Continuous opinion dynamics of multidimensional allocation problems under bounded confidence: More dimensions lead to better chances for consensus

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Continuous opinion dynamics of multidimensional allocation problems under bounded confidence: More dimensions lead to better chances for consensus

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Abstract

We study multidimensional continuous opinion dynamics, where opinions are nonnegative vectors which components sum up to one. Examples of such opinions are budgets or other allocation vectors which display a distribution of a fixed amount of ressource to n projects.
We use the opinion dynamics models of Deffuant-Weisbuch and Hegselmann-Krause, which both extend naturally to more dimensional opinions. They both rely on bounded confidence of the agents and differ in their communication regime. We show detailed simulation results regarding $n = 2, \dots, 8$ and the bound of confidence ε. Number, location and size of opinion clusters in the stabilized opinion profiles are of interest.
Known differences of both models repeat under higher opinion dimensions: Higher number of clusters and more minor clusters in the Deffuant-Weisbuch model, meta-stable states in the Hegselmann-Krause model. But surprisingly, higher dimensions lead to better chances for a vast majority consensus even for lower bounds of confidence. On the other hand, the number of minority clusters rises with n, too.

Keywords: Continuous opinion dynamics, multidimensional opinion, consensus, simplex, budget discussion

1 Introduction

Consider a group of m agents, each having an opinion about a certain issue. Consider that each opinion is a n-dimensional nonnegative vector of real numbers, which components sum up to one, e.g. the allocation of a fixed amount of money to n projects or the probabilities one admits to n propositions. The example we want to stress here is that each opinion is a budget plan proposal of an agent. So each agent has a proposal how to distribute a fixed amount of money to n departments. Such opinions should be held by members of a parliament which have to discuss on a states budget or by citizens of a town which have to work in a participatory budgeting process on the cities budget plan Participatory budgeting has been invented in 1989 in the Brazilian city of Porto Alegre and is now tested in several local communities all around the world (de Sousa Santos, 1998; Lorenz *et al.*, 2005). One argument for participatory budgeting is, that it fosters social consensus. This simulation gives some evidence that raising the number of decision parameter may indeed foster the evolution of consensus in the discussion.

The group of agents (politicians or citizens) is to find an agreement about the allocation. We suppose that each agent is willing to revise his allocation vector by taking the opinion vectors (budget plan proposals) of other competent agents into consideration. A competent agent in the view of one agent should be an agent with an opinion which is in a measurable way[1] not more than ε away from his own opinion. ε is called the *bound of confidence*. This process of repeated discussing and revising of opinions is called *continuous*[2] *opinion dynamics under bounded confidence*.

Continuous opinion dynamics of one-dimensional opinions under bounded confidence has been studied recently under different communication regimes. In the model of Deffuant-Weisbuch (DW) (Deffuant *et al.*, 2000; Weisbuch *et al.*, 2002) two random agents meet in each time step and compromise at the arithmetic mean[3] of their two opinions if their difference in opinion is beneath the bound of confidence ε. In the model of Hegelmann-Krause (HK) (Krause, 2000; Hegselmann *et al.*, 2002) all agents revise their opinions at the same time. Obviously, both models differ only in their communication regime. They have been generalized into one model by Urbig and Lorenz (Urbig *et al.*, 2004). To some extend each model represents the extrem point of possible communication regimes. While the HK model relies on the full knowledge of every opinion (e.g. in a general meeting where everyone communicates his opinion), the DW model relies totally on 'gossip' in random pairwise encounters of agents. We will use both models in our analysis, suggesting that real communication behavior lies somewhere in between. Thus, we hope that these two models serve as extreme points regarding communication structure.

Both models have been analysed roughly for heterogeneous bounds of confidence (Weisbuch *et al.*, 2002; Lorenz, 2003b) and on different network topologies (Amblard *et al.*, 2004; Fortunato, 2005). The DW model has been extended e. g. to extremism (Deffuant *et al.*, 2002) and to limited verbalisation capabilities of the agents (Urbig, 2003). In (Hegselmann, 2004) there are several extensions for the HK model. In contrast to this mass of extensions, studies of the multidi-

[1] the Euclidean distance in this study
[2] 'Continuous' refers to the type of opinions not to the time.
[3] With $\mu = 0.5$.

mensional case are rare, although both models extend naturally to real-valued vectors. The only mathematical thing to define new is the distance measure for vectors.

A short review of what exists with more dimensions. The vector opinions in (Weisbuch *et al.*, 2002) are bit-strings. The same holds for the Axelrod's famous model of cultural dissemination (Axelrod, 1997; M.F. Laguna *et al.*, 2003)[4]. Thus, these models have no continuous opinion space and regard different opinion issues. (Jager *et al.*, 2005; Urbig *et al.*, 2005) made interesting studies with two dimensional opinions but focussing on specific interplays between two opinion dimensions.

The naturally extended models have been studied in (Lorenz, 2003a; Fortunato *et al.*, 2005) but not under the budgeting restriction of sum-one vectors. Probably, most simulators hesitate to simulate the multidimensional case, because modelling and systematic characterisation of the parameter space gets indistinct. The only expedient onedimensional opinion spaces are intervals (e.g. $[0, 1]$). But in higher dimensions every convex set seems expedient. The same holds for the distance measure regarding the bound of confidence.

We decided to use the nonnegative sum-one vectors as opinion space (known as the unit simplex) for three reasons. First, we have a an important application with the issue of budgeting. Second, the unit simplex is the most simple polygon in higher dimension. For a fixed dimension, it has the lowest possible number of edges and faces. While opinion dynamics is known to be driven from the borders of the opinion space, we regard the unit simplex as the most simple case under maximal number of dimension, a structural argument.

And third we can derive the use of the arithmetic mean in multidimensional opinion dynamics from the axiomatisation of allocation aggregations in (Lehrer *et al.*, 1981)(page 112): If there are at least three departments and if the aggregation function assigns the allocation to each department purely as a function of the allocations to that department by the agents (irrelevance of alternatives), and respects their agreement in assigning a department the amount zero (zero unanimity), then the aggregation function is a weighted arithmetic mean.

In our models the revision an agent does with his opinion is an allocation aggregation of the opinions of others. Thus, if we extend our models to more than two dimensions and restrict them to allocation problems, we got a rationale to use the arithmetic mean by the axiomatisation (which we had not in the onedimensional case!). (See (Hegselmann, 2004) for other aggregation methods in the onedimensional case).

In the next section we will define the models (extended to multidimensionality) and give a review and comparison about the facts we know about the two onedimensional models. Section 3 shows the simulation results and gives some explanations by analysis of the geometry of the opinion space. Section 4 gives conclusions.

One disclaimer in advance: The simulation of these models is not to forecast

[4] In Axelrod's model a higher number of cultural features fosters the evolution of cultural homogeneity. Similarities to our results are discussed in section 4.

real opinion dynamics, but to give qualitative hints about underlying inherent dynamics of opinion formation.

2 The models and what we know

General definitions Let $\underline{m} := \{1, \dots, m\}$ be the set of agents and $\underline{n} := \{1, \dots, n\}$ the set of opinion dimensions. We define the *k-Simplex* $\Delta^k := \{x \in \mathbb{R}^{k+1} \mid x \geq 0, \sum_{i=1}^{k+1} x_i = 1\}$ which is our *opinion space*. Attention, the k-Simplex is a subset of $\mathbb{R}^{k+1}$, this is a mathematical convention because the k-Simplex has affine dimension k. This is clear due to the fact that every k dimensions automatically define the last dimension due to $x_{k+1} = 1 - (x_1 + \cdots + x_k)$. Thus, an opinion space with n opinion dimensions is Δ^{n-1}.

We will call $x^i \in \Delta^{n-1}$ the opinion of agent $i \in \underline{m}$. We call the vector of all opinion vectors $X(t) \in (\Delta^{n-1})^m$ the *opinion profile* at time step t. Thus $x_i^j(t)$ is the opinion of agent i about dimension j at time step t.

For the further development of the formalism it makes sense to think of the vector of vectors X as a matrix $m \times n$, where each row represents an agents opinion. (The opinion is a row vector here.)

Both models need a *bound of confidence* $\varepsilon \in \mathbb{R}_{>0}$ and a measure of distance of two (multidimensional) opinions $x^1, x^2 \in \Delta^n$. In this paper we will use the Euclidean distance $||x^1 - x^2||$ (derived from the Euclidean norm). Thus, $|| \cdot ||$ stands for the Euclidean norm in the respective multidimensional space throughout the paper. Surely, other distance measures are possible and interesting for further research.

We will define for both models the *process of continuous opinion dynamics* as a sequence of opinion profiles.

Definition 1 (Deffuant-Weisbuch Model) [5] *Given an initial profile* $X(0) \in (\Delta^{n-1})^m$ *and a bound of confidence* $\varepsilon \in \mathbb{R}_{>0}$ *we define the* Deffuant-Weisbuch process of opinion dynamics *as the random process* $(X(t))_{t \in \mathbb{N}_0}$ *that chooses in each time step* $t \in \mathbb{N}_0$ *two random*[6] *agents* $i, j \in \underline{m}$ *which perform the action*

$$x^i(t+1) \quad = \quad \begin{cases} \frac{1}{2}(x^i(t) + x^j(t)) & \text{if } ||x^i(t) - x^j(t)|| \leq \varepsilon \\ x^i(t) & \text{otherwise.} \end{cases}$$

The same for $x^j(t+1)$ *with* i *and* j *interchanged.*

Definition 2 (Hegselmann-Krause Model) [7] *Given a bound of confidence* $\varepsilon \in \mathbb{R}_{>0}$ *we define for an opinion profile* $X \in (\Delta^{n-1})^m$ *the* confidence matrix $A(X, \varepsilon)$ *as*

$$a_{ij}(X, \varepsilon) := \begin{cases} \frac{1}{\#I(i,X)} & \text{if } j \in I(i, X) \\ 0 & \text{otherwise,} \end{cases}$$

[5]This is a natural multidimensional extension of the basic version of the model in (Weisbuch *et al.*, 2002) with $\mu = 0.5$.

[6]With 'random' we mean 'random and equally distributed in the respective space'.

[7]This a natural multidimensional extension of the model in (Hegselmann *et al.*, 2002).

with $I(i,X) := \{j \in \underline{n} \,|\, ||x^i - x^j|| \le \varepsilon\}$*. ("#" stands for the number of elements.)*

Given an initial opinion profile $X(0) \in \mathbb{R}^n$*, we define the* Hegselmann-Krause process of opinion dynamics *as a sequence of opinion profiles* $(X(t))_{t\in\mathbb{N}_0}$ *recursively defined through*

$$X(t+1) = A(X(t),\varepsilon)X(t)$$

It has been proved analytically in (Lorenz, 2005b) that both processes converge to a stabilized opinion profile for every initial condition. Thus each process of continuous opinion dynamics converges to a stabilised opinion profile.[8]

One wants to know now how the stabilised profile looks like. But this depends heavily on the initial profile and for the DW process also on the choice of agents who meet. Thus, simulation has to play its role. Actually, both models have been studied more by simulation then analytically.

Facts about the one-dimensional case In nearly every study one takes initial profiles with random and equally distributed opinions between zero and one.

To circumvent the specific properties of one random setting one studies a set of several random initial profiles and computes all stabilised profiles. This set of stabilised profiles is analysed for clusters (e.g. opinion blocks or, political parties). The parameters of most interest are the number, the location and the size of clusters.

In Figure 1 we summarize what we know about the two models.

Description of figure 1 We give an example process with $m = 200$ agents and the bifurcation diagram dervied from the governing interactive Markov chain for each model. (See (Lorenz, 2005a; Lorenz, 2006) for details about the interactive Markov chain.) We set the opinion space to the line $[(0,1),(1,0)]$ which is equal to Δ^1 and has a Euclidean length $\sqrt{2}$ to get a good comparison with further figures.[9]

A quick description how the dynamic works: Agents will move to opinion regions with a high density of agents. Thus, in our case of uniform distribution the dynamic will start at the border of the opinion space with agents moving towards the center, creating a higher density there and thus attracting other agents even from the center.

We plot three confidence intervals around selected opinions in each of the two example processes to give an impression how far the confidence of one agent reaches. The vertical line in the bifurcation diagrams represents the ε-value (0.23) of the example.

[8] The first proof of stabilisation has been done in (Dittmer, 2001), but it holds only in the one-dimensional case.

[9] For comparison of the results with (Weisbuch *et al.*, 2002; Hegselmann *et al.*, 2002) we have to respect that the length of the opinion space there is one. We can compare our ε with $\sqrt{2}\varepsilon$ in (Weisbuch *et al.*, 2002; Hegselmann *et al.*, 2002).

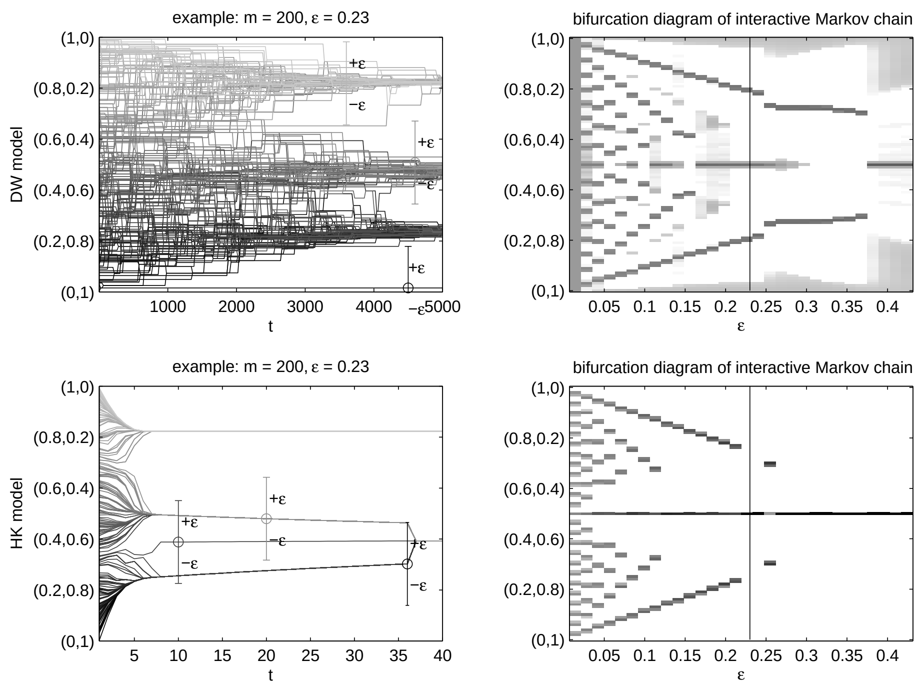

Figure 1: Overview about the one-dimensional case (two opinion dimensions in the allocation case) by example processes and bifurcation diagrams derived for interactive Markov chains.

The bifurcation diagrams we display are not as usual in bifurcation theory but 'reverse'. We have to read them from right to left to say that clusters bifurcate (or split) into more clusters. The ε-value where this happens is called a *bifurcation point*. The diagrams are very familiar with the bifurcation diagram in (Ben-Naim *et al.*, 2003) (which is not reverse). There is also a relationship to the figures in (Hegselmann *et al.*, 2002). In application we often look for the *consensus brink*[10] which is the bifurcation point form consensus to polarization. With this term in mind it is more natural to think of rising ε.

One observation in the bifurcation diagrams of figure 1 is that the consensus brink is significantly lower in the HK model as in the WD model. The following simulation study will show that the consensus brink can be lowered in both models by raising the number of dimensions under the restriction to allocation problems.

3 The impact of multidimensionality

Simulation setup The basis for our simulation results are 200 random opinion profiles out of $(\Delta^1)^{200}, \dots, (\Delta^7)^{200}$, so $n = 2, \dots, 8$. Choosing a random opinion uniformly distributed in a simplex Δ^k is not totally trivial. To ensure a uniform distribution and to avoid distortions by normalisation of random vectors we take a random vector out of $[0,1]^k$ and compute one more entry as 1 minus the sum of all other entries, but only if the sum of all former components is less or equal than 1. If we fail, we try it again.[11]

Further on, we explore the parameter space $\varepsilon = 0.15, \overset{+0.01}{\dots}, 0.42$. Thus, we ran 200 simulations runs for each pair (n, ε). Additionally, we took for $n = 3$ a set of 4000 initial profiles to compute the stabilized profiles for selected values of ε to start the exploration of more opinion dimensions.

Figures 2, 3, 4 and 5 show visualisations of relevant properties of the set of stabilised profiles. In the following we will deliver descriptions of the figures, a summary about the dynamics with three departments ($n = 3$) and a summary about the impact the number of departments n.

Description of Figure 2 We see the opinion space Δ^2 divided into several sub-triangles. The gray-scale of one triangle stands for the number of opinions of all 4000 stabilised profiles which are in that region (black is high, white low, and a spot is zero). The gray scale is only relative in each figure. It shows attractive regions for clusters. Additionally, we got two histograms in each subfigure. The histogram #clusters shows how often a stabilized profile got a specific number of clusters in all 4000 runs. It is important to notice that we count each cluster even isolated outliers. In the histogram about clustersizes we group all agents by the size of the cluster where they belong to. The bins are $1-10, 11-20, \dots, 191-200$.

[10] The term goes back to (Hegselmann *et al.*, 2004).

[11] Obviously, the probability for success is getting rapidly low for rising k. For that reason we had to restrict us at $n = 8$ for computation time reasons.

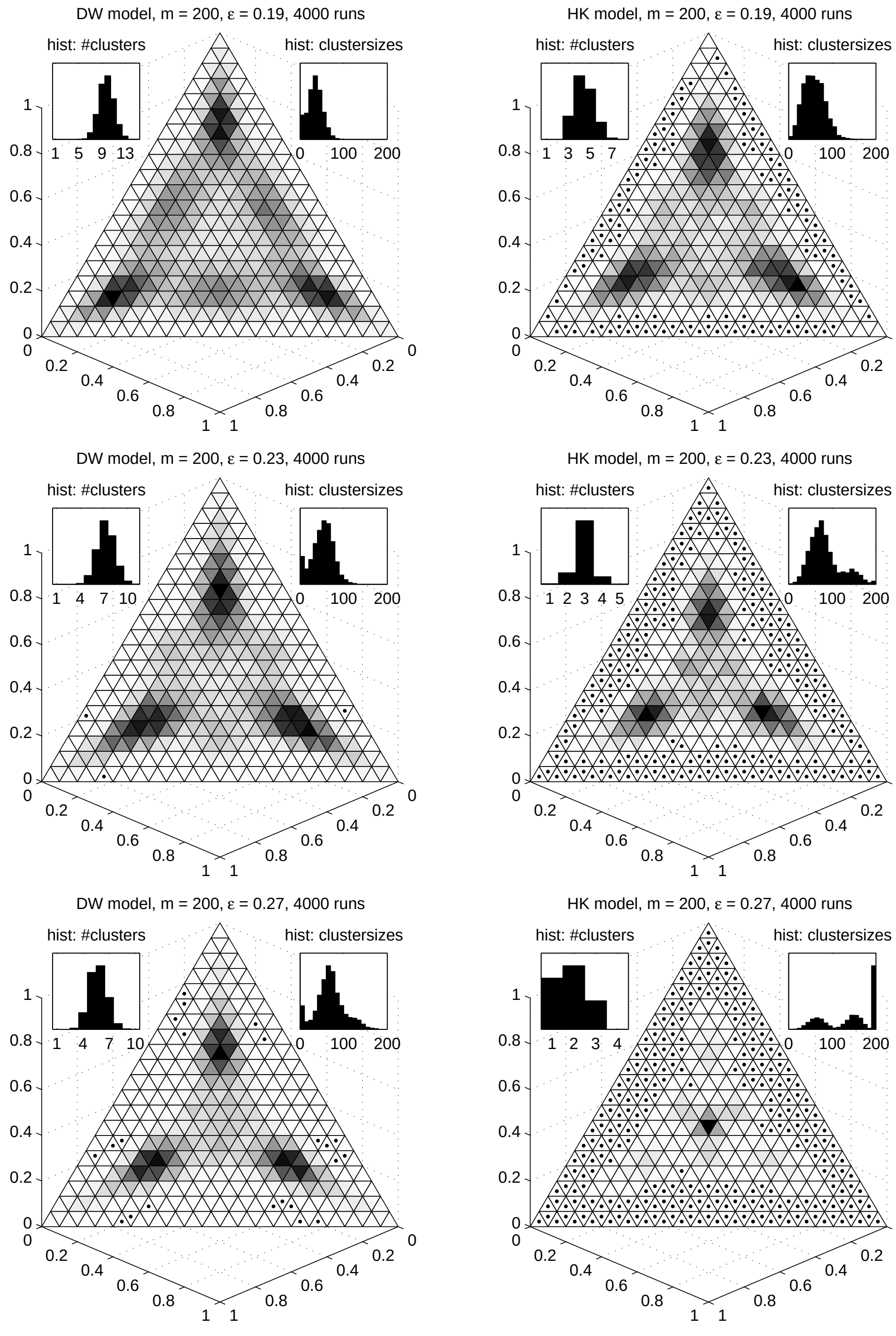

Figure 2: Overview about location and sizes of clusters with 3 opinion dimensions for selected ε-values.

Summary about dynamics with three departments ($n = 3$) Most of the results can be derived from figure 2.

1. The characteristic polarization into two big clusters for a great ε-interval in Δ^1 extends to a characteristic polarization into three big clusters in Δ^2. Each cluster represents a budget plan proposal like: "The biggest part for one department (around 70%) and the rest equally for the two others."

2. A polarization into two opinion clusters is also possible. It occurs by the union of two of the three characteristic clusters, but we can not predict which two clusters unite. In this situation we have two third of the agents saying about "40%/40%/20%" and the other third "15%/15%/70%".

3. Higher numbers of big clusters and more chances for minorities to survive in the DW model as reported in the one dimensional model occur also in Δ^2.

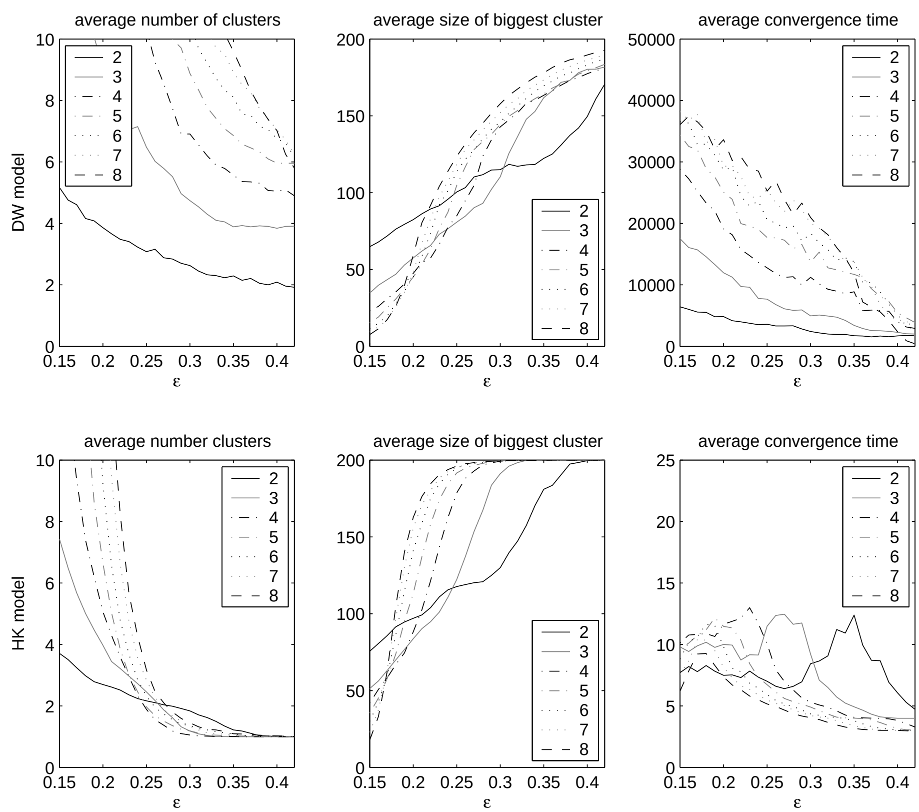

Figure 3: Average number of clusters, size of the biggest cluster and convergence time for $n = 2, \ldots, 8$.

Description of Figure 3 We collect some aggregated data about the 200 stabilised profiles for each pair (n, ε) and the two models. The left hand figure column shows the average number of clusters (arithmetic mean over all 200 runs). We count every cluster, even if it is only an isolated agent. Thus, another interesting parameter is the average size of the biggest cluster, which gives an impression how many agents have found a common agreement. This is shown in the central column figure. The right hand column shows average time to reach stabilisation. Observing stabilisation in the HK model is easy by checking if something has changed in one step. This does not work in the DW model. We use this: Every 2000 time steps we check the opinion profile for clusters (a set of directly or indirectly connected agents), if in each cluster the maximal distance of agents is below ε, we stop the time. Fortunately, we can compute the final location of the cluster by building the arithmetic mean (see (Urbig *et al.*, 2004) for details). Thus the convergence times in the figure is counted only to the time when clusters have built which can not split anymore, and the time we measure may be at least 1999 time steps to long.

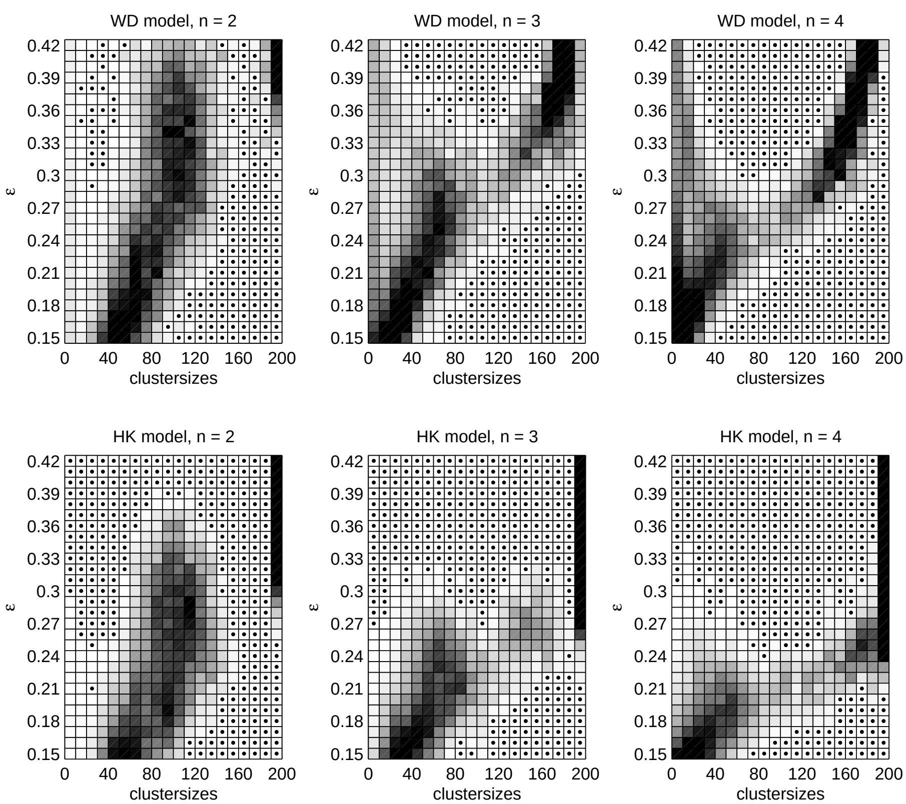

Figure 4: Histograms of the sizes of clusters for $n = 2, 3, 4$.

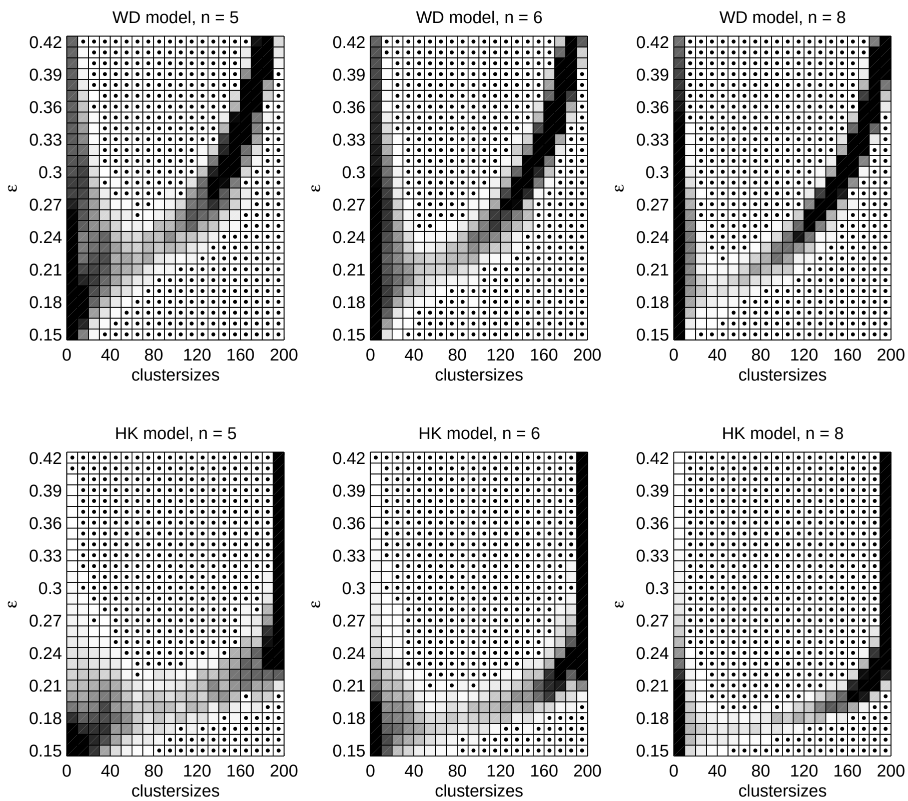

Figure 5: Histograms of the sizes of clusters for $n = 5, 6, 8$.

Description of Figure 4 and 5 In each row we got the histogram about the clustersizes. We group all agents by the size of the cluster where they belong to. The bins are $1-10, 11-20, \dots, 191-200$. This is the same as the histogram in figure 2 at the right hand side of each subfigure. The gray-scale of each histogram is only relative in each row. Black is a high number of agents in clusters of that size. White is a low number and a dot is no clusters of that size. We omit the subfigure for $n = 7$ it has no additional information.

The impact of the number of departments (n) This paragraph summarizes the key results of this paper. The conclusions can be derived from the figures 3, 4 and 5.

1. Raising the numbers of departments leads to more minor clusters not only in the DW model, but also in the HK model.
2. If we regard the existence of a cluster with a vast majority of agents (e.g. more than 150, so 75%) as a *majority-consensus* we can say that the majority-consensus brink is sinking with rising n. But sinking slows down.
3. The ε-interval between majority consensus and total plurality of opinions is getting shorter with rising n.
4. The convergence times in the HK model give a hint that we may reach meta-stable states with long convergence times in the ε-interval of majority-consensus close to the majority-consensus brink.

About the multidimensional opinion space In this paragraph we give some geometrical facts about the opinion space Δ^k which shed some light on the lowering of the majority consensus brink with rising n.

According to (Buchholz, 1992) the k-dimensional volume of Δ^k is

$$\mathrm{Vol}\Delta^k = \frac{\sqrt{k+1}}{k!} = \frac{\sqrt{k+1}}{\Gamma(k+1)}.$$

(The Γ-function is an extension of $n!$ to real numbers. It holds $\Gamma(n+1) = n!$.) The n-dimensional volume of the ball B^n_ε(regarding the Euclidean norm) with radius ε (our area of confidence) is

$$\mathrm{Vol}B^n_\varepsilon = \varepsilon^n \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}.$$

For that reason it holds that for each ε there is an n such that $\mathrm{Vol}B^n_\varepsilon > \mathrm{Vol}\Delta^k$ which means that the area of confidence is bigger than the whole opinion space. To give some numerical values: $\varepsilon = 0.23$ it is for $n \geq 10$, for $\varepsilon = 0.15$ it is for $n \geq 21$. That explains why we get that strong impact on majority-consensus with rising n.

On the other hand the Euclidean distance from one extreme opinion, e.g. $(1, 0, \dots, 0)$, to the barycenter of the opinion space $\frac{1}{n}(1, \dots, 1)$ (which is the attractive point for consensus) is $\sqrt{(1 - \frac{1}{n})^2 + (n-1)\frac{1}{n^2}}$ and is thus converging to 1 for $n \to \infty$ (which means that it comes infinitely close to the barycenter of each $n-1$-dimensional face of Δ^n). This explains the growing probability for extremists to get cut of the others during the process.

Colloquial: The opinion space of allocation vectors transforms with growing number of departments to a space with greater compromising opportunities in the center but growing hiding-places for extremists.

Actually, in our random profiles of Δ^7 some extreme agents were disconnected from the other agents from the very beginning (for the lower ε-values).

Thus, the geometry of the simplex opinion space is a major source of the phenomena described above. Performing the same analysis with e.g. a cubic opinion space will probably not show such drastic effects when raising the dimension of the cube (for some hints see (Lorenz, 2003a).

4 Conclusion

We come back to the issue of social consensus. From our simulation results one may conclude that a public debate and decision about a continuous issue may come to a social majority consensus more likely if we give not only two opposite possibilities but more to allocate the resources between. This holds under a gossip like communication (DW) as well as under a meeting driven communication (HK).

Main assumptions of the models were bounded confidence of all agents, arithmetic averaging as opinion aggregation and a uniform distribution of opinions at the beginning. This is obviously not the case in each real situations when budget discussion begins. Long time ideologies, hard positions regarding certain departments and pre-clustering by political parties and lobby groups certainly play important roles. Probably, most of these issues do not have a good impact on the chances of social consensus. Simplifying the opinion space by projecting it to a lower dimensional (e.g. twodimensional) opinion space as it is often done by mass media or populists, may, due to our results, significantly lower chances of reaching a social consensus.

A main impact factor is the structure of the opinion space itself. Thus, the restriction to a fixed amount of money to distribute (without permitting negative amounts) has a surprisingly strong force to attract agents to the consensual center. Colloquial: The fact that one can only suggest a higher budget for one department if one lowers the budgets of other departments produces an opinion space which is more appropriate to foster social consensus.

Nevertheless, the impact of fostering social majority consensus by raising the number of dimensions is sinking if we have already a multidimensional opinion space. Raising from 7 to 8 dimensions has not the same impact as raising from 2 to 3. The drawback of raising the number of dimensions is that we open new space for extremists in each corner, which gets more and more loosely connected to the center. Thus, the social majority consensus is getting more and more a slim majority consensus, especially if we rely only on gossip dynamics.

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