Robot Long Jump

March 18, 2023

Question link

1 Variables

X(0): Position at time 0 = 0
X(1): Position at time 1 = X(0) + U(1) = U(1)
...
X(t): Position at time t = X(t-1) + U(t) = X(t-2) + U(t-1) + U(t) = $\sum_{i=1}^{t} U(i)$

Robot has a-strategy which means if $X(t) > a$ and X(t-1) was less than a, then the robot jumps. The sub-goal of the question is to find the ideal value of a ($0 \leq a \leq 1$) that maximizes the expected score of the robot.

Irwin Hall Distribution:

$$Y_n = \sum_{i=1}^{n} U(i) = \frac{1}{(n-1)!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (x-i)_+^{n-1}$$

$$(x-i)_+^{n-1} = \begin{cases} x-i & \text{for } x \geq i \\ 0 & \text{for } x < i \end{cases}$$

2 Probability Calculations

2.1 Probability that Robot with strategy-a scores 0

$$P = P\Big(X(2) \geq 1, X(1) \leq a\Big) + P\Big(X(3) \geq 1, X(2) \leq a\Big) + ...$$

$$= \sum_{i=1}^{\infty} T_i \qquad \text{where} \quad T_i = P\Big(X(i+1) \geq 1, X(i) \leq a\Big)$$

$$T_i = P\Big(X(i+1) \geq 1, X(i) \leq a\Big) = P\Big(X(i) + U_{i+1} \geq 1, X(i) \leq a\Big) = \int_0^a f(x_i) \int_{1-x_i}^{1} du_{i+1}\, dx_i$$

$$= \int_0^a \frac{1}{(i-1)!} \binom{i}{0} x_i^{i-1} [1 - (1 - x_i)]\, dx_i$$

$$= \frac{1}{(i-1)!} \int_0^a x_i^i \, dx_i$$

$$= \frac{i.a^{i+1}}{(i+1)!}$$

$$T_i = \frac{a^{i+1}}{i!}\Big(1 - \frac{1}{i+1}\Big)$$

$$\sum_{i=1}^{\infty} T_i = \frac{a^2}{1!}\Big(1 - \frac{1}{2}\Big) + \frac{a^3}{2!}\Big(1 - \frac{1}{3}\Big) + ...$$

$$\sum_{i=1}^{\infty} T_i = \frac{a^2}{1!} - \frac{a^2}{2!} + \frac{a^3}{2!} - \frac{a^3}{3!} + \frac{a^4}{3!} - \frac{a^4}{4!} + ...$$

$$= \frac{a^2}{1!} + (a-1)\Big(\frac{a^2}{2!} + \frac{a^3}{3!} + ...\Big)$$

$$= a^2 + (a-1)\Big(e^a - a - 1\Big) = (a-1).e^a - (a^2-1) + a^2 = 1 - (1-a).e^a$$

$$P\Big(\text{Robot with a-strategy scoring 0}\Big) = 1 - (1-a).e^a \quad (\text{say S}\infty\ )$$

2.2 Distribution of score with a-strategy

$$Score = \begin{cases} 0 & \text{w.p. S}\infty \\ X(2) \quad | \quad a < X(1) < 1 & \text{w.p. } P(a < X(1) < 1) \\ X(3) \quad | \quad a < X(2) < 1, 0 < X(1) < a & \text{w.p. } P(a < X(2) < 1, 0 < X(1) < a) \\ ... & \\ X(i+1) \quad | \quad a < X(i) < 1, 0 < X(i-1) < a & \text{w.p. } P(a < X(i) < 1, 0 < X(i-1) < a) \end{cases}$$

Lets find general $Z_i = X(i+1) \quad | \quad a < X(i) < 1, 0 < X(i-1) < a$:

$$F(z_i) = \frac{P\Big(X(i-1) + U_i + U_{i+1} \le z \quad , \quad a \le X(i-1) + U_i \le 1 \quad , \quad 0 \le X(i-1) \le a\Big)}{P\Big(a \le X(i-1) + U_i \le 1 \quad , \quad 0 \le X(i-1) \le a\Big)}$$

Let:
$X(i-1) = X$
$U_i \qquad = U_1$
$U_{i+1} \quad = U_2$

$$\text{Denominator } = P\Big(a \le X + U_1 \le 1 \quad , \quad 0 \le X \le a\Big)$$

$$= \int_0^a f(x) \int_{a-x}^{1-x} du_1\, dx$$

$$= \frac{1}{(i-2)!}\int_0^a x^{i-2}[(1-x) - (a-x)]\, dx$$

$$= \frac{1-a}{(i-2)!}.\frac{1}{i-1}[a^{i-1} - 0]$$

$$= \frac{(1-a).a^{i-1}}{(i-1)!}$$

$$\text{Numerator} = P\Big(X + U_1 + U_2 \le z \quad , \quad a \le X + U_1 \le 1 \quad , \quad 0 \le X \le a\Big)$$

To calculate integral for numerator, I will have to break into 3 ranges for z:

1. If $a < z < 1$:

$$\text{Numerator} = P\Big(X + U_1 + U_2 \le z \quad , \quad a \le X + U_1 \le z \quad , \quad 0 \le X \le a\Big)$$

$$= \int_0^a f(x) \int_{a-x}^{z-x} \int_0^{z-x-u_1} du_2 \, du_1 \, dx$$

$$= \int_0^a f(x) \int_{a-x}^{z-x} [z - x - u_1] \, du_1 \, dx$$

$$= \frac{1}{(i-2)!} \int_0^a x^{i-2} \Big((z-x).[z-a] - \frac{1}{2}\Big[(z-a).(z+a-2x)\Big]\Big) \, dx$$

$$= \frac{(z-a)}{(i-2)!} \int_0^a x^{i-2} \Big(\frac{2.(z-x) - (z+a-2x)}{2} \Big]\Big) \, dx$$

$$= \frac{(z-a)}{(i-2)!} \int_0^a x^{i-2} \Big(\frac{z-a}{2} \Big) \, dx$$

$$= \frac{(z-a)^2}{2.(i-2)!} \Big[\frac{a^{i-1}}{(i-1)} \Big] = \frac{(z-a)^2.a^{i-1}}{2.(i-1)!}$$

$$F(z) = \frac{\text{Numerator}}{\text{Denominator}}$$
$$= \frac{(z-a)^2}{2.(1-a)}$$
$$f(z) = \frac{(z-a)}{(1-a)}$$

2. If $1 < z < 1 + a$:

$$\text{Numerator} = P\Big(X + U_1 + U_2 \le z \quad , \quad a \le X + U_1 \le 1 \quad , \quad 0 \le X \le a\Big)$$

$$= \int_0^a f(x) \int_{a-x}^{1-x} \int_0^{z-x-u_1} du_2 \, du_1 \, dx$$

...after integrating...

$$F(z) = \frac{2z - 1 - a}{2}$$
$$f(z) = 1$$

3. If $1 + a < z < 2$:

$$\text{Numerator} = P\Big(X + U_1 + U_2 \leq z \quad , \quad a \leq X + U_1 \leq 1 \quad , \quad 0 \leq X \leq a\Big)$$

$$= \int_0^a f(x) \int_{a-x}^{1-x} \int_0^{min(z-x-u_1,1)} du_2\, du_1\, dx$$

For $z - (x + u_1) \leq 1 \quad => \quad (x + u_1) \geq z - 1:$ upper bound for $u_2 = z - x - u_1$
For $z - (x + u_1) \geq 1 \quad => \quad (x + u_1) \leq z - 1:$ upper bound for $u_2 = 1$

$$= \int_0^a f(x) \int_{z-1-x}^{1-x} \int_0^{z-x-u_1} du_2\, du_1\, dx \quad + \quad \int_0^a f(x) \int_{a-x}^{z-1-x} \int_0^1 du_2\, du_1\, dx$$

...after integrating...

$$F(z) = \frac{4z - z^2}{2(1-a)} \quad - \quad \frac{(1+a)}{(1-a)}$$

$$f(z) = \frac{(2-z)}{(1-a)}$$

Final pdf for Z_i is:

$$f(z) = \begin{cases} \frac{z-a}{1-a} & a \leq z \leq 1 \\ 1 & 1 \leq z \leq 1 + a \\ \frac{2-z}{1-a} & 1 + a \leq z \leq 2 \end{cases}$$

Important thing to note is pdf of Z_i is independent of i!

Hence, final distribution for score can be simply written as:

$$Score = \begin{cases} 0 & \text{w.p. S}\infty \\ Z & \text{w.p. } 1 - S\infty \end{cases}$$

Note:- I also tallied the remaining probabilities $P(a < X(1) < 1) + P(a < X(2) < 1, 0 < X(1) < a) + P(a < X(3) < 1, 0 < X(2) < a) + ...$ As expected, this came out as $1 - S\infty$.

3 Calculating a

$$E[Score] = 0.S_\infty + E[Z].(1 - S_\infty)$$

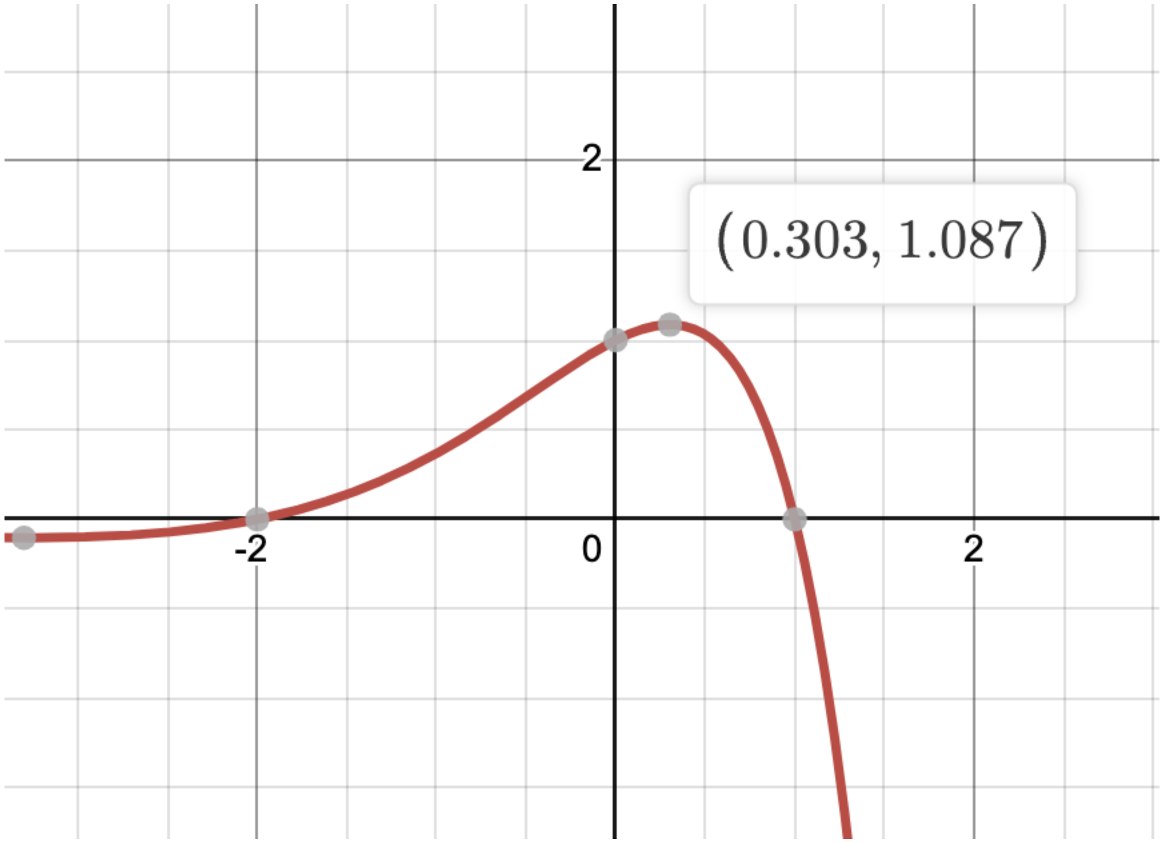
$$E[Z] = \frac{1}{(1-a)} \int_a^1 z(z-a)\, dz \quad + \quad \int_1^{1+a} z\, dz \quad + \quad \frac{1}{(1-a)} \int_{1+a}^2 z(2-z)\, dz$$

...after integrating...

$$E[Z] = \frac{a+2}{2}$$

Now, we can simply maximize $E[Score]$ for value of a. Function to maximize:

$$G = \frac{(a+2).(1-a).e^a}{2}$$



I got the maximum expected score at $a = 0.303$ with maximum expected score $= 1.087$.