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ZIENTZIA ETA TEKNOLOGIA FAKULTATEA
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Master in Quantum Science and Technology

Master's thesis

Interactions of Chern-Simons vortices: a systematic study

Asier Izquierdo

Supervisors:

Betti Hartmann
Joanes Lizarraga
Jon Urrestilla

Department of Theoretical Physics
Faculty of Science and Technology
University of the Basque Country UPV/EHU

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Chapter 1

Introduction and objectives

1.1 Introduction

Topological defects are stable regular classical solutions to underlying quantum field theories. These defects emerge in theories which have phase transitions with spontaneous symmetry breaking (SSB)[1] as a common denominator.

They typically consist of physical structures with energy confined in a region of space-time where the "old" phase is trapped. The study of these defects can show valuable information about the SSB scheme. Another usual property of the defects is that, due to their topological nature, they carry a quantized magnetic flux, as we shall see soon.

Monopoles, strings and domain walls are an example of the sort of defects that can be formed. This names refer, respectively, to point-like, one- or two-dimensional topological defects in $(3+1)$ space-time dimensions [1, p. 351, l. 11-12]. In the forthcoming sections we will focus on one of the most studied topological defect: strings (or vortices[2]). As the name indicates, these are one dimensional objects whose matter density is concentrated in a line.

Topological defects –in particular, cosmic strings– appear in broadly differing fields of physics, such as cosmology or condensed matter physics. Briefly explained, in the former context, cosmic strings were possibly produced during the phase transitions that took place in the early stages of the evolution of the universe, at a very high mass scale. They initially gained interest because they offered a potential alternative to the cosmological inflation for the origin of the primeval density inhomogenities that seeded, via gravitational instability, large-scale structure in the universe [2, p. 68, l. 26-31]. Measurements rule out pure topological defect models as candidates for the origin of initial density perturbations leading to structure formation, but their CMB predictions and Gravitational Wave predictions have been and are studied constantly.

Cosmic strings or vortex lines are ubiquitous, as well, in condensed matter systems. In this framework, there is a fundamental distinction between type I and type II superconductors. Type I superconductors show a Meissner effect, i.e., if the applied magnetic field is not as strong as to abruptly destroy superconductivity, magnetic field lines are completely expelled from the main

[1]See Appendix B for a brief explanation on SSB in the Abelian-Higgs (AH) model.

[2]Sometimes vortices and strings are treated as synonyms in the literature. Some other times, vortices refer to the $(2+1)$ dimensional counterpart of the $(3+1)$ dimensional string; that is, to the cross sections, perpendicular to the axis of symmetry, of strings. Also, the name "cosmic string" is used, in cosmological settings or in order to distinguish from the fundamental strings or F-strings. We will use the names "vortex", "string" and "cosmic string" indistinctively.

body of the metal [3, p. 12, Fig. 1.5]. However, for type II superconductors the transition to the normal state is not sudden. Instead of one, two critical values for external magnetic fields can be identified. Between them, type II superconductors allow magnetic field partially penetrate (incomplete Meissner effect) with a quantized flux, but superconductivity is maintained. The superconductor is in a mixed state of superconductivity and normal conductivity, called vortex state or Shubnikov phase [3, p. 12] and features the formation of magnetic field vortices or flux tubes, first considered theoretically by Abrikosov.

The simplest and most common model to describe the formation of this kind of string-like topological defects is the Abelian Higgs (AH) model, which will be addressed in a following chapter. In this theory, string solutions can be embedded, and considerable work has been carried out in obtaining their properties, such as stability with respect to various parameters, interactions between them and decay channels.

Unlike other topological defects, vortices do not admit finite-energy electrically charged generalizations. However, extending the AH model with a Chern-Simons (CS) term, the vortices acquire electrical charge keeping the energy finite, even though this only works in odd dimensional space-times.

The CS term that is added to the usual AH model arose first in pure mathematics as the CS form, which in turn was an application of the mathematical machinery developed by Maxwell to explain electromagnetism. One unusual feature of the CS term is that even though it is not invariant under gauge transformations, it yields consistent physical equations of motion.

Regarding the relevance of this field of investigation, as introduced earlier for cosmic strings in general, the AH vortex model plays important roles in a wide range of areas, from cosmology to superconductivity. We will focus on the latter context, since the CS form will be added to the AH model and observed symmetry properties of high temperature superconductors under parity transformations suggest interactions modeled by the CS theory [4, p. 3464, l. 4-6], which is P-nonconserving. Moreover, AHCS vortices must be studied in odd space-time dimensions and we will work in two spatial dimensions. Therefore, we hope the results given might contribute to a better understanding of the interaction between AHCS vortices, which would also help clarify the properties of these high T superconductive systems [4, p. 3444, l. 18-19].

Beyond the scope of this work, its application in the planar (or two spatial dimensional) condensed matter physics phenomenon known as Quantum Hall Effect (QHE) [5, p. 86, Table 1][6, p. 12, l. 2-4] is a further reason for considering Chern-Simons theory a field of interest.

1.2 Objectives

Our primary aim in this work is to understand the properties of Abelian-Higgs-Chern-Simons (AHCS) vortices, which have not been studied as widely as in the Abelian-Higgs (AH) case, by numerically constructing solutions to the corresponding equations of motion.

Precisely, we want to investigate systematically the string energy of the AHCS model as a function of the parameters that govern the theory, which include the CS term. The main question we want to address is the question of stability of the AHCS strings; i.e., whether strings can interact with each other to form bound states, or whether strings break into other strings, in a similar manner as the type I and II cases mentioned earlier.

This constitutes a subject which has not been addressed yet in the existing literature. Some studies on the forces acting between AHCS vortices have been done in [4], but a systematic study does not yet exist. Our aim here is to provide this systematic study.

For this purpose, we solve the corresponding equations of motion of the string as a function of the different parameters. We scan parameter space by fixing some of the parameters alternatively, and leaving the other ones free. With the solutions obtained, we compute the energies of the strings for each case and compare them. With this, we obtain the regions where strings are attractive (type I), repulsive (type II), and the boundary between them.

In this work we used the conventions found in Appendix A.1. In the following section, global strings are explained to help understand Section 3, where the AH is introduced as the simplest theory used to model (local) cosmic strings. We study certain general properties which emerge from this theory and Abrikosov-Nielsen-Olesen (ANO) strings, and set the stage for introducing the AHCS model. In fact, the AHCS model is presented in Section 4, by first introducing the CS term and then coupling it to the AH model. Various properties concerning the AHCS strings are discussed in some detail. In Section 5, we briefly describe the numerical methods used to solve the field equations of AHCS strings and present the numerical results obtained in this work. We conclude in Section 6.

Chapter 2

Global strings

Some general remarks are suitable in order to pave our way, prior to the properties introduced in the next chapter. Topological defects can be either global or local. Precisely, in the AH model, the symmetry of the theory is promoted from global, $U(1)_G$, to local, $U(1)_L$, so we will present global strings. This kind of models are invariant under the global phase transformation

$$\phi(x) \xrightarrow{U(1)_G} e^{i\alpha}\phi(x), \tag{1}$$

where $\phi(x) \in \mathbb{C}$ is a complex-valued scalar field, and $\alpha \in \mathbb{R}$ is a phase that is equal at all space-time points –id est, it is constant, and hence "global"–. The theory is described by the following Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi(x)\right)^* \left(\partial^\mu \phi(x)\right) + \mathcal{V}(|\phi(x)|), \tag{2}$$

where the self-interaction potential energy density is defined as

$$\mathcal{V}(|\phi(x)|) = \frac{\lambda}{4}\left(|\phi(x)|^2 - \eta^2\right)^2, \tag{3}$$

with $\lambda \in \mathbb{R}_+$, generally referred to as "Higgs potential" or "Mexican hat" potential, which triggers the SSB. $\eta \in \mathbb{R}_+$ is the modulus of the true ground or vacuum state of the scalar field $\phi(x)$. It is straightforward to see that this Lagrangian density is invariant under the global phase transformation $U(1)_G$, Eq. (1).

The field configuration of minimum energy is found by minimizing the scalar quartic potential of the theory, $d\mathcal{V}(|\phi(x)|)/d(|\phi(x)|) = 0$ and $d^2\mathcal{V}(|\phi(x)|)/d^2(|\phi(x)|) > 0$[3]. The potential minimum has constant solutions $|\phi(x)| = \eta$. Therefore, the vacuum manifold $\mathcal{M}$ corresponding to a theory with the potential density defined in Eq. (3) is isomorphic to the circle S^1,

$$\mathcal{M} = \{\phi(x) \in \mathbb{C} |\, \phi(x)\phi^*(x) - \eta = 0\} \cong S^1. \tag{4}$$

This can be seen more clearly if we define $\phi(x) = \mathrm{Re}\left(\phi(x)\right) + i\mathrm{Im}\left(\phi(x)\right) \equiv \phi_1(x) + i\,\phi_2(x)$, where $\phi_1, \phi_2 \in \mathbb{R}$. It can be easily noted that the Higgs potential density minima lie on a circle of radius $|\phi(x)| = \eta$ in the $\big(\phi_1(x), \phi_2(x)\big)$ plane:

$$|\phi(x)|^2 = \phi_1^2(x) + \phi_2^2(x) = \eta^2, \tag{5}$$

[3] It can be checked from this condition that $\lambda > 0$. Otherwise, $|\phi(x)| = \eta$ would be the maxima, and, $\phi(x) = 0$, a minimum. There would be no SSB!

which form a set of degenerate vacua related to each other by rotation.

Hence, the magnitude of the radius of the minima is well defined, only a certain value minimizes the potential energy density, $|\phi(x)| = \eta$. However, the phase of the ground states $\phi(x) = \eta e^{i\alpha}$, $\alpha \in \mathbb{R}$, is not defined, remains arbitrary. That is, the value of α is irrelevant because of invariance of the Lagrangian density under $U(1)_G$, Eq. (1). Even the ground states are determined by α, the energy of the quartic potential changes just under different values of $|\phi(x)|$.

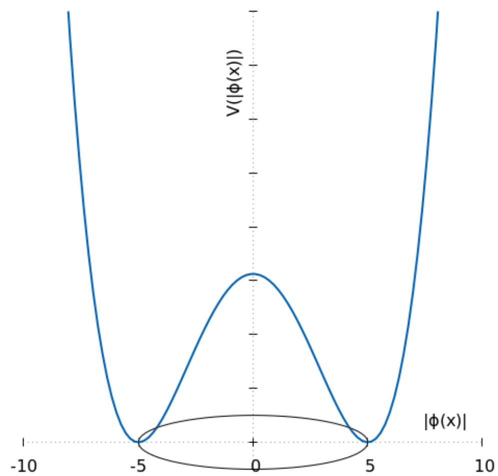

Figure 1: "Mexican hat" potential density, Eq. (3). The set of points for which $|\phi(x)|^2 = \eta^2$ conform the minima of the potential. We have chosen $\eta = 5$ in this plot.

The topology of the space of ground states is a fundamental aspect of topological defects. In fact, their formation is based on the homotopy properties of the vacuum manifold. If the symmetry breaking of a Lie group G down to a subgroup H of G, $H \subset G$ [2, p. 69, l. 26-27], i.e., the transition $G \mapsto H$, is considered –which means the theory is invariant under the continuous group G, but the vacuum states are invariant only under $H \subset G$–, the formation of cosmic strings is possible provided the manifold of vacua $\mathcal{M}$ is not simply connected –equivalently, if $\mathcal{M}$ contains non-contractible loops, loops which cannot be continuously shrunk into a point within $\mathcal{M}$–[2, p. 69, l. 34-37][7, p. 312, l. 7-12].

When the asymptotic field configuration of the Higgs field, $\phi(r \to \infty)$, winds around this circle of vacua, Eq. (4), strings form [1, p. 360, l. 9-10]. Since the vacuum manifold $\mathcal{M}$ is isomorphic to the circle group S^1, a winding configuration cannot be extended continuously inwards from $r \to \infty$ to the entire (x, y) plane while remaining in $\mathcal{M}$. Therefore, taking also into account the continuity of the Higgs field, $\phi(x)$ must have a zero somewhere in the OXY plane [1, p. 360, l. 11-15]. In three spatial dimensions, the continuous lines of zeros signal the position of the string (a sheet in space-time). The topological nature of defects makes the zeros of the scalar field form a continuum, so that strings are infinite, do not have an end.

The group space of $U(1)$ is a circle S^1. This is, elements of $U(1)$, since single valued, may be written as $e^{i\theta} = e^{i(\theta+2\pi)}$, where the space of all values of θ can be considered a line with $\theta = 0$ identified with $\theta = 2\pi$, id est, a circle S^1. The boundary value of the field, $\phi(r \to \infty) = f(r \to \infty)\, e^{in\theta} = e^{in\theta}$, $n \in \mathbb{Z}$, is also a circle S^1, with $r \to \infty$ and $\theta = [0, 2\pi)$. Accordingly, $\phi(r \to \infty)$ may be regarded as defining a loop in $\mathcal{M}$, a map from the boundary circle S^1 in physical space onto the $U(1)$ group space circle S^1 of the vacuum manifold $\mathcal{M}$, $\phi(r \to \infty) : S^1 \to \mathcal{M} \cong S^1$, specified by the winding number[4] $n \in \mathbb{Z}$. That is, in cylindrical or polar coordinates, analyzing the the behavior of the scalar field at infinity reduces to the mapping of a circle, in coordinate space, onto another, in field space.

[4]Winding number or vortex topological charge of the scalar field, which tells the number of times the phase winds around. $n = 0$ corresponds to there being no vortex. The winding number is positive for loops clockwise and negative for counterclockwise. All our equations and results are independent of the sign of n. For sinplicity, we will choose $n > 0$.

Chapter 3

Abelian-Higgs model

Let us introduce the Abelian-Higgs (AH) model, the most common theory for describing strings. We demand that the complete AH Lagrangian be invariant under local phase transformations. In other words, the AH model is described by the local version of the Lagrangian for global strings.

The local phase transformation differs from the global phase transformation (1) in the sense that the phase α is a function of space-time points $x \equiv x^\mu = (t, \vec{x})$, $\alpha \equiv \alpha(x)$. Therefore, $\partial_\mu \alpha(x) \neq 0$ and, effectively, demanding invariance under local gauge transformations means requiring the Lagrangian density which describes AH strings be invariant under the $U(1)_L$ gauge transformations

$$\phi(x) \xrightarrow{U(1)_L} \phi(x)\, e^{i\, e\alpha(x)} \text{ (local phase trans.)}, \quad A_\mu(x) \xrightarrow{U(1)_L} A_\mu(x) + \partial_\mu \alpha(x), \quad \alpha(x), e \in \mathbb{R}. \tag{6}$$

Since the free Lagrangian density (2) is not locally phase invariant because

$$\partial_\mu \phi(x) \xrightarrow{U(1)_L} \partial_\mu \big(\underbrace{\phi(x)\, e^{i\, e\alpha(x)}}_{(6a)}\big) = \big(\partial_\mu\, \phi(x)\big)\, e^{i\, e\alpha(x)} + i\, e\big(\partial_\mu \alpha(x)\big)\phi(x)\, e^{i\, e\alpha(x)}, \tag{7}$$

the $\partial_\mu \alpha(x)$ in Eq. (6b), via the gauge covariant derivative,

$$\begin{aligned} &D_\mu(x)\phi(x) = \partial_\mu \phi(x) - i\, eA_\mu(x)\phi(x) \xrightarrow{U(1)_L} \\ &\underbrace{e^{i\, e\alpha(x)}\, \partial_\mu \phi(x) + e^{i\, e\alpha(x)}\, i\, e\big(\partial_\mu \alpha(x)\big)\phi(x)}_{(7)} - i\, e\big(\underbrace{A_\mu(x) + \partial_\mu \alpha(x)}_{(6b)}\big)\, \underbrace{\phi(x) e^{i\, e\alpha(x)}}_{(6a)} \\ &= e^{i\, e\alpha(x)}\, D_\mu(x)\phi(x) \end{aligned} \tag{8}$$

is the responsible for compensating for the extra term in Eq. (7). That is, the partial derivatives ∂_μ in Eq. (2) must be replaced by the $U(1)_L$-covariant derivatives $D_\mu(x)$.

That is, we are obliged to add a new field, $A_\mu(x)$, which changes $D_\mu(x)$ in a particular manner, in coordination with the local phase transformation of $\phi(x)$, in order to soak up the term $\partial_\mu \alpha(x)$. This new field $A^\mu(x) \in \mathbb{R}$ is the $U(1)_L$ vector gauge field.

The AH Lagrangian density must include a kinetic term for this new field $A_\mu(x)$, so we include the Maxwell Lagrangian density term $\frac{1}{4}\, F_{\mu\nu}(x)\, F^{\mu\nu}(x)$, where $F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ denotes the electromagnetic field strength tensor. $F^{\mu\nu}(x)$ is invariant under Eq. (6b),

$$\begin{aligned} &F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) \xrightarrow{U(1)_L} \\ &\partial^\mu \big(\underbrace{A^\nu(x) + \partial^\nu \alpha(x)}_{(6b)}\big) - \partial^\nu \big(\underbrace{A^\mu(x) + \partial^\mu \alpha(x)}_{(6b)}\big) = F^{\mu\nu}(x), \end{aligned} \tag{9}$$

because the partial derivatives of continuously differentiable fields can be commuted, $[\partial^\mu, \partial^\nu]\alpha(x) = 0$. However, $A^\mu(x)\, A_\mu(x)$ is not invariant under transformation (6b). Therefore, in order for local gauge invariance not to be lost, the Lagrangian does not include a mass term explicitly –quadratic self-interacting term, not containing derivatives– for the field $A^\mu(x)$, $\frac{1}{2} m_A^2\, A^\mu(x)\, A_\mu(x)$. In other words, local gauge symmetry $U(1)_L$ requires the photon corresponding to the field $A^\mu(x)$ to be massless, $m_A = 0$ [8, p. 359, l. 25-27]. However, after spontaneous symmetry breaking from $U(1)$ symmetry to $\mathbb{1}$ –see Appendix B–, the mass of the vector field will acquire a non-zero value via Higgs mechanism.

To sum up, first, we demand the Lagrangian (2) be invariant under local phase transformations of $\phi(x)$. Then, in order to counter the second term in Eq. (7), the vector gauge field $A^\mu(x)$ is introduced, which changes according to the rule (6b), and every partial derivative (∂_μ) is replaced by the $U(1)_L$-covariant derivative $D_\mu(x)$. This results in the AH theory, described by the Lagrangian density

$$\mathcal{L}_{\mathrm{AH}}(A_\mu(x), \phi(x); \lambda) = \frac{1}{4}\, F_{\mu\nu}(x)\, F^{\mu\nu}(x) + \frac{1}{2}\big(D_\mu(x)\, \phi(x)\big)^* D^\mu(x)\, \phi(x) + \mathcal{V}(|\phi(x)|), \tag{10}$$

where $x \in \mathbb{R}^{3+1}$. $\phi(x) \in \mathbb{C}$ is a complex-valued electrically charged $(e > 0)$ scalar field, also known as the Higgs field, coupled minimally both to the electromagnetic field $A^\mu(x)$, by the $U(1)$ gauge covariant derivative $D_\mu(x) \equiv \partial_\mu - i\, eA_\mu(x)$, and to itself, by the potential energy density $\mathcal{V}(|\phi(x)|)$ defined in Eq. (3). We will make $\eta \equiv 1$ throughout for simplicity, without loss of generality. This can be done because η can be taken as a common factor of the Lagrangian density, with the proper rescaling,

$$\phi(x) \to \eta\phi(x), \quad A_\mu(x) \to \eta A_\mu(x), \quad x \to \frac{x}{\eta}. \tag{11}$$

The equations of motion –see Appendix A.3.2– associated with the Lagrangian density (10) read

$$(98)^* \quad \leftrightarrow \quad \big(D_\mu(x) D^\mu(x) \phi(x)\big) - \lambda\big(|\phi(x)|^2 - 1\big)\phi(x) = 0, \tag{12}$$

$$(99) \quad \leftrightarrow \quad \partial_\mu F^{\mu\nu}(x) = \frac{ie}{2}\Big(\phi^*(x) D^\nu(x)\phi(x) - \phi(x)\big(D^\nu(x)\phi(x)\big)^*\Big); \tag{13}$$

and the total energy, [5, p. 92, Eq. (2.1.13)]

$$E = \int d^3x\, \mathcal{H} \overset{(95)}{=} \int d^3x \left\{ \frac{1}{2}|D_0(x)\, \phi(x)|^2 + \frac{1}{2}|D_i(x)\, \phi(x)|^2 + \frac{1}{2}\Big(E^2 + B^2\Big) + \mathcal{V}(|\phi(x)|) \right\}, \tag{14}$$

where the electric and magnetic fields, respectively, are given by $E_i = F_{0i}$ and $B^k = \frac{1}{2}\epsilon^{ijk} F_{ij}$.

3.1 Abrikosov-Nielsen-Olesen (ANO) vortices

A solution for the Abelian-Higgs model is the Abrikosov-Nielsen-Olesen (ANO) string, an infinitely long, axially symmetric string stretching along the z-axis in $d = (3 + 1)$. Let us consider the static, radial ansatz in cylindrical coordinates, $x^\mu = (t, r, \theta, z)$, which originate from the center of the string ($r = 0$), with $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$ and metric tensor $(g_{\mu\nu}) = \mathrm{diag}(-1, 1, r^2, 1)$,

$$A_t = A_r = A_z = 0, \quad A_\theta = \frac{1}{e}\left(n - P(r)\right), \quad \phi(r,\theta) = f(r)e^{in\theta}. \tag{15}$$

$P(r)$,$f(r) \in \mathbb{R}$ and $r \in [0, \infty)$ $\{\theta \in [0, 2\pi)\}$ is the radial {angular} coordinate on the (x, y) plane in physical space. $n \in \mathbb{Z}$ is the winding number of the Higgs field.

The choice of Eq. (15) is given by the $U(1)_L$ invariance of the Lagrangian density. For instance, we could have started from $A_0 = C_1 \neq 0$ and then use $\mathcal{L}_{\mathrm{AH}}$'s invariance under Eq. (6b) to obtain $A_0 = 0$ in Eq. (15),

$$A_0 = C_1 \xrightarrow{U(1)_L} A_0 + \partial_0\alpha(x) = 0 \quad \leftrightarrow \quad \partial_0\alpha(x) = -A_0 = -C_1. \tag{16}$$

However, since the solutions are static, the symmetry under which the Lagrangian density of the theory remains invariant must also be time independent. Therefore, the local phase $\alpha(x)$ present in the local gauge transformations (6) must be time independent, $\partial_t\alpha(x) = 0$. Then, the gauge symmetry that applies to the vector field $A_\mu(x)$, Eq. (6b), becomes

$$A_0(x) \xrightarrow{U(1)_L} A_0(x), \quad \text{and} \quad A_i(x) \xrightarrow{U(1)_L} A_i(x) + \partial_i\alpha(x). \tag{17}$$

Notice that, since $A_t(x) = 0$, ANO vortices (15) are electrically neutral solutions,

$$E_i(x) = \boxed{F_{i0}(x) = 0} = \Big(\partial_i A_0(x) - \partial_0 A_i(x)\Big) \overset{(15)}{=} 0 - \frac{\partial_0}{e}\left(n - P(r)\right) = 0. \tag{18}$$

However, as an effect of introducing the electromagnetic field $A_\mu(x)$ into the theory, the ANO vortices possess a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ inside their core, whose only surviving component is in the z direction, since $A_t(x) = A_z(x) = 0$ and all other fields are independent of t and z,

$$\begin{aligned}\mathbf{B} = \nabla \times \mathbf{A} &= \left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\hat{z} \\ &\overset{(15)}{=} \frac{1}{r}\left(\frac{d(rA_\theta(r))}{dr}\right)\hat{z} = \frac{n - (rP(r))'}{er}\hat{z}.\end{aligned} \tag{19}$$

The string energy per unit length or energy density of such configuration (static and z-independent) reads

$$\frac{E}{l} = \int d^2x\, \mathcal{H} \overset{(14)}{=} \int d^2x \left\{ \frac{1}{2}|D_i(x)\,\phi(x)|^2 + \frac{1}{2}B^2 + \mathcal{V}(|\phi(x)|) \right\}. \tag{20}$$

The magnetic flux through the OXY plane is quantized (in units of $2\pi/e$),

$$\Phi_B = \int_{\mathbb{R}^2} d^2x\, B = \int_{\mathbb{R}^2} (\nabla \times \mathbf{A}) \cdot d\mathbf{S} \overset{(1)}{=} \oint_{\mathcal{C}=S^1_\infty} \mathbf{A} \cdot d\mathbf{l} \overset{(15)}{=} \int_0^{2\pi} A_\theta(r)\, d\theta \overset{(2)}{=} \int_0^{2\pi} \frac{1}{e}\partial_\theta \chi(\theta)\, d\theta$$
$$\overset{(3)}{=} \frac{2\pi n}{e}. \tag{21}$$

In (21(1)), Stokes theorem $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{l} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$ is used, where the integration $\oint_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{l}$ is performed round the circle S^1 or contour $\mathcal{C}$ at spatial infinity ($r \to \infty$), S^1_∞. From Eq. (27), we must have $\phi(r \to \infty) \overset{(15c)}{=} e^{i\chi(\theta)}$ and $D_\theta(x)\phi(x) = (i(\partial_\theta \chi(\theta)) - ieA_\theta(r))e^{i\chi(\theta)} \xrightarrow{r\to\infty} 0$. Thus, as employed in (21(2)), $eA_\theta(r) = \partial_\theta \chi(\theta)$. Also, since the scalar field $\phi(x)$ must be single valued, from the exponential of Eq. (15c), $\chi(k+2\pi) - \chi(k) = 2\pi n$, $k \in \mathbb{Z}$, for winding number $n \in \mathbb{Z}$, as used in (21(3)) for $k = 0$.

Let us introduce the following change of variable and couplings, respectively,

$$x = \sqrt{2\lambda}r, \quad \beta \equiv 2\alpha^2 \equiv \frac{m_s^2}{m_v^2} = \frac{2\lambda}{e^2}, \tag{22}$$

where $\beta \geq 0$ is the Higgs or scalar field self-coupling constant λ in units of $e^2/2$, the parameter that measures the square of the ratio of the Higgs $\phi(x)$ or scalar field mass, m_s, and the gauge $A^\mu(x)$ or vector field mass, m_v, after the spontaneous symmetry breaking from $U(1)_L$ to $\mathbb{1}$. β distinguishes –see Fig. 3– type I ($\beta < 1$) and type II ($\beta > 1$) superconductors.

Recall that, in the AH model, the electron charge e acts as the scalar-gauge coupling constant. On the other hand, $\lambda \in \mathbb{R}_+$ is the scalar field self-coupling constant. Both have been synthesized, in Eq. (22b), in a unique coupling constant β. Rigorously speaking, the parameter β has its origin in the AH model, not in the ANO ansatz (15).

The result of substituting ansatz (15) in the field equations (12-13) is the following system of coupled ordinary differential equations (in rescaled coordinates, Eq. (22)),

$$P''(x) = \frac{P'(x)}{x} + \frac{P(x)f^2(x)}{\alpha^2}, \tag{23}$$

$$\frac{f'(x)}{x} + f''(x) - \frac{P^2(x)f(x)}{x^2} - \frac{1}{2}\big(f^2(x) - 1^2\big)f(x) = 0, \tag{24}$$

where prime denotes derivative with respect to the x-coordinate. These field equations need boundary conditions.

In order to ensure the regularity of the function f and P at $x = 0$,

$$A^\theta(x)|_{r=0} = g^{\theta\theta} A_\theta(x)|_{r=0} \overset{(15b)}{=} \frac{1}{r^2\, e}(n - P(r))\Big|_{r=0} \neq \infty \quad \leftrightarrow \quad \boxed{P(0) = n}, \tag{25}$$

which, combined with the third term of Eq. (24), leads to $f(r = 0) = 0$,

$$(24)|_{x=0} \neq \infty \quad \leftrightarrow \quad \frac{f'(x)}{x}, \frac{P^2(x) f(x)}{x^2} = 0 \quad \xleftrightarrow{(25)} \quad \boxed{f(0) = 0}. \tag{26}$$

The latter can also be inferred from definition (15c), not to make $\phi(r = 0, \theta) = f(r = 0)e^{in\theta}$ uncertain, since the coordinate θ, being an angle, is not defined for $r = 0$.

Differential equations of the Nielsen-Olesen strings, (23-24), are coupled and non-linear and depend on two parameters: (β, n). The winding number n has its origin in the ANO ansatz (15) and is introduced in the system of equations through the boundary conditions, see Eq. (25).

More boundary conditions can be obtained from the requirement of finite string energy density (20), which demands

$$D_i(x)\phi(x),\ |\phi(x)|^2 - 1,\ F_{ij}(x) \xrightarrow{r\to\infty} 0, \tag{27}$$

which means that at large distance from the center in the plane, $r \to \infty$, the string configuration minimizes the scalar potential, $\mathcal{V}(|\phi(x)|)|_{r\to\infty} = 0$, and the behavior of the field $A_\mu(x)$ is such that at $D_i(x)\,\phi(x)\Big|_{r\to\infty} = 0$ and $F_{ij}(x)\Big|_{r\to\infty} = 0$:

$$\diamond \quad \mathcal{V}(|\phi(x)|)\Big|_{r\to\infty} = 0 \quad \leftrightarrow \quad |\phi(r\to\infty)| = 1 \overset{(15c)}{=} \Big|f(r\to\infty)\, e^{in\theta}\Big| = \boxed{f(r\to\infty) = 1}, \tag{28}$$

$$\begin{aligned} \diamond \quad D_\mu(x)\,\phi(x)\Big|_{r\to\infty} = 0 \quad &\xleftrightarrow{(15)} \quad \Big(\partial_\theta - i\, e\frac{n - P(r\to\infty)}{e}\Big)\, f(r\to\infty)\, e^{in\theta} \\ &\overset{(28)}{=} \Big(in - i\big(n - P(r\to\infty)\big)\Big)\, 1\, e^{in\theta} = 0 \quad \leftrightarrow \quad \boxed{P(r\to\infty) = 0}. \end{aligned} \tag{29}$$

Boundary condition (28) indicates that the Higgs field $\phi(x)$ converges towards its vacuum expectation value $|\phi(x)| = 1 \in \mathcal{M}$.

To sum up, the boundary conditions for the equations (23-24) are $P(r = 0) = n$, $f(r = 0) = 0$, $P(\to\infty) \to 0$ and $f(\to\infty) \to 1$. The region between this boundary values must be computed numerically, as it will be explained in more detail.

The core width of both $f(x)$ and $P(x)$ gets wider as β augments. However the width of the gauge field $P(x)$ core seems to increase more rapidly than the core of the scalar field $f(x)$, which is coherent with the fact that increasing β makes strings more repulsive due to longer-range magnetic lines.

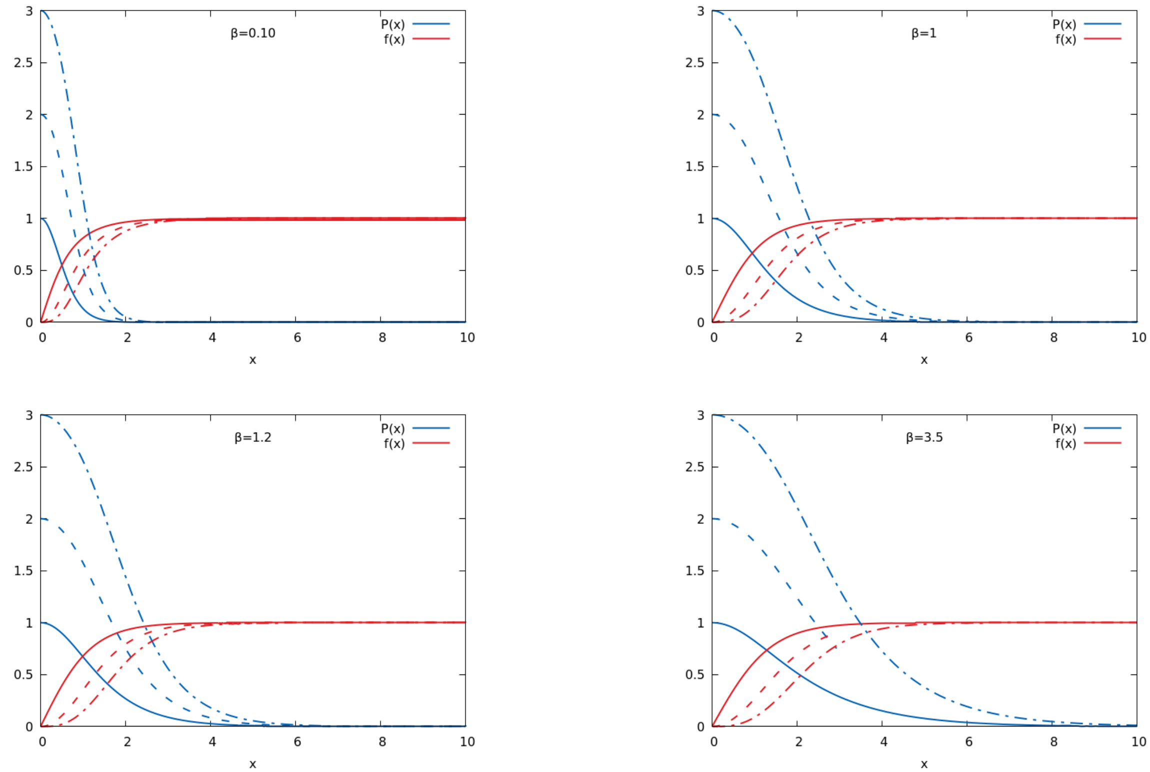

Figure 2: Profiles of the scalar field $f(x)$ and gauge field $P(x)$ on dependence of the distance from the core of the string in units of $1/\sqrt{2\lambda}$, $x = \sqrt{2\lambda}r$ for $n = 1, 2, 3$ –solid, dashed and dotted-dashed, respectively– and several values of β: β =0.1, 1.00, 1.2, 3.5.

3.2 Stability

As first exposed in a seminal paper by Bogomol'nyi, the solutions of the equations of motion are stable provided the energy is bounded from below and the energy of the minimum string energy configuration is equal to this bound. Here, stability refers to straight strings with winding number $n > 1$ not being likely to split into n strings of unit winding number[5].

We will see that this lower bound is a multiple of the winding number n, $E = kn$, $k \in \mathbb{R}$. Moreover, the value of the Bogomol'nyi energy bound corresponds to the critical value of the coupling constant β in Eq. (22), $\beta = 1$, which separates the type I and type II superconducting regimes. For, $\beta = 1$, as it will be proved, the field equations can be reduced from second to first order ODEs, the latter known as Bogomol'nyi equations. [6, p. 7, l. 10-15]

First, notice that the static and axially symmetric nature of ansatz (15) implies, respectively,

$$\begin{aligned} D_0(x)\,\phi(x) &\overset{(15)}{=} \big(\partial_t - 0\big)\, f(r)e^{in\theta} = 0, \quad \text{and} \\ D_3(x)\,\phi(x) &\overset{(15)}{=} \big(\partial_z - 0\big)\, f(r)e^{in\theta} = 0. \end{aligned} \tag{30}$$

[5] Strings with negative winding number can decay into $|n|$ strings with winding number $n = -1$.

Hence, the following useful identity can be obtained, where the colors indicate the terms that cancel out, but must be introduced to achieve the final result,

$$|D_\mu(x)\,\phi(x)|^2 \overset{(30)}{=} |D_1(x)\,\phi(x)|^2 + |D_2(x)\,\phi(x)|^2$$
$$= \Big(D_1(x)\phi(x)\big(D_1(x)\phi(x)\big)^* - iD_1(x)\phi(x)\big(D_2(x)\phi(x)\big)^* + iD_2(x)\phi(x)\big(D_1(x)\phi(x)\big)^*$$
$$+ D_2(x)\phi(x)\big(D_2(x)\phi(x)\big)^*\Big) - i\big(D_1(x)\phi(x)\big)^* D_2(x)\phi(x) + i\big(D_2(x)\phi(x)\big)^* D_1(x)\phi(x)$$
$$= \Big|\big(D_1(x) + iD_2(x)\big)\phi(x)\Big|^2 - i\Big(\big(D_1(x)\phi(x)\big)^* D_2(x)\phi(x) - \big(D_2(x)\phi(x)\big)^* D_1(x)\phi(x)\Big)$$
$$= \Big|\big(D_1(x) + iD_2(x)\big)\phi(x)\Big|^2 - i\Big(\underbrace{-\phi^*(x)\partial_1\big(D_2(x)\phi(x)\big) + i\,e\phi^*(x)A_1(x)D_2(x)\phi(x)}_{-\phi^*(x)D_1(x)D_2(x)\phi(x)}$$
$$\underbrace{+\phi^*(x)\partial_2\big(D_1(x)\phi(x)\big) - i\,e\phi^*(x)A_2(x)D_1(x)\phi(x)}_{\phi^*(x)D_2(x)D_1(x)\phi(x)} + \big(\partial_1\phi^*(x)\big)D_2(x)\phi(x)$$
$$+\phi^*(x)\partial_1\big(D_2(x)\phi(x)\big) - \big(\partial_2\phi^*(x)\big)D_1(x)\phi(x) - \phi^*(x)\partial_2\big(D_1(x)\phi(x)\big)\Big)$$

$$= \Big|\big(D_1(x) + iD_2(x)\big)\phi(x)\Big|^2 + i\phi^*(x)\,[D_1(x), D_2(x)]\,\phi(x)$$
$$- i\Big(\partial_1\big(\phi^*(x)D_2(x)\phi(x)\big) - \partial_2\big(\phi^*(x)D_1(x)\phi(x)\big)\Big)$$
$$\overset{(1)}{=} |(D_1(x) + iD_2(x))\,\phi(x)|^2 + \underbrace{\phi^*(x)\,i\big(-iB\big)\phi(x)}_{B|\phi(x)|^2} + \epsilon^{ijk}\partial_i J_j, \tag{31}$$

where in (31(1)) we used

$$[D_1(x), D_2(x)]\,\phi(x) = \underbrace{\big(\partial_1 - i\,eA_1(x)\big)}_{D_1(x)}\underbrace{\big(\partial_2 - i\,eA_2(x)\big)}_{D_2(x)}\phi(x) - \underbrace{\big(\partial_2 - i\,eA_2(x)\big)}_{D_2(x)}\underbrace{\big(\partial_1 - i\,eA_1(x)\big)}_{D_1(x)}\phi(x)$$
$$= i\Big(- \underbrace{\partial_1 A_2(x)}_{\partial_1 A_2(x) - \cancel{A_2(x)\partial_1}} - \cancel{A_1(x)\partial_2} + \underbrace{\partial_2 A_1(x)}_{\partial_2 A_1(x) + \cancel{A_1(x)\partial_2}} + \cancel{A_2(x)\partial_1}\Big)\phi(x)$$
$$= -i\big(\underbrace{1}_{\epsilon^{120}}\partial_1 A_2(x) \underbrace{-}_{\epsilon^{210}} \partial_2 A_1(x)\big)\phi(x) = -i\big(\epsilon^{ij0}\partial_i A_j\big)\phi(x) \overset{(90)}{=} -iB\phi(x), \tag{32}$$

and $J_i = -i\phi^*(x)\,D_i(x)\,\phi(x)$. (33)

Therefore, the energy density (20) can be rewritten as

$$\frac{E}{l} = \int_{\mathbb{R}^2} d^2x\,\mathcal{H} \overset{(31)}{=} \int_{\mathbb{R}^2} d^2x\,\Big\{\frac{1}{2}\left(|(D_1(x) + iD_2(x))\,\phi(x)|^2 + B|\phi(x)|^2 + \epsilon^{ijk}\partial_i J_j\right)$$
$$+ \frac{1}{2}B^2 + \mathcal{V}\left(|\phi(x)|\right)\Big\} \overset{(1)}{=} \int_{\mathbb{R}^2} d^2x\,\Big\{\frac{1}{2}\,|(D_1(x) + iD_2(x))\,\phi(x)|^2$$
$$+ \frac{B}{2}|\phi(x)|^2 + \frac{B^2}{2} + \frac{e^2\beta}{8}\big(|\phi(x)|^2 - 1\big)^2\Big\}, \tag{34}$$

where, in (34(1)) we substituted $\mathcal{V}(|\phi(x)|) \overset{(10)}{=} \frac{\lambda}{4}\left(|\phi(x)|^2 - 1^2\right)^2$, $\lambda \overset{(22b)}{=} e^2\beta/2$, and

$$\int_{\mathbb{R}^2} d^2x\, \epsilon^{ijk}\partial_i J_j \overset{(1)}{=} \oint_{r\to\infty} d\vec{x}\cdot\vec{J} \overset{(15,33)}{=} \oint dx^\theta\left(-i\phi^*(r)D_\theta\phi(r)\right)_{r\to\infty} \overset{(29)}{=} 0. \tag{35}$$

In (35(1)) we used Stokes theorem. The line integral of the vector current $\vec{J}$ vanishes because spatial components of $U(1)_L$-covariant derivatives have to be zero at spatial infinity, from Eq. (29), since we are considering only finite energy configurations. If we use

$$\begin{aligned}\frac{B}{2}|\phi(x)|^2 + \frac{B^2}{2} &= \frac{1}{2}\left\{B\left(|\phi(x)|^2 - 1\right) + B^2\right\} + \frac{B}{2}\\ &= \frac{1}{2}\underbrace{\left\{B\left(|\phi(x)|^2 - 1\right) + B^2 + \frac{e^2}{4}\left(|\phi(x)|^2 - 1\right)^2\right\}}_{\left\{B+\frac{e}{2}\left(|\phi(x)|^2-1\right)\right\}^2} - \frac{e^2}{8}\left(|\phi(x)|^2 - 1\right)^2 + \frac{B}{2}.\end{aligned} \tag{36}$$

and $\int_{\mathbb{R}^2} d^2x\, B = 2\pi n$, for the last term of Eq. (36), when integrated, Eq. (34) reads

$$\boxed{\begin{aligned}\frac{E}{l} = \pi n + \int_{\mathbb{R}^2} d^2x &\left\{\frac{1}{2}\left|\left(D_1(x) + iD_2(x)\right)\phi(x)\right|^2\right.\\ &\left.+ \frac{1}{2}\left(B + \frac{e}{2}\left(|\phi(x)|^2 - 1\right)\right)^2 + \frac{e^2}{8}(\beta - 1)\left(|\phi(x)|^2 - 1\right)^2\right\}\end{aligned}}. \tag{37}$$

Hence, since all summands except the last one are always positive,

$$\frac{E}{l} \geq \pi n \quad \leftrightarrow \quad (\beta - 1) \geq 0 \quad \leftrightarrow \quad \beta \geq 1. \tag{38}$$

That is, string energy per unit length is bounded from below.

Therefore, when $\beta = 1$, since the configuration under consideration equals the Bogomol'ny bound in Eq. (38), $E/l = n\pi$, strings are stable for any winding number n. In that case, as it can be noted in Fig. 3, n strings of winding number $n = 1$ or one string of winding number n have the same energy, so they are stable. Being rigorous, splitting in strings of lower winding number would not cost any energy to the string of $n > 1$, but strings are considered stable in this case.

If $\beta > 1$, from Eq. (38), the energy is bounded from below, $E/l > n\pi$. However, for $\beta > 1$, there does not exist a static solution with $E/l > n\pi$, since requiring, from Eq. (37), $B + \frac{e}{2}\left(|\phi(x)|^2 - 1\right) = 0$ and $|\phi(x)|^2 - 1 = 0$ implies $B = 0$, which is not allowed due to the conservation of the magnetic flux. This has an effect on the stability of strings: when $\beta > 1$, vortices split. Precisely, higher winding vortices, $n > 1$, break into n vortices with repeal each other, each with a unit of magnetic flux ($\Phi_{\mathrm{B}} = 2\pi n/e$, $n = 1$). Strings with unit winding number, $n = 1$, are stable because they cannot decay into anything else, even if their energy is higher than the Bogomol'ny bound.

As aforementioned, for the critical value $\beta = 1 = \frac{m_s^2}{m_v^2}$, where, from Eq. (22b), $m_s = m_v$ the equality in the energy (per unit length) bound holds, $E/l|_{\beta=1} = \pi n$. From Eq. (34), this is fulfilled if and only if

- $$\left|\left(D_1(x) + iD_2(x)\right)\phi(x)\right|^2 = 0 \quad \leftrightarrow \quad \Big(D^1(x) + iD^\theta(x)\Big)\phi(x) = 0$$
$$= \Big(g^{11}\Big(\partial_1 - i\,eA_1(x)\Big) + ig^{22}\Big(\partial_2 - i\,eA_2(x)\Big)\Big)\phi(x) = 0$$
$$\stackrel{(15)}{=} \Big(\big(f'(r) - 0\big) + \frac{i}{r}\big(in - i(n - P(r))\big)f(r)\Big)\cancel{e^{in\theta}} = 0 = f'(r) - \frac{P(r)f(r)}{r},$$
$$\stackrel{(22a)}{=} \boxed{f'(x) - \frac{P(x)f(x)}{x}}, \quad \text{and} \tag{39}$$

- $$B + \frac{e}{2}\big(|\phi(x)|^2 - 1\big) = 0 \quad \stackrel{(1)}{\longleftrightarrow} \quad P'(r) + \frac{e^2}{2}r\big(f^2(r) - 1\big) = 0 \quad \stackrel{(22a)}{\longleftrightarrow}$$
$$\boxed{P'(x) + \frac{x}{2}\big(f^2(x) - 1\big) = 0}, \tag{40}$$

where in (40(1)) we used

$$B \stackrel{(90)}{=} \epsilon^{ij0}\partial_i A_j \stackrel{(15)}{=} \epsilon^{120}\partial_1 A_2 + \cancelto{0}{\epsilon^{220}}\partial_2 A_2 + \epsilon^{320}\cancelto{0}{\partial_3 A_2} = \frac{\text{sign}(g)}{e\sqrt{g}}\partial_1\big(n - P(r)\big) = -\frac{(-P'(r))}{er}. \tag{41}$$

Thus, in the limit $\beta = 1$, known as Bogomol'nyi-Prasad-Sommerfield (BPS) limit, the equations of motion (23-24) reduce to first order equations (39-40), the so-called Bogomol'nyi equations.

For peace of mind, we want to check the second order equations of motion (23-24), in the case that β equals 1, can be obtained using only BPS equations (39-40):

- $$(39)' = \Big(f'(x) - \frac{P(x)f(x)}{x} = 0\Big)' = f''(x) + \Big(\frac{P(x)f(x)}{x^2} - \frac{P'(x)f(x)}{x} - \frac{P(x)f'(x)}{x}\Big) = 0$$
$$= f''(x) + \underbrace{\left(\frac{f'(x)x}{\cancel{f(x)}}\right)}_{(39)}\frac{\cancel{f(x)}}{x^2} - \underbrace{\left(-\frac{1}{2}\cancel{x}\big(f^2(x) - 1\big)\right)}_{(40)}\frac{f(x)}{\cancel{x}} - \frac{P(x)}{x}\underbrace{\left(\frac{P(x)f(x)}{x}\right)}_{(39)} \tag{42}$$
$$= f''(x) + \frac{f'(x)}{x} + \frac{f(x)}{2}\big(f^2(x) - 1\big) - \frac{P^2(x)f(x)}{x^2} = 0 = (24) \quad \text{QED}, \tag{43}$$

- $$(40)' = P''(x) + \frac{1}{2}\big(f^2(x) - 1\big) + xf(x)f'(x) = 0 \tag{44}$$
$$= P''(x) + \underbrace{\left(-\frac{P'(x)}{x}\right)}_{(40)} + xf(x)\underbrace{\left(\frac{P(x)f(x)}{x}\right)}_{(39)} \tag{45}$$
$$= P''(x) - \frac{P'(x)}{x} + f^2(x)P(x) = 0 = (23) \quad \text{QED}. \tag{46}$$

In Section 5 we will attempt to study numerically whether AHCS vortices can form bound states. Bound state configurations can be formed by two attractive strings, and are usually studied using higher winding solutions (also denoted "excited states"), i.e., solutions with $n > 1$. The string energy per unit winding E/n, as compared to the energy of a single vortex, $n = 1$, gives a good indication on whether n vortices would form a bound state (for $E/n < E(n = 1)$) or not (for $E/n > E(n = 1)$).

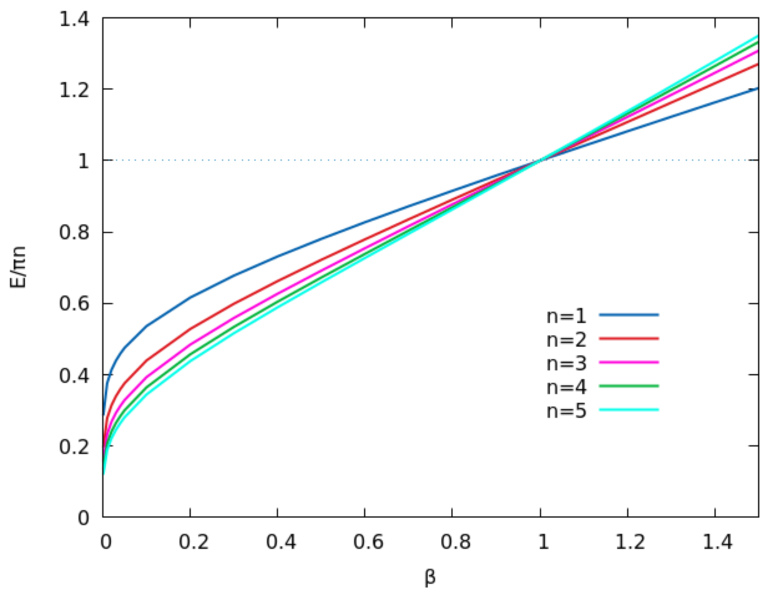

Figure 3: String energy per winding number n (in units of π), $E/\pi n$, as a function of the free parameter β, for the range $\beta \in [0, 1.5]$, for several values of n: $n = 1, 2, 3, 4, 5$, in the Abelian-Higgs model. For $n > 1$ and $\beta < 1$ ($\beta > 1$), strings attract (repel) each other, because it takes more (less) energy to form n strings of $n = 1$, $E/(n = 1) > (<) E/n$. In contrast, at critical coupling $\beta = 1$ (BPS limit), it is equal for all n, $E/\pi n = 1$, so interactions between vortices disappear.

As mentioned, for infinite straight strings, the most stable or minimum energy configuration for a given winding number n at $n > 1$ can be one string of winding number n or n strings of unit winding number $n = 1$. The result, as you can see in Fig. 3, depends on the free parameters of the theory, (β, n). There are two sorts of forces: the scalar field $\phi(x)$ pushes the strings on top of each other in order to minimize the quantity of minima and hence the potential energy; on the other hand, the magnetic field lines related to the vector gauge field $A_\mu(x)$ want to spread out, leading to a repulsive force. The field with the lower mass, i.e., the longer-range force, dominates.

Therefore, when the gauge mass exceeds the scalar mass, $m_v > m_s$ ($\beta = m_s^2/m_v^2 < 1$), the scalar field $\phi(x)$ dominates; thus, energy decreases if strings merge, so they tend to unify. Conversely, for $m_v < m_s$ ($\beta = m_s^2/m_v^2 > 1$), the vector field $A_\mu(x)$ dominates, so strings with $n > 1$ decay into n strings of $n = 1$. These two cases correspond to type I ($\beta < 1$) and type II ($\beta > 1$) superconductors in condensed matter theory, respectively. [7, p. 313, l. 11-16]

Namely, for $\beta < 1$ (type I), the string energy per winding number n decreases with n (left-hand side of Fig. 3), which means that the n strings will attract each other in order to merge into an structure that has less energy; i.e., strings are stable (in other words, potentially form bound states, $E/n < E(n = 1)$) for any n if $\beta < 1$. However, for $\beta > 1$ (type II), the energy per winding number n increases with n (right-hand side of Fig. 3), so being together costs more energy to the n strings and they repeal one another; i.e., for $\beta > 1$, a string with $n > 1$ is unstable to splitting into n strings with vortex topological charge $n = 1$.

We see, therefore, that, as mentioned in Section 2, string formation is possible provided the manifold of vacua $\mathcal{M}$ is not simply connected, but is a necessary, but not sufficient condition for the existence of stable strings. Id est, it does not guarantee the existence of a stable vortex solution. in the AH theory, although all strings with non-zero winding number are non-contractible, we have seen there are no stable vortices with winding number greater than one in the type II ($\beta > 1$) superconductive region.

3.3 Quest for electrically charged vortices

The AH model admits vortices with magnetic fields. It is then natural to enquire: are electrically charged AH strings allowed? They are not, since they lead to infinite energy (per unit length) [9, p. 2232, Eq. (C6)]:

$$Q \equiv \int_{\mathbb{R}^3} d^3x\, \vec{\nabla} \cdot \vec{E} = \int_{S(\infty)} d\vec{S} \cdot \vec{E} - \int_{S(0)} d\vec{S} \cdot \vec{E}, \tag{47}$$

where $S(\infty)$ and $S(0)$ denote cylindrical surfaces at $r = \infty$ and $r = \epsilon$ ($\epsilon \to 0$), respectively. Divergence theorem is used in the second equality. Since $|\vec{E}|(r \to \infty) \to 0$, the charge is nonzero only if $|\vec{E}| \sim 1/r$[6], which implies $|\vec{E}|(r \to 0) \to \infty$.

Still, electrically charged vortices of finite string energy in the $(2+1)$-dimensional AH model do exist, just if the Chern-Simons (CS) term is added [10] –see Section 4–.

[6] Id est, $E_i \overset{(89)}{=} F_{i0} = \partial_i A_0(r) \sim 1/r \leftrightarrow A_0 \sim \log r$.

Chapter 4

Abelian-Higgs-Chern-Simons model

As mentioned, adding the CS term allows AH vortices, otherwise divergent in energy per unit lenght, to have electric charge Q. For our $U(1)$ gauge theory, the Chern-Simons Lagrangian density term takes the form [5, p. 84, Eq. (1.1.3)] [11, p. 9, Eq. (21)]

$$\mathcal{L}_{\mathrm{CS}}(A_\mu(x);\kappa) = -\frac{\kappa}{4}\epsilon^{\lambda\mu\nu}A_\lambda(x)F_{\mu\nu}(x), \tag{48}$$

where $\kappa \in \mathbb{R}_+$ is the Chern-Simons coupling constant or topological mass of the gauge field [10, p. 420, Eq. (1)], which determines the relative strength of the Chern-Simons dynamics, and, $\epsilon^{\lambda\mu\nu}$, the Levi-Civita tensor in $(2+1)$ dimensional space-time.

The CS Lagrangian density (48) is *topological*, i.e., it does not depend on the metric of the background space-time manifold, it depends only on its topology. The indices in $\mathcal{L}_{\mathrm{CS}}$ are contracted with the permutation tensor $\epsilon^{\lambda\mu\nu}$, not with the metric $g_{\mu\nu}$.

The simplest (Abelian) Chern-Simons model is the Abelian-Higgs-Chern-Simons (AHCS) model or Paul-Khare (PK) model, which is the usual Abelian-Higgs model (10) supplemented by the term of Chern-Simons form (48), defined on $(2+1)$ space-time dimensions, with Minkowski metric $(\eta_{\mu\nu}) = \mathrm{diag}(-1,\,1,\,1)$. The Lagrangian density of the model can be expressed formally as [10, p. 420, Eq. (1)][5, p. 90, Eq. (2.1.1)]

$$\begin{aligned}\mathcal{L}_{\mathrm{AHCS}}(A_\mu(x),\phi(x);\lambda,\kappa) &= \mathcal{L}_{\mathrm{AH}}(A_\mu(x),\phi(x);\lambda) + \mathcal{L}_{\mathrm{CS}}(A_\mu(x);\kappa) = (10) + (48)\\ &= \frac{1}{4}\left(F_{\mu\nu}(x)\right)^2 + \frac{1}{2}|D_\mu(x)\,\phi(x)|^2 + \frac{\lambda}{4}\left(\phi^*(x)\,\phi(x) - \eta^2\right)^2 - \frac{\kappa}{4}\epsilon^{\lambda\mu\nu}A_\lambda(x)F_{\mu\nu}(x).\end{aligned} \tag{49}$$

Notice that the Chern-Simons interaction is admitted in gauge theories with odd spacetime dimensions, but has no counterpart in even space-time dimensions. This is the reason why, when adding the CS form to the AH model in Eq. (49), the space-time manifold was reduced from three to two spatial dimensions, $(3+1) \to (2+1)$.

Unlike the terms in $\mathcal{L}_{\mathrm{AH}}$ (10), $\mathcal{L}_{\mathrm{CS}}$ (48) is *not* invariant under the local gauge transformation (6), [5, p. 84, Eq. (1.1.4)]

$$\mathcal{L}_{\mathrm{CS}} \xrightarrow{U(1)_L,\,(6\mathrm{b}),\,(9)} -\frac{\kappa}{4}\epsilon^{\sigma\mu\nu}\big(A_\sigma(x) + \partial_\sigma\alpha(x)\big)F_{\mu\nu}(x) \overset{(48)}{=} \mathcal{L}_{\mathrm{CS}} \underbrace{-\frac{\kappa}{4}\partial_\sigma\Big(\epsilon^{\sigma\mu\nu}\alpha(x)F_{\mu\nu}(x)\Big)}_{\delta\mathcal{L}_{\mathrm{CS}}}. \tag{50}$$

Nevertheless, the change in the CS Lagrangian density under a $U(1)$ gauge transformation is a total space-time derivative, so its equations of motion

$$
\begin{aligned}
&-\frac{\kappa}{4}\epsilon^{\alpha\beta\gamma}\left\{\partial_\mu \frac{\partial}{\partial\big(\partial_\mu A_\nu(x)\big)} - \frac{\partial}{\partial A_\nu(x)}\right\}\left\{A_\alpha(x)\big(\overbrace{\partial_\beta A_\gamma(x) - \partial_\gamma A_\beta(x)}^{F_{\beta\gamma}(x)}\big)\right\} = 0 \quad \leftrightarrow \\
&-\frac{\kappa}{4}\epsilon^{\alpha\beta\gamma}\left\{\partial_\mu\big(A_\alpha(x)(\delta^\mu_\beta\delta^\nu_\gamma - \delta^\mu_\gamma\delta^\nu_\beta)\big) - \delta^\nu_\alpha F_{\beta\gamma}(x)\right\} = 0 \quad \leftrightarrow \\
&-\frac{\kappa}{4}\Big\{\big(\epsilon^{\alpha\mu\nu}\partial_\mu A_\alpha(x) - \epsilon^{\alpha\nu\mu}\partial_\mu A_\alpha(x)\big) - \epsilon^{\nu\beta\gamma}F_{\beta\gamma}(x)\Big\} = 0 \quad \leftrightarrow \\
&-\frac{\kappa}{2}\underbrace{\epsilon^{\alpha\mu\nu}}_{\epsilon^{\nu\alpha\mu}}\underbrace{F_{\mu\alpha}(x)}_{(-F_{\alpha\mu}(x))} = 0 \quad \leftrightarrow \quad \boxed{\frac{\kappa}{2}\epsilon^{\nu\alpha\mu}F_{\alpha\mu}(x) = 0},
\end{aligned}
\tag{51}
$$

are gauge invariant, from Eq. (9).

On the other hand, the equations of motion –see Appendix A.3.3– corresponding to the whole AHCS theory (49) read

$$
(12) = \big(D_\mu(x)D^\mu(x)\phi(x)\big) - \lambda\big(|\phi(x)|^2 - 1\big)\phi(x) = 0, \tag{52}
$$

$$
(101) \quad \leftrightarrow \quad \partial_\mu F^{\mu\nu}(x) + \frac{\kappa}{2}\epsilon^{\nu\alpha\mu}F_{\alpha\mu}(x) = j^\nu(x) = \frac{ie}{2}\Big(\phi^*(x)D^\nu(x)\phi(x) - \phi(x)\big(D^\nu(x)\phi(x)\big)^*\Big); \tag{53}
$$

and the total energy (in this model, remember, in $(2+1)$d) is

$$
E = \int d^2x\left\{\frac{1}{2}|D_i(x)\,\phi(x)|^2 + \frac{1}{2}\Big(E^2 + B^2\Big) + \mathcal{V}(|\phi(x)|) - \frac{1}{2}e^2A_0^2(x)|\phi(x)|^2 + \kappa A_0(x)B\right\}, \tag{54}
$$

where the electric and magnetic fields, respectively, are given by $E_i = F_{0i}$ and $B^k = \frac{1}{2}\epsilon^{ijk}F_{ij}$. Notice that, since the CS term does not couple to the scalar field, the equation of motion corresponding to the Higgs field $\phi(x)$, Eq. (53), is exactly the same as in the AH model.

Solutions to the CS theory can be sought along the same lines as in the ANO ansatz (15). Here, they are also radial and static, but with $A_0 = V(r)$ and reducing from $(3+1)$d cylindrical coordinates to $(2+1)$d polar coordinates $x^\mu = (t, r, \theta)$, with $x = r\cos\theta$, $y = r\sin\theta$ and metric tensor $(g_{\mu\nu}) = \mathrm{diag}(-1, 1, r^2)$. As mentioned, $A_0 = V(r)$ is introduced because we look for vortices with electrical charge Q, but if the AH model was not supplemented with the CS form –recall Section 3.3– it would lead to infinite energy density, which is not allowed. On the other hand, the reduction from $(3+1)$ to $(2+1)$ is just to make CS indices match up. The new ansatz reads

$$
A_t = V(r), \quad A_r = 0, \quad A_\theta = \frac{1}{e}\left(n - P(r)\right), \quad \phi(r,\theta) = f(r)e^{in\theta}, \tag{55}
$$

with winding number $n \in \mathbb{Z}$, and $V(r)$,$P(r)$,$f(r) \in \mathbb{R}$.

Then, the energy of a static, radial configuration is obtained substituting ansatz (55) in Eq. (54)

$$
\begin{aligned}
E &= \int_{\mathbb{R}^2} d^2x\ \mathcal{H} = 2\pi \int_{\mathbb{R}_+} dr\, r\, \mathcal{H} \\
&= 2\pi \int_{\mathbb{R}_+} rdr \left\{ \frac{1}{2}(-ieV(r)\, f(r))^2 + \frac{1}{2}(f'(r))^2 + \frac{1}{2}\frac{f^2(r)}{r^2}(P(r))^2 \right. \\
&+ \left. \frac{1}{2}\Big((V'(r))^2 + \frac{(-P'(r))^2}{e^2}\Big) + \frac{\lambda}{4}\left(f^2(r)-1\right)^2 - \frac{1}{2}e^2V^2(r)f^2(r) + \kappa V(r)\frac{(-P'(r))}{e} \right\} \\
&\overset{(55,22)}{=} \pi \int_{\mathbb{R}_+} dx\, x \left(V'^2(x) + \alpha^2\frac{P'^2(x)}{x^2} + f'^2(x) + \frac{P^2(x)f^2(x)}{x^2} - \frac{V^2(x)f^2(x)}{\alpha^2} + \right. \\
&\left. \frac{1}{2}\left(f^2(x)-1\right)^2 - 2\gamma\frac{V(x)P'(x)}{x} \right)
\end{aligned}
\tag{56}
$$

As we shall show later, if the CS term is added to the AH Lagrangian density, the corresponding equations of motion demand $A_0 = V(x) \neq 0$ for a non-zero coupling of the CS interaction, $\kappa \neq 0$. In Eq. (56) we have used

$$
B = \frac{1}{2}\epsilon^{ij0}F_{ij} = \frac{1}{2}\left(\epsilon^{120}F_{12} + \epsilon^{210}F_{21}\right) = \frac{1}{2}\left(\epsilon^{120}F_{12} + (-\epsilon^{120})(-F_{12})\right) = \epsilon^{120}F_{12} \tag{57}
$$

$$
\begin{aligned}
&\overset{(83)}{=} F_{12} = F_{r\theta} = \partial_r A_\theta - \partial_\theta A_r \overset{(55)}{=} \partial_r\left(\frac{n-P(r)}{e}\right) - 0 = -\frac{P'(r)}{e} = -\frac{1}{e}\frac{dP(x)}{dx}\frac{dx}{dr} \\
&\overset{(22a)}{=} -\frac{\sqrt{2\lambda}}{e}\frac{dP(x)}{dx} \overset{(22b)}{=} -\sqrt{\beta}P'(x).
\end{aligned}
\tag{58}
$$

For ensuring the stability of the ground states, the integrand of the total energy must be positive definite. The positive definiteness of the energy of the AHCS vortices is indeed ensured. As the functions and parameters present in Eq. (56) are real and positive, only minus signed terms are relevant in order to prove the above statement,

$$
\begin{aligned}
\tilde{E} &= \int_{\mathbb{R}_+} x\,dx \left[-2\gamma\frac{V(x)P'(x)}{x} - \frac{V^2(x)f^2(x)}{\alpha^2}\right] \overset{(62)}{=} \int_{\mathbb{R}_+} x\,dx\left[-\gamma\frac{V(x)P'(x)}{x} - \frac{(xV'(x))'}{x}V(x)\right] \\
&\int_{\mathbb{R}_+} x\,dx\left[-\gamma\frac{V(x)P'(x)}{x}\right] - V(x)xV'(x)\Big|_0^\infty + \int_{\mathbb{R}_+} dx\, xV'^2(x) \\
&\overset{(69,\,63)}{=} \int_{\mathbb{R}_+} x\,dx\left[-\gamma\frac{V(x)P'(x)}{x} + V'^2(x)\right],
\end{aligned}
\tag{59}
$$

and, as it can be noted in Fig. 5, $P(x)$ is a decreasing function for $\mathrm{dom}(P(x)) = [0,\infty)$, so $P'(x) < 0$ and then the AHCS energy is *positive definite*, $E > 0$.

After rescaling (22) and the substitution of $\gamma \equiv \frac{\kappa}{e}$ and ansatz (55), the resulting equations of motion that the functions f, P and V must satisfy are

$$(106) \quad \leftrightarrow \quad \left(\frac{P'(x)}{x}\right)' = \frac{\gamma}{\alpha^2}V'(x) + \frac{f^2(x)P(x)}{\alpha^2 x}, \tag{60}$$

$$(107) \quad \leftrightarrow \quad \frac{(xf'(x))'}{x} = \frac{1}{2}f(x)\left(f^2(x)-1\right) + \frac{f(x)V^2(x)}{\alpha^2} + \frac{f(x)P^2(x)}{x^2}, \tag{61}$$

$$(104) \quad \leftrightarrow \quad \frac{(xV'(x))'}{x} = \frac{f^2(x)V(x)}{\alpha^2} + \gamma\frac{P'(x)}{x}. \tag{62}$$

where prime denotes derivative with respect to the x-coordinate. As expected, for $V(x) = 0$ and $\kappa = 0$, these reduce to the equations of the Abelian-Higgs model. Also, as said in advance, from Eq. (62), it can be deduced that $A_0(r)$ must be nonzero for $\gamma \neq 0$.

As it happened also for the AH theory, no exact analytical solution has yet been found for the equations of motion of the AHCS model (60-62). They have to be determined numerically, subject to a given set of appropriate boundary conditions.

The general solution of a differential equation of order m depends on m parameters, m integration constants. Then, we need $3 \times (m = 2) = 6$ boundary conditions. They are obtained as follows:

- At the origin ($r = 0$), the fields must be non-singular, so, to preserve the continuity of $\phi(x)$ and $A^\mu(x)$ at the origin, we have the following behavior, respectively,

- $$(62)\Big|_{x=0} \quad \leftrightarrow \quad \left\{\frac{(xV'(x))'}{x} = \frac{f^2(x)V(x)}{\alpha^2} + \gamma\frac{P'(x)}{x}\right\}\Big|_{x=0} \neq \infty$$
$$\xleftrightarrow{(65)} \quad \frac{V'(x)}{x}\Big|_{x=0} = 0 \quad \leftrightarrow \quad \boxed{V'(0) = 0}. \tag{63}$$

- $$A^\theta(x)|_{r=0} = g^{22}A_2(x)|_{r=0} \overset{(55c)}{=} \frac{n - P(r)}{e\,r^2}\Big|_{r=0} \neq \infty$$
$$\leftrightarrow \quad n - P(0) = 0 \quad \leftrightarrow \quad \boxed{P(0) = n}. \tag{64}$$

- $$(61)\Big|_{x=0} \quad \leftrightarrow \quad \left(\frac{(xf'(x))'}{x} = \frac{1}{2}f(x)\left(f^2(x)-1\right) - \frac{f(x)V^2(x)}{\alpha^2} + \frac{f(x)P^2(x)}{x^2}\right)\Big|_{x=0} \neq \infty$$
$$\leftrightarrow \frac{f'(x)}{x}, \frac{f(x)P^2(x)}{x^2}\Big|_{x=0} \neq \infty \quad \xleftrightarrow{(64)} \quad \boxed{f(0) = 0}, \quad \boxed{f'(0) = 0}. \tag{65}$$

  In fact, as in the AH model, $f(r = 0) = 0$ could have been deduced before Eq. (65), since, from ansatz (55d), $\phi(r,\theta) = f(r)e^{in\theta}$, whose polar angle θ is not defined in the origin ($r = 0$)

of the coordinate system. Then, $f(r=0)=0$ is a necessary choice not to make $\phi(r=0,\theta)$ uncertain. Besides, the boundary conditions (63) and (64) can be partially checked using

$$
\begin{aligned}
(60)\Big|_{x=0} \quad &\leftrightarrow \quad \left\{\left(\frac{P'(x)}{x}\right)' = \frac{\gamma}{\alpha^2}V'(x) + \frac{f^2(x)P(x)}{\alpha x}\right\}\Big|_{x=0} \neq \infty \\
\xleftrightarrow{(65)} \quad & P'(0), V'(0) = 0 \quad \to \quad P(0), V(0) = \text{const.}
\end{aligned} \tag{66}
$$

- For large-r asymptotics ($r=\infty$), that is, at large distances from the axis of symmetry, we will require, as in the AH model, all terms in the total energy, Eq. (56), be null. In fact, although there are negative terms in Eq. (56) that could be suspected of lowering the energy for spatial infinity, we have proved the energy is positive-definite. Therefore, the asymptotic form of the fields at infinity is

$$\diamond \quad \frac{1}{2}\left(f^2(x\to\infty)-1\right)^2 = 0 \quad \leftrightarrow \quad \boxed{f(x\to\infty)=1}; \quad f'(x\to\infty)=0, \tag{67}$$

$$\diamond \quad \frac{P^2(x)f^2(x)}{x^2}\Big|_{x\to\infty} = 0 \quad \xleftrightarrow{(67)} \quad \boxed{P(x\to\infty)=0}; \quad \frac{P'^2(x)}{x^2}\Big|_{x\to\infty} = 0 = P'(x\to\infty)=0, \tag{68}$$

$$\diamond \quad V^2(x)f^2(x)\Big|_{x\to\infty} \quad \xleftrightarrow{(67)} \quad \boxed{V(x\to\infty)=0}; \quad V'(x\to\infty)=0. \tag{69}$$

From Eq. (67), $f(r\to\infty)=1\in\mathcal{M}$, it can be noted, as in the AH theory, that a necessary condition for a configuration to have finite string energy is that the asymptotic Higgs field must lie entirely in the manifold of degenerate vacua $\mathcal{M}$, or, which is the same once ansatz (55d) has been implemented, $\phi(x)$ must tend to a pure phase at infinity, $\phi(r\to\infty,\theta)\to e^{in\theta}$.

The magnetic field reads

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\hat{z} \overset{(55)}{=} \frac{n-(rP(r))'}{er}\hat{z}, \tag{70}$$

which is the same as in the AH theory, Eq. (19), and, also as in the AH model, its flux is quantized in units of $2\pi/e$ [4, p. 3456, Eq. (2.2.1)][10, p. 421, Eq. (6)].

What is different from the AH model is that, since $A_t(x) \overset{(55a)}{=} V(r)\neq 0$ is admitted, the vortex has an electric field, as well,

$$E_i = F_{i0} = \partial_i A_0 - \partial_0 A_i = \partial_i V(r) = \partial_r V(r) = \boxed{V'(r) = E_r}, \tag{71}$$

which is also quantized, but in units of $\kappa 2\pi/e$, and hence, remarkably, related to the magnetic flux, as follows from the integration of the Gauss law, i.e., the field equation for the gauge field $A_\mu(x)$, Eq. (101), with $\nu = 0$

$$
\begin{aligned}
Q &= \int d^2x \, \left(\partial_i E^i - \kappa B\right) \overset{(53)}{=} \int d^2x \, j^0 \overset{(1)}{=} -e^2 \int_{\mathbb{R}} 2\pi r \, dr \, V(r) f^2(r) \overset{(22)}{=} -\frac{1}{\alpha} \int_{\mathbb{R}} 2\pi x \, dx \, V(x) f^2(r) \\
&\overset{(62)}{=} -\int_{\mathbb{R}} 2\pi x \, dx \left(\frac{(xV(x))'}{x} - \gamma \frac{P'(x)}{x} \right) = -2\pi \left(xV(x) - \frac{\kappa}{e} P(x) \right) \Big|_0^\infty \\
&\overset{(2)}{=} -2\pi \left(0 - \frac{\kappa}{e}(0 - n) \right) = -\kappa \frac{2\pi n}{e} = -\kappa \Phi_B,
\end{aligned} \tag{72}
$$

where, comparing the first and last equality, definition $\Phi_B = \int_{\mathbb{R}^2} d^2x \, B$, used in Eq. (21), can be checked. In (72(1)) we employed

$$
\begin{aligned}
j^0 &\overset{(103)}{=} \frac{ie}{2} \left(\phi^*(x) \partial^0 \phi(x) - \phi(x) \partial^0 \phi^*(x) - 2i \, e A^0(x) |\phi(x)|^2 \right) = \frac{ie}{2} (-2i \, e(-V(r)) f^2(r)) \\
&= -e^2 V(r) f^2(r),
\end{aligned} \tag{73}
$$

and, in (72(2)), boundary conditions (64), (68) and (69).

Besides, from this definition of the electric charge, Eq. (72), since its explicit dependence on the coefficient of the CS term κ, it is straightforward to check that in the absence of a term of the CS form, i.e., for the AH model ($\kappa = 0$), the electric charge is not supported, $Q = 0$.

The equations of motion (60-62), together with the boundary conditions (65-69), must be solved numerically. We will employ the COLSYS solver developed by Ascher et al. [12].

Chapter 5

Numerical results

The equations of motion (60-62) are solved numerically, subject to the boundary conditions (65-69), employing the COLSYS solver [12].

In COLSYS package, the problem is solved on a sequence of adaptive meshes until the required accuracy is reached, when the deviation from the true solution is below a prescribed error tolerance selected by the user. Adaptive meshes adjust the precision in specific regions which need more accuracy, while leaving other areas at resolution. COLSYS works very well when the initial approximate solution is close to the true solution.

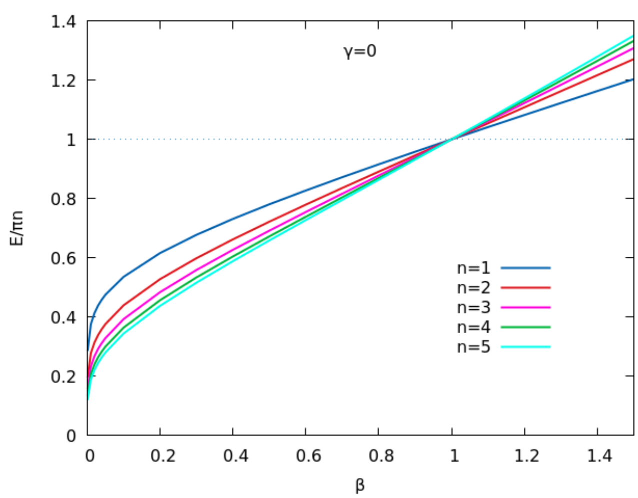

Figure 4: String energy per winding number n in units of π, $E/\pi n$, as a function of the parameter α, for $n = 1, 2, 3, 4, 5$ in the Abelian-Higgs model ($\gamma = \kappa/e = 0$, $A_0 = 0$), as appeared in Fig. 4, but with the CS coupling explicitly expressed. For $n > 1$ and $\beta < 1$ ($\beta > 1$), strings attract (repel) each other, since $E/(n = 1) > (<)E/n$. For $\beta = 1$, $E/(n = 1) = E/n$, so they do not interact, they do not merge neither split.

Using COLSYS to solve our equations of motions, we will study if AHCS vortices form bound states. Bound states, as anticipated in Section 3.2, can be formed provided $E/n < E(n = 1)$. This E/n is the string energy (in two spatial dimensions) with respect to the winding number of the string, since otherwise it would always yield $E(n_1) > E(n_2)$ for $n_1 > n_2 > 0$. We therefore have that if, e.g., $E/(n = 2) > E(n = 1)$, the energy of the $n = 2$ string is greater than the sum of the two $n = 1$ strings in which it splits.

Our theory will depend in one more parameter than the AH model, γ; that is, it will depend on

(β, n, γ). Moreover, we have a new kind of force. For the AH model, we saw that there were two sorts of forces: the force due to the scalar field $\phi(x)$, which pushes the strings on top of each other, and the force due to the magnetic fields, which is repulsive. Nevertheless, in the AHCS model we have one more field, the electric field $E_r = V'(r)$. We will analyse what kind of force the latter exerts. One could wonder whether the $\vec{E}$ field produced by this CS term would conspire against the magnetic field $\vec{B}$, and thus make the force less repulsive; or, on the contrary, it would align itself with $\vec{B}$ and contribute to the repulsive force.

First, we plot the profiles of the gauge $P(x)$ and scalar $f(x)$ functions and $V(x)$ as a function of the distance from the axis of symmetry of the string in units of $1/\sqrt{2\lambda}$, $x = \sqrt{2\lambda}r$ –recall Eq. (22a)–. We do this for several values of the free parameters, in a series of plots. In all of these plots, we display the profiles for three winding numbers corresponding to $n = 1$ (solid), $n = 2$ (dashed) and $n = 3$ (dotted-dashed). In Fig. 5 we show the profiles for $\beta = 0.1$, $n = 1, 2, 3$ and $\gamma = 0, 0.1, 0.5, 1, 1.5, 5$. Fig. 6 is the analogous of Fig. 5, but with $\beta = 1$, and, so is Fig. 7, but with $\beta = 3.5$.

The core width of both the scalar $f(x)$ and gauge $P(x)$ fields gets wider as β augments. As a function of γ both $f(x)$ and $P(x)$ also grow, but the core width of the gauge field $P(x)$ increases more rapidly than the core of the scalar field $f(x)$ –note the point where the $P(x)$ crosses the $f(x)$ of the same line type, i.e., same winding number–. The fact that the $P(x)$ increases faster as a function of γ suggests the electric field influences on making the the forces between strings more repulsive.

The attraction or repulsion can be better studied using the energy, rather than using the profile functions introduced above. The most meaningful comparison would be the energy per winding number n, E/n. In this way, for example, we can compare the energy of n strings with winding number one with one string with winding number n.

We compute the string energy (normalized by the winding number) as a function of the coupling constants β and κ, projecting them in an energy-coupling plane, i.e., fixing the coupling constants alternatively. Besides, the evolution of the parameter space position of the boundary between type I (attractive) and type II (repulsive) superconductive regimes is studied. This approach should give a conclusive result on the effect a term of Chern-Simons form has when added to the standard Abelian-Higgs (AH) theory.

In Fig. 8 we show the energy per winding number n (in units of π), $E/n\pi$, for several fixed values κ (or, equivalently, the CS coupling constant κ in units of e, $\gamma = \kappa/e$) and n, as a function of β. As it can be inferred from this plots, the presence of the CS term always makes the vortices repel one another. This can be seen noting the place where the three lines (corresponding to different n) cross for every γ.

For the $\gamma = 0$ (that is, the AH model, Fig. 4, or, its tantamount, the first plot in Fig. 8), this crossing happens at $\beta = 1$. For values of β lower than that, strings attract; in other words, $E/n\pi$ is lower for larger n. So, for example, two $n = 1$ strings require more energy than one $n = 2$ string. As γ is increased, the crossing of the lines takes place at lower β (the higher the γ, the lower the β), thus decreasing the attractive region. This occurs until eventually strings repel for every value of β. Note also that $E/n\pi$ is monotonically increasing with respect to β.

Figure 9 shows the analogous case, plotting $E/n\pi$ for several fixed n and β, and varying γ. Here, it is maybe even more apparent that the attractive region is shrinking until it disappears completely. Clearly, also, for $\beta > 1$, there is no attractive region; i.e., the CS does not change the behavior for that region.

For instance, in the plot corresponding to fixed $\beta = 1$ in Fig. 9, it can be discerned that for $\gamma = 0$ –i.e., when the AHCS model reduces to the ordinary AH– the string energy per winding number is the same for all winding numbers –as should be–. However, for bigger values of γ, energy per unit winding number, $E/(n = 1)$, stays below the rest, below energy per winding number $n > 1$, $E/(n = 1) < E/n$, thus making splitting into strings of $n = 1$ the most probable option, being the configuration that requires the smallest amount of energy. That is, growing γ makes strings unbounded. Note also that, as for β, the string energy per winding number (in units of π), $E/\pi n$, increases monotonically with γ, for all winding numbers n.

Lastly, in Fig. 10, we have obtained the lines in the (β, γ) parameter space plane where the system goes from attractive to repulsive. In other words, we have obtained the lines which are the analogous to the Bogomol'ny bound for the AH case.

In order to do that, we have calculated the values of β and γ for which the energies per winding number (in units of π), $E/n\pi$, for different winding number n, are equal. In Fig. 8 and Fig. 9, this correspond to the points where the $E/n\pi$ curves cross.

Once again, it can be clearly seen that as γ grows, the equality line happens for lower and lower values of β, until $\beta = 0$ (beyond which our model makes no sense). The CS interaction makes the point in parameter space for which string energies per different winding number are equal –or, equivalently, for which there is no interaction between vortices; id est, the boundary between type I (attractive strings) and type II (repulsive strings)– approach $\beta = 0$. Notice from this figure how increasing γ reduces the value of β– from $\beta = 1$ to $\beta = 0$, the minimum possible value of β, and eventually the whole region corresponds to repulsive strings. Once $\beta = 0$ (which means either $\lambda = 0$ or $e = \infty$) is reached, we exclude the results for bigger values of γ.

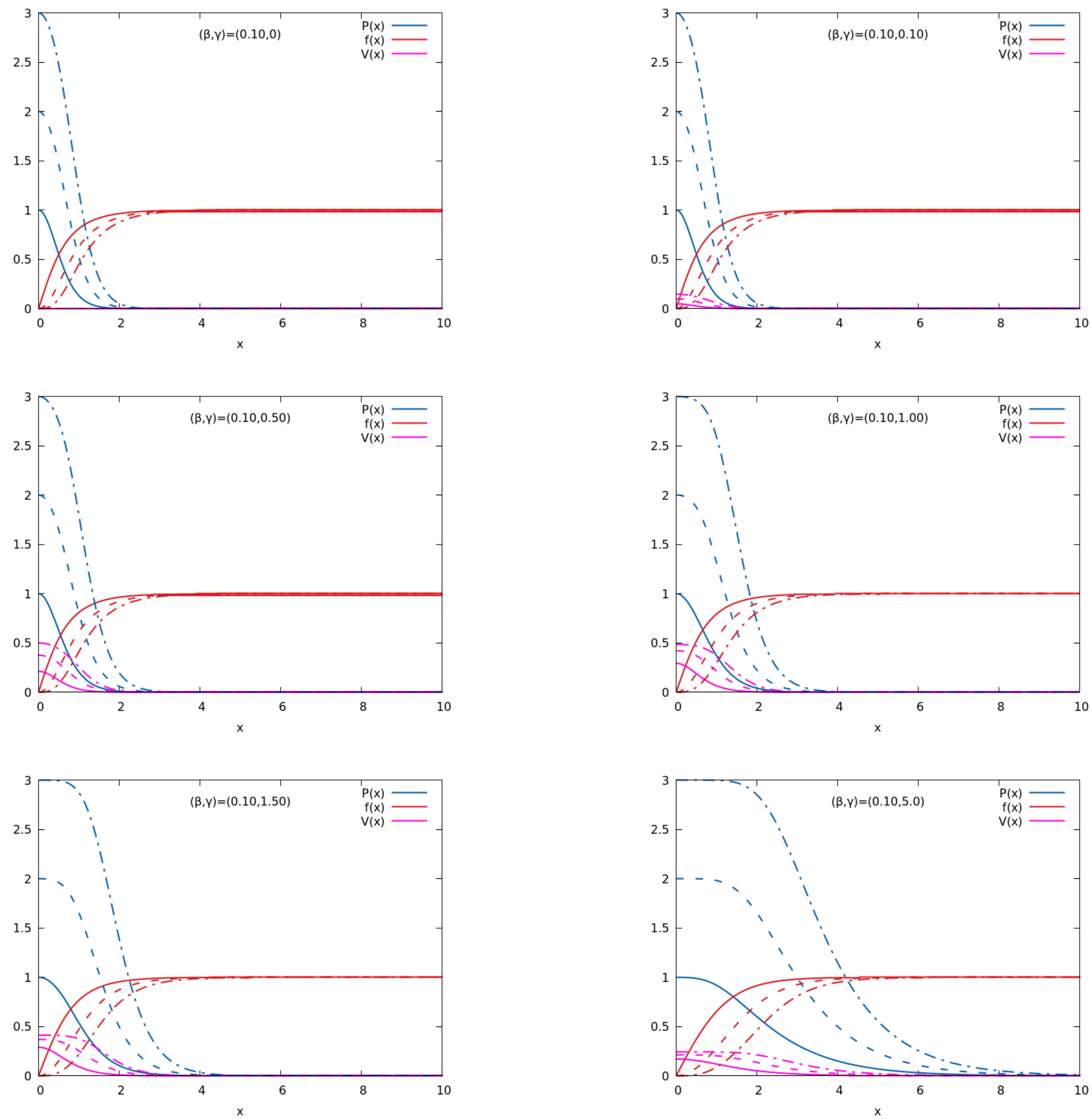

Figure 5: Profiles of the gauge and scalar field functions –$P(x)$ and $f(x)$, respectively– and $V(x)$ as a function of the distance from the center of the vortex (in units of $1/\sqrt{2\lambda}$), $x = \sqrt{2\lambda}r$ for $n = 1, 2, 3$ –solid, dashed and dotted-dashed, respectively–, $\beta = 0.1$ and several values of γ: $\gamma = 0$, 0.1, 0.5, 1, 1.5, 5.

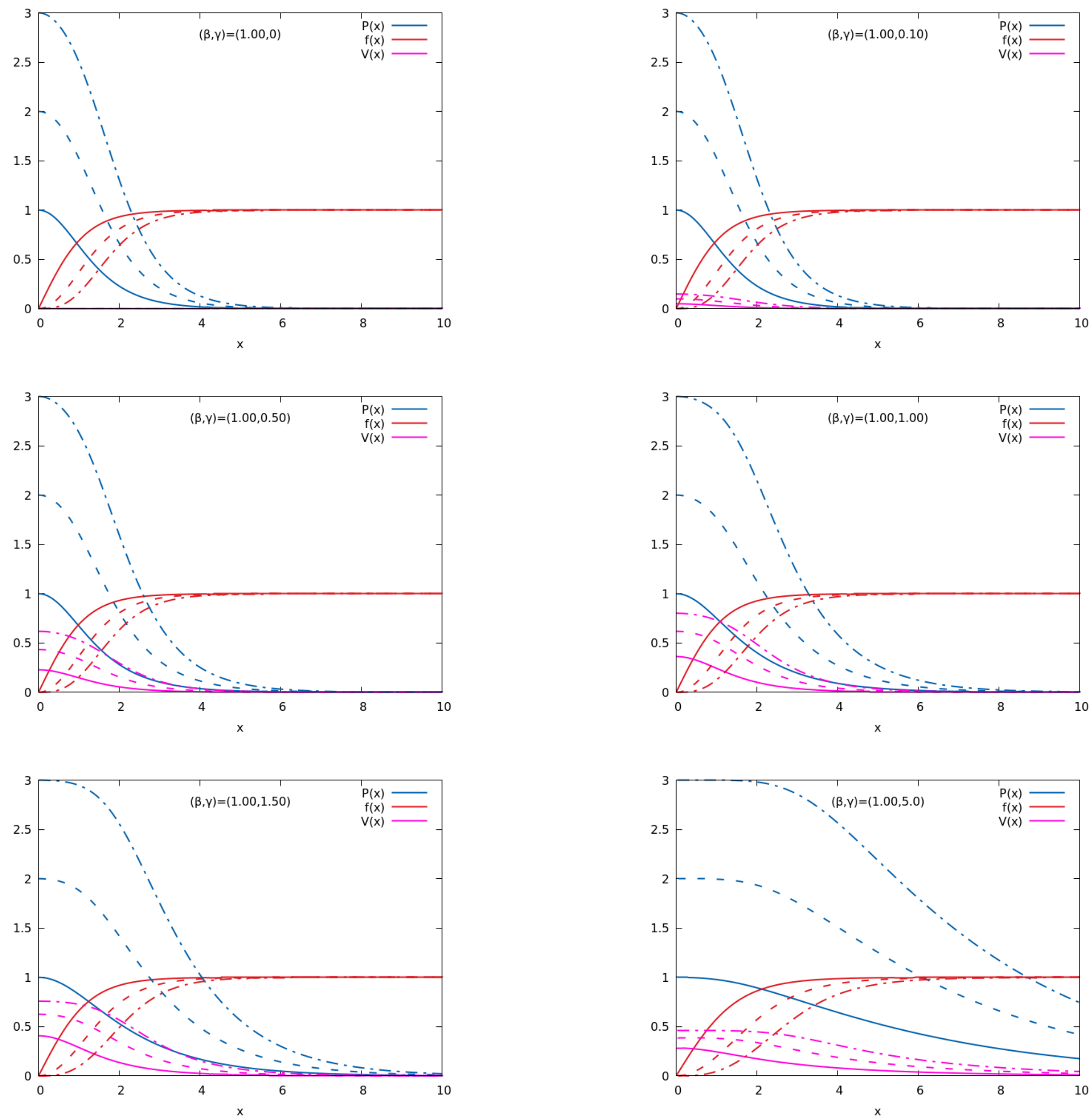

Figure 6: Profiles of the gauge and scalar field functions –$P(x)$ and $f(x)$, respectively– and $V(x)$. Same as Fig. 5, but for $\beta = 1.00$.

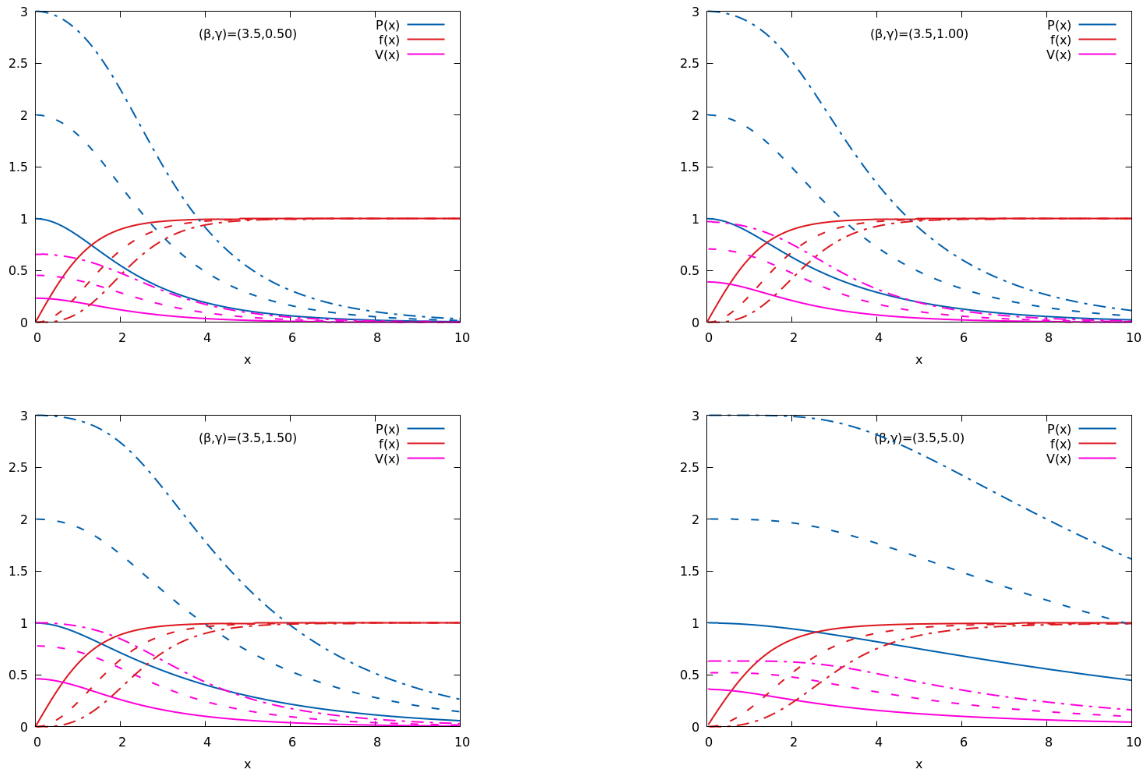

Figure 7: Profiles of the gauge and scalar field functions $-P(x)$ and $f(x)$, respectively– and $V(x)$. Same as Fig. 5 and Fig. 6, but for $\beta = 3.5$. The plots for $\beta = 3.5$ and $\gamma = 0$, 0.1 are not showed due to computational errors.

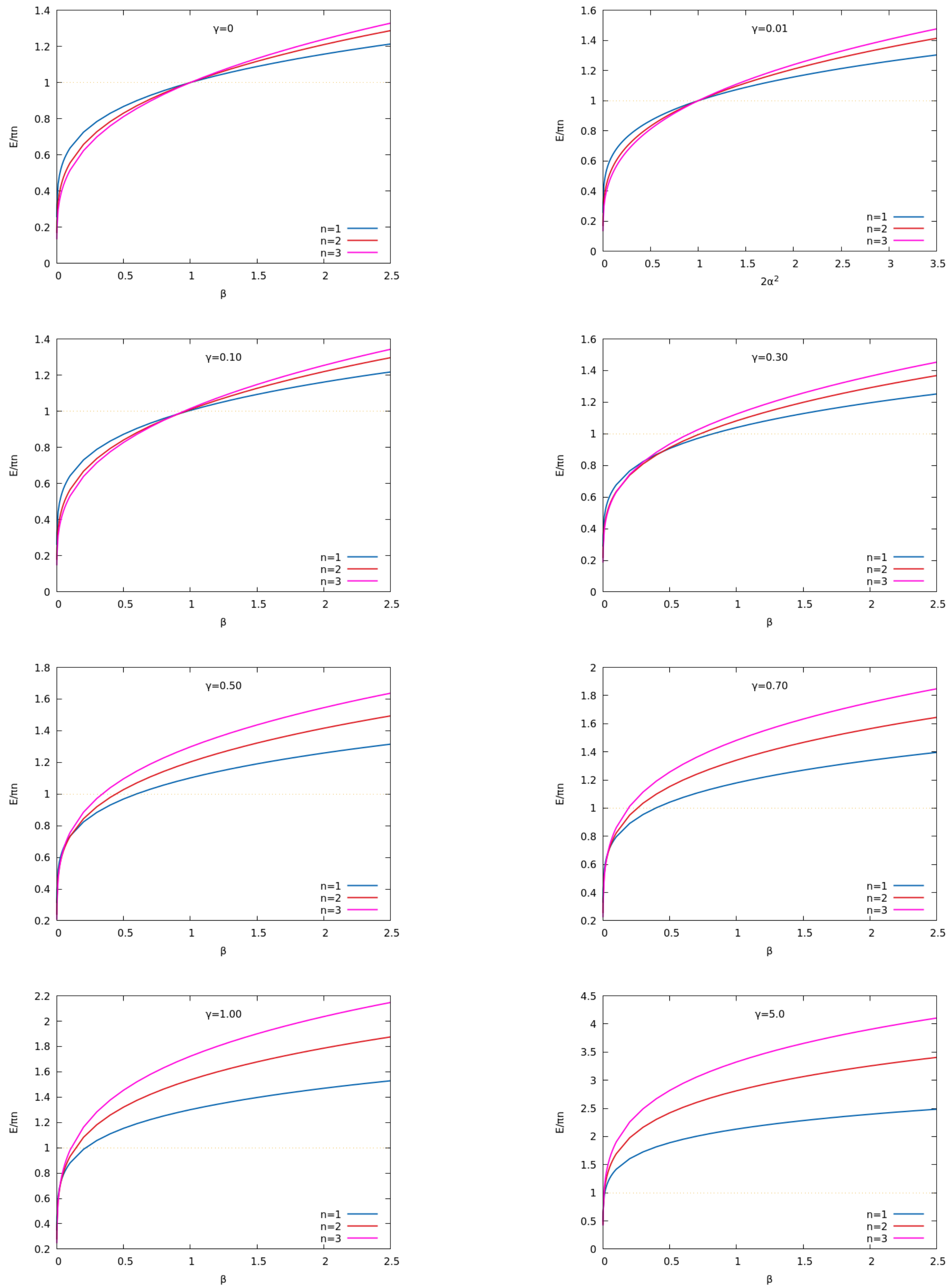

Figure 8: The energy of strings (in two spatial dimensions) per winding number n in units of π, $E/\pi n$, for several fixed values of the CS coupling γ: $\gamma = 0$, 0.01, 0.1, 0.3, 0.5, 0.7, 1, 5; and n: $n = 1, 2, 3$, as a function of the coupling constant β.

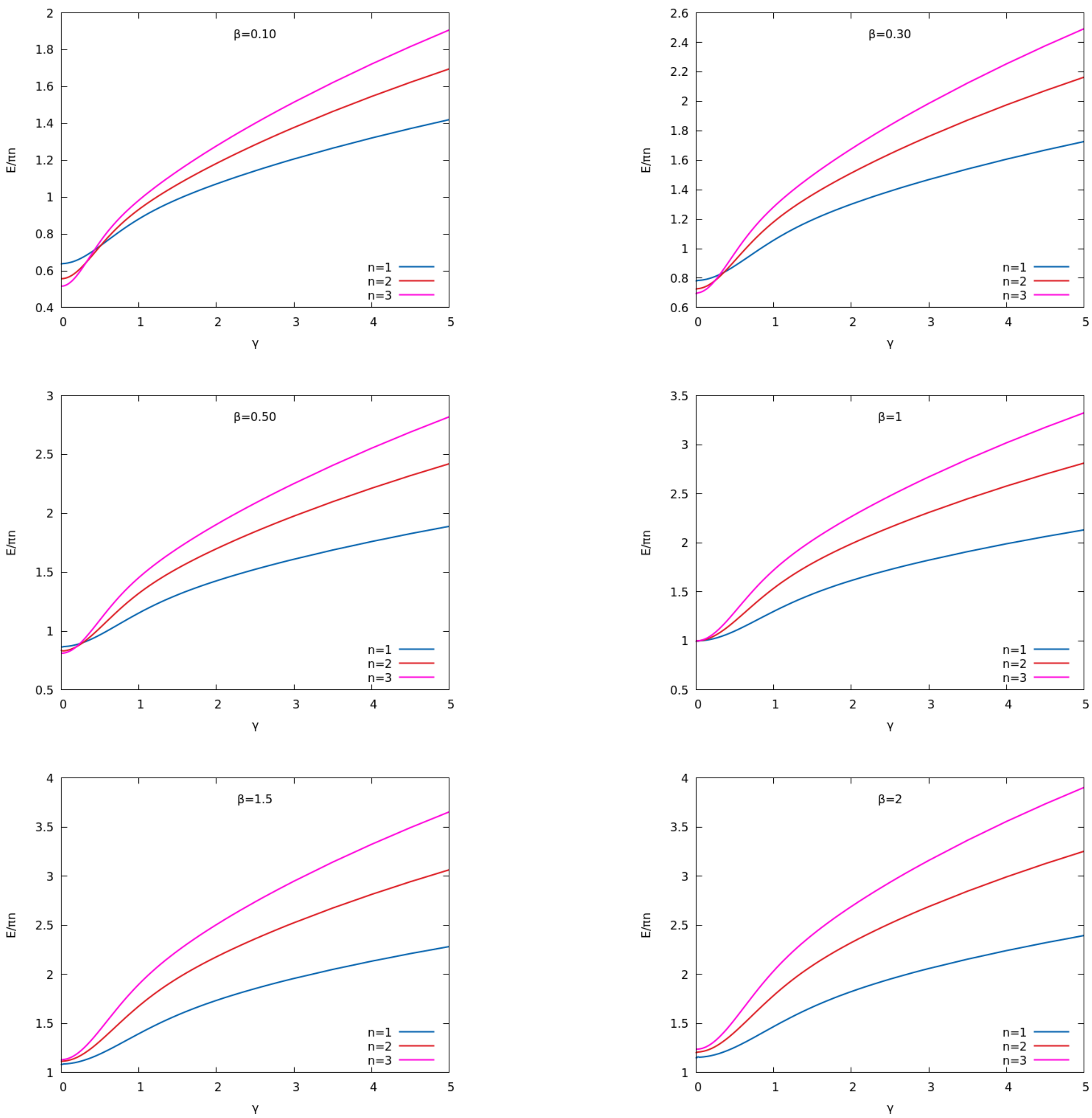

Figure 9: String energy per winding number n in units of π, $E/\pi n$, for several fixed values of the free parameter β: $\beta = 0.1,\ 0.3,\ 0.5,\ 1,\ 1.5,\ 2$; and n: $n = 1, 2, 3$, as as function of the CS coupling constant in units of e, γ.

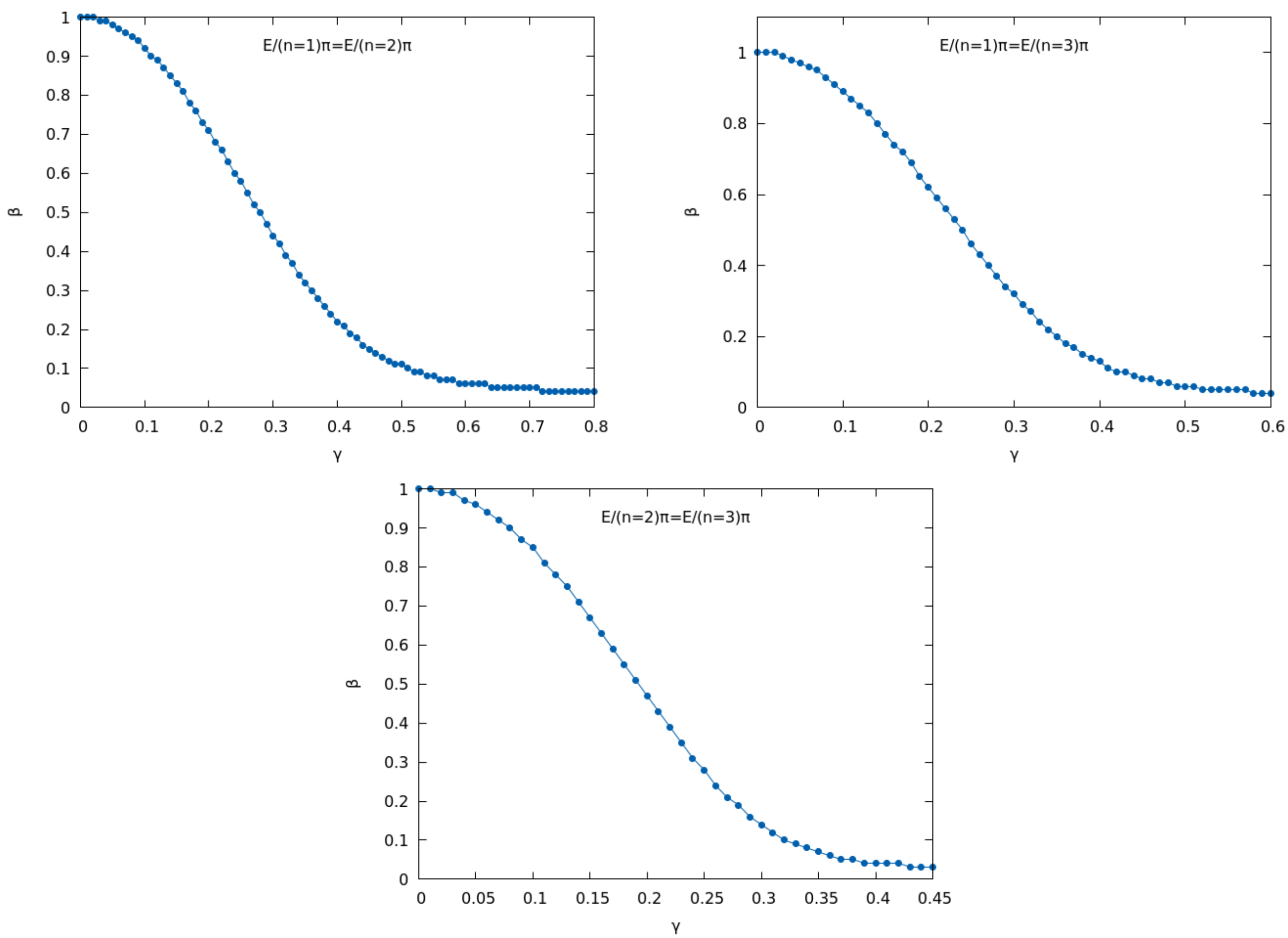

Figure 10: Coupling constant β in dependence on γ for which the string energies per winding number (in units of π), $E/\pi n$, are equal for $n = 1, 2$; $n = 1, 3$ and $n = 2, 3$, respectively.

Chapter 6

Conclusion and outlook

Let us summarize the main results of this paper and give some closing remarks with which this work could be ended.

In this work we have introduced, promoting the $U(1)_G$ symmetry of global straight strings to $U(1)_L$ invariance, the AH model, the simplest string forming model, which carries, as proved, a quantized magnetic flux, but is electrically neutral. We have showed it is governed by a single parameter β, defined as the square of the ratio of the masses of the AH theory after the spontaneous breaking from $U(1)_L$ symmetry to $\mathbb{1}$.

Next, ANO string solutions, static and axially symmetric, have been embedded to the Abelian-Higgs model. They are responsible for introducing the winding number n in the theory. Numerically constructing solutions for the equations of motions and studying the energy of the strings normalized by the corresponding winding number, it has been showed that, in the type II superconductive region ($\beta > 1$), strings with winding number greater than one, $n > 1$, are unstable into splitting.

The impossibility of endowing AH strings with electric charge Q has been reasoned. It has been stressed that, nevertheless, augmenting the Abelian-Higgs model with a term of the CS form (AHCS) enables strings to have electric charge, while keeping their total energy (in two spatial dimensions) finite. This electric charge is quantized, as the magnetic flux, and, in turn, related to the latter by the coupling constant of the CS term, κ. It has been pointed out the fact that, however, when adding the CS term, our manifold must be odd-dimensional. In this case, it has been reduced from $(3 + 1)$ space-time dimensions to $(2 + 1)$d.

In the same lines as realized with the AH model, the study of interactions of AHCS strings has been carried out. To this end, we extensively simulated the (β, n, κ) parameter space.

As aforementioned, AH strings are known to have a very different behavior depending on the parameter β. For $\beta < 1$ (type I) strings are likely to join in order to form higher winding strings; e.g., two $n = 1$ strings would form a $n = 2$ string. In other words, $n = 2$ strings, in this particular example, and $n > 1$ strings, in general, are stable for $\beta < 1$. On the other hand, for $\beta > 1$ (type II), the opposite occurs. Strings are prone to separating into n strings of unit winding number. That is, $n > 1$ strings are unstable for $\beta > 1$. The boundary of type I and II superconductive regions is $\beta = 1$, where this kind of straight strings stay the same, do not interact.

We wanted to address the stability or interaction of AHCS strings. Does the CS term, which creates electric field $\vec{E}$, attract or repel? In other words, does it conspire against the magnetic field $\vec{B}$, or, by contrast, does the CS interaction support the repulsive forces between strings?

Taking as a basis the performed analysis, it can be certainly stated that, when the AH Lagrangian density is augmented by a CS interaction term, the repulsive force between strings always increases. That is, the type I region shrinks as the CS coupling κ grows. In fact, for $\kappa > e/2$, strings always repel, irrespective of β. In addition, it can be remarked that, as for β, the string energy per winding number (in units of π), $E/\pi n$, increases monotonically with γ.

We also estimated the vale of the AH coupling β for each κ that would be the equivalent of the $\beta = 1$ boundary point in the AH model. Id est, we have precised the (β, κ) point in parameter space for which strings do not repel or attract one another. We have done this for $n = 1$ and $n = 2$; $n = 1$ and $n = 3$; $n = 2$ and $n = 3$. In other words, we have mapped the parameter space (β, κ) for different winding numbers n.

In this regard, the initial objective of studying the interactions of AHCS cosmic strings has been fulfilled. It could be suggested this added repulsion may have its origin on the screening effect of the electric fields belonging to the AHCS strings [4, p. 3444, l. 23-24], a unique feature not shared by AH vortices, being the latter, as mentioned, electrically neutral.

With regard to future research, we suggest extending this study to non-Abelian gauge theories with the Chern-Simons term. Moreover, a competing scalar field could be added to Abelian as well as to non-Abelian Chern-Simons vortices in order to discuss its consequences, since there has been renewed interest [13, p. 1, l. 10-12] in the study of models in which there are two scalars –here, the Higgs field $\phi(x)$ and another scalar, $\chi(x)$, competing with it– with different ground states.

For the Abelian case, we could augment the AHCS vortices studied in our work with the competing scalar field $\chi(x) \in \mathbb{C}$. The theory (in $(2 + 1)$d, remember) would be described by the action

$$S_{\mathrm{AHCS}\chi} = \int d^3x \, \mathcal{L}_{\mathrm{AHCS}\chi}, \tag{74}$$

where the Lagrangian density reads

$$\mathcal{L}_{\mathrm{AHCS}\chi} = \mathcal{L}_{\mathrm{AHCS}} + \frac{1}{2} \left(\partial_\mu \chi(x)\right) \left(\partial^\mu \chi(x)\right)^* + \mathcal{V}(|\phi(x)|, |\chi(x)|), \tag{75}$$

with the potential energy density given by

$$\mathcal{V}(|\phi(x)|, |\chi(x)|) = \frac{\lambda_\phi}{4} \left(|\phi(x)|^2 - \eta_\phi^2\right) + \frac{\lambda_\chi}{4} |\chi(x)|^2 \left(|\chi(x)|^2 - 2\eta_\chi^2\right)^2 + \frac{\lambda_{\phi\chi}}{2} |\phi(x)|^2 |\chi(x)|^2, \tag{76}$$

which is a model analogue to the superconducting vortices of Witten [14, p. 559, Eq. (2)].

On the other hand, for the non-Abelian case, $SU(N = 2)$ Chern-Simons-Higgs theory can be the model augmented by the mentioned scalar field $\chi(x) \in \mathbb{C}$ that competes with the Higgs one –i.e.,

$N = 2$ scalars in total–. The Lagrangian density that describes this theory would be [13, p. 2, Eq. (1)][13, p. 3, Eq. (13)]

$$\begin{aligned}\mathcal{L}_{\mathrm{CSH}\chi} = & \frac{1}{2} D_\mu(x)\vec{\phi}(x) \cdot D^\mu(x)\vec{\phi}(x) + \frac{1}{2}\partial_\mu \vec{\chi}(x) \cdot \partial^\mu \vec{\chi}(x) \\ & + \frac{\kappa}{4}\epsilon^{\mu\nu\lambda}\left(\vec{F}_{\mu\nu}(x) \cdot \vec{A}_\lambda(x) - \frac{e}{3}\vec{A}_\mu(x) \cdot \left(\vec{A}_\nu(x) \times \vec{A}_\lambda(x)\right)\right) - \mathcal{V}(|\vec{\phi}(x)|, |\vec{\chi}(x)|),\end{aligned} \tag{77}$$

where the field strength $\vec{F}_{\mu\nu}(x)$ and $SU(2)$-covariant derivatives $D_\mu(x)$ can be written as

$$\vec{F}_{\mu\nu}(x) = \partial_\mu \vec{A}_\nu(x) - \partial_\nu \vec{A}_\mu(x) + e\vec{A}_\mu(x) \times \vec{A}_\nu(x), \tag{78}$$

$$D_\mu(x)\vec{\phi}(x) = \partial_\mu \vec{\phi}(x) + e\vec{A}_\mu(x) \times \vec{\phi}(x), \tag{79}$$

and the potential density is given by [13, p. 2, Eq. (8)][13, p. 3, Eq. (14)]

$$\mathcal{V}(|\vec{\phi}(x)|, |\vec{\chi}(x)|) = \frac{\lambda^2}{2}|\vec{\phi}(x)|^2\left(|\vec{\phi}(x)|^2 - \eta^2\right)^2 + \gamma\left[\left(|\vec{\phi}(x)|^2 - \mu^2\right)|\vec{\chi}(x)|^2 + \beta|\vec{\chi}(x)|^4\right], \tag{80}$$

$\gamma, \mu \in \mathbb{R}_+$. The gauge field $\vec{A}_\mu(x)$ (μ =0, 1, 2) obeys the $\mathfrak{g} = \mathfrak{su}(2)$ Lie algebra,

$$A_\mu = A_\mu^a T^a \equiv \vec{A}_\mu \cdot \vec{T}, \quad \text{with generators } T^a \text{ satisfying} \tag{81}$$

$$[T^a, T^b] = i\epsilon^{abc}T^c, \quad \{T^a, T^b\} = \mathrm{Tr}(T^a T^b) = \frac{\delta^{ab}}{2}, \quad a = 1, 2, 3. \tag{82}$$

This thesis will be part of an article related to this briefly presented works.

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Appendix A

Conventions and definitions

A.1 Conventions

The convention here used is the following:
• Natural units, i.e., Planck's constant and the speed of light are set to one, $\hbar = c = 1$.
• The notation (x) is shorthand for all space-time coordinates (x^0, x^i), i =1, 2, ..., $d-1$, where $x^0 = t$. It should be clear from the context whenever the x-coordinate is meant.
• Einstein summation notation over repeated indices is implied, $\sum_\mu a^\mu a_\mu \equiv a^\mu a_\mu$.
• Complex conjugation is indicated with the symbol $(^*)$.
• Partial derivatives, $_{,\mu} \equiv \partial_\mu = \frac{\partial}{\partial x^\mu}$.
• In $(3+1)$ dimensions, we work with a metric of signature $(-+++)$.
In $d = (2+1)$, with a metric signature $(-++)$.
• (3+1) space-time indices are denoted by Greek letters, μ, ν, ... =0, 1, 2, 3.
In (2+1), μ, ν, ... =0, 1, 2.
• In (3+1) space-time dimensions, Latin letters i, j, ... = 1, 2, 3 label spatial components.
In (2+1), i, j, ... = 1, 2.

• $$\varepsilon^{012} = 1. \tag{83}$$

A.2 Definitions

• The metric tensor $g_{\mu\nu}$ and inverse metric tensor $g^{\mu\nu}$ –for which $g_{\mu\nu}g^{\nu\gamma} = \delta_\mu{}^\gamma$ and $g_{\mu\nu}g^{\mu\nu} = \mathbb{1}_4$– in polar coordinates, respectively, read
$$(g_{\mu\nu}) = \mathrm{diag}(-1, 1, r^2), \quad (g^{\mu\nu}) = (g_{\mu\nu})^{-1} = \mathrm{diag}(-1, 1, r^{-2}). \tag{84}$$

• The metric tensor $g_{\mu\nu}$ and inverse metric tensor $g^{\mu\nu}$ in cylindrical coordinates are
$$(g_{\mu\nu}) = \mathrm{diag}(-1, 1, r^2, 1), \quad (g^{\mu\nu}) = (g_{\mu\nu})^{-1} = \mathrm{diag}(-1, 1, r^{-2}, 1). \tag{85}$$

• The general form of the Laplacian for a scalar function $\phi(x)$ is
$$\partial_\mu \partial^\mu \phi(x) = \frac{1}{\sqrt{g}} \partial_\mu \big(\sqrt{g} g^{\mu\nu} \partial_\nu \phi(x)\big). \tag{86}$$

In cylindrical coordinates,

$$\partial_\mu\partial^\mu\phi(x) \overset{(86)}{=} \frac{1}{r}\Big(\partial_t\big(r\,(-1)\,\partial_t\phi(x)\big) + \partial_r\big(r\,1\,\partial_r\phi(x)\big) + \partial_\theta\big(r\frac{1}{r^2}\partial_\theta\phi(x) + \partial_z\big(r\,1\,\partial_z\phi(x)\big)\Big)$$
$$= \Big(-(\partial_t)^2 + \underbrace{\frac{\partial_r}{r} + (\partial_r)^2}_{\frac{\partial_r}{r}(r\partial_r)} + \frac{(\partial_\theta)^2}{r^2} + (\partial_z)^2\Big)\phi(x), \tag{87}$$

where $g = \det\left(g^{\mu\nu}\right) \overset{(85)}{=} \det\big(\mathrm{diag}(-1,1,r^2,1)\big) = |-r^2| = r^2$.

$$\text{In polar coordinates,}\quad \partial_\mu\partial^\mu\phi(x) \overset{(86)}{=} \Big(-(\partial_t)^2 + \frac{\partial_r}{r}(r\partial_r) + \frac{(\partial_\theta)^2}{r^2}\Big)\phi(x). \tag{88}$$

- $$\boxed{F_{i0}} = \partial_i A_0 - \partial_0 A_i \overset{(1)}{=} \partial_i\big(-\phi\big) - \partial_t A_i = \Big(\nabla\big(-\phi\big) - \partial_t\vec{A}\Big)_i = \boxed{E_i}, \tag{89}$$

as stated in Ref. [5, p. 84, Footnote 1], and where in (89(1)) we recall $A_\mu = \big(-\phi, \vec{A}\big)$.

- $$\boxed{\vec{B}} = B^k\,\hat{e}_k = \epsilon^{ijk}\partial_i A_j\,\hat{e}_k = \boxed{\nabla\times\vec{A}}. \tag{90}$$

$$\text{For } k=0,\quad B = |(90)|_{k=0}| = B^{k=0} = \epsilon^{ij0}\partial_i A_j \equiv \epsilon^{ij}\partial_i A_j,\quad \text{from Ref. [5, p. 84, Footnote 1].} \tag{91}$$

- $$\epsilon^{ijk}F_{ij} = \epsilon^{ijk}\left(\partial_i A_j - \partial_j A_i\right) = \epsilon^{ijk}\partial_i A_j - \left(-\epsilon^{jik}\right)\partial_j A_i = \epsilon^{ijk}\partial_i A_j + \underbrace{\epsilon^{jik}\partial_j A_i}_{\epsilon^{ijk}\partial_i A_j,\,\text{dummy } i\leftrightarrow j}$$
$$= 2\epsilon^{ijk}\partial_i A_j \overset{(90)}{=} 2B^k \quad\leftrightarrow\quad \boxed{B^k = \frac{1}{2}\epsilon^{ijk}F_{ij}}, \tag{92}$$

$$\text{Then, (92)}\quad\leftrightarrow\quad \epsilon_{ljk}\big(2B^k = \epsilon^{ijk}F_{ij}\big) = 2\epsilon_{ljk}B^k = \epsilon_{ljk}\epsilon^{ijk}F_{ij} = \epsilon_{klj}\epsilon^{kij}F_{ij}$$
$$\overset{(1)}{=} \big(\delta_l{}^i\,\underbrace{\delta_j{}^j}_{3} - \underbrace{\delta_l{}^j\delta_j{}^i}_{\delta_l{}^i}\big)F_{ij} = 2\delta_l{}^iF_{ij} = 2F_{lj} \quad\leftrightarrow\quad \boxed{F_{lj} = \epsilon_{ljk}B^k}, \tag{93}$$

$$\text{where in (93(1)) we used } \epsilon_{123}\,\epsilon^{156} = \delta_2{}^5\delta_3{}^6 - \delta_2{}^6\delta_3{}^5. \tag{94}$$

A.3 Field theory calculations

The following are established results used commonly in field theory calculations. The reason for including them here is to show that they have been studied and reproduced by the student. This was a necessary step to obtain the equations of motion of the CS term.

A.3.1 Hamiltonian density of the AH model

The Hamiltonian density $\mathcal{H}$ for the AH theory is the following,

$$\begin{aligned}\mathcal{H} &= \frac{\partial\mathcal{L}}{\partial\Big(\partial_0\phi(x)\Big)}\Big(\partial_0\phi(x)\Big)+\frac{\partial\mathcal{L}}{\partial\Big(\partial_0\phi^*(x)\Big)}\Big(\partial_0\phi^*(x)\Big)+\frac{\partial\mathcal{L}}{\partial\Big(\partial_0 A_\nu(x)\Big)}\Big(\partial_0 A_\nu(x)\Big)-\mathcal{L}\\
&= \frac{1}{2}\frac{\partial|D_0\phi(x)|^2}{\partial\Big(\partial_0\phi(x)\Big)}\Big(\partial_0\phi(x)\Big)+\frac{1}{2}\frac{\partial|D_0\phi(x)|^2}{\partial\Big(\partial_0\phi^*(x)\Big)}\Big(\partial_0\phi^*(x)\Big)+\frac{1}{4}\frac{\partial\Big(F_{\mu\nu}F^{\mu\nu}\Big)}{\partial\Big(\partial_0 A_\nu(x)\Big)}\Big(\partial_0 A_\nu(x)\Big)-\mathcal{L}\\
&= \frac{1}{2}\Big(D^0\phi(x)\Big)^*\Big(\partial_0\phi(x)\Big)+\frac{1}{2}\Big(D^0\phi(x)\Big)\Big(\partial_0\phi^*(x)\Big)+\frac{1}{4}\Big(4F^{0\nu}\Big)\Big(\partial_0 A_\nu(x)\Big)-\mathcal{L}\\
&= \frac{1}{2}\Big(D^0\phi(x)\Big)^*\Big(\partial_0\phi(x)\Big)+\frac{1}{2}\Big(D^0\phi(x)\Big)\Big(\partial_0\phi^*(x)\Big)-\underbrace{F^{0\nu}\Big(\partial_0 A_\nu(x)\Big)}_{-E^2}-\frac{1}{4}\underbrace{F_{\mu\nu}F^{\mu\nu}}_{\overset{(1)}{=}2(E^2-B^2)}\\
&\quad-\frac{1}{2}|D_\mu(x)\,\phi(x)|^2-\mathcal{V}(|\phi(x)|)\\
&= \frac{1}{2}|D_0(x)\,\phi(x)|^2+\frac{1}{2}|D_i(x)\,\phi(x)|^2+\frac{1}{2}\Big(E^2+B^2\Big)+\mathcal{V}(|\phi(x)|),
\end{aligned}\tag{95}$$

where in (95(1)) we used

$$\begin{aligned}\boxed{F_{\mu\nu}F^{\mu\nu}} &= F_{0\nu}F^{0\nu}+F_{i\nu}F^{i\nu}=\Big(0+\underbrace{F_{0i}}_{E_i}\overbrace{\underbrace{g^{00}}_{-1}g^{ii}\underbrace{F_{0i}}_{E_i}}^{F^{0i}}\Big)+\Big(\underbrace{F_{i0}}_{(-F_{0i})}\underbrace{F^{i0}}_{(-F^{0i})}+\underbrace{F_{ij}}_{\overset{(93)}{=}B^k\epsilon_{ijk};}\underbrace{F^{ij}}_{\overset{(93)}{=}B_k\epsilon^{ijk}}\Big)\\
&= -2E_iE^i+B^kB_k=\boxed{2(B^2-E^2)}.
\end{aligned}\tag{96}$$

A.3.2 Field equations of the AH model

The equations of motion are obtained using the Euler-Lagrange equations, for the corresponding field $\varphi(x) = \phi(x), A_\nu(x)$,

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \varphi(x)\right)} - \frac{\partial \mathcal{L}}{\partial \varphi(x)} = 0, \tag{97}$$

- For $\phi(x)$, [5, p. 91, Eq. (2.1.3)]

$$\begin{aligned}
&\partial_\mu \underbrace{\frac{\partial}{\partial\left(\partial_\mu \phi(x)\right)} \frac{1}{2}\Big(\overbrace{g^{\alpha\beta}\big(D_\alpha(x)\phi(x)\big)^*}^{\left(D^\beta(x)\phi(x)\right)^*}\overbrace{\big(\partial_\beta\phi(x) - i\,eA_\beta(x)\phi(x)\big)}^{D_\beta(x)\phi(x)}\Big)}_{g^{\alpha\beta}\left(D_\alpha(x)\phi(x)\right)^*\left(\delta^\mu_\beta - 0\right) = \left(D^\mu(x)\phi(x)\right)^*} \\
&- \Big\{\underbrace{\frac{1}{2}\overbrace{g^{\alpha\beta}\big(D_\alpha(x)\phi(x)\big)^*}^{\left(D^\beta(x)\phi(x)\right)^*}\overbrace{\big(0 - i\,eA_\beta(x)\big)}^{\partial\left(D_\beta(x)\phi(x)\right)/\partial\phi(x)}}_{\left(D^\mu(x)\phi(x)\right)^*\left(-i\,eA_\mu(x)\right),\,\text{dummy } \beta\to\mu} + \overbrace{\frac{\lambda}{4}2\big(\phi(x)\phi^*(x) - \eta^2\big)\phi^*(x)}^{\partial\mathcal{V}(|\phi(x)|)/\partial\phi(x)}\Big\} \\
&= \boxed{\frac{1}{\cancel{2}}\underbrace{\big(\partial_\mu + i\,eA_\mu(x)\big)\big(D^\mu(x)\phi(x)\big)^*}_{\left(D_\mu(x)D^\mu(x)\phi(x)\right)^*} - \frac{\lambda}{\cancel{2}}\big(|\phi(x)|^2 - \eta^2\big)\phi^*(x) = 0}.
\end{aligned} \tag{98}$$

- For $A^\mu(x)$, [5, p. 91, Eq. (2.1.4)]

$$\begin{aligned}
&\frac{1}{4}\partial_\mu \underbrace{\underbrace{\underbrace{\frac{\partial}{\partial(\partial_\mu A_\nu(x))}\Big(F_{\alpha\beta}\,\overbrace{F_{\gamma\rho}\,g^{\alpha\gamma}g^{\beta\rho}}^{F^{\alpha\beta}}\Big)}_{g^{\alpha\gamma}g^{\beta\rho}\left(\left(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha\right)F_{\gamma\rho} + F_{\alpha\beta}\left(\left(\delta^\mu_\gamma\delta^\nu_\rho - \delta^\mu_\rho\delta^\nu_\gamma\right)\right)\right)}}_{\left(g^{\mu\gamma}g^{\nu\rho} - g^{\nu\gamma}g^{\mu\rho}\right)F_{\gamma\rho} + F_{\alpha\beta}\left(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\nu}g^{\beta\mu}\right)}}_{F^{\mu\nu} - F^{\nu\mu} + F^{\mu\nu} - F^{\nu\mu} \overset{(1)}{=} 4F^{\mu\nu}(x)} - \frac{1}{2}\frac{\partial}{\partial A_\nu(x)}\Big\{\underbrace{g^{\alpha\beta}\big(\partial_\beta + i\,eA_\beta(x)\big)\phi^*(x)}_{\left(D^\alpha(x)\phi(x)\right)^*}\underbrace{\big(\partial_\alpha - i\,eA_\alpha(x)\big)\phi(x)}_{D_\alpha(x)\phi(x)}\Big\} \\
&= \frac{1}{\cancel{4}}\partial_\mu\Big(\cancel{4}F^{\mu\nu}(x)\Big) - \frac{1}{2}g^{\alpha\beta}\Big\{\big(0 + i\,e\delta^\nu_\beta\big)\phi^*(x)D_\alpha(x)\phi(x) + \big(D_\beta(x)\phi(x)\big)^*\big(0 - i\,e\delta^\nu_\alpha\big)\phi(x)\Big\} = 0 \\
&= \boxed{\partial_\mu F^{\mu\nu}(x) - \underbrace{\frac{ie}{2}\Big(\phi^*(x)D^\nu(x)\phi(x) - \phi(x)\big(D^\nu(x)\phi(x)\big)^*\Big)}_{j^\nu(x)} = 0},
\end{aligned} \tag{99}$$

where in (99(1)) we used that the electromagnetic strength tensor $F_{\mu\nu}(x)$ is antisymmetric,

$$F^{\nu\mu}(x) = \partial^\nu A^\mu(x) - \partial^\mu A^\nu(x) = -\Big(\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)\Big) = -F^{\mu\nu}(x). \tag{100}$$

A.3.3 Field equations of the AHCS model

The equations of motion corresponding to the AHCS model (48) are Eq. (98) itself, because the Chern-Simons term is not reliant on the scalar field $\phi(x)$, and, from Eq. (99) and Eq. (51), [5, p. 91, Eq. (2.1.4)]

$$\boxed{\partial_\mu F^{\mu\nu}(x) - \frac{ie}{2}\Big(\phi^*(x)D^\nu(x)\phi(x) - \phi(x)\big(D^\nu(x)\phi(x)\big)^*\Big) + \frac{\kappa}{2}\epsilon^{\nu\alpha\mu}F_{\alpha\mu}(x) = 0}. \tag{101}$$

$$(101) = \partial_\mu F^{\mu\nu}(x) - \frac{ie}{2}\Big(\underbrace{\phi^*(x)\underbrace{D^\nu\phi(x)}_{\partial^\nu\phi(x)-i\,eA^\nu(x)\phi(x)} - \phi(x)\underbrace{\big(D^\nu(x)\phi(x)\big)^*}_{\partial^\nu\phi^*(x)+i\,eA^\nu(x)\phi^*(x)}}_{\phi^*(x)\partial^\nu\phi(x)-\phi(x)\partial^\nu\phi^*(x)-2i\,eA^\nu(x)|\phi(x)|^2}\Big) + \frac{\kappa}{2}\epsilon^{\nu\alpha\mu}F_{\alpha\mu}(x) = 0, \tag{102}$$

where the current j^ν is given by [4, p. 3446, Eq. (2.4)]

$$j^\nu \equiv \frac{ie}{2}\left(\phi^*(x)\partial^\nu\phi(x) - \phi(x)\partial^\nu\phi^*(x) - 2i\,eA^\nu(x)|\phi(x)|^2\right). \tag{103}$$

Substituting ansatz (55) renders the field equations corresponding to the gauge field $A_\mu(x)$, Eq. (101), the following, where prime denotes derivative with respect to the variable in parenthesis,

- $$\begin{aligned}
(102)_{\nu=0} &\overset{(55)}{=} \partial_\mu\Big(\partial^\mu \underbrace{A^t(r)}_{g^{00}A_0(r)\overset{(84b)}{=}-A_0(r)} - \underbrace{\partial^t A^\mu(r)}_{0}\Big) - \frac{ie}{2}\Big(0-0-2i\,e\underbrace{A^t(r)}_{g^{00}A_0(r)=-A_0(r)}|\phi(r,\theta)|^2\Big)\\
&\quad + \underbrace{\frac{\kappa}{2}\Big(\underbrace{\epsilon^{012}F_{12}}_{\epsilon^{012}(\partial_1A_2(r))} + \underbrace{\epsilon^{021}F_{21}}_{(-\epsilon^{012})(-\partial_1A_2(r))}\Big)}_{=\kappa\frac{\mathrm{sign}(g)}{\sqrt{r^2}}\varepsilon_{012}\partial_1A_2=-\frac{\kappa}{r}\partial_1\left(\frac{n-P(r)}{e}\right)} = 0\\
&= \partial_\mu\partial^\mu\left(-A_0(r)\right) - e^2\left(-A_0(r)\right)f^2(r) - \frac{\kappa}{r}\left(\frac{0-P'(r)}{e}\right)\\
&\overset{(88)}{=} \left(-0^2 + \frac{\partial_r}{r} + (\partial_r)^2 + \frac{0^2}{r^2}\right)(-V(r)) + e^2V(r)f^2(r) + \frac{\kappa}{r}\frac{P'(r)}{e} \quad \leftrightarrow\\
&= -\left(\frac{V'(r)}{r} + V''(r)\right) + e^2V(r)f^2(r) - \frac{\kappa}{e}\frac{P'(r)}{r} = 0\\
&\overset{(22a)}{=} -\left(\frac{V'(x)}{x} + V''(x)\right) + e^2\frac{V(x)}{(\sqrt{\lambda})^2}f^2(x) + \frac{\kappa}{e}\frac{P'(x)}{x} = 0\\
&\overset{(22b\text{-}c)}{=} \boxed{-\frac{(xV'(x))'}{x} + \frac{V(x)}{\alpha^2}f^2(x) + \gamma\frac{P'(x)}{x} = 0}, \quad \text{known as Gauss law}
\end{aligned} \tag{104}$$

  [4, p. 3442, Eq. (1.3a)], is the field equation with respect to $A_0(x)$.

- $(102)_{\nu=1} \overset{(55)}{=} \partial_\mu\big(\partial^\mu \underbrace{A^r}_{0} - \partial^r A^\mu(r)\big) - \frac{ie}{2}\big(\phi^*(r,\theta) \underbrace{\partial^1}_{g^{11}\partial_1 \overset{(84)}{=} \partial_1} \phi(r,\theta) - \phi(r,\theta)\partial^1\phi^*(r,\theta) - 0\big) + 0$

$$
\begin{aligned}
&= -\partial_\mu \partial^r A^\mu(r) + ie^2\big(\underbrace{f(r)\,f'(r) - f(r)\,f'(r)}_{0}\big) = 0 \quad \leftrightarrow \quad \partial_\mu \partial^r A^\mu(r) = 0 \\
&= \partial_t\Big(\underbrace{g^{rr}}_{1}\partial_r\big(\underbrace{g^{tt}}_{-1} A_t(r)\big)\Big) + \partial_r\Big(\underbrace{g^{rr}}_{1}\partial_r\big(\underbrace{g^{rr}}_{1}\underbrace{A_r}_{0}\big)\Big) + \partial_\theta\Big(\underbrace{g^{rr}}_{1}\partial_r\big(\underbrace{g^{\theta\theta}}_{r^{-2}}\frac{(n-P(r))}{e}\big)\Big) \\
&= \partial_t(-V'(r)) + (\partial_r)^2(0) + \partial_\theta\left(-2\frac{n-P(r)}{er^3} - \frac{P'(r)}{er^2}\right) \quad \leftrightarrow \quad 0 = 0, \qquad (105)
\end{aligned}
$$

- $(102)_{\nu=2} \overset{(55)}{=} \partial_\mu\big(\partial^\mu \underbrace{A^\theta(r)}_{g^{22}A_2(r) \overset{(84)}{=} r^{-2}A_2(r)} - \underbrace{\partial^\theta A^\mu(r)}_{0}\big)$

$$
\begin{aligned}
&- \frac{ie}{2}\Big(\phi^*(r,\theta) \underbrace{\partial^\theta}_{g^{22}\partial_2 \overset{(84)}{=} r^{-2}\partial_2} \phi(r,\theta) - \phi(r,\theta)\,\partial^\theta\phi^*(r,\theta) - 2i\,e \underbrace{A^\theta(r)}_{g^{22}A_2(r) \overset{(84)}{=} r^{-2}A_2(r)} |\phi(r,\theta)|^2\Big) \\
&\underbrace{+ \frac{\kappa}{2}\Big(\underbrace{\epsilon^{201}F_{01}}_{\epsilon^{201}(-\partial_1 A_0)} + \underbrace{\epsilon^{210}F_{10}}_{-\epsilon^{201}(\partial_1 A_0)}\Big)}_{\kappa\frac{\text{sign}(g)}{\sqrt{r^2}}\varepsilon_{012}(-\partial_1 A_0) = +\frac{\kappa}{r}\partial_1 A_0} = 0 \\
&= \partial_\mu \partial^\mu \left(\frac{A^\theta(r)}{r^2}\right) - \frac{ie}{2r^2} f^2(r)\big(in - (-in)\big) - e^2\frac{A_2(r)}{r^2} f^2(r) + \frac{\kappa}{r}V'(r) \\
&\overset{(88)}{=} \left(-0 + \frac{\partial_r}{r} + (\partial_r)^2 + 0\right)\frac{n-P(r)}{er^2} + \cancel{\frac{ne}{r^2}f^2(r)} - e^2\frac{\cancel{n} - P(r)}{er^2} f^2(r) + \frac{\kappa}{r}V'(r) \\
&\overset{\cdot er^2}{\longleftrightarrow} \quad -P''(r) + \frac{P'(r)}{r} + 2e^2 P(r) f^2(r) - 2\kappa e r V'(r) = 0 \\
&\xleftrightarrow{(22\text{a}),\cdot(2\lambda)^{-1}} -P''(x) + \frac{P'(x)}{x} + \frac{e^2}{\lambda}P(x)f^2(x) + \frac{\kappa e}{\lambda}xV'(x) = 0 \\
&\overset{(22\text{b-c})}{=} \boxed{-P''(x) + \frac{P'(x)}{x} + \frac{P(x)f^2(x)}{\alpha^2} + \frac{\gamma x V'(x)}{\alpha^2} = 0}. \qquad (106)
\end{aligned}
$$

On the other hand, upon utilizing ansatz (55), the field equation belonging to the scalar field $\phi(x)$, Eq. (98), can be recast as,

$$
\begin{aligned}
(98) = \quad & \underbrace{\Big(D_\mu(x)D^\mu(x)\phi(x)\Big)^*}_{\left\{\partial_\mu\partial^\mu + i\,e\left(\partial_\mu A^\mu(x) + \underbrace{A^\mu(x)\partial_\mu + A_\mu(x)\partial^\mu}_{2A^\mu(x)\partial_\mu}\right) - e^2 A_\mu(x)A^\mu(x)\right\}\phi^*(x)} \qquad -\lambda\Big(|\phi(x)|^2 - 1^2\Big)\phi^*(x) = 0 \\
& \overset{(55,\,87)}{=} \Bigg\{\Bigg(-0^2 + \frac{\partial_r}{r} + (\partial_r)^2 + \frac{(\partial_\theta)^2}{r^2}\Bigg) + i\,e\Big(\underbrace{\partial_t A^t(r)}_{0} + 2A^t(r)\underbrace{\partial_t}_{0}\Big) - e^2 A_t(r) \underbrace{A^t(r)}_{g^{00}A_0(r)\overset{(84)}{=}-A_0(r)} \\
& + i\,e\Big(\underbrace{\partial_\theta A^\theta(r)}_{0} + 2\underbrace{A^\theta(r)}_{g^{22}A_2(r)=r^{-2}A_2(r)}\underbrace{\partial_\theta}_{-in}\Big) - A_\theta(r)\underbrace{A^\theta(r)}_{g^{22}A_2(r)=r^{-2}A_2(r)}\Bigg)\Bigg\} f(r)e^{-in\theta} \\
& - \lambda\Big(f^2(r) - 1^2\Big)f(r)e^{-in\theta} = 0 \quad \overset{e^{+in\theta}}{\longleftrightarrow} \quad \frac{f'(r)}{r} + f''(r) + e^2\left(A_0(r)\right)^2 f(r) \\
& + \frac{1}{r^2}\Bigg(\underbrace{-n^2 + ie\,2A_2(r)(-in) - e^2\left(A_2(r)\right)^2}_{-n^2+2n\cancel{e}\frac{n-P(r)}{\cancel{e}}-\cancel{e}^2\frac{n^2+P^2(r)-2nP(r)}{\cancel{e}^2}=-P^2(r)}\Bigg)f(r) - \lambda\Big(f^2(r) - 1^2\Big)f(r) \\
& = \frac{f'(r)}{r} + f''(r) + e^2V^2(r)f(r) - \frac{P^2(r)f(r)}{r^2} - \lambda\Big(f^2(r) - 1^2\Big)f(r) = 0 \\
& \overset{(22a)}{=} (\sqrt{2\lambda})^2\frac{f'(x)}{x} + (\sqrt{2\lambda})^2 f''(x) + e^2V^2(x)f(x) - \frac{(\sqrt{2\lambda})^2}{x^2}P^2(x)f(x) - \lambda\Big(f^2(x) - 1^2\Big)f(x) \\
& \leftrightarrow \boxed{\frac{f'(x)}{x} + f''(x) + \frac{V^2(x)f(x)}{\alpha^2} - \frac{P^2(x)f(x)}{x^2} - \frac{1}{2}\Big(f^2(x) - 1^2\Big)f(x) = 0}.
\end{aligned}
\tag{107}
$$

Appendix B

Spontaneous symmetry breaking (SSB)

Spontaneous symmetry breaking is a common denominator of topological defects. Moreover, in our case, one of the free parameters of the AHCS theory will be, β, Eq. (22b), the square ratio of the scalar field mass, m_s, and the vector field mass, m_v, –here they will be denoted m_ρ and m_A, respectively– in the AH model, after the spontaneous symmetry breaking of the $U(1)_L$ symmetry. We consider therefore that a brief explanation of SSB is in order.

The physical fields (here the scalar field $\phi(x)$), which are excitations above the vacuum, are realized by being re-expressed as deviations about one particular ground state, i.e., by moving a chosen ground state "off center" [8, p. 374, l. 3]. The ground state of the AH model is degenerate, there are infinitely many possible ground states in the vacuum manifold $\mathcal{M}$, the circle S^1 in Eq. (5), all related by a rotational symmetry. Choosing one is enough. We pick a minimum, e.g. $\phi_1(x) = \eta$ and $\phi_2(x) = 0$[7], and introduce the new fields $\rho(x)$ $\xi(x)$, which are perturbations in the magnitude and phase of $\phi(x)$, respectively, about the chosen vacuum state:

$$\phi(x) = \big(\eta + \rho(x)\big)\, e^{i\xi(x)} \in \mathbb{C}, \quad \rho(x), \xi(x) \in \mathbb{R}, \tag{108}$$

where $|\rho(x)|, |\xi(x)| \ll \eta$. That is, the complex field $\phi(x)$ is expressed in terms of the real scalar fields $\rho(x)$ and $\xi(x)$.

Rewriting the Lagrangian density (10) in terms of these new field variables, we find

$$\begin{aligned}
\mathcal{L} &\overset{(108)}{=} \frac{1}{4} F_{\mu\nu}(x)\, F^{\mu\nu}(x) \\
&+ \frac{1}{2} \underbrace{\underbrace{\underbrace{\underbrace{\big(\partial_\mu - i\, eA_\mu(x)\big)\, \big(\eta + \rho(x)\big)}_{\Big(\underbrace{\partial_\mu \eta}_{0+\eta\partial_\mu} + \underbrace{\partial_\mu \rho(x)}_{\partial_\mu\rho(x)+\rho(x)\partial_\mu} -i\, eA_\mu(x)\big(\eta+\rho(x)\big)\Big)} \quad e^{-i\xi(x)}}_{e^{-i\xi(x)}\Big(\eta\big(-i\partial_\mu\xi(x)\big)+\partial_\mu\rho(x)+\rho(x)\big(-i\partial_\mu\xi(x)\big)-i\, eA_\mu(x)\big(\eta+\rho(x)\big)\Big)} \quad \big(\partial^\mu + eiA^\mu(x)\big)\, \big(\big(\eta + \rho(x)\big)\, e^{i\xi(x)}}_{\Big|\partial_\mu\rho(x)+i\big(\eta+\rho(x)\big)\big(\partial_\mu\xi(x)+eA_\mu(x)\big)\Big|^2}}_{\partial_\mu\rho(x)\partial^\mu\rho(x)+\big(\eta^2+\rho^2(x)+2\eta\rho(x)\big)\big(\partial_\mu\xi(x)\partial^\mu\xi(x)+e^2A_\mu(x)A^\mu(x)+2e\partial_\mu\xi(x)A^\mu(x)\big)}
\end{aligned}$$

[7] That is, we will choose $\alpha = 0$, i.e., we will consider the solution $\phi = \eta \in \mathbb{R}$, but the general solution $\phi = \eta e^{i\alpha}$ can be generated by applying the phase transformation (6).

$$
\begin{aligned}
&+\underbrace{\frac{\lambda}{4}\Big(\big(\eta+\rho(x)\big)^2-\eta^2\Big)^2}_{\frac{\lambda}{4}\rho^4(x)+\lambda\eta\rho^3(x)+\lambda\eta^2\rho^2(x)} \\
&=\underbrace{\frac{1}{4}\,F_{\mu\nu}\,(x)\,F^{\mu\nu}(x)}_{\text{kinetic term of } A^\mu(x)}+\underbrace{\frac{\eta^2e^2}{2}A_\mu(x)A^\mu(x)}_{\text{mass term of } A^\mu(x)} \\
&+\underbrace{\frac{1}{2}\partial_\mu\rho(x)\partial^\mu\rho(x)}_{\text{kinetic term of } \rho(x)}+\underbrace{\lambda\eta^2\rho^2(x)}_{\text{mass term of } \rho(x)}+\underbrace{\frac{1}{2}\eta^2\partial_\mu\xi(x)\partial^\mu\xi(x)}_{\text{kinetic term of } \xi(x)}+\underbrace{0}_{\text{mass term of } \xi(x)} \\
&+\underbrace{\eta^2e\partial_\mu\xi(x)A^\mu(x)+\mathcal{O}\left(\rho(x)A_\mu(x)A^\mu(x),\rho(x)\xi^2(x),\rho^3(x),...\right)}_{\mathcal{L}_{\text{int.}}}
\end{aligned}
\tag{109}
$$

In this way, the $U(1)_L$ symmetry in the original theory, Eq. (10), that is, invariance under rotations in $(\phi_1(x),\,\phi_2(x))$ space, where $\phi(x)=\phi_1(x)+i\,\phi_2(x)$, has been broken, $U(1)_L\mapsto\mathbb{1}$, by the arbitrary selection of a particular (asymmetrical) vacuum state. For example, Eq. (10) is even in $\phi(x)$ i.e. invariant as $\phi(x)\to-\phi(x)$, but the reformulated Lagrangian (109) is not even in its fields.

This happened because neither of this vacua is invariant under $U(1)_L$, each ground state on its own does not share the full symmetry of the original Lagrangian density. Though the Lagrangian density is invariant with respect to a phase transformation, the vacuum state of the Higgs field is not. Rather, under phase transformation, which is a rotation in the complex plane of $\phi(x)$, it changes into another ground state, since the degenerate vacua are connected by the symmetry operations of the theory. The collection of all ground states displays the symmetry of the original Lagrangian density, but to discern the particle spectrum we are obliged to work with one of them, and that spoils the symmetry. This is called "spontaneous" symmetry breaking because no external agency is responsible, the Higgs potential $\mathcal{V}(|\phi(x)|)$ itself breaks the local gauge invariance by inducing an asymmetry in the vacuum. [8, p. 375]

Let us calculate the mass spectrum, that is, obtain the masses corresponding to the fields from the coefficients of the field^2 terms. The term proportional to $A_\mu^2(x)=A_\mu(x)\,A^\mu(x)$ indicates that the "photon", the quantum of the electrodynamic field $A^\mu(x)$, whose presence promotes invariance under $U(1)_G$ to $U(1)_L$ invariance, has acquired mass ($m_A=+e\eta$) as a result of SSB ("Higgs mechanism"), by absorbing the Goldstone boson[8] of the global case. The mass of $A^\mu(x)$ comes from the term proportional to $\phi^*(x)\phi(x)A_\mu(x)A^\mu(x)$ in the original Lagrangian, Eq. (10), which, once $\phi(x)$ is substituted as fluctuations around the vacuum state, Eq. (108), gives a contribution of a constant multiplied by $A_\mu(x)A^\mu(x)$; id est, a mass term of $A^\mu(x)$.

The third and fourth summands of Eq. (109), $(1/2)\partial_\mu\rho(x)\partial^\mu\rho(x)+\lambda\eta^2\rho^2(x)$, constitute a free Klein-Gordon Lagrangian for the scalar field $\rho(x)$, which carries a mass $m_\rho=\sqrt{2\lambda}\eta$. The fifth term, $(1/2)\eta^2\partial_\mu\xi(x)\partial^\mu\xi(x)$, is a free Lagrangian for the $\xi(x)$, which is massless ($m_\xi=0$), because there is no term with $\xi^2(x)$.

[8]Goldstone's theorem states that spontaneous breaking of a continuous global symmetry entails the appearance one or more massless scalar particles ("Goldstone bosons") [8, p. 377, l. 19-23][15, p. 286, l. 11-12].

Intuitively, the latter is related to the fact that there is no resistance to field excitations that change $\xi(x)$, moving perpendicularly to the radial direction (rotations in a constant $(\phi_1(x), \phi_2(x))$ plane, along the circular valley of minima $|\phi(x)| = \eta$, staying in the bottom of the potential density). This behaviour is related to the fact that the Lagrangian density, and accordingly also the potential density $\mathcal{V}(|\phi(x)|)$, is invariant under phase transformations (6a), which correspond to rotations around the point $\phi(x) = 0$. On the other hand, radial fluctuations mean taking a different potential density $\mathcal{V}(|\phi(x)|)$ value and, being approximately parabolic in the vicinity of the minimum, they encounter a restoring force, it costs potential energy to displace $\rho(x)$ [8, p. 377, Footnote 1][15, p. 286, l. 4-7].

The rest of the terms, gathered in $\mathcal{L}_{\text{int.}}$, define couplings. The term $\eta^2\partial_\mu\xi(x)A^\mu(x)$, which leads to a vertex in which $\xi(x)$ turns into an $A^\mu(x)$, suggests $\xi(x)$ is a spurious field [8, p. 380, l. 1-5]. This is the reason $\xi(x)$ is known as a gauge artifact or nonphysical "ghost" particle rather than an actual field.

However, the unwanted[9] "Goldstone boson" $\xi(x)$ and the non-physical coupling term $\eta^2\partial_\mu\xi(x)A^\mu(x)$ can be totally eliminated exploiting the local gauge invariance of the original Lagrangian (10). That is, we can apply a gauge phase transformation (6a) with $e\alpha(x) = -\xi(x) \in \mathbb{R}$, under which the unbroken Lagrangian (10) is invariant, before inserting Eq. (108) in (10), and eliminate the $e^{i\xi(x)}$ in Eq. (108),

$$\phi(x) \xrightarrow{U(1)_L|_{e\alpha(x)=-\xi(x)}} \phi(x)\, e^{i(-\xi(x))} \overset{(108)}{=} \big(\eta + \rho(x)\big)\, e^{i\xi(x)}\, e^{i(-\xi(x))} = \big(\eta + \rho(x)\big). \tag{110}$$

This means the auxiliary field $\xi(x)$ only contains gauge degrees of freedom, it does not have physical significance. In fact, a massless scalar field has not yet been observed.

Inserting Eq. (110) in the original Lagrangian density (10), we obtain the same as in Eq. (109), but with $\xi(x) = 0$. That is, the Lagrangian takes the same form in terms of the new field variables as it did in terms of the old ones (this is another interpretation of what $\mathcal{L}$ being invariant means) [8, p. 380, l. 15-17]. In this way, we eliminate the unwanted "Goldstone boson" $\xi(x)$ and the nonphysical term $\eta^2\partial_\mu\xi(x)A^\mu(x)$ in Eq. (109), and we are left with the massive scalar field $\rho(x)$ (the "Higgs" particle) and the massive vector gauge field $A^\mu(x)$. We have selected a convenient gauge, Eq. (110), and rewritten the fields in terms of fluctuations about a particular ground state η, but both Lagrangian densities, Eq. (10) and Eq. $(109)|_{\xi(x)=0}$, describe the same physical system.

Scalar fields (massive or massless) have one degree of freedom [15, p. 294, l. 25-26]. On the other hand, a massless vector field $A_\mu(x)$ carries two degrees of freedom (transverse polarizations). Therefore, before SSB: 2×1 (from $\phi_1(x)$ and $\phi_2(x)$) $+ 2$ (from massless $A_\mu(x)$) $= 4$. When the gauge field $A^\mu(x)$ eats the Goldstone boson after SSB, it picks up an extra degree of freedom (longitudinal polarization state) [8, p. 381, l. 1-2][15, p. 294, l. 27-28] as well as the aforementioned mass. Consequently, 1 (from $\rho(x)$) $+ 3$ (from massive $A_\mu(x)$) $= 4$, the number of degrees of freedom is preserved.

[9] Unwanted because Goldstone bosons appear as a consequence of spontaneously breaking a continuous global (not local, as in this case) symmetry, as stated in Footnote 8.