



Simulation and implementation of a Conformal Finite Difference Time Domain method

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English Abstract

Some differential equations in the literature present arduous work to find the associated solution, even in some cases, the solution to said systems turn out to be impossible to find through analytical methods. In this situation, numerical methods plays an important role since they allow us to solve the system of interest through discrete operations with a low numerical error involved.

Among the numerous existing techniques to solve electromagnetism problems, the Finite Difference in Time Domain method (FDTD) stands out, however, when we consider complicated geometries, it is necessary to refine the method in search of better efficiency, that is where we can introduce the Conformal Finite Difference in Time Domain method (CFDTD), which can be studied as the modification of the FDTD by introducing a Perfect Electric Conductor (PEC) volume into the geometry to consider.

In this work, a simulation and implementation of the CFDTD method is made in both one and two dimensions, in the last one, considering a line or an area of PEC that interrupts the spatial mesh worked. The codes worked out were prepared with test-oriented development in the python language, these can be found in the associated GitHub repository presented in annexes.

Resumen en Español

Algunas ecuaciones diferenciales en la literatura presentan un trabajo arduo para encontrar la solución asociada, incluso en algunos casos, la solución a dicho sistema resulta ser imposible de encontrar a través de métodos analíticos. Ante esta situación los métodos numéricos juegan un papel importante ya que nos permiten resolver el sistema de interés a través de operaciones discretas con un bajo error numérico de por medio.

Entre las diversas técnicas existentes para poder resolver problemas de electromagnetismo destaca el método de diferencias finitas en el dominio del tiempo (FDTD por sus siglas en inglés), sin embargo, al momento de considerar geometrías complicadas, es necesario refinar el método en búsqueda de una mayor eficiencia, allí es donde se puede introducir la técnica conforme de diferencias finitas (CFDTD), la cual puede ser estudiada como la modificación de FDTD al introducir un volumen de conductor eléctrico perfecto (PEC) en la geometría a considerar.

En el presente trabajo se realiza una simulación e implementación del método CFDTD tanto en una como en dos dimensiones, en este último caso, considerando una línea o un área de PEC que interrumpen en el mallado. Los códigos trabajados fueron realizados con desarrollo orientado por tests en el lenguaje python, estos pueden encontrar en el repositorio de GitHub asociado presentado en anexos.

1 | Introduction

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2 | Maxwell's equations and the FDTD method

Let's first start by introducing the Maxwell's equation of electromagnetism and the basic notions of the FDTD algorithm in one and two dimensions in the free space case.

2.1 Introduction to the finite differences

We want to find a function that is the solution to a specific differential equation, however, this is a hard problem in general and only rarely can an analytic formula be found for the solution. A finite difference method proceeds by replacing the derivatives in the differential equation with finite differences approximations [1, 2]. For example, let's consider the Taylor approximation for $f(x + h)$ and $f(x - h)$

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \mathcal{O}(h^3) = f(x) + hf'(x) + \mathcal{O}(h^2), \quad (2.1)$$

$$f(x - h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) + \mathcal{O}(h^3) = f(x) - hf'(x) + \mathcal{O}(h^2), \quad (2.2)$$

in both equations it is possible to isolate the derivative, then we obtain:

$$f'(x) = \frac{f(x + h) - f(x)}{h} + \mathcal{O}(h), \quad (2.3)$$

$$f'(x) = \frac{f(x) - f(x - h)}{h} + \mathcal{O}(h). \quad (2.4)$$

If we ignore the order h terms, we obtain the first order approximation for the derivative of the function with an error proportional to h . However, if we want to improve and reduce the error to order h^2 , it's necessary to introduce the central finite difference approximation. If we consider $h = \Delta x/2$ and subtract the equations 2.1 and 2.2

we can obtain the central finite difference as it follows

$$f'(x) = \frac{f\left(x + \frac{\Delta x}{2}\right) - f\left(x - \frac{\Delta x}{2}\right)}{\Delta x} + \mathcal{O}(\Delta x^2), \quad (2.5)$$

we obtain then the approximation searched by ignoring the quadratic order terms. Since the error decreases faster in this case for smaller Δx , the equation will be more efficient to work with, for this reason, this approximation will be used for the discretization of the Maxwell's equations.

2.2 One dimensional discrete Maxwell equations

First let's remember the time-dependent Maxwell's curl equations for free space [3, 4]

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\epsilon_0} \nabla \times \mathbf{H}, \quad (2.6)$$

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu_0} \nabla \times \mathbf{E}, \quad (2.7)$$

here \mathbf{E} and \mathbf{H} are vectors in three dimensions, with all the components being functions that depend of the spatial coordinates. For the one-dimensional case we can assume that the only non zero components of \mathbf{E} and \mathbf{H} are E_x and H_y respectively, then, the previous equations become

$$\frac{\partial E_x}{\partial t} = -\frac{1}{\epsilon_0} \frac{\partial H_y}{\partial z}, \quad (2.8)$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0} \frac{\partial E_x}{\partial z}. \quad (2.9)$$

These equations represents a plane wave traveling through the z direction. Taking the central difference approximation discussed above for both the temporal and spatial derivatives we obtain [5]

$$\frac{E_x^{n+\frac{1}{2}}(k) - E_x^{n-\frac{1}{2}}(k)}{\Delta t} = -\frac{1}{\epsilon_0} \frac{H_y^n(k + \frac{1}{2}) - H_y^n(k - \frac{1}{2})}{\Delta x}, \quad (2.10)$$

$$\frac{H_y^{n+1}(k + \frac{1}{2}) - H_y^n(k + \frac{1}{2})}{\Delta t} = -\frac{1}{\mu_0} \frac{E_x^{n+\frac{1}{2}}(k + 1) - E_x^{n+\frac{1}{2}}(k)}{\Delta x}. \quad (2.11)$$

In these two equations, the time step is represented by the superscripts (n) while the argument inside functions represent the spatial step (k), so the current time and distance are given by $t = \Delta t \cdot n$ and $z = \Delta x \cdot k$. Finally, we can rearrange the last equations to obtain the next iterative equations

$$E_x^{n+1/2}(k) = E_x^{n-1/2}(k) - \frac{\Delta t}{\varepsilon_0 \cdot \Delta x} \left[H_y^n \left(k + \frac{1}{2} \right) - H_y^n \left(k - \frac{1}{2} \right) \right], \quad (2.12)$$

$$H_y^{n+1} \left(k + \frac{1}{2} \right) = H_y^n \left(k + \frac{1}{2} \right) - \frac{\Delta t}{\mu_0 \cdot \Delta x} [E_x^{n+1/2}(k+1) - E_x^{n+1/2}(k)]. \quad (2.13)$$

It's important to notice that this formulation assume that the electric and magnetic fields are interleaved in both space and time, this is illustrated in the Figure 2.1.

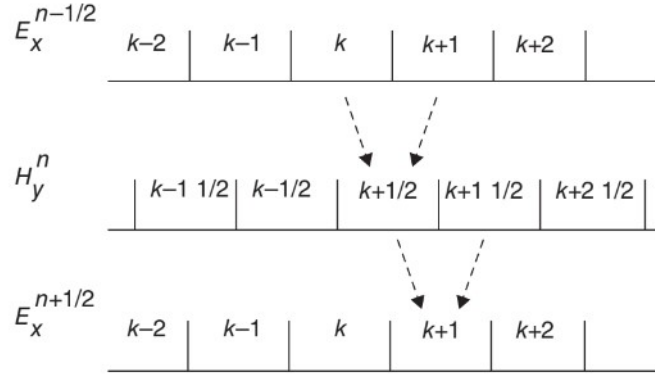


Figure 2.1: Interleaving of the electric and magnetic fields in the FDTD formulation. Image taken from [5].

2.3 Two dimensional discrete Maxwell equations

In two dimensional problems, the third dimension is invariant [6], for this reason, it's convenient to separate the fields in two groups and only work with one of those; the first one is the transversal magnetic mode (TM), which is composed of H_x , H_y and E_z , and the other group is the transversal electric mode (TE), composed of E_x , E_y and H_z . In this work we only consider the TE mode.

Considering again the time-dependent Maxwell's curl equations for the free space

and by introducing the TE mode, we can obtain

$$\frac{\partial H_z}{\partial t} = -\frac{1}{\mu_0} \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right), \quad (2.14)$$

$$\frac{\partial E_x}{\partial t} = \frac{1}{\epsilon_0} \frac{\partial H_z}{\partial y}, \quad (2.15)$$

$$\frac{\partial E_y}{\partial t} = -\frac{1}{\epsilon_0} \frac{\partial H_z}{\partial x}. \quad (2.16)$$

We can again use the central difference approximation but now with two spatial steps Δx and Δy to obtain the next iterative equation for the magnetic field

$$\begin{aligned} H_z^{n+1} \left(i + \frac{1}{2}, j + \frac{1}{2} \right) &= H_z^n \left(i + \frac{1}{2}, j + \frac{1}{2} \right) - \frac{\Delta t}{\mu_0 \Delta x} (E_y^{n+1/2}(i+1, j) - E_y^{n+1/2}(i, j)) \\ &\quad + \frac{\Delta t}{\mu_0 \Delta y} (E_x^{n+1/2}(i, j+1) - E_x^{n+1/2}(i, j)), \end{aligned} \quad (2.17)$$

and the next ones for the electric fields

$$E_x^{n+1}(i, j) = E_x^n(i, j) + \frac{\Delta t}{\epsilon_0 \Delta y} \left[H_z^{n+1/2} \left(i, j + \frac{1}{2} \right) - H_z^{n+1/2} \left(i, j - \frac{1}{2} \right) \right], \quad (2.18)$$

$$E_y^{n+1}(i, j) = E_y^n(i, j) + \frac{\Delta t}{\epsilon_0 \Delta x} \left[H_z^{n+1/2} \left(i + \frac{1}{2}, j \right) - H_z^{n+1/2} \left(i - \frac{1}{2}, j \right) \right]. \quad (2.19)$$

2.4 Stability in the FDTD method

We have seen that the central difference approximation used converges to the analytic solution with quadratic order, in this case, we have errors similar to $\mathcal{O}(\Delta t(\Delta t^2 + \Delta x^2))$ in one dimension and $\mathcal{O}(\Delta t(\Delta t^2 + \Delta x^2 + \Delta y^2))$ in the two dimension case, however, there are some restrictions in order to guarantee the convergence.

First let's start with the restriction of the time step Δt . The maximum value this parameter can have is determined by the *CFL* condition, the physical meaning of this condition states that the electromagnetic wave must not pass through more than one cell in just one time step [7], and this have mathematical sense since the central difference approximation only considers the nearest neighbors to estimate the evolution of the cell. In general, for a 3D rectangular grid, we have

$$c\Delta t \leq \left(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2} \right)^{-1/2} \equiv d, \quad (2.20)$$

where c represents the speed of the electromagnetic wave (speed of light in the free space case). We can convert the inequality into an equality by multiplying by a constant k less than 1 on the right side, so we finally have

$$\Delta t = \frac{kd}{c}. \quad (2.21)$$

Finally, the spatial step can not be selected at random either. The fundamental restriction is that the cell size must be smaller than the smallest wavelength of the electromagnetic wave [7]. A frequently used rule states that we need to have at least 10 cells per wavelength.

3 | Conformal extension of the FDTD method

Using the FDTD method with rectangular grid to analyze objects with curved metallic surfaces not only introduces errors due to inaccurate approximation of the geometry [8]. If we want to extend the FDTD method to solve curved surfaces or irregularities composed of perfect electric conductor, it's necessary to introduce the Conformal Finite Difference in Time Domain method (CFDTD). In this chapter we introduce the CFDTD basics.

3.1 Introduction to CFDTD

The CFDTD algorithm was first introduced to make a FDTD analysis of a curved two dimensional PEC using a locally conformal grid [9]. For two dimensional problems we can consider for example a quadrant of the cross section of a circular resonator shown in Figure 3.1; we can see that the magnetic fields over the cells that don't intersect with the curve can be solved with the regular FDTD method, however, we can't use the same algorithm for the others cells; this is because some neighbour nodes of the electric field discretization are no longer able to interact due to the presence of the PEC surface between them. For this reason, we need to introduce another formulation for the FDTD that can be generalized and used for the CFDTD case.

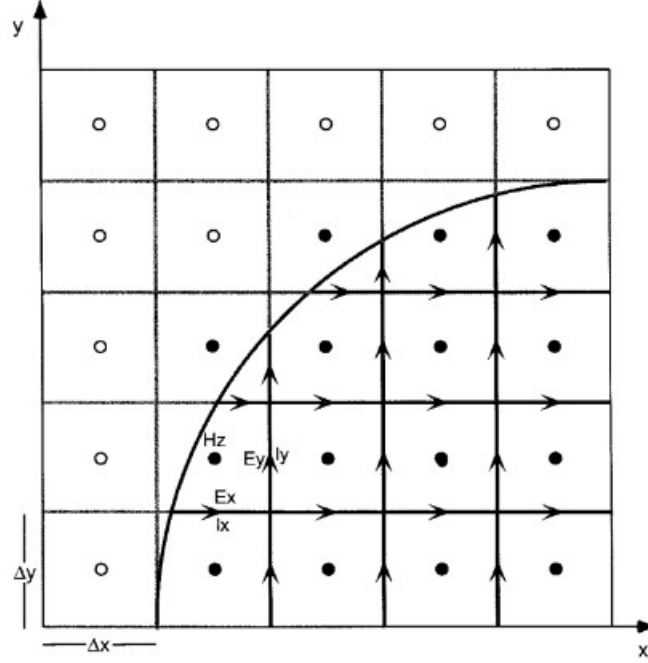


Figure 3.1: Example of a rectangular mesh on a curved PEC surface. Image taken from [10].

3.2 The Dey-Mittra algorithm

Let's consider a two dimensional rectangular grid discretization same as the FDTD case. Now, let's consider the Faraday equation in the integral form for the free space given by

$$\oint_{\partial S} \mathbf{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \mu_0 \mathbf{H} \cdot d\vec{s}. \quad (3.1)$$

If we assume that the electric fields are constant along the edges between the nodes, that the magnetic fields are also constant in the worked cell and if we are working on the TE mode, we can obtain from the equation 3.1 the following relation

$$\begin{aligned} -\mu_0 * A(i, j) \frac{\partial}{\partial t} H_z \left(i + \frac{1}{2}, j + \frac{1}{2} \right) = & E_x(i, j) * l_x(i, j) - E_x(i, j + 1) * l_x(i, j + 1) \\ & + E_y(i + 1, j) * l_y(i + 1, j) - E_y(i, j) * l_y(i, j), \end{aligned} \quad (3.2)$$

where $l_x(i, j)$ and $l_y(i, j)$ represents the length of the edge where is assumed the electric node $E_x(i, j)$ and $E_y(i, j)$ respectively, and $A(i, j)$ the area where the magnetic

field $H_z(i + 1/2, j + 1/2)$ is placed. We also assumed that the area is constant in the time so it can be separated from the time derivative. Finally we take the central difference approximation for the time derivative and rearranging we obtain

$$\begin{aligned}
 H_z^{n+1} \left(i + \frac{1}{2}, j + \frac{1}{2} \right) &= H_z^n \left(i + \frac{1}{2}, j + \frac{1}{2} \right) - \frac{\Delta t}{\mu_0 * A(i, j)} \times \\
 &\times [E_x^{n+1/2}(i, j) * l_x(i, j) - E_x^{n+1/2}(i, j + 1) * l_x(i, j + 1) \\
 &+ E_y^{n+1/2}(i + 1, j) * l_y(i + 1, j) - E_y^{n+1/2}(i, j) * l_y(i, j)].
 \end{aligned} \tag{3.3}$$

In the uniform rectangular grid case, if there is no curved surface in the mesh, we have $l_x = \Delta x$, $l_y = \Delta y$ and $A = \Delta x \Delta y$ for any pair i, j , then, by replacing these values into the equation 3.3, we can obtain again the magnetic field iterative formula presented in the equation 2.17.

The great advantage of this formulation is the fact that we can use it to solve curved PEC surfaces with rectangular grids like the one shown in the Figure 3.1. Indeed, as said before, the cells that don't intersect with the curved surface can be solved as stated, on the other hand, in cells with non-empty intersections, we need to consider in the path integral of the electric field another component corresponding to the value of this field over the part of the curved surface inside the cell, however, the field over this path is always zero, for this reason, the construction of the iterative formula of the magnetic field remains valid, but it must be taken into account that the lengths and areas in these cells don't have the usual values now. This formulation is known as the Dey-Mitra algorithm in two dimensions [9].

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