Proof of Eq. (B.6), $\hat{\rho} = \frac{1}{2}(\hat{1}_2 + \vec{r} \cdot \vec{\sigma})$.

From Eq. (B.2) we have, in the $\{|0\rangle, |1\rangle\}$ basis,

$$|\hat{\sigma} \cdot \hat{r}; \pm\rangle \equiv |\psi(\theta, \varphi)\rangle = \cos(\theta/2)\,|0\rangle + \sin(\theta/2)e^{i\varphi}\,|1\rangle \doteq \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\varphi} \end{pmatrix}. \tag{0.1}$$

- The density matrix for a pure ensemble is $\hat{\rho} = |\psi\rangle\langle\psi|$ [10, p. 182, Eq. (3.4.12)]. Hence,

$$\begin{aligned}
\hat{\rho} = |\psi\rangle\langle\psi| &\doteq \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\varphi} \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2)e^{-i\varphi} \end{pmatrix} \\
&= \begin{pmatrix} \cos^2(\theta/2) & \cos(\theta/2)\sin(\theta/2)e^{-i\varphi} \\ \cos(\theta/2)\sin(\theta/2)e^{i\varphi} & \sin^2(\theta/2) \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1+\cos(\theta) & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & 1-\cos(\theta) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & \sin\theta\cos\varphi - i\sin\theta\sin\varphi \\ \sin\theta\cos\varphi + i\sin\theta\sin\varphi & 1-\cos\theta \end{pmatrix} \\
&= \frac{1}{2}\left[\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \sin\theta\cos\varphi\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin\theta\sin\varphi\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos\theta\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \\
&\frac{1}{2}\left[\hat{1}_2 + \sin\theta\cos\varphi\,\hat{\sigma}_1 + \sin\theta\sin\varphi\,\hat{\sigma}_2 + \cos\theta\,\hat{\sigma}_3\right] \\
&= \frac{1}{2}\left[\hat{1}_2 + (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\cdot(\sigma_1, \sigma_2, \sigma_3)\right] \\
&\overset{(1)}{=} \frac{1}{2}\left[\hat{1}_2 + (r_1, r_2, r_3)\cdot(\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)\right] = \frac{1}{2}\left[\hat{1}_2 + \vec{r}\cdot\hat{\vec{\sigma}}\right],
\end{aligned} \tag{0.2}$$

  where in (1) we defined $(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \equiv (r_1, r_2, r_3) = \vec{r}$.

As stated, for pure states,

$$\begin{aligned}
|\vec{r}|^2 = r_1^2 + r_2^2 + r_3^2 &= \sin^2\theta\,\cos^2\varphi + \sin^2\theta\,\sin^2\varphi + \cos^2\theta = \sin^2\theta\,(\cos^2\varphi + \sin^2\varphi) + \cos^2\theta \\
&= \sin^2\theta + \cos^2\theta = 1 \;\leftrightarrow\; |\vec{r}| = +\sqrt{1} = 1.
\end{aligned} \tag{0.3}$$

That is, for pure states the *Bloch vector* or *Polarization vector* $\vec{r} = [\hat{\sigma}_i]_{\hat{\rho}}\,\vec{n}_i$ is equal to the radius of unit length $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$ from the origin to the surface of the Bloch sphere.

Then, at least for pure state density matrices on the Bloch sphere,

$$\hat{\rho} = (\hat{1}_2 + \vec{r}\cdot\vec{\sigma})/2 \tag{0.4}$$