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The Shape Jacobian of a Manipulator with Hyper Degrees of Freedom

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Abstract

The Shape Jacobian which is the counterpart of the manipulator Jacobian plays a key role to control the shape of a manipulator with extraordinarily many degrees of freedom. In this paper, we show some significant properties of the Shape Jacobian; the structure, boundedness, determinant and singularity, in geometric flavor.

1 Introduction

A kinematic degrees of freedom of a manipulator is one of the important factors which indicates task capability of the manipulator. For this reason, a manipulator with extraordinarily many degrees of freedom has been considered from all angles [1, 2, 3, 9]. In this paper, such a manipulator is called a *Hyper Degrees of Freedom manipulator* (HDOF manipulator, for short).

The manipulator Jacobian in one of the most important tool in conventional robot control theory. It is defined as the Jacobian matrix of forward kinematics which describes the relation between joint angles and tip position and orientation. When a manipulator has a lot of kinematic degrees of freedom, it is reasonable to control the shape of a manipulator, that is, we need to change our viewpoint from tip control to shape control. In shape control, it is the Shape Jacobian that corresponds to the manipulator Jacobian [3]. In this paper, we show some important properties of the Shape Jacobian.

In Section 2, we give descriptions of a kinematics of an HDOF manipulator and a parameterized spatial curve. In Section 3, we review the definition of the Shape Jacobian proposed by the authors in order to control the shape of a manipulator. In Section 4, we show significant properties of the Shape Jacobian in geometric flavor. In Section 5, we summarize the results in this paper.

2 Preliminaries

Suppose that an HDOF manipulator satisfies the following assumption (Figure 1):

Assumption 1 (HDOF Manipulators)
An HDOF manipulator has

1. a serial rigid chain structure, and
2. 2-degree-of-freedom(2DOF) revolute joints.

□

The above assumption is satisfied by manipulators developed in [6] and [7], [8].

Under Assumption 1, by the coordinate setting method shown in [4], the kinematics of the manipulator can be expressed as

$$\begin{aligned} \boldsymbol{\Phi}_i &= \boldsymbol{\Phi}_{i-1}\boldsymbol{R}_{w,i}, & (1)\\ \boldsymbol{R}_{w,i} &= \boldsymbol{R}(\boldsymbol{a}_{s,i},\theta_{s,i})\boldsymbol{R}(\boldsymbol{a}_{m,i},\theta_{m,i}), & (2)\\ \boldsymbol{p}_i &= \boldsymbol{p}_{i-1}+l_i\boldsymbol{\Phi}_i\boldsymbol{e}_x, \quad , \quad i=1,\cdots,n & (3)\end{aligned}$$

where $\boldsymbol{\Phi}_i \in SO(3)$ is the frame attached to the i-th link, $\boldsymbol{R}_{w,i} \in SO(3)$ is the rotation matrix of the i-th 2DOF revolute joint, $\boldsymbol{R}(\boldsymbol{a},\theta) \in SO(3)$ the rotaion matrix about a unit-length axis $\boldsymbol{a} \in \Re^3$ through an angle $\theta \in [\,-\pi\ \pi)$, $\boldsymbol{a}_{s,i}$, $\boldsymbol{a}_{m,i} \in \Re^3$ are the unit-length and constant rotational axes of the joint, $\theta_{s,i}$, $\theta_{m,i} \in [\,-\pi\ \pi)$ are the rotaional angles of the joint, $\boldsymbol{p}_i \in \Re^3$ is the position of the link, l_i is the constant link length and $\boldsymbol{e}_x := [\,1\ 0\ 0\,]^T$. Define $\boldsymbol{\theta} := [\theta_{s,1}\ \theta_{m,1}\ \cdots \theta_{s,n}\ \theta_{m,n}]^T \in \Re^{2n}$. See [4] for more detail.

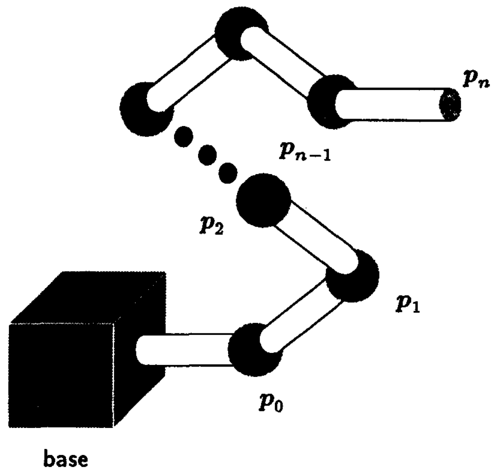

Figure 1: Manipulator with hyper degrees of freedom

In order to prescribe a desired shape for an HDOF manipulator, we use a parametrized spatial curve $c : \Re \rightarrow \Re^3$ with the following assumption.

Assumption 2 (Spatial Curves)
A spatial curve $c : \Re \rightarrow \Re^3$ has the following properties:

1. Mapping c is of class C^2 in $\Re$.
2. Any tangent vectors on a curve are normalized, i.e., $\forall\sigma$, $\left\|\frac{dc}{d\sigma}(\sigma)\right\|_2 = 1$ where $\|\cdot\|_2$ denotes the euclidean vector norm.
3. The curvature of a curve is bounded.

$\square$

Frenet-Serret formula expresses the relationship between a parametrized curve and geometric values, the curvature and torsion, as follows:

$$\frac{d\Phi}{d\sigma}(\sigma) = \Phi(\sigma)\,[\omega(\sigma)\times], \tag{4}$$

$$\omega(\sigma) := \begin{bmatrix} \tau(\sigma) \\ 0 \\ \kappa(\sigma) \end{bmatrix}, \tag{5}$$

$$\frac{dc}{d\sigma}(\sigma) = \Phi(\sigma)e_x, \tag{6}$$

where $\Phi(\sigma) \in SO(3)$ is the Frenet frame at σ, $\kappa : \Re \rightarrow \Re_+$ and $\tau : \Re \rightarrow \Re$ is the curvature and torsion of the curve respectively. For a vector $a = [a_x\ a_y\ a_z]^T \in \Re^3$, $[a\times]$ is defined as

$$[a\times] := \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}. \tag{7}$$

Note that

$$[a\times]\, b = a \times b, \tag{8}$$

where the symbol ' $\times$ ' denotes an outer product in euclidean space.

We make the following assumption on the relation between an HDOF manipulator and a spatial curve:

Assumption 3
The given curve $c : \Re \rightarrow \Re^3$ passes through the position of the base link, p_0, i.e., there exists a real number σ_0 such that $c(\sigma_0) = p_0$ Without loss of generality, we set $\sigma_0 = 0$. $\square$

3 Definition

Define $p(\theta) \in \Re^{3n}$ by arranging all the link positions in a row as

$$p(\theta) := \begin{bmatrix} p_1(\theta) \\ \vdots \\ p_n(\theta) \end{bmatrix}. \tag{9}$$

Vector $p(\theta)$ includes complete information on all the corners of the chain of line segments which expresses the kinematic feature of an HDOF manipulator. Define also $p_d(\sigma) \in \Re^{3n}$ for a given curve $c : \Re \rightarrow \Re^3$ as

$$p_d(\sigma) := \begin{bmatrix} c(\sigma_1) \\ \vdots \\ c(\sigma_n) \end{bmatrix}, \tag{10}$$

where $\sigma_i \in \Re$ $(i = 1, \cdots, n)$ is a curve parameter corresponding to the position of the i-th link, and $\sigma := [\sigma_1 \cdots \sigma_n]^T \in \Re^n$. Further define $e(\theta, \sigma) \in \Re^{3n}$ as the difference between them, i.e.,

$$e(\theta, \sigma) := p(\theta) - p_d(\sigma). \tag{11}$$

We can express that the manipulator forms the shape of a given curve by $e(\theta, \sigma) = \mathbf{o}$. Vector e is interpreted as the shape error between the manipulator and curve †.

† Strictly speaking, $e(\theta, \sigma) = 0$ is a necessary condition in order that an HDOF manipulator is 'fitted' to a curve. See [3] or [5] for more detail.

Suppose that $\boldsymbol{\theta}$ and $\boldsymbol{\sigma}$ are time functions and differentiable with respect to time t in $\Re_+$. Then, the derivative of $\boldsymbol{e}$ becomes

$$\dot{\boldsymbol{e}}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}, \tag{12}$$

where $\boldsymbol{q} := \left[\boldsymbol{\theta}^T \boldsymbol{\sigma}^T\right]^T \in \Re^{3n}$ is called the extended joint angles and $\boldsymbol{J}(\boldsymbol{q}) \in \Re^{3n\times 3n}$ is the *Shape Jacobian* which is defined as

$$\boldsymbol{J}(\boldsymbol{q}) := \left[\begin{array}{cc} \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{\theta}}(\boldsymbol{\theta}) & -\frac{\partial \boldsymbol{p}_d}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}) \end{array}\right]. \tag{13}$$

The Shape Jacobian contains information on a given curve. Due to this, the Shape Jacobian becomes square. That is a key that we can obtain important properties of the Shape Jacobian we will see next.

4 Properties

First, we see an essential structure of the Shape Jacobian. Next, boundedness of the Shape Jacobian is considered. Then, by use of the structure, the determinant and singularity are analyzed.

4.1 Lower Triangular Structure

Let $\boldsymbol{I}_{i,j} \in \Re^{3n\times 3n}$ be the elementary matrix exchanging the i-th line for the j-th. This matrix has properties such that

$$\begin{aligned} \boldsymbol{I}_{i,j}{}^{-1} &= \boldsymbol{I}_{i,j}{}^{T}, & (14)\\ \det \boldsymbol{I}_{i,j} &= -1. & (15) \end{aligned}$$

Using this elementary matrix, define $\boldsymbol{P} \in \Re^{3n\times 3n}$ as

$$\begin{aligned} \boldsymbol{P} &:= (\boldsymbol{I}_{2n,2n+1}\boldsymbol{I}_{2n-1,2n}\cdots \boldsymbol{I}_{3,4})\\ &\quad \cdot(\boldsymbol{I}_{2n+1,2n+2}\boldsymbol{I}_{2n,2n+1}\cdots \boldsymbol{I}_{6,7})\\ &\quad \cdots(\boldsymbol{I}_{2n+i-1,2n+i}\boldsymbol{I}_{2n+i-2,2n+i-1}\cdots \boldsymbol{I}_{3i,3i+1})\\ &\quad \cdots(\boldsymbol{I}_{3n-2,3n-1}\boldsymbol{I}_{3n-3,3n-2})\\ &= \prod_{i=1}^{n-1}\prod_{j=1}^{2(n-i)} \boldsymbol{I}_{2n+i-j,2n+i-j+1}. \qquad (16) \end{aligned}$$

The number of the elementary matrices appeared in the definition $\boldsymbol{P}$ is

$$\sum_{i=1}^{n-1} 2(n-i) = 2\left\{\sum_{i=1}^{n-1}(n-i)\right\}, \tag{17}$$

which is always even. Thus, $\boldsymbol{P}$ has properties:

$$\begin{aligned} \boldsymbol{P}^{-1} &= \boldsymbol{P}^{T}, & (18)\\ \det \boldsymbol{P} &= 1, & (19) \end{aligned}$$

i.e., $\boldsymbol{P} \in SO(3n)$. Furthermore, using this $\boldsymbol{P}$, define $\bar{\boldsymbol{J}}(\boldsymbol{q}) \in \Re^{3n\times 3n}$ as

$$\bar{\boldsymbol{J}}(\boldsymbol{q}) := \boldsymbol{J}(\boldsymbol{q})\boldsymbol{P}. \tag{20}$$

Then, $\bar{\boldsymbol{J}}(\boldsymbol{q})$ can be represented as the following lower-triangular 3×3-block matrix:

$$\bar{\boldsymbol{J}} = \begin{bmatrix} \boldsymbol{J}_{11} & & & \\ \boldsymbol{J}_{21} & \boldsymbol{J}_{22} & & \mathbf{O} \\ \vdots & \vdots & \ddots & \\ \boldsymbol{J}_{n1} & \boldsymbol{J}_{n2} & \cdots & \boldsymbol{J}_{nn} \end{bmatrix}, \tag{21}$$

where $\boldsymbol{J}_{ij} \in \Re^{3\times 3}$ is defined as

$$\boldsymbol{J}_{ij} := \begin{cases} \left[\begin{array}{ccc} \frac{\partial \boldsymbol{p}_i}{\partial \theta_{s,j}}(\boldsymbol{\theta}) & \frac{\partial \boldsymbol{p}_i}{\partial \theta_{m,j}}(\boldsymbol{\theta}) & -\frac{d\boldsymbol{c}}{d\sigma}(\sigma_i) \end{array}\right], & i = j, \\ \left[\begin{array}{ccc} \frac{\partial \boldsymbol{p}_i}{\partial \theta_{s,j}}(\boldsymbol{\theta}) & \frac{\partial \boldsymbol{p}_i}{\partial \theta_{m,j}}(\boldsymbol{\theta}) & \mathbf{o} \end{array}\right], & i > j. \end{cases} \tag{22}$$

Property 1 (structure)
The Shape Jacobian can be expressed as a 3×3-block lower-triangular matrix by an even-numbered sorting of its columns. □

It is important to note that partial derivatives of link positions in joint angles are represented by the outer products of vectors as

$$\begin{aligned} \frac{\partial \boldsymbol{p}_i}{\partial \theta_{s,j}} &= (\boldsymbol{\Phi}_{j-1}\boldsymbol{a}_{s,j}) \times (\boldsymbol{p}_i - \boldsymbol{p}_{j-1}), & (23)\\ \frac{\partial \boldsymbol{p}_i}{\partial \theta_{m,j}} &= (\boldsymbol{\Phi}_{j}\boldsymbol{a}_{m,j}) \times (\boldsymbol{p}_i - \boldsymbol{p}_{j-1}). & (24) \end{aligned}$$

4.2 Boundedness

Let $\|\cdot\|_2$ denote the euclidean norm for vectors, or the spectral norm for matrices (i.e., the matrix norm induced by the euclidean vector norm). The spectral norm of the Shape Jacobian is evaluated as

$$\begin{aligned} \|\boldsymbol{J}(\boldsymbol{q})\|_2 &\le \sqrt{3n}\,\|\boldsymbol{J}(\boldsymbol{q})\|_1\\ &= \sqrt{3n}\max\left\{ \max_{x,j}\sum_{i=j}^{n}\left\|\frac{\partial \boldsymbol{p}_i}{\partial \theta_{x,j}}(\boldsymbol{\theta})\right\|_1, \right.\\ &\qquad \left. \max_{j}\left\|-\frac{d\boldsymbol{c}}{d\sigma}(\sigma_j)\right\|_1 \right\}, \qquad (25) \end{aligned}$$

where $x \in \{s, m\}$ and $\|\cdot\|_1$ means the 1-norm for vectors or its induced matrix norm. From (23) and

(24), we obtain

$$\begin{aligned}\left\|\frac{\partial \boldsymbol{p}_i}{\partial \theta_{x,j}}(\boldsymbol{\theta})\right\|_1 &\le \sqrt{3}\left\|\frac{\partial \boldsymbol{p}_i}{\partial \theta_{x,j}}(\boldsymbol{\theta})\right\|_2 \\ &\le \sqrt{3}\left\|\boldsymbol{p}_i - \boldsymbol{p}_{j-1}\right\|_2 \\ &\le \sqrt{3}\sum_{k=j}^{i} l_k. \qquad (26)\end{aligned}$$

Assumption 2-2 allows us to have the following inequality:

$$\begin{aligned}\left\|-\frac{dc}{d\sigma}(\sigma_i)\right\|_1 &\le \sqrt{3}\left\|\frac{dc}{d\sigma}(\sigma_i)\right\|_2 \\ &\le \sqrt{3}. \qquad (27)\end{aligned}$$

Therefore,

$$\begin{aligned}\|\boldsymbol{J}(\boldsymbol{q})\|_2 &= \sqrt{3n}\max\left\{\max_j \sum_{i=j}^{n}\left(\sqrt{3}\sum_{k=j}^{i} l_k\right), \sqrt{3}\right\} \\ &= 3\sqrt{n}\max\left\{\max_j \sum_{i=j}^{n}\sum_{k=j}^{i} l_k, 1\right\} \\ &\le 3\sqrt{n}\max\left\{\sum_{i=1}^{n}\sum_{k=1}^{i} l_k, 1\right\}. \qquad (28)\end{aligned}$$

Next we see the bound of the time-derivative of the Shape Jacobian which is defined as

$$\begin{aligned}&\dot{\boldsymbol{J}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ &:= \frac{d}{dt}\boldsymbol{J}(\boldsymbol{q}) \\ &= \Big[\ \frac{\partial \boldsymbol{J}}{\partial \theta_{s,1}}(\boldsymbol{q})\dot{\boldsymbol{q}}\ \ \frac{\partial \boldsymbol{J}}{\partial \theta_{m,1}}(\boldsymbol{q})\dot{\boldsymbol{q}}\ \ \cdots\ \ \frac{\partial \boldsymbol{J}}{\partial \theta_{s,n}}(\boldsymbol{q})\dot{\boldsymbol{q}} \\ &\quad \frac{\partial \boldsymbol{J}}{\partial \theta_{m,n}}(\boldsymbol{q})\dot{\boldsymbol{q}}\ \ \frac{\partial \boldsymbol{J}}{\partial \sigma_1}(\boldsymbol{q})\dot{\boldsymbol{q}}\ \cdots\ \frac{\partial \boldsymbol{J}}{\partial \sigma_n}(\boldsymbol{q})\dot{\boldsymbol{q}}\ \Big]. \qquad (29)\end{aligned}$$

The spectral norm can be evaluated as

$$\begin{aligned}\left\|\dot{\boldsymbol{J}}(\boldsymbol{q}, \dot{\boldsymbol{q}})\right\|_2 &\le \sqrt{3n}\left\|\dot{\boldsymbol{J}}(\boldsymbol{q}, \dot{\boldsymbol{q}})\right\|_1 \\ &= \sqrt{3n}\max\Big\{\ \max_{k,x}\left\|\frac{\partial \boldsymbol{J}}{\partial \theta_{x,k}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right\|_1, \\ &\qquad \max_k\left\|\frac{\partial \boldsymbol{J}}{\partial \sigma_k}(\boldsymbol{q})\dot{\boldsymbol{q}}\right\|_1\ \Big\} \\ &\le 3n\max\Big\{\ \max_{k,x}\left\|\frac{\partial \boldsymbol{J}}{\partial \theta_{x,k}}(\boldsymbol{q})\right\|_1, \\ &\qquad \max_k\left\|\frac{\partial \boldsymbol{J}}{\partial \sigma_k}(\boldsymbol{q})\right\|_1\ \Big\}\|\dot{\boldsymbol{q}}\|_2. \qquad (30)\end{aligned}$$

By differentiating (23) and (24), the second-order partial derivatives of $\boldsymbol{p}_i$ with respect to joint angles can be evaluated as

$$\left\|\frac{\partial^2 \boldsymbol{p}_j}{\partial \theta_{y,k}\partial \theta_{x,i}}(\boldsymbol{\theta})\right\|_1 \le 2\sqrt{3}\sum_{p=j}^{i} l_p, \qquad (31)$$

Thus, we obtain

$$\begin{aligned}&\max_{k,x}\left\|\frac{\partial \boldsymbol{J}}{\partial \theta_{x,k}}(\boldsymbol{q})\right\|_1 \\ &= \max_{j,k,x,y}\sum_{i=\max\{j,k\}}^{n}\left\|\frac{\partial^2 \boldsymbol{p}_j}{\partial \theta_{y,k}\partial \theta_{x,i}}(\boldsymbol{\theta})\right\|_1 \\ &\le 2\sqrt{3}\max_{j,k}\sum_{i=\max\{j,k\}}^{n}\sum_{p=j}^{i} l_p \\ &\le 2\sqrt{3}\sum_{i=1}^{n}\sum_{p=1}^{i} l_p. \qquad (32)\end{aligned}$$

By Frenet-Serret formula and Assumption 2-3, we obtain

$$\begin{aligned}\max_k\left\|\frac{\partial \boldsymbol{J}}{\partial \sigma_k}(\boldsymbol{q})\right\|_1 &= \max_k\left\|-\frac{d^2\boldsymbol{c}}{d\sigma^2}(\sigma_k)\right\|_1 \\ &\le \max_k \sqrt{3}\left\|\kappa(\sigma_k)\boldsymbol{\Phi}(\sigma_k)e_y\right\|_2 \\ &\le \sqrt{3}\sup_\sigma \kappa(\sigma). \qquad (33)\end{aligned}$$

Therefore,

$$\begin{aligned}&\left\|\dot{\boldsymbol{J}}(\boldsymbol{q}, \dot{\boldsymbol{q}})\right\|_2 \\ &\le 3\sqrt{3}n\max\Big\{\ 2\sum_{i=1}^{n}\sum_{p=1}^{i} l_p, \sup_\sigma \kappa(\sigma)\ \Big\}\|\dot{\boldsymbol{q}}\|_2. \qquad (34)\end{aligned}$$

Property 2 (boundedness)
The spectral norm of the Shape Jacobian is bounded from a positive real constant related to the link length and the number of degrees of freedom, i.e.,

$$\forall \boldsymbol{q} \in \Re^{3n} \quad \|\boldsymbol{J}(\boldsymbol{q})\|_2 \le J_M, \qquad (35)$$

where

$$J_M := 3\sqrt{n}\max\left\{\sum_{i=1}^{n}\sum_{k=1}^{i} l_k, 1\right\}. \qquad (36)$$

Moreover, the spectral norm of the time-derivative of the Shape Jacobian, $\dot{\boldsymbol{J}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) := \frac{d}{dt}\{\boldsymbol{J}(\boldsymbol{\theta})\}$, is bounded from the product of the spectral norm of the extended joint angular velocities $\dot{q}$ and a positive real constant related to the link length, the number of degrees of freedom and the supremum of curvature of a given curve, i.e.,

$$\forall q, \dot{q} \in \Re^{3n} \quad \|\dot{J}(q, \dot{q})\| \leq J_H \|\dot{q}\|, \tag{37}$$

where

$$J_H := 3\sqrt{3} n \max \left\{ 2 \sum_{i=1}^{n} \sum_{p=1}^{i} l_p, \sup_{\sigma} \kappa(\sigma) \right\}. \tag{38}$$

□

4.3 Determinant

The lower-triangular structure of the Shape Jacobian allows us to calculate its determinant as follows:

$$\begin{aligned}
&\det \boldsymbol{J}(q) \\
&= \det \bar{\boldsymbol{J}}(q) \\
&= \det \begin{bmatrix} \boldsymbol{J}_{11} & & & \\ \boldsymbol{J}_{21} & \boldsymbol{J}_{22} & & \\ \vdots & \vdots & \ddots & \\ \boldsymbol{J}_{n1} & \boldsymbol{J}_{n2} & \cdots & \boldsymbol{J}_{nn} \end{bmatrix} \\
&= \prod_{i=1}^{n} \det \boldsymbol{J}_{ii} \\
&= \prod_{i=1}^{n} \det \left[\frac{\partial \boldsymbol{p}_i}{\partial \theta_{s,i}} \quad \frac{\partial \boldsymbol{p}_i}{\partial \theta_{m,i}} \quad -\frac{dc}{d\sigma}(\sigma_i) \right] \\
&= \prod_{i=1}^{n} \left\{ \left(\frac{\partial \boldsymbol{p}_i}{\partial \theta_{m,i}} \times \frac{\partial \boldsymbol{p}_i}{\partial \theta_{s,i}} \right)^T \frac{dc}{d\sigma}(\sigma_i) \right\}.
\end{aligned}$$

Taking $j = i$ in (23) and (24), the outer product of the partial derivatives of $\boldsymbol{p}_i$ with respect to $\theta_{s,i}$ and $\theta_{m,i}$ is calculated as

$$\begin{aligned}
&\frac{\partial \boldsymbol{p}_i}{\partial \theta_{m,i}} \times \frac{\partial \boldsymbol{p}_i}{\partial \theta_{s,i}} \\
&= \{ \boldsymbol{\Phi}_i \boldsymbol{a}_{m,i} \times (\boldsymbol{p}_i - \boldsymbol{p}_{i-1}) \} \times \\
&\quad \{ \boldsymbol{\Phi}_{i-1} \boldsymbol{a}_{s,i} \times (\boldsymbol{p}_i - \boldsymbol{p}_{i-1}) \} \\
&= \det [\ \boldsymbol{\Phi}_i \boldsymbol{a}_{m,i} \quad (\boldsymbol{p}_i - \boldsymbol{p}_{i-1}) \quad (\boldsymbol{p}_i - \boldsymbol{p}_{i-1})\ ] \\
&\quad \boldsymbol{\Phi}_{i-1} \boldsymbol{a}_{s,i} \\
&\quad - \det [\ \boldsymbol{\Phi}_i \boldsymbol{a}_{m,i} \quad (\boldsymbol{p}_i - \boldsymbol{p}_{i-1}) \quad \boldsymbol{\Phi}_{i-1} \boldsymbol{a}_{s,i}\ ] \\
&\quad (\boldsymbol{p}_i - \boldsymbol{p}_{i-1}) \\
&= \det [\ (\boldsymbol{p}_i - \boldsymbol{p}_{i-1}) \quad \boldsymbol{\Phi}_i \boldsymbol{a}_{m,i} \quad \boldsymbol{\Phi}_{i-1} \boldsymbol{a}_{s,i}\ ] \\
&\quad (\boldsymbol{p}_i - \boldsymbol{p}_{i-1}) \\
&= \det [\ l_i \boldsymbol{e}_x \quad \boldsymbol{a}_{m,i} \quad \boldsymbol{R}^T(\boldsymbol{a}_{m,i}, \theta_{m,i}) \boldsymbol{a}_{s,i}\ ] \\
&\quad (\boldsymbol{p}_i - \boldsymbol{p}_{i-1}).
\end{aligned} \tag{39}$$

Therefore, we obtain

$$\begin{aligned}
&\det \boldsymbol{J}(q) \\
&= \prod_{i=1}^{n} \Big\{ \det [\ l_i \boldsymbol{e}_x \quad \boldsymbol{a}_{m,i} \quad \boldsymbol{R}^T(\boldsymbol{a}_{m,i}, \theta_{m,i}) \boldsymbol{a}_{s,i}\ ] \\
&\quad (\boldsymbol{p}_i - \boldsymbol{p}_{i-1})^T \frac{dc}{d\sigma}(\sigma_i) \Big\},
\end{aligned} \tag{40}$$

Property 3 (determinant)
The determinant of the Shape Jacobian can be expressed as

$$\det \boldsymbol{J}(q) = \prod_{i=1}^{n} \{ v_i(\theta_{m,i}) w_i(\boldsymbol{\theta}, \sigma_i) \}, \tag{41}$$

where

$$v_i(\theta_{m,i}) := \det [\ l_i \boldsymbol{e}_x \quad \boldsymbol{a}_{m,i} \quad \boldsymbol{R}^T(\boldsymbol{a}_{m,i}, \theta_{m,i}) \boldsymbol{a}_{s,i}\ ], \tag{42}$$

$$w_i(\boldsymbol{\theta}, \sigma_i) := (\boldsymbol{p}_i - \boldsymbol{p}_{i-1})^T \frac{dc}{d\sigma}(\sigma_i). \tag{43}$$

□

Note that v_i and w_i have geometric meanings. The value $|v_i|$ can be interpreted as the volume of the parallelopiped with edges $l_i \boldsymbol{e}_x$, $\boldsymbol{a}_{m,i}$ and $\boldsymbol{R}^T(\boldsymbol{a}_{m,i}, \theta_{m,i}) \boldsymbol{a}_{s,i}$. The value w_i is the inner product of difference vector, $(\boldsymbol{p}_i - \boldsymbol{p}_{i-1})$, and tangent of a given curve at σ_i, $\frac{dc}{d\sigma}(\sigma_i)$.

4.4 Singularity

Since the determinant of the Shape Jacobian has a product form, we immediately obtain useful expression of the singularity by solving equation $\det \boldsymbol{J}(\boldsymbol{\theta}) = 0$.

Property 4 (Singularity)
The Shape Jacobian is singular if and only if there exists a positive integer, $i \in \{1, \cdots, n\}$, such that at least one of the following two conditions holds:

1. Consider the three directions in euclidean space; the directions of the length of the i-th link and the two rotational axes of the i-th joint. At least two among the directions align, i.e.,

$$\det [\ \boldsymbol{e}_x \quad \boldsymbol{a}_{m,i} \quad \boldsymbol{R}^T(\boldsymbol{a}_{m,i}, \theta_{m,i}) \boldsymbol{a}_{s,i}\ ] = 0. \tag{44}$$

2. The directions of the i-th link length and the tangent at the point corresponding to the i-th link position are orthogonal, i.e.,

$$e_x{}^T \left(\Phi_i{}^T \frac{dc}{d\sigma}(\sigma_i) \right) = 0. \qquad (45)$$

□

There are three remarks on this property. First, the singularity condition of the Shape Jacobian can be described by completely separated n conditions for each link, joint and curve parameter. This is very helpful for the calculation. Second, each completely separated condition is further devided into two distinguished parts. One part, described by (44), is only related to the mechanical structure and joint angles, while the other, (45), depends on the tangent of the curve. Finally, the derived conditions (44) and (45) have geometric meanings in euclidean space. That is, they can be explained by the geometric relation of vectors, inner and outer products in euclidean space. The first is alignment condition among three vectors, e_x, $a_{m,i}$ and $R^T(a_{m,i}, \theta_{m,i})a_{s,i}$. Note that these vectors correspond to the directions of the link and the two rotational axes of the joint respectively (see Figure 2). The second is orthogonality condition between two vectors, e_x and $\Phi_i{}^T \frac{dc}{d\sigma}(\sigma_i)$, which are coresspond to the link length direction and the tangent on a curve at σ_i.

5 Conclusion

In this paper, we have shown the following properties of the Shape Jacobian:

1. The Shape Jacobian has a structure of lower-triangular of 3×3-block matricies by appropriate sorting of the columns.

2. The spectral norm of the Shape Jacobian is bounded from a positive constant. Moreover, the norm of the time-derivative is also bounded from the product of the norm of the extended joint anglular velocities and a positive constant.

3. The determinant of the Shape Jacobian consists of the product of the signed volume of a parallelopiped and an inner product of vectors.

4. The singularity of the Shape Jacobian can be interpreted as an alignment and an orthogonality of vectors.

These properties of the Shape Jacobian are very useful for the dynamics-based shape control of HDOF manipulators, which can be seen in [3].

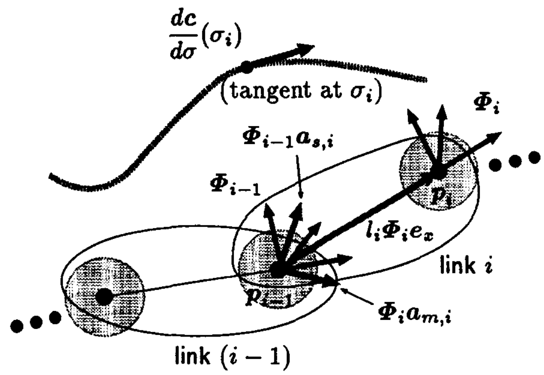

Figure 2: Geometric meanings of singularity

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