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ABSTRACT

In the compromise model of continuous opinions proposed by Deffuant et al., the states of two agents in a network can start to converge if they are neighbors and if their opinions are sufficiently close to each other, below a given threshold of tolerance ϵ. In directed networks, if agent i is a neighbor of agent j, j need not be a neighbor of i. In Watts–Strogatz networks we performed simulations to find the averaged number of final opinions $\langle F\rangle$ and their distribution as a function of ϵ and of the network structural disorder. In directed networks $\langle F\rangle$ exhibits a rich structure, being larger than in undirected networks for higher values of ϵ, and smaller for lower values of ϵ.

1. Introduction

In many social, biological, and economic systems there are complex networks that include directed links acting only in one direction, outwards or inwards. That is, there are processes in which agents do not act in a symmetrical way. Therefore, studying and comparing different dynamics in similar undirected (UN) and directed networks (DN) is an important task. In social systems such as families, clans, schools, etc. there exist directed or asymmetrical processes like copying or imitating, giving orders, teaching, etc. Similar mechanisms occur in automatized or control systems regarding flux of information or signals.

We can visualize asymmetric cases with arrows pointing in only one direction to indicate the lack of reciprocity or bilaterality. Only one direction instead of two is equivalent to cutting links. Therefore, when some agents are chosen in a network, it is possible that they remain totally passive, i.e. do not change their state.

Our network models are based on the Watts–Strogatz structure [1] for small-world phenomena, where in DN there is only one link instead of bi-directional double links connecting the same two vertices or nodes in the corresponding UN.

We will now briefly mention some representative works on directed networks. Sanchez et al. [2] investigated the effect of directed links on the behavior of a simple spin-like model evolving on a small-world network, leading to a phase diagram including first- and second-order phase transitions out of equilibrium. The majority-vote model has been studied with noise on directed random graphs [3]. Inspired by the food web theory in ecosystems, Morelli [4] studied the fraction of basal, top nodes and node level distributions in directed networks based on the Watts–Strogatz model, gave analytical expressions for the fraction of basal and top nodes for the model, and studied the node level distributions with numerical simulations. When a naive spreading process starts in a directed network, the interplay of shortcuts and unfavorable bonds on the small-world properties is studied in simple models of small-world networks with directed links [5], leading to general results for small-world networks with directed links.

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On the other hand, convergence (or divergence) of ideas or opinions among participants of a debate is a very important social process. In practice, in opinion models it is plausible to assume that discussions take place when the opinions of the people involved are sufficiently close to each other, a process called bounded confidence. Then agents can negotiate their difference to try to reach consensus, or at least, to share similar, if not equal, opinions. In the Deffuant et al. model [6] (hereafter referred to, for simplicity, as Deffuant model), a simple opinion dynamics was proposed for the full graph, which we employ here in both UN and DN within the Watts–Strogatz model. Some models of opinion dynamics, including Deffuant model, have been reviewed in Ref. [7,8]. We now mention some papers based on this model. The effect of varying the number of peers met at one time, for different population sizes, and the effects of changing the self-support in the Deffuant model was studied by Urbig et al. [9]. Huet et al. [10] added a rejection mechanism into a 2D bounded confidence model based on the Deffuant model, yielding metastable clusters, which maintain themselves through opposite influences of competitor clusters. The Deffuant model has been studied also in scale free network topology [11–15]. Guo and Cai [11] took into account the heterogeneous distribution of connectivity degree in the Deffuant model where the convergence parameter of an agent is a function of the degree of its interaction partner, and in particular, they assumed that convergence is faster in agents interacting with more connected or "famous" agents ("celebrity" effect). Within the Deffuant model Groeber et al. [16] combined an agent's behavior and the mean behavior of her in-group (agent's past interaction partners in her neighborhood) to foster consensus and to yield new local clusters or cultures. Within the same spirit of incorporating the states of each agent's neighborhood, the transition from invasion to coexistence in nonlinear voter models has been investigated [17] to yield three regimes: complete invasion; random coexistence; and correlated coexistence. In Ref. [18], a node will convert to its opposite opinion, if it is in the local minority opinion, but in contrast with the majority-voter model, the opinion of each node itself is included with its neighbors. Then, because of the clustering (community support) of agents holding the same opinion, these clusters cannot be invaded by the other opinion, analogously to incompressible fluids, a fact that allows us to map this opinion clustering behavior to a known physics percolation problem [19]. In a discrete Deffuant model on a directed Barabasi–Albert, it is shown in Ref. [13] that it is difficult to reach absolute consensus. In a similar network [14], a multi-layer model representing various age levels was employed and advertising effects were included. In the Deffuant model, extremists were defined [20] as individuals with a very low uncertainty in their opinion states, and located at the extremes of the initial opinion distribution. In Ref. [21], the role of network topology on extremism propagation in small-world networks was investigated and a drift to a single extreme appeared only beyond a critical level of connectivity, which decreases when the randomness increases. Assmann [22] investigated agents with different random and systematic opinion qualities in the Deffuant model.

In this paper, we investigate the effect of directed links on the behavior of a simple continuous opinion model, the Deffuant model, evolving on disordered networks. Our simulations could help understand opinion formation in networks with less number of links due to lack of reciprocity and disordered heterogeneity. After reviewing the two main elements of our model; the dynamics of Deffuant model and secondly, the directed networks, here we combine them to simulate the Deffuant model in *directed* Watts–Strogatz networks for small-world phenomena. In Section 2 we present the general opinion dynamics in the Deffuant model and the construction of the directed network (topology). Section 3 is devoted to results and discussion of the differences of the Deffuant model in UN and DN regarding clustering, convergence time, distribution of final opinions and conservation of opinion. In Section 4 we present our summary and conclusions.

2. The model

In the Deffuant model [6], a simple opinion dynamics was proposed for the fully connected graph, where all nodes are connected among them (or any agent or node has all nodes as neighbors). Here we apply the notion of neighbors connected by (bi-directional or uni-directional) links. In the case of directed networks j is neighbor of i only if there exists a link from i to j. The dynamics is summarized as follows: On any issue, each opinion is represented by a continuous number x chosen between 0.0 and 1.0, without loss of generality. One selects an agent i and then with the same probability for all its neighbors (if there are any) one selects one of them called j.

Then between the selected neighbors, if the difference of the opinions $x_i(t)$ and $x_j(t)$ exceeds the threshold, openness or tolerance, ϵ, nothing happens (here t is a *dynamical* time that labels and orders the time steps of the iterative process), but if, $|x_i(t) - x_j(t)| < \epsilon$, then:

$$\begin{aligned} x_i(t+1) &= x_i(t) + \mu[x_j(t) - x_i(t)], \\ x_j(t+1) &= x_j(t) + \mu[x_i(t) - x_j(t)]. \end{aligned} \tag{1}$$

which means that their opinions get closer, as measured by the parameter $\mu(\mu \in [0, \frac{1}{2}])$, This process can be labeled as a "negotiation".

This procedure is repeated until all opinion values do not change, and then it is said that convergence is reached. It is usual to use the same approaching parameter μ for all agents. If $\mu = \frac{1}{2}$ then both opinions $x_i(t)$ and $x_j(t)$ become the same in one step at their midpoint and consensus of the two agents involved is then very quick and complete. Obviously the underlying topology of the system defines the neighborhood of every agent. When convergence is reached, after employing iteratively the Eq. (1) procedure, we define a cluster as the set of nodes sharing the same final opinions.

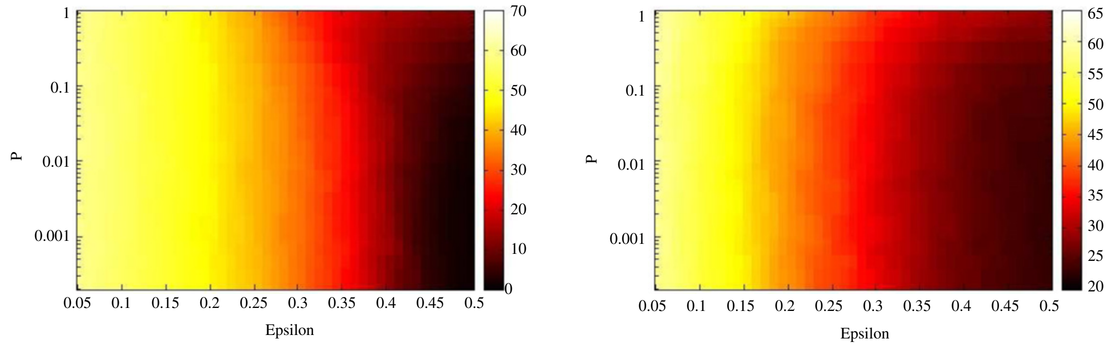

Fig. 1. Final opinions averaged over 100 realizations. $\langle F_{UN}\rangle$ (UN) on the left panel and $\langle F_{DN}\rangle$ (DN) on the right panel, for $N = 100$, $k = 2$.

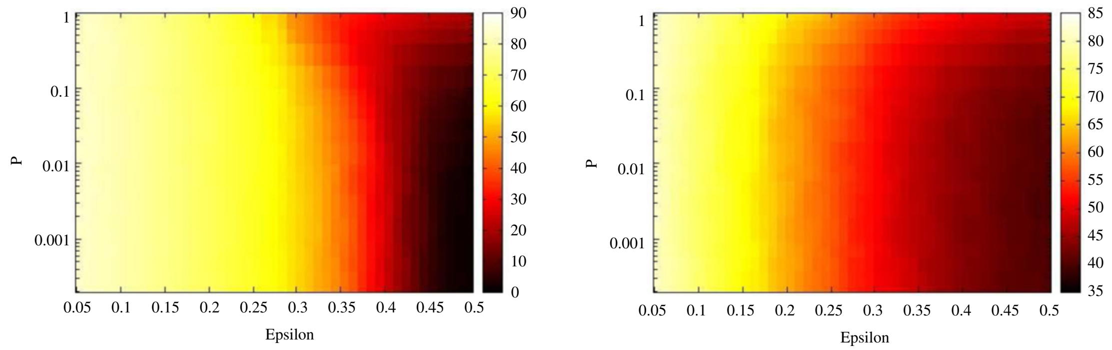

Fig. 2. Final opinions averaged over 100 realizations. $\langle F_{UN}\rangle$ on the left panel and $\langle F_{DN}\rangle$ on the right panel, for $N = 200$, $k = 2$.

A common procedure to create a disordered network, known as the Watts and Strogatz model (WSM) [1], is to start from a regular structure with N nodes (in our case nodes on a circle connected with nearest neighbors) of connectivity k, and then remove each link with probability p, reconnecting it at random. The random rewiring process introduces pNk long links which connect nodes that otherwise were connecting only nearest neighbors. The resulting network from the above procedure is an undirected (or more properly called in this case bi-directional) network, because each link connects or acts in both directions. For the DN case each bi-directional link is reemplaced by a line with only one arrow, whose direction is randomly selected. Therefore, if directedness (or presence of arrows) is not taken into account, the directed network (DN) has exactly the same structure as the undirected one (UN).

3. Results and discussion

In Figs. 1 and 2 we show the number of final states $\langle F\rangle$ for networks with $N = 100$ and $N = 200$, respectively, averaged over 100 realizations. We notice that in both UN and DN, $\langle F\rangle$ exhibits a much stronger dependence on the tolerance ϵ than on the disorder parameter p. As in any bounded confidence model, increasing ϵ leads to a monotonous decrease in $\langle F\rangle$, an effect that is expected to be more pronounced in the case of the UN, due to a larger (double) number of links that lead to a more refined convergence process. In contrast, there is a very weak dependence of $\langle F\rangle$ on p in both UN and DN. To compare the results of Figs. 1 and 2, in Fig. 3 we plot the difference of final opinions $\langle F_{DN}\rangle - \langle F_{UN}\rangle$ for $N = 100$ and $N = 200$, which present the same qualitative behavior. As mentioned above, this difference as function of ϵ is expected to be positive and large due to larger number of links in the UN. This is the case for large values of ϵ, but surprisingly, the difference exhibits small negative values for lower values of ϵ.

To understand this behavior we get more detailed information by calculating the time evolution of the system and distributions of the final opinions as a function of ϵ and p. We plot in Fig. 4 the time evolution of the number of final opinions $\langle F\rangle$ averaged over 100 realizations for a set of different parameters. Here each time unit is defined in terms of the system size as N dynamical steps or sequential loops of Eq. (1). Notice that the convergence time is much larger in the UN than in the DN, and this difference grows with ϵ. The presence of less number of links in the DN cause a much faster convergence, whereas the presence of disorder slightly affects the convergence time in both UN and DN.

In Fig. 5 (UN) and Fig. 6 (DN) we show examples of final opinion distributions in the interval [0,1] for two different values of tolerance, $\epsilon = 0.15$ and 0.4. Here we show results for $N = 1000$, but the same results are obtained for other network sizes, like $N = 200$ and $N = 500$.

Results in UN exhibit a more smoother behavior as compared with those of the DN (with larger number of links), in the sense that the DN curves show a more regular and symmetrical behavior in [0,1] due to slower and better convergence.

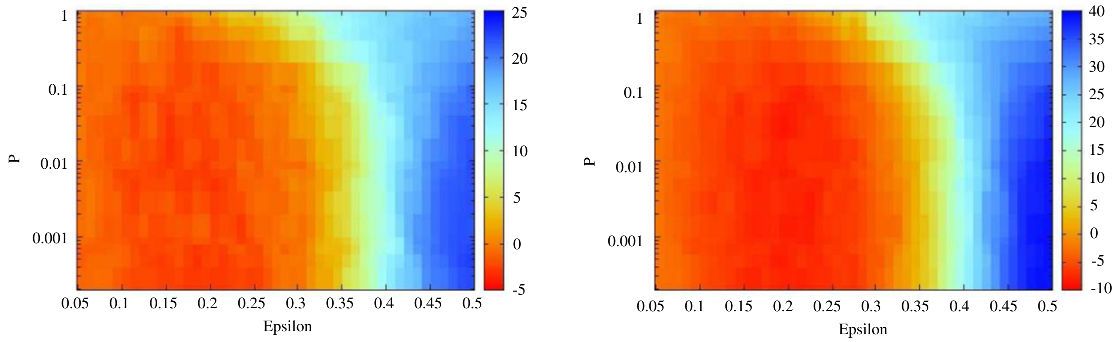

Fig. 3. $\langle F_{DN} \rangle - \langle F_{UN} \rangle$. For $N = 100$ on the left panel and $N = 200$ on the right panel, averaged over 100 realizations and $k = 2$.

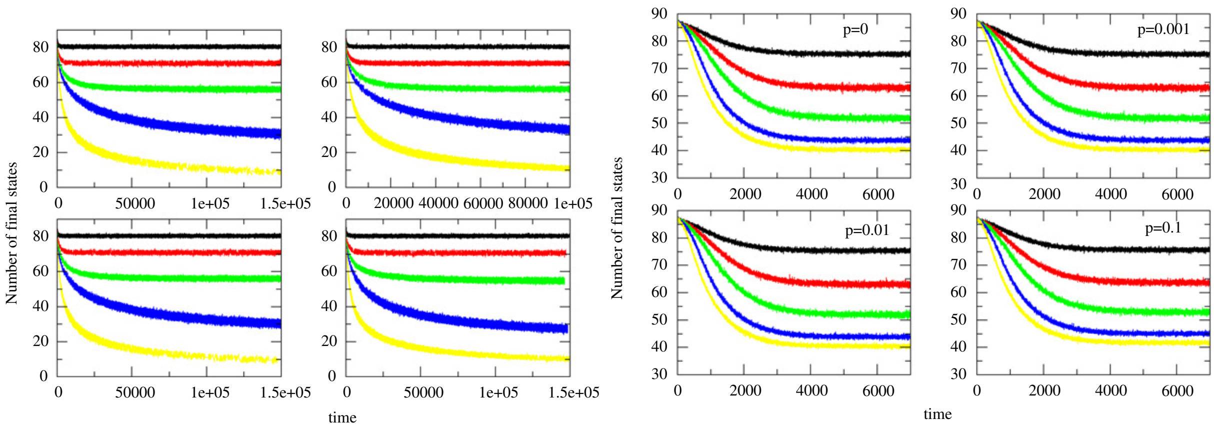

Fig. 4. Time evolution of the number of final opinions, for $N = 200$, $k = 20$, averaged over 100 realizations. UN on the left panel and DN on the right panel. For disorder values of $p = 0$ (top left), $p = 0.001$ (top right), $p = 0.01$ (lower left) and $p = 0.1$ (lower right). In each graph black corresponds to $\epsilon = 0.1$, red $\epsilon = 0.2$, green $\epsilon = 0.3$, blue $\epsilon = 0.4$ and yellow $\epsilon = 0.5$ (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

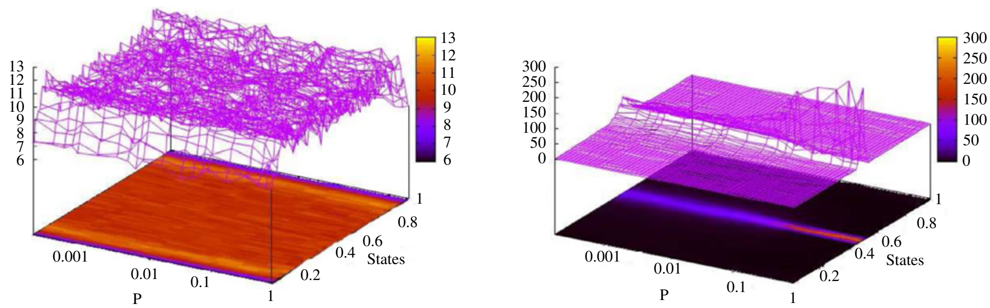

Fig. 5. Distribution of final opinions for UN networks, $N = 1000$, averaged over 100 realizations, for $\epsilon = 0.15$ on the left panel and $\epsilon = 0.4$ on the right panel.

As expected, for high values of tolerance, the system converges to only one cluster with the average value of all states, but surprisingly for small values of the tolerance, clusters with opinion values closer to the opinion edges 0 and 1 interact less than those near the middle, that have neighbors with similar opinions at both their right and left flanks. This effect is more pronounced in the DN, for which even some clusters are lost in the middle of [0,1] for values of ϵ less than 0.3, approximately. If we roughly divide the interval [0,1] into three sections, it is the middle section that tends to be emptied because there always exist opinions at both right and left ranges, while opinions at the edge sections are more restricted to interact with neighbors on one side, rather than in a less symmetric opinion space. Since in the DN a fewer number of links yield much shorter convergence times, then nodes around the middle in opinion space have less opportunities to come back

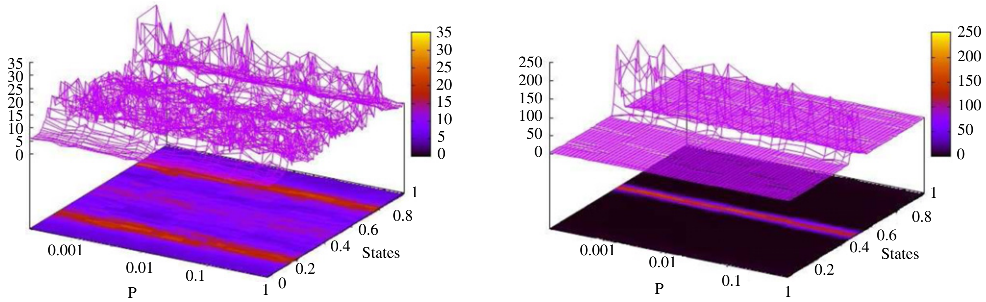

Fig. 6. Distribution of final opinions for DN networks, $N = 1000$, averaged over 100 realizations, for $\epsilon = 0.15$ on the left panel and $\epsilon = 0.4$ on the right panel.

to the middle opinions after interacting with agents with more extreme opinions. The fact that clusters around the middle loose members or strength was already observed in UN on the curves for the population of equally spaced adjacent opinion clusters or "parties" by Ben-Naim et al. [23], who employed a density-based dynamics for a density function to determine the agents' density in the opinion space. Here, besides this depopulating effect, we also found that in the DN some final opinions do not survive at all, which explains the negative values in $\langle F_{DN}\rangle - \langle F_{UN}\rangle$ in Fig. 4.

To summarize the dependence of our results on ϵ, we emphasize that for higher values of ϵ there are more nodes or agents in the DN that are isolated or jammed than in the UN case. Then the number of final opinions $\langle F\rangle$ in DN is much larger, as compared with the UN, because of the relative absence of links. That is in DN there are more isolated agents or pathway endings. The larger number of final opinion in the DN resembles results in UN models that include many extremists or "stubborn" [21] agents with small capacity to change opinion. Of course, the underlying mechanisms are different; in the former case (DN), there are less links than in the usual UN, and in the latter UN case agents behavior is not homogeneous by construction. In the region of lower ϵ, the fact that $\langle F_{DN}\rangle - \langle F_{UN}\rangle$ is negative was explained in terms of the irregularity and loss of clusters in the DN, relative to the UN.

The dependence of both $\langle F_{DN}\rangle$ and $\langle F_{UN}\rangle$ on p is more much smaller than on ϵ. Figs. 1 and 2 show how the increase of disorder (increasing p) favors long-range dissemination and convergence in the case of $\langle F_{UN}\rangle$ and when ϵ also increases. In general, it is known that there is a competition of local and global dissemination in complex networks, since increasing disorder favors long-range dissemination. The presence of a small fraction of "short cuts" connecting otherwise distant points, drastically reduces the average shortest distance between any pair of nodes in the network, keeping the clustering high [5,11]. In our dynamics, i.e. the Deffuant mode, the effect of the short cuts seems to be less important than, for instance, in the Ising model in small-world topologies, for which, when directed links are considered, not only the critical temperature changes but the nature of the transition also switches [2]. In the Axelrod model of cultural dissemination, the same qualitative result as for the equilibrium Ising model was found, in the sense that the small-world connectivity favors homogeneous or ordered states [24].

On the other hand, when p increases and ϵ is large, $\langle F_{DN}\rangle$ may slightly decrease, in contrast to $\langle F_{UN}\rangle$, showing that short cuts are not so efficient in the DN in favoring convergence in the Deffuant model. However, the behavior of $\langle F_{DN}\rangle - \langle F_{UN}\rangle$ is more similar to that of $\langle F_{UN}\rangle$.

Finally, in our simulations we found that the value of the final opinions in each cluster for both UN and DN was the *exact* average of the initial opinions for that cluster, regardless of the random process to reach the final consensus within each final cluster C. In a similar direction, Jacobmeier [25] employed the Deffuant model on a directed Barabasi–Albert network with discrete opinions, and pointed out that the resulting opinion distribution converged towards the average value of the initial opinion distribution, which was identified as a guide of opinion forming. Ben-Naim et al. [23] also showed how opinion is conserved in the particular case $\mu = \frac{1}{2}$ with a density formalism. Here, for any value μ, we outline a proof of how the conservation of opinion leads to clusters with the final average of the initial opinion values for both UN and DN. Of course, this conservation property is a consequence of the symmetrical approach process; that is, this property arises from the same values of the "approaching" parameter μ for both agents i and j in Eq. (1).

To prove that all connected nodes will converge to the average of the initial opinions, we start with the simple case $\epsilon = 1$, for which all agents are capable of interacting and converging to the final consensus value A. Let $X^F = \sum_{\alpha=1}^{N} x_\alpha^F = NA$ be the final total sum of opinions after convergence, where superindices I and F indicate initial or final state values. From Eq. (1), since in each step $x_i(t)+x_j(t) = x_i(t+1)+x_j(t+1)$ is conserved, then the initial total sum of opinions $X^I = \sum_{\alpha=1}^{N} x_\alpha^I$ is equal to X^F. Thus the final opinion satisfies $A = \frac{1}{N}\sum_{\alpha=1}^{N} x_\alpha^I$. When $\epsilon < 1$ we need to resort to a cumbersome double mathematical induction method over both the number of nodes inside and outside a cluster, but we sketch here the main ideas. At any step of the simulation, the system may be broken into subsets because some interactions might be restricted. At any given time, when a node l interacts with an element s of a new cluster with size M (while its former cluster may or may not disappear), if the node l only interacts with the same node s always – a very improbable case – the proof is trivial, but in general, there is a competition between different clusters that can attract a given node. We prove in the Appendix how the conservation of opinion holds when this competition mechanism applies for the simplified case of three nodes aligned in a simple array. Then, taking into account the fact that in the competition process opinion is always conserved, we can apply the same procedure shown above (as if $\epsilon = 1$) for all the nodes in the subset with $M + 1$ elements.

This conservation law obviously arises from Eq. (1), where for all values of μ the approach process is symmetric. There is no empirical evidence that the approach process is exactly symmetric in social systems, but it could be regarded as a useful rule to simplify simulations. Two different random values of μ_i and μ_j in a pair-wise interaction is another alternative.

4. Conclusions

It is important to study directed links in networks because many real system relationships and dynamics are not bilateral or reciprocal. Here, we first constructed a directed network (DN) from a bi-directional network UN, by simply deleting half of the links of the UN, to obtain a DN sharing the same structure. Then we performed simulations within the Deffuant model of negotiators to find averaged distributions of final opinions $\langle F\rangle$ as function of the tolerance ϵ and of the network disorder parameter p in both UN and DN. Then, by comparing the results of the Deffuant dynamics in these networks we find a rich structure of the averaged difference of clusters or final opinions, $\langle F_{DN}\rangle - \langle F_{UN}\rangle$. The dependence of $\langle F\rangle$ on ϵ in both UN and DN was much stronger than on p. Obviously, as in all bounded confidence models, increasing ϵ leads to a monotonous decrease of $\langle F\rangle$ in each network, but $\langle F_{DN}\rangle - \langle F_{UN}\rangle$, exhibits a rich structure, since it was found to be large and positive for large ϵ, but small and negative for smaller values of ϵ. The former case follows logically from the fact that, by construction, the DN have less (half) number of links, as compared to the UN, making the convergence process to find consensus more difficult in the DN. The presence of disorder has only a small effect on the convergence time in both UN and DN, and the larger number of links causes a much faster convergence in DN. To understand why $\langle F_{DN}\rangle - \langle F_{UN}\rangle$ could be negative, it was helpful to plot the corresponding distributions of $\langle F\rangle$ in the interval [0,1] to find that the DN distributions are less homogeneous than the analogous UN distributions. Also for low ϵ the DN distributions suffer a relative loss in the number and strength of the middle opinions, as compared to the UN. Finally, in both UN and DN we show how final opinions in each cluster are the exact average of the initial opinions. This conservation property is just a consequence of the two-agent symmetrical approach process built into the model. Such a mechanism can be changed by assigning two different random values for the approach parameters in each pair-wise interaction without affecting the main results.

We hope that this work may contribute to the advancement of more realistic models and applications of opinion dynamics in directed networks.

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Appendix

If we have three nodes in a row labeled a, b and c (with b in the middle) with initial opinions a_0, b_0 and c_0, we first rewrite these quantities in terms of the average of the initial opinions:

$$\begin{aligned}
a_0 &= \frac{2^1+1}{3[2^1]}\{a_0+b_0+c_0\} - \frac{1}{[2^1]}\{-a_0+b_0+c_0\} \\
&= \frac{1}{3}\{a_0+b_0+c_0\} - \frac{2}{3[2^1]}\{-2a_0+b_0+c_0\}, \\
b_0 &= \frac{2^1+1}{3[2^1]}\{a_0+b_0+c_0\} - \frac{1}{[2^1]}\{a_0-b_0+c_0\} \\
&= \frac{1}{3}\{a_0+b_0+c_0\} - \frac{2}{3[2^1]}\{a_0-2b_0+c_0\}, \\
c_0 &= \frac{2^1+1}{3[2^1]}\{a_0+b_0+c_0\} - \frac{1}{[2^1]}\{a_0+b_0-c_0\} \\
&= \frac{1}{3}\{a_0+b_0+c_0\} - \frac{2}{3[2^1]}\{a_0+b_0-2c_0\}.
\end{aligned} \tag{2}$$

Then we start the simulation by choosing first node a and its nearest neighbor node b. Here, for simplicity, the approaching parameter μ is $\frac{1}{2}$, then both opinions a_1, b_1 become the same at the middle point in this step;

$$\begin{aligned}
a_1 &= b_1 = \frac{a_0}{2} + \frac{b_0}{2} \\
&= \frac{1}{3}\{a_0 + b_0 + c_0\} - \frac{1}{3[2^1]}\{-2a_0 + b_0 + c_0\} - \frac{1}{3[2^1]}\{a_0 - 2b_0 + c_0\} \\
&= \frac{1}{3}\{a_0 + b_0 + c_0\} + \frac{1}{3[2^1]}\{a_0 + b_0 - 2c_0\}, \\
c_1 &= c_0 = \frac{1}{3}\{a_0 + b_0 + c_0\} - \frac{2}{3[2^1]}\{a_0 + b_0 - 2c_0\}.
\end{aligned} \tag{3}$$

In the next simulation step, node c and its neighbor b are chosen

$$\begin{aligned}
a_2 &= a_1 = \frac{1}{3}\{a_0 + b_0 + c_0\} + \frac{2}{3[2^2]}\{a_0 + b_0 - 2c_0\} \\
b_2 &= c_2 = \frac{b_1}{2} + \frac{c_1}{2} = \frac{1}{3}\{a_0 + b_0 + c_0\} + [1-2]\frac{1}{3[2^2]}\{a_0 + b_0 - 2c_0\} \\
&= \frac{1}{3}\{a_0 + b_0 + c_0\} - \frac{1}{3[2^2]}\{a_0 + b_0 - 2c_0\},
\end{aligned} \tag{4}$$

to obtain

$$\begin{aligned}
a_{2n} &= \frac{1}{3}\{a_0 + b_0 + c_0\} + \frac{F}{[2^{2n}]}\{a_0 + b_0 - 2c_0\}; \quad n = 1, 2, 3, \ldots \\
b_{2n} &= c_{2n} = \frac{1}{3}\{a_0 + b_0 + c_0\} + \frac{G}{[2^{2n}]}\{a_0 + b_0 - 2c_0\}; \quad n = 1, 2, 3, \ldots
\end{aligned} \tag{5}$$

with $F = \frac{2}{3}$ and $G = -\frac{1}{3}$. Notice how rapidly (faster than a geometrical series) the last terms go to zero in these equations.

References

[1] D.J. Watts, S.H. Strogatz, Nature 393 (1998) 409.
[2] A.D. Sanchez, J.M. Lopez, M.A. Rodriguez, Phys. Rev. Lett. 88 (2002) 048701.
[3] F.W.S. Lima, A.O. Sousa, M.A. Sumuor, Phys. A. 387 (2008) 3503.
[4] L.G. Morelli, Phys. Rev. E 67 (2003) 066107.
[5] A. Ramezanpour, V. Karimipour, Phys. Rev. E 66 (2002) 036128.
[6] G. Deffuant, D. Neau, F. Amblard, G. Weisbuch, Adv. Complex Syst. 3 (2000) 87;
G. Deffuant, F. Amblard, G. Weisbuch, T. Faure, J. Artif. Soc. Soc. Simul. 5 (2002) 4;
G. Weisbuch, G. Deffuant, F. Amblard, J.P. Nadal, Complexity 7 (2002) 55.
[7] J. Lorenz, J. Lorenz, Int. J. Mod. Phys. C 18 (2007) 1819.
[8] C. Castellano, S. Fortunato, V. Loreto, Rev. Mod. Phys. 81 (2009) 591.
[9] D. Urbig, J. Lorentz, H. Herzberg, J. Artif. Soc. Soc. Simul. 11 (2008) 4.
[10] S. Huet, G. Deffuant, W. Jager, Adv. Complex Syst. 11 (2008) 529.
[11] L. Guo, X. Cai, Commun. Comput. Phys. 5 (2009) 1045.
[12] G. Weisbuch, Eur. Phys. J. B 38 (2004) 339.
[13] D. Jacobmeier, Int. J. Mod. Phys. C 16 (2005) 633.
[14] A.U. Stauffer, D. Sousa, A. Schulze, J. Artif. Soc. Simulation 7 (2004).
[15] S. Fortunato, Int. J. Mod. Phys. C 15 (2004) 1301.
[16] P. Groeber, F. Schweitzer, J. Artif. Soc. Soc. Simul. 12 (2/4) (2009).
[17] F. Schweitzer, L. Behera, Eur. Phys. J. B 67 (2009) 301–318.
[18] J. Shao, S. Havlin, H.E. Stanley, Phys. Rev. Lett. 103 (2009) 018701.
[19] S. Schwarzer, S. Havlin, A. Bunde, Phys. Rev. E 59 (1999) 3262.
[20] G. Deffuant, F. Amblard, G. Weisbuch, T. Faure, JASSS 5 (2002) 4.
[21] F. Amblard, G. Deffuant, Physica A 343 (2004) 725.
[22] P. Assmann, Int. J. Mod. Phys. C 15 (2004) 1439.
[23] Ben-Naim, E.P. Krapivsky, S. Redner, Physica D 183 (2003) 190.
[24] K. Klemm, V.M. Eguiluz, R. Toral, M. San Miguel, Phys. Rev. E 67 (2003) 026120.
[25] D. Jacobmeier, Int. J. Mod. Phys. C 17 (2006) 1801.