Clusters and the entropy in opinion dynamics on complex networks

5 authors, including:

Wenchen Han
Sichuan Normal University
19 PUBLICATIONS   95 CITATIONS

SEE PROFILE

Feng Yue e
Jiangsu Aviation Technical College
7 PUBLICATIONS   61 CITATIONS

SEE PROFILE

Changwei Huang
Guangxi University
25 PUBLICATIONS   161 CITATIONS

SEE PROFILE

Clusters and the entropy in opinion dynamics on complex networks

Wenchen Han

College of Physics and Electronic Engineering, Sichuan Normal University, Chengdu, 610101, China

Yuee Feng

College of Arts and Sciences, Jiangsu Aviation Technical College, Zhenjiang, 212134, China

Xiaolan Qian

School of Media and Engineering, Communication University of Zhejiang, Hangzhou, 310018, China

Qihui Yang

Department of Electrical and Computer Engineering,
Kansas State University, Manhattan, Kansas 66506, USA

Changwei Huang[*]

School of Computer, Electronics and Information, Guangxi University, Nanning, 530004, China

In this work, we investigate a heterogeneous population in the modified Hegselmann-Krause opinion model on complex networks. We introduce the Shannon information entropy about all relative opinion clusters to characterize the cluster profile in the final configuration. Independent of network structures, there exists the optimal stubbornness of one subpopulation for the largest number of clusters and the highest entropy. Besides, there is the optimal bounded confidence (or subpopulation ratio) of one subpopulation for the smallest number of clusters and the lowest entropy. However, network structures affect cluster profiles indeed. A large average degree favors consensus for making different networks more similar with complete graphs. The network size has limited impact on cluster profiles of heterogeneous populations on scale-free networks but has significant effects upon those on small-world networks.

I. INTRODUCTION

The field of multiagent network dynamics, which can reveal the emergence of collective behavior characteristics based on local individual interaction rules, has been investigated by researchers from a vast range of disciplines in recent years [1–3]. Agents' opinions can change in time subject to the interactions between the neighboring agents as well as the global feedback and external factors. The opinion dynamics characterized with statistical physics has been developed in the social opinion research. Opinion dynamics models are classified into discrete opinion dynamics models and continuous ones. More specifically, opinions are modeled as variables, discrete or continuous. In the discrete case, binary opinion models are most studied in terms of their analogy with spin systems. Well-known discrete opinion dynamics models include the voter model [4], the Sznajd model [5], the Galam majority-rule model [6], and the nonconsensus opinion model [7, 8]. Among the continuous opinion dynamics models, the Deffuant-Weisbuch model [9] and the Hegselmann-Krause (HK) model [10] have attracted much attention. In the final stationary state, opinion clusters can be one (consensus), two (polarization), or many (fragmentation).

Opinions are continuous variables in the interval $[0, 1]$

in an HK model. The evolution is treated as a series of discrete time steps. At each time step, an agent updates her opinion by adopting the average of her neighbors' opinions and her own. Two agents are neighbors only when the difference of their opinions is below the threshold, i.e., the bounded confidence. In a homogeneous population, where agents are with a same set of parameters, the consensus is the result of agents with a high bounded confidence, while a fragmentation is caused by a low bounded confidence. On the other hand, agents with heterogeneous bounded confidences could help agents to reach consensus [11, 12]. Fu *et al.* found that agents with high bounded confidence could not contribute to forging consensus and might lead to fragmentation [13]. While Han *et al.* [14] found whether agents would reach consensus depended upon not only the bounded confidence but also the stubbornness, which measured the extent of an agent's insistence on her own opinion [15–18].

The network structure provides a background for the opinion dynamics system, in which nodes stand for agents and links connecting the nodes represent the possible interactions between agents. The network structure can be a complete graph, a random network [19], a scale-free network [20], or a small-world network [21]. However, it is known that network structures affect dynamical processes on networks [22–24]. Results of complete graphs and square lattices are similar for large bounded confidence values, except for remaining a few extreme opinions [25]. Jalili studied the social power for the consensus in opinion dynamics on complex networks, found that giv-

ing the social power to nodes in small-world networks cannot significantly affect the consensus while hubs with social power in scale-free networks can [26]. Even for small-world networks, directed networks and bi-directed networks show different effects on opinion dynamics [27]. Bagnoli and Rechtman studied a society with conformist and reasonable contrarian agents, and they found systems on small-world networks and scale-free networks showed some similarities due to the long-range connectivity [28, 29].

In this work, we considered a heterogeneous population on different complex networks, where agents in a same subpopulation shared a same set of parameters (i.e., bounded confidence and stubbornness) and different subpopulations were with different sets of parameters. We paid our attention to the cluster profile while varying one of the parameters in this heterogeneous population. The cluster profile contained the information about the number of opinion clusters and the relative sizes of all clusters, and it implied the Shannon information entropy about the sizes of clusters. We found that the cluster profile was the trade-off of competition effects of agent's parameters. Furthermore, we also investigated the effect of the average degree of networks and the effect of the network size on the cluster profile. A large average degree made networks behave like a complete graph, supporting fewer clusters, a larger size of the largest cluster, and a low entropy. Besides, the network size had less impact on cluster profiles of populations on scale-free networks but significant impacts on those on small-world networks, due to the strongly connected local structures.

II. MODEL

We consider a heterogeneous population composed of N agents on complex networks. Agent i is placed on the node i in the network. A_{ij} is an element of the adjacency matrix for the network structure. $A_{ij} = A_{ji} = 1$ denotes node i and node j are connected and that agent i and agent j may be neighbors. $A_{ij} = A_{ji} = 0$ indicates that node i and node j are disconnected and that agent i and agent j cannot be neighbors. Additionally, self-edges are avoided ($A_{ii} = 0$). The opinion of agent i at discrete time t is described by a continuous variable $x_i(t)$, satisfying $0 \leq x_i(t) \leq 1$. The evolution of the opinion of agent i follows the updating rule

$$x_i(t+1) = \begin{cases} \alpha_i x_i(t) + \frac{(1-\alpha_i)}{||N_i(t)||} \sum_{j \in N_i(t)} x_j(t), & ||N_i(t)|| > 0, \\ x_i(t), & ||N_i(t)|| = 0. \end{cases}$$
(1)

α_i, which is in the interval $[0, 1]$, describes the stubbornness of agent i, suggesting the extent of agent i insisting on her own opinion. $\alpha_i = 0$ means agent i is a conformist, just following other neighbors' opinions, while agent i with $\alpha_i = 1$ is a zealot, only sticking to her own opinion and neglecting others [30]. Note that agent i is

more stubborn than agent j when α_i is larger than α_j. $N_i(t)$ for agent i at time t is the set of neighbor agents, whose opinions satisfy $|x_j(t) - x_i(t)| < \sigma_i$ and $A_{ij} = 1$. $|| N_i(t) ||$ is the cardinality of $N_i(t)$. Bounded confidence σ reflects the psychological concept of selective exposure, which refers to an individual's tendency to favor information that supports her opinion while neglecting conflicting arguments, with $\sigma \in (0, 1]$. We name agents with high σ as open-minded ones and agents with low σ as close-minded ones.

In this work, the total population is composed of M subpopulations. Each subpopulation with $\rho_l N$ agents, with the ratio $\rho_l \in [0, 1]$ and $\sum_{l=1}^{l=M} \rho_l = 1$, takes the bounded confidence σ_l and the stubbornness α_l. The subpopulation with σ_l and α_l is denoted as C_l for convenience. When opinions of all agents stop evolving, we say the system reaches its steady state. Then we monitor the number of opinion clusters, N_c, where connecting agents within a same cluster share a same opinion. Unconnected agents with a same opinion belong to different clusters. So it is with connected agents with different opinions. S_i is the relative size of an opinion cluster, namely the ratio between the number of agents in the cluster and the number of agents in the system. The relative size of all opinion clusters can be sorted in a descending order, i.e., $1 \geq S_1 \geq S_2 \geq \cdots \geq S_{N_c} > 0$. Additionally, $\{S_1, S_2, \cdots, S_{N_c}\}$ is an opinion cluster profile. We introduce the Shannon information entropy H [31] as

$$H = -\sum_{i=1}^{N_c} S_i \log_2 S_i.$$
(2)

The Shannon information entropy, analogy with Boltzmann Entropy [28, 29], describes the information gain of a certain opinion cluster profile. When $N_c = 1$ for the existence of only one cluster and $S_1 = 1$, no information can be gained ($H = 0$). If the opinions are scattered (e.g., $N_c = N$ and $S_i = 1/N$ for $i = 1, 2, \cdots, N$), much information can be gained ($H = \log_2(N)$). For similar cluster sizes, a cluster profile with more clusters corresponds to a higher H. If two opinion cluster profiles share a same number of clusters, the profile composed of less heterogeneous clusters, where some clusters are with similar sizes, are with a higher H. In this work, we consider the connecting structure between agents as complex networks, including random networks (RNs), scale-free networks (SFNs), and small-world networks (SWNs) with rewiring possibility 0.1. All the networks are connected and have N nodes and the $NK/2$ links, where K is the average degree. We consider the population size $N = 1000$ and the average degree $K = 20$ unless specified. The simulation results are averaged over 100 realizations with uniform random initial conditions in opinions. The opinions of agents are updated synchronously.
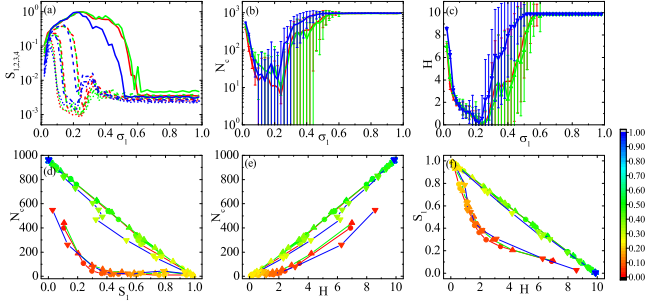
FIG. 1: The dependent relations for varying σ_1 when $\rho_1 = 0.9$, $\alpha_1 = 0.8$, $\rho_2 = 0.1$, $\alpha_2 = 0$, and $\sigma_2 = 0.04$. The red line/circle dots for random networks, the green line/up-triangles dots for scale-free networks, the blue line/down-triangles dots for small-world networks. (a) the dependence of four largest clusters $S_{1,2,3,4}$ (solid, dash, dash-dot, short dash lines) on σ_1. (b) the dependence of the number of clusters N_c on σ_1. (c) the dependence of the Shannon information entropy H on σ_1. (d) the relation of N_c against S_1. (e) the relation of N_c against H. (f) the relation of S_1 against H. In panels (d-f), the colors of dots for the values of σ_1 linearly.

FIG. 2: The dependent relations for varying σ_1 when $\rho_1 = 0.9$, $\alpha_1 = 0$, $\rho_2 = 0.1$, $\alpha_2 = 0.8$, and $\sigma_2 = 0.3$. The red line/circle dots for random networks, the green line/up-triangles dots for scale-free networks, the blue line/down-triangles dots for small-world networks. (a) the dependence of four largest clusters $S_{1,2,3,4}$ (solid, dash, dash-dot, short dash lines) on σ_1. (b) the dependence of the number of clusters N_c on σ_1. (c) the dependence of the Shannon information entropy H on σ_1. (d) the relation of N_c against S_1. (e) the relation of N_c against H. (f) the relation of S_1 against H. In panels (d-f), the colors of dots for the values of σ_1 linearly.

III.   SIMULATION RESULTS

In a homogeneous population ($M = 1$), the stubbornness α only affects the transition time when the bounded confidence σ is fixed [32]. A larger α is for a longer transition process. Increasing σ of agents always leads to a smaller number of opinion clusters N_c and a larger size of the largest opinion cluster S_1 when the bounded confidence is smaller than the threshold ($\sigma_c = 0.25$). Beyond σ_c, only one opinion cluster exists. It is also known that the dependence of N_c (or S_1) on σ is caused by the competition effects of agents' bounded confidences and stubbornness in a heterogeneous population ($M \geq 2$) on complete graphs [14].

In the following, we will report the results in a heterogeneous opinion population on complex networks (RNs, SFNs, SWNs) and pay attention to the cluster profile in the final configuration, which can be depicted by N_c, $S_i, i = 1, 2, \cdots, N_c$, and H. We consider a heterogeneous population just composed of two subpopulations $M = 2$, where $\rho_1 N$ agents are with σ_1 and α_1, $\rho_2 N$ agents are with σ_2 and α_2, and $\rho_1 + \rho_2 = 1$.

In Fig. 1, C_1 agents are more stubborn than close-minded C_2 agents ($\rho_1 = 0.9$, $\alpha_1 = 0.8$, $\rho_2 = 0.1$, $\alpha_2 = 0.0$, and $\sigma_2 = 0.04$), where C_2 subpopulation tends to form about 10 clusters when only C_2 agents are on a complete graph. In Fig. 1 (a), increasing the bounded confidence of C_1 agents makes S_1 first increase, then decrease, and remain constant finally ($\sigma_1 > 0.6$), while it makes N_c decrease, increase, then remain constant, as shown in Fig. 1 (b). Interestingly, when the C_1 agents are much open-minded (e.g., $\sigma_1 = 0.8$), all clusters are small and the size of the largest opinion cluster is even smaller than that when C_1 agents are close-minded (e.g., $\sigma_1 = 0.1$).
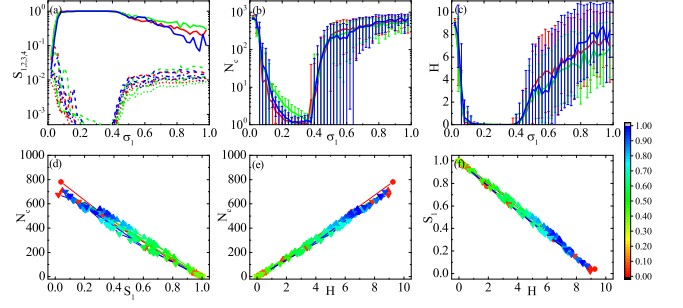
A high σ leads to fragmentation [13, 14]. However, in this case, this is caused by not only the competition effect between σ and α but also the connection constrains of network structures, which can be verified in the following discussion. However, the dependence of S_1 on σ_1 and the dependence of N_c on σ_1 are not simply negative correlated, as shown in Fig. 1 (d). For example, S_1 increases while N_c almost remains when $0.1 \leq \sigma_1 \leq 0.2$. This means that only considering S_1 and neglecting the evolution of other opinion clusters cannot explain the behavior of N_c, which counts up all clusters. That's because S_1 keeps increasing along with the decreases of S_4, S_3, and S_2 when N_c almost remains. The Shannon information entropy is a good measure for taking all clusters into account. It encodes not only the number but also the heterogeneity of all opinion clusters. In Fig. 1 (c), H decreases when S_1 increases along with S_4, S_3, S_2 decreasing for $\sigma_1 \in (0.1, 0.2)$. H shows some similar features but not exactly the same with N_c, which is not a strict linear relationship as shown in Fig. 1 (e). Figure 1 (f) shows the relation of S_1 against H is a roughly negative linear one, suggesting that a same H corresponds to different S_1 and vice versa. Figure 1 (d-f) suggests that one of H, N_c, and S_1 corresponds to different cluster profiles and that only one of H, N_c, and S_1 is not sufficient to describe a cluster profile. Additionally, Fig. 1 shows cluster profiles on different networks share many characteristics.

Figure 2 shows the results of C_1 agents ($\alpha_1 = 0.0$) with stubborn but open-minded C_2 agents ($\alpha_2 = 0.8$ and $\sigma_2 = 0.3$), where C_2 agents can achieve consensus when only C_2 agents exists on a complete graph. The dependence of S_1, N_c, and H on σ_1 is qualitatively the same as that of the population in Fig. 1 regardless of dif-
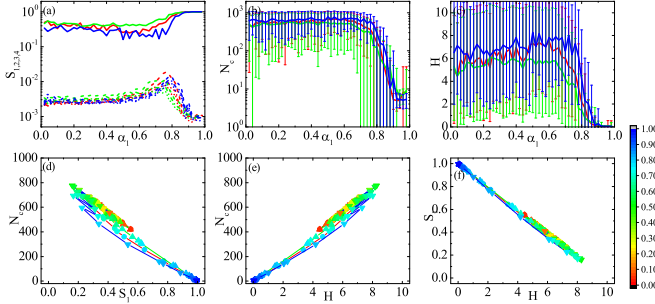
FIG. 3: The dependent relations for varying α_1 when $\rho_1 = 0.9$, $\sigma_1 = 0.3$, $\rho_2 = 0.1$, $\alpha_2 = 0$, and $\sigma_2 = 0.04$. The red line/circle dots for random networks, the green line/up-triangles dots for scale-free networks, the blue line/down-triangles dots for small-world networks. (a) the dependence of four largest clusters $S_{1,2,3,4}$ (solid, dash, dash-dot, short dash lines) on σ_1. (b) the dependence of the number of clusters N_c on σ_1. (c) the dependence of the Shannon information entropy H on σ_1. (d) the relation of N_c against S_1. (e) the relation of N_c against H. (f) the relation of S_1 against H. In panels (d-f), the colors of dots for the values of α_1 linearly.

FIG. 4: The dependent relations for varying ρ_1 when $\alpha_1 = 0$, $\sigma_1 = 0.3$, $\rho_2 = 1 - \rho_1$, $\alpha_2 = 0.8$, and $\sigma_2 = 0.04$. The red line/circle dots for random networks, the green line/up-triangles dots for scale-free networks, the blue line/down-triangles dots for small-world networks. (a) the dependence of four largest clusters $S_{1,2,3,4}$ (solid, dash, dash-dot, short dash lines) on σ_1. (b) the dependence of the number of clusters N_c on σ_1. (c) the dependence of the Shannon information entropy H on σ_1. (d) the relation of N_c against S_1. (e) the relation of N_c against H. (f) the relation of S_1 against H. In panels (d-f), the colors of dots for the values of ρ_1 linearly.

ferent values of α_1, α_2, and σ_2. However, it is a little different from the result when a heterogeneous population sitting on complete graphs [14], where increasing σ_1 always leads to consensus when the other subpopulation agents are open-minded. A steep decrease in H suggests a quick process of clusters merging when $\sigma_1 \in (0.02, 0.1)$, presented by the largest cluster increasing with the vanishing of small clusters. However, further increasing σ_1 even leads to a smaller S_1, a larger N_c, and a higher H in surprise, when $\sigma_1 > 0.4$, evolving on complex networks. These results are constrained by connections of network structures. Different from what is shown in Fig. 1, N_c against S_1 (Fig. 2 (d)) and S_1 against H (Fig. 2 (f)) exhibit highly negative linear relations and N_c against H (Fig. 2 (e)) is with a highly linear relation. These high correlations imply that one of H, N_c, and S_1 is enough to describe a cluster profile, in this case. Moreover, the dependent relations show no difference between populations on networks.

By varying α_1, cluster profiles for open-minded agents ($\sigma_1 = 0.3$) interacting with not stubborn and close-minded agents ($\sigma_2 = 0.04$, $\alpha_2 = 0$) are shown in Fig. 3. Slightly different from what happens on complete graphs, where increasing the stubbornness of open-minded agents always leads to smaller clusters and a larger largest cluster [14], on complex networks S_1 decreases then increases while N_c and H increases then decreases when α_1 increases. When C_1 agents are not so stubborn, the decrease of S_1 is caused by the local equilibrium, where C_1 agents quickly adjust their opinions to the average opinion of their neighbors. While C_1 are very stubborn, they wait their neighbors to adjust their opinions, which favors the cluster merging process and leads to a larger largest clusters. We have also checked the results of some other
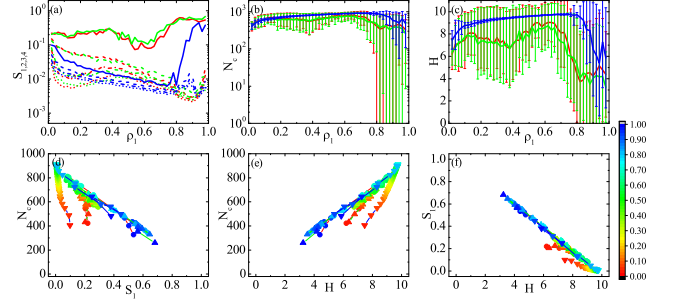
heterogeneous population. The (negative) linear relation of N_c against H (N_c against S_1 or S_1 against H) is robust. It can be inferred that one of H, N_c, and S_1 is sufficient for describing cluster profiles when varying the stubbornness of one subpopulation. Furthermore, the dependent relations on α_1 is almost independent of network structures.

The ratios of subpopulations affect the influence of the dependent relations, as shown in Fig. 4, where the open-minded subpopulation C_1 ($\alpha_1 = 0$, $\sigma_1 = 0.3$) is with a more stubborn close-minded subpopulation C_2 ($\alpha_1 = 0.8$, $\sigma_1 = 0.04$). In this case, cluster profiles on different complex networks share some characteristics, while cluster profiles of populations on SWNs show some differences from those on RNs or SFNs. In Fig. 4 (a), when ρ_1 is rather small, the four largest clusters are with similar sizes in spite of network structures. Increasing the ratio of C_1 subpopulation ρ_1 when $\rho_1 < 0.8$, the size of the largest cluster S_1 almost remains along with other clusters shrinking on RNs or SFNs, while the largest cluster together with other clusters shrinks on SWNs. When ρ_1 further increases, S_1 continues to increase and it increases to around 0.5. Furthermore, the number of clusters N_c when populations evolve on SWNs is comparable larger than that on RNs or SFNs in Fig. 4 (b). So it is with the entropy H in Fig. 4 (c). Although the roughly linear dependent relations remain, differences between populations on SWNs and those on RNs or SFNs are more obvious, as shown in Fig.4 (d-f). On SWNs, the population will show more clusters with smaller S_1, higher H, and larger N_c. This is caused by the strongly connected local structures, causing local equilibrium, in SWNs.

The results above show the effects of parameters of agents on cluster profiles of heterogeneous populations.
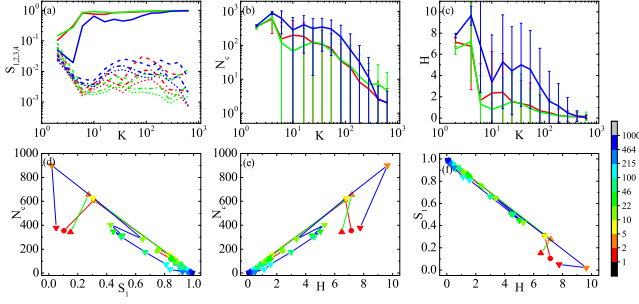
FIG. 5: The dependent relations for varying K when $\rho_1 = 0.9$, $\alpha_1 = 0.8$, $\sigma_1 = 0.3$, $\rho_2 = 0.1$, $\alpha_2 = 0$, and $\sigma_2 = 0.04$, with the population size $N = 1000$. The red line/circle dots for random networks, the green line/up-triangles dots for scale-free networks, the blue line/down-triangles dots for small-world networks. (a) the dependence of four largest clusters $S_{1,2,3,4}$ (solid, dash, dash-dot, short dash lines) on σ_1. (b) the dependence of the number of clusters N_c on σ_1. (c) the dependence of the Shannon information entropy H on σ_1. (d) the relation of N_c against S_1. (e) the relation of N_c against H. (f) the relation of S_1 against H. In panels (d-f), the colors of dots for the values of K logarithmically.

FIG. 6: The dependent relations for varying N when $\rho_1 = 0.9$, $\alpha_1 = 0.8$, $\sigma_1 = 0.3$, $\rho_2 = 0.1$, $\alpha_2 = 0$, and $\sigma_2 = 0.04$, with the average degree $K = 20$. The red line/circle dots for random networks, the green line/up-triangles dots for scale-free networks, the blue line/down-triangles dots for small-world networks. (a) the dependence of four largest clusters $S_{1,2,3,4}$ (solid, dash, dash-dot, short dash lines) on σ_1. (b) the dependence of the relative number of clusters N_c/N on σ_1. (c) the dependence of the Shannon information entropy H on σ_1. (d) the relation of N_c/N against S_1. (e) the relation of N_c/N against H. (f) the relation of S_1 against H. In panels (d-f), the colors of dots for the values of N logarithmically.

It is convinced that cluster profiles are the trade-off of competition between all parameters [14] and that features of cluster profiles are almost independent of network structures when only one parameter of agents varies. However, network structures should have some effects on cluster profiles, as known in Ref. [22–24] and shown in Fig. 4. In the following, two parameters of network structures, the average degree K and the network size N, are concerned.

We first study the influence of the average degree K on these dependent relations. S_1 increases while N_c and H decrease when K increases, as shown in Fig. 5 (a-c). The difference between SWNs and RNs or SFNs still remains when K is small, but it is vanishing when K is rather large. It should be noticed that when K increases with N fixed the difference between different network structures vanishes. When the network connectivity is high $(K/N \to 1)$, all network structures will show the characteristics of complete graphs. The results in Fig. 5 (d-f) confirm this, with all dot-lines for different networks converging to a same point, which almost corresponds to the result for the population evolving on complete graphs.

We also investigate the effects of the network size N. In this case, we monitor the relative number of clusters N_c/N in place of the absolute number of clusters N_c. Despite of network structures, S_1 is decreasing while N_c/N and H are increasing when N increases. Thanks to the strongly connected local structures, the variation of H on SWNs is more obvious than that on RNs or SFNs. Although the linear relations are robust, the differences between the networks are shown in Fig. 6 (d-f) directly. The increase of N shows the least effect on SFNs and the most effects on SWNs, which suggests that effects of strongly connected local structure in SWNs are strongly affected by N.
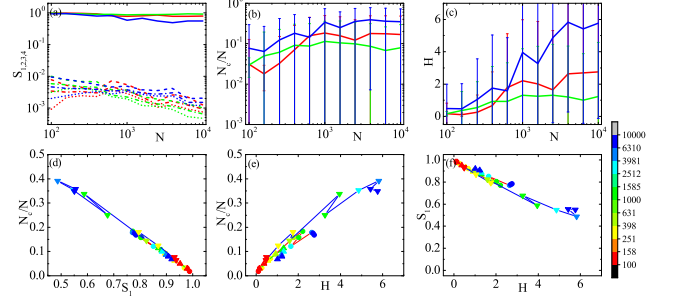
IV. CONCLUSION AND DISCUSSION

In this work, we studied cluster profiles of a heterogeneous population obeying the modified Hegselmann-Krause opinion dynamic rule on complex networks. The cluster profile could be described by not only the number of clusters, but also the relative size of the largest cluster, as well as the Shannon information entropy about relative sizes of all clusters. We found the cluster profile was the trade-off of competition between all parameters of this heterogeneous population, including the stubbornness, the bounded confidence, and the subpopulation ratio. Except that even if agents in another subpopulation were open-minded, increasing the bounded confidence could lead to more clusters and a smaller size of the largest cluster on complex networks, which was quite similar with that on complete graphs [14]. Optimal bounded confidence of one subpopulation existed for the fewest clusters, the largest size of largest cluster, and the lowest information gain. While increasing the stubbornness of one subpopulation led to the most clusters and the highest entropy. Increasing the ratio of open-minded agents was not always beneficial to the fewest clusters and the lowest entropy. Moreover, we found that the dependent relations between H, N_c, and S_1 were almost independent of network structures when only one of parameters varied, including bounded confidence, stubbornness, and subpopulation ratio.

Furthermore, we also investigated effects of parame-

ters of network structures on cluster profiles, such as the average degree and the network size. The difference between network structures remained when the average degree was small, while it vanished when the average degree was large enough. The large average degree made networks similar with a complete graph, where cluster profiles on it had a larger S_1, a smaller N_c, and a lower H. As for the network size, cluster profiles of heterogeneous populations on scale-free networks shared many similar features, while those on small-world networks showed more differences, and those on random networks were in the medium. These different results between different network structures were caused by the strongly connected local structures in small-world networks, hubs-leaf structures in scale-free networks, and random connection structure in random networks. These two findings showed that network structures could affect the evolution of opinion dynamics indeed.

Our work sheds light upon the cluster profile of a heterogeneous population on complex networks. Cluster profiles on different networks have many similar features but also display some differences. However, the conclusion in this work is just from one perspective. It is worthy studying the mechanisms to forge consensus from other perspectives and in detail. For example, it may be interesting to study a heterogeneous population on some special networks, such as networks with symmetry [33].

[1] C. Castellano, S. Fortunato, V. Loreto, "Statistical physics of social dynamics," Review of Modern Physics, vol. 81, no. 2, pp. 591-646, 2009.

[2] Z. Wang, C. T. Bauch, S. Bhattacharyya et al., "Statistical physics of vaccination," Physics Reports, vol. 664, pp. 1-113, 2016.

[3] Serge Galam, "Unavowed abstention can overturn poll predictions," Fronters in Physics, vol. 6, Article ID 24, 2018.

[4] R. Holley, T. Liggett, "Ergodic theorems for weakly interacting infinite systems and the voter model," The Annals of Probability, vol. 3, no. 4 pp. 643-663, 1975.

[5] K. Sznajd-Weron, J. Sznajd, "Opinion evolution in closed community," International Journal of Modern Physics C, vol. 11, no. 6, pp. 1157-1165, 2000.

[6] Serge Galam, "Minority opinion spreading in random geometry," The European Physical Journal B, vol. 25, no. 4, pp. 403-406, 2002.

[7] J. Shao, S. Havlin, H. E. Stanley, "Dynamic opinion model and invasion percolation," Physical Review Letters, vol. 103, no. 1, Article ID 018701, 2009.

[8] H. Yang, L. Huang, "Opinion percolation in structured population," Computer Physics Communications, vol. 192, pp. 129-142, 2015.

[9] G. Deffuant, D. Neau, F. Amblard et al, "Mixing beliefs among interacting agents," Advances in Complex Systems, vol. 03, no. 01n04, pp. 87-98, 2000.

[10] R. Hegselmann, U. Krause, "Opinion dynamics and bounded confidence models, Analysis and Simulation," Journal of Artificial Societies Social Simulation, vol. 5, no. 3, Artical ID 2, 2002.

[11] Jan Lorenz, "Heterogeneous bounds of confidence: Meet, discuss and find consensus!" Complexity, vol. 15, no. 4, pp. 43-52, 2009.

[12] M. Pineda, G. M. Buenda, "Mass media and heterogeneous bounds of confidence in continuous opinion dynamics," Physica A, vol. 420, pp. 73-84, 2015.

[13] G. Fu, W. Zhang, Z. Li, "Opinion dynamics of modified HegselmannCKrause model in a group-based population with heterogeneous bounded confidence," Physica A, vol. 419, pp. 558-565, 2015.

[14] W. Han, C. Huang, J. Yang, "Opinion clusters in a modified Hegselmann-Krause model with heterogeneous bounded confidences and stubbornness," Physica A, vol. 531, Artical ID 121791, 2019.

[15] J. Ghaderi, R. Srikant, "Opinion dynamics in social networks with stubborn agents: Equilibrium and convergence rate," Automatica, vol. 50, no. 12, pp. 3209-3215, 2014.

[16] A. Olshevsky, J. Tsitsiklis, "Convergence speed in distributed consensus and averaging," SIAM Journal on Control and Optimization, vol. 48 no. 1, pp. 33-55, 2009.

[17] B. Chazelle, C. Wang, "Inertial Hegselmann-Krause systems," IEEE Transactions on Automatic Control, vol. 62, no. 8, pp. 3905-3913, 2017.

[18] X. Chen, S. Zhao, W. Li, "Opinion dynamics model based on cognitive styles: Field-dependence and field independence," Complexity, vol. 2019, Artical ID 2864124, 2019.

[19] P. Erdös, A. Rényi, "On the evolution of random graphs," Publ. Math. Inst. Hung. Acad. Sci. vol. 5, pp. 17-60, 1960.

[20] A.-L. Barabási, R. Albert, "Emergence of Scaling in Random Networks," Science, vol. 286, no. 5439, pp. 509-512, 1999.

[21] D. J. Watts, S. H. Strogatz, "Collective dynamics of small-world networks," Nature, vol. 393, no. 6684, pp. 440-442, 1998.

[22] M. A. Porter, J. P. Gleeson, "Dynamical systems on networks: A tutorial," Frontiers in Applied Dynamical Systems: Reviews and Tutorials, Vol. 4, Springer International Publishing Switzerland, 2016.

[23] X. F. Meng, R. A. Van Gorder , M. A. Porter, "Opinion formation and distribution in a bounded-confidence model on various networks," Physical Review E, vol. 97, no. 2, Artical ID 022312, 2018.

[24] O. AskariSichani, M. Jalili, "Influence maximization of informed agents in social networks," Applied Mathematics and Computation, vol. 254, pp. 229-239, 2015.

[25] G. Weisbuch, G. Deffuant, F. Amblard et al., "Meet, discuss, and segregate!" Complexity, vol. 7, no. 3, pp. 55-63, 2002.

[26] Mahdi Jalili, "Social power and opinion formation in complex networks," Physica A, vol. 392, no. 4, pp. 959-966, 2013.

[27] Y. Gandica, M. del Castillo-Mussot, G. J. Vzquez et al., "Continuous opinion model in small-world directed networks," Physica A, vol. 389, no. 4, pp. 5864-5870, 2010.

[28] F. Bagnoli, R. Rechtman, "Topological bifurcations in a model society of reasonable contrarians," Physical Re-

view E, vol. 88, no. 6, Article ID 062914, 2013.

[29] F. Bagnoli, R. Rechtman, "Bifurcations in models of a society of reasonable contrarians and conformists," Physical Review E, vol. 92, no. 4, Article ID 042913, 2015.

[30] S. Wang, L. Rong, J. Wu, "Bistability and multistability in opinion dynamics models," Applied Mathematics and Computation, vol. 289, pp. 388-395, 2016.

[31] C. E. Shannon, "A mathematical theory of communication," The Bell System Technical Journal, vol. 27, no. 3,

pp. 379-423, 1948.

[32] Jan Lorenz, "Continuous opinion dynamics under bounded confidence: a survey," International Journal of Modern Physics C, vol. 18, no. 12, pp. 1819-1838, 2007.

[33] I. Klickstein, L. Pecora, F.Sorrentino, "Symmetry induced group consensus," Chaos, vol. 29, no. 7, Artical ID 073101, 2019.