



Order preservation in a generalized version of Krause's opinion dynamics model

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ARTICLE INFO

Article history:

Received 31 August 2007

Received in revised form 11 March 2008

Available online 18 May 2008

Keywords:

Opinion dynamics

Continuous opinions

Social systems

Multi-agent systems

ABSTRACT

Krause's model of opinion dynamics has recently been the subject of several studies, partly because it is one of the simplest multi-agent systems involving position-dependent changing topologies. In this model, agents have an opinion represented by a real number and they update it by averaging those agent opinions distant from their opinion by less than a certain interaction radius. Some results obtained on this model rely on the fact that the opinion orders remain unchanged under iteration, a property that is consistent with the intuition in models with simultaneous updating on a fully connected communication topology.

Several variations of this model have been proposed. We show that some natural variations are not order preserving and therefore cause potential problems with the theoretical analysis and the consistence with the intuition. We consider a generic version of Krause's model parameterized by an "influence function" that encapsulates most of the variations proposed in the literature. We then derive a necessary and sufficient condition on this function for the opinion order to be preserved.

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1. Introduction

Dynamics of opinions and propagation of beliefs are the object of many studies in the literature. Agents have an opinion which can be a continuous value [2,3,7,11,21] or restricted to discrete or even binary sets [13,23]. The evolution of the agents' opinions is influenced by the opinions of other agents, which typically are their neighbors on some fixed graph [1,7,23] or are randomly selected at each iteration [6,16]. The originality of Krause's model proposed in 1997 [17], and also known as Hegselmann-Krause model after [14], is that the interaction graph is not fixed or randomly defined, but depends on the agents opinions in a deterministic way: two agents influence each other if their opinion are not too different. Formally, agents have a value $x_i \in \mathcal{X}$ interpreted as their opinion on some subject, and they update it synchronously at every time-step by taking a new opinion x'_i defined by

$$x'_i = \frac{\sum_{j: |x_i - x_j| \leq r} x_j}{|\{j: |x_i - x_j| \leq r\}|}, \quad (1)$$

where the vision range r is a pre-specified constant, and $|\{j: |x_i - x_j| \leq r\}|$ is the number of agents whose opinions differ from x_i by at most r . Note that this model presents similarities with the non-deterministic model of Deffuant et al. [6]. Krause's model has recently been the subject of a wide study [4,5,9,10,12,14,15,18,20] due inter-alia to the fact that it is one

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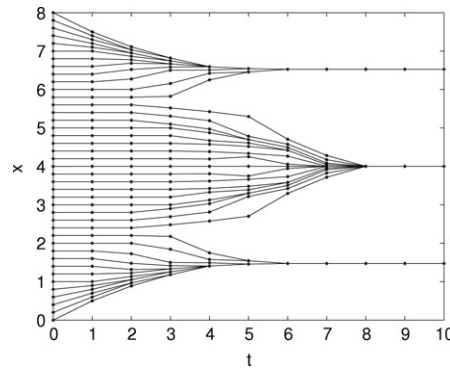


Fig. 1. Evolution of 41 opinions initially equidistantly distributed on $[0, 8]$ during 10 iterations. At each time t , each opinion is obtained from those at time $t - 1$ by the model given in (1), with $r = 1$.

of the simplest multi-agent model involving position-dependent topologies, much simpler for example than the famous Vicsek swarming model [24]. Fig. 1 shows an example of opinions evolving according to (1) for 10 iterations. The opinions converge in finite time to opinion clusters separated by more than r as shown in [8] and also in [19] in a more general context. The exact distance between clusters at equilibrium has actually a more complex behavior, which is studied for example in Refs. [4,5,15]. One can also see that the opinion order is preserved at each iteration, if $x_i \leq x_j$ then $x'_i \leq x'_j$. This property proved in Ref. [18] is consistent with the intuition that agent opinions that evolve in a one-dimensional space according to the same rules and whose communications are not artificially restrained to an arbitrary topology should not cross each other. Moreover, the proofs of several important properties of Krause's model explicitly use the fact that the opinion order is preserved [4,5,8,15].

Several extensions of Krause's model have been introduced in the literature. Asymmetric behaviors are for example considered in [14]. An agent i takes j into account if $x_j \in [x_i - r_l, x_i + r_r]$, and the update rule is then

$$x'_i = \frac{\sum_{j: -r_l \leq x_j - x_i \leq r_r} x_j}{|\{j : -r_l \leq x_j - x_i \leq r_r\}|}. \quad (2)$$

Fig. 2(b) shows the evolution of opinions according to this model for $r_l = \frac{3}{4}$ and $r_r = \frac{5}{4}$. Unsurprisingly, opinions converge to clusters with higher value than in the symmetric case $r_l = r_r = r = 1$ represented in Fig. 2(a) for comparison. In another extension of the model proposed in Ref. [22], the author proposes to weight the agent opinions depending on the distance separating them. One can for example weight by 1 the opinions that are at distances between $\frac{1}{10}$ and 1 and by 5 those at distance at most $\frac{1}{10}$:

$$x'_i = \frac{\sum_{j: \frac{1}{10} < |x_i - x_j| \leq 1} x_j + \sum_{j: |x_i - x_j| \leq \frac{1}{10}} 5x_j}{|\{j : \frac{1}{10} < |x_i - x_j| \leq 1\}| + 5|\{j : |x_i - x_j| \leq \frac{1}{10}\}|}. \quad (3)$$

Opinions evolving according to this rule are represented in Fig. 2(c), where it can be seen that convergence is slower than with the usual model (1). Although this model seems to be a natural one, Fig. 3 shows that it does not necessarily preserve the opinion order. This renders its potential validity questionable, as the possibility for opinions evolving according to the same rules to cross each other can be subject to debate. Also, it makes the analysis of such a model more challenging. The analysis of the initial model (1) in Refs. [4,5] uses indeed extensively the order preservation property, and its adaptation to models such as (3) can thus be uneasy.

In order to characterize those variations of Krause's model that preserve the opinion order, we consider the following generic model already suggested in Ref. [22] and encapsulating all value-independent variations of Krause's model. The generic update rule is

$$x'_i = \frac{\sum_j f(x_j - x_i) x_j}{\sum_j f(x_j - x_i)}, \quad (4)$$

where $f : \mathcal{R} \rightarrow \mathcal{R}^+$ is a nonnegative function whose support¹ is a positive length interval containing 0. We call the function f an *influence function*. In particular, the model (1) is obtained by taking $f_1 = \chi_{[-r, r]}$ as influence function, where the χ_S

¹ The support of a non-negative function is the set on which it takes positive values.

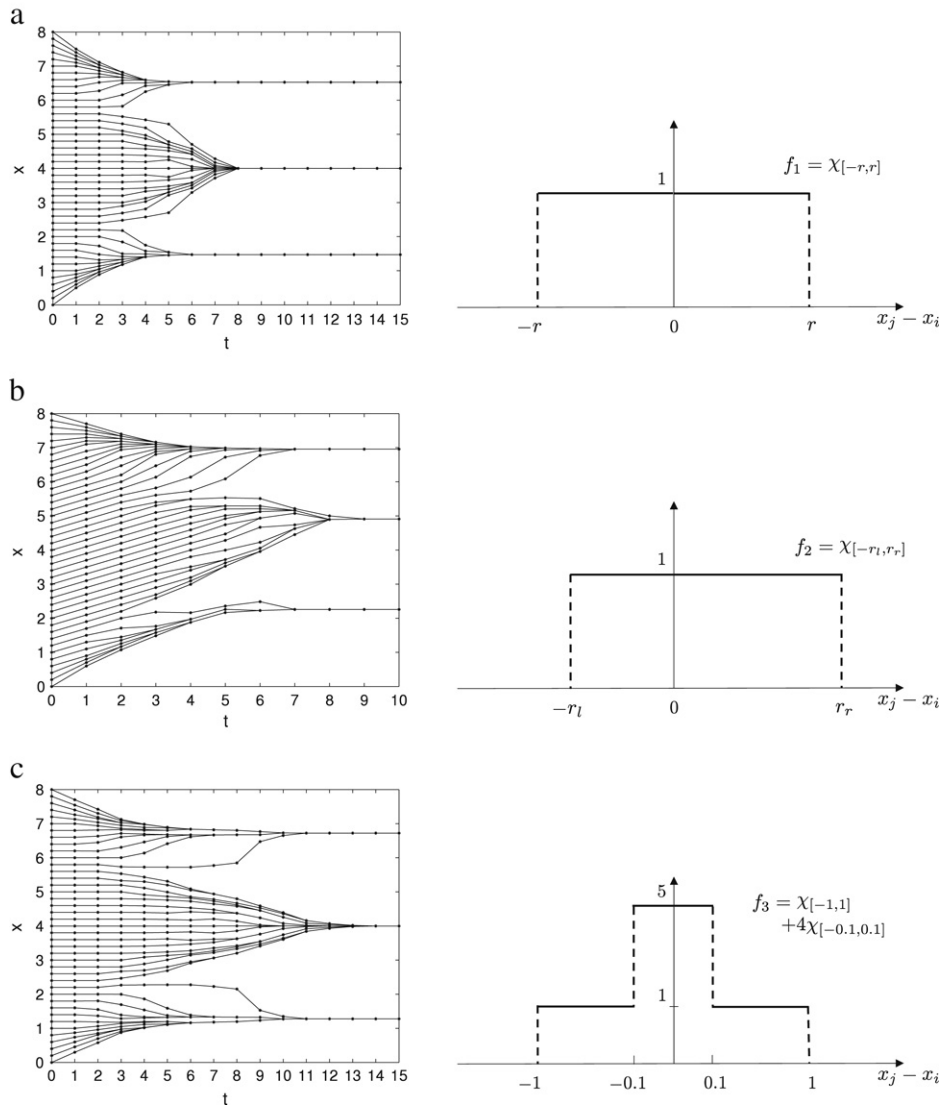


Fig. 2. Evolution with time of 41 opinions initially equidistantly distributed on $[0, 8]$. The opinions follow the model (1) with $r = 1$ in (a), the model (2) with $r_l = \frac{5}{4}$, $r_r = \frac{3}{4}$ in (b) and the model (3) in (c). The influence functions describing the respective models are also represented.

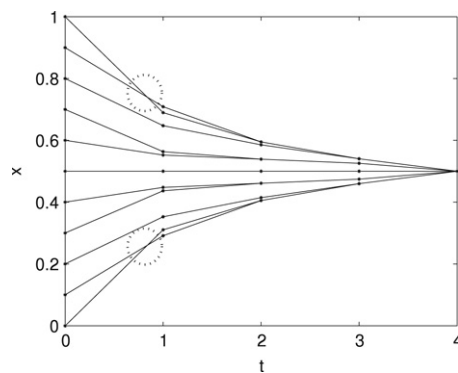


Fig. 3. Evolution with time of 11 opinions initially equidistantly distributed on $[0, 1]$ and following the model (3). The order of opinions is not preserved between $t = 0$ and $t = 1$.

defined for any set S is the *indicator function* of S , that takes the value 1 on S and 0 everywhere else. The asymmetric model (2) corresponds to $f_2 = \chi_{[-r_l, r_r]}$, and the model (3) to $f_3 = \chi_{[-1, 1]} + 4\chi_{[-\frac{1}{10}, \frac{1}{10}]}$ as represented in Fig. 2. Lorenz also proposes in Ref. [22] a time varying function $f(y) = e^{-|y|^t}$, but we do not consider time-varying functions here. Note finally that the idea of describing various extensions of an opinion dynamics model via an influence function has also been applied to the model introduced by Deffuant et al. in Ref. [6] as mentioned in Ref. [25].

We say that an influence function is *order preserving* if for any state vector $x \in \mathfrak{R}^n$, the updated vector obtained by Eq. (4) satisfies $x'_i \leq x'_j$ for any i, j such that $x_i \leq x_j$. Note that when a function is not order preserving, there may very well exist many different initial conditions such that no crossing takes place along the evolution of the system. However, there always exists at least one initial state vector $x \in \mathfrak{R}^n$ such that the order of two opinions is inverted after one opinion update. If no such vector exist, then the function is order preserving. The function f_3 in Fig. 2 is for example not order preserving as shown in Fig. 3, while it can be proved that the functions f_1 and f_2 are order preserving.

We give in Section 2 a simple necessary and sufficient condition for a function to be order preserving. We show in Section 3 how this condition relates to the log-concavity of the influence function. We close the paper in Section 4 with concluding remarks and the mention of two open questions.

2. Algebraic condition for order preservation

We first consider a very simple system with $2n + 2$ agents. The agents 1 and 2 have opinions $x_1 = a$ and $x_2 = b$ respectively, for some $b > a$. Among the remaining $2n$ agents, n have an opinion $a + b$ and n others an opinion $a + b + c$ for some $c > 0$. We suppose that 1 and 2 both take all other agents into account, that is, $f(a), f(b), f(a + c), f(b + c) > 0$. If n is sufficiently large, we can neglect the agents 1 and 2 in the computation of x'_1 and x'_2 , which according to (4) are given by

$$x'_1 \simeq \frac{nf(b)(a + b) + nf(b + c)(a + b + c)}{nf(b) + nf(b + c)} = a + b + \frac{c}{1 + f(b)/f(b + c)},$$

$$x'_2 \simeq \frac{nf(a)(a + b) + nf(a + c)(a + b + c)}{nf(a) + nf(a + c)} = a + b + \frac{c}{1 + f(a)/f(a + c)}.$$

So if $\frac{f(a)}{f(a + c)} > \frac{f(b)}{f(b + c)}$, then $x'_1 > x'_2$ although $x_1 \leq x_2$. As a result, if f is order preserving, for any $a < b$ and $c > 0$, there holds $\frac{f(a + c)}{f(a)} \geq \frac{f(b + c)}{f(b)}$, for otherwise we could build the example above.

To see that this simple condition is also sufficient for order preservation, we now consider another system of n agents among which we select two agents p and q such that $x_q \geq x_p$ (we may possibly chose $p = q$). We suppose that $f(x_i - x_p) > 0$ and $f(x_i - x_q) > 0$ for all agents i . Since the system (4) is translation-invariant we assume that all x_i are nonnegative, and we relabel the agents in such a way that $x_1 \leq x_2 \leq \dots \leq x_n$. The updated values of x_p and x_q are

$$x'_p = \frac{\sum_{i=1}^n f(x_i - x_p)x_i}{\sum_{i=1}^n f(x_i - x_p)}, \quad \text{and} \quad x'_q = \frac{\sum_{i=1}^n f(x_i - x_q)x_i}{\sum_{i=1}^n f(x_i - x_q)}.$$

As a consequence, $x'_q \geq x'_p$ holds if

$$\left(\sum_{i=1}^n f(x_i - x_q)x_i \right) \left(\sum_{i=1}^n f(x_i - x_p) \right) \geq \left(\sum_{i=1}^n f(x_i - x_p)x_i \right) \left(\sum_{i=1}^n f(x_i - x_q) \right)$$

holds. This can be rewritten as

$$\begin{aligned} & \left(\sum_{i=1}^{n-1} f(x_i - x_q)x_i \right) \left(\sum_{i=1}^{n-1} f(x_i - x_p) \right) + f(x_n - x_q)x_n \left(\sum_{i=1}^{n-1} f(x_i - x_p) \right) \\ & + f(x_n - x_p) \left(\sum_{i=1}^{n-1} f(x_i - x_q)x_i \right) + f(x_n - x_q)f(x_n - x_p)x_n \\ & \geq \left(\sum_{i=1}^{n-1} f(x_i - x_p)x_i \right) \left(\sum_{i=1}^{n-1} f(x_i - x_q) \right) + f(x_n - x_p)x_n \left(\sum_{i=1}^{n-1} f(x_i - x_q) \right) \\ & + f(x_n - x_q) \left(\sum_{i=1}^{n-1} f(x_i - x_p)x_i \right) + f(x_n - x_p)f(x_n - x_q)x_n. \end{aligned}$$

For $n = 1$ (and thus $a = b = 1$), this relation reduces to $f(0)^2 x_1 \geq f(0)^2 x_1$ and is trivially satisfied. Suppose now that it holds for $n - 1$, then it also holds for n provided that

$$\begin{aligned}
& f(x_n - x_q)x_n \left(\sum_{i=1}^{n-1} f(x_i - x_p) \right) + f(x_n - x_p) \left(\sum_{i=1}^{n-1} f(x_i - x_q)x_i \right) \\
& \geq f(x_n - x_p)x_n \left(\sum_{i=1}^{n-1} f(x_i - x_q) \right) + f(x_n - x_q) \left(\sum_{i=1}^{n-1} f(x_i - x_p)x_i \right)
\end{aligned} \tag{5}$$

holds. Reorganizing the terms of (5) and dividing them by $f(x_n - x_p)f(x_n - x_q)x_n > 0$ yields

$$\sum_{i=1}^{n-1} \left(\frac{f(x_i - x_p)}{f(x_n - x_p)} - \frac{f(x_i - x_q)}{f(x_n - x_q)} \right) \geq \sum_{i=1}^{n-1} \frac{x_i}{x_n} \left(\frac{f(x_i - x_p)}{f(x_n - x_p)} - \frac{f(x_i - x_q)}{f(x_i - x_n)} \right).$$

Since all x_i are nonnegative and no greater than x_n , it is sufficient for this relation to hold that $\frac{f(x_i - x_p)}{f(x_n - x_p)} \geq \frac{f(x_i - x_q)}{f(x_n - x_q)}$ holds for all i . Since $x_p \leq x_q$, the latter is always true if f is such that $\frac{f(a+c)}{f(a)} \geq \frac{f(b+c)}{f(b)}$ holds for any $b \geq a$ and $c \geq 0$ for which $f(a), f(b), f(a+c), f(b+c) > 0$. It suffices indeed to take $a = x_i - x_q$, $b = x_i - x_p$ and $c = x_n - x_i$.

Suppose now that there is some i for which $f(x_i - x_p) > 0$ and/or $f(x_i - x_q) > 0$ does not hold. Let J_p be the set of agents i such that $f(x_i - x_p) > 0$, J_q the corresponding set for x_q and $I = J_p \cap J_q$. If $I = \emptyset$, then any value of J_q is larger than all values of J_p as the support of f is an interval, so that $x'_q \geq x'_p$ trivially holds. If $I = J_p \cup J_q$, we have seen that the necessary condition for order preservation is sufficient for $x'_q \geq x'_p$ to hold. Finally, observe that the presence of agents in $J_q \setminus I$ or in $J_p \setminus I$ only increases x'_q or decreases x'_p , so that this condition is still sufficient for $x'_q \geq x'_p$ to hold. We have thus proved the following result:

Theorem 1. An influence function $f : \mathfrak{R} \rightarrow \mathfrak{R}^+$ is order preserving if and only if

$$\frac{f(a+c)}{f(a)} \geq \frac{f(b+c)}{f(b)} \tag{6}$$

holds for all $a \leq b, c \geq 0$ such that $f(a), f(b), f(a+c), f(b+c) > 0$

Note that the model of Krause can be extended to continuous distribution of opinions [4,5,12,15,21] Theorem 1 can also be proved for such systems, replacing sums by integrals, and assuming that f is measurable.

To understand the intuitive meaning of this result, consider three agents i, p and q , with $x_i < x_p < x_q$. The values $f(x_p - x_i)$ and $f(x_q - x_i)$ represent the “importance” given respectively to p and q by i when computing its new opinion, that is, the weight given by i to the opinions of p and q respectively. The ratio $\frac{f(x_p - x_i)}{f(x_q - x_i)}$ is thus large if i discriminates q with respect to p , i.e. gives much more importance to p than to q , and small otherwise. Observe now that an agent having an opinion $x_i - c$ would have a “discriminating ratio” $\frac{f(x_p - x_i + c)}{f(x_q - x_i + c)}$. Taking $a = x_p - x_i$ and $b = x_q - x_i$, one can verify that the condition of Theorem 1 implies that an agent with an opinion $x_i - c$ should discriminate more p from q than an agent with opinion x_i . Since this is true for any c and x_i , the condition of Theorem 1 means that the more remote the opinion of an agent is, the more it should discriminate p from q (with $x_p < x_q$). This may seem surprising as one would expect that an agent having an opinion very different from x_p and x_q should treat them more equally than an agent having an opinion close to one of them.

Using Theorem 1, one can see that the function f_3 in Fig. 2 is not order preserving, as observed in Fig. 3. Take indeed $a = 0$, $b = 0.5$ and $c = 0.2$. We have $a \leq b, c \geq 0$, and there holds

$$\frac{f_3(a+c)}{f_3(a)} = \frac{f_3(0.2)}{f_3(0)} = \frac{1}{5} < \frac{1}{1} = \frac{f_3(0.7)}{f_3(0.5)} = \frac{f_3(b+c)}{f_3(b)},$$

so that f_3 does not satisfy the condition of Theorem 1. It will be seen later that no “reasonable function” is order preserving if it is discontinuous somewhere in the interior of its domain. One can prove on the other hand that the functions f_1 and f_2 in Fig. 2 do satisfy these conditions and are thus order preserving. Proving that a function satisfy these conditions or finding a, b and c that invalid them may however not always be trivial. For this reason, we show in the next section that these conditions can be re-expressed in term of the concavity of $\log f$.

3. Log-concavity of influence functions

Remember that a function g is *concave* if for any x, y in its domain, and any $\lambda \in [0, 1]$, there holds

$$g(\lambda x + (1 - \lambda)y) \leq \lambda g(x) + (1 - \lambda)g(y).$$

In other words, g is concave if for any x and y , the line between $(x, g(x))$ and $(y, g(y))$ remains below the curve of g and does not cross it. Fig. 4 shows examples of concave and non concave functions. A concave function is always continuous on the interior of its domain, that is, everywhere except possibly on the frontier of its domain of definition. When a function is differentiable, it is concave if and only if its derivative is non-increasing. And when it is twice differentiable, it is concave if and only if its second derivative is non-positive.

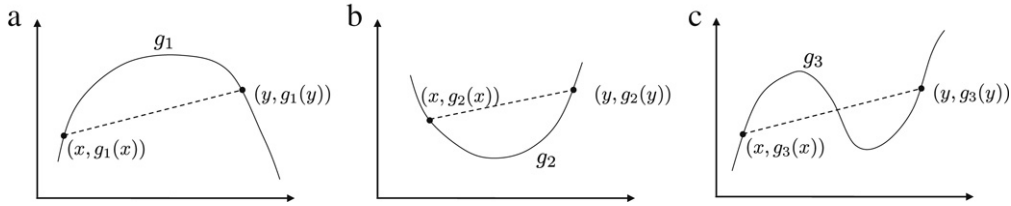


Fig. 4. Examples of concave and of non-concave functions. The function g_2 is not concave because for the x and y represented, the line between $(x, g_2(x))$ and $(y, g_2(y))$ is above the curve of g_2 . The function g_3 is not concave either because the represented line crosses the curve. On the other hand, g_1 is concave, as for any x and y , the line between $(x, g_1(x))$ and $(y, g_1(y))$ remains below the curve of g_1 .

We now analyze how the algebraic condition of [Theorem 1](#) can be related to the concavity of $\log f$, which is often simpler to check. Note that in the sequel we always implicitly assume that the points at which $\log f$ is evaluated belong to the support of f .

Let us first assume that f (and thus also $\log f$) is differentiable. Taking the logarithm of (6), we see that f is order preserving if and only if, for any $c > 0$, $a \leq b$, there holds

$$\log f(a + c) - \log f(a) \geq \log f(b + c) - \log f(b). \quad (7)$$

Taking the limit for $c \rightarrow 0$, this condition implies that $(\log f)'(a) \geq (\log f)'(b)$ for all $a \leq b$, and thus that $\log f$ is concave since its derivative is non-increasing. Similarly, if $\log f$ is concave, there holds $(\log f)'(a) \geq (\log f)'(b)$ for all $a \leq b$, and one can then show by integrating $(\log f)'$ that the condition (7) holds for any $c > 0$, so that f is order preserving. As a result, a differentiable function is order preserving if and only if it is *log-concave*, that is, if and only if its logarithm is concave.

Many natural influence functions are however not differentiable or continuous everywhere (see for example [Fig. 2\(c\)](#)). We consider thus now an influence function f on which no smoothness assumption is made. Suppose first $\log f$ is concave. Then for $b > a$ and $c > 0$, there hold

$$\begin{aligned} \log f(a + c) &\geq \frac{b - a}{b - a + c} \log f(a) + \frac{c}{b - a + c} \log f(b + c), \\ \log f(b) &\geq \frac{c}{b - a + c} \log f(a) + \frac{b - a}{b - a + c} \log f(b + c). \end{aligned}$$

Adding these two inequalities leads to the relation $\log f(a + c) - \log f(a) \geq \log f(b + c) - \log f(b)$, or equivalently $\frac{f(a+c)}{f(a)} \geq \frac{f(b+c)}{f(b)}$. So it follows from [Theorem 1](#) that a log-concave influence function, that is a function whose logarithm is concave, is always order preserving.

Conversely, let now $x < y < z$ be arbitrary points of its support such that $(y - x)/(z - y)$ is rational. There exist thus two integers m, n such that

$$\frac{z - y}{n} = \frac{y - x}{m} =: c > 0.$$

If f is order preserving, it follows from [Theorem 1](#) that

$$\log f(a + c) - \log f(a) \geq \log f(b + c) - \log f(b).$$

holds for any $a < b$ in the support of f . So we have

$$\begin{aligned} \log f(y) - \log f(x) &= \sum_{j=1}^m (\log f(x + jc) - \log f(x + (j-1)c)) \geq m (\log f(y + c) - \log f(y)), \\ \log f(z) - \log f(y) &= \sum_{j=1}^n (\log f(y + jc) - \log f(y + (j-1)c)) \leq n (\log f(y + c) - \log f(y)), \end{aligned}$$

which implies that

$$\log f(y) \geq \frac{n}{n+m} \log f(x) + \frac{m}{n+m} \log f(z) = \frac{z-y}{z-x} \log f(x) + \frac{y-x}{z-x} \log f(z). \quad (8)$$

If this holds for any x, y and z , then $\log f$ is concave. But remember that (8) only holds for those x, y, z for which $(y-x)/(z-x)$ is rational. Nevertheless, if f is continuous on its support, its continuity together with (8) implies that it is concave. And, the following proposition proved in [Appendix A](#) shows that every “reasonable” order preserving influence function is continuous on its support:

Proposition 1. *Let f be an order preserving influence function that is discontinuous at one point of its support's interior. Then f admits a positive lower bound on no positive length interval, and is as a consequence discontinuous everywhere on its support.*

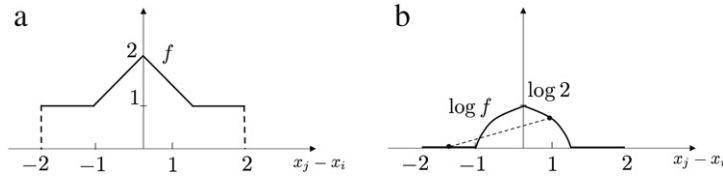


Fig. 5. The function defined by $f(x) = \chi_{[-2,2]}(x) \max(1, 1 - |x|)$ and represented in (a) is not order preserving because its logarithm represented in (b) is not concave, as shown by the dashed line.

The results of this section are summarized in the following theorem.

Theorem 2. Let $f : \mathfrak{X} \rightarrow \mathfrak{X}^+$ be an influence function. If $\log f$ is concave, then f is order preserving. And if f is order preserving and admits a positive lower bound on at least one positive-length interval or is continuous at one point of its support, then $\log f$ is concave.

Using the fact that the logarithm of a concave function is also concave, we obtain the following Corollary.

Corollary 1. Let $f : \mathfrak{X} \rightarrow \mathfrak{X}^+$ be an influence function. If the restriction of f to its support (i.e. the set on which it takes positive values) is concave, then it is order preserving.

A consequence of Theorem 2 is that no function discontinuous on the interior of its domain (and admitting a positive lower bound on at least one interval) is order preserving. Such function is indeed never concave. No function similar to f_3 in Fig. 2, or containing a gap, is thus order preserving. Similarly, the function defined by $f(x) = \chi_{[-2,2]}(x) \max(2 - |x|, 1)$ represented in Fig. 5 is not order preserving either because its logarithm is not concave. Based on Theorem 2, one can actually show that no non-concave piecewise linear function is order preserving.

On the other hand, influence functions such as $\max(1 - x^2, 0)$ or $\chi_{[-\pi/2, \pi/2]}(x) \cos(x)$ are concave on their support, and it follows then from Corollary 1 that they are order preserving. For the same reasons, the functions f_1 and f_2 in Fig. 2 are also order preserving. Functions such as e^{-x^2} and even e^x are not concave, but their respective logarithm $-x^2$ and x are concave, so that they are also order preserving by Theorem 2.

4. Conclusions and open questions

We have shown that an influence function is order preserving if it is log-concave, and that an order preserving function is log-concave unless it is discontinuous at every point of its support and admits a positive lower bound on no positive length-interval. The existence of such order preserving functions that are not log-concave however remains open. Besides, Krause's model has also been defined for two or more dimensional spaces, to which the order preservation property cannot be extended. Log-concave functions might however have a more generic property in those spaces, which would imply order preservation for one-dimensional spaces. Finally, since the order preservation property is widely used in the mathematical analysis of Krause's initial model, it would be interesting to see how exactly the main features of the system are affected when an influence function is selected.

Acknowledgments

This research was supported by the Concerted Research Action (ARC) "Large Graphs and Networks" of the French Community of Belgium and by the Belgian Programme on Interuniversity Attraction Poles initiated by the Belgian Federal Science Policy Office. The scientific responsibility rests with the author. Julien Hendrickx holds a F.R.S.-FNRS fellowship (Belgian Fund for Scientific Research).

Appendix. Proof of Proposition 1

Let f be an order preserving function and call S_f its support. Suppose that f is not continuous at some x_0 in the interior of S_f . We prove that this implies the unboundedness of $\log f$ on all positive length intervals in S_f . In particular, we show that $\log f$ is unbounded on $[x - \epsilon, x + \epsilon] \cap S_f$ for any $\epsilon > 0$ and $x \in S_f$, and therefore continuous at no point of S_f . This implies that f is also continuous nowhere on S_f , and admits a positive lower bound on no positive length interval.

Let $\Delta(x) = \frac{\log f(x) - \log f(x_0)}{x - x_0}$. The discontinuity of $\log f$ at x_0 implies that $\Delta(x)$ is unbounded on any open interval containing x_0 . We suppose that it takes arbitrary large positive values on any such interval. If it is not the case, then it necessarily takes arbitrary large negative values, and a similar argument can be applied. Consider an arbitrary large M and a interval $I \subset S_f$ of positive length $|I|$ with $\sup I < x_0$, where by $\sup I$ we denote the supremum of I . The following construction is illustrated in Fig. 6. There is a $x_M \in (x_0 - |I|, x_0 + |I|)$ such that $\left| \frac{M + \log f(x_0)}{\sup I - x_0} \right| < \Delta(x_M)$. Consider now the sequence

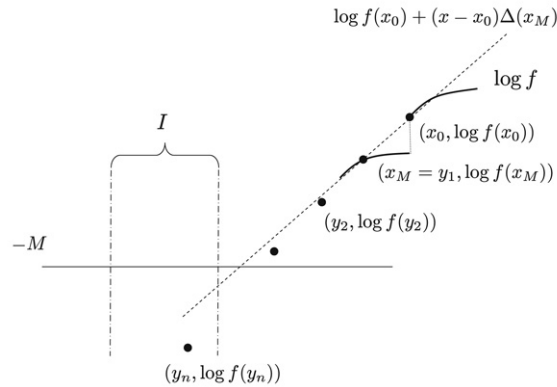


Fig. 6. Illustration of the construction in the proof of Proposition 1. All $\log f(y_i)$ must be below $\log f(x_0) + (y_i - x_0)\Delta(x_M)$.

of points defined by $y_0 = x_0$ and $y_i = y_{i-1} - |x_M - x_0|$. Since all y_i are smaller than or equal to x_0 , it follows from Theorem 1 that $\frac{f(y_{i-1})}{f(y_i)} \geq \frac{f(x_M)}{f(x_0)}$ holds if $x_M > x_0$ and $\frac{f(y_{i-1})}{f(y_i)} \geq \frac{f(x_0)}{f(x_M)}$ holds if $x_M < x_0$. In both cases, this implies that $\log f(y_{i-1}) - \log f(y_i) \geq |\log f(x_M) - \log f(x_0)|$ and thus that

$$\log f(y_i) \leq \log f(x_0) - i |\log f(x_M) - \log f(x_0)| = \log f(x_0) - (x_0 - y_i)\Delta(x_M).$$

Since $|x_M - x_0| < |I|$, there is a n such that $y_n \in I$. For this y_n , there holds $x_0 - y_n \geq |x_0 - \sup I| \geq \frac{M + \log f(x_0)}{\Delta(x_M)}$. It follows then from the inequality above that

$$\log f(y_n) \leq \log f(x_0) - (x_0 - y_n)\Delta(x_M) \leq \log f(x_0) - |x_0 - \sup I| \Delta(x_M) < -M.$$

Therefore, $\log f$ takes arbitrary large negative values on any positive length interval I with $\sup I < x_0$.

Consider now a $x_1 < x_0$. For any δ , $\log f$ takes arbitrary large negative values on $[x_1, x_1 + \delta]$, and therefore so does $\Delta_1(x) := \frac{\log f(x) - \log f(x_1)}{x - x_1}$. It follows then from a similar argument as above that $\log f$ admits no lower bound on any positive length interval I with $\inf I > x_1$, and thus that it does not admit any lower bound on any positive length interval contained in S_f since every such interval contains at least a subinterval I with $\inf I > x_1$ or with $\sup I < x_0$.

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