1 Proof of $|\hat{\sigma} \cdot \hat{r}; \pm\rangle \equiv |\psi(\theta, \varphi)\rangle = \cos(\theta/2)\,|0\rangle + \sin(\theta/2) e^{i\varphi}\,|1\rangle$.

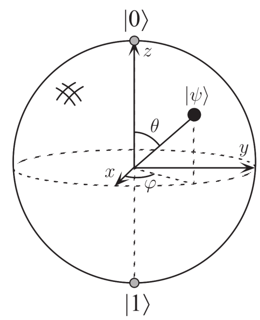

Figure 1.1: Bloch sphere representation of a *qubit* $|\psi\rangle$, as a function of the angles θ and φ.

In the Bloch sphere representation of a qubit, Fig. 1.1,

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z} = (\sin\theta\cos\varphi\,\hat{x}, \sin\theta\sin\varphi\,\hat{y}, \cos\theta\,\hat{z})\,. \tag{1.1}$$

Hence,[1]

$$\hat{\vec{\sigma}} \cdot \hat{r} = (\hat{\sigma}_1\,\hat{x}, \hat{\sigma}_2\,\hat{y}, \hat{\sigma}_3\,\hat{z}) \cdot (\sin\theta\cos\varphi\,\hat{x}, \sin\theta\sin\varphi\,\hat{y}, \cos\theta\,\hat{z}) = \hat{\sigma}_1 \sin\theta\cos\varphi + \hat{\sigma}_2 \sin\theta\sin\varphi + \hat{\sigma}_3\cos\theta\,. \tag{1.2}$$

From the definition of the Pauli matrices [10, p. 169, Eq. (3.2.32)], in the $\{|\pm\rangle_z\} \equiv \{|0\rangle\,, |1\rangle\}$ computational basis,

$$\begin{aligned}
\hat{\vec{\sigma}} \cdot \hat{r} &= \hat{\sigma}_1 \sin\theta\cos\varphi + \hat{\sigma}_2 \sin\theta\sin\varphi + \hat{\sigma}_3\cos\theta \\
&\doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin\theta\cos\varphi + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sin\theta\sin\varphi + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cos\theta \\
&= \begin{pmatrix} \cos\theta & \sin\theta\cos\varphi - i\,\sin\theta\sin\varphi \\ \sin\theta\cos\varphi + i\,\sin\theta\sin\varphi & -\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}.
\end{aligned} \tag{1.3}$$

In order to find the eigenvalues (λ) and the pertaining eigenvectors corresponding to the operator $\hat{\vec{\sigma}} \cdot \hat{r}$ ($|\hat{\vec{\sigma}} \cdot \hat{r}; \pm\rangle$),

$$\begin{aligned}
|\hat{\vec{\sigma}} \cdot \hat{r} - \lambda\hat{1}| \doteq \begin{vmatrix} \cos\theta - \lambda & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta - \lambda \end{vmatrix} &= (\cos\theta - \lambda)(-\cos\theta - \lambda) - \sin^2\theta = -\cos^2\theta - \lambda\cos\theta \\
+ \lambda\cos\theta + \lambda^2 - \sin^2\theta &= -\cos^2\theta - \sin^2\theta + \lambda^2 = \lambda^2 - 1 = 0 \;\leftrightarrow\; \lambda = \pm 1,
\end{aligned} \tag{1.4}$$

as we could have inferred directly from $\hat{\vec{\sigma}} \cdot \hat{r}\,|\hat{\vec{\sigma}} \cdot \hat{r}; \pm\rangle = \pm\,|\hat{\vec{\sigma}} \cdot \hat{r}; \pm\rangle = \lambda\,|\hat{\vec{\sigma}} \cdot \hat{r}; \pm\rangle \leftrightarrow \lambda = \pm 1$.

Let as find the eigenvectors of $\hat{\vec{\sigma}} \cdot \hat{r}$, $|\hat{\vec{\sigma}} \cdot \hat{r}\rangle$. If we substitute $\lambda = \pm 1$ in Eq. (1.4),

$$\begin{pmatrix} \cos\theta \mp 1 & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \mp 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 = \begin{pmatrix} (\cos\theta \mp 1)c_1 + (e^{-i\varphi}\sin\theta)c_2 \\ (e^{i\varphi}\sin\theta)c_1 + (-\cos\theta \mp 1)c_2 \end{pmatrix}. \tag{1.5}$$

[1] $\hat{\vec{\sigma}} \cdot \hat{r}$ is an operator, the *hat* in $\hat{\vec{\sigma}}$ means that it is an operator, and the hat in $\hat{r}$, that it is a unitary vector.

The 2 rows we have for each eigenvalue λ in the column matrix in the right-hand side of Eq. (1.5) should be equivalent. From the first row, $c_2 = -\frac{(\cos\theta \mp 1)c_1}{e^{-i\varphi}\sin\theta} = \frac{\pm 1 - \cos\theta}{\sin\theta} e^{i\varphi} c_1$. So, if we choose $c_1 = \sin\theta$,

$$|\hat{\vec{\sigma}} \cdot \hat{r}; \pm\rangle \doteq N \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = N \begin{pmatrix} \sin\theta \\ (\pm 1 - \cos\theta) e^{i\varphi} \end{pmatrix}, \tag{1.6}$$

subject to the normalization condition

$$\begin{aligned} 1 = \langle \hat{\vec{\sigma}} \cdot \hat{r}; \pm | \hat{\vec{\sigma}} \cdot \hat{r}; \pm \rangle &\doteq N^* \begin{pmatrix} \sin\theta & (\pm 1 - \cos\theta) e^{-i\varphi} \end{pmatrix} N \begin{pmatrix} \sin\theta \\ (\pm 1 - \cos\theta) e^{i\varphi} \end{pmatrix} \\ &= N^* N (\sin^2\theta + (\pm 1 - \cos\theta)^2) = |N|^2 (\sin^2\theta + (\pm 1 - \cos\theta)^2) \end{aligned} \tag{1.7}$$

$$\leftrightarrow \quad |N| = \frac{1}{\sqrt{(\sin^2\theta + (\pm 1 - \cos\theta)^2}} \quad \leftrightarrow \quad N = \frac{1}{\sqrt{(\sin^2\theta + (\pm 1 - \cos\theta)^2}} e^{i\delta} \quad : \quad \delta \in \mathbb{R}.$$

We will choose the phase δ so that N is real and positive. For example, if we choose $\delta = 0$, Eq. (1.6) becomes

$$|\hat{\vec{\sigma}} \cdot \hat{r}; \pm\rangle \doteq \frac{1}{\sqrt{(\sin^2\theta + (\pm 1 - \cos\theta)^2}} \begin{pmatrix} \sin\theta \\ (\pm 1 - \cos\theta) e^{i\varphi} \end{pmatrix}. \tag{1.8}$$

Careful! In Fig. 1.1, we use $|+\rangle_z \equiv |0\rangle \equiv |\hat{\sigma}_z; +\rangle$ as a *reference axis*. Then, since we use the coordinate system chosen in Fig. 1.1, we should take the upper sign. Thus, Eq. (1.8) becomes

$$|\hat{\vec{\sigma}} \cdot \hat{r}; +\rangle \doteq \frac{1}{\sqrt{(\sin^2\theta + (+1 - \cos\theta)^2}} \begin{pmatrix} \sin\theta \\ (+1 - \cos\theta) e^{i\varphi} \end{pmatrix}, \tag{1.9}$$

where

- For the first row of Eq. (1.8),

$$\begin{aligned} N\,c_1 &= \frac{\sin\theta}{\sqrt{\sin^2\theta + (1 - \cos\theta)^2}} = \frac{2\sin(\theta/2)\cos(\theta/2)}{\sqrt{(2\sin(\theta/2)\cos(\theta/2))^2 + (2\sin^2(\theta/2))^2}} = \frac{2\sin(\theta/2)\cos(\theta/2)}{\sqrt{4\sin^2(\theta/2)(\cos^2(\theta/2) + \sin^2(\theta/2))}} \\ &= \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin(\theta/2)} = \cos(\theta/2). \end{aligned} \tag{1.10}$$

- For the second row of Eq. (1.8),

$$N\,c_2 = \frac{1 - \cos\theta}{\sin\theta} e^{i\varphi} c_1 = \frac{1 - \cos\theta}{\sin\theta} e^{i\varphi} \cos(\theta/2) = \frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} e^{i\varphi} \cos(\theta/2) = \sin(\theta/2) e^{i\varphi}. \tag{1.11}$$

Thus, we obtain the desired normalized eigenket that represents a qubit in the Bloch sphere in Fig. 1.1,

$$\boxed{|\hat{\vec{\sigma}} \cdot \hat{r}; \pm\rangle = \cos(\theta/2)\,|0\rangle + \sin(\theta/2) e^{i\varphi}\,|1\rangle} \tag{1.12}$$

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