CSCI 5454: Algorithms: Exam 2 makeup

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May 3, 2023

Problem 1

To get a perfectly balanced tree when inserting the numbers 1 to n, we would first have to insert the median element in the range 1 to n, let that be m. So the root of the tree is m and the left child would be the median element of the range 1 to m and the right child would be the median element of the range m to n. Then recurse in a similar way for the left and right child nodes until we reach a range with only 1 element.

we have $n = 2^k - 1$, and P_k be the probability to get a balanced tree with $2^k - 1$ nodes, then
P_k = P(select median)*P(balanced left sub-tree|median is selected)*P(balanced right sub-tree|median is selected)
both to the left and right of the median there are $2^{k-1} - 1$ elements and the probability of making a balanced tree with them would be P_{k-1}
$P_k = 1/n * P_{k-1} * P_{k-1}$ with $P_1 = 1$ (tree with one node is already perfectly balanced)
$P_k = \frac{1}{2^k - 1} * (P_{k-1}^2)$
Based on the above recurrence calculating P_5

$$
\begin{aligned}
&P_2 = 1/3 * 1^2 = 1/3 \\
&P_3 = 1/7 * (1/3)^2 \\
&P_4 = 1/15 * (1/7)^2 * (1/3)^4 \\
&P_5 = 1/31 * (1/15)^2 * (1/7)^4 * (1/3)^8 = 1/109876902975
\end{aligned}
$$

The probability of getting a perfectly balanced tree when inserting 1...31 nodes is 1/109876902975.

Problem 2

2.1 Part A

P(one sweet not being consume for k days) $= 1/2^k$ (getting tails on each of the k days)
P(consuming the sweet in k days) $= 1 - 1/2^k$
P(consuming all n sweets in k days) $= (1 - 1/2^k)^n$
P(not consuming all n sweets in k days) $= 1 - (1 - 1/2^k)^n$

2.1 Part B

Let X be a random variable indicating the time taken to eat the sweet.
From the 2a we have $P(X \geq k) = 1 - (1 - 1/2^k)^n$
The expected time to eat all the sweets is:

$$E(X) = P(X \geq 1) + P(X \geq 2) + P(X \geq 3) + ...$$
$$E(X) = [1 - (1 - 1/2)^n] + [1 - (1 - 1/2^2)^n] + [1 - (1 - 1/2^3)^n] + ...$$
$$E(X) = \sum_{k=1}^{\infty} 1 - (1 - \frac{1}{2^k})^n$$

2.1 Part C

Applying Bernoulli's inequality to the series in 2.2 we get:

$$(1 - \frac{1}{2^k})^n \geq (1 - \frac{n}{2^k})$$
$$1 - (1 - \frac{1}{2^k})^n \leq 1 - (1 - \frac{n}{2^k}) \quad \text{\# simplifying}$$
$$1 - (1 - \frac{1}{2^k})^n \leq \frac{n}{2^k} \quad \text{\# summation on both sides}$$
$$\sum_{k=1}^{\infty} 1 - (1 - \frac{1}{2^k})^n \leq \sum_{k=1}^{\infty} \frac{n}{2^k}$$
$$\sum_{k=1}^{log_2 n} 1 - (1 - \frac{1}{2^k})^n + \sum_{k=log_2 n+1}^{\infty} 1 - (1 - \frac{1}{2^k})^n \leq \sum_{k=1}^{log_2 n} \frac{n}{2^k} + \sum_{k=log_2 n+!}^{\infty} \frac{n}{2^k}$$

For k in $[1, log_2 n]$, $1 - (1 - \frac{1}{2^k})^n \leq 1$

$$1 - (1 - \frac{1}{2^k})^n \leq 1$$
$$\sum_{k=1}^{log_2 n} 1 - (1 - \frac{1}{2^k})^n \leq \sum_{k=1}^{log_2 n} 1$$
$$\sum_{k=1}^{log_2 n} 1 - (1 - \frac{1}{2^k})^n \leq log_2 n$$

Solving $\sum_{k=log_2 n+1}^{\infty} \frac{n}{2^k}$ using infinite GP sum

$$S = \frac{n}{2n} + \frac{n}{2^2 n} + \frac{n}{2^3 n} + ...$$
$$S = \frac{1/2}{1 - 1/2}$$
$$S = 1$$

Thus, $\sum_{k=1}^{\infty} 1 - (1 - \frac{1}{2^k})^n \leq log_2 n + 1$