6yy}S_5^2S_{66}\dot\theta_3\dot\theta_6+\mathrm{I}_{6xx}S_{66}\dot\theta_3\dot\theta_6-\mathrm{I}_{6yy}S_{66}\dot\theta_3\dot\theta_6-4\mathrm{COM}_{6x}C_6L_2m_6\dot\theta_3\dot\theta_6+4\mathrm{COM}_{6y}S_6L_2m_6\dot\theta_3\dot\theta_6$$

$$-2\mathrm{I}_{6xy}C_6^2\dot\theta_4\dot\theta_6-2\mathrm{I}_{6xy}C_6^2S_5^2\dot\theta_4\dot\theta_6+2\mathrm{I}_{6xy}S_5^2S_6^2\dot\theta_4\dot\theta_6+2\mathrm{I}_{6xy}S_6^2\dot\theta_4\dot\theta_6+2\mathrm{I}_{6yz}C_6S_{55}\dot\theta_4\dot\theta_6+2\mathrm{I}_{6xz}S_{55}S_6\dot\theta_4\dot\theta_6$$

$$+\mathrm{I}_{6xx}S_5^2S_{66}\dot\theta_4\dot\theta_6-\mathrm{I}_{6yy}S_5^2S_{66}\dot\theta_4\dot\theta_6+\mathrm{I}_{6xx}S_{66}\dot\theta_4\dot\theta_6-\mathrm{I}_{6yy}S_{66}\dot\theta_4\dot\theta_6-4\mathrm{COM}_{6x}C_6L_2m_6\dot\theta_4\dot\theta_6$$

$$+4\mathrm{COM}_{6y}S_6L_2m_6\dot\theta_4\dot\theta_6-2\mathrm{I}_{6xx}S_5S_6^2\dot\theta_5\dot\theta_6+2\mathrm{I}_{6yy}S_5S_6^2\dot\theta_5\dot\theta_6+2\mathrm{I}_{6xx}C_6^2S_5\dot\theta_5\dot\theta_6-2\mathrm{I}_{6yy}C_6^2S_5\dot\theta_5\dot\theta_6$$

$$+2\mathrm{I}_{6zz}S_5\dot\theta_5\dot\theta_6+4\mathrm{I}_{6xy}S_5S_{66}\dot\theta_5\dot\theta_6+4\mathrm{COM}_{6y}C_6S_5L_2m_6\dot\theta_5\dot\theta_6+4\mathrm{COM}_{6x}S_5S_6L_2m_6\dot\theta_5\dot\theta_6$$

$$+2L_1\left(\left(\mathrm{COM}_{4y}C_4+\mathrm{COM}_{4x}S_4\right)m_4\dot\theta_4\left(2\dot\theta_3+\dot\theta_4\right)+m_5\left(\left(\mathrm{COM}_{5z}C_4+L_2C_4+S_4\left(\mathrm{COM}_{5x}C_5\right.\right.\right.\right.$$

$$\left.\left.-\mathrm{COM}_{5y}S_5\right)\right){\dot\theta_4}^2+2C_4\left(\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5\right)\dot\theta_5\dot\theta_4+S_4\left(\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5\right){\dot\theta_5}^2$$

$$\left.+2\dot\theta_3\left(\left(\mathrm{COM}_{5z}C_4+L_2C_4+S_4\left(\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5\right)\right)\dot\theta_4+C_4\left(\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5\right)\dot\theta_5\right)\right)$$

$$+m_6\Big(\left(C_4\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)+S_4\left(\mathrm{COM}_{6z}S_5+C_5\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right)+C_4L_2\right){\dot\theta_4}^2$$

$$-2\Big(C_4\left(\mathrm{COM}_{6z}C_5+S_5\left(\mathrm{COM}_{6y}S_6-\mathrm{COM}_{6x}C_6\right)\right)\dot\theta_5-\left(C_4C_5\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)\right.$$

$$\left.+S_4\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right)\dot\theta_6\Big)\dot\theta_4+\mathrm{COM}_{6x}C_5C_6S_4{\dot\theta_5}^2+\mathrm{COM}_{6z}S_4S_5{\dot\theta_5}^2-\mathrm{COM}_{6y}C_5S_4S_6{\dot\theta_5}^2$$

$$+\mathrm{COM}_{6y}C_4C_6{\dot\theta_6}^2+\mathrm{COM}_{6x}C_5C_6S_4{\dot\theta_6}^2+\mathrm{COM}_{6x}C_4S_6{\dot\theta_6}^2-\mathrm{COM}_{6y}C_5S_4S_6{\dot\theta_6}^2$$

$$-2\mathrm{COM}_{6y}C_6S_4S_5\dot\theta_5\dot\theta_6-2\mathrm{COM}_{6x}S_4S_5S_6\dot\theta_5\dot\theta_6+2\dot\theta_3\left(\left(C_4\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)+S_4\left(\mathrm{COM}_{6z}S_5\right.\right.\right.$$

$$\left.\left.+C_5\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right)+C_4L_2\right)\dot\theta_4-C_4\left(\mathrm{COM}_{6z}C_5+S_5\left(\mathrm{COM}_{6y}S_6-\mathrm{COM}_{6x}C_6\right)\right)\dot\theta_5\quad\ldots$$

$$+ (C_4C_5 (\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6) + S_4 (\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6)) \dot{\theta}_6))))) + S_2 ((\mathrm{COM}_{3x}C_3$$

$$-\mathrm{COM}_{3y}S_3) m_3 + (\mathrm{COM}_{4x}C_{34} - \mathrm{COM}_{4y}S_{34} + C_3L_1) m_4 + \mathrm{COM}_{5x}C_{34}C_5m_5 - \mathrm{COM}_{5z}S_{34}m_5$$

$$-\mathrm{COM}_{5y}C_{34}S_5m_5 + C_3L_1m_5 - S_{34}L_2m_5 + \mathrm{COM}_{6x}C_3C_4C_5C_6m_6 - \mathrm{COM}_{6y}C_4C_6S_3m_6$$

$$-\mathrm{COM}_{6y}C_3C_6S_4m_6 - \mathrm{COM}_{6x}C_5C_6S_3S_4m_6 + \mathrm{COM}_{6z}C_{34}S_5m_6 - \mathrm{COM}_{6y}C_3C_4C_5S_6m_6$$

$$-\mathrm{COM}_{6x}C_4S_3S_6m_6 - \mathrm{COM}_{6x}C_3S_4S_6m_6 + \mathrm{COM}_{6y}C_5S_3S_4S_6m_6 + C_3L_1m_6 - S_{34}L_2m_6) \ddot{d}_1$$

$$+\frac{1}{4} (4\mathrm{I}_{5xy}C_3C_4C_5^2 - 4\mathrm{I}_{6xz}C_3C_4C_6C_5^2 - 4\mathrm{I}_{5xy}S_3S_4C_5^2 + 4\mathrm{I}_{6xz}C_6S_3S_4C_5^2 + 4\mathrm{I}_{6yz}C_3C_4S_6C_5^2$$

$$-4\mathrm{I}_{6yz}S_3S_4S_6C_5^2 - 4\mathrm{I}_{5yz}C_4S_3C_5 + 4\mathrm{I}_{6yz}C_4C_6S_3C_5 - 4\mathrm{I}_{5yz}C_3S_4C_5 + 4\mathrm{I}_{6yz}C_3C_6S_4C_5$$

$$+4\mathrm{I}_{6xz}C_4S_3S_6C_5 + 4\mathrm{I}_{6xz}C_3S_4S_6C_5 - 4\mathrm{I}_{5xy}C_3C_4S_5^2 + 4\mathrm{I}_{6xz}C_3C_4C_6S_5^2 + 4\mathrm{I}_{5xy}S_3S_4S_5^2$$

$$-4\mathrm{I}_{6xz}C_6S_3S_4S_5^2 + 4\mathrm{I}_{6xy}C_4S_3S_5S_6^2 + 4\mathrm{I}_{6xy}C_3S_4S_5S_6^2 + \mathrm{I}_{6xx}C_3C_4S_{55}S_6^2 - \mathrm{I}_{6yy}C_3C_4S_{55}S_6^2$$

$$-\mathrm{I}_{6xx}S_3S_4S_{55}S_6^2 + \mathrm{I}_{6yy}S_3S_4S_{55}S_6^2 - 4\mathrm{I}_{3xz}C_3 - 4\mathrm{I}_{4xz}C_3C_4 + 4\mathrm{I}_{3yz}S_3 + 4\mathrm{I}_{4yz}C_4S_3 + 4\mathrm{I}_{4yz}C_3S_4$$

$$+4\mathrm{I}_{4xz}S_3S_4 - 4\mathrm{I}_{6xy}C_4C_6^2S_3S_5 - 4\mathrm{I}_{5xz}C_4S_3S_5 - 4\mathrm{I}_{6xy}C_3C_6^2S_4S_5 - 4\mathrm{I}_{5xz}C_3S_4S_5 - \mathrm{I}_{6xx}C_3C_4C_6^2S_{55}$$

$$+\mathrm{I}_{6yy}C_3C_4C_6^2S_{55} - 2\mathrm{I}_{5xx}C_3C_4S_{55} + 2\mathrm{I}_{5yy}C_3C_4S_{55} - \mathrm{I}_{6xx}C_3C_4S_{55} - \mathrm{I}_{6yy}C_3C_4S_{55} + 2\mathrm{I}_{6zz}C_3C_4S_{55}$$

$$+\mathrm{I}_{6xx}C_6^2S_3S_4S_{55} - \mathrm{I}_{6yy}C_6^2S_3S_4S_{55} + 2\mathrm{I}_{5xx}S_3S_4S_{55} - 2\mathrm{I}_{5yy}S_3S_4S_{55} + \mathrm{I}_{6xx}S_3S_4S_{55} + \mathrm{I}_{6yy}S_3S_4S_{55}$$

$$-2\mathrm{I}_{6zz}S_3S_4S_{55} - 4\mathrm{I}_{6yz}C_3C_4S_5^2S_6 + 4\mathrm{I}_{6yz}S_3S_4S_5^2S_6 + 2\mathrm{I}_{6xx}C_4S_3S_5S_{66} - 2\mathrm{I}_{6yy}C_4S_3S_5S_{66}$$

$$+2\mathrm{I}_{6xx}C_3S_4S_5S_{66} - 2\mathrm{I}_{6yy}C_3S_4S_5S_{66} - 2\mathrm{I}_{6xy}C_3C_4S_{55}S_{66} + 2\mathrm{I}_{6xy}S_3S_4S_{55}S_{66}$$

$$-4S_{34}L_2 ((\mathrm{COM}_{5y}C_5 + \mathrm{COM}_{5x}S_5) m_5 - (\mathrm{COM}_{6z}C_5 + S_5 (\mathrm{COM}_{6y}S_6 - \mathrm{COM}_{6x}C_6)) m_6)$$

$$-4C_3L_1 (\mathrm{COM}_{4z}m_4 - (\mathrm{COM}_{5y}C_5 + \mathrm{COM}_{5x}S_5) m_5 + (\mathrm{COM}_{6z}C_5 + S_5 (\mathrm{COM}_{6y}S_6$$

$$-\mathrm{COM}_{6x}C_6)) m_6)) \ddot{\theta}_2 + \frac{1}{4} (4 (m_4 + m_5 + m_6) L_1^2 + 8 ((\mathrm{COM}_{4x}C_4 - \mathrm{COM}_{4y}S_4) m_4 + (-\mathrm{COM}_{5z}S_4$$

$$-L_2S_4 + C_4 (\mathrm{COM}_{5x}C_5 - \mathrm{COM}_{5y}S_5)) m_5 + (-S_4 (\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6) + C_4 (\mathrm{COM}_{6z}S_5$$

$$+C_5 (\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6)) - S_4L_2) m_6) L_1 + \frac{1}{2} (8\mathrm{I}_{3zz} + 8\mathrm{I}_{4zz} + 4\mathrm{I}_{5xx} + 4\mathrm{I}_{5yy} + 2\mathrm{I}_{6xx} + 2\mathrm{I}_{6yy}$$

$$+4\mathrm{I}_{6zz} - 2(2\mathrm{I}_{5xx} - 2\mathrm{I}_{5yy} + \mathrm{I}_{6xx} + \mathrm{I}_{6yy} - 2\mathrm{I}_{6zz})C_{55} + (\mathrm{I}_{6yy} - \mathrm{I}_{6xx})C_{55-66} - 4\mathrm{I}_{6yz}C_{55-6} + 2\mathrm{I}_{6xx}C_{66}$$

$$-2\mathrm{I}_{6yy}C_{66} - \mathrm{I}_{6xx}C_{5566} + \mathrm{I}_{6yy}C_{5566} + 4\mathrm{I}_{6yz}C_{556} - 8\mathrm{I}_{5xy}S_{55} + 2\mathrm{I}_{6xy}S_{55-66} + 4\mathrm{I}_{6xz}S_{55-6} + 4\mathrm{I}_{6xy}S_{66}$$

$$-2\mathrm{I}_{6xy}S_{5566} + 4\mathrm{I}_{6xz}S_{556}) + 4L_2^2 (m_5 + m_6) + 8L_2 (\mathrm{COM}_{5z}m_5 + (\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6) m_6)) \ddot{\theta}_3$$

$$+\frac{1}{4} \Big(4 (m_5 + m_6) L_2^2 + 8 (\mathrm{COM}_{5z}m_5 + (\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6) m_6) L_2 + \frac{1}{2} (8\mathrm{I}_{4zz} + 4\mathrm{I}_{5xx} + 4\mathrm{I}_{5yy}$$

$$+2\mathrm{I}_{6xx} + 2\mathrm{I}_{6yy} + 4\mathrm{I}_{6zz} - 2(2\mathrm{I}_{5xx} - 2\mathrm{I}_{5yy} + \mathrm{I}_{6xx} + \mathrm{I}_{6yy} - 2\mathrm{I}_{6zz})C_{55} + (\mathrm{I}_{6yy} - \mathrm{I}_{6xx})C_{55-66}$$

$$-4\mathrm{I}_{6yz}C_{55-6} + 2\mathrm{I}_{6xx}C_{66} - 2\mathrm{I}_{6yy}C_{66} - \mathrm{I}_{6xx}C_{5566} + \mathrm{I}_{6yy}C_{5566} + 4\mathrm{I}_{6yz}C_{556} - 8\mathrm{I}_{5xy}S_{55} + 2\mathrm{I}_{6xy}S_{55-66}$$

$$+4\mathrm{I}_{6xz}S_{55-6} + 4\mathrm{I}_{6xy}S_{66} - 2\mathrm{I}_{6xy}S_{5566} + 4\mathrm{I}_{6xz}S_{556}) + 4L_1 ((\mathrm{COM}_{4x}C_4 - \mathrm{COM}_{4y}S_4) m_4$$

$$+ (-\mathrm{COM}_{5z}S_4 - L_2S_4 + C_4 (\mathrm{COM}_{5x}C_5 - \mathrm{COM}_{5y}S_5)) m_5 + (-S_4 (\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6)$$

$$+C_4 (\mathrm{COM}_{6z}S_5 + C_5 (\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6)) - S_4L_2) m_6)) \ddot{\theta}_4 + \Big(\mathrm{I}_{6xy}S_5C_6^2 - \mathrm{I}_{6yz}C_5C_6 \ \ldots$$

$$+\mathrm{I}_{6yy}S_5S_6C_6 - \mathrm{I}_{6xy}S_5S_6^2 + \mathrm{I}_{5yz}C_5 + \mathrm{I}_{5xz}S_5 - \mathrm{I}_{6xz}C_5S_6 - \frac{1}{2}\mathrm{I}_{6xx}S_5S_{66} - S_4L_1\left(\left(\mathrm{COM}_{5y}C_5\right.\right.$$
$$+\mathrm{COM}_{5x}S_5)\,m_5 - \left(\mathrm{COM}_{6z}C_5 + S_5\left(\mathrm{COM}_{6y}S_6 - \mathrm{COM}_{6x}C_6\right)\right)m_6\right) + L_2\left(\left(\mathrm{COM}_{5y}C_5\right.\right.$$
$$+\mathrm{COM}_{5x}S_5)\,m_5 - \left(\mathrm{COM}_{6z}C_5 + S_5\left(\mathrm{COM}_{6y}S_6 - \mathrm{COM}_{6x}C_6\right)\right)m_6\right)\right)\ddot{\theta}_5 + \left(\mathrm{I}_{6zz}C_5 + \left(\mathrm{COM}_{6y}C_6\right.\right.$$
$$+\mathrm{COM}_{6x}S_6)\,L_2m_6C_5 + \mathrm{I}_{6xz}C_6S_5 - \mathrm{I}_{6yz}S_5S_6 + \left(C_4\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right) - C_5S_4\left(\mathrm{COM}_{6y}C_6\right.\right.$$
$$\left.\left.+\mathrm{COM}_{6x}S_6)\right)L_1m_6\right)\ddot{\theta}_6\Big) \tag{A.5c}$$

$$\tau_4 = 10^{-6}\left(-9810\left(\left(\mathrm{COM}_{4y}C_{34} + \mathrm{COM}_{4x}S_{34}\right)m_4 + \left(\mathrm{COM}_{5z}C_{34} + S_{34}\left(\mathrm{COM}_{5x}C_5 - \mathrm{COM}_{5y}S_5\right)\right)m_5\right.\right.$$
$$\left.+\left(C_{34}\left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right) + S_{34}\left(\mathrm{COM}_{6z}S_5 + C_5\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right)\right)\right)m_6\right)$$
$$+\frac{1}{32}\Big(\Big(16S_{3344}\left(m_5+m_6\right)L_2^2 - 8\left(2\left(\mathrm{COM}_{5x}C_{3344-5} + \mathrm{COM}_{5x}C_{33445} - 2\mathrm{COM}_{5z}S_{3344}\right.\right.$$
$$+\mathrm{COM}_{5y}S_{3344-5} - \mathrm{COM}_{5y}S_{33445})\,m_5 + (-2\mathrm{COM}_{6x}C_{3344-6} + \mathrm{COM}_{6x}C_{3344-56} + \mathrm{COM}_{6x}C_{33445-6}$$
$$+2\mathrm{COM}_{6x}C_{33446} + \mathrm{COM}_{6x}C_{33446-5} + \mathrm{COM}_{6x}C_{334456} - 2\mathrm{COM}_{6z}S_{3344-5} + 2\mathrm{COM}_{6z}S_{33445}$$
$$-2\mathrm{COM}_{6y}S_{3344-6} + \mathrm{COM}_{6y}S_{3344-56} + \mathrm{COM}_{6y}S_{33445-6} - 2\mathrm{COM}_{6y}S_{33446} - \mathrm{COM}_{6y}S_{33446-5}$$
$$-\mathrm{COM}_{6y}S_{334456})\,m_6)\,L_2 - 32\mathrm{I}_{4xy}C_{3344} + 8\mathrm{I}_{5xy}C_{3344-55} - 16\mathrm{I}_{5xz}C_{3344-5} - 8\mathrm{I}_{5xy}C_{334455}$$
$$-16\mathrm{I}_{5xz}C_{33445} - 8\mathrm{I}_{6xy}C_{3344-566} - 8\mathrm{I}_{6xy}C_{33445-66} + 12\mathrm{I}_{6xy}C_{3344-66} - 4\mathrm{I}_{6xz}C_{3344-556}$$
$$+2\mathrm{I}_{6xy}C_{3344-5566} + 8\mathrm{I}_{6xz}C_{3344-56} + 2\mathrm{I}_{6xy}C_{334455-66} - 8\mathrm{I}_{6xz}C_{33445-6} + 4\mathrm{I}_{6xz}C_{334455-6}$$
$$-12\mathrm{I}_{6xy}C_{334466} - 4\mathrm{I}_{6xz}C_{334466-55} - 2\mathrm{I}_{6xy}C_{334466-55} - 8\mathrm{I}_{6xz}C_{33446-5} - 2\mathrm{I}_{6xy}C_{33445566}$$
$$+8\mathrm{I}_{6xz}C_{334456} + 4\mathrm{I}_{6xz}C_{3344556} - 8\mathrm{I}_{6xy}C_{334466-5} - 8\mathrm{I}_{6xy}C_{3344566} + 16\mathrm{I}_{4xx}S_{3344} - 16\mathrm{I}_{4yy}S_{3344}$$
$$+8\mathrm{I}_{5xx}S_{3344} + 8\mathrm{I}_{5yy}S_{3344} - 16\mathrm{I}_{5zz}S_{3344} - 4\mathrm{I}_{6xx}S_{3344} - 4\mathrm{I}_{6yy}S_{3344} + 8\mathrm{I}_{6zz}S_{3344} + 4\mathrm{I}_{5xx}S_{3344-55}$$
$$-4\mathrm{I}_{5yy}S_{3344-55} + 2\mathrm{I}_{6xx}S_{3344-55} + 2\mathrm{I}_{6yy}S_{3344-55} - 4\mathrm{I}_{6zz}S_{3344-55} - 16\mathrm{I}_{5yz}S_{3344-5} + 4\mathrm{I}_{5xx}S_{334455}$$
$$-4\mathrm{I}_{5yy}S_{334455} + 2\mathrm{I}_{6xx}S_{334455} + 2\mathrm{I}_{6yy}S_{334455} - 4\mathrm{I}_{6zz}S_{334455} + 16\mathrm{I}_{5yz}S_{33445} - 4\mathrm{I}_{6xx}S_{3344-566}$$
$$+4\mathrm{I}_{6yy}S_{3344-566} - 4\mathrm{I}_{6xx}S_{33445-66} + 4\mathrm{I}_{6yy}S_{33445-66} + 6\mathrm{I}_{6xx}S_{3344-66} - 6\mathrm{I}_{6yy}S_{3344-66}$$
$$-4\mathrm{I}_{6yz}S_{3344-556} + \mathrm{I}_{6xx}S_{3344-5566} - \mathrm{I}_{6yy}S_{3344-5566} + 8\mathrm{I}_{6yz}S_{3344-56} + \mathrm{I}_{6xx}S_{334455-66}$$
$$-\mathrm{I}_{6yy}S_{334455-66} - 8\mathrm{I}_{6yz}S_{33445-6} + 4\mathrm{I}_{6yz}S_{334455-6} + 6\mathrm{I}_{6xx}S_{334466} - 6\mathrm{I}_{6yy}S_{334466} + 4\mathrm{I}_{6yz}S_{334466-55}$$
$$+\mathrm{I}_{6xx}S_{334466-55} - \mathrm{I}_{6yy}S_{334466-55} + 8\mathrm{I}_{6yz}S_{33446-5} + \mathrm{I}_{6xx}S_{33445566} - \mathrm{I}_{6yy}S_{33445566} - 8\mathrm{I}_{6yz}S_{334456}$$
$$-4\mathrm{I}_{6yz}S_{3344556} + 4\mathrm{I}_{6xx}S_{334466-5} - 4\mathrm{I}_{6yy}S_{334466-5} + 4\mathrm{I}_{6xx}S_{3344566} - 4\mathrm{I}_{6yy}S_{3344566}$$
$$-16L_1\left(2S_3\left(\mathrm{COM}_{4x}C_{34} - \mathrm{COM}_{4y}S_{34}\right)m_4 + S_3\left(\mathrm{COM}_{5x}C_{34-5} + \mathrm{COM}_{5x}C_{345} - 2\mathrm{COM}_{5z}S_{34}\right.\right.$$
$$+\mathrm{COM}_{5y}S_{34-5} - \mathrm{COM}_{5y}S_{345} - 2S_{34}L_2)\,m_5 - \Big(S_4\Big(2\mathrm{COM}_{6z}S_5S_3^2 + 2C_5\left(\mathrm{COM}_{6x}C_6\right.$$
$$-\mathrm{COM}_{6y}S_6)\,S_3^2 + \mathrm{COM}_{6y}C_6S_{33} + \mathrm{COM}_{6x}S_{33}S_6\Big) + C_4\Big(2\mathrm{COM}_{6x}S_6S_3^2 + C_6\Big(2\mathrm{COM}_{6y}S_3^2$$
$$-\mathrm{COM}_{6x}C_5S_{33}) + S_{33}\left(\mathrm{COM}_{6y}C_5S_6 - \mathrm{COM}_{6z}S_5\right)) + 2S_3S_{34}L_2)\,m_6)\Big)\,\dot{\theta}_2{}^2 + 4\left(\left(4(2\mathrm{I}_{5zz} + \mathrm{I}_{6xx}\right.\right.$$
$$+\mathrm{I}_{6yy})C_{34} - 4\mathrm{I}_{5xx}C_{34-55} + 4\mathrm{I}_{5yy}C_{34-55} - 2\mathrm{I}_{6xx}C_{34-55} - 2\mathrm{I}_{6yy}C_{34-55} + 4\mathrm{I}_{6zz}C_{34-55} \quad \ldots$$

$$+8\mathrm{I}_{5yz}C_{34-5}-8\mathrm{I}_{5yz}C_{345}-4\mathrm{I}_{5xx}C_{3455}+4\mathrm{I}_{5yy}C_{3455}-2\mathrm{I}_{6xx}C_{3455}-2\mathrm{I}_{6yy}C_{3455}+4\mathrm{I}_{6zz}C_{3455}$$
$$-2\mathrm{I}_{6xx}C_{34-66}+2\mathrm{I}_{6yy}C_{34-66}+2\mathrm{I}_{6xx}C_{34-566}-2\mathrm{I}_{6yy}C_{34-566}+2\mathrm{I}_{6xx}C_{345-66}-2\mathrm{I}_{6yy}C_{345-66}$$
$$-\mathrm{I}_{6xx}C_{3455-66}+\mathrm{I}_{6yy}C_{3455-66}+4\mathrm{I}_{6yz}C_{34-556}-4\mathrm{I}_{6yz}C_{34-56}+4\mathrm{I}_{6yz}C_{345-6}-4\mathrm{I}_{6yz}C_{3455-6}$$
$$-4\mathrm{I}_{6yz}C_{346-55}-4\mathrm{I}_{6yz}C_{346-5}+4\mathrm{I}_{6yz}C_{3456}+4\mathrm{I}_{6yz}C_{34556}-2\mathrm{I}_{6xx}C_{3466}+2\mathrm{I}_{6yy}C_{3466}-\mathrm{I}_{6xx}S_{3466-55}$$
$$+\mathrm{I}_{6yy}S_{3466-55}-2\mathrm{I}_{6xx}C_{3466-5}+2\mathrm{I}_{6yy}C_{3466-5}-2\mathrm{I}_{6xx}C_{34566}+2\mathrm{I}_{6yy}C_{34566}-\mathrm{I}_{6xx}C_{34-5566}$$
$$+\mathrm{I}_{6yy}C_{34-5566}-\mathrm{I}_{6xx}C_{345566}+\mathrm{I}_{6yy}C_{345566}+8\mathrm{I}_{5xy}S_{34-55}-8\mathrm{I}_{5xz}S_{34-5}-8\mathrm{I}_{5xz}S_{345}-8\mathrm{I}_{5xy}S_{3455}$$
$$+4\mathrm{I}_{6xy}S_{34-66}-4\mathrm{I}_{6xy}S_{34-566}-4\mathrm{I}_{6xy}S_{345-66}+2\mathrm{I}_{6xy}S_{3455-66}-4\mathrm{I}_{6xz}S_{34-556}+4\mathrm{I}_{6xz}S_{34-56}$$
$$-4\mathrm{I}_{6xz}S_{345-6}+4\mathrm{I}_{6xz}S_{3455-6}-4\mathrm{I}_{6xz}S_{346-55}-4\mathrm{I}_{6xz}S_{346-5}+4\mathrm{I}_{6xz}S_{3456}+4\mathrm{I}_{6xz}S_{34556}-4\mathrm{I}_{6xy}S_{3466}$$
$$-2\mathrm{I}_{6xy}S_{3466-55}-4\mathrm{I}_{6xy}S_{3466-5}-4\mathrm{I}_{6xy}S_{34566}+2\mathrm{I}_{6xy}S_{34-5566}-2\mathrm{I}_{6xy}S_{345566}-16S_{34}L_2\left(\left(\mathrm{COM}_{5x}C_5\right.\right.$$
$$\left.\left.-\mathrm{COM}_{5y}S_5\right)m_5+\left(\mathrm{COM}_{6z}S_5+C_5\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right)m_6\right)\dot{\theta}_5+4\left(C_4\left(2S_3S_5\left(\mathrm{I}_{6zz}+(\mathrm{I}_{6xx}\right.\right.\right.$$
$$\left.-\mathrm{I}_{6yy})C_{66}+2\mathrm{I}_{6xy}S_{66}\right)-C_3\left(4\mathrm{I}_{6yz}C_6S_5^2+4\mathrm{I}_{6xz}S_6S_5^2+2\mathrm{I}_{6xy}C_{66}S_{55}-\mathrm{I}_{6xx}S_{55}S_{66}+\mathrm{I}_{6yy}S_{55}S_{66}\right)\Big)$$
$$+S_4\left(2C_3S_5\left(\mathrm{I}_{6zz}+(\mathrm{I}_{6xx}-\mathrm{I}_{6yy})C_{66}+2\mathrm{I}_{6xy}S_{66}\right)+S_3\left(4\mathrm{I}_{6yz}C_6S_5^2+4\mathrm{I}_{6xz}S_6S_5^2+2\mathrm{I}_{6xy}C_{66}S_{55}\right.\right.$$
$$\left.\left.-\mathrm{I}_{6xx}S_{55}S_{66}+\mathrm{I}_{6yy}S_{55}S_{66}\right)\right)+4S_{34}S_5\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)L_2m_6\right)\dot{\theta}_6\Big)\dot{\theta}_2+8\left(4L_1\left(\left(\mathrm{COM}_{4y}C_4\right.\right.\right.$$
$$\left.+\mathrm{COM}_{4x}S_4\right)m_4+\left(\mathrm{COM}_{5z}C_4+L_2C_4+S_4\left(\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5\right)\right)m_5+\left(C_4\left(\mathrm{COM}_{6y}C_6\right.\right.$$
$$\left.\left.\left.+\mathrm{COM}_{6x}S_6\right)+S_4\left(\mathrm{COM}_{6z}S_5+C_5\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right)+C_4L_2\right)m_6\right)\dot{\theta}_3{}^2+\left(\left(-8\mathrm{I}_{5xy}C_{55}\right.\right.$$
$$+2\mathrm{I}_{6xy}C_{55-66}+4\mathrm{I}_{6xz}C_{55-6}-2\mathrm{I}_{6xy}C_{5566}+4\mathrm{I}_{6xz}C_{556}+4\mathrm{I}_{5xx}S_{55}-4\mathrm{I}_{5yy}S_{55}+2\mathrm{I}_{6xx}S_{55}+2\mathrm{I}_{6yy}S_{55}$$
$$\left.-4\mathrm{I}_{6zz}S_{55}+\mathrm{I}_{6xx}S_{55-66}-\mathrm{I}_{6yy}S_{55-66}+4\mathrm{I}_{6yz}S_{55-6}+\mathrm{I}_{6xx}S_{5566}-\mathrm{I}_{6yy}S_{5566}-4\mathrm{I}_{6yz}S_{556}\right)\dot{\theta}_5$$
$$+4\Big(2\mathrm{I}_{6xy}C_{66}S_5^2-\mathrm{I}_{6xx}S_{66}S_5^2+\mathrm{I}_{6yy}S_{66}S_5^2-\mathrm{I}_{6yz}C_6S_{55}-\mathrm{I}_{6xz}S_{55}S_6+2\left(\mathrm{COM}_{6x}C_6\right.$$
$$\left.-\mathrm{COM}_{6y}S_6\right)L_2m_6\right)\dot{\theta}_6\Big)\dot{\theta}_3+2\left(\left(2S_5\left(-\mathrm{I}_{5yz}+\mathrm{I}_{6yz}C_6+\mathrm{I}_{6xz}S_6\right)+C_5\left(2\mathrm{I}_{5xz}+2\mathrm{I}_{6xy}C_{66}+(\mathrm{I}_{6yy}\right.\right.\right.$$
$$\left.-\mathrm{I}_{6xx})S_{66}\right)+2L_2\left(\left(\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5\right)m_5+\left(\mathrm{COM}_{6z}S_5+C_5\left(\mathrm{COM}_{6x}C_6\right.\right.\right.$$
$$\left.\left.\left.-\mathrm{COM}_{6y}S_6\right)\right)m_6\right)\right)\dot{\theta}_5{}^2-2S_5\left(\mathrm{I}_{6zz}+(\mathrm{I}_{6xx}-\mathrm{I}_{6yy})C_{66}+2\mathrm{I}_{6xy}S_{66}+2\left(\mathrm{COM}_{6y}C_6\right.\right.$$
$$\left.\left.+\mathrm{COM}_{6x}S_6\right)L_2m_6\right)\dot{\theta}_6\dot{\theta}_5-2\left(S_5\left(\mathrm{I}_{6yz}C_6+\mathrm{I}_{6xz}S_6\right)-C_5\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)L_2m_6\right)\dot{\theta}_6{}^2$$
$$+\dot{\theta}_4\Big(\left(-4\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)C_5^2+4S_5^2\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)+S_{55}\left(2\mathrm{I}_{5xx}-2\mathrm{I}_{5yy}+\mathrm{I}_{6xx}\right.\right.$$
$$\left.\left.+\mathrm{I}_{6yy}-2\mathrm{I}_{6zz}+(\mathrm{I}_{6xx}-\mathrm{I}_{6yy})C_{66}+2\mathrm{I}_{6xy}S_{66}\right)\right)\dot{\theta}_5+2\Big(2\mathrm{I}_{6xy}C_{66}S_5^2-\mathrm{I}_{6xx}S_{66}S_5^2+\mathrm{I}_{6yy}S_{66}S_5^2$$
$$\left.-\mathrm{I}_{6yz}C_6S_{55}-\mathrm{I}_{6xz}S_{55}S_6+2\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)L_2m_6\right)\dot{\theta}_6\Big)\Big)\Big)\Big)+S_2\left(\mathrm{COM}_{4x}C_{34}m_4\right.$$
$$-\mathrm{COM}_{4y}S_{34}m_4+\mathrm{COM}_{5x}C_{34}C_5m_5-\mathrm{COM}_{5z}S_{34}m_5-\mathrm{COM}_{5y}C_{34}S_5m_5-S_{34}L_2m_5$$
$$+\mathrm{COM}_{6z}C_{34}S_5m_6+\mathrm{COM}_{6y}\left(S_4\left(C_5S_3S_6-C_3C_6\right)-C_4\left(C_6S_3+C_3C_5S_6\right)\right)m_6$$
$$\left.+\mathrm{COM}_{6x}\left(C_3\left(C_4C_5C_6-S_4S_6\right)-S_3\left(C_5C_6S_4+C_4S_6\right)\right)m_6-S_{34}L_2m_6\right)\ddot{d}_1+\frac{1}{4}\left(S_3\left(S_4\left(-4\left(\mathrm{I}_{5xy}\right.\right.\right.\right.$$
$$\left.-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)C_5^2-\mathrm{I}_{6xx}S_{55}S_6^2+\mathrm{I}_{6yy}S_{55}S_6^2+4\mathrm{I}_{4xz}+\mathrm{I}_{6xx}C_6^2S_{55}-\mathrm{I}_{6yy}C_6^2S_{55}+2\mathrm{I}_{5xx}S_{55}$$
$$-2\mathrm{I}_{5yy}S_{55}+\mathrm{I}_{6xx}S_{55}+\mathrm{I}_{6yy}S_{55}-2\mathrm{I}_{6zz}S_{55}+4S_5^2\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)+2\mathrm{I}_{6xy}S_{55}S_{66}\Big)\ \ldots$$

$$
\begin{aligned}
&+2C_4\left(2\mathrm{I}_{4yz}+C_5\left(-2\mathrm{I}_{5yz}+2\mathrm{I}_{6yz}C_6+2\mathrm{I}_{6xz}S_6\right)-S_5\left(2\mathrm{I}_{5xz}+2\mathrm{I}_{6xy}C_{66}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})S_{66}\right)\right)\\
&-C_3\left(C_4\left(-4\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)C_5^2-\mathrm{I}_{6xx}S_{55}S_6^2+\mathrm{I}_{6yy}S_{55}S_6^2+4\mathrm{I}_{4xz}+\mathrm{I}_{6xx}C_6^2S_{55}-\mathrm{I}_{6yy}C_6^2S_{55}\right.\right.\\
&\left.+2\mathrm{I}_{5xx}S_{55}-2\mathrm{I}_{5yy}S_{55}+\mathrm{I}_{6xx}S_{55}+\mathrm{I}_{6yy}S_{55}-2\mathrm{I}_{6zz}S_{55}+4S_5^2\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)+2\mathrm{I}_{6xy}S_{55}S_{66}\right)\\
&+2S_4\left(-2\mathrm{I}_{4yz}+2C_5\left(\mathrm{I}_{5yz}-\mathrm{I}_{6yz}C_6-\mathrm{I}_{6xz}S_6\right)+S_5\left(2\mathrm{I}_{5xz}+2\mathrm{I}_{6xy}C_{66}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})S_{66}\right)\right)\\
&-4S_{34}L_2\left(\left(\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5\right)m_5-\left(\mathrm{COM}_{6z}C_5+S_5\left(\mathrm{COM}_{6y}S_6-\mathrm{COM}_{6x}C_6\right)\right)m_6\right)\ddot{\theta}_2\\
&+\frac{1}{4}\Bigg(4\left(m_5+m_6\right)L_2^2+8\left(\mathrm{COM}_{5z}m_5+\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)m_6\right)L_2+\frac{1}{2}\left(8\mathrm{I}_{4zz}+4\mathrm{I}_{5xx}+4\mathrm{I}_{5yy}\right.\\
&+2\mathrm{I}_{6xx}+2\mathrm{I}_{6yy}+4\mathrm{I}_{6zz}-2(2\mathrm{I}_{5xx}-2\mathrm{I}_{5yy}+\mathrm{I}_{6xx}+\mathrm{I}_{6yy}-2\mathrm{I}_{6zz})C_{55}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})C_{55-66}\\
&-4\mathrm{I}_{6yz}C_{55-6}+2\mathrm{I}_{6xx}C_{66}-2\mathrm{I}_{6yy}C_{66}-\mathrm{I}_{6xx}C_{5566}+\mathrm{I}_{6yy}C_{5566}+4\mathrm{I}_{6yz}C_{556}-8\mathrm{I}_{5xy}S_{55}+2\mathrm{I}_{6xy}S_{55-66}\\
&\left.+4\mathrm{I}_{6xz}S_{55-6}+4\mathrm{I}_{6xy}S_{66}-2\mathrm{I}_{6xy}S_{5566}+4\mathrm{I}_{6xz}S_{556}\right)+4L_1\left(\left(\mathrm{COM}_{4x}C_4-\mathrm{COM}_{4y}S_4\right)m_4\right.\\
&+\left(-\mathrm{COM}_{5z}S_4-L_2S_4+C_4\left(\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5\right)\right)m_5+\left(-S_4\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)\right.\\
&\left.\left.+C_4\left(\mathrm{COM}_{6z}S_5+C_5\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right)-S_4L_2\right)m_6\right)\ddot{\theta}_3+\frac{1}{8}\left(8\left(m_5+m_6\right)L_2^2\right.\\
&+16\left(\mathrm{COM}_{5z}m_5+\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)m_6\right)L_2+8\mathrm{I}_{4zz}+4\mathrm{I}_{5xx}+4\mathrm{I}_{5yy}+2\mathrm{I}_{6xx}+2\mathrm{I}_{6yy}+4\mathrm{I}_{6zz}\\
&-2(2\mathrm{I}_{5xx}-2\mathrm{I}_{5yy}+\mathrm{I}_{6xx}+\mathrm{I}_{6yy}-2\mathrm{I}_{6zz})C_{55}-\mathrm{I}_{6xx}C_{55-66}+\mathrm{I}_{6yy}C_{55-66}-4\mathrm{I}_{6yz}C_{55-6}+2\mathrm{I}_{6xx}C_{66}\\
&-2\mathrm{I}_{6yy}C_{66}-\mathrm{I}_{6xx}C_{5566}+\mathrm{I}_{6yy}C_{5566}+4\mathrm{I}_{6yz}C_{556}-8\mathrm{I}_{5xy}S_{55}+2\mathrm{I}_{6xy}S_{55-66}+4\mathrm{I}_{6xz}S_{55-6}+4\mathrm{I}_{6xy}S_{66}\\
&\left.-2\mathrm{I}_{6xy}S_{5566}+4\mathrm{I}_{6xz}S_{556}\right)\ddot{\theta}_4+\Bigg(C_5\left(\mathrm{I}_{5yz}-\mathrm{I}_{6yz}C_6-\mathrm{I}_{6xz}S_6\right)+\frac{1}{2}S_5\left(2\mathrm{I}_{5xz}+2\mathrm{I}_{6xy}C_{66}+(\mathrm{I}_{6yy}\right.\\
&\left.-\mathrm{I}_{6xx})S_{66}\right)+L_2\left(\left(\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5\right)m_5-\left(\mathrm{COM}_{6z}C_5+S_5\left(\mathrm{COM}_{6y}S_6-\mathrm{COM}_{6x}C_6\right)\right)m_6\right)\Bigg)\ddot{\theta}_5\\
&+\left(\mathrm{I}_{6zz}C_5+\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)L_2m_6C_5+S_5\left(\mathrm{I}_{6xz}C_6-\mathrm{I}_{6yz}S_6\right)\right)\ddot{\theta}_6\Bigg)
\end{aligned}
\tag{A.5d}
$$

$$
\begin{aligned}
\tau_5=10^{-6}&\left(-9810C_{34}\left(\left(\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5\right)m_5-\left(\mathrm{COM}_{6z}C_5+S_5\left(\mathrm{COM}_{6y}S_6-\mathrm{COM}_{6x}C_6\right)\right)m_6\right)\right.\\
&+\frac{1}{32}\left(\left(32S_3S_{34}L_1\left(\left(\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5\right)m_5-\left(\mathrm{COM}_{6z}C_5+S_5\left(\mathrm{COM}_{6y}S_6-\mathrm{COM}_{6x}C_6\right)\right)m_6\right)\right.\right.\\
&-8C_{34}\left(S_3\left(S_4\left(-4\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)C_5^2+4S_5^2\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)+S_{55}\left(2\mathrm{I}_{5xx}-2\mathrm{I}_{5yy}\right.\right.\right.\right.\\
&\left.\left.+\mathrm{I}_{6xx}+\mathrm{I}_{6yy}-2\mathrm{I}_{6zz}+(\mathrm{I}_{6xx}-\mathrm{I}_{6yy})C_{66}+2\mathrm{I}_{6xy}S_{66}\right)\right)+C_4\left(C_5\left(-4\mathrm{I}_{5yz}+4\mathrm{I}_{6yz}C_6+4\mathrm{I}_{6xz}S_6\right)\right.\\
&\left.\left.-2S_5\left(2\mathrm{I}_{5xz}+2\mathrm{I}_{6xy}C_{66}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})S_{66}\right)\right)\right)+C_3\left(C_4\left(4\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)C_5^2-4S_5^2\left(\mathrm{I}_{5xy}\right.\right.\right.\\
&\left.\left.-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)-S_{55}\left(2\mathrm{I}_{5xx}-2\mathrm{I}_{5yy}+\mathrm{I}_{6xx}+\mathrm{I}_{6yy}-2\mathrm{I}_{6zz}+(\mathrm{I}_{6xx}-\mathrm{I}_{6yy})C_{66}+2\mathrm{I}_{6xy}S_{66}\right)\right)\\
&\left.+2S_4\left(C_5\left(-2\mathrm{I}_{5yz}+2\mathrm{I}_{6yz}C_6+2\mathrm{I}_{6xz}S_6\right)-S_5\left(2\mathrm{I}_{5xz}+2\mathrm{I}_{6xy}C_{66}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})S_{66}\right)\right)\right)\\
&\left.-4S_{34}L_2\left(\left(\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5\right)m_5-\left(\mathrm{COM}_{6z}C_5+S_5\left(\mathrm{COM}_{6y}S_6-\mathrm{COM}_{6x}C_6\right)\right)m_6\right)\right)\dot{\theta}_2{}^2\\
&-4\left(\left(8\mathrm{I}_{5zz}C_{34}+4\mathrm{I}_{6xx}C_{34}+4\mathrm{I}_{6yy}C_{34}-4\mathrm{I}_{5xx}C_{34-55}+4\mathrm{I}_{5yy}C_{34-55}-2\mathrm{I}_{6xx}C_{34-55}-2\mathrm{I}_{6yy}C_{34-55}\right.\right.\\
&+4\mathrm{I}_{6zz}C_{34-55}+8\mathrm{I}_{5yz}C_{34-5}-8\mathrm{I}_{5yz}C_{345}-4\mathrm{I}_{5xx}C_{3455}+4\mathrm{I}_{5yy}C_{3455}-2\mathrm{I}_{6xx}C_{3455}-2\mathrm{I}_{6yy}C_{3455}\\
&+4\mathrm{I}_{6zz}C_{3455}-2\mathrm{I}_{6xx}C_{34-66}+2\mathrm{I}_{6yy}C_{34-66}+2\mathrm{I}_{6xx}C_{34-566}-2\mathrm{I}_{6yy}C_{34-566}+2\mathrm{I}_{6xx}C_{345-66}\quad\ldots
\end{aligned}
$$

$$-2\mathrm{I}_{6yy}C_{345-66}-\mathrm{I}_{6xx}C_{3455-66}+\mathrm{I}_{6yy}C_{3455-66}+4\mathrm{I}_{6yz}C_{34-556}-4\mathrm{I}_{6yz}C_{34-56}+4\mathrm{I}_{6yz}C_{345-6}$$
$$-4\mathrm{I}_{6yz}C_{3455-6}-4\mathrm{I}_{6yz}C_{346-55}-4\mathrm{I}_{6yz}C_{346-5}+4\mathrm{I}_{6yz}C_{3456}+4\mathrm{I}_{6yz}C_{34556}-2\mathrm{I}_{6xx}C_{3466}+2\mathrm{I}_{6yy}C_{3466}$$
$$-\mathrm{I}_{6xx}S_{3466-55}+\mathrm{I}_{6yy}S_{3466-55}-2\mathrm{I}_{6xx}C_{3466-5}+2\mathrm{I}_{6yy}C_{3466-5}-2\mathrm{I}_{6xx}C_{34566}+2\mathrm{I}_{6yy}C_{34566}$$
$$-\mathrm{I}_{6xx}C_{34-5566}+\mathrm{I}_{6yy}C_{34-5566}-\mathrm{I}_{6xx}C_{345566}+\mathrm{I}_{6yy}C_{345566}+8\mathrm{I}_{5xy}S_{34-55}-8\mathrm{I}_{5xz}S_{34-5}-8\mathrm{I}_{5xz}S_{345}$$
$$-8\mathrm{I}_{5xy}S_{3455}+4\mathrm{I}_{6xy}S_{34-66}-4\mathrm{I}_{6xy}S_{34-566}-4\mathrm{I}_{6xy}S_{345-66}+2\mathrm{I}_{6xy}S_{3455-66}-4\mathrm{I}_{6xz}S_{34-556}$$
$$+4\mathrm{I}_{6xz}S_{34-56}-4\mathrm{I}_{6xz}S_{345-6}+4\mathrm{I}_{6xz}S_{3455-6}-4\mathrm{I}_{6xz}S_{346-55}-4\mathrm{I}_{6xz}S_{346-5}+4\mathrm{I}_{6xz}S_{3456}$$
$$+4\mathrm{I}_{6xz}S_{34556}-4\mathrm{I}_{6xy}S_{3466}-2\mathrm{I}_{6xy}S_{3466-55}-4\mathrm{I}_{6xy}S_{3466-5}-4\mathrm{I}_{6xy}S_{34566}+2\mathrm{I}_{6xy}S_{34-5566}$$
$$-2\mathrm{I}_{6xy}S_{345566}+16C_3L_1\left(\left(\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5\right)m_5+\left(\mathrm{COM}_{6z}S_5+C_5\left(\mathrm{COM}_{6x}C_6\right.\right.\right.$$
$$\left.\left.\left.-\mathrm{COM}_{6y}S_6\right)\right)m_6\right)-16S_{34}L_2\left(\left(\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5\right)m_5+\left(\mathrm{COM}_{6z}S_5+C_5\left(\mathrm{COM}_{6x}C_6\right.\right.\right.$$
$$\left.\left.\left.-\mathrm{COM}_{6y}S_6\right)\right)m_6\right)\right)\dot{\theta}_3+\left(4(2\mathrm{I}_{5zz}+\mathrm{I}_{6xx}+\mathrm{I}_{6yy})C_{34}-4\mathrm{I}_{5xx}C_{34-55}+4\mathrm{I}_{5yy}C_{34-55}-2\mathrm{I}_{6xx}C_{34-55}\right.$$
$$-2\mathrm{I}_{6yy}C_{34-55}+4\mathrm{I}_{6zz}C_{34-55}+8\mathrm{I}_{5yz}C_{34-5}-8\mathrm{I}_{5yz}C_{345}-4\mathrm{I}_{5xx}C_{3455}+4\mathrm{I}_{5yy}C_{3455}-2\mathrm{I}_{6xx}C_{3455}$$
$$-2\mathrm{I}_{6yy}C_{3455}+4\mathrm{I}_{6zz}C_{3455}-2\mathrm{I}_{6xx}C_{34-66}+2\mathrm{I}_{6yy}C_{34-66}+2\mathrm{I}_{6xx}C_{34-566}-2\mathrm{I}_{6yy}C_{34-566}$$
$$+2\mathrm{I}_{6xx}C_{345-66}-2\mathrm{I}_{6yy}C_{345-66}-\mathrm{I}_{6xx}C_{3455-66}+\mathrm{I}_{6yy}C_{3455-66}+4\mathrm{I}_{6yz}C_{34-556}-4\mathrm{I}_{6yz}C_{34-56}$$
$$+4\mathrm{I}_{6yz}C_{345-6}-4\mathrm{I}_{6yz}C_{3455-6}-4\mathrm{I}_{6yz}C_{346-55}-4\mathrm{I}_{6yz}C_{346-5}+4\mathrm{I}_{6yz}C_{3456}+4\mathrm{I}_{6yz}C_{34556}$$
$$-2\mathrm{I}_{6xx}C_{3466}+2\mathrm{I}_{6yy}C_{3466}-\mathrm{I}_{6xx}S_{3466-55}+\mathrm{I}_{6yy}S_{3466-55}-2\mathrm{I}_{6xx}C_{3466-5}+2\mathrm{I}_{6yy}C_{3466-5}$$
$$-2\mathrm{I}_{6xx}C_{34566}+2\mathrm{I}_{6yy}C_{34566}-\mathrm{I}_{6xx}C_{34-5566}+\mathrm{I}_{6yy}C_{34-5566}-\mathrm{I}_{6xx}C_{345566}+\mathrm{I}_{6yy}C_{345566}$$
$$+8\mathrm{I}_{5xy}S_{34-55}-8\mathrm{I}_{5xz}S_{34-5}-8\mathrm{I}_{5xz}S_{345}-8\mathrm{I}_{5xy}S_{3455}+4\mathrm{I}_{6xy}S_{34-66}-4\mathrm{I}_{6xy}S_{34-566}-4\mathrm{I}_{6xy}S_{345-66}$$
$$+2\mathrm{I}_{6xy}S_{3455-66}-4\mathrm{I}_{6xz}S_{34-556}+4\mathrm{I}_{6xz}S_{34-56}-4\mathrm{I}_{6xz}S_{345-6}+4\mathrm{I}_{6xz}S_{3455-6}-4\mathrm{I}_{6xz}S_{346-55}$$
$$-4\mathrm{I}_{6xz}S_{346-5}+4\mathrm{I}_{6xz}S_{3456}+4\mathrm{I}_{6xz}S_{34556}-4\mathrm{I}_{6xy}S_{3466}-2\mathrm{I}_{6xy}S_{3466-55}-4\mathrm{I}_{6xy}S_{3466-5}-4\mathrm{I}_{6xy}S_{34566}$$
$$+2\mathrm{I}_{6xy}S_{34-5566}-2\mathrm{I}_{6xy}S_{345566}-16S_{34}L_2\left(\left(\mathrm{COM}_{5x}C_5-\mathrm{COM}_{5y}S_5\right)m_5+\left(\mathrm{COM}_{6z}S_5\right.\right.$$
$$\left.\left.\left.+C_5\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right)m_6\right)\right)\dot{\theta}_4+8\left(C_3\left(S_4\left((\mathrm{I}_{6xx}-\mathrm{I}_{6yy})S_{66}-2\mathrm{I}_{6xy}C_{66}\right)+C_4\left(2S_5\left(\mathrm{I}_{6xz}C_6\right.\right.\right.\right.$$
$$\left.\left.\left.-\mathrm{I}_{6yz}S_6\right)+C_5\left(\mathrm{I}_{6zz}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})C_{66}-2\mathrm{I}_{6xy}S_{66}\right)\right)\right)-S_3\left(C_4\left(2\mathrm{I}_{6xy}C_{66}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})S_{66}\right)\right.$$
$$\left.\left.\left.+S_4\left(2S_5\left(\mathrm{I}_{6xz}C_6-\mathrm{I}_{6yz}S_6\right)+C_5\left(\mathrm{I}_{6zz}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})C_{66}-2\mathrm{I}_{6xy}S_{66}\right)\right)\right)\right)\dot{\theta}_6\right)\dot{\theta}_2+8\left(\left(4\left(\mathrm{I}_{5xy}\right.\right.\right.$$
$$\left.-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)C_5^2-4\mathrm{I}_{5xy}S_5^2+4\mathrm{I}_{6xz}C_6S_5^2+\mathrm{I}_{6xx}S_{55}S_6^2-\mathrm{I}_{6yy}S_{55}S_6^2-\mathrm{I}_{6xx}C_6^2S_{55}+\mathrm{I}_{6yy}C_6^2S_{55}$$
$$-2\mathrm{I}_{5xx}S_{55}+2\mathrm{I}_{5yy}S_{55}-\mathrm{I}_{6xx}S_{55}-\mathrm{I}_{6yy}S_{55}+2\mathrm{I}_{6zz}S_{55}-4\mathrm{I}_{6yz}S_5^2S_6-2\mathrm{I}_{6xy}S_{55}S_{66}$$
$$+4C_4L_1\left(\left(\mathrm{COM}_{5y}C_5+\mathrm{COM}_{5x}S_5\right)m_5-\left(\mathrm{COM}_{6z}C_5+S_5\left(\mathrm{COM}_{6y}S_6-\mathrm{COM}_{6x}C_6\right)\right)m_6\right)\right)\dot{\theta}_3{}^2$$
$$+2\left(\left(4\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)C_5^2-4S_5^2\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)-S_{55}\left(2\mathrm{I}_{5xx}-2\mathrm{I}_{5yy}+\mathrm{I}_{6xx}+\mathrm{I}_{6yy}\right.\right.\right.$$
$$\left.\left.-2\mathrm{I}_{6zz}+(\mathrm{I}_{6xx}-\mathrm{I}_{6yy})C_{66}+2\mathrm{I}_{6xy}S_{66}\right)\right)\dot{\theta}_4+2\left(C_5\left(2\mathrm{I}_{6yz}S_6-2\mathrm{I}_{6xz}C_6\right)+S_5\left(\mathrm{I}_{6zz}+(\mathrm{I}_{6yy}\right.\right.$$
$$\left.\left.\left.\left.-\mathrm{I}_{6xx})C_{66}-2\mathrm{I}_{6xy}S_{66}\right)\right)\dot{\theta}_6\right)\dot{\theta}_3+\left(4\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)C_5^2-4S_5^2\left(\mathrm{I}_{5xy}-\mathrm{I}_{6xz}C_6+\mathrm{I}_{6yz}S_6\right)\right.$$
$$\left.-S_{55}\left(2\mathrm{I}_{5xx}-2\mathrm{I}_{5yy}+\mathrm{I}_{6xx}+\mathrm{I}_{6yy}-2\mathrm{I}_{6zz}+(\mathrm{I}_{6xx}-\mathrm{I}_{6yy})C_{66}+2\mathrm{I}_{6xy}S_{66}\right)\right)\dot{\theta}_4{}^2-4\left(2C_5\left(\mathrm{I}_{6xz}C_6\quad\ldots\right.\right.$$

$$
\begin{aligned}
&-\mathrm{I}_{6yz}S_6) - S_5\left(\mathrm{I}_{6zz} + (\mathrm{I}_{6yy} - \mathrm{I}_{6xx})C_{66} - 2\mathrm{I}_{6xy}S_{66}\right)\right)\dot{\theta}_4\dot{\theta}_6 - 4\dot{\theta}_6\Big(\left(2\mathrm{I}_{6xy}C_{66} + (\mathrm{I}_{6yy} - \mathrm{I}_{6xx})S_{66}\right)\dot{\theta}_5\\
&+\left(\mathrm{I}_{6xz}C_6 - \mathrm{I}_{6yz}S_6\right)\dot{\theta}_6\Big)\Big)\Big) + \big((S_2S_{34}\left(\mathrm{COM}_{6z}C_5 + S_5\left(\mathrm{COM}_{6y}S_6 - \mathrm{COM}_{6x}C_6\right)\right) - C_2\left(\mathrm{COM}_{6z}S_5\right.\\
&\left.+C_5\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right)\right))m_6 - (S_2S_{34}\left(\mathrm{COM}_{5y}C_5 + \mathrm{COM}_{5x}S_5\right) + C_2\left(\mathrm{COM}_{5x}C_5\right.\\
&\left.-\mathrm{COM}_{5y}S_5\right))m_5\big)\ddot{d}_1 - \frac{1}{2}\left(2\mathrm{I}_{6xy}C_3C_4C_5C_6^2 - \mathrm{I}_{6xx}C_4S_3C_6^2 + \mathrm{I}_{6yy}C_4S_3C_6^2 - \mathrm{I}_{6xx}C_3S_4C_6^2\right.\\
&+\mathrm{I}_{6yy}C_3S_4C_6^2 - 2\mathrm{I}_{6xy}C_5S_3S_4C_6^2 + 2\mathrm{I}_{6yz}C_3C_4S_5C_6 - 2\mathrm{I}_{6yz}S_3S_4S_5C_6 - 2\mathrm{I}_{6xy}C_3C_4C_5S_6^2\\
&+\mathrm{I}_{6xx}C_4S_3S_6^2 - \mathrm{I}_{6yy}C_4S_3S_6^2 + \mathrm{I}_{6xx}C_3S_4S_6^2 - \mathrm{I}_{6yy}C_3S_4S_6^2 + 2\mathrm{I}_{6xy}C_5S_3S_4S_6^2 + 2\mathrm{I}_{5xz}C_3C_4C_5\\
&+2\mathrm{I}_{5zz}C_4S_3 + \mathrm{I}_{6xx}C_4S_3 + \mathrm{I}_{6yy}C_4S_3 + 2\mathrm{I}_{5zz}C_3S_4 + \mathrm{I}_{6xx}C_3S_4 + \mathrm{I}_{6yy}C_3S_4 - 2\mathrm{I}_{5xz}C_5S_3S_4\\
&-2\mathrm{I}_{5yz}C_3C_4S_5 + 2\mathrm{I}_{5yz}S_3S_4S_5 + 2\mathrm{I}_{6xz}C_3C_4S_5S_6 - 2\mathrm{I}_{6xz}S_3S_4S_5S_6 - \mathrm{I}_{6xx}C_3C_4C_5S_{66}\\
&+\mathrm{I}_{6yy}C_3C_4C_5S_{66} - 2\mathrm{I}_{6xy}C_4S_3S_{66} - 2\mathrm{I}_{6xy}C_3S_4S_{66} + \mathrm{I}_{6xx}C_5S_3S_4S_{66} - \mathrm{I}_{6yy}C_5S_3S_4S_{66}\\
&+2S_3L_1\left(\left(\mathrm{COM}_{5x}C_5 - \mathrm{COM}_{5y}S_5\right)m_5 + \left(\mathrm{COM}_{6z}S_5 + C_5\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right)\right)m_6\right)\\
&\left.+2C_{34}L_2\left(\left(\mathrm{COM}_{5x}C_5 - \mathrm{COM}_{5y}S_5\right)m_5 + \left(\mathrm{COM}_{6z}S_5 + C_5\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right)\right)m_6\right)\right)\ddot{\theta}_2\\
&+\Big(\mathrm{I}_{6xy}S_5C_6^2 - \mathrm{I}_{6yz}C_5C_6 + \mathrm{I}_{6yy}S_5S_6C_6 - \mathrm{I}_{6xy}S_5S_6^2 + \mathrm{I}_{5yz}C_5 + \mathrm{I}_{5xz}S_5 - \mathrm{I}_{6xz}C_5S_6 - \frac{1}{2}\mathrm{I}_{6xx}S_5S_{66}\\
&-S_4L_1\left(\left(\mathrm{COM}_{5y}C_5 + \mathrm{COM}_{5x}S_5\right)m_5 - \left(\mathrm{COM}_{6z}C_5 + S_5\left(\mathrm{COM}_{6y}S_6 - \mathrm{COM}_{6x}C_6\right)\right)m_6\right)\\
&+L_2\left(\left(\mathrm{COM}_{5y}C_5 + \mathrm{COM}_{5x}S_5\right)m_5 - \left(\mathrm{COM}_{6z}C_5 + S_5\left(\mathrm{COM}_{6y}S_6 - \mathrm{COM}_{6x}C_6\right)\right)m_6\right)\right)\ddot{\theta}_3 + \left(C_5\left(\mathrm{I}_{5yz}\right.\right.\\
&\left.-\mathrm{I}_{6yz}C_6 - \mathrm{I}_{6xz}S_6\right) + \frac{1}{2}S_5\left(2\mathrm{I}_{5xz} + 2\mathrm{I}_{6xy}C_{66} + (\mathrm{I}_{6yy} - \mathrm{I}_{6xx})S_{66}\right) + L_2\left(\left(\mathrm{COM}_{5y}C_5 + \mathrm{COM}_{5x}S_5\right)m_5\right.\\
&\left.\left.-\left(\mathrm{COM}_{6z}C_5 + S_5\left(\mathrm{COM}_{6y}S_6 - \mathrm{COM}_{6x}C_6\right)\right)m_6\right)\right)\ddot{\theta}_4 + \frac{1}{2}\left(2\mathrm{I}_{5zz} + \mathrm{I}_{6xx} + \mathrm{I}_{6yy} + (\mathrm{I}_{6yy} - \mathrm{I}_{6xx})C_{66}\right.\\
&\left.-2\mathrm{I}_{6xy}S_{66}\right)\ddot{\theta}_5 - \left(\mathrm{I}_{6yz}C_6 + \mathrm{I}_{6xz}S_6\right)\ddot{\theta}_6\Big)
\end{aligned}
\tag{A.5e}
$$

$$
\begin{aligned}
\tau_6 = 10^{-6}&\left(-9810\left(C_{34}C_5\left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right) + S_{34}\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right)\right)m_6\right.\\
&+\left(-\mathrm{COM}_{6x}C_6S_2S_3S_4 + \mathrm{COM}_{6y}S_2S_3S_6S_4 + \mathrm{COM}_{6y}C_2C_6S_5 + \mathrm{COM}_{6x}C_2S_5S_6\right.\\
&-C_4C_5S_2S_3\left(\mathrm{COM}_{6y}C_6 + \mathrm{COM}_{6x}S_6\right) + C_3S_2\left(C_4\left(\mathrm{COM}_{6x}C_6 - \mathrm{COM}_{6y}S_6\right) - C_5S_4\left(\mathrm{COM}_{6y}C_6\right.\right.\\
&\left.\left.\left.+\mathrm{COM}_{6x}S_6\right)\right)\right)\ddot{d}_1m_6 + \frac{1}{32}\left(\left(8\mathrm{I}_{6xy}C_{3344-566} + 8\mathrm{I}_{6xy}C_{33445-66} - 12\mathrm{I}_{6xy}C_{3344-66} + 2\mathrm{I}_{6xz}C_{3344-556}\right.\right.\\
&-2\mathrm{I}_{6xy}C_{3344-5566} - 4\mathrm{I}_{6xz}C_{3344-56} - 4\mathrm{I}_{6xy}C_{55-66} - 2\mathrm{I}_{6xy}C_{334455-66} + 4\mathrm{I}_{6xz}C_{33445-6} - 4\mathrm{I}_{6xz}C_{55-6}\\
&-2\mathrm{I}_{6xz}C_{334455-6} + 8\mathrm{I}_{6xy}C_{66} - 12\mathrm{I}_{6xy}C_{334466} - 2\mathrm{I}_{6xz}C_{334466-55} - 2\mathrm{I}_{6xy}C_{334466-55} - 4\mathrm{I}_{6xz}C_{33446-5}\\
&-4\mathrm{I}_{6xy}C_{5566} - 2\mathrm{I}_{6xy}C_{33445566} + 4\mathrm{I}_{6xz}C_{334456} + 4\mathrm{I}_{6xz}C_{556} + 2\mathrm{I}_{6xz}C_{3344556} - 8\mathrm{I}_{6xy}C_{334466-5}\\
&-8\mathrm{I}_{6xy}C_{3344566} + 4\mathrm{I}_{6xx}S_{3344-566} - 4\mathrm{I}_{6yy}S_{3344-566} + 4\mathrm{I}_{6xx}S_{33445-66} - 4\mathrm{I}_{6yy}S_{33445-66}\\
&-6\mathrm{I}_{6xx}S_{3344-66} + 6\mathrm{I}_{6yy}S_{3344-66} + 2\mathrm{I}_{6yz}S_{3344-556} - \mathrm{I}_{6xx}S_{3344-5566} + \mathrm{I}_{6yy}S_{3344-5566}\\
&-4\mathrm{I}_{6yz}S_{3344-56} - 2\mathrm{I}_{6xx}S_{55-66} + 2\mathrm{I}_{6yy}S_{55-66} - \mathrm{I}_{6xx}S_{334455-66} + \mathrm{I}_{6yy}S_{334455-66} + 4\mathrm{I}_{6yz}S_{33445-6}\\
&-4\mathrm{I}_{6yz}S_{55-6} - 2\mathrm{I}_{6yz}S_{334455-6} - 4\mathrm{I}_{6xx}S_{66} + 4\mathrm{I}_{6yy}S_{66} + 6\mathrm{I}_{6xx}S_{334466} - 6\mathrm{I}_{6yy}S_{334466}\ \ldots
\end{aligned}
$$

$$
\begin{aligned}
&+2\mathrm{I}_{6yz}S_{334466-55}+\mathrm{I}_{6xx}S_{334466-55}-\mathrm{I}_{6yy}S_{334466-55}+4\mathrm{I}_{6yz}S_{33446-5}+2\mathrm{I}_{6xx}S_{5566}-2\mathrm{I}_{6yy}S_{5566}\\
&+\mathrm{I}_{6xx}S_{33445566}-\mathrm{I}_{6yy}S_{33445566}-4\mathrm{I}_{6yz}S_{334456}-4\mathrm{I}_{6yz}S_{556}-2\mathrm{I}_{6yz}S_{3344556}+4\mathrm{I}_{6xx}S_{334466-5}\\
&-4\mathrm{I}_{6yy}S_{334466-5}+4\mathrm{I}_{6xx}S_{3344566}-4\mathrm{I}_{6yy}S_{3344566}+32S_3\left(S_3\left(C_4C_5\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)\right.\right.\\
&\left.+S_4\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right)+C_3\left(C_5S_4\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)+C_4\left(\mathrm{COM}_{6y}S_6\right.\right.\\
&\left.\left.\left.-\mathrm{COM}_{6x}C_6\right)\right)\right)L_1m_6-32C_{34}\left(C_3\left(C_4\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)-C_5S_4\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)\right)\right.\\
&\left.-S_3\left(C_4C_5\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)+S_4\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right)\right)L_2m_6\Big)\dot{\theta}_2{}^2\\
&-32\left(S_5\left(-2\mathrm{I}_{6xy}C_3C_4C_5C_6^2+\mathrm{I}_{6xx}C_4S_3C_6^2-\mathrm{I}_{6yy}C_4S_3C_6^2+\mathrm{I}_{6xx}C_3S_4C_6^2-\mathrm{I}_{6yy}C_3S_4C_6^2\right.\right.\\
&+2\mathrm{I}_{6xy}C_5S_3S_4C_6^2-2\mathrm{I}_{6yz}C_3C_4S_5C_6+2\mathrm{I}_{6yz}S_3S_4S_5C_6+2\mathrm{I}_{6xy}C_3C_4C_5S_6^2-\mathrm{I}_{6xx}C_4S_3S_6^2\\
&+\mathrm{I}_{6yy}C_4S_3S_6^2-\mathrm{I}_{6xx}C_3S_4S_6^2+\mathrm{I}_{6yy}C_3S_4S_6^2-2\mathrm{I}_{6xy}C_5S_3S_4S_6^2+\mathrm{I}_{6zz}C_4S_3+\mathrm{I}_{6zz}C_3S_4\\
&-2\mathrm{I}_{6xz}C_3C_4S_5S_6+2\mathrm{I}_{6xz}S_3S_4S_5S_6+\mathrm{I}_{6xx}C_3C_4C_5S_{66}-\mathrm{I}_{6yy}C_3C_4C_5S_{66}+2\mathrm{I}_{6xy}C_4S_3S_{66}\\
&+2\mathrm{I}_{6xy}C_3S_4S_{66}-\mathrm{I}_{6xx}C_5S_3S_4S_{66}+\mathrm{I}_{6yy}C_5S_3S_4S_{66}-2C_3\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)L_1m_6\\
&\left.+2S_{34}\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)L_2m_6\right)\dot{\theta}_3-S_5\left(C_3\left(C_4\left(2S_5\left(\mathrm{I}_{6yz}C_6+\mathrm{I}_{6xz}S_6\right)\right.\right.\right.\\
&\left.\left.+C_5\left(2\mathrm{I}_{6xy}C_{66}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})S_{66}\right)\right)-S_4\left(\mathrm{I}_{6zz}+(\mathrm{I}_{6xx}-\mathrm{I}_{6yy})C_{66}+2\mathrm{I}_{6xy}S_{66}\right)\right)-S_3\left(C_4\left(\mathrm{I}_{6zz}+(\mathrm{I}_{6xx}\right.\right.\\
&\left.\left.\left.-\mathrm{I}_{6yy})C_{66}+2\mathrm{I}_{6xy}S_{66}\right)+S_4\left(2S_5\left(\mathrm{I}_{6yz}C_6+\mathrm{I}_{6xz}S_6\right)+C_5\left(2\mathrm{I}_{6xy}C_{66}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})S_{66}\right)\right)\right)\\
&\left.-2S_{34}\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)L_2m_6\right)\dot{\theta}_4+\left(C_3\left(S_4\left(2\mathrm{I}_{6xy}C_{66}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})S_{66}\right)+C_4\left(2S_5\left(\mathrm{I}_{6yz}S_6\right.\right.\right.\right.\\
&\left.\left.\left.\left.-\mathrm{I}_{6xz}C_6\right)-C_5\left(\mathrm{I}_{6zz}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})C_{66}-2\mathrm{I}_{6xy}S_{66}\right)\right)\right)+S_3\left(C_4\left(2\mathrm{I}_{6xy}C_{66}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})S_{66}\right)\right.\right.\\
&\left.\left.+S_4\left(2S_5\left(\mathrm{I}_{6xz}C_6-\mathrm{I}_{6yz}S_6\right)+C_5\left(\mathrm{I}_{6zz}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})C_{66}-2\mathrm{I}_{6xy}S_{66}\right)\right)\right)\dot{\theta}_5\Big)\dot{\theta}_2+16\left(\left(-2\mathrm{I}_{6xy}C_{66}S_5^2\right.\right.\\
&+\mathrm{I}_{6xx}S_{66}S_5^2-\mathrm{I}_{6yy}S_{66}S_5^2+\mathrm{I}_{6yz}C_6S_{55}+\mathrm{I}_{6xz}S_{55}S_6+2\left(C_4C_5\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)\right.\\
&\left.\left.+S_4\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)\right)L_1m_6-2\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)L_2m_6\right)\dot{\theta}_3{}^2+2\left(\left(-2\mathrm{I}_{6xy}C_{66}S_5^2\right.\right.\\
&\left.+\mathrm{I}_{6xx}S_{66}S_5^2-\mathrm{I}_{6yy}S_{66}S_5^2+\mathrm{I}_{6yz}C_6S_{55}+\mathrm{I}_{6xz}S_{55}S_6-2\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)L_2m_6\right)\dot{\theta}_4\\
&\left.+\left(2C_5\left(\mathrm{I}_{6xz}C_6-\mathrm{I}_{6yz}S_6\right)-S_5\left(\mathrm{I}_{6zz}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})C_{66}-2\mathrm{I}_{6xy}S_{66}\right)\right)\dot{\theta}_5\right)\dot{\theta}_3+\frac{1}{4}\left(4S_5\left(2C_5\left(\mathrm{I}_{6yz}C_6\right.\right.\right.\\
&\left.\left.+\mathrm{I}_{6xz}S_6\right)+S_5\left((\mathrm{I}_{6xx}-\mathrm{I}_{6yy})S_{66}-2\mathrm{I}_{6xy}C_{66}\right)\right)-8\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)L_2m_6\Big)\dot{\theta}_4{}^2+\left(2\mathrm{I}_{6xy}C_{66}\right.\\
&\left.\left.\left.+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})S_{66}\right)\dot{\theta}_5{}^2+2\left(2C_5\left(\mathrm{I}_{6xz}C_6-\mathrm{I}_{6yz}S_6\right)-S_5\left(\mathrm{I}_{6zz}+(\mathrm{I}_{6yy}-\mathrm{I}_{6xx})C_{66}-2\mathrm{I}_{6xy}S_{66}\right)\right)\dot{\theta}_4\dot{\theta}_5\right)\right)\\
&+\left(-\mathrm{I}_{6xz}C_3C_4C_5C_6+\mathrm{I}_{6yz}C_4S_3C_6+\mathrm{I}_{6yz}C_3S_4C_6+\mathrm{I}_{6xz}C_5S_3S_4C_6+\mathrm{I}_{6zz}C_3C_4S_5-\mathrm{I}_{6zz}S_3S_4S_5\right.\\
&+\mathrm{I}_{6yz}C_3C_4C_5S_6+\mathrm{I}_{6xz}C_4S_3S_6+\mathrm{I}_{6xz}C_3S_4S_6-\mathrm{I}_{6yz}C_5S_3S_4S_6+S_3S_5\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)L_1m_6\\
&\left.+C_{34}S_5\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)L_2m_6\right)\ddot{\theta}_2+\left(\mathrm{I}_{6zz}C_5+\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)L_2m_6C_5\right.\\
&\left.+\mathrm{I}_{6xz}C_6S_5-\mathrm{I}_{6yz}S_5S_6+\left(C_4\left(\mathrm{COM}_{6x}C_6-\mathrm{COM}_{6y}S_6\right)-C_5S_4\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)\right)L_1m_6\right)\ddot{\theta}_3\\
&+\left(\mathrm{I}_{6zz}C_5+\left(\mathrm{COM}_{6y}C_6+\mathrm{COM}_{6x}S_6\right)L_2m_6C_5+S_5\left(\mathrm{I}_{6xz}C_6-\mathrm{I}_{6yz}S_6\right)\right)\ddot{\theta}_4-\left(\mathrm{I}_{6yz}C_6+\mathrm{I}_{6xz}S_6\right)\ddot{\theta}_5\\
&+\mathrm{I}_{6zz}\ddot{\theta}_6\Big)
\end{aligned}
\tag{A.5f}
$$

Appendix B

CORRECTIONS AND MODIFICATIONS

- Modified Equation (3.15) from

$$\theta_{4a} = \arcsin\left(L_1^2 + L_2^2 - \frac{\left|P_x^2 + P_y^2 + (P_z - d_1)^2\right|}{2L_1L_2}\right) \tag{B.1a}$$

$$\theta_{4b} = \pi - \arcsin\left(L_1^2 + L_2^2 - \frac{\left|P_x^2 + P_y^2 + (P_z - d_1)^2\right|}{2L_1L_2}\right) \tag{B.1b}$$