Proof of Eq. (B.2), $|\hat{\sigma}\cdot\hat{r};\pm\rangle \equiv |\psi(\theta,\varphi)\rangle = \cos(\theta/2)\,|0\rangle + \sin(\theta/2)e^{i\varphi}\,|1\rangle$.

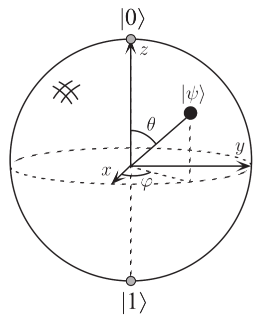

Figure 0.1: Bloch sphere representation of a *qubit*.

In the Bloch sphere representation of a qubit, Fig. 0.1,

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z} = (\sin\theta\cos\varphi\,\hat{x}, \sin\theta\sin\varphi\,\hat{y}, \cos\theta\,\hat{z})\,. \tag{0.1}$$

Hence,[1]

$$\hat{\vec{\sigma}}\cdot\hat{r} = (\hat{\sigma}_1\,\hat{x}, \hat{\sigma}_2\,\hat{y}, \hat{\sigma}_3\,\hat{z})\cdot(\sin\theta\cos\varphi\,\hat{x}, \sin\theta\sin\varphi\,\hat{y}, \cos\theta\,\hat{z}) = \hat{\sigma}_1\sin\theta\cos\varphi + \hat{\sigma}_2\sin\theta\sin\varphi + \hat{\sigma}_3\cos\theta\,. \tag{0.2}$$

From the definition of the Pauli matrices [10, p. 169, Eq. (3.2.32)], in the $\{|\pm\rangle_z\} \equiv \{|0\rangle, |1\rangle\}$ computational basis,

$$\begin{aligned}\hat{\vec{\sigma}}\cdot\hat{r} &= \hat{\sigma}_1\sin\theta\cos\varphi + \hat{\sigma}_2\sin\theta\sin\varphi + \hat{\sigma}_3\cos\theta \\ &\doteq \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\sin\theta\cos\varphi + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\sin\theta\sin\varphi + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cos\theta \\ &= \begin{pmatrix}\cos\theta & \sin\theta\cos\varphi - i\,\sin\theta\sin\varphi\\ \sin\theta\cos\varphi + i\,\sin\theta\sin\varphi & -\cos\theta\end{pmatrix} = \begin{pmatrix}\cos\theta & e^{-i\varphi}\sin\theta\\ e^{i\varphi}\sin\theta & -\cos\theta\end{pmatrix}.\end{aligned} \tag{0.3}$$

In order to find the eigenvalues (λ) and the pertaining eigenvectors corresponding to the operator $\hat{\vec{\sigma}}\cdot\hat{r}$ ($|\hat{\vec{\sigma}}\cdot\hat{r};\pm\rangle$),

$$\begin{aligned}|\hat{\vec{\sigma}}\cdot\hat{r} - \lambda\hat{1}| \doteq \begin{vmatrix}\cos\theta-\lambda & e^{-i\varphi}\sin\theta\\ e^{i\varphi}\sin\theta & -\cos\theta-\lambda\end{vmatrix} &= (\cos\theta-\lambda)(-\cos\theta-\lambda) - \sin^2\theta = -\cos^2\theta - \lambda\cos\theta \\ &+ \lambda\cos\theta + \lambda^2 - \sin^2\theta = -\cos^2\theta - \sin^2\theta + \lambda^2 = \lambda^2 - 1 = 0 \;\leftrightarrow\; \lambda = \pm 1,\end{aligned} \tag{0.4}$$

as we could have infered directly from $\hat{\vec{\sigma}}\cdot\hat{r}\,|\hat{\vec{\sigma}}\cdot\hat{r};\pm\rangle = \pm\,|\hat{\vec{\sigma}}\cdot\hat{r};\pm\rangle = \lambda\,|\hat{\vec{\sigma}}\cdot\hat{r};\pm\rangle \leftrightarrow \lambda = \pm 1$.

Let as find the eigenvectors of $\hat{\vec{\sigma}}\cdot\hat{r}$, $|\hat{\vec{\sigma}}\cdot\hat{r}\rangle$. If we substitute $\lambda = \pm 1$ in Eq. (0.4),

$$\begin{pmatrix}\cos\theta\mp 1 & e^{-i\varphi}\sin\theta\\ e^{i\varphi}\sin\theta & -\cos\theta\mp 1\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = 0 = \begin{pmatrix}(\cos\theta\mp 1)c_1 + (e^{-i\varphi}\sin\theta)c_2\\ (e^{i\varphi}\sin\theta)c_1 + (-\cos\theta\mp 1)c_2\end{pmatrix}. \tag{0.5}$$

[1] $\hat{\vec{\sigma}}\cdot\hat{r}$ is an operator, the *hat* in $\hat{\vec{\sigma}}$ means that it is an operator, and the hat in $\hat{r}$, that it is a unitary vector.

The 2 rows we have for each eigenvalue λ in the column matrix in the right-hand side of Eq. (0.5) should be equivalent. From the first row, $c_2 = -\frac{(\cos\theta \mp 1)c_1}{e^{-i\varphi}\sin\theta} = \frac{\pm 1 - \cos\theta}{\sin\theta} e^{i\varphi} c_1$. So, if we choose $c_1 = \sin\theta$,

$$|\hat{\vec{\sigma}} \cdot \hat{r}; \pm\rangle \doteq N \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = N \begin{pmatrix} \sin\theta \\ (\pm 1 - \cos\theta) e^{i\varphi} \end{pmatrix}, \tag{0.6}$$

subject to the normalization condition

$$\begin{aligned} 1 = \langle \hat{\vec{\sigma}} \cdot \hat{r}; \pm | \hat{\vec{\sigma}} \cdot \hat{r}; \pm\rangle &\doteq N^* \begin{pmatrix} \sin\theta & (\pm 1 - \cos\theta) e^{-i\varphi} \end{pmatrix} N \begin{pmatrix} \sin\theta \\ (\pm 1 - \cos\theta) e^{i\varphi} \end{pmatrix} \\ &= N^* N (\sin^2\theta + (\pm 1 - \cos\theta)^2) = |N|^2 (\sin^2\theta + (\pm 1 - \cos\theta)^2) \end{aligned} \tag{0.7}$$

$$\leftrightarrow \quad |N| = \frac{1}{\sqrt{(\sin^2\theta + (\pm 1 - \cos\theta)^2}} \quad \leftrightarrow \quad N = \frac{1}{\sqrt{(\sin^2\theta + (\pm 1 - \cos\theta)^2}} e^{i\delta} \quad : \quad \delta \in \mathbb{R}.$$

We will choose the phase δ so that N is real and positive. For example, if we choose $\delta = 0$, Eq. (0.6) becomes

$$|\hat{\vec{\sigma}} \cdot \hat{r}; \pm\rangle \doteq \frac{1}{\sqrt{(\sin^2\theta + (\pm 1 - \cos\theta)^2}} \begin{pmatrix} \sin\theta \\ (\pm 1 - \cos\theta) e^{i\varphi} \end{pmatrix}. \tag{0.8}$$

Careful! In Fig. 0.1, we use $|+\rangle_z \equiv |0\rangle \equiv |\hat{\sigma}_z; +\rangle$ as a *reference axis*. Then, in order to derive Eq. (B.2), which corresponds to the coordinate system chosen in Fig. 0.1, we should take the upper sign. Thus, Eq. (0.8) becomes

$$|\hat{\vec{\sigma}} \cdot \hat{r}; +\rangle \doteq \frac{1}{\sqrt{(\sin^2\theta + (+1 - \cos\theta)^2}} \begin{pmatrix} \sin\theta \\ (+1 - \cos\theta) e^{i\varphi} \end{pmatrix}, \tag{0.9}$$

where

- For the first row of Eq. (0.8),

$$\begin{aligned} N\,c_1 &= \frac{\sin\theta}{\sqrt{\sin^2\theta + (1 - \cos\theta)^2}} = \frac{2\sin(\theta/2)\cos(\theta/2)}{\sqrt{(2\sin(\theta/2)\cos(\theta/2))^2 + (2\sin^2(\theta/2))^2}} = \frac{2\sin(\theta/2)\cos(\theta/2)}{\sqrt{4\sin^2(\theta/2)(\cos^2(\theta/2) + \sin^2(\theta/2))}} \\ &= \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin(\theta/2)} = \cos(\theta/2). \end{aligned} \tag{0.10}$$

- For the second row of Eq. (0.8),

$$N\,c_2 = \frac{1 - \cos\theta}{\sin\theta} e^{i\varphi} c_1 = \frac{1 - \cos\theta}{\sin\theta} e^{i\varphi} \cos(\theta/2) = \frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} e^{i\varphi} \cos(\theta/2) = \sin(\theta/2) e^{i\varphi}. \tag{0.11}$$

Thus, we obtain the desired normalized eigenket that represents a qubit in the Bloch sphere in Fig. 0.1,

$$\boxed{|\hat{\vec{\sigma}} \cdot \hat{r}; \pm\rangle = \cos(\theta/2)\,|0\rangle + \sin(\theta/2) e^{i\varphi}\,|1\rangle} \tag{0.12}$$