

1 Proof of $\hat{\rho} = \frac{1}{2}(\hat{1}_2 + \vec{r} \cdot \vec{\sigma})$.

As we have proved¹, in the $\{|0\rangle, |1\rangle\}$ basis,

$$|\hat{\sigma} \cdot \hat{r}; \pm\rangle \equiv |\psi(\theta, \varphi)\rangle = \cos(\theta/2) |0\rangle + \sin(\theta/2)e^{i\varphi} |1\rangle \doteq \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\varphi} \end{pmatrix}. \quad (1.1)$$

- The density matrix for a pure ensemble is $\hat{\rho} = |\psi\rangle \langle\psi|$ [10, p. 182, Eq. (3.4.12)]. Hence,

$$\begin{aligned} \hat{\rho} &= |\psi\rangle \langle\psi| \doteq \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\varphi} \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2)e^{-i\varphi} \end{pmatrix} \\ &= \begin{pmatrix} \cos^2(\theta/2) & \cos(\theta/2)\sin(\theta/2)e^{-i\varphi} \\ \cos(\theta/2)\sin(\theta/2)e^{i\varphi} & \sin^2(\theta/2) \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 + \cos(\theta) & \sin \theta e^{-i\varphi} \\ \sin \theta e^{i\varphi} & 1 - \cos(\theta) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + \cos \theta & \sin \theta \cos \varphi - i \sin \theta \sin \varphi \\ \sin \theta \cos \varphi + i \sin \theta \sin \varphi & 1 - \cos \theta \end{pmatrix} \\ &= \frac{1}{2} \left[\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \sin \theta \cos \varphi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin \theta \sin \varphi \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos \theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \\ &= \frac{1}{2} [\hat{1}_2 + \sin \theta \cos \varphi \hat{\sigma}_1 + \sin \theta \sin \varphi \hat{\sigma}_2 + \cos \theta \hat{\sigma}_3] \\ &= \frac{1}{2} [\hat{1}_2 + (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \cdot (\sigma_1, \sigma_2, \sigma_3)] \\ &\stackrel{(1)}{=} \frac{1}{2} [\hat{1}_2 + (r_1, r_2, r_3) \cdot (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)] = \frac{1}{2} [\hat{1}_2 + \vec{r} \cdot \vec{\sigma}], \end{aligned} \quad (1.2)$$

where in (1) we defined $(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \equiv (r_1, r_2, r_3) = \vec{r}$.

As stated, for pure states,

$$\begin{aligned} |\vec{r}|^2 &= r_1^2 + r_2^2 + r_3^2 = \sin^2 \theta \cos^2 \varphi + \sin^2 \theta \sin^2 \varphi + \cos^2 \theta = \sin^2 \theta (\cos^2 \varphi + \sin^2 \varphi) + \cos^2 \theta \\ &= \sin^2 \theta + \cos^2 \theta = 1 \leftrightarrow |\vec{r}| = +\sqrt{1} = 1. \end{aligned} \quad (1.3)$$

That is, for pure states the *Bloch vector* or *Polarization vector* $\vec{r} = [\hat{\sigma}_i]_{\hat{\rho}} \vec{n}_i$ is equal to the radius of unit length $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$ from the origin to the surface of the Bloch sphere.

Then, at least for pure state density matrices on the Bloch sphere,

$$\hat{\rho} = (\hat{1}_2 + \vec{r} \cdot \vec{\sigma})/2 \quad (1.4)$$

¹See the proof for Eq. (1.1) on the author's page.

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