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On the Hegselmann-Krause conjecture in opinion dynamics

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We give an elementary proof of a conjecture by Hegselmann and Krause in opinion dynamics, concerning a symmetric bounded confidence interval model: If there is a truth and all individuals take each other seriously by a positive amount bounded away from zero, then all truth seekers will converge to the truth. Here truth seekers are the individuals which are attracted by the truth by a positive amount. In the absence of truth seekers it was already shown by Hegselmann and Krause that the opinions of the individuals converge.

1. Introduction

We answer in the affirmative a conjecture posed by Hegselmann and Krause about the long-term behavior of opinions in a finite group of individuals, some of them attracted to the truth, the so-called *truth seekers*. Our contribution: Under mild assumptions, the opinions of all truth seekers converge to the truth, despite being distracted by individuals not attracted to the truth, the *ignorants*.

The underlying model for opinion dynamics is the *bounded-confidence model*: Opinions, which themselves are represented by real numbers in the unit interval, are influenced by the opinions of others by means of averaging, but only if not too far away. This bounded-confidence model (formal definitions below) was first suggested by Krause in 1997. It received a considerable amount of attention in the artificial societies and social simulation community [1–6].

The concept of truth seekers was invented in 2006 by Hegselmann and Krause [3], along with a philosophical discussion about the scientific context with respect to the notion of truth. We blind out the philosophical discussions here and focus on the resulting dynamical system, governed by difference equations that we find interesting in their own right.

The opinions of *truth seekers* are not only attracted by opinions of others; they are additionally attracted by a constant number, the truth. The resulting opinion is weighted average of the result of the original bounded-confidence dynamics and the truth. Individuals not attracted by the truth in this sense are *ignorants*. In their paper, Hegselmann and Krause show that if all individuals are truth seekers – no matter how small the weight –, then (the opinions of) all the individuals converge to consensus on the truth value.

The question we answer in this paper arises when some of the individuals are ignorants, i.e., the weight of the influence of the truth is zero for them. Numerous simulation experiments led Hegselmann and Krause to the conjecture, that still the opinions of all the truth seekers finally end up at the truth. However, a proof of this fact could not be found so far. Evidence by simulation only, however, bears the risk of numerical artefacts – very much so in the non-continuous bounded-confidence model. Therefore, it is desirable to provide mathematically rigid proofs of structural properties of bounded-confidence dynamics.

Allthough the conjecture may seem self-understood at first glance because of the contraction property

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of the system dynamics for truth seekers, a second look on the situation reveals that the conjecture and its confirmation in this paper are far from trivial: several innocent-looking generalizations of the conjecture are actually false, as we will show below in the technical parts of the paper. Relying on intuition only is dangerous.

Even in the affirmative cases, convergence turns out to be quite slow in general and far from monotone. The main difficulty is the following: the convergence of truth seekers heavily depends on their long-term influence on ignorants. Depending on the configuration of ignorants and the parameters of the system, there are arbitrarily many iterations in which the truth seekers deviate from the truth. The crucial observation is that, during these iterations, the configuration of ignorants is somehow "improved" because the truth seekers attract them.

After all, the proof is elementary but extremely technical. We introduce some structures like the *confidence graph*, that might prove useful also in other contexts. Other structures we need are rather special, probably with limited use beyond this paper. It would, therefore, be desirable to find a more elegant proof, revealing the reason why the conjecture is true. For example: find a suitable Lyapunov function. The examples we give as we go along in the proof, however, indicate that a certain amount of complexity has to be captured by the arguments because the line between true and false conjectures is extremely thin.

2. Formal problem statement

Suppose there is a set $[n] := \{1, \ldots, n\}$ of individuals with opinions $x_i(t) \in [0, 1]$ at time t for all $i \in [n]$, $t \in \mathbb{N}$. The abstract truth is modeled as a constant over time, denoted by $h \in [0, 1]$. The opinion of an individual $i \in [n]$ is influenced in a time step t only by those individuals which have a similar opinion, more precisely which have an opinion in the *confidence interval* of $x_i(t)$.

Definition 2.1: For $x \in [0, 1]$ and a parameter $\varepsilon \geq 0$ we define the *confidence set of value x at time t* as

$$I_x^{\varepsilon}(t) := \{j \in [n] \mid |x - x_j(t)| \leq \varepsilon\}.$$

As a shorthand we define $I_i^{\varepsilon}(t) := I_{x_i(t)}^{\varepsilon}(t)$ for any $i \in [n]$.

The update of the opinions is modeled as a weighted arithmetic mean of opinions in the confidence set and a possible attraction towards the truth.

Definition 2.2: A *weighted arithmetic mean symmetric bounded confidence opinion system* (WASBOCOS) is a tupel

$$(n, h, \varepsilon, \alpha, \beta; \alpha_i(t), \beta_{ij}(t), x_i(0)),$$

where

- $n \in \mathbb{N}$ ist the number of *individuals*,
- $h \in [0, 1]$ ist the *truth*,
- $\varepsilon \in [0, 1]$ is the *bounded confidence radius*,
- $\alpha \in (0, 1]$ is a lower bound for the weight of the truth for truth seekers,
- $\beta \in (0, \frac{1}{2}]$ is a lower bound for the weight of opinions in the bounded confidence interval,
- $\alpha_i(t) \in [\alpha, 1]$ or $\alpha_i(t) = 0$ for all $t \in \mathbb{N}$ is the actual weight of the truth for truth seeker i at time step t,
- $\beta_{ij}(t) \in [\beta, 1 - \beta]$ with $\sum_{j=1}^{n} \beta_{ij}(t) = 1$ for all $i \in [n]$ and for all $t \in \mathbb{N}$ is the weight of opinion j in the view of agent i,
- $x_i(0) \in [0, 1]$ is the starting opinion of Individual i.

The *bounded confidence dynamics* on such a system is defined by simultaneous updates of the opinions in

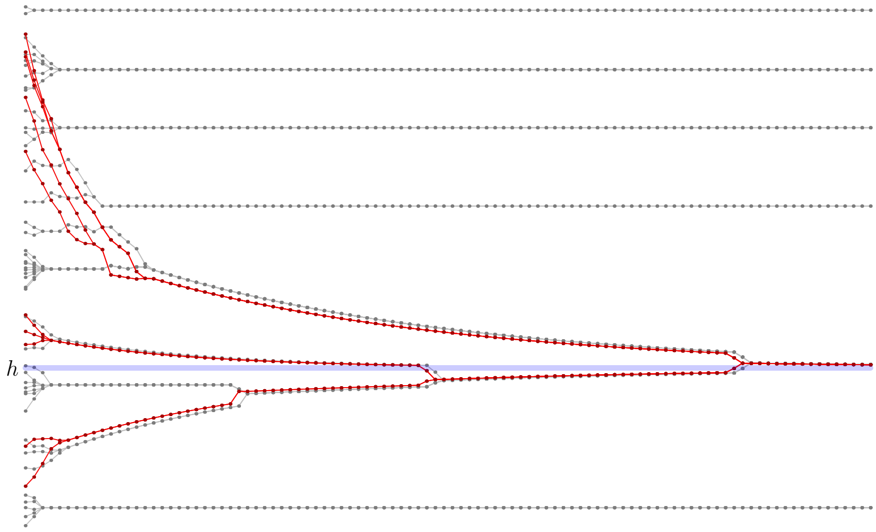

Figure 1. A sketch of the behaviour of a more or less random (WASBOCOS). Opinions are depicted by height, time goes from left to right. The truth seekers' trajectories are darker. All truth seekers apparently converge to the truth (indicated by the thicker horizontal line).

the following form:

$$x_i(t+1) := \alpha_i(t) \cdot h + \Big(1 - \alpha_i(t)\Big) \frac{\sum\limits_{j \in I_i^{\varepsilon}(t)} \beta_{ij}(t) x_j(t)}{\sum\limits_{j \in I_i^{\varepsilon}(t)} \beta_{ij}(t)}. \tag{1}$$

Individuals are members of the index set $[n]$. *Truth seekers* are members of the set $K := \{k \in [n] \mid \alpha_k(t) \geq \alpha\, \forall t \in \mathbb{N}\}$. All other individuals, i.e., those with $\alpha_i(t) = 0$ for all t, are called *ignorants*; their set is denoted by $\overline{K}$.

See Figure 1 for a sketch of a typical set of trajectories. Remark: The term *symmetric* in the notion of a (WASBOCOS) refers to the confidence radius, not to the weights that individuals assign to other individuals' opinions. The main result we wish to prove is the following:

Theorem 2.3 : *(Generalized Hegselmann-Krause Conjecture) All truth seekers in an (WASBOCOS) Ω converge to the truth h. Formally, for each $\gamma > 0$ and each Ω there exists a $T(\gamma, \Omega)$ so that we have $|x_k(t) - h| < \gamma$ for all $k \in K$ and all $t \geq T(\gamma, \Omega)$.*

Note that we use $\gamma > 0$ in the statement of convergence instead of $\varepsilon > 0$ because ε is traditionally used for the bounded confidence radius.

It is important that convergence is not just implied by the contraction property of the dynamics with ignorants ignored. Ignorants and where their opinions are make a huge difference (see Figure 2 for an example).

It would be nice if one could derive a bound on the speed of convergence, i.e., a bound on $T(\gamma, \Omega)$, in terms of the structural parameters ε, α, β, and n. Unfortunately, this is not possible. The speed of convergence is not determined by the structural parameters alone. This can be seen in the following simple example.

Example 2.4 Consider a (WASBOCOS) with truth $h = \varepsilon$, $\varepsilon > 0$, $\alpha_1(t) = \alpha$, $\alpha_2(t) = 0$, $\beta_{ij}(t) = \frac{1}{2}$, $\beta = \frac{1}{2}$, $x_1(0) = 2\varepsilon$, $x_2(0) = \tilde{\varepsilon}$, where $\varepsilon > \tilde{\varepsilon} > 0$. Let $T \in \mathbb{N}$ be the smallest integer so that $(1-\alpha)^T \varepsilon \leq \tilde{\varepsilon}$. Then by induction we have $x_1(t) = \varepsilon + (1-\alpha)^t \varepsilon$ and $x_2(t) = x_2(0) = \tilde{\varepsilon}$ for all $t \leq T$. So truth seeker 1 seems to monotonically converge to the truth, but at time $T+1$ we have $x_1(T+1) = \alpha\varepsilon + \frac{1-\alpha}{2}\left(\varepsilon + (1-\alpha)^T \varepsilon + \tilde{\varepsilon}\right) \leq \frac{1+\alpha}{2} \cdot \varepsilon + \tilde{\varepsilon}$.

See Figure 3 for a sketch of the situation. Since we may choose $\tilde{\varepsilon}$ arbitrarily small, we find the following: in general, we can not expect that for every $\gamma > 0$ there is a $T(\gamma, \varepsilon, \alpha, \beta, n)$ such that for all $t \geq T(\gamma, \varepsilon, \alpha, \beta, n)$

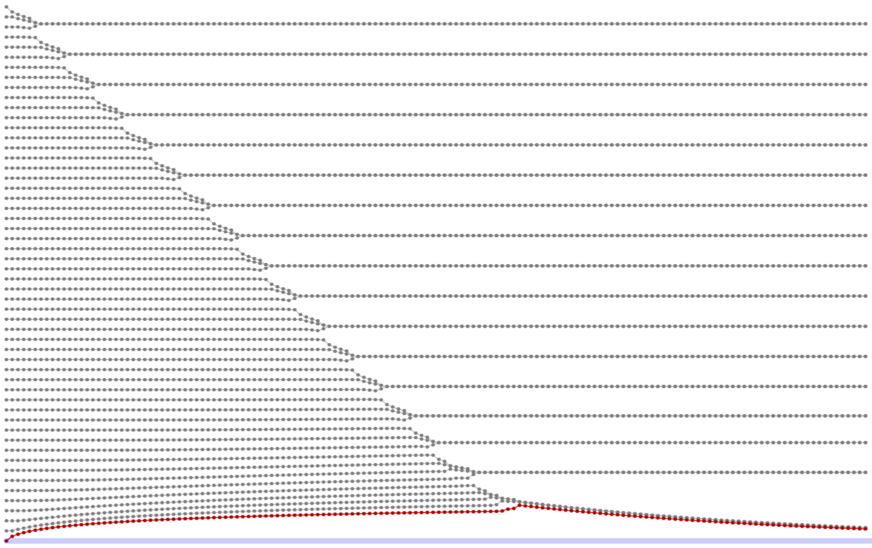

Figure 2. The only truth seeker is located at the bottom on the truth; it gets attracted away from the truth for quite some time; eventually, the ignorants either are "converted" or left behind. Thus, we cannot expect monotone convergence of the truth seeker farthest from the truth, i.p, reaching the truth once does not imply convergence to the truth.

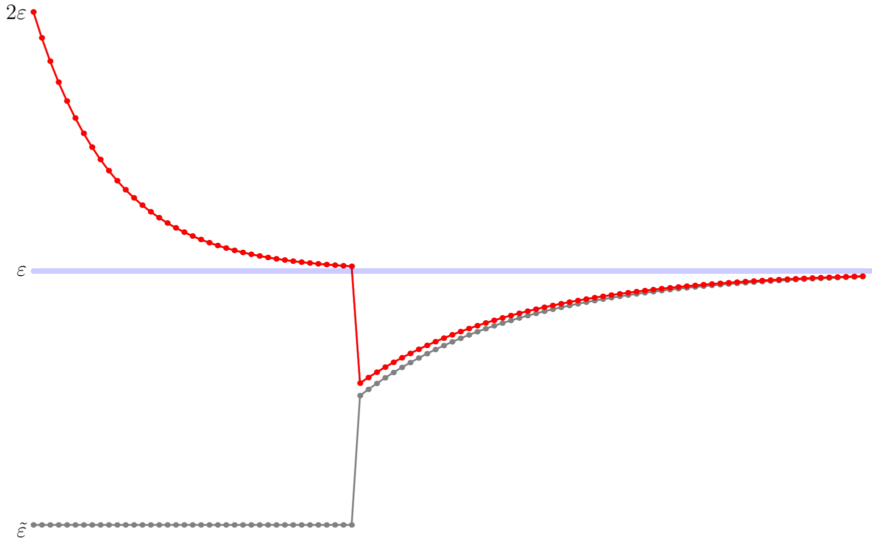

Figure 3. A sketch of interrupted convergence: a lonely truth seeker starting at 2ε seems to monotonically converge to the truth right away, but suddenly its confidence interval picks up the ignorant on the other side of the truth, and the truth seeker gets distracted. However, finally the ignorant gets distracted himself, and convergence is eventually established.

we have $|x_k(t) - h| < \gamma$ for each truth seeker $k \in K$. But we may have *interrupted convergence*: In a first phase, the truth seekers come arbitrarily close to the truth in time only dependent on the structural parameters; then, they may temporarily get distracted at some point; finally, they converge to the truth in time depending only on the structural parameters *and the time of distraction*. This can be formalized as follows:

Definition 2.5: Given ε, α, β, n, we say that truth seekers $k \in K$ are (1-fold) *interrupted convergent to the truth*, if for each $\gamma > 0$ there exist two functions $T_1^s(\gamma, \varepsilon, \alpha, \beta, n)$ and $T_2^s(\gamma, \varepsilon, \alpha, \beta, n, T_1^e)$, so that for each (WASBOCOS) Ω, with structural parameters ε, α, β and n, there exists an $T_1^e \in \mathbb{N}$ satisfying

$$\forall k \in K,\ \forall t \in [T_1^s(\gamma, \varepsilon, \alpha, \beta, n), T_1^e] : |x_k(t) - h| < \gamma,$$
$$\forall k \in K,\ \forall t \geq T_2^s(\gamma, \varepsilon, \alpha, \beta, n, T_1^e) : |x_k(t) - h| < \gamma.$$

Theorem 2.3 is now a corollary of the following substantially strengthened Theorem:

Theorem 2.6 : *All truth seekers in an (WASBOCOS) Ω are (1-fold) interrupted convergent to the truth.*

Originally Hegselmann and Krause considered the (WASBOCOS) model for $\alpha_i(t) \in \{0, \alpha\}$ and $\beta_{ij}(t) = \frac{1}{n}$. In the case of complete absence of truth seekers they have already proved, that the opinion of each individual converges, as can be expected, not necessarily to the truth. In fact in general the individuals form several clusters, where two individuals of different clusters converge to different opinions.

We give an example without truth seekers where the individuals will converge to five different clusters.

Example 2.7 Consider a (WASBOCOS) with $\alpha_i(t) = 0$ (no truth seekers), $\beta = \beta_{ij}(t) = \frac{1}{n}$, $n = 12$; the values of α and h do not matter. The starting positions are given by

$$x_1(0) = x_2(0) = 0,\ x_3(0) = \varepsilon,\ x_4(0) = 2\varepsilon,\ x_5(0) = 3\varepsilon,\ x_6(0) = x_7(0) = 4\varepsilon,$$
$$x_8(0) = 5\varepsilon,\ x_9(0) = 6\varepsilon,\ x_{10}(0) = 7\varepsilon,\ \text{and}\, x_{11}(0) = x_{12}(0) = 8\varepsilon,$$

see Figure 4.

In Table 1 we give the complete dynamics of the opinions of all 12 individuals over time until the opinion of every individual has converged. For brevity we write x_i instead of $x_i(t)$. After three time steps, see Figure 5 for the dynamics, we have reached a stable state, see Figure 6 for the resulting positions of the individuals.

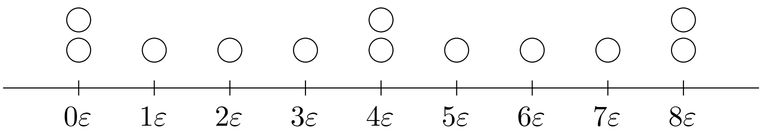

Figure 4. Starting positions of the individuals in Example 2.7.

t	$x_1 = x_2$	x_3	x_4	x_5	$x_6 = x_7$	x_8	x_9	x_{10}	$x_{11} = x_{12}$
0	0ε	1ε	2ε	3ε	4ε	5ε	6ε	7ε	8ε
1	$\frac{1}{3}\varepsilon$	$\frac{3}{4}\varepsilon$	2ε	$\frac{13}{4}\varepsilon$	4ε	$\frac{19}{4}\varepsilon$	6ε	$\frac{29}{4}\varepsilon$	$\frac{23}{3}\varepsilon$
2	$\frac{17}{36}\varepsilon$	$\frac{17}{36}\varepsilon$	2ε	$\frac{15}{4}\varepsilon$	4ε	$\frac{17}{4}\varepsilon$	6ε	$\frac{271}{36}\varepsilon$	$\frac{271}{36}\varepsilon$
3	$\frac{17}{36}\varepsilon$	$\frac{17}{36}\varepsilon$	2ε	4ε	4ε	4ε	6ε	$\frac{271}{36}\varepsilon$	$\frac{271}{36}\varepsilon$

Table 1. The dynamics of Example 2.7 in numbers.

We remark that for symmetric weights $\beta_{ij}(t) = \beta_{ji}(t)$ one can easily show that in the absence of truth seekers the dynamics becomes stable after a finite number of time steps. In the case of asymmetric weights $\beta_{ij}(t) \neq \beta_{ji}(t)$ we only have convergence, but need not reach a stable state after an arbitrary, problem dependent, but finite number of time steps, as illustrated in the following example.

Example 2.8 Consider a (WASBOCOS) with $\alpha_i(t) = 0$ (no truth seekers), $n = 2$, $x_1(0) = 0$, $x_2(0) = \varepsilon$, $\beta_{11}(t) = \frac{2}{3}$, $\beta_{12}(t) = \frac{1}{3} = \beta$, $\beta_{21}(t) = \frac{1}{2}$, $\beta_{22}(t) = \frac{1}{2}$; the values of α and h do not matter.

One can easily verify, e.g., by induction, that we have

$$x_1(t) = \left(\frac{2}{5} - \frac{2}{5 \cdot 6^t}\right) \cdot \varepsilon \text{ and } x_2(t) = \left(\frac{2}{5} + \frac{3}{5 \cdot 6^t}\right) \cdot \varepsilon$$

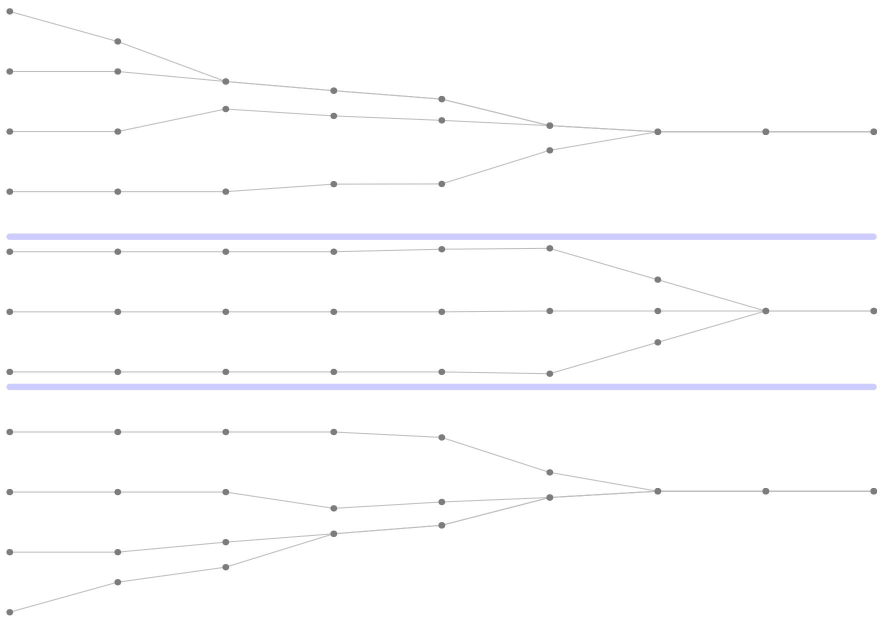

Figure 5. The dynamics in Example 2.7.

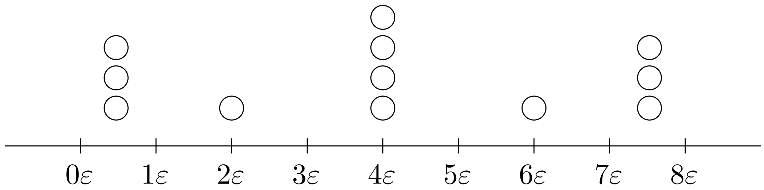

Figure 6. Final positions of the individuals in Example 2.7.

for all $t \in \mathbb{N}$. So we have $|x_1(t) - x_2(t)| = \frac{1}{6^t} \cdot \varepsilon > 0$ but clearly the opinions of the two individuals converge to $\frac{2}{5}$.

All stated insights with the absence of truth seekers were known so far. It becomes a bit more interesting if we allow truth seekers, i.e., if we consider a general (WASBOCOS).

Example 2.9 Consider a (WASBOCOS) with $\alpha_1(t) = \alpha$, $\alpha_i(t) = 0$ for $i \neq 1$, $\beta_{ij}(t) = \frac{1}{n}$, $h = \frac{1}{2}\varepsilon$, $x_1(0) = 0$, and $x_i(0) = \varepsilon$ for $i \neq 1$. The opinion u_t of the truth seeker 1 at time t and the opinion v_t of the other ignorants at time $t > 0$ are given by

$$u_t = \left[\frac{1}{2} - \alpha\left(\frac{1}{2} - \frac{1}{n}\right)\left(1 - \frac{\alpha}{n}\right)^{t-1}\right]\varepsilon,$$
$$v_t = \left[\frac{1}{2} + \left(\frac{1}{2} - \frac{1}{n}\right)\left(1 - \frac{\alpha}{n}\right)^{t-1}\right]\varepsilon$$

respectively. This can be verified, e.g., by induction. We see that the opinions of the truth seekers, and here also those of the ignorants, converge to the truth $h = \frac{1}{2}\varepsilon$.

Note the opinions of ignorants may in general fail to converge to the truth as one can see by adding some further ignorants with $\tilde{x}_i(0) = 3\varepsilon$.

As our analytical investigation of the previous example was rather technical, we also depict the situation for special values $n = 6$ and $\alpha = \frac{2}{3}$ in Figure 7. We sketch the truth seeker by a dark line and the ignorants by a grey line.

One can easily imagine more complicated configurations as in Example 2.9 where one has little chance and willingness to describe the situation analytically. Our result Theorem 2.3 states that – whatever the

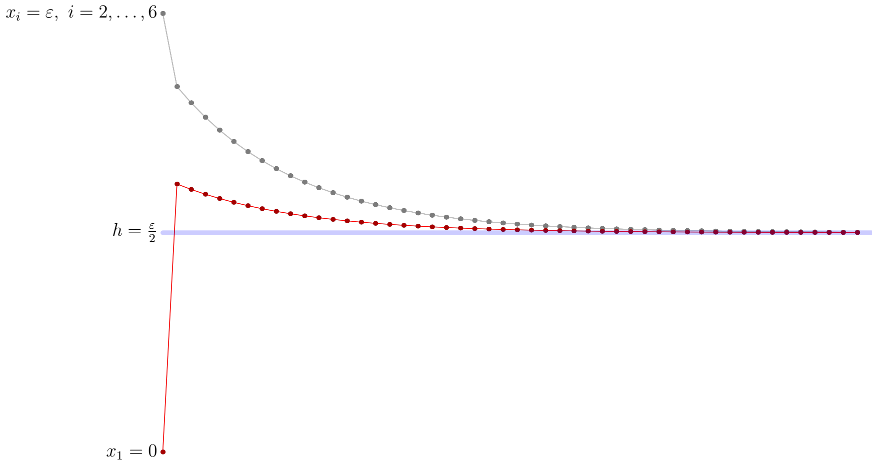

Figure 7. The dynamics in example 2.9.

parameters of a (WASBOCOS) are – the opinions of the truth seekers converge to the truth. This settles an open conjecture of Hegselmann and Krause.

3. The crucial objects

To get a first impression of what we may expect in terms of convergence we consider a lonely truth seeker, i.e., $n = 1$.

Lemma 3.1: *For a lonely truth seeker $i = 1$ we have*

$$|x_i(t+r) - h| \leq |x_i(t) - h| \cdot (1-\alpha)^r.$$

Proof:

$$|x_i(t+1) - h| = |x_i(t) - h| \cdot (1 - \alpha_i(t)) \leq |x_i(t) - h| \cdot (1-\alpha).$$

□

Clearly this bound is tight. Similar to this very special situation of a lonely truth seeker is the case $\varepsilon = 0$, so that we now assume $\varepsilon > 0$ for the remaining part of this article.

In order to describe the states of the discrete time dynamical system with more than one truth seeker, we look at the truth seekers with the most extreme opinions.

Definition 3.2: We define $\tilde{u}(t) \in K$ as the lexicographically smallest truth seeker which fullfills $x_{\tilde{u}(t)}(t) \geq h$ and $x_{\tilde{u}(t)}(t) \geq x_k(t)$ for all $k \in K$. If there is no truth seeker with opinion greater or equal to the truth h we set $\tilde{u}(t) = 0$. In order to avoid case distinctions, we define $x_0(t') := h$ for all $t' \in \mathbb{N}$. Similar we define $\tilde{l}(t)$ as the lexicographically smallest truth seeker that fullfills $x_{\tilde{l}(t)}(t) \leq h$ and $x_{\tilde{l}(t)}(t) \leq x_k(t)$ for all $k \in K$. Again, we set $\tilde{l}(t) = 0$ if there is no such truth seeker.

Due to the *symmetrical* – one could say *fair* – definition of the confidence set, the confidence structure between the individuals can be described as a simple graph with loops.

Definition 3.3: The *confidence graph* $\mathcal{G}(t)$ with vertex set $V(t)$ and edge set $E(t)$, of a configuration $x(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^n$ and the additional is defined as follows:

$$V(t) := [n] \cup \{0\},$$
$$E(t) := \{\{i,j\} \in \tbinom{V}{2} \mid |x_i(t) - x_j(t)| \leq \varepsilon\}.$$

For $i \in V(t)$ let $C_i(t)$ be the set of vertices in the connectivity component of vertex i in $\mathcal{G}(t)$.

Because we want to keep track of the individuals which can influence the truth seekers in the future, we give a further definition for individuals, which is similar to Definition 3.2 for truth seekers.

Definition 3.4: We define $\hat{u}(t) \in C_{\tilde{u}(t)}(t)$ as the lexicographically smallest individual with $x_{\hat{u}(t)}(t) \geq x_c(t)\ \forall c \in C_{\tilde{u}(t)}(t)$ and $\hat{l}(t) \in C_{\tilde{l}(t)}(t)$ as the lexicographically smallest individual with $x_{\hat{l}(t)}(t) \leq x_c(t)\ \forall c \in C_{\tilde{l}(t)}(t)$ for all $t \in \mathbb{N}$.

The opinions of $\hat{u}(t)$ and $\hat{l}(t)$ form an interval $[x_{\hat{l}(t)}(t), x_{\hat{u}(t)}(t)]$ called the *hope interval* which is crucial for our further investigations. To prove the main theorem we will show that the length of this hope interval converges to zero.

In Figure 8, we have depicted a configuration to illustrate Definition 3.2 and Definition 3.4. In particular, we have $\tilde{l} = 4$, $\tilde{u} = 9$, $\hat{l} = 2$, and $\hat{u} = 12$. Individual 1 is *lost* and not contained in the hope interval, because there is no path in $\mathcal{G}$ from 1 to $\tilde{l} = 4$. So we already know that the opinion of Individual 1 will not converge to the truth.

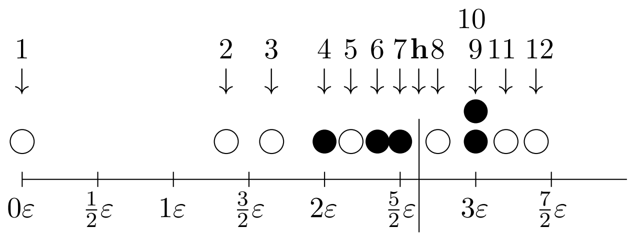

Figure 8. Illustration of Definition 3.2 and Definition 3.4.

In the configuration depicted in Figure 9 we have $\tilde{l} = 2$, $\tilde{u} = 0$, $\hat{l} = 2$, and $\hat{u} = 5$.

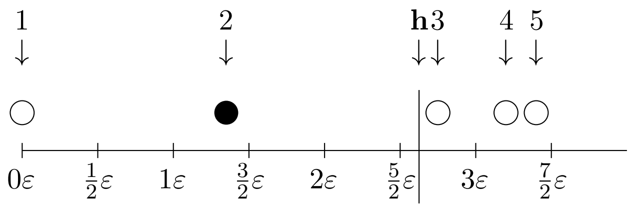

Figure 9. Illustration of a special case in Definition 3.2.

Note that the weights $\beta_{ij}(t)$ may be assymmetric. Thus, the sequence of the opinions of the individuals may reorder during the time steps. As an example, consider, e.g., three ignorants with starting positions $x_1(0) = 1\varepsilon$, $x_2(0) = \frac{3}{2}\varepsilon$, and $x_3(0) = 2\varepsilon$. The weights may be given as $\beta_{11}(0) = 0.01$, $\beta_{12}(0) = 0.01$, $\beta_{13}(0) = 0.98$, $\beta_{21}(0) = 0.98$, $\beta_{22}(0) = 0.01$, $\beta_{23}(0) = 0.01$, $\beta_{31}(0) = 0.4$, $\beta_{32}(0) = 0.4$, and $\beta_{33}(0) = 0.2$. After one time step the new opinions are given by $x_1(1) = 1.985\varepsilon$, $x_2(1) = 1.015\varepsilon$, and $x_3(1) = 1.4\varepsilon$. We remark that it is possible to achieve every ordering of the three opinions in one time step by choosing suitable weights $\beta_{ij}(t)$ in this example. Nevertheless we have the following straight-forward lemma:

Lemma 3.5: *Let i be an ignorant, $l \in I_i^\varepsilon(t)$ be an individual with smallest opinion and $u \in I_i^\varepsilon(t)$ be an individual with largest opinion then we have $x_i(t+1) \in [x_l(t), x_u(t)]$.*

Proof: This follows directly from the system dynamics in Equation (1). $\square$

For truth seekers we have a similar lemma:

Lemma 3.6: *Let i be a truth seeker, $l \in I_i^\varepsilon(t)$ be an individual with smallest opinion, and let $u \in I_i^\varepsilon(t)$ be an individual with largest opinion. For $x_i(t) \leq h$ we have $x_i(t+1) \in [x_l(t), \max(h, x_u(t))]$ and for $x_i(t) \geq h$ we have $x_i(t+1) \in [\min(h, x_l(t)), x_u(t)]$.*

Our goal is to prove that the length of the hope interval converges to zero. To this end, we show first that the length does not increase after an iteration of Equation (1).

Lemma 3.7: *For all time steps $t \in \mathbb{N}$ we have $x_{\hat{u}(t+1)}(t+1) \leq x_{\hat{u}(t)}(t)$ and $x_{\hat{l}(t+1)}(t+1) \geq x_{\hat{l}(t)}(t)$.*

Proof: We only prove the last inequality since the proof is symmetric for the first inequality. Due to Definition 3.4, we have $x_{\hat{l}(t+1)}(t+1) \leq h$ and $x_{\hat{l}(t)}(t) \leq h$. By $\mathcal{L}(t)$ we denote the set of individuals with opinion strictly smaller than $x_{\hat{l}(t)}(t)$. That is, $\mathcal{L}(t') := \{i \in [n] \mid x_i(t') < x_{\hat{l}(t')}(t')\}$ for all $t' \geq t$. We remark that, by definition, $\mathcal{L}(t')$ does not contain a truth seeker. We set $\mathcal{U}(t') := [n] \backslash \mathcal{L}(t')$; this set contains the remaining individuals.

Let u be an individual in $\mathcal{L}(t)$ with the largest opinion. By applying Lemma 3.5 we get $x_i(t+1) \leq x_u(t)$ for all $i \in \mathcal{L}(t)$. Now let l (e.g., $l = \hat{l}(t)$) be an individual in $\mathcal{U}(t)$ with smallest opinion then by applying Lemma 3.5 and Lemma 3.6 we receive $x_i(t+1) \geq x_l(t)$ for all $i \in \mathcal{U}(t)$. Thus, we have $\hat{l}(t+1) \in \mathcal{U}(t)$ and so $x_{\hat{l}(t+1)}(t+1) \geq x_{\hat{l}(t)}(t)$ follows. □

In the remaining part of this article we prove that the length of the hope interval $|x_{\hat{u}(t)}(t) - x_{\hat{l}(t)}(t)|$ converges (in some special sense) to zero, as t tends to infinity.

4. Proof of the Generalized Hegselmann-Krause Conjecture

One difficulty in the proof arises from the fact that convergence happens in two phases: in a first phase, the hope interval becomes sufficiently small so that the confidence graph is the complete graph. Then, it may happen that truth seekers approaching the truth from one side get distracted to the other side of the truth. At that point, however, the confidence structure is so simple that all individuals in the hope interval converge to the truth. Since all truth seekers are in the hope interval at all times, this proves the theorem. Where exactly we split the phases is a technical decisison.

First, we show that after a finite number T_1 of time steps, depending only on n, ε, α, and β, the hope interval $[x_{\hat{l}(T_1)}(T_1), x_{\hat{u}(T_1)}(T_1)]$ is contained in the interval $[h-\varepsilon-\frac{\varepsilon\alpha\beta}{12}, h+\varepsilon+\frac{\varepsilon\alpha\beta}{12}]$. Therefore, we introduce the following notion.

Definition 4.1: A *good iteration* is an iteration where for $1 \leq r \leq 3$ one of the following conditions is fullfilled:

(1) the number of individuals in the hope interval decreases,
(2) the opinion of $\hat{l}(t+r)$ reaches or passes $h - \varepsilon - \frac{\varepsilon\alpha\beta}{12}$,
(3) the opinion of $\hat{u}(t+r)$ reaches or passes $h + \varepsilon + \frac{\varepsilon\alpha\beta}{12}$,
(4) $|x_{\hat{u}(t+r)}(t+r) - x_{\hat{u}(t)}(t)| \geq \frac{\varepsilon\alpha\beta^2}{12}$,
(5) $|x_{\hat{l}(t+r)}(t+r) - x_{\hat{l}(t)}(t)| \geq \frac{\varepsilon\alpha\beta^2}{12}$.

Clearly, there is only a finite number of good iterations. We may choose $T_1 = 3 \cdot \left(n + 2 \cdot 1 + 2 \cdot \frac{12}{\varepsilon\alpha\beta^2}\right)$. We formulate the next two lemmas only for the lower bound $x_{\hat{l}(t)}(t)$ because analog arguments hold for

$x_{\hat{u}(t)}(t)$. As a shorthand we define $d(i,j,t) := |x_i(t) - x_j(t)|$. For each point in time t we define the sets

$$\mathcal{N}(t) := \left\{ i \in [n] \mid d(\hat{l}(t), i, t) \in \left[0, \frac{\varepsilon\alpha\beta}{12}\right) \right\},$$

$$\mathcal{M}(t) := \left\{ i \in [n] \mid d(\hat{l}(t), i, t) \in \left[\frac{\varepsilon\alpha\beta}{12}, \varepsilon\right] \right\}, \text{ and}$$

$$\mathcal{F}(t) := \left\{ i \in [n] \mid x_i(t) - x_{\hat{l}(t)}(t) > \varepsilon \right\}.$$

Lemma 4.2: *If $x_{\hat{l}(t)} \leq h - \varepsilon$ and $\mathcal{M}(t) \neq \emptyset$ then there is a good iteration after 1 step.*

Proof: We assume that there is an individual $j \in \mathcal{M}(t)$, i.e., $d(\hat{l}(t), j, t) \in \left[\frac{\varepsilon\alpha\beta}{12}, \varepsilon\right]$. For the evaluation of Equation (1) for elements of $\mathcal{N}(t)$, $\mathcal{M}(t)$, or $\mathcal{F}(t)$ we do not need to consider the opinion of individuals in $[n] \backslash (\mathcal{N}(t) \cup \mathcal{M}(t) \cup \mathcal{F}(t))$. Let i be an element of $\mathcal{N}(t)$ with opinion $x_i(t) = x_{\hat{l}(t)} + \delta$, where $0 \leq \delta < \frac{\varepsilon\alpha\beta}{12}$. Let us first assume that i is an ignorant. Due to Individual j we have

$$\begin{aligned} x_i(t+1) &\geq x_i(t) - \underbrace{\delta\,(1-2\beta)}_{\text{individuals in } \mathcal{N}(t)\backslash\{i\}} + \underbrace{0 \cdot \beta}_{i} + \underbrace{\left(\frac{\varepsilon\alpha\beta}{12} - \delta\right) \cdot \beta}_{j} \\ &\geq x_{\hat{l}(t)} + \frac{\varepsilon\alpha\beta^2}{12}. \end{aligned}$$

For a truth seeker we similarly get

$$\begin{aligned} x_i(t+1) &\geq x_i(t) + \alpha\varepsilon + (1-\alpha)\left(-\delta\,(1-2\beta) + \left(\frac{\varepsilon\alpha\beta}{12} - \delta\right) \cdot \beta\right) \\ &\geq x_{\hat{l}(t)} + \frac{\varepsilon\alpha\beta^2}{12}. \end{aligned}$$

Now let i be an element of $\mathcal{M}(t) \cup \mathcal{F}(t)$ with $x_i(t) = x_{\hat{l}(t)} + \delta$ where $\delta \geq \frac{\varepsilon\alpha\beta}{12}$. In any case ($i$ being a truth seeker or an ignorant) we have

$$x_i(t+1) \geq x_{\hat{l}(t)} + \delta - \underbrace{\delta(1-\beta)}_{\text{individuals with smaller opinion than } i} + \beta \cdot 0 \geq x_{\hat{l}(t)} + \frac{\varepsilon\alpha\beta^2}{12}.$$

$\square$

Lemma 4.3: *If $x_{\hat{l}(t)} < h - \varepsilon - \frac{\varepsilon\alpha\beta}{12}$ then after at least 3 time steps we have a good iteration.*

Proof: Due to Lemma 4.2 we can assume $\mathcal{M}(t) = \mathcal{M}(t+1) = \mathcal{M}(t+2) = \emptyset$. We can also assume

$$\left| x_{\hat{l}(t)}(t) - x_{\hat{l}(t+1)}(t+1) \right| < \frac{\varepsilon\alpha\beta^2}{12},$$

$$\left| x_{\hat{l}(t+1)}(t+1) - x_{\hat{l}(t+2)}(t+2) \right| < \frac{\varepsilon\alpha\beta^2}{12}, \text{ and}$$

$$d\left(\hat{l}(t+1), 0, t+1\right) > \varepsilon + \frac{\varepsilon\alpha\beta}{12}$$

since otherwise we have a good iteration in at most 2 time steps. At first we claim $\mathcal{N}(t+1) \cap K = \emptyset$. If at time t there is a truth seeker $i \in \mathcal{N}(t) \cap K$ then we have

$$\begin{aligned} x_i(t+1) &\geq x_{\hat{l}(t)}(t) + \alpha\varepsilon - \frac{(1-\alpha)(1-\beta)\varepsilon\alpha\beta}{12} \\ &\geq x_{\hat{l}(t)}(t) + \frac{\varepsilon\alpha\beta^2}{12} + \frac{\varepsilon\alpha\beta}{12} \\ &\geq x_{\hat{l}(t+1)}(t+1) + \frac{\varepsilon\alpha\beta}{12}. \end{aligned}$$

So the only truth seekers that have a chance to move into the set $\mathcal{N}(t+1)$ could be those of the set $\mathcal{F}(t)$. So let truth seeker i be in the set $\mathcal{F}(t) \cap K$, with $x_i(t) = x_{\hat{l}(t)}(t) + \delta$, where $\varepsilon < \delta < \varepsilon + \frac{\varepsilon\alpha\beta}{12}$. (Truth seekers where $\delta \geq \varepsilon + \frac{\varepsilon\alpha\beta}{12}$ are ruled out by Lemma 3.6.) We have

$$\begin{aligned} x_i(t+1) &\geq \underbrace{x_{\hat{l}(t)}(t) + \varepsilon}_{\leq x_i(t)} - (1-\alpha)(1-2\beta) \\ &\geq x_{\hat{l}(t)}(t) + \varepsilon\alpha \\ &\geq x_{\hat{l}(t)}(t) + \frac{\varepsilon\alpha\beta^2}{12} + \frac{\varepsilon\alpha\beta}{12} \\ &\geq x_{\hat{l}(t+1)}(t+1) + \frac{\varepsilon\alpha\beta}{12}. \end{aligned}$$

Similarly, we can deduce $\mathcal{N}(t+2) \cap K = \emptyset$. Now we can assume that the individuals of $\mathcal{N}(t+1)$, who are all ignorants, are in the hope interval at time $t+1$, since otherwise we would have a good iteration after 1 time step. So there exist individuals $i \in \mathcal{N}(t+1)$ and $j \in \mathcal{F}(t+1)$ with $|x_i(t+1) - x_j(t+1)| \leq \varepsilon$. We set $x_i(t+1) = x_{\hat{l}(t+1)}(t+1) + \delta$, where $0 \leq \delta \leq \frac{\varepsilon\alpha\beta}{12}$ and calculate

$$\begin{aligned} x_i(t+2) &\geq x_i(t+1) - (1-2\beta)\delta + \underbrace{\beta \cdot 0}_{i} + \underbrace{\beta\left(\varepsilon - \frac{\varepsilon\alpha\beta}{12}\right)}_{j} \\ &\geq x_{\hat{l}(t+1)}(t+1) + \frac{\varepsilon\alpha\beta^2}{12} + \frac{\varepsilon\alpha\beta}{12} \\ &\geq x_{\hat{l}(t+2)}(t+2) + \frac{\varepsilon\alpha\beta}{12}. \end{aligned}$$

For the other direction we have

$$\begin{aligned} x_i(t+2) &\leq x_i(t+1) - \underbrace{\beta\delta}_{\hat{l}(t+1)} + (1-2\beta)\varepsilon \\ &\leq x_{\hat{l}(t+1)}(t+1) + \frac{\varepsilon\alpha\beta}{12} + \varepsilon - 2\beta\varepsilon \\ &\leq x_{\hat{l}(t+1)}(t+1) + \frac{\varepsilon\alpha\beta^2}{12} + \varepsilon \\ &\leq x_{\hat{l}(t+2)}(t+2) + \varepsilon. \end{aligned}$$

Thus, $i \in \mathcal{M}(t+2)$, which results in a good iteration in three time steps. □

Thus, we can conclude:

Corollary 4.4: *After a finite number $T_1(\varepsilon, n, \alpha, \beta)$ of steps we have $x_{\hat{l}(T_1)}(T_1) \ge h - \varepsilon - \frac{\varepsilon\alpha\beta}{12}$ and $x_{\hat{u}(T_1)}(T_1) \le h + \varepsilon + \frac{\varepsilon\alpha\beta}{12}$.*

Due to Lemma 3.1 there can not exist a general bound on the convergence that does not depend on α. We consider the two side lengths $\ell_2(t) := |x_{\hat{u}(t)}(t) - h|$ and $\ell_1(t) := |x_{\hat{l}(t)}(t) - h|$ of the hope interval. Clearly $\ell_1(t)$ and $\ell_2(t)$ are not increasing due to Lemma 3.7. For $t \ge T_1$ we have $\ell_1(t), \ell_2(t) \le \varepsilon + \frac{\varepsilon\alpha\beta}{12}$

Lemma 4.5: *If $\ell_1(t) + \ell_2(t) \le \varepsilon$ then we have*

$$(\ell_1(t+2) + \ell_2(t+2)) \le (\ell_1(t) + \ell_2(t)) \cdot \left(1 - \frac{\alpha\beta}{2}\right).$$

Proof: Let us assume, without loss of generality, that $\ell_1(t) \ge \ell_2(t)$. At first we consider the case $\ell_2(t) > 0$. If i is an ignorant with $x_i(t) = h - \ell_1(t) + \delta$ then we have

$$\begin{aligned} x_i(t+1) &\ge h - \ell_1(t) + \delta - (1-2\beta)\delta + \beta(\ell_1(t) + \ell_2(t) - \delta) \\ &\ge h - (1-\beta)\ell_1(t). \end{aligned}$$

For a truth seeker i with $x_i(t) = h - \ell_1(t) + \delta$ we have

$$\begin{aligned} x_i(t+1) \ge\ & h - \ell_1(t) + \delta - \alpha(\delta - \ell_1(t)) - (1-\alpha)(1-2\beta)\delta + \\ & (1-\alpha)\beta(\ell_1(t) + \ell_2(t) - \delta) \\ \ge\ & h - \ell_1(t) + \beta\delta(1-\alpha) + \alpha\ell_1(t)(1-\beta) + \beta\ell_2(t)(1-\alpha) + \beta\ell_1(t) \\ \ge\ & h - (1-\beta)\ell_1(t). \end{aligned}$$

Similarly we obtain $x_i(t+1) \le h + (1-\beta)\ell_2(t)$ in both cases.

Next we consider the case $\ell_1(t) > \ell_2(t) = 0$ and $l_i(t+1) > l_i(t) \cdot \left(1 - \frac{\alpha}{2}\right)$. Let i be an arbitrary truth seeker with opinion $x_i(t) = h - \ell_1(t) + \delta$. We have

$$\begin{aligned} x_i(t+1) &\ge h - \ell_1(t) + \delta + \alpha(\ell_1(t) - \delta) - (1-\alpha)(1-\beta)\delta \\ &\ge h - \ell_1(t) + \alpha\ell_1(t). \end{aligned}$$

Thus, we have $x_i(t+1) \ge h - \ell_1(t+1) + \frac{\alpha}{2} \cdot \ell_1(t)$. If j is an ignorant with $x_j(t+1) = h - \ell_1(t+1) + \delta$, then we have

$$\begin{aligned} x_j(t+2) &\ge h - \ell_1(t+1) + \delta - (1-2\beta)\delta + \beta\left(\frac{\alpha}{2} \cdot \ell_1(t) - \delta\right) \\ &\ge h - \ell_1(t) + \frac{\alpha\beta\ell_1(t)}{2}. \end{aligned}$$

For an arbitrary truth seeker j we have

$$\begin{aligned} x_i(t+2) &\ge h - \ell_1(t+1) + \alpha\ell_1(t+1) \\ &\ge h - \ell_1(t) + \frac{\alpha\beta\ell_1(t)}{2}. \end{aligned}$$

Thus, in all cases we have $(\ell_1(t+2) + \ell_2(t+2)) \le (\ell_1(t) + \ell_2(t)) \cdot \left(1 - \frac{\alpha\beta}{2}\right)$. □

This states that once the length of the hope interval becomes at most ε its length converges to zero.

Lemma 4.6: *Let $t \geq T_1$. If there exists an individual i with $\frac{\alpha\beta\ell_1(t)}{12} \leq d(\hat{l}(t), i, t) \leq \varepsilon$, then we have $\ell_1(t+1) \leq \ell_1(t) \cdot \left(1 - \frac{\alpha\beta^2}{12}\right)$. If there exists an individual i with $\frac{\alpha\beta\ell_2(t)}{12} \leq d(\hat{u}(t), i, t) \leq \varepsilon$, then we have $\ell_2(t+1) \leq \ell_2(t) \cdot \left(1 - \frac{\alpha\beta^2}{12}\right)$.*

Proof: Due to symmetry it suffices to prove the first statement. Let j be an ignorant with $x_j(t) = h - \ell_1(t) + \delta$, where $\delta \geq 0$. We have

$$\begin{aligned} x_j(t+1) &\geq h - \ell_1(t) + \delta - (1-2\beta)\delta + \underbrace{\beta\left(\frac{\alpha\beta\ell_1(t)}{12} - \delta\right)}_{i} \\ &\geq h - \left(1 - \frac{\alpha\beta^2}{12}\right)\ell_1(t). \end{aligned}$$

For a truth seeker j with $x_j(t) = h - \ell_1(t) + \delta$, $\delta \geq 0$ we have

$$\begin{aligned} x_j(t+1) &\geq h - \ell_1(t) + \delta + \alpha(\ell_1(t) - \delta) - (1-\alpha)(1-2\beta)\delta + \\ &\quad \underbrace{(1-\alpha)\beta\left(\frac{\alpha\beta\ell_1(t)}{12} - \delta\right)}_{i} \\ &\geq h - \ell_1(t) + \beta\delta(1-\alpha) + \alpha\ell_1(t)\left(1 - \frac{\alpha\beta^2}{12}\right) + \frac{\alpha\beta^2\ell_1(t)}{12} \\ &\geq h - \left(1 - \frac{\alpha\beta^2}{12}\right)\ell_1(t). \end{aligned}$$

□

For transparency we introduce the following six sets:

$$\begin{aligned} \mathcal{N}_1(t) &:= \left\{ i \in [n] \mid d(\hat{l}(t), i, t) < \frac{\alpha\beta\ell_1(t)}{12} \right\}, \\ \mathcal{N}_2(t) &:= \left\{ i \in [n] \mid d(\hat{u}(t), i, t) < \frac{\alpha\beta\ell_2(t)}{12} \right\}, \\ \mathcal{M}_1(t) &:= \left\{ i \in [n] \mid \frac{\alpha\beta\ell_1(t)}{12} \leq d(\hat{l}(t), i, t) \leq \varepsilon \right\}, \\ \mathcal{M}_2(t) &:= \left\{ i \in [n] \mid \frac{\alpha\beta\ell_2(t)}{12} \leq d(\hat{u}(t), i, t) \leq \varepsilon \right\}, \\ \mathcal{F}_1(t) &:= \left\{ i \in [n] \mid d(\hat{l}(t), i, t) > \varepsilon,\, x_i(t) \leq h + \ell_2(t) \right\}, \\ \mathcal{F}_2(t) &:= \{ i \in [n] \mid d(\hat{u}(t), i, t) > \varepsilon,\, x_i(t) \geq h - \ell_1(t) \}. \end{aligned}$$

With this the individuals of the hope interval are partitioned into

$$\mathcal{N}_1(t) \cup \mathcal{M}_1(t) \cup \mathcal{F}_1(t) = \mathcal{N}_2(t) \cup \mathcal{M}_2(t) \cup \mathcal{F}_2(t).$$

Lemma 4.7: *If for $k \in \{1, 2\}$ and $t \geq T_1$ there exists an ignorant $i \in \mathcal{N}_k(t)$ and an individual $j \in \mathcal{F}_k(t)$*

with $|x_i(t) - x_j(t)| \le \varepsilon$ *then* $l_k(t+2) \le l_k(t) \cdot \left(1 - \frac{\alpha\beta^2}{12}\right)$.

Proof: If $l_k(t+1) > l_k(t) \cdot \left(1 - \frac{\alpha\beta^2}{12}\right)$, then it is easy to check that the influence of Individual j suffices to put ignorant i in set $\mathcal{M}_k(t+1)$. In this case we can apply Lemma 4.6 □

Lemma 4.8: *If* $\mathcal{N}_k(t+1) \cap K \neq \emptyset$ *and* $t \ge T_1$ *then* $l_k(t+1) \le l_k(t) \cdot \left(1 - \frac{\alpha}{2}\right)$ *for* $k \in \{1,2\}$.

Proof: Due to symmetry it suffices to consider $k = 1$. So let i be a truth seeker with $i \in \mathcal{N}_1(t+1)$. We set $x_i(t) = h - \ell_1(t) + \delta$ and calculate

$$\begin{aligned} x_i(t+1) &\ge h - \ell_1(t) + \delta + \alpha(\ell_1(t) - \delta) - (1-\alpha)(1-\beta)\delta \\ &\ge h - (1-\alpha)\ell_1(t). \end{aligned}$$

□

Lemma 4.9: *For* $t \ge T_1$ *we have* $l_k(t+3) \le l_k(t) \cdot \left(1 - \frac{\alpha\beta^2}{12}\right)$ *for at least one* $k \in \{1,2\}$.

Proof: Due to Lemma 4.8 we can assume $\mathcal{N}_k(t+1) \cap K = \emptyset$. At time $t+1$ there must be a truthseeker i. Without loss of generality, we assume $x_i(t) \le h$ and $i = \tilde{l}(t+1)$. Due to Lemma 4.6 we can assume $i \in \mathcal{F}_1(t+1)$. Now let j_1 be the ignorant with smallest opinion fulfilling $d(i, j_1, t+1) \le \varepsilon$. If $j_1 \in \mathcal{N}_1(t+1)$ then we can apply Lemma 4.7 with j_1 and i. Otherwise we let j_2 be the ignorant with smallest opinion fulfilling $d(j_1, j_2, t+1) \le \varepsilon$. So we have $d(j_2, i, t+1) > \varepsilon$ and $j_2 \in \mathcal{N}_1(t+1)$. Thus, we can apply Lemma 4.7 with j_2 and j_1. □

Lemma 4.10: *If* $l_k(t) > \varepsilon$ *and* $t \ge T_1$ *then we have* $l_k(t+3) - \varepsilon \le (l_k(t) - \varepsilon) \cdot \left(1 - \frac{\alpha\beta^2}{12}\right)$ *or* $l_k(t+3) \le \varepsilon$.

Proof: Due to Lemma 4.8, we can assume $\mathcal{N}_k(t+1) \cap K = \emptyset$ and, due to Lemma 4.6, we can assume $\mathcal{M}_k(t+1) = \emptyset$. Due to symmetry, we only consider the case $k = 1$. Let $i \in \mathcal{N}_1(t+1)$ the ignorant with largest opinion $x_i(t+1)$, meaning that $d(\hat{l}(t+1), i, t+1)$ is maximal.

If there exists an individual $j \in \mathcal{F}_1(t+1)$ with $d(i, j, t+1) \le \varepsilon$, then we can apply Lemma 4.7. If no such individual j exists then we must have $d(i, 0, t+1) \le \varepsilon$ or $\ell_1(t+1) = 0$. So only the first case remains. We set $\delta = d(\hat{l}(t+1), i, t+1) \ge \varepsilon - \ell_1(t+1)$. Let $h \in \mathcal{N}_1(t+1)$ be an ignorant with $x_h(t+1) = x_{\hat{l}(t+1)}(t+1) + \mu$, where $0 \le \mu \le \delta$. For time $t+2$ we get

$$\begin{aligned} x_h(t+2) &\ge x_{\hat{l}(t+1)}(t+1) + \mu - (1-2\beta)\mu + \beta(\delta - \mu) \\ &\ge x_{\hat{l}(t+1)}(t+1) + \beta\delta \\ &\ge x_{\hat{l}(t+1)}(t+1) + \beta(\varepsilon - \ell_1(t+1)). \end{aligned}$$

□

From Lemma 4.9 and Lemma 4.10 we conclude:

Corollary 4.11: *There exists a finite number* $T_2(\varepsilon, n, \alpha, \beta)$ *so that we have*

$$\ell_1(t) + \ell_2(t) \le \varepsilon + \frac{\varepsilon\alpha^2\beta^3}{60}, \quad \text{and} \quad \min(\ell_1(t), \ell_2(t)) \le \frac{\varepsilon\alpha^2\beta^3}{60}$$

for all $t \ge T_2(\varepsilon, n, \alpha, \beta)$.

We would like to remark that, e.g., $T_2(\varepsilon, n, \alpha, \beta) = T_1(\varepsilon, n, \alpha, \beta) + \frac{36}{\alpha^2\beta^4}$ suffices.

Lemma 4.12: *For each* $t \ge T_2(\varepsilon, n, \alpha, \beta)$ *we have*

$$\ell_1(t+3) + \ell_2(t+3) \le \varepsilon$$

or

$$d(k,0,t) \le \frac{\varepsilon\alpha^2\beta^3}{60} \cdot \left(1 - \frac{\alpha\beta}{2}\right)^{\left\lfloor \frac{t-T_2}{2} \right\rfloor}$$

for all truth seekers $k \in K$.

Proof: Without loss of generality, we assume $\ell_1(t) \le \frac{\varepsilon\alpha^2\beta^3}{60}$ and prove the statement by induction on t. Due to Lemma 4.6 and Lemma 4.8, we can assume $\mathcal{M}_2(t+r) = \mathcal{N}_2(t+r) \cap K = \emptyset$ for $r \in \{0,1\}$ since otherwise we would have $\ell_1(t+3) + \ell_2(t+3) \le \varepsilon$. Thus, we have $d(k,0,t+r) \le \frac{\varepsilon\alpha^2\beta^3}{60}$ for all $k \in K$ and $K \subseteq \mathcal{F}_2(t+r)$.

Due to Lemma 4.6 for $r \in \{0,1\}$, the individuals in $\mathcal{F}_2(t+r)$ are not influenced by the individuals in $\mathcal{F}_2(t+r)$ since otherwise we would have $\ell_1(t+3) + \ell_2(t+3) \le \varepsilon$. Thus, we can apply Lemma 4.5 for the individuals in $\mathcal{F}_2(t)$. □

From the previous lemmas we can conclude Theorem 2.3 and Theorem 2.6.

Proof: (Proof of Theorems 2.3 and 2.6.) After a finite time $T_2(\varepsilon, n, \alpha, \beta)$ we are in a *nice* situation as described in Lemma 4.11. If we have $\ell_1(T_2+3) + \ell_2(T_2+3) \le \varepsilon$ then we have an ordinary convergence of the truth seekers being described in Lemma 4.5. Otherwise we have $d(k,0,T_2) \le \frac{\varepsilon\alpha^2\beta^3}{60}$ for all truth seekers $k \in K$. Due to Lemma 4.12 and Lemma 4.5 either we have

$$d(k,0,t) \le \frac{\varepsilon\alpha^2\beta^3}{60} \cdot \left(1 - \frac{\alpha\beta}{2}\right)^{\left\lfloor \frac{t-T_2}{2} \right\rfloor}$$

for all truth seekers $k \in K$ and all $t \ge T_2$, or there exists an $S \in \mathbb{N}$, such that we have

(1) $d(k,0,t) \le \frac{\varepsilon\alpha^2\beta^3}{60} \cdot \left(1 - \frac{\alpha\beta}{2}\right)^{\left\lfloor \frac{t-T_2}{2} \right\rfloor}$ for all $T_2 \le t \le S$,

(2) $d(k,0,t) \le \varepsilon \left(1 - \frac{\alpha\beta}{2}\right)^{\left\lfloor \frac{t-S-3}{2} \right\rfloor}$ for all $t \ge S+3$,

for all $k \in K$. The latter case is 1-fold interrupted convergence. Thus, the Hegselmann-Krause Conjecture is proven. □

5. Remarks

In this section we would like to generalize the Hegselmann-Krause Conjecture and show up which requirements can not be weakened.

Lemma 5.1: *A finite number n of individuals and symmetric confidence intervals are necessary for a convergence of the truth seekers.*

Proof: Infinitely many ignorants can clearly hinder a truth seeker in converging to the truth. If the confidence intervals are not symmetric then it is easy to design a situation where some ignorants are influencing a truth seeker which does not influence the ignorants, so that the truth seeker has no chance to converge to the truth. □

Lemma 5.2: *The condition $\beta_{ij}(t) \ge \beta > 0$ is necessary for a convergence of the truth seekers.*

Proof: If we only require $\beta_{ij}(t) > 0$, then we have the following example: $n = 2$, $x_1(0) = 1 - \frac{1}{5}\varepsilon$, $x_2(0) = 1 - \varepsilon$, $\alpha_1(t) = \frac{1}{5}$, $\alpha_2(t) = 0$, $\beta_{11}(t) = \left(\frac{1}{2}\right)^{t+1}$, $\beta_{12}(t) = 1 - \left(\frac{1}{2}\right)^{t+1}$, $\beta_{21}(t) = \left(\frac{1}{2}\right)^{t+1}$, $\beta_{22}(t) = 1 - \left(\frac{1}{2}\right)^{t+1}$, and $h = 1$. By a straight forward calculation we find that $|x_1(t) - h| \ge \frac{1}{2}\varepsilon$ for $t \ge 1$. □

We remark that conditions like $\beta_{ij}(t) + \beta_{ij}(t+1) \ge 2\beta$ would also not force a convergence of the truth seekers in general. One might consider an example consisting of two ignorants with starting positions

$h \pm \frac{7}{10}\varepsilon$ and a truth seeker k with starting position $h - \frac{1}{5}\varepsilon$. We may choose suitable $\beta_{ij}(t)$ and $\alpha_i(t)$ so that we have $|h - x_k(t)| \geq \frac{1}{5}\varepsilon$ for all t, $h - x_k(t) \geq \frac{1}{5}\varepsilon$ for even t and $x_k(t) - h \geq \frac{1}{5}\varepsilon$ for odd t.

For the next lemma we need a generalization of Definition 2.5.

Definition 5.3: Given ε, α, β, n, we say that the truth seekers $k \in K$ are r-fold interrupted convergent, if for each $\gamma > 0$ there exists $r+1$ functions $T_i^s(\gamma, \varepsilon, \alpha, \beta, n, T_{i-1}^e)$, $i = 1, \ldots, r+1$, so that for each (WASBOCOS) Ω with structural parameters ε, α, β and n there exist $T_i^e \in \mathbb{N}$, $i = 1, \ldots, r$ satisfying

$$\forall k \in K,\, \forall t \in [T_i^s(\gamma, \varepsilon, \alpha, \beta, n, T_{i-1}^s), T_i^e] : \, |x_k(t) - h| < \gamma$$

for $i = 1, \ldots, r$, where $T_0^e = 0$, and

$$\forall k \in K,\, \forall t \geq T_{r+1}^s(\gamma, \varepsilon, \alpha, \beta, n, T_r^e) : \, |x_k(t) - h| < \gamma.$$

Lemma 5.4: *The condition $\alpha_i(t) = 0$ for all $i \in \overline{K}$ is necessary for Theorem 2.6. If it is dropped then the truth seekers are not $(|\overline{K}| - 1)$-fold convergent in general.*

Proof: At first we remark that it clearly suffices to have $\alpha_i(t) = 0$ for all $i \in \overline{K}$ only for all $t \geq T$, where T is a fix integer. W.l.o.g. we assume $T = 0$ and consider the following example: $h = 1$, $x_i(0) = 1 - 2i\varepsilon$, $1 \in K$, $1 \neq i \in \overline{K}$, $\beta_{ij}(t) = \beta$, $\alpha_i(t) = \alpha$ for the truth seekers, and $\alpha_i(t) = 0$ for the ignorants until we say otherwise. Let there be a given $\gamma > 0$ being sufficiently small. There exists a time T_1 until $x_1(T_1) < 1 - \gamma$. Up to this time no other individual has changed its opinion. After time $T_1 + 1$ we suitably choose $\alpha_2(t)$ so that we have $\frac{1}{2}\varepsilon \leq x_1(\tilde{T}_1) - x_2(\tilde{T}_1) \leq \varepsilon$. So at time $\tilde{T}_1 + 1$ the convergence of truth seeker 1 is interrupted the first time. After that we may arrange it that x_1 and x_2 get an equal opinion and will never differ in there opinion in the future. Now there exists a time T_2 until $x_2(T_2) = x_1(T_2) < 1 - \gamma$ and we may apply our construction described above again. Thus, every ignorant $i \in \overline{K}$ may cause an interruption of the convergence of the truth seekers. □

Conjecture 5.5 If we drop the condition $\alpha_i(t) = 0$ for all $i \in \overline{K}$ in Theorem 2.6 then we have $(|\overline{K}|)$-fold convergence of the truth seekers.

The Hegselmann-Krause Conjecture might be generalized to opinions in $\mathbb{R}^m$ instead of $\mathbb{R}$ when we use a norm instead of $|\cdot|$ in the definition of the update formula. Using our approach to prove this m-dimensional conjecture would become very technical, so new ideas and tools are needed. We give an even stronger conjecture:

Conjecture 5.6 The m-dimensional generalized Hegselmann-Krause Conjecture holds and there exists a function $\phi(\Omega, \gamma)$ so that the truth seekers in an arbitrary generalized (WASBOCOS) Ω are $\phi(\Omega, \gamma)$-fold interrupted convergent in ε, α, β, and n.

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