Vector Opinion Dynamics in a Bounded Confidence Consensus Model

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VECTOR OPINION DYNAMICS
IN A BOUNDED CONFIDENCE CONSENSUS MODEL

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We study the continuum opinion dynamics of the compromise model of Krause and Hegselmann for a community of mutually interacting agents, by solving numerically a rate equation. The opinions are here represented by bidimensional vectors with real-valued components. We study the situation starting from a uniform probability distribution for the opinion configuration and for different shapes of the confidence range. In all cases, we find that the thresholds for consensus and cluster merging either coincide with their one-dimensional counterparts, or are very close to them. The symmetry of the final opinion configuration, when more clusters survive, is determined by the shape of the opinion space. If the latter is a square, which is the case we consider, the clusters in general occupy the sites of a square lattice, although we sometimes observe interesting deviations from this general pattern, especially near the center of the opinion space.

Keywords: Sociophysics; Monte Carlo simulations.

1. Introduction

If two individuals are supporters of the same football team, they may like to discuss about football; on the other hand, if two persons have opposite political views, they will hardly discuss about politics. Suppose now to have two individuals who are fans of the same football team but support opposite political parties: will they talk to each other or not? Will their affinity in football win over their contrasts in politics? The question sounds ambiguous: some pairs of people will interact, others will not. What we would like to stress here is the rather obvious fact that the tendency of people to effectively communicate with each other is usually favoured by affinities and hindered by contrasts. So far so good, this is basic sociology. Indeed, Axelrod [1] had this in mind when he proposed his model for the dissemination of culture, to explain how different cultural islands could be generated from a local tendency to convergence. For Axelrod, "culture" is modelized as an array of features, each feature being specified by "traits", which are expressed by integers. The number of features or dimensions is nothing but the number of components of a vector, and two persons interact if and only if they share at least one common feature (i.e. the same value of the corresponding vector component). In this model, two persons are the culturally closer the more features they have in common, and the number of common features is related to the probability of the two individuals to interact. Starting from the Axelrod model, a number of simple agent-based models have been devised, mostly by physicists. This should not be surprising, as these models are essentially cellular automata, which physicists are used to play with. Among these models we mention those of Sznajd [2], Deffuant et al. [3], Krause and Hegselmann [4], Wu and Huberman [5], Pluchino et al. [6]. For a review of the recent research activity in this field we refer to [7,8].

In this way proper physical questions could be addressed, which concern not just the behaviour of single individuals but the collective behaviour of a community. In fact, if the behaviour of a person is of little relevance for quantitative scientific investigations, and essentially unpredictable, the global organization of many mutually interacting subjects presents general patterns which go beyond specific individual attributes and may emerge in several different contexts. One can then hope that quantities like averages and statistical distributions may characterize not just specific situations but large classes of systems. This explains why in the last years physicists tried to use their expertise to investigate social systems, where the elementary interacting objects are the people (or "agents") and the "charge" is represented by their opinion, although the latter is strictly speaking not a measurable quantity.

In all above-mentioned models opinions are modelized as numbers, integer or real. One starts by assigning randomly a number to every agent of the system. Then the dynamics starts to act, and the agents rearrange their opinion variables, due to mutual discussions. At some stage, the system reaches a configuration which is stable under the dynamics; this final configuration may represent consensus, with all agents sharing the same opinion, polarization, with two main clusters of opinions

("parties") or fragmentation, where several opinion clusters survive. However, as we explained at the beginning, a discussion between two persons is not simply stimulated by their common view/preference about a specific issue, but it in general depends on the global affinity of the two persons, which is influenced by several factors. So, for a more realistic modeling of opinion dynamics, one should represent the opinions/attitudes like vectors, as Axelrod did, and not like scalars. In a forthcoming paper [9], K. Sznajd-Weron and J. Sznajd assign two Ising-like spin variables to each agent, referring to the opinion about a personal issue and an economic one. In a very recent work on the Deffuant model [10], the opinion has several components which are integers and the agents sit on the sites of a Barabási-Albert network. However, a systematic study of the problem of multidimensional opinion dynamics is still missing, and the aim of this paper is to try to fill up this gap.

In this paper we deal precisely with this kind of problem; opinions are bidimensional vectors $\vec{S} = (x, y)$, and the components can take any real value in some finite interval (e.g. $[0 : 1]$). Instead of formulating a new opinion dynamics, it is for us more important to check what happens if we use one of the existing models. In this way one can compare the results for vector opinions with those relative to standard scalar opinions. We chose to adopt the opinion dynamics proposed by Krause and Hegselmann [4] (KH), which has recently been subject of intense investigations [11,12]. The KH model is based on bounded confidence, i.e. on the presence of a parameter ϵ, called confidence bound, which expresses the compatibility of agents. If the opinions of two agents i and j differ by less than ϵ, their positions are close enough to allow for a discussion between i and j which eventually leads to a variation of their opinions, otherwise the two agents do not interact with each other. A society, or community, is modelized as a graph, where the vertices represent the agents and the edges relationships between agents. So we say that two agents are friends and could eventually talk to each other if there is an edge joining the two corresponding vertices (in graph language, if the two vertices are neighbours).

The dynamics of the model is very simple: one chooses at random one of the agents and checks how many neighbours of the selected agent are compatible with it. Next, the agent takes the average opinion of its compatible neighbours. The procedure is repeated by selecting at random another agent and so on. The type of final configuration reached by the system depends on the value of the confidence bound ϵ. For a society where everybody talks to everybody else, consensus is reached for $\epsilon > \epsilon_c$, where $\epsilon_c \sim 0.2$ [12]. Whereas one may expect that the value of the consensus threshold be strictly related to the type of graph adopted to modelize society, it turns out that it can take only one of two possible values, $\epsilon_c \sim 0.2$ and $1/2$, depending on whether the average degree (= number of neighbours) of the graph diverges or stays finite when the number of vertices goes to infinity [12].

Here we want to apply the dynamics of the KH model to vector opinions, using a continuum opinion distribution and integrating the corresponding rate equation. We focus on a society where everybody talks to everybody else, because in this case the evolution of the system can be described by a simple rate equation, that one

can easily solve numerically. This procedure has already been used to investigate the compromise model of Deffuant et al. with one-dimensional opinions [13]. The advantages of the evolution equation over Monte Carlo simulations are that one can (in principle) deal with a system with arbitrarily many agents and can better resolve the crucial steps of the time evolution of the system, especially when opinion clusters merge. Furthermore, as we will see, the final cluster configurations obtained for a continuum distribution will be much more symmetric and regular.

2. The model

The opinion space is represented by the points (x, y) of a bidimensional manifold, that for us is a square: $[0, 1] \times [0, 1]$. The continuum distribution of the opinions among the agents is expressed by the function $P(x, y, t)$, such that $P(x, y, t)dx\,dy$ represents the fraction of agents whose opinions have components in the intervals $[x, x + dx]$ and $[y, y + dy]$. The integral of the distribution $P(x, y, t)$ over the whole opinion space is of course one. The dynamics of KH, as well as that of Deffuant et al., can simply be extended to the multi-dimensional case. We must however be careful to the definition of bounded confidence, and the corresponding parametrization. The crucial point is the concept of "closeness" between agents. Shall one consider the two opinion components independently or jointly? In the first case one can assume that two agents are compatible if either the x- or the y-components of their opinions are close enough to each other, even if the other components are far apart. However, in this case, the presence of several different issues would not represent a change with respect to the standard situation with scalar opinions. In fact, the issues can be considered separately and each opinion component would evolve independently of the others, so that one would just have to compose the results obtained for each single issue. As we said in the introduction, this is not what we would like to have, and it is not what happens in society. The closeness of the agents depends on the general affinity between them, and the affinity has to do with all issues. So, two agents are compatible if both their opinion components are close to each other. The shape of the confidence range, i. e. of the set of points in the opinion space which represent all opinions compatible with that of some agent, can be arbitrarily chosen: we took the two simplest cases of the square and the circle.

We have then two possible scenarios:

1. squared opinion space, squared confidence range;

2. squared opinion space, circular confidence range.

The first scenario is illustrated in Fig. 1: the large square is the opinion space, the black dot is the opinion $\mathbf{S}$ of an agent A and the smaller square centered at $\mathbf{S}$ is the (symmetric) confidence range of the agent. All agents whose opinions lie within the square are compatible with A and can interact with it. For the second scenario, the square is replaced by a circle of radius ϵ. We now come to the evolution equation
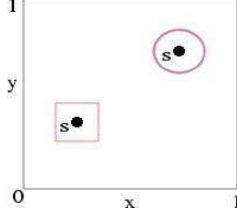
Fig. 1. Simplest scenarios for our model: the opinion space is a square and the confidence range can be a square or a circle.

of the system. We start by examining the continuum version of the standard one-dimensional KH model, in order to test the goodness of our procedure for the numerical solution of the rate equation, by making comparisons with known results from Monte Carlo simulations. After that, we turn to the bidimensional case.

3. Results for scalar opinions

The opinion space is the range $[0,1]$, the confidence bound ϵ. The opinion distribution among the agents is given by the function $P(x,t)$. The rate equation of the KH model is

$$\frac{\partial}{\partial t}P(x,t) = \int_0^1 dx_1\, P(x_1,t)\Big[\delta\Big(x - \frac{\int_{x_1-\epsilon}^{x_1+\epsilon} dx'\, x'\, P(x',t)}{\int_{x_1-\epsilon}^{x_1+\epsilon} dx'\, P(x',t)}\Big) - \delta(x - x_1)\Big] \quad (1)$$

The two δ's in the equation represent the only two contributions to the variation of $P(x,t)$ around the opinion x in the time interval $[t, t+dt]$. In fact, take all agents in the range $[x_1, x_1 + dx_1]$, with x_1 in $[0,1]$. The complicated ratio of integrals inside the first δ is nothing but the average opinion $\bar{x}$ of all agents whose opinions are compatible with x_1. As we explained above, $\bar{x}$ is then precisely the new opinion of the agents. So, whenever $\bar{x} = x$, there will be new agents with opinion in the range $[x, x + dx]$. On the other hand, if $x_1 = x$, the final opinion will be in general different from x, so those agents will leave the range $[x, x + dx]$. The total balance of the two contributions is expressed by the integration over all values of x_1, which cover the opinion range $[0,1]$.

We immediately see from the equation that the dynamics conserves the total population of agents, $N(t) = \int_0^1 P(x,t)dx$, as it should be for the physical interpretation. In fact, if we integrate both sides of Eq. (1) over x on the opinion range $[0,1]$, we obtain

$$\frac{\partial}{\partial t}\int_0^1 dx\, P(x,t) = \int_0^1 \int_0^1 dx\, dx_1\, P(x_1,t)\Big[\delta\Big(x - \bar{x}(x_1,t)\Big) - \delta(x - x_1)\Big] \quad (2)$$

where, for simplicity, we indicate with $\bar{x}(x_1, t)$ the ratio of integrals within the first δ.

If we perform the integral over the variable x on the right hand side, we see that the only dependence on x is contained in the two δ's; by integrating them in the range $[0, 1]$ they give equal contributions but opposite in sign $(1 - 1 = 0)$. The derivative with respect to time of the norm $N(t)$ (left-hand side) is then zero and the norm keeps its initial value $N(0)$ during the whole evolution (we set $N(0) = 1$).

It is not so straigthforward, though relatively simple, to check that also the first moment of the opinion distribution, i. e. the average value of the opinion of the system, is conserved by the dynamics. For this purpose one should multiply both sides of Eq. (1) by x and integrate over x in $[0, 1]$,

$$\frac{\partial}{\partial t} \int_0^1 dx \, x \, P(x, t) = \int_0^1 \int_0^1 dx \, dx_1 \, x \, P(x_1, t) \Big[\delta\Big(x - \bar{x}(x_1, t)\Big) - \delta(x - x_1) \Big]. \quad (3)$$

Again, we first perform the integration over the variable x on the right-hand side, and we obtain

$$\frac{\partial}{\partial t} \int_0^1 dx \, x \, P(x, t) = \int_0^1 dx_1 \, P(x_1, t) \Big[\bar{x}(x_1, t) - x_1 \Big]. \quad (4)$$

We remark that the KH dynamics, as well as the Deffuant dynamics, is symmetric with respect to the transformation $x \to 1 - x$, i. e. the two wings of the opinion range are perfectly equal, and remain equal during the whole evolution of the system. Let us now focus on the integral on the right-hand side of Eq. (4): we will show that it vanishes. We just need to transform the integration variable x_1 in $1 - x_1$. We get

$$\int_0^1 dx_1 \, P(x_1, t) \Big[\bar{x}(x_1, t) - x_1 \Big] = - \int_1^0 dx_1 \, P(1 - x_1, t) \Big[\bar{x}(1 - x_1, t) - (1 - x_1) \Big]. \quad (5)$$

Due to the above-mentioned symmetry, we have

$$P(1 - x_1, t) = P(x_1, t) \quad (6)$$

and

$$\bar{x}(1 - x_1, t) = 1 - \bar{x}(x_1, t), \quad (7)$$

so we finally obtain

$$\int_0^1 dx_1 \, P(x_1, t) \Big[\bar{x}(x_1, t) - x_1 \Big] = \int_1^0 dx_1 \, P(x_1, t) \Big[\bar{x}(x_1, t) - x_1 \Big], \quad (8)$$

which means that both integrals must yield zero. The conservation of the average opinion of the community, which also holds in the model of Deffuant [13], has

several important consequences: for instance, if we start from a uniform opinion distribution, $P(x, t = 0) = \text{const}$, for which the average opinion is $1/2$, an eventual consensus can only be reached at the central opinion $1/2$. Moreover, as the flat distribution keeps the symmetry with respect to the central opinion $1/2$, the final configuration of the system will be characterized by a symmetric pattern of opinion clusters to the right and the left of $1/2$; if the number of clusters is odd, there will necessarily be a cluster sitting exactly at $x = 1/2$.

Let us now come back to the rate equation (1). In order to test numerically the behavior of the one dimensional KH model in the limit of a continuum distribution of opinions $P(x, t)$, we have integrated the rate equation using a standard fourth order Runge-Kutta algorithm. The opinion range $[0, 1]$ has been discretized in 1000 bins and the accuracy in $P(x, t)$ was of 10^{-9}. In all simulations we started from a flat distribution, $P(x, t = 0) = \text{const}$, as one usually does for these simple consensus models. In this way, we assume that all opinions are equiprobable, as it could well be in a community before people begin to talk to each other. The dynamics runs until the distribution $P(x, t)$ reaches a stationary state (with an accuracy of 10^{-5}) for a given value of the confidence bound.
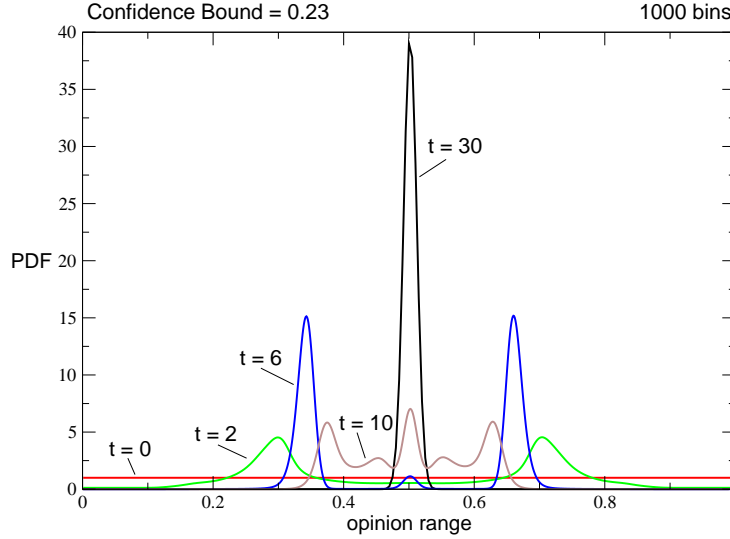

Fig. 2. Time evolution of the opinion distribution for $\epsilon = 0.23$. Initially the opinions condense in a two-clusters structure, then the two clusters slowly approach each other and merge, due to the presence of few agents with opinions lying near the center. After a long time, all agents end up in a single big cluster centered at $1/2$ (consensus).

Fig. 2 shows the time evolution of the opinion distribution, for $\epsilon = 0.23$. One clearly sees how the inhomogeneities at the two edges of the opinion range create a perturbation which determines variations of the density profile, with initial peaks

forming close to the edges (not reported in the plot) and others which successively form towards the middle. This is not a peculiar feature of the KH model; the time evolution of the opinion dynamics of Deffuant et al. reveals the same pattern [13]. For the chosen value of the confidence bound, two main peaks form rather quickly. However the configuration is not stable, and after a longer time the two peaks fuse in a big cluster centered at the central opinion $1/2$. By looking at Fig. 2, we see that initially the two peaks are separated by a distance which exceeds the confidence bound, so the corresponding agents are incompatible with each other and should not interact. Why is then the two-peak configuration unstable in this case? The reason has to do with the features of KH dynamics and it can be better explained if we speak of single agents instead of opinion distributions. In the KH model one agent is updated at a time; as we start from a flat opinion distribution, there will be, at least initially, agents with opinions lying near the central opinion $1/2$. Little by little these agents will accept the opinion of the cluster which lies closest to it. But suppose that one agent A lies between two different clusters of agents, C_1 and C_2, centered at the opinions s_1 and s_2, respectively; suppose as well that ϵ is large enough for A to interact with the agents of C_1 and C_2, but smaller than $|s_1 - s_2|$. In this case, when we come to update the opinion of the agent A, the latter will take the average of the opinions of all agents in C_1 and C_2, which lies between s_1 and s_2. On the other hand, when we update an agent B in C_1, it will be of course compatible with all other agents in C_1, but also with A though not with C_2. So, when one calculates the average of the opinions compatible with that of B, it will not be exactly s_1, but it will depart from s_1 by a tiny amount towards s_A. The same happens for the agents of C_2 too. In conclusion, agent A will keep lying between the two large clusters, and the latter will slightly move towards each other due to the intermediation of A. At some stage, the distance of the two clusters will become smaller than the confidence bound ϵ, and all agents of the system will interact with each other, so they will all take the average opinion $1/2$ of the whole system. The process we have described shows that the KH dynamics spontaneously creates a sort of hierarchy among agents that otherwise behave in a perfectly identical way. Agent A could be a political leader which brings two parties to a mutual agreement. As we have seen, the large clusters move very slowly; in the limit of infinite agents, the presence of a finite number of intermediary agents will lead a pair of clusters to merge only after an infinitely long time. This happens not only at the consensus threshold, but for all values of ϵ for which pairs of clusters fuse according to the mechanism we have described, like the transition from three to two final opinion clusters, from four to three, etc.

In the model of Deffuant et al., where any agent interacts at a time with only one of its compatible agents, not with all of them, there cannot be intermediary agents. This implies that clusters can never interact with each other if their separation in opinion space exceeds the confidence bound. This is actually the reason why the
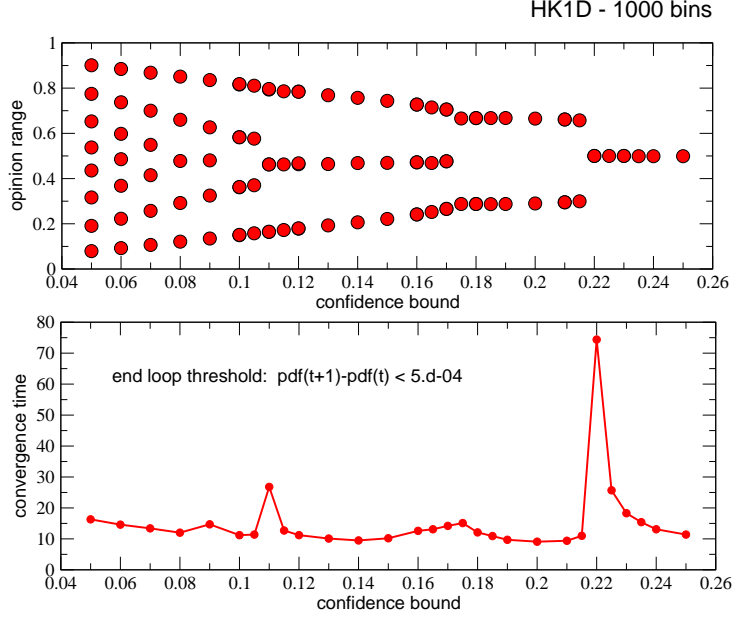
Fig. 3. (Top) Final configuration of the system as a function of ϵ. The circles indicate the positions of the surviving clusters in opinion space. (Bottom) Variation of the convergence times with ϵ. The narrow peaks are located in proximity of the thresholds corresponding to cluster merging, at which the convergence time diverges.

consensus threshold in the model of KH is much lower than in Deffuant.[*]

Fig. 3 (top) shows the position of the final clusters in opinion space as a function of ϵ. The threshold for consensus and that for the transition from three to two final clusters are consistent with those determined by means of Monte Carlo simulations. Notice the correspondence of the thresholds for cluster merging with the peaks in the convergence time (bottom).

4. Results for vector opinions

Now that we have tested the reliability of the numerical solution of the rate equation, we can proceed with the multidimensional case. The generalization of Eq. (1) is straightforward. For the sake of compactness, we will use the following vector notation. The opinion vector **S** is represented as $\vec{x}$, an n-dimensional vector with components x_1, x_2, ... , x_n, describing all points of the hypercube $[0,1]^n$. The opinion distribution can be written as $P(\vec{x},t)$. In this way the rate equation can be

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[*]The consensus threshold for Deffuant, in a society where everybody speaks with everybody else, is $\epsilon_c \sim 0.27$, in the sense that for $\epsilon > \epsilon_c$ most agents belong to one large cluster and the others are either isolated or form tiny groups. For $\epsilon > 1/2$, instead, all agents end up in the same cluster, also for other graph topologies [14].

immediately written

$$\frac{\partial}{\partial t}P(\vec{x},t) = \int_0^1 d\vec{x}_1 \, P(\vec{x}_1,t)\Big[\delta\Big(\vec{x} - \frac{\int_{\Omega(x_1)} d\vec{x}_0 \, \vec{x}_0 \, P(\vec{x}_0,t)}{\int_{\Omega(x_1)} d\vec{x}_0 \, P(\vec{x}_0,t)}\Big) - \delta(\vec{x} - \vec{x}_1)\Big] \qquad (9)$$

Beware of the meaning of the symbols: the integral on the right-hand side is a multiple n-dimensional integral and $\delta(\vec{x}) = \delta(x_1)\,\delta(x_2)\,...\delta(x_n)$. In the first multidimensional δ, to get the i-th component of the ratio of integrals one proceeds as follows. The integrals must be both calculated within the hypervolume $\Omega(x_1)$, which for us can be a hyperbox/hypersphere centered at $\vec{x}_1$ and with side/diameter 2ϵ. For the i-th component of the ratio, one has to replace the term $\vec{x}_0$ inside the integral at the numerator with the corresponding i-th component x_{0i}. It takes but a little work to show that the dynamics of Eq. (9), analogously as we have seen for scalar opinions, conserves the zeroth and first moments of the opinion distribution.

Eq. (9) is valid for an arbitrary number of dimensions. We will deal here only with the bidimensional case, essentially for two reasons:

- It is computationally cheap; if we discretize each opinion component in N intervals, the hypercube becomes a grid with N^n points, and as the number of operations is proportional to the number of points, the procedure becomes considerably slower for higher dimensions.

- It is easy to present the results in figures, with three-dimensional or contour plots.

4.1. *Squared confidence range*

Let us start with a squared opinion range. We solved numerically Eq. (9) for a few values of the confidence bound ϵ, here representing the half side of the square. As in the one-dimensional case, also this time a fourth order Runge-Kutta integrator has been used and the simulations started from a flat ditribution, $P(x,y,t=0) =$ const. The squared (x,y) opinion space has been reduced to a grid of 100×100 bins (200×200 bins have been also used in some simulations, in order to better estimate the consensus threshold) and the accuracy in $P(x,y,t)$ was of 10^{-9}. Even in this case, the dynamics runs until the distribution $P(x,y,t)$ reaches a stationary state for a given value of the confidence bound.
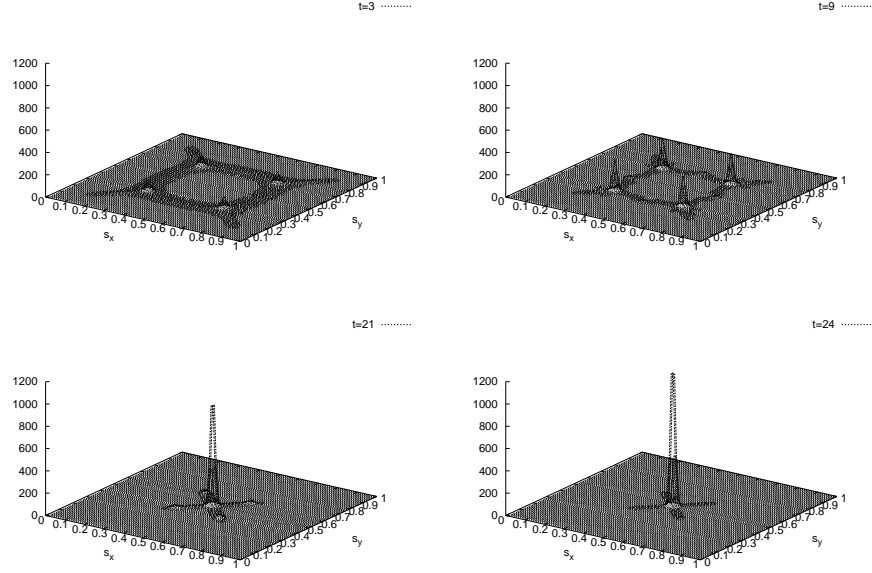
Fig. 4. KH dynamics with bidimensional real opinions and squared confidence range (from top left to bottom right: t=3,9,21,24). The initial opinion distribution is uniform and the confidence bound $\epsilon = 0.22$.
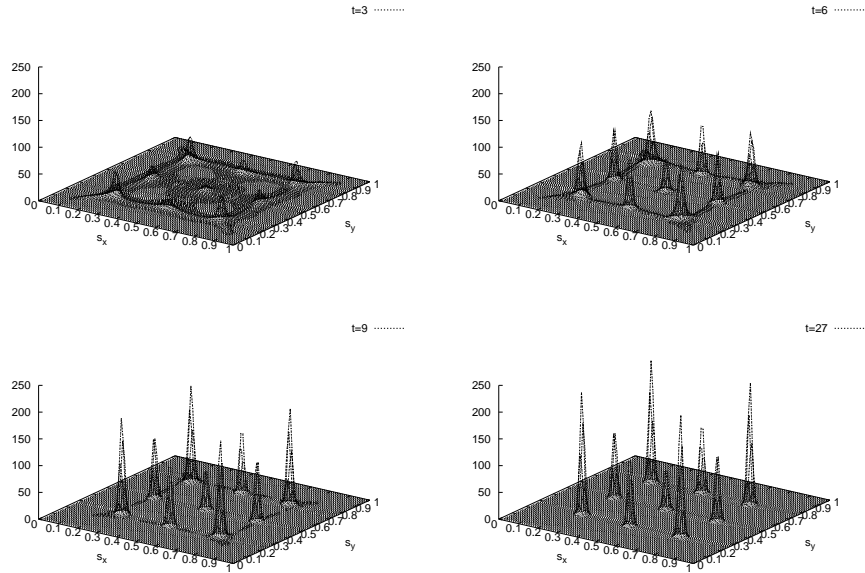

Fig. 5. As Fig. 4 but for $\epsilon = 0.15$ and t=3,6,9,27.

Fig. 4 shows how an initially flat probability distribution evolves with time, for $\epsilon = 0.22$. We see that four major clusters are formed quite early, but after a sufficiently long time they fuse to a single central cluster. The reason of such long-lived unstable states is again the fact that clusters can interact with each other through intermediary agents, as we have seen for scalar opinions. If the confidence bound is small, we expect that many clusters survive. Fig. 5 shows that this is indeed the case. We notice the regular structure of the final configuration; both the opinion space and the confidence range are squares, and this symmetry is reflected in the opinion configurations, where the clusters sit on the sites of a square lattice. From the figure one can see that the masses of the clusters are not equal. The four clusters near the vertices of the opinion space are the largest, followed by the four clusters near the centers of the edges of the square; the central cluster is the smallest.

We have also tried to estimate the value of the consensus threshold ϵ_c. Our result, $\epsilon_c = 0.215$, is consistent with the corresponding value for standard one-dimensional opinions (the consistency refers to the estimates determined through the rate equation; Monte Carlo simulations deliver values closer to 0.2 [12]).

Fig. 6 shows the final opinion configurations of the system for several values of ϵ. The figures are contour plots of the opinion distribution after many iterations. We notice the symmetry of the configurations: the clusters sit on the sites of a square lattice. We also find interesting variations of this scheme, however. As a matter of fact, we remark that in some cases also small clusters survive, which lie on the sites of the dual lattice. In particular, when a small cluster lies exactly on the center, it is likely to act as intermediate of the four large clusters which lie closest to it, so that in the long run they fuse in a large central cluster, which explicitly breaks the lattice symmetry of the configuration (as for $\epsilon = 0.10$, for instance). Such an anomalous feature can be viewed as an example of partial consensus below the critical treshold.

We have compared the final configuration of each opinion component with that of the one-dimensional dynamics corresponding to the same value of ϵ, for several values of ϵ. Most comparisons we have performed show a good similarity, with just a few exceptions, which means that the single opinion components - for a squared opinion range - evolve almost independently of each other. The bidimensional dynamics is then, in most cases, effectively one-dimensional.
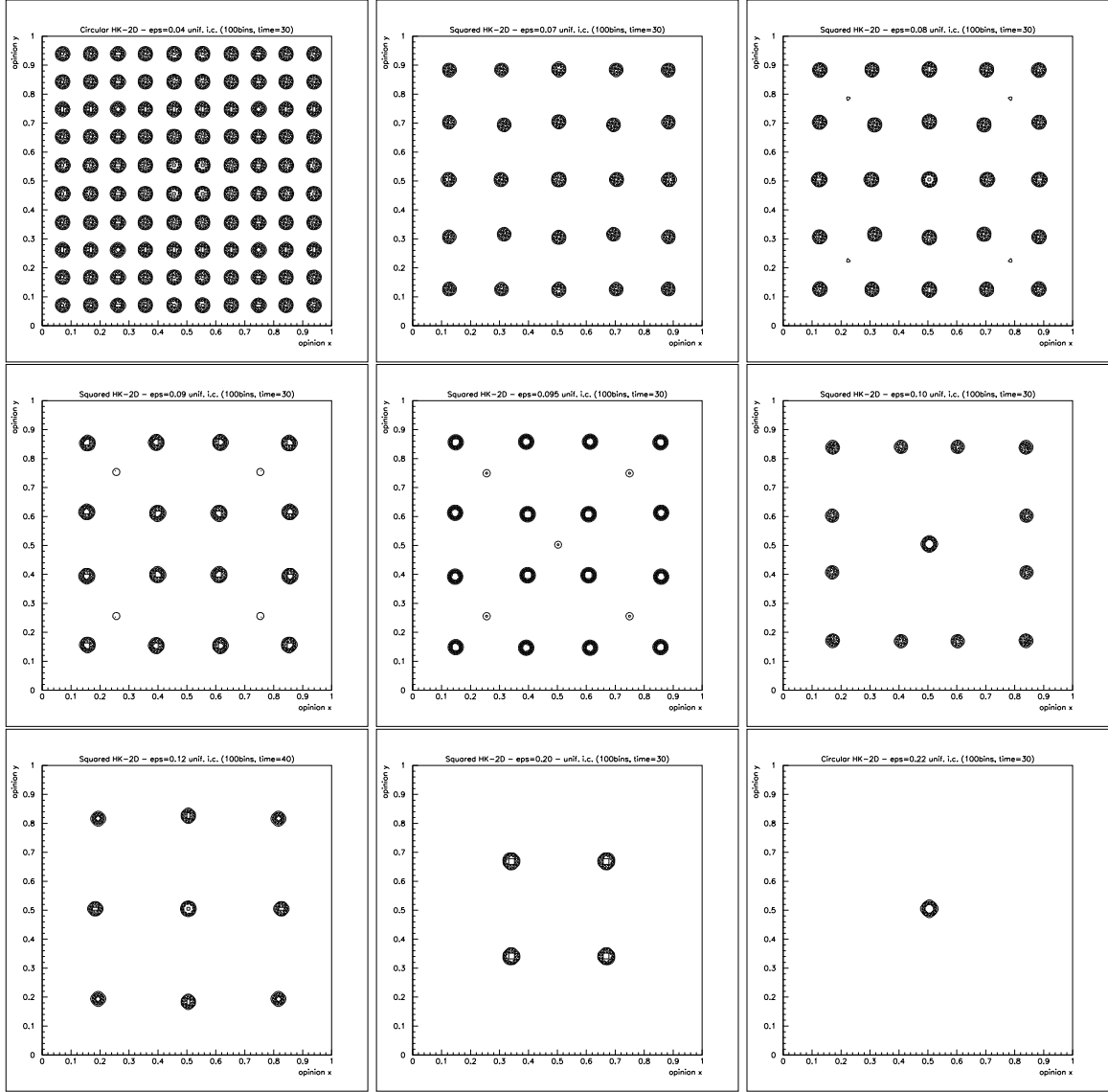
Fig. 6. Final configurations of the KH model with bidimensional opinions and squared confidence range. From top left to bottom right: ϵ=0.04,0.07,0.08; 0.09, 0.095,0.10; 0.12, 0.20,0.22.

4.2. *Circular confidence range*

Let us now examine the situation when the confidence range is a circle of radius ϵ. In this case the two components x, y are necessarily correlated, thus we would expect appreciable changes, both in the values of the thresholds for cluster merging and in the symmetry of the final opinion configurations.
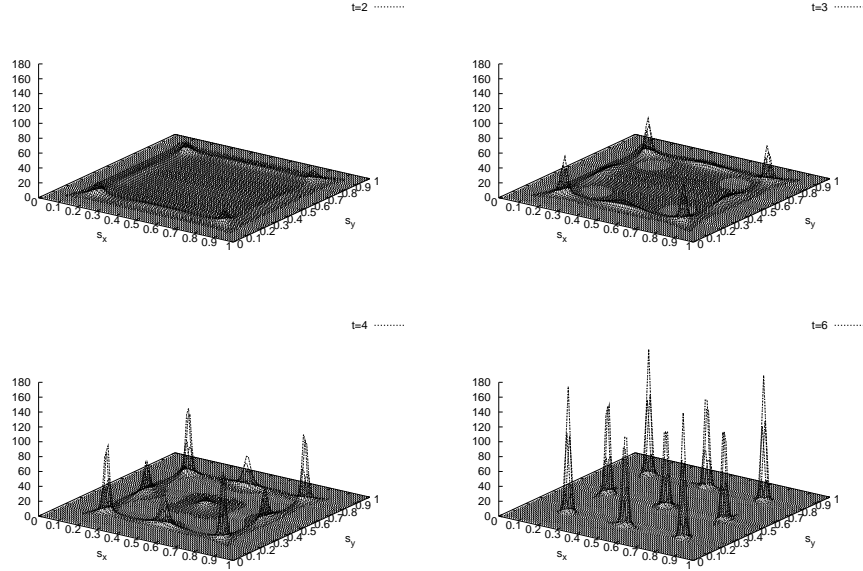

Fig. 7. As Fig. 5 (ϵ=0.15) but for circular confidence range and t=2,3,4,6.

In Fig. 7 we take four pictures of the dynamics of the system for $\epsilon = 0.15$. The pattern looks very much the same as in the corresponding Fig. 5. The number of clusters, their ordering in opinion space and the ratio of the cluster masses are the same. Other trials for different values of the confidence bound show that this is not a coincidence: the circular confidence range does not change much the situation. In particular, the consensus threshold is only slightly higher than in the previous case, about 0.23. There are two reasons for that:

- the surface of the circle is close to that of the square with the same linear dimension (the ratio is $\pi/4 \sim 0.8$), so that the sets of compatible agents in the two cases considerably overlap;

- the dynamics always starts from the edges of the opinion space, where the opinion distribution is necessarily inhomogeneous, so that it is essentially the shape of the opinion space which rules the symmetry of the resulting opinion landscape.

To have an overview of the situation, we report a series of contour plots relative to the final stage of the evolution at various ϵ, like in Fig. 6. The resulting Fig. 8 confirms that the clusters indeed sit on the sites of a square lattice, as we have seen above. There are also important differences, however. Particularly striking is the occasional existence of groups of four clusters near the center, which lie closer to each other as compared to the other clusters (see, for instance, the patterns corresponding to $\epsilon = 0.08$ and $\epsilon = 0.12$). Moreover, as we have seen for the case of the squared confidence range, sometimes smaller clusters survive on (some) sites of the dual lattice, especially at the center of the opinion space (like for $\epsilon = 0.10$ and $\epsilon = 0.20$).

It is also interesting to show the cluster formation in the case in which a group of four clusters near the center appears and remains stable in time. In Fig.9 a sequence of six contour plots calculated at different times is shown for $\epsilon = 0.08$. As one can see, the first snapshot confirms the previous statement that the symmetry of the opinion configurations is fixed by the shape of the opinion space. Going on, one can observe the progressive merging of the pairs of clusters with reciprocal distance less than the confidence bound radius. Finally, in the last picture, the survival of the four "anomalous" central clusters indicates that these clusters lie exactly at the border of the confidence range and this is clearly an effect of the circular shape of the confidence range (with a squared range, these four clusters would be attracted towards the center). We could also interpret it as a typical effect of the interdependence between the two components of the opinion vector.

5. Conclusions

We have extended the continuum opinion dynamics of Krause-Hegselmann to the case in which the opinion is not just a scalar but a vector with real-valued components. The extension is straightforward, with some freedom in the choice of the shape of the confidence range and the opinion space. Here we took a square for the opinion space and a square and a circle for the confidence range. We investigated a community where everybody talks to everybody else, and analyzed the dynamics and the final opinion configurations by solving numerically a rate equation. We found that if we project the final opinion configurations on any component, the number of clusters is essentially the same as in the one-dimensional opinion dynamics. The consensus threshold is slightly larger for the circular confidence range because the area spanned by the circle is smaller than that spanned by the square with the same linear dimension, but the two values are close to each other and to the one-dimensional consensus threshold. The structure formed by the centers of the final opinion clusters is a regular square lattice, but for special values of the confidence bound peculiar patterns also occur: survival of small clusters on the sites of the dual lattice, merging of the innermost four clusters into a large central one, existence of a compact group of four clusters near the center.
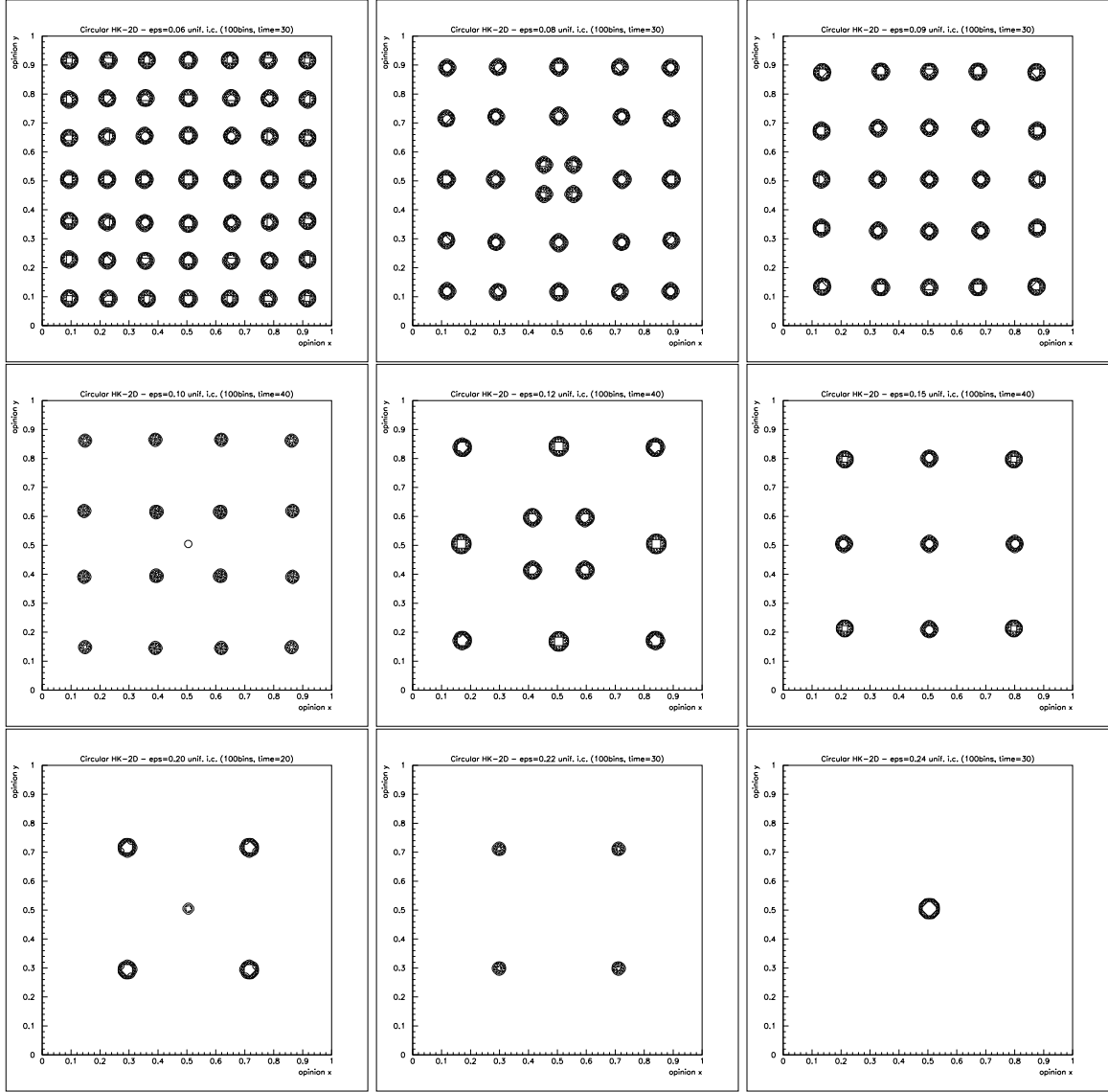
Fig. 8. As in Fig. 6, but for circular confidence range. From top left to bottom right: ϵ=0.06,0.07,0.09; 0.10, 0.12,0.15;0.20,0.22,0.24.
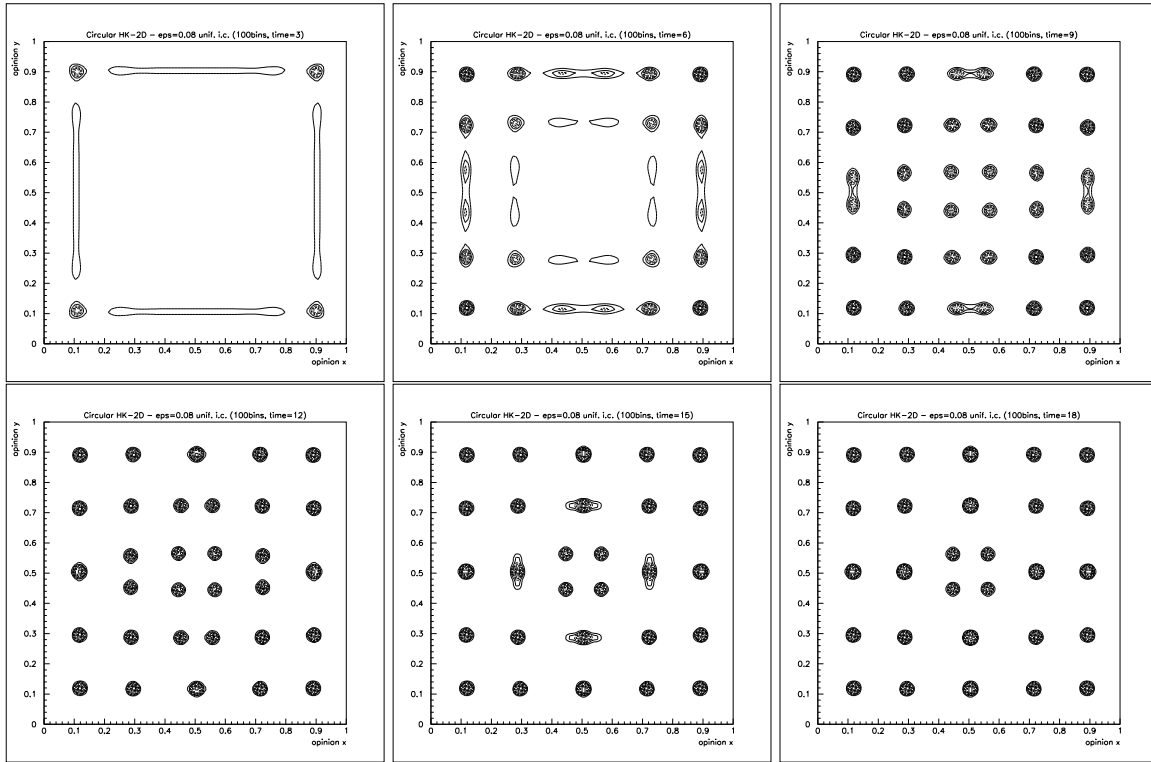
Fig. 9. Six snapshots show the temporal evolution of cluster formation and merging for circular confidence range and $\epsilon = 0.08$. From top left to bottom right: t=3, 6, 9; 12, 15, 18.

Summing up, we have found that vector opinion dynamics induces no significative variation in the evolution of the system that cannot be deduced by combining the results of the simple one-dimensional dynamics of the single opinion components. We studied here just the case of bidimensional opinions, but we do not expect big changes for a higher number of opinion components. However, we should not forget that we investigated a particularly simple model on a complete graph starting from a uniform opinion distribution, and this is at best only a zero*th*-order approximation of what happens for real systems.

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