8.9 FORMULAE FOR POWER OPTIONS

An option with a payoff that depends on the asset price at expiry raised to some power is called a **power option**. Suppose that it has a payoff

$$\text{Payoff}(S^\alpha)$$

we can find a simple formula for the value of the option if we have a simple formula for an option with payoff given by

$$\text{Payoff}(S). \tag{8.9}$$

This is because of the lognormality of the underlying asset.

Writing

$$\mathscr{S} = S^\alpha$$

the Black–Scholes equation becomes, in the new variable $\mathscr{S}$,

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\alpha^2\sigma^2\mathscr{S}^2\frac{\partial^2 V}{\partial \mathscr{S}^2} + \alpha\left(\tfrac{1}{2}\sigma^2(\alpha-1)+r\right)\mathscr{S}\frac{\partial V}{\partial \mathscr{S}} - rV = 0.$$

Thus whatever the formula for the option value with simple payoff (8.9), the formula for the power version has S^α instead of S and adjustment made to σ, r and D.

8.10 THE log CONTRACT

The **log contract** has the payoff

$$\log(S/E).$$

The theoretical fair value for this contract is of the form

$$a(t) + b(t)\log(S/E).$$

Substituting this expression into the Black–Scholes equation results in

$$\dot{a} + \dot{b}\log(S/E) - \tfrac{1}{2}\sigma^2 b + (r-D)b - ra - rb\log(S/E) = 0,$$

where $\dot{}$ denotes d/dt. Equating terms in $\log(S/E)$ and those independent of S results in

$$b(t) = e^{-r(T-t)} \quad \text{and} \quad a(t) = \left(r - D - \tfrac{1}{2}\sigma^2\right)(T-t)e^{-r(T-t)}.$$

The two arbitrary constants of integration have been chosen to match the solution with the payoff at expiry.

This value is rather special in that the dependence of the option price on the underlying asset, S, and the volatility, σ, uncouples. One term contains S and no σ and the other contains σ and no S. We briefly saw in Chapter 7 the concept of vega hedging to eliminate volatility risk. It is conceivable, even though not entirely justifiably, that the simplicity of the log contract value makes it a useful weapon for hedging other contracts against fluctuations in volatility. Having said that, it's not exactly a highly liquid contract.

The log contract payoff can be positive or negative depending on whether $S > E$ or $S < E$. If we modify the payoff to be

$$\max(\log(S/E), 0)$$

then we have a genuine 'option' which may or may not be exercised. The value of this option is

$$e^{-r(T-t)}\sigma\sqrt{T-t}\,N'(d_2) + e^{-r(T-t)}\left(\log(S/E) + \left(r - D - \tfrac{1}{2}\sigma^2\right)(T-t)\right)N(d_2).$$

8.11 SUMMARY

In this chapter I made some very simple generalizations to the Black–Scholes world. I showed the effect of discretely paid dividends on the value of an option, deriving a jump condition by a no-arbitrage argument. Generally, this condition would be applied numerically and its implementation is discussed in Chapter 78. I also showed how time-dependent parameters can be incorporated into the pricing of simple vanilla options.

FURTHER READING

- See Merton (1973) for the original derivation of the Black–Scholes formulae with time-dependent parameters.
- For a model with stochastic dividends, see Geske (1978).
- The practical implications of discrete dividend payments are discussed by Gemmill (1992).
- See Neuberger (1994) for further info on the log contract.