When open mindedness hinders consensus

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When open mindedness hinders consensus

Hendrik Schawe[1,*] **and Laura Hernández**[1,+]

[1]Laboratoire de Physique Théorique et Modélisation, UMR-8089 CNRS, CY Cergy Paris Université, France
[*]hendrik.schawe@u-cergy.fr
[+]laura.hernandez@u-cergy.fr

ABSTRACT

We perform a detailed study of the Hegselmann-Krause bounded confidence opinion dynamics model with heterogeneous confidence ε_i drawn from uniform distributions in different intervals $[\varepsilon_l, \varepsilon_u]$. The phase diagram reveals a highly complex and nonmonotonous behaviour, with a re-entrant consensus phase in the region where fragmentation into multiple distinct opinions is expected for the homogeneous case. A careful exploration of the phase diagram, along with an extensive finite-size analysis, allows us to identify the mechanism leading to this counter-intuitive behaviour. This systematic study over system sizes which go well beyond those of previous works, is enabled by an efficient algorithm presented in this article.

1 Introduction

Opinion formation and its dissemination within a society, a recurrent subject of study in social sciences, is often addressed through statistical analyses of data collected by the means of surveys or polls. By following the evolution of the statistical outcomes over time, it is possible to obtain some information about the dynamical processes underlying these phenomena. This broadly applied approach implicitly assumes a full mixing of the population, neglecting the role of the structure of interactions among social actors. However, the key role of such interactions had already been acknowledged long ago by E. Durkheim[1], who coined the notion of *social facts*. A social fact is a property characterizing the whole society instead of the individuals, which emerges as an outcome of the dynamics governed by the interactions among them. Very early works have collected data about these interactions in very small societies, using graph theory to represent them. J. Scott[2] gives a nice historical overview of network development in social sciences. Nowadays, the development of mathematical models along with the continuous growth of data collection on human activities proliferated in particular due to the rise of online social networks, allow us to investigate different dynamical processes of opinion dynamics.

The rationale behind most opinion dynamics models is grounded in *Social Influence Theory*[3,4] which assumes that as agents interact they may influence each other, making their opinions more alike. Accordingly, the dynamical rules that govern the evolution of the opinion of each agent, are often based on functions which aggregate the opinions of a given set of actors. Recently a complete mathematical classification of the possible outcomes of the dynamics based on aggregation functions in the case of discrete opinion variables has been obtained[5,6].

The mathematical description of the structure of social interactions allows to identify the neighbourhood of a given agent (the set of other agents directly connected to it) and it is now clear that this structure is relevant. Different neighbourhood choices along with the type of influence they have on each social actor have been considered, for example the adoption of the opinion of a randomly chosen neighbour[7], the adoption of the neighbours' majoritarian opinion[8], or of agents whose opinion on other topics are already near the one of the target agent[9].

Among all the studied models[10,11], those representing the agents' opinion by a continuous variable are well suited to describe situations where the opinion on a particular problem is gradually built through the exchanges among the social actors. In particular *bounded confidence* models consider that each agent will only interact with those agents whose opinions are already close to theirs and will not interact with the others. Noteworthy, as agents' opinions evolve with time, the network of social ties is not fixed.

The best studied bounded confidence models are the *Deffuant-Weisbuch* (DW) model[12] and the *Hegselmann-Krause* (HK) model[13]. In both models the opinion of agent i is coded in a continuous variable and each agent may be influenced only by others whose opinion differs from his at most by a quantity called *confidence* ε_i. Unlike the DW model, which considers pairwise interactions, in the HK model agents are synchronously influenced by all others within their confidence range.

The most studied variant of the latter is the case of homogeneous confidence $\varepsilon_i = \varepsilon$, where large populations will always converge towards a uniform opinion, if the confidence is above a threshold $\varepsilon_c \approx 0.2$[13,14].

However, a society is not a homogeneous collection of individuals. Some of them are *open minded* and this property may be modelled by relatively large values of their confidence ε_i, accordingly the *closed minded* ones will have low values

of their confidence ε_i. Therefore heterogeneity of the confidence of single agents is an obvious and well motivated addition to the model. Until now heterogeneity in the HK model was mainly studied by introducing multiple subpopulations, each of them characterized by a specific confidence value, and formed by a homogeneous set of agents.[15–18] Some other works study systems with random confidence, with ε_i drawn from some distribution[19–21], or from different distributions for multiple subpopulations[22]. Those studies are usually performed by the means of multi-agent simulations on small statistical samplings (50-100 samples), where each realization represents also a very small system (typically a few hundred agents). An alternative method, the multiple chain Markov model[15], has been proposed in order to obtain the properties of the infinite system. However, there, opinions are discretised which is known to to cause deviations from the HK model[23] and the probabilities of each opinion are derived from an initial set of agents. Nevertheless, these works seemed to indicate that heterogeneity indeed leads to some surprising nonmonotonous effects in the dynamical outcomes. In particular, the existence of consensus in regions where the confidence of the agents is below the critical threshold of the homogeneous case have been reported[17].

In this article, we present a systematic study of the phase diagram of the HK heterogeneous model in the parameter space given by the possible lower and upper bounds of the confidence values of the agents, $(\varepsilon_l, \varepsilon_u)$. We carefully sample the parameter space for large systems and we are able to obtain very good statistics for the studied quantities. Furthermore, we study the finite-size effects on the dynamics of the model, which reveals complex details of the consensus landscape that only become apparent at system sizes that are larger than those studied before. A careful study of the dynamical trajectories in the opinion space allows us to explain the re-entrant phase observed in the phase diagram. Finally, we introduce an algorithm that allows us to study system sizes and sampling statistics that go well beyond the present state of the art.

2 Models and Methods

We study the Hegselmann-Krause model (HK), which describes a compromise dynamic under bounded confidence. Each of n *agents i* is endowed of a continuous variable $x_i(t)$ representing *opinion* and a fixed *confidence* ε_i, modeling the heterogeneous idiosyncrasies of the agents. The *neighbours* of agent i are all agents j with opinions inside the interval $[x_i - \varepsilon_i, x_i + \varepsilon_i]$, i.e.

$$I(i, \vec{x}) = \left\{ 1 \leq j \leq n \big| |x_i - x_j| \leq \varepsilon_i \right\}. \tag{1}$$

Note that every agent is a neighbour of itself. In each time step an agent i talks to all its neighbours j and adopts the average opinion of all neighbours, i.e.

$$x_i(t+1) = |I(i, \vec{x}(t))|^{-1} \sum_{j \in I(i, \vec{x}(t))} x_j(t). \tag{2}$$

This update is performed synchronously for all agents, although a sequential random update is possible. The latter typically leads to longer convergence times, as single agents are left behind and may, for small values of ε, persist as isolated clusters in the final state. However, besides these effects, the observations which we describe in the following sections are qualitatively robust against the update schedule.

The dynamic leads to a stable configuration, where the agents are converged to one or several opinions. The groups of agents ending in the same opinion are usually called *clusters*. The situation in which only one giant cluster exists is called *consensus*, two clusters are called *polarization* and more are called *fragmentation*. For the homogeneous case it is well known that above a critical $\varepsilon_c \approx 0.2$[10] large systems will always converge to consensus and never reach consensus below this threshold.

While the original HK model uses homogeneous confidence $\varepsilon_i = \varepsilon$, the only modification we apply to this model, which will lead to surprisingly complex behaviour as we will see later, is to draw heterogeneous ε_i from an i.i.d. uniform distribution bounded by two parameters $\varepsilon_l, \varepsilon_u$ with $\varepsilon_l \leq \varepsilon_u$. The choice of the parameter $\varepsilon_l \geq 0$ determines how closed minded the most closed minded agents are and $\varepsilon_u \leq 1$ determines how open minded the most open minded agents are.

We are mainly interested in the effects of heterogeneity on consensus for the whole $(\varepsilon_l, \varepsilon_u)$ space. To evaluate this, we look at the relative size of the largest cluster S averaged over all simulated realizations $\langle S \rangle$. A value near 1 indicates consensus, smaller values indicate polarization or fragmentation.

One difficulty in simulating the HK dynamics is that the run time scales in the number of agents as $\mathcal{O}(n^2)$, such that simulation of large system sizes becomes infeasible quickly. There are attempts to invent algorithms which are faster, but they come with trade offs, such as not simulating the actual dynamic, but generating an approximation for the $n \to \infty$ case[24] or necessitating a discretization of the model[23].

We will here introduce an algorithm, which is much faster than the naive approach for typical realizations, while preserving the continuous character down to the precision of the data types used. This enables us to simulate larger sizes with far better statistics than other contemporary studies of the HK model. Its fundamental idea is that to update agent i we do only have to look at the agents within its confidence interval $[x_i - \varepsilon_i, x_i + \varepsilon_i]$, which are typically far fewer than n for small values of ε_i – but in the order of $\mathcal{O}(n)$ in the worst case.

To achieve this in an efficient way, we maintain a *binary search tree* (BST) [25, p. 286] of all opinions $x_j(t)$[1]. We use this datastructure to efficiently find the agent with the smallest opinion in the range $[x_i - \varepsilon_i, x_i + \varepsilon_i]$ and traverse the tree in order until we find the first node outside of this range, to calculate $x_i(t+1)$.

Since BSTs can only store unique elements, we have to handle the case of multiple agents having the same opinion within precision of the used datatype. Therefore, we save at each node of the tree, additionally to the opinion, a *counter*, which keeps track of the number of agents which have this opinion and are therefore represented by this node. Correspondingly this counter has to be handled as a weight when calculating the average.

When the opinion of an agent is updated, it also has to be updated in the tree by removing its former opinion (or decreasing its counter by one) and inserting its current opinion (or increasing its counter by one). Both operations can be performed in time $\mathcal{O}(\log(n))$. This algorithm therefore has a typical time complexity for one step between $\Omega(n \log(n))$ in the best case and $\mathcal{O}(n^2)$ in the worst case.

Note that the performance of this algorithm benefits from two independent effects. First, for agents with small confidence ε_i we profit from the reduced number of opinions we need to look at to determine the average opinion of its neighbours. Second, the HK model tends to form clusters (within the chosen numerical precision) quickly, especially for large ε_i. Since clusters are represented in the BST as a single node with a high counter, the computational cost to calculate the average opinion is greatly reduced.

3 Results

We simulated the system for a broad range of system sizes $64 \leq n \leq 131072$ and we carefully explored the parameter space of confidence intervals. For each run, we have drawn the confidence parameter of each agent ε_i uniformly from an interval bounded by $(\varepsilon_l, \varepsilon_u) \in [0, 0.35] \times [0, 1]$. The simulations are run until the opinions, represented with 32 bit IEEE 754 float datatypes, converge. The convergence criterion we use here requires that the sum of the changes over all the agents is below a threshold, i.e. $\sum_{i=1}^{n} |x_i(t-1) - x_i(t)| < 10^{-4}$.

Clusters are composed of agents holding the same opinion within a tolerance range of 10^{-4}. We have checked that the results are robust when using different clustering criteria, e.g. binning the opinion space.

Figure 1 shows a heat map of the average size of the largest cluster $\langle S \rangle$ for each of the 8224 points that we have simulated in the parameter space. The results shown by the heat map at each point $(\varepsilon_l, \varepsilon_u)$ correspond to the average over 1000 simulated samples, which differ in their initial conditions although all represent the same society, where the heterogeneous confidence parameters of the agents ε_i have been uniformly drawn from the same interval. Note that the aspect ratio is not unity.
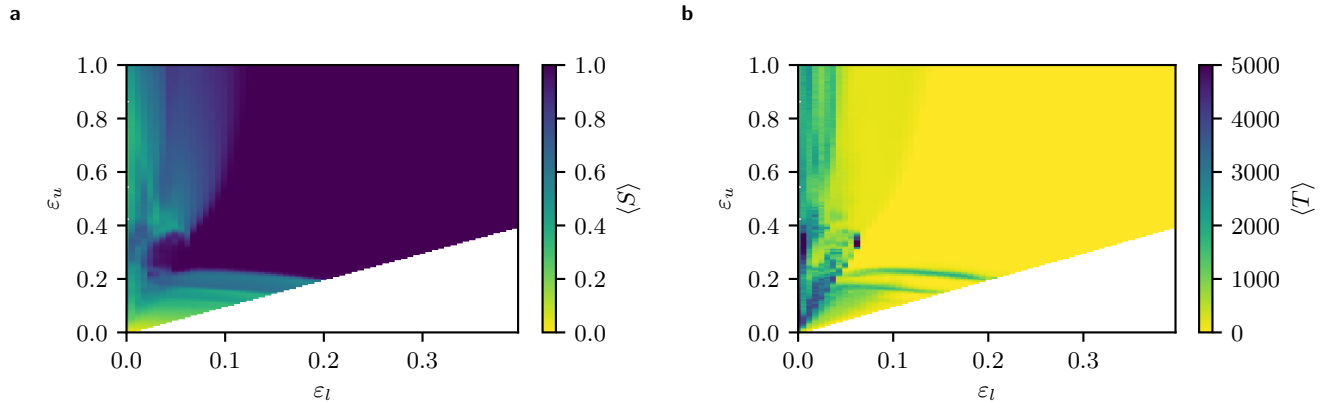

Figure 1. Left: Average relative size of the largest cluster $\langle S \rangle$. Right: Average convergence time $\langle T \rangle$. These data are collected for 8224 pairs of $(\varepsilon_l, \varepsilon_u)$ for a system of $n = 16384$ agents and averaged over 1000 realizations. Note that three parameter pairs at $\varepsilon_l = 0$ did not converge in reasonable computing time for all realizations. To avoid selection bias, they are therefore omitted and marked in white.

The region not shown here ($[0.35, 1] \times [0, 1]$) corresponds to values of confidence intervals that always lead to consensus. This is expected, as both bounds are far above the critical value of the homogeneous case. The white triangle in the lower right

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[1]Technically, we use a B-tree [25, p. 484], which is a self-balancing generalization of a BST and better suited to iterate over contiguous ranges of entries.

corner consists of impossible intervals, where the lower bound would be higher than the upper one. The diagonal elements correspond to the homogeneous case, i.e. $\varepsilon_l = \varepsilon_u$. One can see that on this diagonal, $\langle S \rangle$ changes from 1 to approximately $1/2$ around $\varepsilon_l = \varepsilon_u = 0.2$, showing the transition from consensus to polarization that has been found in the homogeneous HK model[10].

The interesting results are situated on the left of the map, where a *re-entrant consensus region* occurs around the point $(0.05, 0.3)$. This reveals a nonmonotonous behaviour of the system as the fraction of confident agents increases in the population. Let us examine the map by considering a fixed value of $\varepsilon_l = 0.05$ while varying the upper value ε_u. We start at the homogeneous case $\varepsilon_u = 0.05$, where we find fragmentation. As ε_u increases, a dark region is encountered showing strong consensus. This is expected as more agents with an increasing confidence (having a larger number of neighbours) enter the system, contributing to integrate agents with low confidence into the consensus group. However, as ε_u increases further, consensus surprisingly disappears although the fraction of confident agents and the magnitude of their confidence is even larger.
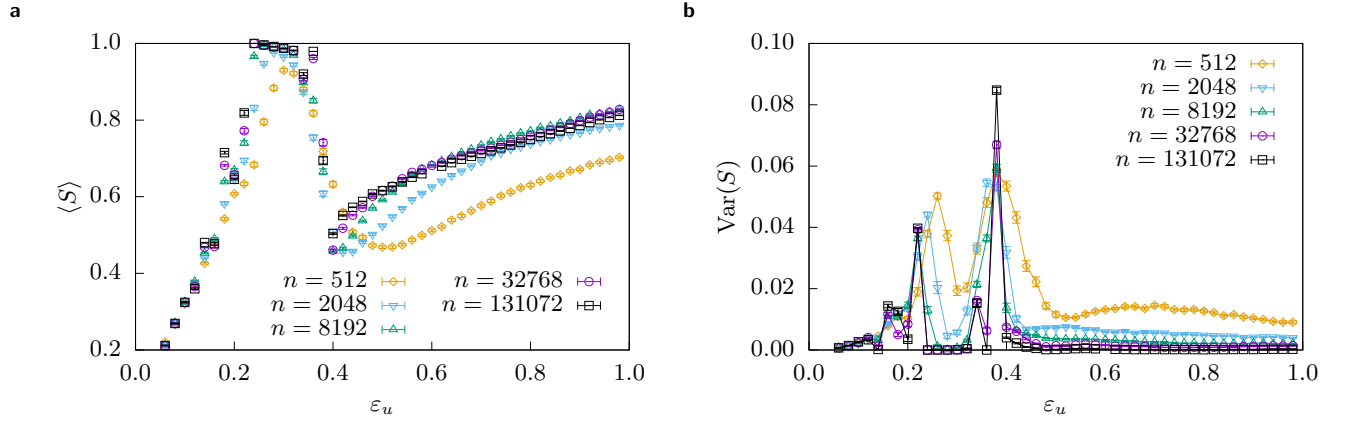

a

b

Figure 2. Left: Average size of the largest cluster $\langle S \rangle$. Fixed $\varepsilon_l = 0.05$ for varying values of $\varepsilon_u \in [\varepsilon_l, 1]$; this corresponds to a straight vertical line in Fig. 1 through the re-entrant consensus phase. The consensus region is robust with increasing system sizes. Also note that the behaviour is highly complex as the behaviour with increasing system size is not always monotonous, e.g. at $\varepsilon_u = 0.2$ or $\varepsilon_u = 0.42$. Right: Variance of the size of the largest cluster $\mathrm{Var}(S)$. The transition points into consensus and out of consensus can be located at the peaks of the variance. Using the finite-size behaviour, we estimate them for $\varepsilon_l = 0.05$ at $\varepsilon_u = 0.22(1)$ and $\varepsilon_u = 0.38(1)$. Lines are just guides to the eye.

The left panel of Fig. 2 shows the evolution of the order parameter $\langle S \rangle$ with ε_u for a vertical cut in the region of the re-entrant phase of the heatmap shown in Fig. 1 and described in the previous paragraph in higher detail and for different system sizes. For a system that contains very closed minded agents, when the fraction of open minded agents increases, the size of the largest cluster also increases leading to consensus, however as the fraction of confident agents increases further, consensus is lost.

The right of Fig. 2 shows the fluctuation of the order parameter $\langle S \rangle$. There are two peaks visible, which indicate the location of the two transitions, in and out of the re-entrant phase. Interestingly, the finite-size behaviour signals a real double transition as the peaks increase and separate with increasing size, suggesting that this is not just a finite-size effect, but that it remains in the thermodynamic limit, i.e. the $n \to \infty$ limit. Using these data, we estimate the re-entrant phase to span the interval $\varepsilon_u \in [0.22(1), 0.32(1)]$ for fixed $\varepsilon_l = 0.05$.

In order to explain this paradoxical result, a careful examination of the dynamic behaviour of the agents' opinions is necessary. In Fig. 3 we show the evolution of opinions for systems corresponding to three different values of ε_u in the re-entrant region. The top row corresponds to averages over 10000 realizations of the initial conditions, while the bottom row shows, as examples, the evolution of opinions of a single realization each.

The mechanism leading to the observed behaviour is rooted in the different characteristic times that open and closed minded agents need to join a majoritarian opinion strand. The latter take more time to reach the consensus opinion as they need to meet agents within their narrow confidence interval. This is at the origin of the bell shaped structures in the bottom left panel of Fig. 3

These structures enable closed minded agents from the whole region over which the bell spans, to join a strand and evolve. If the strand counts enough open minded agents, they can pull it towards a consensus opinion, bringing with them the closed minded agents that have an opinion close to theirs. This is what happens in the bottom middle panel of Fig. 3 at $\varepsilon_u = 0.3$. Also the signature of this structure is clearly visible in the average over many realization in the top middle panel. The two smaller

strands visible in this panel contain only 3% of the agents. As it is very frequent that the central cluster converges to one of them, the averaged picture shows an apparent diffusion of the central opinion for long times. Moreover we expect the small clusters to vanish in the thermodynamic limit, since the peak of Fig. 2a shows an upward trend for increasing system sizes.
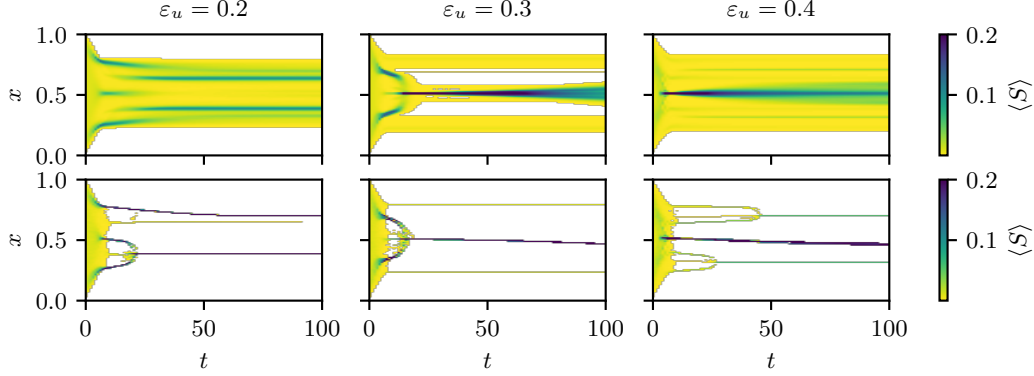

Figure 3. We look for fixed $\varepsilon_l = 0.05$ and $n = 16384$ at three values of $\varepsilon_u \in \{0.2, 0.3, 0.4\}$ (from left to right) corresponding to one value left of the peak, one at the peak and one right of the peak of Fig. 2. The images show which fraction of agents have a given opinion x for each time step. Note that the colour scale is truncated at 0.3 to generate images with a good contrast in the interesting region at small times, such that the darkest colour can also represent values larger than 0.3. The time axis is also truncated to focus on the most interesting region of small times, i.e. not the whole range until reaching a stationary state is visualized. Top: aggregated statistics over 10000 realizations of the initial conditions. Bottom: single trajectories.

Note that agent i can in each time step move at most ε_i, such that closed minded agents need to see other agents for at least a few iterations to be able to change their opinion from one extreme to the consensus opinion at 0.5. More open minded agents (like those that appear when increasing ε_u) will evolve very quickly to a central opinion, because they can interact with a large fraction of the other agents and therefore are able to jump directly into the centre. As a consequence closed minded agents are left behind and are not able to join the consensus opinion. This is the situation depicted in the right panels of Fig. 3.

The case of $\varepsilon_u = 0.4$, shown on the right of Fig. 3, seems to show that the central strand contains about 90%, which is an apparent contradiction with the mean cluster size $\langle S \rangle \approx 0.5$ shown in Fig. 1. The solution to this discrepancy is that the central strand located at an opinion $x \approx 0.44$ splits in the stationary state to three very close clusters, therefore there are 5 clusters in total, which are composed of different groups of agents. One of them contains about 50% of the agents, mainly having large confidence intervals, able to interact with agents of the small strands located at $x \approx 0.32$ and $x \approx 0.68$. Another cluster contains about 30% of the agents, which can interact only with the bottom strand at $x \approx 0.32$ (and all agents of the central strand), but not with the upper one at $x \approx 0.68$, such that it will converge to a slightly lower opinion. The last cluster contains about 10% of the agents which can only interact with those in the central strand (their opinion being in the middle of the other two clusters forming the central violet strand). Therefore, the size of the largest cluster of this realization is $S \approx 0.5$, the value observed in the valley of Fig. 2. Note that this effect can easily be missed when using a more discrete cluster criterion, like binning with too few bins.

Figure 1b confirms these results: average convergence time in the region of re-entrant consensus is much larger than on neighbouring regions, illustrating the extra time needed by closed minded agents to join the consensus strand *pulled* by open minded agents.

Interestingly, the bell shapes observed in the evolution of opinions, which help to integrate isolated agents into a single strand, are observed for or all the parameters shown here. However, when ε_u is either too low or too high, these structures are mainly formed by agents with very low confidence, in the first case just because they are the majority of the population and in the second because agents with large confidence have already joined the main strand. For intermediate values of ε_u, these bell structures contain both open and closed minded agents, and the former may bring the latter into the final consensus strand.

Another peculiarity visible in Fig. 1 is the extremely complex behaviour of $\langle S \rangle$. For example at $\varepsilon_u \approx 0.16$ there is a local maximum. This is reminiscent of an effect called "consensus strikes back" by Ref.[26] for a related "interacting Markov chain" (and not agent-based) HK model. There, a consensus phase was observed for the homogeneous case for $0.152 \leq \varepsilon \leq 0.174$. We show finite size data in Fig. 4, where one bound (either ε_l or ε_u) is fixed to 0.16 while the other bound is indicated in the horizontal axis, to test whether we can see a signature of this effect in the agent-based HK model. Indeed, we can identify an upward trend in $\langle S \rangle$, which becomes even stronger with added inhomogeneities at and slightly below 0.16.

Although we do not attempt to extrapolate this to larger sizes, due to the nonmonotonous behaviour observed in this region
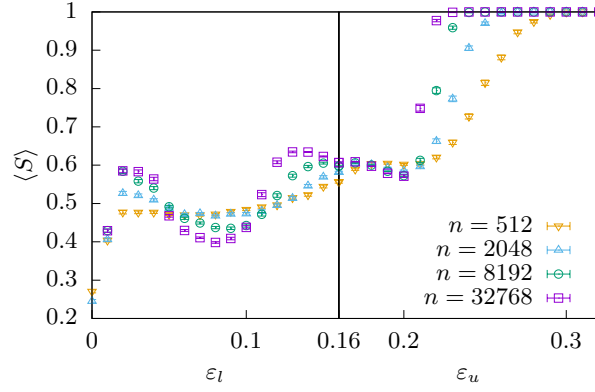
Figure 4. Average size of the largest cluster $\langle S \rangle$. The left half shows $\varepsilon_u = 0.16$ fixed and ε_l varying, the right half shows $\varepsilon_l = 0.16$ fixed and ε_u varying. The position marked by the vertical line is the homegeneous case $\varepsilon_l = \varepsilon_u = 0.16$. This corresponds to a straight line in Fig. 1 reflected at the diagonal.

of the parameter space, our results indicate that there is indeed a trend of increasing cluster size located approximately in the region where the "consensus strikes back" effect was observed in the homogeneous case for an related model[26]. However, also with heterogenity, which we showed to be beneficial for consensus, the agent-based HK model does not show consensus at the system sizes we observed.

4 Conclusions

We have performed a very detailed characterisation of the phase diagram of the heterogeneous Hegselmann-Krause model by means of an efficient algorithm that allows the simulation of large samples. In this way we could obtain very good statistics and investigate finite-size effects overcoming the size limitations of previous works. Our results reveal a nonmonotonous behaviour with a re-entrant consensus phase in the region of the parameter space where fragmentation is expected.

Previously, the phase diagram of the HK model or closely related models with two different values of the bounded confidence coexisting in the population has been studied, also revealing a nonmonotonous behaviour depending on the amount of open and closed minded agents[15,17]. Here, by the means of the simulation of large systems, we characterise the rich behaviour of the fully heterogeneous HK model, and we identify the region $[\varepsilon_l, \varepsilon_u]$, where the consensus phase clearly enters in the region, in which one would not expect consensus based on the behaviour of the homogeneous case. In particular, we find that increasing the proportion of open minded agents may lead to a loss of consensus, provided closed minded agents are still in the system. We were able to explain this counter-intuitive observation with a careful study of the opinion evolution. Its origin is the slow movement of closed minded agents in the opinion space. When the system contains a large fraction of very open minded agents, who converge very quickly to a majoritarian opinion, the closed minded agents are left behind and full consensus is precluded. On the contrary it is easier to reach consensus when the closed minded agents coexist with a fraction of moderately open minded one, who converge to the majoritarian opinion much slower. This relative slow convergence allows for the interaction with the closed minded agents who are slowly dragged to the majoritarian opinion.

We believe that the introduction of quenched disorder by increasing the complexity of the HK model offers a lot of potential for further studies. Variations of the model that introduce a network of social ties, which limits the possible neighbourhoods of any agent, or the integration of cost that the agents must bear for changing their opinion, are work in progress.

Acknowledgments

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