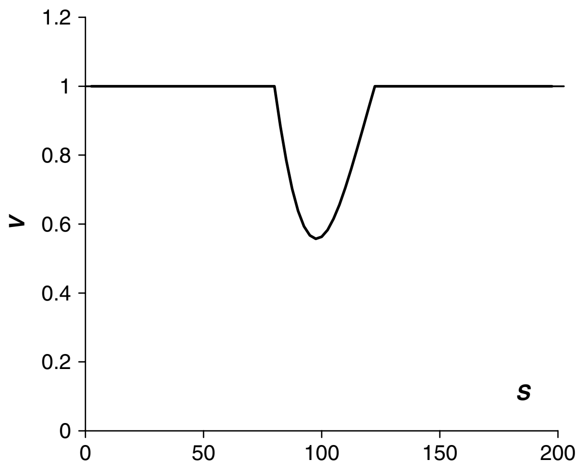

Figure 9.8 The value of a double one-touch option.

9.8.1 Bermudan Options

It is common for contracts that allow early exercise to permit the exercise only at certain specified times, and not at *all* times before expiry. For example, exercise may only be allowed on Thursdays between certain times. An option with such intermittent exercise opportunities is called a **Bermudan option**. All that this means mathematically is that the constraint (9.4) is only 'switched on' at these early exercise dates. The pricing of a such a contract numerically is, as we shall see, no harder than the pricing of American options when exercise is permitted at all times.

This situation can be made more complicated by the dependence of the exercise dates on a second asset. For example, early exercise is permitted only when a second asset is above a certain level. This makes the contract a multi-asset contract, see Chapter 11.

9.8.2 Make Your Mind Up

In some contracts the decision to exercise must be made before exercise takes place. For example, we must give two weeks' warning before we exercise, and we cannot change our mind. This contract is not hard to value theoretically. Suppose that we must give a warning of time τ. If at time t we decide to exercise at time $t+\tau$ then on exercise we receive a certain deterministic amount. To make the analysis easier to explain, assume that there is no time dependence in this payoff, so that on exercise we receive $P(S)$. The value of this payoff at a time τ earlier is $V^\tau(S,\tau)$ where $V^\tau(S,t)$ is the solution of

$$\frac{\partial V^\tau}{\partial t}+\tfrac{1}{2}\sigma^2S^2\frac{\partial^2V^\tau}{\partial S^2}+rS\frac{\partial V^\tau}{\partial S}-rV^\tau=0$$

with

$$V^\tau(S,0)=P(S).$$

This would have to be modified if the problem were time-inhomogeneous.

Obviously, we have

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \le 0.$$

Because $V^\tau(S, \tau)$ is the value of the contract at decision time if we have decided to exercise then our early-exercise constraint becomes

$$V(S, t) \ge V^\tau(S, \tau).$$

As an example, suppose that we get a payoff of $S - E$, this is $P(S)$. Note that there is no $\max(\cdot)$ function in this; we have said we will exercise and exercise we must, even if the asset is out of the money. The function $V^\tau(S, \tau)$ is clearly $S - Ee^{-r\tau}$ so that our **make-your-mind-up option** satisfies the constraint

$$V(S, t) \ge S - Ee^{-r\tau}.$$

A further complication is to allow one change of mind. That is, we say we will exercise in two weeks' time, but when that date comes we change our mind, and do not exercise. But the next time we say we will exercise, we must. This is also not too difficult to price theoretically.

The trick is to introduce two functions for the option value, $V_0(S, t)$ and $V_1(S, t)$. The former is the value before making the first decision to exercise, the latter is the value having made that decision but having changed your mind. We also need $V_0^\tau(S, t', t)$ and $V_1^\tau(S, t)$. The latter is simply the earlier V^τ. The former is slightly more complicated. In $V_0^\tau(S, t', t)$ the t' represents the time at which the option will be exercised or exercise is declined. The t represents the time before that date.

The problem for V_1 is exactly the same as for the basic make-your-mind-up option i.e.

$$\frac{\partial V_1}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + rS\frac{\partial V_1}{\partial S} - rV_1 \le 0$$

with

$$V_1(S, t) \ge V_1^\tau(S, \tau)$$

where

$$\frac{\partial V_0^\tau}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_0^\tau}{\partial S^2} + rS\frac{\partial V_0^\tau}{\partial S} - rV_0^\tau = 0$$

with

$$V_1^\tau(S, 0) = P(S).$$

The function $V_0^\tau(S, t', t)$ satisfies

$$\frac{\partial V_0^\tau}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_0^\tau}{\partial S^2} + rS\frac{\partial V_0^\tau}{\partial S} - rV_0^\tau = 0$$

(with time derivatives with respect to t and *not* t') with

$$V_0^\tau(S, t', 0) = \max(P(S), V_1(S, t')).$$

Then we have

$$\frac{\partial V_0}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_0}{\partial S^2} + rS\frac{\partial V_0}{\partial S} - rV_0 \leq 0$$

with the optimality constraint

$$V_0(S,t) \geq V_0^{\tau}(S, t+\tau, \tau).$$

Obviously, we can introduce more levels if we are permitted to change our minds a specified number of times.

In Part Two we will see many problems where we must introduce more than one function to value a single contract.

9.9 OTHER ISSUES

The pricing of American options and all the issues that this raises are important for many reasons. Some of these we describe here, but we will come back to the ideas again and again.

9.9.1 Non-linearity

The pricing of American options is a non-linear problem because of the free boundary. There are other non-linear problems in finance, some are non-linear because of the free boundary and some because the governing differential equation is itself non-linear. Non-linearity can be important for several reasons. Most obviously, non-linear problems are harder to solve than linear problems, usually requiring numerical solution.

Non-linear governing equations are found in Chapter 48 for models of pricing with transaction costs, Chapter 52 for uncertain parameter models, Chapter 58 for models of market crashes, and Chapter 59 for models of options used for speculative purposes.

9.9.2 Free-boundary Problems

Free-boundary problems, in other contexts, will be found scattered throughout the book. Again, the solution must almost always be found numerically. As an example of a free-boundary problem that is not quite an American option (but is similar), consider the **instalment option**. In this contract the owner must keep paying a premium, on prescribed dates, to keep the contract alive. If the premium is not paid then the contract lapses. Consider two cases, the first is when the premium is paid out continuously day by day, and the second, more realistic case, is when the premium is paid at discrete intervals. Part of the valuation is to decide whether or not it is worth paying the premium, or whether the contract should be allowed to lapse.

First, consider the case of continuous payment of a premium. If we pay out a constant rate $L\,dt$ in a time step dt to keep the contract alive then we must solve

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV - L \leq 0.$$

The term L represents the continual input of cash. But we would only pay the premium if it is, in some sense, 'worth it.' As long as the contract value is positive, we should maintain the payments. If the contract value ever goes negative, we should let the contract lapse. However, we can do better than this. If we impose the constraint

$$V(S,t) \geq 0,$$

with continuity of the delta, and let the contract lapse if ever $V = 0$ then we give our contract the *highest value possible*. This is very much like the American option problem, but now we must optimally cease to pay the premium (instead of optimally exercising).

Now let us consider the more realistic discrete payment case. Suppose that payments of L (not $L\,dt$) are made discretely at time t_i. The value of the contract must increase in value from before the premium is paid to just after it is paid. The reason for this is clear. Once we have paid the premium on date t_i we do not have to worry about handing over any more money until time t_{i+1}. The rise in value exactly balances the premium, L:

$$V(S, t_i^-) = V(S, t_i^+) - L,$$

where the superscripts $+$ and $-$ refer to times just after and just before the premium is paid. But we would only hand over L if the contract would be worth more than L at time t_i^+. Thus we arrive at the jump condition

$$V(S, t_i^-) = \max(V(S, t_i^+) - L, 0).$$

If $V(S, t_i^+) \leq L$ then it is optimal to discontinue payment of the premiums.

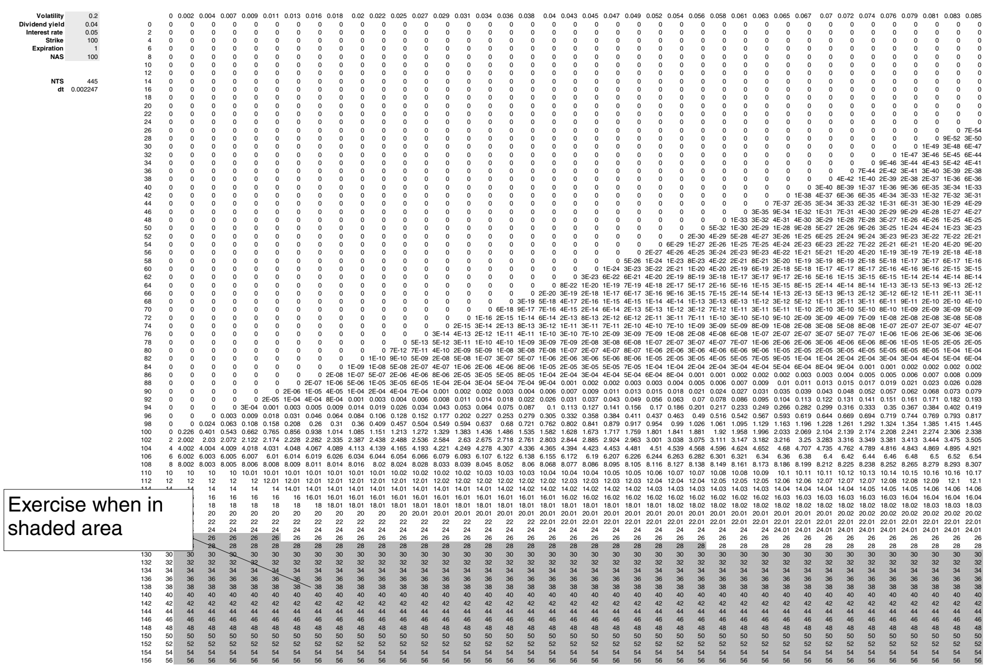

Figure 9.9 Spreadsheet showing the value of an American call option on a stock paying dividends. Shaded area is where you should exercise.

In practice, the premium L is chosen so that the value of the contract at initiation is exactly equal to L. This means that the start date is just like any other payment date.

9.9.3 Numerical Solution

Although free-boundary problems must usually be solved numerically, this is not difficult as we shall see in later chapters. We solve the relevant equation by either a finite-difference method or the binomial method.

The other numerical method that I describe is the Monte Carlo simulation. If there is any early-exercise feature in a contract this makes solution by Monte Carlo more complicated. I discuss this issue in Chapter 80.

The next three figures, Figures 9.9, 9.10, and 9.11, show the output of an explicit finite-difference method for the value of American call and put and Bermudan put respectively. You can't read the numbers but you can see that Excel's conditional formatting has been used to show the regions where the option value and the payoff are the same. Here you should exercise the option. In these figures time is in the top row. The first long column on the left is the stock price. Time goes from right to left, so that the second long column is the payoff. The second row down represents option value when stock price is zero, so the further down the row the higher the stock price.

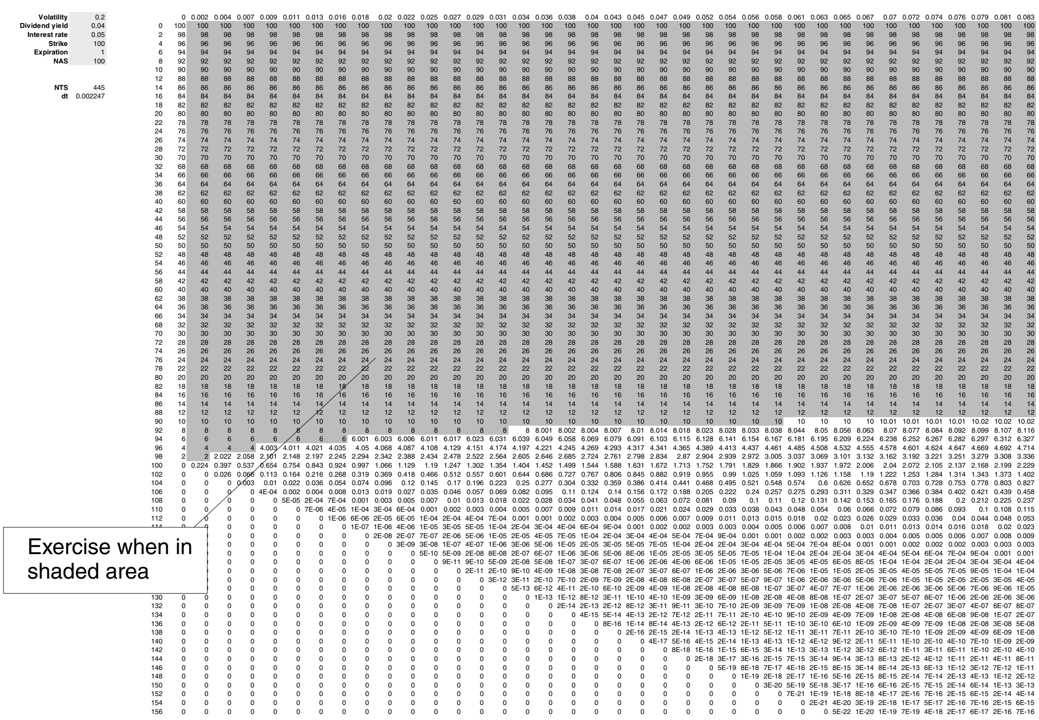

Figure 9.10 Spreadsheet showing the value of an American put option. Shaded area is where you should exercise.