Simulation and implementation of a Conformal Finite Difference Time Domain method

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English Abstract

Some differential equations in the literature present arduous work to find the associated solution, even in some cases, the solution to said systems turn out to be impossible to find through analytical methods. In this situation, numerical methods plays an important role since the allow us to solve the system of interest through discrete operations with a low numerical error involved.

Among the numerous existing techniques to solve electromagnetism problems, the Finite Difference in Time Domain method (FDTD) stands out, however, when we consider complicated geometries, it is necessary to refine the method in search of better efficiency, that is where we can introduce the Conformal Finite Difference in Time Domain method (CFDTD), which can be studied as the modification of the FDTD by introducing a Perfect Electric Conductor (PEC) volume into the geometry to consider.

In this work, a simulation and implementation of the CFDTD method is made in both one and two dimensions, in the last one, considering a line or an area of PEC that interrupts the spatial mesh worked. The codes worked out were prepared with test-oriented development in the python language, these can be found in the associated GitHub repository presented in annexes.

Resúmen en Español

Algunas ecuaciones diferenciales en la literatura presentan un trabajo arduoso para encontrar la solución asociada, incluso en algunos casos, la solución a dicho sistema resulta ser imposible de encontrar a través de métodos analíticos. Ante esta situación los métodos numéricos juegan un papel importante ya que nos permiten resolver el sistema de interés a través de operaciones discretas con un bajo error numérico de por medio.

Entre las diversas técnicas existentes para poder resolver problemas de electromagnetismo destaca el método de diferencias finitas en el dominio del tiempo (FDTD por sus siglas en inglés), sin embargo, al momento de considerar geometrías complicadas, es necesario refinar el método en búsqueda de una mayor eficiencia, allí es donde se puede introducir la técnica conforme de diferencias finitas (CFDTD), la cual puede ser estudiada como la modificación de FDTD al introducir un volumen de conductor eléctrico perfecto (PEC) en la geometría a considerar.

En el presente trabajo se realiza una simulación e implementación del método CFDTD tanto en una como en dos dimensiones, en este último caso, considerando una línea o un área de PEC que interrumpen en el mallado. Los códigos trabajados fueron realizados con desarrollo orientado por tests en el lenguaje python, estos pueden encontrar en el repositorio de GitHub asociado presentado en anexos.

1 Introduction

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2 Maxwell's equations and the FDTD method

Let's first start by introducing the Maxwell's equation of electromagnetism and the basic notions of the FDTD algorithm in one and two dimensions in the free space case.

2.1 Introduction to the finite differences

We want to find a function that is the solution to a specific differential equation, however, this is a hard problem in general and only rarely can an analytic formula be found for the solution. A finite difference method proceeds by replacing the aderivatives in the differential equation with finite differences aproximations [1, 2]. For example, let's consider the Taylor approximation for $f(x+h)$ and $f(x-h)$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \mathcal{O}(h^3) = f(x) + hf'(x) + \mathcal{O}(h^2), \quad (2.1)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) + \mathcal{O}(h^3) = f(x) - hf'(x) + \mathcal{O}(h^2), \quad (2.2)$$

in both equations it is possible to isolate the derivative, then we obtain:

$$f'(x) = \frac{f(x+h) + f(x)}{h} + \mathcal{O}(h), \quad (2.3)$$

$$f'(x) = \frac{f(x) - f(x-h)}{h} + \mathcal{O}(h). \quad (2.4)$$

If we ignore the order h terms, we obtain the first order approximation for the derivative of the function with an error proportional to h. However, if we want to improve and reduce the error to order h^2, it's necessary to introduce the central finite difference approximation. If we consider $h = \Delta x/2$ and substract the equations 2.1 and 2.2

we can obtain the central finite difference as it follows

$$f'(x) = \frac{f\left(x + \dfrac{\Delta x}{2}\right) - f\left(x - \dfrac{\Delta x}{2}\right)}{\Delta x} + \mathcal{O}(\Delta x^2), \tag{2.5}$$

we obtain then the approximation searched by ignoring the cuadratic order tearms. Since the error decreases faster in this case for smaller Δx, the equation will be more efficient to work with, for this reason, this approximation will be used for the discretization of the Maxwell's equations.

2.2 One dimensional discrete Maxwell equations

First let's remember the time-dependent Maxwell's curl equations for free space [3, 4]

$$\frac{\partial \boldsymbol{E}}{\partial t} = \frac{1}{\varepsilon_0}\nabla \times \boldsymbol{H}, \tag{2.6}$$
$$\frac{\partial \boldsymbol{H}}{\partial t} = -\frac{1}{\mu_0}\nabla \times \boldsymbol{E}, \tag{2.7}$$

here $\boldsymbol{E}$ and $\boldsymbol{H}$ are vectors in three dimensions, with all the components being functions that depend of the spatial coordinates. For the one-dimensional case we can assume that the only non zero components of $\boldsymbol{E}$ and $\boldsymbol{H}$ are E_x and H_y respectively, then, the previous equations become

$$\frac{\partial E_x}{\partial t} = -\frac{1}{\epsilon_0}\frac{\partial H_y}{\partial z}, \tag{2.8}$$
$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0}\frac{\partial E_x}{\partial z}. \tag{2.9}$$

These equations represents a plane wave traveling through the z direction. Taking the central difference approximation discused above for both the temporal and spatial derivatives we obtain [5]

$$\frac{E_x^{n+\frac{1}{2}}(k) - E_x^{n-\frac{1}{2}}(k)}{\Delta t} = -\frac{1}{\epsilon_0}\frac{H_y^n\left(k+\frac{1}{2}\right) - H_y^n\left(k-\frac{1}{2}\right)}{\Delta x}, \tag{2.10}$$
$$\frac{H_y^{n+1}\left(k+\frac{1}{2}\right) - H_y^n\left(k+\frac{1}{2}\right)}{\Delta t} = -\frac{1}{\mu_0}\frac{E_x^{n+\frac{1}{2}}(k+1) - E_x^{n+\frac{1}{2}}(k)}{\Delta x}. \tag{2.11}$$

In these two equations, the time step is represented by the superscripts (n) while the argument inside functions represent the spatial step (k), so the current time and distance are given by $t = \Delta t \cdot n$ and $z = \Delta x \cdot k$. Finally, we can rearrenge the last equations to obtain the next iterative equations

$$E_x^{n+1/2}(k) = E_x^{n-1/2}(k) - \frac{\Delta t}{\varepsilon_0 \cdot \Delta x}\left[H_y^n\left(k+\frac{1}{2}\right) - H_y^n\left(k-\frac{1}{2}\right)\right], \tag{2.12}$$

$$H_y^{n+1}\left(k+\frac{1}{2}\right) = H_y^n\left(k+\frac{1}{2}\right) - \frac{\Delta t}{\mu_0 \cdot \Delta x}\left[E_x^{n+1/2}(k+1) - E_x^{n+1/2}(k)\right]. \tag{2.13}$$

It's important to notice that this formulation assume that the electric and magnetic fields are interleaved in both space and time, this is illustrated in the Figure 2.1.

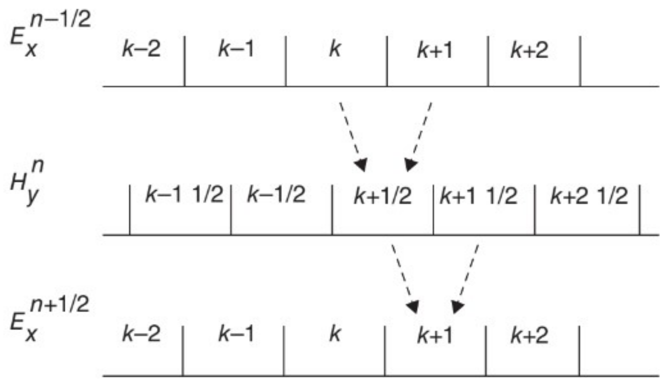

Figure 2.1: Interleaving of the electric and magnetic fields in the FDTD formulation. Image taken from [5].

2.3 Two dimensional discrete Maxwell equations

In two dimensional problems, the third dimension is invariant [6], for this reason, it's convenient to separate the fields in two groups and only work with one of those; the first one is the transversal magnetic mode (TM), which is composed of H_x, H_y and E_z, and the other group is the transversal electric mode (TE), composed of E_x, E_y and H_z. In this work we only consider the TE mode.

Considering again the time-dependent Maxwell's curl equations for the free space and by introducing the TE mode, we can obtain

$$\frac{\partial H_z}{\partial t} = -\frac{1}{\mu_0}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right), \tag{2.14}$$

$$\frac{\partial E_x}{\partial t} = \frac{1}{\epsilon_0}\frac{\partial H_z}{\partial y}, \tag{2.15}$$

$$\frac{\partial E_y}{\partial t} = -\frac{1}{\epsilon_0}\frac{\partial H_z}{\partial x}. \tag{2.16}$$

We can again use the central difference approximation but now with two spatial steps Δx and Δy to obtain the next iterive equation for the magnetic field

$$\begin{aligned} H_z^{n+1}\left(i+\frac{1}{2}, j+\frac{1}{2}\right) = H_z^{n}\left(i+\frac{1}{2}, j+\frac{1}{2}\right) &- \frac{\Delta t}{\mu_0 \Delta y}\left(E_y^{n+1/2}(i+1,j) - E_y^{n+1/2}(i,j)\right) \\ &+ \frac{\Delta t}{\mu_0 \Delta x}\left(E_x^{n+1/2}(i,j+1) - E_x^{n+1/2}(i,j)\right), \end{aligned} \tag{2.17}$$

and the next ones for the electric fields

$$E_x^{n+1}(i,j) = E_x^{n}(i,j) + \frac{\Delta t}{\epsilon_0 \Delta y}\left[H_z^{n+1/2}\left(i, j+\frac{1}{2}\right) - H_z^{n+1/2}\left(i, j-\frac{1}{2}\right)\right], \tag{2.18}$$

$$E_y^{n+1}(i,j) = E_x^{n}(i,j) + \frac{\Delta t}{\epsilon_0 \Delta x}\left[H_z^{n+1/2}\left(i+\frac{1}{2}, j\right) - H_z^{n+1/2}\left(i-\frac{1}{2}, j\right)\right]. \tag{2.19}$$

2.4 Stability in the FDTD method

We have seen that the central difference approximation used converges to the analytic solution with cuadratic order, in this case, we have errors similar to $\mathcal{O}(\Delta t(\Delta t^2 + \Delta x^2))$ in one dimension and $\mathcal{O}(\Delta t(\Delta t^2 + \Delta x^2 + \Delta y^2))$ in the two dimension case, however, there are some restrictions in order to guarantee the convergence.

First let's start with the restriction of the time step Δt. The maximum value this parameter can have is determined by the *CFL* condition, the physical meaning of this condition states that the electromagnetic wave must not pass through more than one cell in just one time step [7], and this have mathematical sense since the central difference approximation only consideres the nearest neighbors to estimate the evolution of the cell. In general, for a 3D rectangular grid, we have

$$c\Delta t \leq \left(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2}\right)^{-1/2} \equiv d^{-1/2}, \tag{2.20}$$

where c represents the speed of the electromagnetic wave (speed of light in the free space case). We can convert the inequality into an equality by multiplying by a constant k less than 1 on the right side, so we finally have

$$\Delta t = \frac{kd^{-1/2}}{c}. \tag{2.21}$$

Finally, the spatial step can not be selected at random either. The foundamental restriction is that the cell size must be smaller than the smallest wavelenght of the electromagnetic wave [7]. A frequently used rule states that we need to have at least 10 cells per wavelenght.

3 Conformal extension of the FDTD method

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