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Multi-Dimensional Hegselmann-Krause Dynamics

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Abstract— **We consider multi-dimensional Hegselmann-Krause model for opinion dynamics in discrete-time for a set of homogeneous agents. Using dynamic system point of view, we investigate stability properties of the dynamics and show its finite time convergence. The novelty of this work lies in the use of dynamic system approach and the development of Lyapunov-type tools for the analysis of the Hegselmann-Krause model. Furthermore, some new insights and results are provided. The results are valid for any norm that is used to define the neighbor sets.**

I. Introduction

Recently, modeling of social networks has attracted a significant attention and several models for opinion dynamics have been proposed and studied [1], [2]. These models have also found applications in engineered network systems of autonomous agents such as, for example, in robotic networks [3]. One of the commonly used models for opinion dynamics is the Hegselmann-Krause proposed by R. Hegselmann and U. Krause in [1], where agent opinions are scalar valued. The stability and convergence of the Hegselmann-Krause dynamics and its extensions have been investigated in [4], [5], while this dynamics for continuous time and time-varying networks have been proposed in [6]–[8]. The convergence rate of the Hegselmann-Krause model has been investigated in [9] (see also [10]). All of the aforementioned work considers scalar-valued opinions, where the analysis rests on the fact that the Hegselmann-Krause dynamics preserves the initial ordering of the opinion values.

In this paper, we consider the Hegselmann-Krause model extended to the case where the agent opinions are finite-dimensional vectors. In the vector case, we cannot use the line of analysis applicable to the scalar case, as its basic premise that we can order the opinions is violated. To deal with this, we approach the Hegselmann-Krause model by adopting a dynamic system point of view, and by using novel tools and results established in [9] for weighted-averaging dynamics. We develop a new termination criterion for the dynamics. Then, we construct a Lyapunov function based on an adjoint dynamics, and we use this function to establish the finite-time convergence of the Hegselmann-Krause dynamics. While developing these results, we were unaware that the stability and finite time convergence for multi-dimensional model has already been proven by Lorenz in [2], Theorem 3.3.1, where the analysis relies on non-negative matrix theory. While our finite-time convergence result is not new, our approach and analysis of the Hegselmann-Krause dynamics is novel. Besides this, we provide some insights into the dynamics which lead to a new termination criterion (Proposition 2). We also assert the existence of adjoint dynamics, which plays critical role in Lyapunov-based argument for the finite-time convergence (Propositions 3 and 4). Finally, the approach proposed in this paper is better suited for the development of tight bounds on the termination time than an approach based on non-negative matrix theory (the matrix product analysis typically yields a rather pessimistic bounds when the matrices are time-varying).

The paper is organized as follows: In Section II, we discuss discrete-time Hegselmann-Krause dynamics. In Section III, we investigate some properties of this dynamics. In particular, we define and study several types of agents and also, discuss the adjoint dynamics for Hegselmann-Krause dynamics which plays an important role in our development. Using the developed tools, in Section IV we establish a finite-time termination time for the Hegselmann-Krause dynamics. We derive this result by bounding the decrease rate of a quadratic Lyapunov function along the trajectories of the dynamics. In Section V we conclude with a discussion on the established results and some suggestions for further studies.

II. Hegselmann-Krause Opinion Dynamics

We consider the discrete time Hegselmann-Krause opinion dynamics model of [1] extended to the case

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where the agent opinions are vectors. We assume that there are m agents that interact in time, and we use $[m] = \{1, \ldots, m\}$ to denote the agent index set. The interactions occur at time instances denoted by non-negative integers $t = 0, 1, 2, \ldots$. At each time t, agent i has an opinion represented by a row-vector $x_i(t) \in \mathbb{R}^n$. The collection of profiles $\{x_i(t), i \in [m]\}$ is referred to as the *opinion profile* at time t.

Initially, each agent i has an opinion vector $x_i(0) \in \mathbb{R}^n$. At each time t, the agents interact with their neighbors and update opinions. The neighbors of each agent are determined based on the difference of the opinions and a prescribed maximum difference $\epsilon > 0$, termed a *confidence*, which limits the agent interactions in time. More precisely, the set of neighbors of agent i at time t, denoted by $\mathcal{N}_i(t)$, is given as follows:

$$\mathcal{N}_i(t) = \{j \in [m] \mid \|x_i(t) - x_j(t)\| \leq \epsilon\}, \quad (1)$$

where $\|\cdot\|$ is any norm in $\mathbb{R}^n$. Note that $i \in \mathcal{N}_i(t)$ for all $i \in [m]$ and all $t \geq 0$. The opinion profile evolves in time according to the Hegselmann-Krause dynamics:

$$x_i(t+1) = \frac{1}{|\mathcal{N}_i(t)|} \sum_{j \in \mathcal{N}_i(t)} x_j(t) \qquad \text{for all } t \geq 0, \quad (2)$$

where $|\mathcal{N}_i(t)|$ denotes the cardinality of the set $\mathcal{N}_i(t)$. Note that the opinion dynamics is completely determined by the confidence value ϵ and the initial profile $\{x_i(0) \mid i \in [m]\}$.

Before we proceed with our study of the model, let us discuss the notation that we use. The vectors are assumed to be column vectors, unless clearly stated otherwise. We use x' to denote the transpose of a vector x and, similarly, A' for the transpose of a matrix A. We write e to denote a column vector with all entries equal to 1. A vector $v \in \mathbb{R}^m$ is a stochastic (or probability) vector if $v_i \geq 0$ and $v'e = 1$. Given a vector $v \in \mathbb{R}^m$, $\mathrm{diag}(v)$ denotes the $m \times m$ diagonal matrix with v_i, $i \in [m]$, on its diagonal. For a matrix A, we write A_{ij} or $[A]_{ij}$ to denote its ij-th entry. A matrix A is stochastic if $Ae = e$ and $A_{ij} \geq 0$ for all i, j. We write $|S|$ to denote the cardinality of a finite set S.

A. Isolated Agent Groups

At a given time t, let agent sets $S_1, S_2 \subseteq [m]$ consist of connected agents. We say that groups S_1 and S_2 are *isolated from each other if the convex hull of* $\{x_i(t), i \in S_1\}$ *and the convex hull of* $\{x_j(t), j \in S_2\}$ *are at the distance greater than* ϵ. Two isolated groups are illustrated in Figure 1.

We next show that isolated groups never interact.

Proposition 1: The agent groups that are initially isolated will remain isolated at all times.

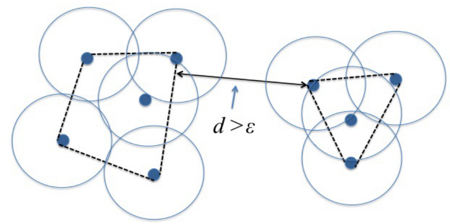

Fig. 1. An illustration of two isolated agent groups in a plane.

Proof: Let S_1 and S_2 be two isolated agent groups at time $t = 0$. At time $t = 1$, each opinion $x_i(1)$ for $i \in S_1$ is a convex combination of the opinions $\{x_i(0), i \in S_1\}$, and therefore the convex hull of $\{x_i(1), i \in S_1\}$ is contained in the convex hull of $\{x_i(0), i \in S_1\}$. By proceeding in this manner, we can conclude that at each time t, the convex hull of opinions $\{x_i(t), i \in S_1\}$ is contained in the convex hull of $\{x_i(0), i \in S_1\}$. The same is true for opinions $\{x_j(t), i \in S_2\}$. Hence, the distance between the convex hulls of $\{x_i(t), i \in S_1\}$ and $\{x_j(t), j \in S_2\}$ is non-decreasing in time. Therefore, at any time t this distance is at least as large the distance between the initial convex hulls of $\{x_i(0), i \in S_1\}$ and $\{x_i(0), i \in S_1\}$, which is larger than ϵ. Thus, the agent groups S_1 and S_2 will be isolated at all times. ■

In light of Proposition 1, it suffices to analyze the stability of an isolated group identified at the initial time. Thus, for the rest of the paper, without loss of generality, we *assume that the initial profile* $\{x_i(0), i \in [m]\}$ *defines a single isolated group*.

III. Properties of Dynamics

Evidently, the dynamics will be stable if the neighbor sets stabilize for all agents, i.e., there is some $T \geq 0$ such that $\mathcal{N}_i(t) = \mathcal{N}_i(t+1)$ for all i and all $t \geq T$. To capture this, we classify agents into those with "small" opinion spread and those with "large" opinion spread. In the next section, we formally define these agent types and investigate their properties.

A. Agent Types

At any time, we distinguish two types of agents, $M(t)$ and $G(t) = [m] \setminus M(t)$, defined as follows:

$$M(t) = \{i \in [m] \mid \max_{j \in \mathcal{N}_i(t)} \|x_i(t) - x_j(t)\| > \frac{\epsilon}{3}\}, \quad (3)$$

$$G(t) = \{i \in [m] \mid \max_{j \in \mathcal{N}_i(t)} \|x_i(t) - x_j(t)\| \leq \frac{\epsilon}{3}\}.$$

To interpret these sets, we view the distance $\max_{j \in \mathcal{N}_i(t)} \|x_i(t) - x_j(t)\|$ as a measure of the local-opinion spread for agent i. Then, the set $M(t)$ consists of the agents that have a large local-opinion spread, while the set $G(t)$ consists of the agents that have a small local-opinion spread. These two sets of agents will play an important role in the subsequent development.

Some important relations for the agents in the set $G(t)$ are provided in the forthcoming lemma.

Lemma 1: There holds for all $t \geq 0$:
(a) $\mathcal{N}_i(t) \subseteq \mathcal{N}_j(t)$ for all $i \in G(t)$ and $j \in \mathcal{N}_i(t)$;
(b) $\mathcal{N}_j(t) = \mathcal{N}_i(t)$ for all $i \in G(t)$ and $j \in \mathcal{N}_i(t) \cap G(t)$;
(c) For every $i, \ell \in G(t)$, either $\mathcal{N}_i(t) = \mathcal{N}_\ell(t)$ or $\mathcal{N}_i(t) \cap \mathcal{N}_\ell(t) = \emptyset$.

Proof: Let $s \in \mathcal{N}_i(t)$. Then, by the triangle inequality for the norm, we have

$$\begin{aligned}\|x_s(t) - x_j(t)\| &\leq \|x_s(t) - x_i(t)\| + \|x_i(t) - x_j(t)\| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon,\end{aligned}$$

where we use $s \in \mathcal{N}_i(t)$, $j \in \mathcal{N}_i(t)$ and $i \in G(t)$. Thus, $\mathcal{N}_i(t) \subseteq \mathcal{N}_j(t)$ showing that part (a) holds. For part (b), note that $i \in \mathcal{N}_j(t)$ and $j \in \mathcal{N}_i(t)$ thus, from part (a) it follows that $\mathcal{N}_i(t) \subseteq \mathcal{N}_j(t)$ and also $\mathcal{N}_j(t) \subseteq \mathcal{N}_i(t)$ and hence, $\mathcal{N}_i(t) = \mathcal{N}_j(t)$.

To show part (c), we distinguish two cases: the case when ℓ and i are neighbors, and the case when ℓ and i are not neighbors. If ℓ and i are neighbors, then we have $\mathcal{N}_i(t) = \mathcal{N}_\ell(t)$ by part (b) since $\ell, i \in G(t)$.

Suppose now that ℓ and i are not neighbors. To arrive at a contradiction assume that $\mathcal{N}_i(t) \cap \mathcal{N}_\ell(t) \neq \emptyset$ and let $s \in \mathcal{N}_i(t) \cap \mathcal{N}_\ell(t)$. Then, since $\ell, i \in G(t)$ it follows

$$\|x_\ell(t) - x_s(t)\| \leq \frac{\epsilon}{3}, \qquad \|x_i(t) - x_s(t)\| \leq \frac{\epsilon}{3}.$$

By the triangle inequality for the norm, we have $\|x_\ell(t) - x_i(t)\| \leq \epsilon$, implying that ℓ and i are neighbors, which is a contradiction. Hence, $\mathcal{N}_i(t) \cap \mathcal{N}_\ell(t) \neq \emptyset$. ■

In the following lemma, we further investigate the properties of the agents with a small local-opinion spread.

Lemma 2: Let $i \in G(t)$ be such that $\mathcal{N}_i(t) \subseteq G(t)$. Then, at time $t+1$, one of the following two cases occurs
(i) $\mathcal{N}_i(t+1) = \mathcal{N}_i(t)$;
(ii) $\mathcal{N}_i(t+1) \supset \mathcal{N}_i(t)$.

Furthermore, in case (ii), if there is an agent $\ell \in G(t)$ such that $\ell \in \mathcal{N}_i(t+1) \setminus \mathcal{N}_i(t)$, then $\ell \in \mathcal{N}_j(t+1)$ for all $j \in \mathcal{N}_i(t)$ and

$$\|x_j(t+1) - x_\ell(t+1)\| > \frac{\epsilon}{3} \qquad \text{for all } j \in \mathcal{N}_i(t).$$

Proof: By Lemma 1(b), it follows that

$$\mathcal{N}_i(t) = \mathcal{N}_j(t) \text{ for every } j \in \mathcal{N}_i(t).$$

Let us use $S = \mathcal{N}_i(t)$ for ease of notation. Thus, at time $t+1$, the agents in S are all neighbors of each other, and they will reach a local-group agreement, denoted by $x_S(t+1)$ i.e.,

$$x_j(t+1) = x_S(t+1) \qquad \text{for all } j \in S. \tag{4}$$

Therefore, $S \subseteq N_i(t+1)$. At the time $t+1$, there are two possibilities: (i) the agent group may remain the same, i.e., $\mathcal{N}_i(t+1) = S$, which means that their group-agreement $x_S(t+1)$ remains isolated from the other agents' opinions; or (ii) the group acquires a new neighbor. In the latter case, there must be an agent ℓ who was outside of the group S at time t, but becomes a member of the group at time $t+1$, i.e., $\ell \in \mathcal{N}_i(t+1) \setminus S$.

Let us now consider case (ii) in detail. For this case to happen, the group-agreement $x_S(t+1)$ must fall within ϵ-distance from the opinion $x_\ell(t+1)$ of an agent ℓ with $\ell \notin S$. Thus, at time $t+1$, there must be an agent ℓ such that $\ell \notin S$ and

$$\|x_\ell(t+1) - x_S(t+1)\| \leq \epsilon,$$

which in view of (4) implies

$$\|x_j(t+1) - x_\ell(t+1)\| \leq \epsilon \qquad \text{for all } j \in S.$$

In other words $\ell \in \mathcal{N}_j(t+1)$ for all $j \in S$. Since ℓ was not a neighbor of any $j \in S$ at time t, we must have

$$\|x_j(t) - x_\ell(t)\| > \epsilon \qquad \text{for all } j \in S.$$

Assume now agent ℓ is such that $\ell \in G(t)$. Then, by the triangle inequality for the norm, we have for all $j \in S$,

$$\begin{aligned}\|x_j(t) - x_\ell(t)\| \leq\ & \|x_j(t) - x_S(t+1)\| \\ &+ \|x_S(t+1) - x_\ell(t+1)\| \\ &+ \|x_\ell(t+1) - x_\ell(t)\|.\end{aligned} \tag{5}$$

Using the convexity of the norm, we further have for all $j \in S$,

$$\begin{aligned}\|x_j(t) - x_S(t+1)\| &\leq \frac{1}{S}\sum_{s \in S}\|x_j(t) - x_s(t)\| \\ &\leq \max_{s \in S}\|x_j(t) - x_s(t)\| \\ &\leq \frac{\epsilon}{3},\end{aligned} \tag{6}$$

where the last inequality follows from $j \in G(t)$ for all $j \in S$. Similarly, since $\ell \in G(t)$, we have

$$\|x_\ell(t) - x_\ell(t+1)\| \leq \frac{\epsilon}{3}. \tag{7}$$

By substituting estimates (6) and (7) in relation (5), we conclude that for all $j \in S$,

$$\|x_j(t) - x_\ell(t)\| \leq \frac{2\epsilon}{3} + \|x_S(t+1) - x_\ell(t+1)\|,$$

which by $\|x_j(t) - x_\ell(t)\| > \epsilon$ implies

$$\epsilon - \frac{2\epsilon}{3} < \|x_S(t+1) - x_\ell(t+1)\|.$$

Thus, since $x_S(t+1)$ represents group S agreement we have

$$\|x_j(t+1) - x_\ell(t+1)\| > \frac{\epsilon}{3} \qquad \text{for all } j \in S. \quad (8)$$

■

B. Termination Criterion

At first, we introduce *the termination time for the dynamics*, which is the instance at which all the agent reach their corresponding steady states. Formally, the termination time T of the dynamics (2) is the smallest $t \geq 0$ such that $x_i(k+1) = x_i(k)$ for all $k \geq t$ and all $i \in [m]$, i.e.,

$$T = \inf_{t\geq 0}\{t \mid x_i(k+1) = x_i(k) \text{ for all } i \in [m] \text{ and } k \geq t\}.$$

At the moment, we are not eliminating the possibility that the time T is infinite i.e., we may have $T = +\infty$.

Using Lemma 2 we establish a *finite-time termination criterion for the multi-dimensional dynamics*. Specifically, we show that the dynamics reaches the stable state when the agents' neighbor sets $\mathcal{N}_i(t), i \in [m]$ become stable and all agents have small local-opinion spread, as seen in the following proposition.

Proposition 2: Suppose that the time $\hat{t}$ is such that

$$G(\hat{t}) = [m] \qquad \text{and} \qquad G(\hat{t}+1) = [m].$$

Then, there holds

$$\mathcal{N}_i(\hat{t}+1) = \mathcal{N}_i(\hat{t}) \qquad \text{for all } i \in [m],$$

and the termination time T of the Hegselmann-Krause dynamics satisfies $T \leq \hat{t}+1$.

Proof: Since $G(\hat{t}) = [m]$, by Lemma 2, it follows that either $\mathcal{N}_i(\hat{t}+1) = \mathcal{N}_i(\hat{t})$ for all $i \in [m]$, or $\mathcal{N}_i(\hat{t}+1) \supset \mathcal{N}_i(\hat{t})$ for some $i \in [m]$. We show that the latter case cannot occur. Specifically, every $\ell \in \mathcal{N}_i(\hat{t}+1) \setminus \mathcal{N}_i(\hat{t})$ must belong to the set $G(\hat{t})$ since $M(\hat{t}) = \emptyset$. Therefore, by Lemma 2 it follows that

$$\|x_j(\hat{t}+1) - x_\ell(\hat{t}+1)\| > \frac{\epsilon}{3} \qquad \text{for all } j \in \mathcal{N}_i(\hat{t}).$$

In particular, this implies $i, \ell \in M(\hat{t}+1)$ which is a contradiction since $M(\hat{t}+1) = \emptyset$ (due to $G(\hat{t}+1) = [m]$). Therefore, we must have $\mathcal{N}_i(\hat{t}+1) = \mathcal{N}_i(\hat{t})$ for all i, which means that each agent group has reached a local group-agreement and this agreement persists at time $\hat{t}+1$ and onward. ■

Observe that Proposition 2 characterizes a situation where we can conclude the termination time is finite. However, the proposition does not guarantee that such a situation is going to occur. In what follows, we develop additional insights and results that will enable us to ensure that the situation described in Proposition 2 will indeed happen.

C. Adjoint Dynamics

Here, we show that the Hegsellman-Krause dynamics has an adjoint dynamics with some special properties. To do so, we first compactly represent the dynamics by defining the agent-interaction matrix $B(t)$ with the entries given as follows:

$$B_{ij}(t) = \begin{cases} \frac{1}{|\mathcal{N}_i(t)|} & \text{if } j \in \mathcal{N}_i(t), \\ 0 & \text{otherwise.} \end{cases}$$

The dynamics in (2) can now be written as:

$$X(t+1) = B(t)X(t) \qquad \text{for all } t \geq 0, \qquad (9)$$

where $X(t)$ is the $m \times n$ matrix with the rows given by the opinion vectors $x_1'(t), \ldots, x_m'(t)$. We say that the Hegselmann-Krause dynamics has an *adjoint dynamics* if there exists a sequence of probability vectors $\{\pi(t)\} \subset \mathbb{R}^m$ such that

$$\pi'(t) = \pi'(t+1)B(t) \qquad \text{for all } t \geq 0.$$

We say that *an adjoint dynamics is uniformly bounded away from zero* if there exists a scalar $p^* \in (0,1)$ such that $\pi_i(t) \geq p^*$ for all $i \in [m]$ and $t \geq 0$.

To show the existence of the adjoint dynamics, we use a variant of Theorem 4.8 in [9], as stated below.

Theorem 1: Let $\{A(t)\}$ be a sequence of stochastic matrices such that the following two properties hold:

(i) There exists a scalar $\gamma \in (0,1]$ such that $A_{ii}(t) \geq \gamma$ for all $i \in [m]$ and $t \geq 0$;

(ii) There is a scalar $\alpha \in (0,1]$ such that for every non-empty $S \subset [m]$ and its complement $\bar{S} = [m] \setminus S$, there holds

$$\sum_{i\in S, j\in \bar{S}} A_{ij}(t) \geq \alpha \sum_{j\in \bar{S}, i\in S} A_{ji}(t) \qquad \text{for all } t \geq 0.$$

Then, the dynamics $z(t+1) = A(t)z(t)$, $t \geq 0$, has an adjoint dynamics $\{\pi(t)\}$ which consists of probability vectors that are uniformly bounded away from zero.

We have the following result as a consequence of Theorem 1.

Proposition 3: The Hegselmann-Krause opinion dynamics has an adjoint dynamics which is uniformly bounded away from zero.

Proof: We show that the conditions of Theorem 1 are satisfied. Condition (i) is obviously satisfied with

$\gamma = \frac{1}{m}$. For condition (ii), we note that for every nonzero entry of $B(t)$, we have

$$B_{ij}(t) = \frac{1}{|\mathcal{N}_i(t)|} \geq \frac{1}{m} \geq \frac{1}{m} \cdot \frac{1}{|\mathcal{N}_j(t)|} = \frac{1}{m} B_{ji}(t).$$

Noting that the preceding relation also holds when $B_{ij}(t) = 0$, we have for all $i, j \in [m]$ and all $t \geq 0$,

$$B_{ij}(t) \geq \frac{1}{m} B_{ji}(t). \qquad (10)$$

Then, for any non-empty $S \subset [m]$, by summing the relations in (10) over $i \in S$ and $j \in \bar{S}$, we see that

$$\sum_{i \in S, j \in \bar{S}} B_{ij}(t) \geq \frac{1}{m} \sum_{j \in \bar{S}, i \in S} B_{ji}(t) \qquad \text{for all } t \geq 0.$$

Thus, the sequence $\{B(t)\}$ satisfies condition (ii) of Theorem 1, and the result follows. ■

By Proposition 3, there is a probability sequence $\{\pi(t)\}$ for which we have

$$\pi'(t) = \pi'(t+1) B(t) \qquad \text{for all } t \geq 0. \qquad (11)$$

The adjoint dynamics is important in our construction of a Lyapunov comparison function, which we use to prove the finite convergence time of the Hegselmann-Krause dynamics, as seen in the next section.

IV. Termination Time

We are now ready to show that the termination time T of the Hegselmann-Krause dynamics is finite. We establish this by constructing a Lyapunov comparison function and showing that the function is decreasing along the trajectories of the opinion dynamics.

At this point, we consider the dynamics defined by the Euclidean norm $\|\cdot\|_2$ in (1). To construct a Lyapunov comparison function we use the adjoint dynamics in (11). The comparison function for the dynamics is a function $V(t)$, which is defined for all $t \geq 0$, as follows

$$V(t) = \sum_{i=1}^{m} \pi_i(t) \|x_i(t) - \pi'(t) X(t)\|_2^2, \qquad (12)$$

where the row $X_{i,:}(t)$ is given by the opinion vector $x_i'(t)$ of agent i and $\pi(t)$ is the adjoint dynamics of (11). For this function, we have following essential relation, which we extensively use in the sequel.

Lemma 3: For any $t \geq 0$, we have

$$V(t+1) = V(t) - \frac{1}{2} \sum_{\ell=1}^{m} \frac{\pi_\ell(t+1)}{|\mathcal{N}_\ell(t)|^2} \sum_{i \in \mathcal{N}_\ell(t)} \sum_{j \in \mathcal{N}_\ell(t)} \|x_i(t) - x_j(t)\|_2^2.$$

Proof: For $\theta \in [n]$, let us define $V^\theta(t) = \sum_{i=1}^{m} \pi_i(t)(x_i^\theta(t) - \pi'(t) X_{:,\theta}(t))^2$, where $x_i^\theta(t)$ is the θth entry of $x_i(t)$ and $X_{:,\theta}(t)$ is the θth column of $X(t)$. In fact, $V^\theta(t)$ is nothing but the restriction of the Lyapunov function $V(t)$ to the θth component of the dynamics. Note, that $X_{:,\theta}(t+1) = B(t) X_{:,\theta}(t)$ for all $\theta \in [n]$.

Since $B(t)$ is a stochastic matrix, by Theorem 1 in [11], we have

$$\begin{aligned} V^\theta(t+1) &= V^\theta(t) - \sum_{i=1}^{m} \sum_{j>i} H_{ij}(t)(x_i^\theta(t) - x_j^\theta(t))^2 \\ &= V^\theta(t) - \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} H_{ij}(t)(x_i^\theta(t) - x_j^\theta(t))^2, \end{aligned} \qquad (13)$$

where $H(t) = B'(t) \mathrm{diag}(\pi(t+1)) B(t)$. Therefore,

$$\begin{aligned} &\sum_{i=1}^{m} \sum_{j=1}^{m} H_{ij}(t)(x_i^\theta(t) - x_j^\theta(t))^2 \qquad (14) \\ &= \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{\ell=1}^{m} \pi_\ell(t+1) B_{\ell i}(t) B_{\ell j}(t)(x_i^\theta(t) - x_j^\theta(t))^2 \\ &= \sum_{\ell=1}^{m} \pi_\ell(t+1) \sum_{i=1}^{m} \sum_{j=1}^{m} B_{\ell i}(t) B_{\ell j}(t)(x_i^\theta(t) - x_j^\theta(t))^2 \\ &= \sum_{\ell=1}^{m} \frac{\pi_\ell(t+1)}{|\mathcal{N}_\ell(t)|^2} \sum_{i \in \mathcal{N}_\ell(t)} \sum_{j \in \mathcal{N}_\ell(t)} (x_i^\theta(t) - x_j^\theta(t))^2, \end{aligned}$$

where the last equality follows from $B_{\ell i}(t) = \frac{1}{|\mathcal{N}_\ell(t)|}$ if $i \in \mathcal{N}_\ell(t)$ and $B_{\ell i}(t) = 0$ otherwise. Thus, by combining (13) and (14), we obtain

$$V^\theta(t+1) = V^\theta(t) - \frac{1}{2} \sum_{\ell=1}^{m} \frac{\pi_\ell(t+1)}{|\mathcal{N}_\ell(t)|^2} \sum_{i \in \mathcal{N}_\ell(t)} \sum_{j \in \mathcal{N}_\ell(t)} (x_i^\theta(t) - x_j^\theta(t))^2.$$

Finally, summing up the above relation for $\theta \in [n]$ and using the properties of the 2-norm, we have

$$\begin{aligned} V(t+1) &= \sum_{\theta=1}^{n} V^\theta(t+1) = \sum_{\theta=1}^{n} V^\theta(t) \\ &\quad - \sum_{\theta=1}^{n} \frac{1}{2} \sum_{\ell=1}^{m} \frac{\pi_\ell(t+1)}{|\mathcal{N}_\ell(t)|^2} \sum_{i \in \mathcal{N}_\ell(t)} \sum_{j \in \mathcal{N}_\ell(t)} (x_i^\theta(t) - x_j^\theta(t))^2 \\ &= V(t) \\ &\quad - \frac{1}{2} \sum_{\ell=1}^{m} \frac{\pi_\ell(t+1)}{|\mathcal{N}_\ell(t)|^2} \sum_{i \in \mathcal{N}_\ell(t)} \sum_{j \in \mathcal{N}_\ell(t)} \|x_i(t) - x_j(t)\|_2^2. \end{aligned}$$

■

Lemma 3 provides a key relation for our subsequent development. In particular, it provides the basis for our proof of the stability of the Hegselmann-Krause dynamic, as seen in the following proposition.

Proposition 4: The Hegselmann-Krause opinion dynamics (9), defined using any norm in (1), reaches its steady state in a finite time.

Proof: By summing the relation of Lemma 3 for $t = 0, 1, \ldots, \tau - 1$ for some $\tau \geq 1$, and by rearranging the terms, we obtain

$$V(\tau) + \frac{1}{2}D(\tau - 1) = V(0).$$

where for $s \geq 0$,

$$D(s) = \sum_{t=0}^{s}\sum_{\ell=1}^{m}\frac{\pi_\ell(t+1)}{|\mathcal{N}_\ell(t)|^2}\sum_{i\in\mathcal{N}_\ell(t)}\sum_{j\in\mathcal{N}_\ell(t)}\|x_i(t)-x_j(t)\|_2^2.$$

Letting $\tau \to \infty$, since $V(\tau) \geq 0$ for all τ, we have

$$\sum_{t=0}^{\infty}\sum_{\ell=1}^{m}\frac{\pi_\ell(t+1)}{|\mathcal{N}_\ell(t)|^2}\sum_{i\in\mathcal{N}_\ell(t)}\sum_{j\in\mathcal{N}_\ell(t)}\|x_i(t)-x_j(t)\|_2^2 < \infty.$$

Thus, we must have

$$\lim_{t\to\infty}\sum_{\ell=1}^{m}\frac{\pi_\ell(t+1)}{|\mathcal{N}_\ell(t)|^2}\sum_{i\in\mathcal{N}_\ell(t)}\sum_{j\in\mathcal{N}_\ell(t)}\|x_i(t)-x_j(t)\|_2^2 = 0.$$

By Proposition 3 the adjoint dynamics $\{\pi(t)\}$ is uniformly bounded away from zero, i.e., $\pi_\ell(t) \geq p^*$ for some $p^* > 0$, and for all $\ell \in [m]$ and $t \geq 0$. Furthermore, $|\mathcal{N}_\ell(t)| \leq m$ for all ℓ and t. Therefore, for every $\ell \in [m]$,

$$\lim_{t\to\infty}\sum_{i\in\mathcal{N}_\ell(t)}\sum_{j\in\mathcal{N}_\ell(t)}\|x_i(t)-x_j(t)\|_2^2 = 0.$$

We further have

$$\begin{aligned}\sum_{i\in\mathcal{N}_\ell(t)}\sum_{j\in\mathcal{N}_\ell(t)}\|x_i(t)-x_j(t)\|_2^2 &\geq \sum_{i\in\mathcal{N}_\ell(t)}\|x_i(t)-x_\ell(t)\|_2^2 \\ &\geq \max_{i\in N_\ell(t)}\|x_i(t)-x_\ell(t)\|_2^2,\end{aligned}$$

where the first inequality follows from the fact $\ell \in \mathcal{N}_\ell(t)$ for all $\ell \in [m]$ and $t \geq 0$. Consequently, for all $\ell \in [m]$,

$$\lim_{t\to\infty}\max_{i\in\mathcal{N}_\ell(t)}\|x_i(t)-x_\ell(t)\|_2^2 = 0.$$

Hence, for every $\ell \in [m]$, there is a time t_ℓ such that

$$\max_{i\in\mathcal{N}_\ell(t)}\|x_i(t)-x_\ell(t)\|_2^2 \leq \frac{\epsilon}{3} \qquad \text{for all } t \geq t_\ell.$$

In view of the definition of the set $M(t)$ in (3), the preceding relation implies that

$$M(t) = \emptyset \qquad \text{for all } t \geq \max_{\ell\in[m]} t_\ell.$$

By Proposition 2, it follows that the termination time T satisfies $T \leq \max_{\ell\in[m]} t_\ell + 1$. Since all norms in $\mathbb{R}^m$ are equivalent, it follows that the Hegselmann-Krause dynamics terminates in a finite time for any norm. ■

V. Discussion

We provided an alternative establishment of a finite termination time for Hegselmann-Krause dynamics in multi-dimensional case. Our method is based on the analysis of a quadratic Lyapunov comparison function for the dynamics, which to the best of the authors' knowledge is a novel method to analyze this dynamics. A generic upper bound $\frac{1}{m^m}$ for the termination time has been established in [9] (Theorem 4.7), which may be improved using the properties of the Hegselmann-Krause dynamics developed in this paper. Another interesting direction to be explored is the stability of the Hegselmann-Krause dynamics for infinitely many agents, which seems possible using the theory established in [12].

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