Entanglement

Entangled states $|\psi\rangle_{AB} = \sum_{i,j} c_{ij} |i\rangle_A |j\rangle_B$ ($c_{ij} \neq a_i b_j$, normalized complex parameters) are not expressible as the product of their constituent states , $|i\rangle_A |j\rangle_B : i,j \in \{0,1\}$. Id est, an entangled state is definite states of a multipartite quantum system as a whole in which neither constituent by itself has a definite state: multiqubit system that cannot be reduced to a product of the single qubits forming the system. In entangled states, single qubits lose their individuality and become a part of the system as a single entity.

They exhibit correlations that have no classical analog. To see this in first hand, consider the Bell state or entangled bipartite state (triplet) of the form

$$|\psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \equiv \frac{|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B}{\sqrt{2}} \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \,, \tag{0.1}$$

as any Bell state, is *maximally entangled*: entangled, because there is no way of expressing it as a list of one-qubit states, and, maximally entangled, because a measurement (in any basis, along any measurement axis at all) of one of the qubits (the system A or B) outputs 0 or 1 with 50% probability. The degree of entanglement can be quantified by the coherence (off-diagonal elements of the density operator), or how mixed the reduced state is. Hence, since Bell states, as mentioned, are maximally entangled, we can predict that this instance gives maximum entropy according to Eq. (C.4); i.e., it will be a completely disordered, decohered or mixed state of the form $\hat{\rho}_A = \frac{1}{d_A}\,\hat{1}_{d_A}$, where d_A is the dimension of the subsystem A ($d_A = 2$ for qubit subsystems):

$$\begin{aligned}
\hat{\rho}_A &= \mathrm{Tr}_B\left(\hat{\rho}_{AB}\right) = \mathrm{Tr}_B\left(|\psi^+\rangle\langle\psi^+|\right) \doteq \frac{1}{2}\mathrm{Tr}_B\,\left[(|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B)(\langle 0|_A \langle 1|_B + \langle 1|_A \langle 0|_B)\right] \\
&= \frac{1}{2}\mathrm{Tr}_B\,\left[|0\rangle_A\langle 0|_A \otimes |1\rangle_B\langle 1|_B + |0\rangle_A\langle 1|_A \otimes |1\rangle_B\langle 0|_B + |1\rangle_A\langle 0|_A \otimes |0\rangle_B\langle 1|_B + |1\rangle_A\langle 1|_A \otimes |0\rangle_B\langle 0|_B\right] \\
&\overset{(1)}{=} \frac{1}{2}\left(|0\rangle_A\langle 0|_A\ \mathrm{Tr}\begin{pmatrix}0&0\\0&1\end{pmatrix} + |0\rangle_A\langle 1|_A\ \mathrm{Tr}\begin{pmatrix}0&0\\1&0\end{pmatrix} + |1\rangle_A\langle 0|_A\ \mathrm{Tr}\begin{pmatrix}0&1\\0&0\end{pmatrix} + |1\rangle_A\langle 1|_A\ \mathrm{Tr}\begin{pmatrix}1&0\\0&0\end{pmatrix}\right) \\
&= \frac{1}{2}\left(|0\rangle_A\langle 0|_A \otimes 1 + 0 + 0 + |1\rangle_A\langle 1|_A \otimes 1\right) = \frac{1}{2}\left(|0\rangle_A\langle 0|_A + |1\rangle_A\langle 1|_A\right) = \frac{1}{2}\,\hat{1}_A = \frac{1}{2}\,\hat{1}_2
\end{aligned} \tag{0.2}$$

where in equality (1) we used $\mathrm{Tr}\,(\hat{A}+\hat{B}) = \mathrm{Tr}\,(\hat{A}) + \mathrm{Tr}\,(\hat{B})$ and [1, p. 105, Eq. (2.178)] $\mathrm{Tr}_B\,(\hat{A}\otimes\hat{B}) = \sum_b (\hat{1}_A \otimes \langle b|)(\hat{A}\otimes\hat{B})(\hat{1}_A \otimes |b\rangle) = (\hat{1}_A\hat{A}\hat{1}_A)\sum_b \langle b|\hat{B}|b\rangle = \hat{A}\ \mathrm{Tr}\,(\hat{B})$. Nevertheless, there is a *correlation*: whenever $|0\rangle_A$ ($|1\rangle_A$) is measured, $|1\rangle_B$ ($|0\rangle_B$) is also measured, with total probability.

All in all, maximally entangled states' individual qubits behave randomly ($\hat{\rho}_A = \hat{\rho}_B = \hat{1}/d$), but, even if it can sound counter-intuitive, they allow prediction about the measurement output of the other qubit in the same basis. Entangled states only exhibit this distinctive combination of perfect individual randomness and strong correlation!