Modifying shallow-water equations as a model for wave-vortex turbulence

Ashwin Vishnu Mohanan*, Pierre Augier^, Erik Lindborg*

* Linne Flow Center, Department of Mechanics, KTH, Stockholm, Sweden

^ LEGI / CNRS, Université Grenoble Alpes, Grenoble, France

11 January 2017 (14:55 - 15:15 hrs), FLOW Meeting 2018
Skytteholm, Stockholm, Sweden

Related research

1. **Augier, P., Mohanan A.V. & Lindborg, E.** *Wave energy cascade in forced-dissipative one-layer shallow-water flows.* **J. Fluid Mech. (to be submitted)**.

2. **Lindborg, E. & Mohanan, A. V.** *A two-dimensional toy model for geophysical turbulence.* **Phys. Fluids (2017)**.

Article [2] selected as featured research by AIP (Nov 22, 2017)

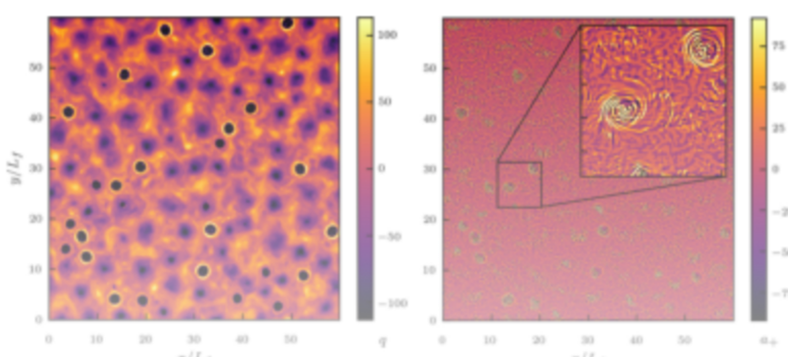
Scilight

22 November 2017

Simple atmospheric model produces hurricanelike vortices

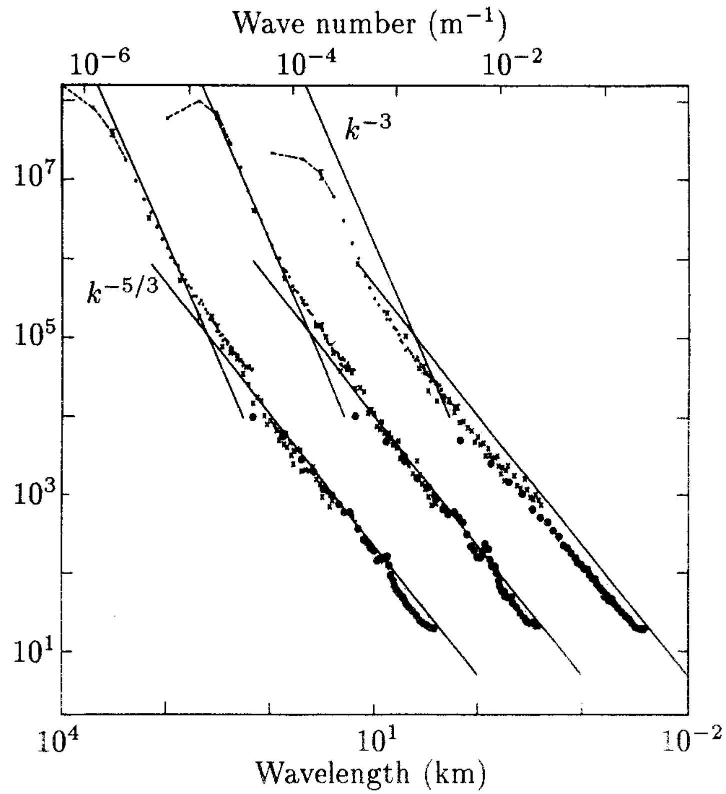
Raima Larter

"Big whirls have little whirls that feed on their velocity; and little whirls have lesser whirls and so on to viscosity."
– Lewis Richardson (1922).



1. Background

1.1. Atmospheric energy spectra from aircraft data: Nastrom and Gage (1985)



- Synoptic scale spectra ($\lambda > 1000$ km) ~ k^{-3}
- Mesoscale spectra ($\lambda = 1$ to 500 km) ~ $k^{-5/3}$

1. Background

1.2. Possible explanations for the mesoscale energy $\boldsymbol{k^{-5/3}}$ spectra

- **Gage (1979) & Lilly (1983)**: *inverse energy cascade* as in **Kraichnan (1967)**
- **Dewan (1979)**:*forward energy cascade* as in **Kolmogorov (1941)**

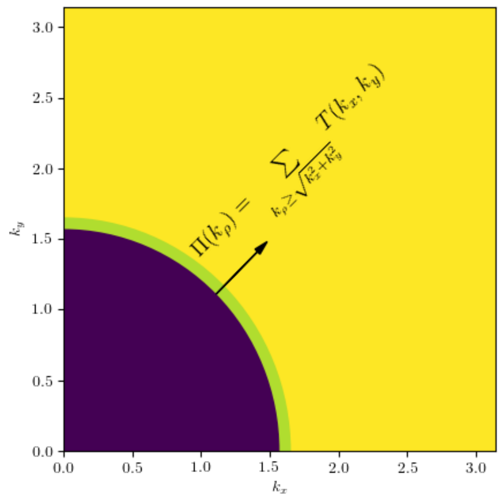
Theoretical predictions for turbulent structures involved

- **Lindborg (2006)** and **Waite & Bartello (2004)**: Stratified turbulence result in thin elongated structures. Vertical length scale $l_v \sim u/N \approx 1km$
- **Callies, Bühler and Ferrari (2016)**: Inertia gravity waves, with frequency $\omega \approx f$. i.e. $l_v \approx$ 100 metres.

1. Background

1.3. Quick recap of turbulence fundamentals

- Kinetic (E_K) and Available Potential Energy (E_A)
- Transfer terms and Spectral energy fluxes



$$T = T_K + T_A$$
$$T_K = -\frac{1}{2} i k_i (\hat{u}_j \widehat{u_i u_j}^* - \hat{u}_j^* \widehat{u_i u_j})$$
$$T_A = -\frac{1}{2} i k_i (\hat{\phi} \widehat{u_i \phi}^* - \hat{\phi}^* \widehat{u_i \phi})$$

$$\Pi = \Pi_K + \Pi_A$$
$$\Pi = \Pi_{VVV} + \Pi_{VVW} + \Pi_{VWW} + \Pi_{WWW}$$

- Normal mode decomposition (for Bousinessq and shallow-water equations)
  - Velocities and the scalar can be decomposed as a sum of **one vortex** mode and **two wave** modes

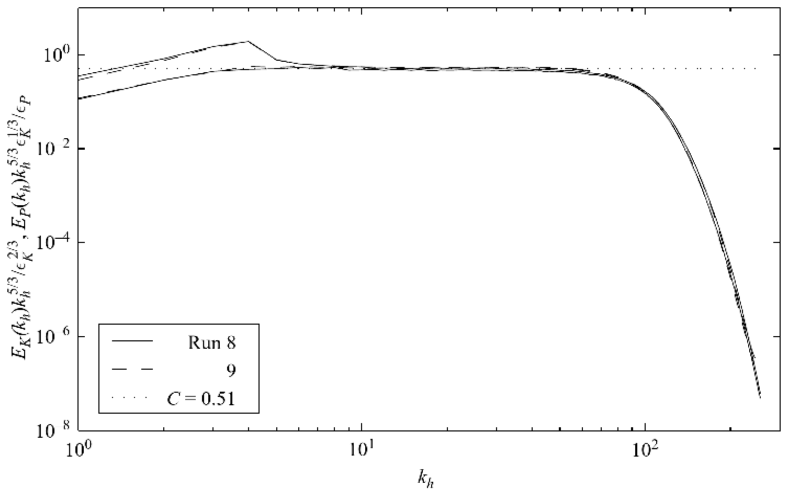
1. Background

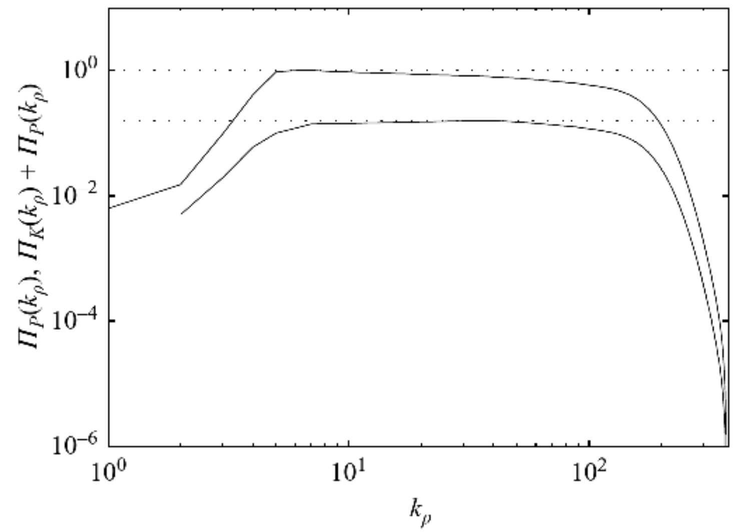
1.3. Stratified Turbulence

- **Lindborg (2006)**: Postulated scaling laws for non-rotating stratified turbulence
  - Length scale:
    - $l_v \sim u/N$
    - $\frac{l_v}{l_h} \sim F_h$, the horizontal Froude number. Typically $F_h << 1$ for strong stratification.
  - Energy spectrum:
    - $E_K(k_h) = C_1 \epsilon_K^{2/3} k_h^{-5/3}$;
    - $E_A(k_h) = C_2 \epsilon_P \epsilon_K^{-1/3} k_h^{-5/3}$
  - Forward energy cascade: $\Pi > 0$

1. Background

1.3. Stratified Turbulence





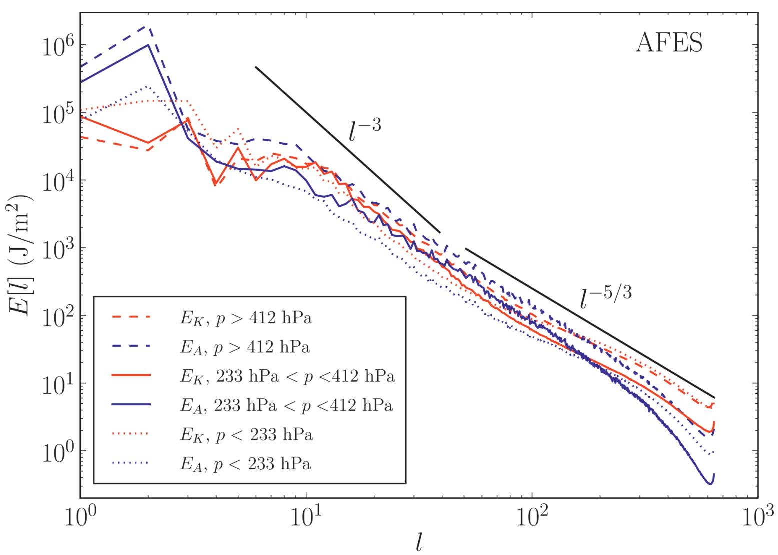
3D Boussinesq equation simulations in **Lindborg (2006)** demonstrated that

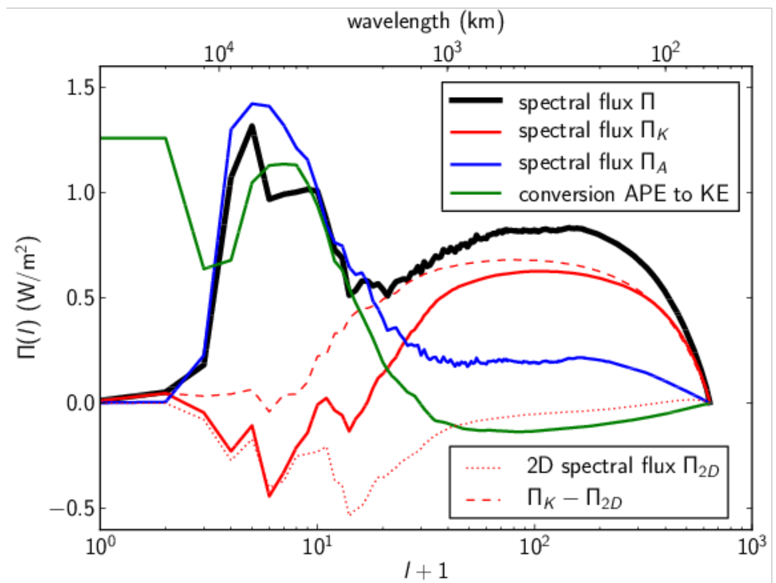
- energy spectra scales as $k^{-5/3}$
- energy flux indicates a **forward energy cascade**

1. Background

1.4. Results from General Circulation Models

- **Augier & Lindborg (2013)**: A GCM called **AFES** can simulate mesoscale energy cascade with coarse vertical resolution: 24 levels!
- Other GCMs (ECMWF) cannot!
- Energy spectra and fluxes computed from spherical harmonics . Spherical harmonic indices l & m correspond to latitude and longitude angles.





1. Background

1.4. Motivation for the present study

Questions and contradictions

- Stratified turbulence predicts a fine vertical resolution to be needed for obtaining $k^{-5/3}$ spectra
- GCM simulations required only 24 pressure levels

- Minimum number of levels required to reproduce $k^{-5/3}$ spectra?
- Is it possible with a single level model? **1-layer Shallow-water equation?**

2. Quest for the simplest model

Why shallow-water equations?

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + f\mathbf{e}_z \times \mathbf{u} = -c^2 \nabla \eta$$

$$\frac{\partial \eta}{\partial t} + \mathbf{u} \cdot \nabla \eta = -(1+\eta)\nabla \cdot \mathbf{u}$$

- Explain many geophysical phenomena, including waves
- Conserves potential vorticity and enstrophy.

Why not shallow-water equations?

- Kinetic energy is not quadratic, but cubic: $E_K = (H+\eta)\frac{\mathbf{u}.\mathbf{u}}{2}$
- Potential enstrophy is not quadratic in general.
- Tendency for waves to develop into shocks giving rise to k^{-2} energy spectra

2. Quest for the simplest model

2.1 Quasi-Geostrophy (QG): Charney (1971)

- In QG limit, potential vorticity can be approximated as $Q = \frac{f+\zeta}{1+\eta} \to q = \zeta + \beta y - f_0 \eta$. Thus QG potential enstrophy, $\Omega = \frac{1}{2} q^2$ is quadratic.
- Inverse energy cascade and forward enstrophy cascade: just like 2D turbulence in **Kraichnan (1971)**.

Shallow water equation is often studied as QG equations:

$$\frac{D}{Dt}\left(\nabla^2 \psi + \beta y - \frac{1}{L_d^2}\psi\right) = \frac{D}{Dt}(\zeta + \beta y - f_0 \eta) = 0$$

Important assumptions:

1. Rossby number, $Ro < 1 \implies$ strong rotation
2. Burger number, $1/Bu = L_d/L < 1 \implies$ planetary scales
3. Variations in coriolis term (β) is small $\implies$ mid-latitudes and above

2. Quest for the simplest model

2.3 Desirable properties for turbulence studies

- Kinetic energy (KE) and Available potential energy (APE) should be **quadratic** and **conserved**
- Potential enstrophy conservation in the QG limit
- No shock formation

2. Quest for the simplest model

2.4. A toy model

(Lindborg and Mohanan 2017): Two simple modifications

1. Replace RHS of the scalar equation:
   $-(1+\eta)\nabla \cdot \mathbf{u}$ with $-\nabla \cdot \mathbf{u}$

2. Replace advective operator:
   $\mathbf{u} \cdot \nabla$ with $\mathbf{u_r} \cdot \nabla$

Helmholtz decomposition:

$$\mathbf{u} = \mathbf{u_r} + \mathbf{u_d}$$

- $\mathbf{u}_r = -\nabla \times (\mathbf{e_z}\Psi)$ is the rotational component
- $\mathbf{u_d} = \nabla\chi$ is the divergent component

with Ψ and χ being the **stream function** and the **velocity potential** respectively.

2. Quest for the simplest model

2.5. The toy model equations

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}_r \cdot \nabla \mathbf{u} + f\mathbf{e}_z \times \mathbf{u} &= -c\nabla\theta \\
\frac{\partial \theta}{\partial t} + \mathbf{u}_r \cdot \nabla \theta &= -c\nabla \cdot \mathbf{u}
\end{aligned}
$$

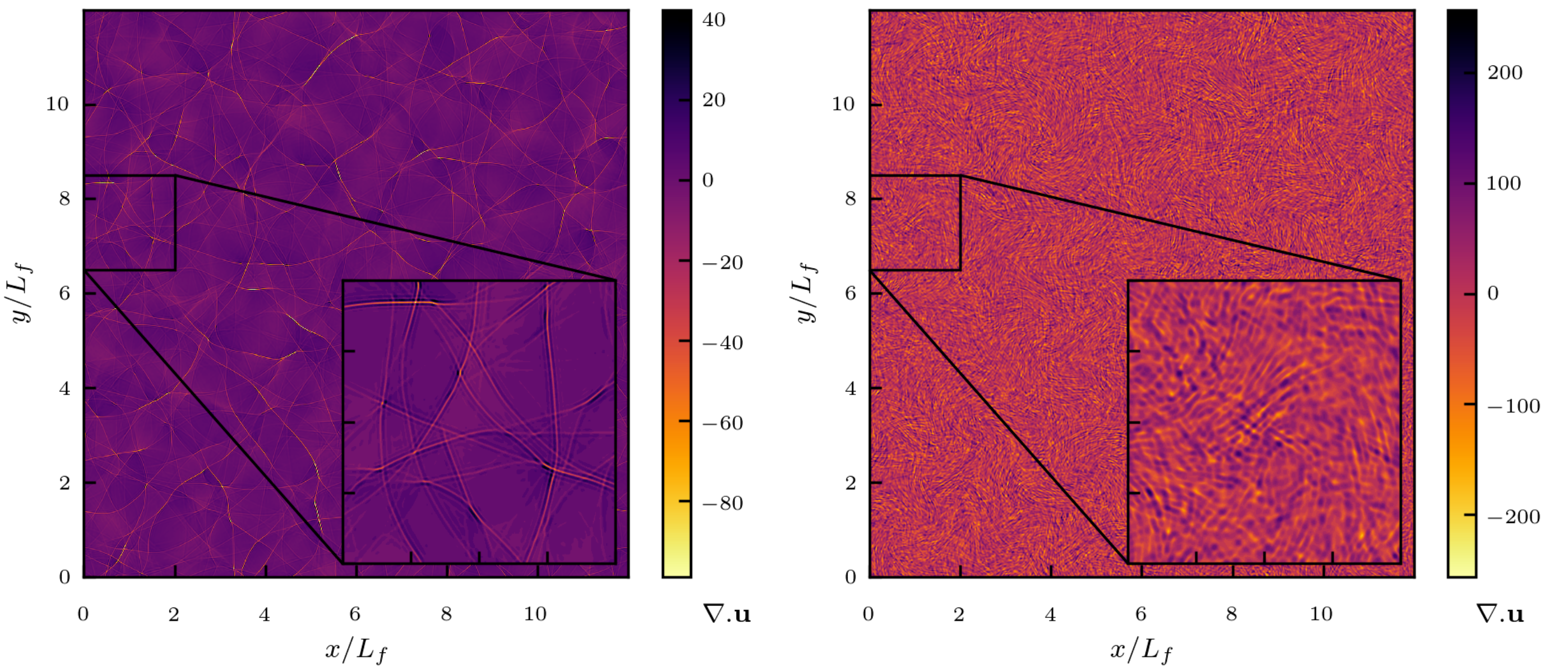
where, $\theta = c\eta$

- **Pros**: No shocks, KE and APE are quadratic and conserved, linearised potential vorticity conserved in the limit $Ro \to 0$: $q = \zeta - f\eta$
- **Cons**: Full potential vorticity Q is not exactly conserved

3. Results

3.1 Divergence fields ($\nabla \cdot \mathbf{u}$)

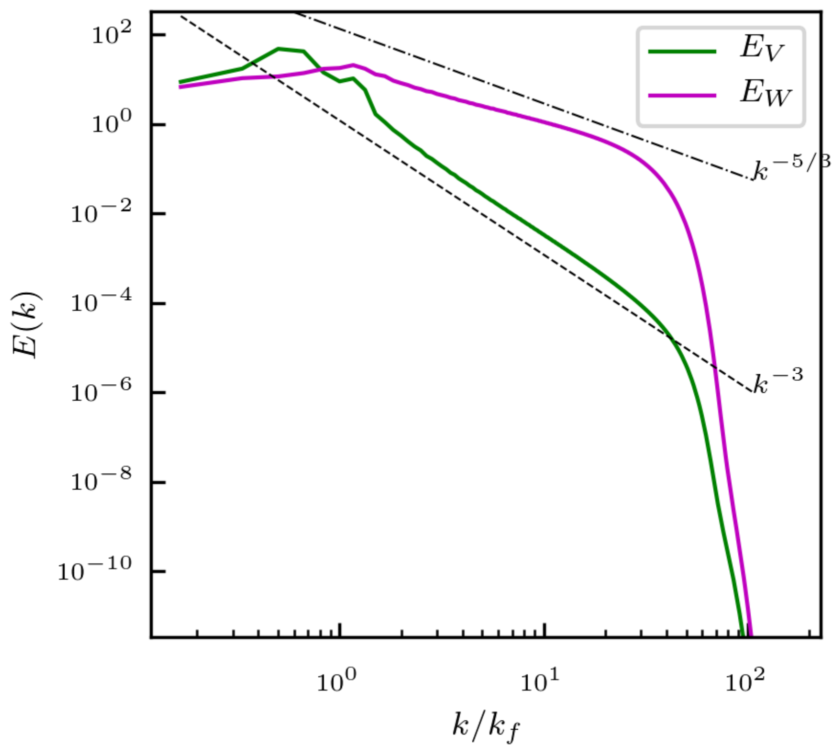
Shallow-water equations and toy model: forcing at $k_f = 6$

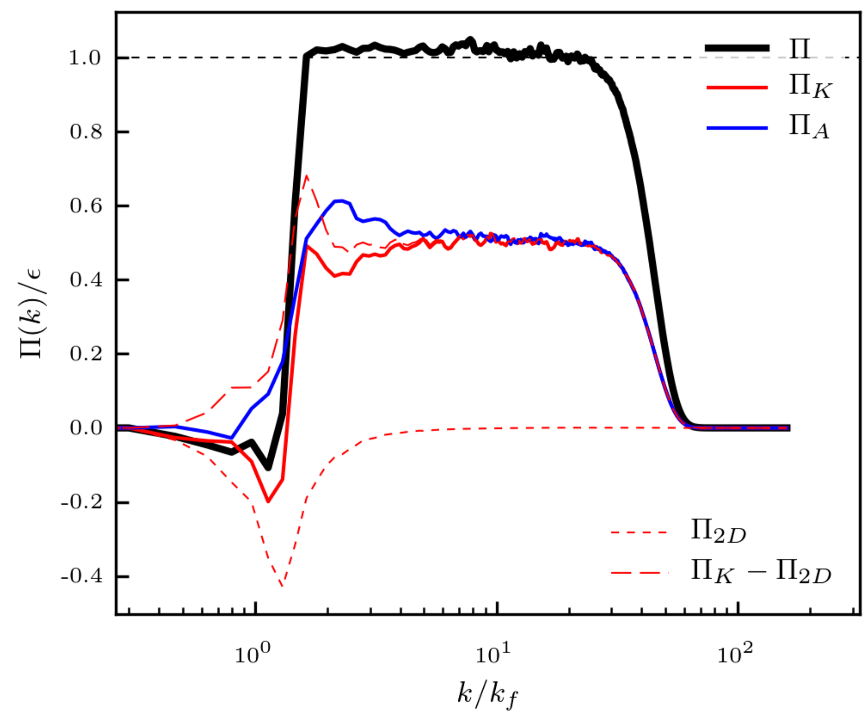


3. Results

3.2 Energy spectra and spectral energy fluxes

Toy model forced at $k_f = 6$

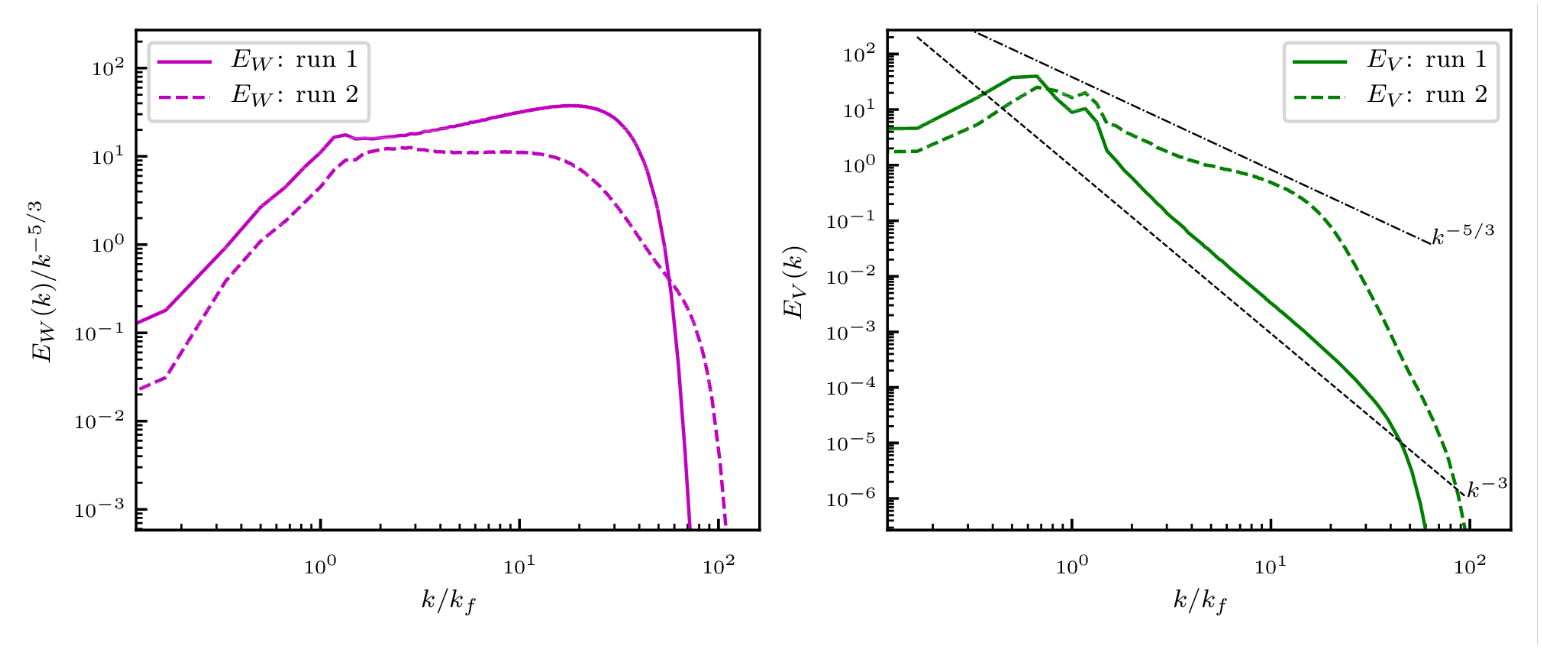




3. Results

3.3 Energy spectra

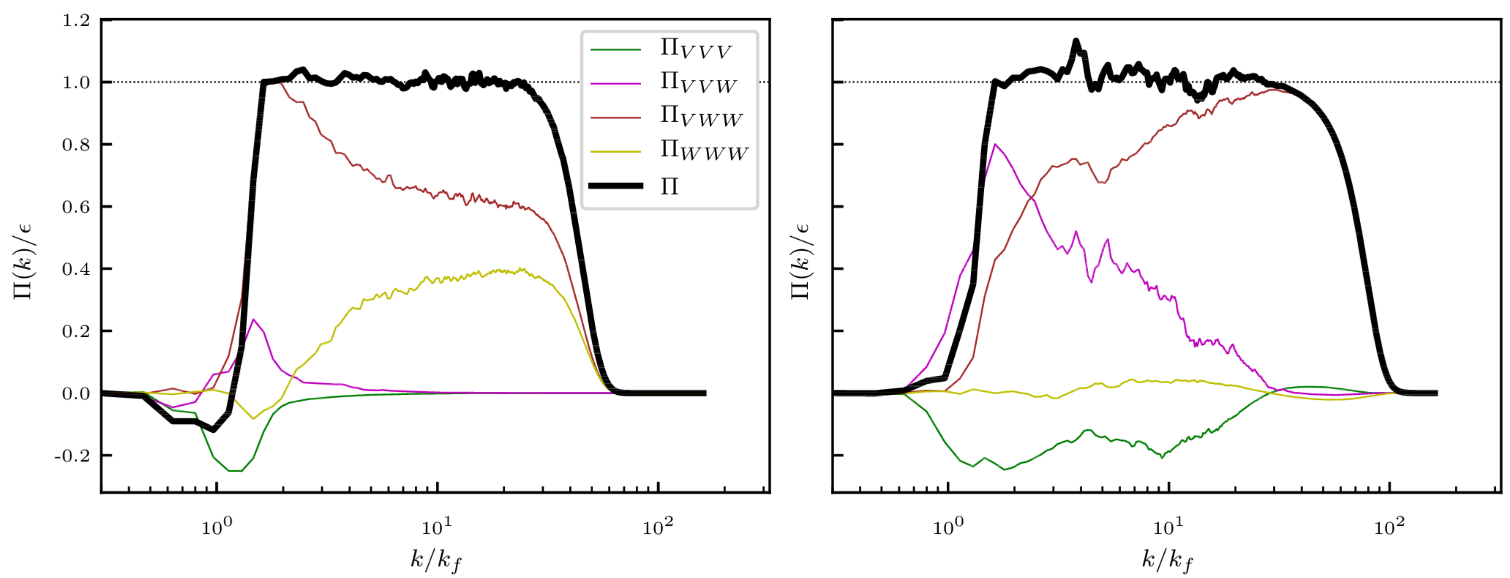
Toy model forced at $k_f = 6$ using different forcing methods



3. Results

3.4 Spectral energy fluxes

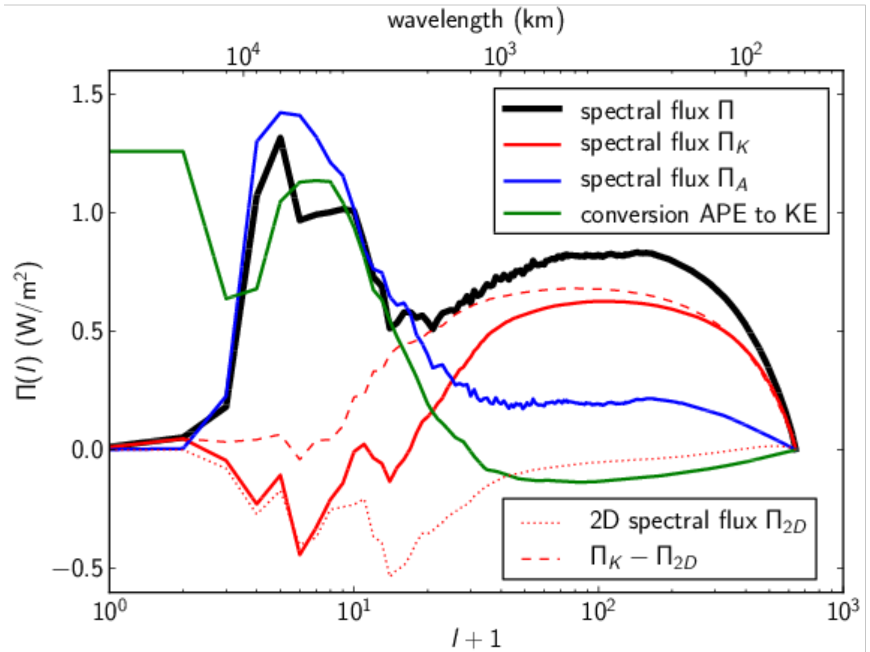
Toy model forced at $k_f = 6$ using different forcing methods

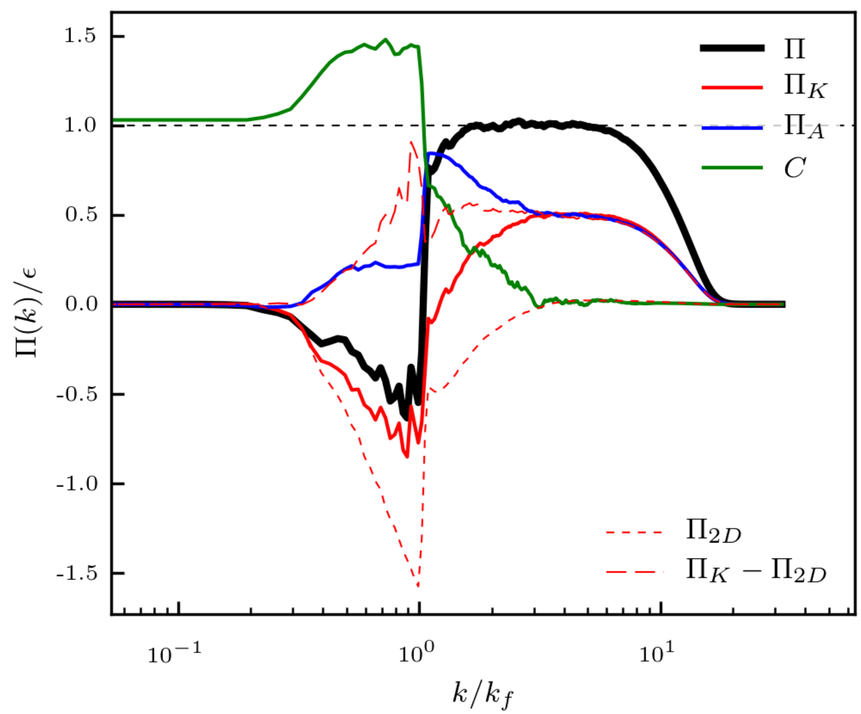


3. Results

3.5 Spectral energy fluxes

Toy-model forced at $k_f = 30$ compared to GCM results from **Augier & Lindborg(2013)**

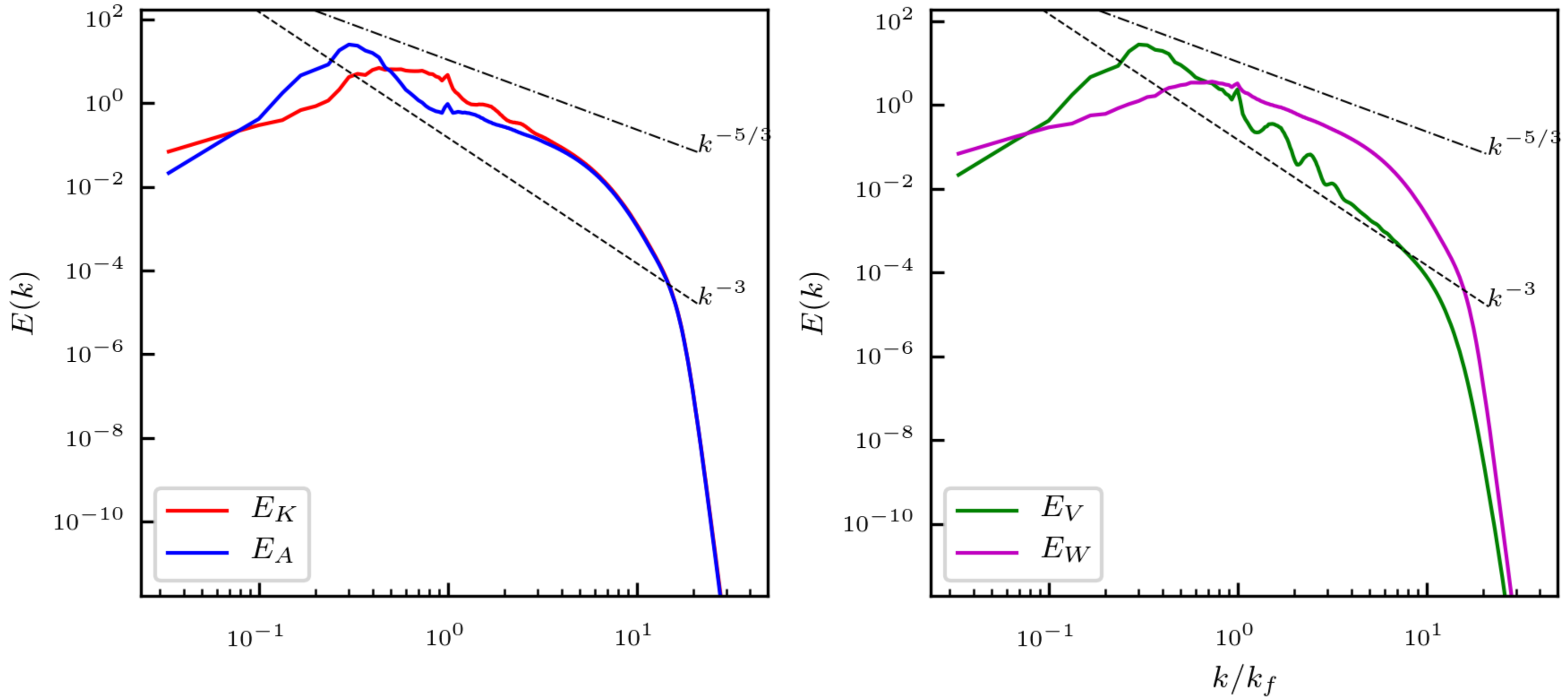




3. Results

3.6 Energy spectra

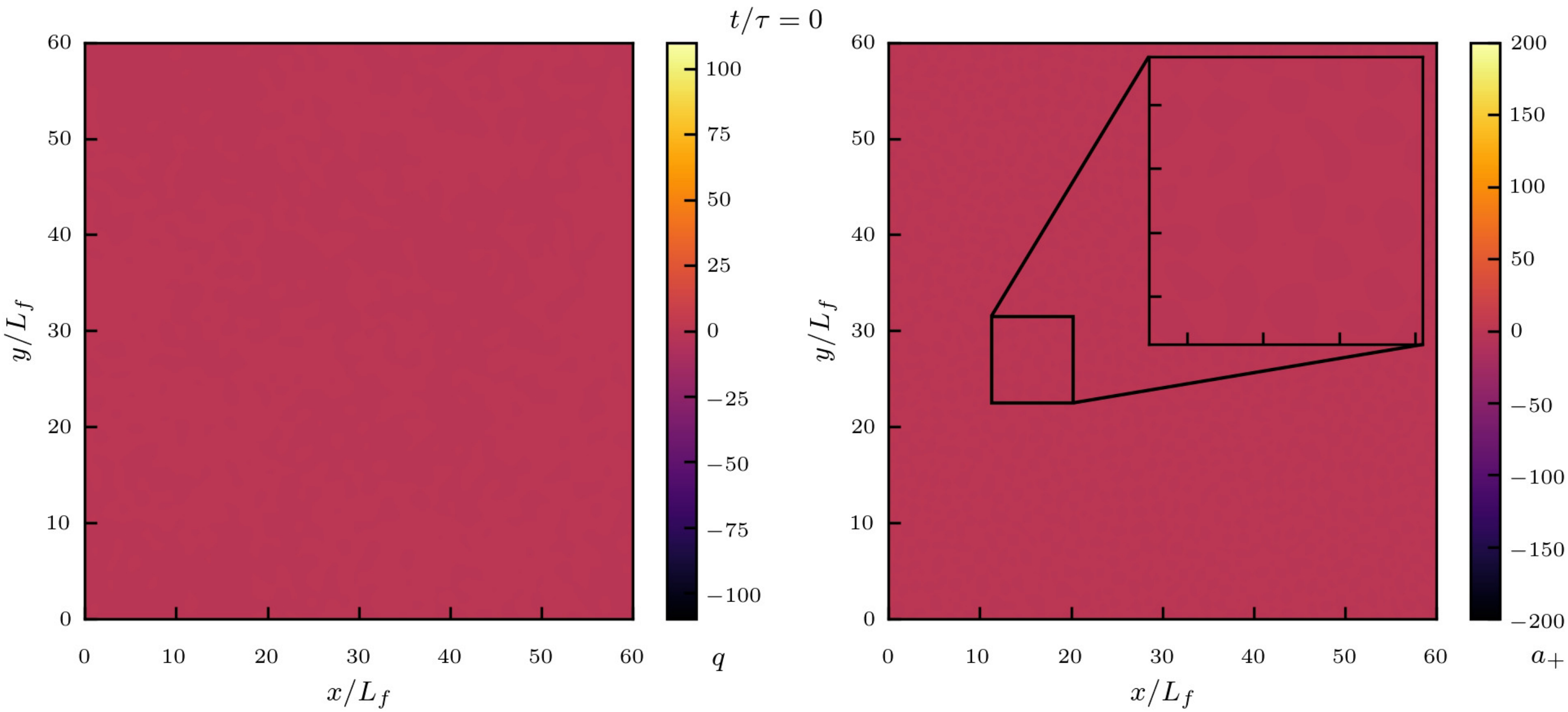
Toy model forced at $k_f = 30$



3. Results

3.7 Potential vorticity (q) and wave (a_+) fields

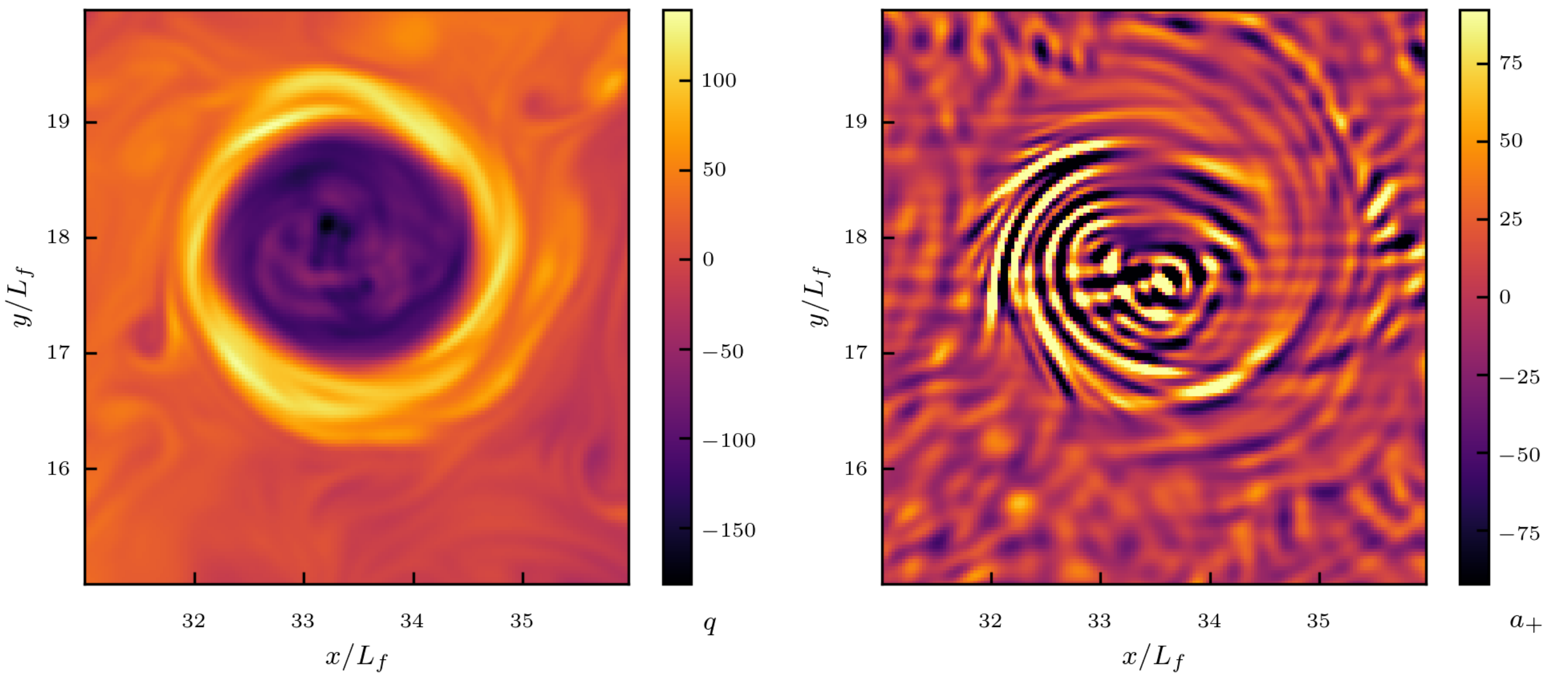
Toy model forced at $k_f = 30$



3. Results

3.8 Potential vorticity (q) and wave (a_+) fields: Anticyclone formation

Toy model forced at $k_f = 30$



4. Outlook

1. Toy model simulations in **beta plane**
2. Large simulation of the toy model over a **sphere**
3. Study of **cyclonic/anticyclonic assymetry** using the toy model

Thank you for your attention!

Open source and reproducible

- **Mohanan, A. V., Bonamy C. & Augier, P.** *FluidSim: modular, object-oriented Python package for CFD simulations* **J. Open Research Software (to be submitted)** | Bitbucket fluiddyn/fluidsim (https://bitbucket.org/fluiddyn/fluidsim) | Github fluiddyn/fluidsim (https://github.com/fluiddyn/fluidsim)

Summary

- Toy model developed by adding two modifications to the 1-layer shallow water equations.
- Able to reproduce $k^{-5/3}$ energy spectra similar to atmospheric mesoscale spectra.
- Conserves K.E., A.P.E. and linear potential enstrophy in the quadratic form: useful in turbulence studies.
- Further reading:
    - **Lindborg, E. & Mohanan, A. V.** *A two-dimensional toy model for geophysical turbulence.* **Phys. Fluids (2017)**

CC-BY-SA: Ashwin Vishnu Mohanan (https://kth.se/profile/avmo)