Statistical Mechanics of Geophysical Flows

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Overview

- Theory
- Results from the literature:
  - 2D Turbulence: Kraichnan (1967), Miller (1990), Robert & Sommeria (1992)
  - Shallow water turbulence: Warn (1986), Renaud et al. (2016)

Theory

2D Euler equation

$$\partial_t \mathbf{u} = -(\mathbf{u}.\nabla)\mathbf{u} - \frac{1}{\rho}\nabla p$$

Useful for studying 2D turbulence.

Quasi geostrophic equation

Quasi-geostrophic equation conserves an approximate *potential vorticity*:

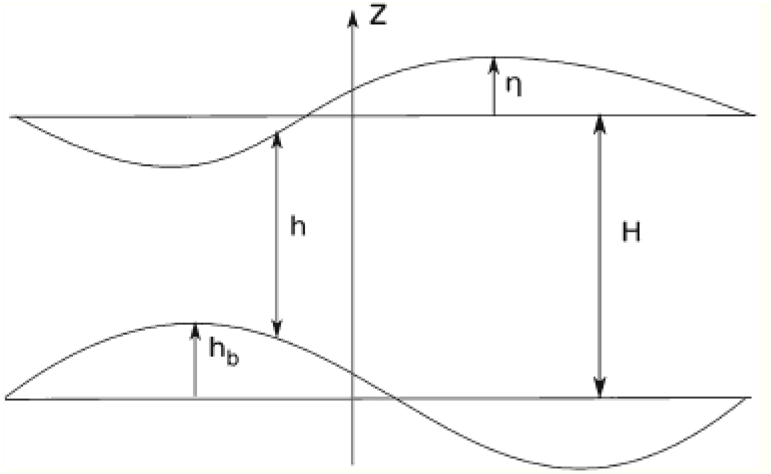
$$\frac{Dq}{Dt} = 0,$$

$$q = \nabla^2\psi + \frac{{f_0}^2}{\tilde{\rho}}\left(\frac{\tilde{\rho}}{N^2}\partial_z\psi\right) + \beta y,$$

Incorporates rotation (Coriolis terms) and stratification (hydrostatic law) in a 2D model. Bridging **ideal 2D turbulence** to **atmospheric turbulence**

1-layer shallow water equations

Fluid surface



Bottom

Simple 2D model, useful for explaining geophysical phenomena.

$$\partial_t \mathbf{u} = -(\mathbf{u}.\nabla)\mathbf{u} - g\nabla(h + h_b) - f\mathbf{e_z} \times \mathbf{u}$$

$$\partial_t h = -\nabla \cdot (h\mathbf{u})$$

- where,
  - $\mathbf{u} =$ horizontal velocity vector,
  - $g =$ acceleration due to gravity (related to wave speed),
  - $f =$ Coriolis parameter (twice the angular velocity of Earth),
  - $h =$ height of fluid

Properties of shallow water equations

- Invariants:
  - mass $h\mathbf{u}$,
  - **energy** $E = E_K + E_P$ (kinetic and potential energy)
  - **potential vorticity**, $q = (\omega + f)/h$ and its higher powers
- Compared to **2D euler equations** or **Quasi-Geostrophic equations** (Euler equations + coriolis force):
  - Admits gravity waves
- Velocity can be split using Helmholtz decomposition:

$$\mathbf{u} = \mathbf{u_r} + \mathbf{u_d}$$

  where,

  - $\mathbf{u}_r = -\nabla \times (\mathbf{e_z}\psi)$ is the rotational component
  - $\mathbf{u_d} = \nabla\phi$ is the divergent component

Results from literature

Common themes and relation between thermodynamic ensembles and geophysical flows

- Similar to:
    - microcanonical ensemble which conserves (N, V, E) and
    - canonical ensemble which conserves (N, V, T),
    - two-dimensional models conserve (mass, energy, vorticity / potential vorticity and its higher powers)
- Statistical mechanics used to theoretically explain:
    - Existence of an equillibrium state in forced-dissipative flows
    - Energy cascade between large and small scales
- Separation of scales: instead of continuum and molecules, between mean flow and turbulent fluctuations

Kraichnan (1967) - Inertial ranges in two-dimensional turbulence

- **2D turbulence** unlike Kolmogorov's **3D turbulence** theory: **vorticity** and **enstrophy** conservation also plays a strong constraint on cascade
- Constant enstrophy flux η would admit an extra inertial range. Dual cascade:

$$E(k) \sim \epsilon^{2/3} k^{-5/3}, \quad E(k) \sim \eta^{2/3} k^{-3}$$

- Cascade Directions
  - Argued by by analysing transfer term which involves triad interactions of wavenumbers (k, p, q)

$$(\partial_t + 2\nu k^2) E(k, t) = \int \int T(k, p, q) dp dq$$

  - $k^{-5/3}$ range: constant energy flux, **inverse** cascade
  - k^{-3} range: constant enstrophy flux, **forward** cascade

Bouchet & Vennaile (2012) - Statistical mechanics of 2D and geophysical flows (review paper)

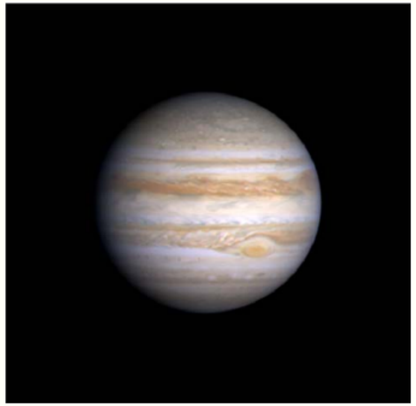


Fig. 10. Observation of the Jovian atmosphere from Cassini (Courtesy of NASA/JPL-Caltech). One of the most striking feature of the Jovian atmosphere is the self-organization of the flow into alternating eastward and westward jets, producing the visible banded structure and the existence of a huge anticyclonic vortex ~20,000 km wide, located around 20 South: the Great Red Spot (GRS). The GRS has a ring structure: it is a hollow vortex surrounded by a jet of typical velocity ~100 m $s^{(-1)}$ and width ~1000 km. Remarkably, the GRS has been observed to be stable and quasi-steady for many centuries despite the surrounding turbulent dynamics. The explanation of the detailed structure of the GRS velocity field and of its stability is one of the main achievement of the equilibrium statistical mechanics of two dimensional and geophysical flows (see Fig. 11 and Section 4).

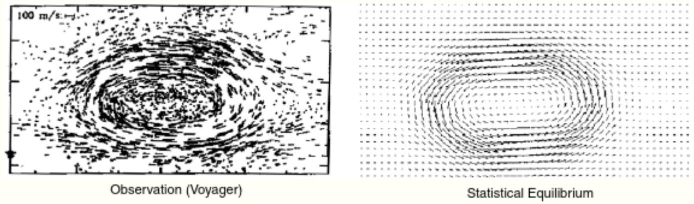


Fig. 11. Left: the observed velocity field is from Voyager spacecraft data, from Dowling and Ingersoll [63]; the length of each line is proportional to the velocity at that point. Note the strong jet structure of width of order *R*, the Rossby deformation radius. Right: the velocity field for the statistical equilibrium model of the Great Red Spot. The actual values of the jet maximum velocity, jet width, vortex width and length fit with the observed ones. The jet is interpreted as the interface between two phases; each of them corresponds to a different mixing level of the potential vorticity. The jet shape obeys a minimal length variational problem (an isoperimetrical problem) balanced by the effect of the deep layer shear.

- **Miller (1990), Robert & Sommeria (1992)**: formulated equillibrium statistical mechanics providing a Liouville theorem and expression for entropy:

  $$S = \int d^2r \int d\sigma \rho \ln \rho$$

  where ρ is the pdf of the enstrophy σ within a distance r .

- **Van der Waals-Cahn-Hilliard model**.
  - Describes phase transitions (equillibria of bubbles, soap films).
  - Explanation for stability of the great red spot of Jupiter: a constant potential vorticity core, surrounded by shear.
  - A variational problem which minimizes free energy

Warn (1986) - Statistical mechanical equilibria of the shallow water equations

- Built upon ideas of Kraichnan (1967) etc. to include effects of small scale inertio-gravity waves or *ageostrophic modes*.

- Considered quadratic constraints constructed from eigenmodes of the linearised shallow-water system.

$$W = \{u, v, \eta\};\ W = \sum_{k,\alpha} A_k^\alpha(t) W_k^\alpha \exp(i\mathbf{k}\cdot\mathbf{r})$$

- Proof for Liouville theorem by expanding the triad interaction coefficients in terms of eigenmodes, which in turn determines rate of change of the coefficients A_k^α.

$$A_k^\alpha = a_k^\alpha + i b_k^\alpha$$

$$\frac{\partial \dot{a}_k^\alpha}{\partial a_k^\alpha} + \frac{\partial \dot{b}_k^\alpha}{\partial b_k^\alpha} = 2\Re \frac{\partial \dot{A}_k^\alpha}{\partial A_k^\alpha} = 0$$

- Studying the evolution of rotational R and inertio-gravity modes G showed using multiple times scales:
  - **Slow manifold**: For short time scales $\tau = \epsilon t$, only surviving resonant interactions are rotational and the motion is quasi-geostrophic
  - **Forward energy cascade**: Evolution into a wave energy cascade and equipartition spectrum requires extraction of energy from the rotational modes. Studied using a Langevin equation of G.

Renaud et al. (2016) - Equilibrium Statistical Mechanics and Energy Partition for the Shallow Water Model

- Generalized and built upon previous studies in the context of shallow water equations.
- Proved Liouville theorem for a triplet h, hu, hv and transformed the result into h, q, μ
- Using a discrete grid and a microcanonical ensemble of microstates in $\{I_n, J_n, h_n, q_n, \mu_n\}$ a macrostate entropy was derived. In the continuous limit:

$$\mathscr{S}[\rho] = -\int \mathrm{d}\mathbf{x}\mathrm{d}\sigma_h\mathrm{d}\sigma_q\mathrm{d}\sigma_\mu \; \sigma_h \rho\left(\mathbf{x}, \sigma_h, \sigma_q, \sigma_\mu\right) \log\left(\frac{\rho\left(\mathbf{x}, \sigma_h, \sigma_q, \sigma_\mu\right)}{\sigma_h^2}\right).$$

  in the absence of height variations / divergence it is equivalent to Miller-Robert-Sommeria's form.

- By constructing the variational problem of entropy maximization under constraints of conserved invariants:

$$\max_{\rho}\left\{\mathscr{S}[\rho] \,\middle|\, \mathscr{E}[\rho] = E, \;\forall k \in \mathbb{N} \quad \mathscr{Z}_k[\rho] = Z_k, \;\forall \mathbf{x} \in \mathscr{D} \quad \bar{\mathrm{I}}(\mathbf{x})[\rho] = 1\right\}.$$

  - Equipartition between kinetic and potential energy
  - Study of two subsystems: mean flow and fluctuations as a coupled variational problem
  - Simplified variational problem on quasi-geostrophic pdf equivalent to Miller-Robert-Sommeria's expression.

Thank you for your attention!

Any questions?

slides will be uploaded: ashwinvis.github.io/talks

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