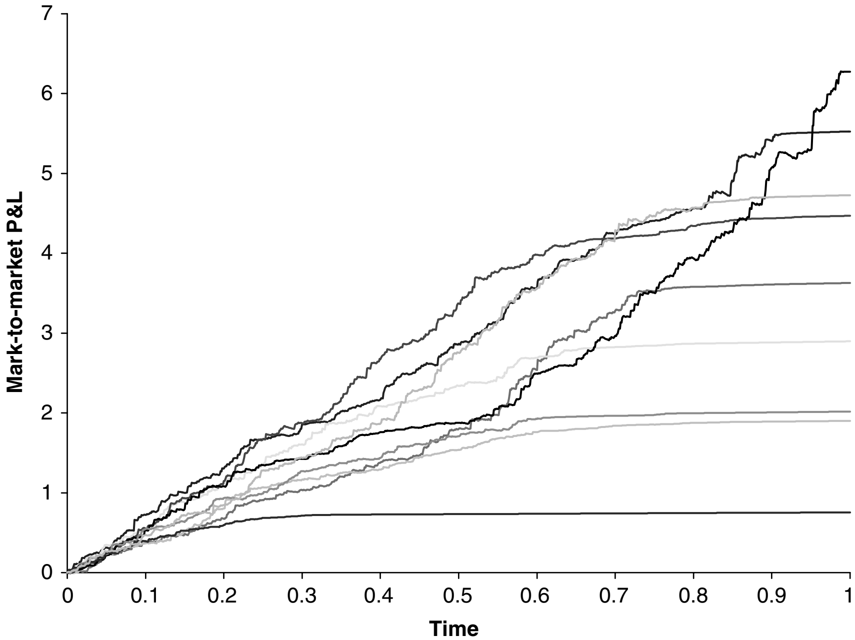

Figure 12.3 P&L for a delta-hedged option on a mark-to-market basis, hedged using implied volatility.

Peter Carr (2005) and Henrard (2001) show that if you hedge using a delta based on a volatility σ_h then the PV of the total profit is given by

$$V(S,t;\sigma_h) - V(S,t;\tilde{\sigma}) + \frac{1}{2}\left(\sigma^2 - \sigma_h^2\right)\int_{t_0}^{T} e^{-r(t-t_0)} S^2 \Gamma^h \, dt, \tag{12.2}$$

where the superscript on the gamma means that it uses the Black–Scholes formula with a volatility of σ_h.

12.5.1 The Expected Profit after Hedging using Implied Volatility

When you hedge using delta based on implied volatility the profit each 'day' is deterministic but the present value of total profit by expiration is path dependent, and given by

$$\frac{1}{2}(\sigma^2 - \tilde{\sigma}^2)\int_{t_0}^{T} e^{-r(s-t_0)} S^2 \Gamma^i \, ds.$$

Introduce

$$I = \frac{1}{2}(\sigma^2 - \tilde{\sigma}^2)\int_{t_0}^{t} e^{-r(s-t_0)} S^2 \Gamma^i \, ds.$$

Since therefore

$$dI = \frac{1}{2}(\sigma^2 - \tilde{\sigma}^2) e^{-r(t-t_0)} S^2 \Gamma^i \, dt$$

we can write down the following partial differential equation for the *real* expected value, $P(S, I, t)$, of I:

$$\frac{\partial P}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + \mu S \frac{\partial P}{\partial S} + \tfrac{1}{2}(\sigma^2 - \tilde{\sigma}^2) e^{-r(t-t_0)} S^2 \Gamma^i \, \frac{\partial P}{\partial I} = 0,$$

with

$$P(S, I, T) = I.$$

Look for a solution of this equation of the form

$$P(S, I, t) = I + H(S, t)$$

so that

$$\frac{\partial H}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 H}{\partial S^2} + \mu S \frac{\partial H}{\partial S} + \tfrac{1}{2}(\sigma^2 - \tilde{\sigma}^2)\, e^{-r(t-t_0)} S^2 \Gamma^i = 0.$$

The source term can be simplified to

$$\frac{E(\sigma^2 - \tilde{\sigma}^2) e^{-r(T-t_0)} e^{-d_2^2/2}}{2\tilde{\sigma}\sqrt{2\pi(T-t)}}.$$

Change variables to

$$x = \log(S/E) + \left(\mu - \tfrac{1}{2}\sigma^2\right)\tau \quad \text{and} \quad \tau = T - t$$

and write

$$H = w(x, \tau).$$

The resulting partial differential equation is then a bit nicer. Details can be found in the appendix to this chapter

After some manipulations we end up with the expected profit initially ($t = t_0$, $I = 0$) being the single integral

$$\frac{E e^{-r(T-t_0)}(\sigma^2 - \tilde{\sigma}^2)}{2\sqrt{2\pi}} \int_{t_0}^{T} \frac{1}{\sqrt{\sigma^2(s-t_0) + \tilde{\sigma}^2(T-s)}}$$
$$\times \exp\left(- \frac{\left(\log(S/E) + \left(\mu - \tfrac{1}{2}\sigma^2\right)(s-t_0) + \left(r - D - \tfrac{1}{2}\tilde{\sigma}^2\right)(T-s)\right)^2}{2(\sigma^2(s-t_0) + \tilde{\sigma}^2(T-s))} \right) ds.$$

Results are shown in the following figures.

In Figure 12.4 is shown the expected profit versus the growth rate μ. Parameters are $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $E = 110$, $T = 1$, $\tilde{\sigma} = 0.2$. Observe that the expected profit has a maximum. This will be at the growth rate that ensures, roughly speaking, that the stock ends up close to at the money at expiration, where gamma is largest. In the figure is also shown the profit to be made when hedging with actual volatility. For most realistic parameters regimes the maximum expected profit hedging with implied is similar to the guaranteed profit hedging with actual.

In Figure 12.5 is shown expected profit versus E and μ. You can see how the higher the growth rate the larger the strike price at the maximum. The contour map is shown in Figure 12.6.

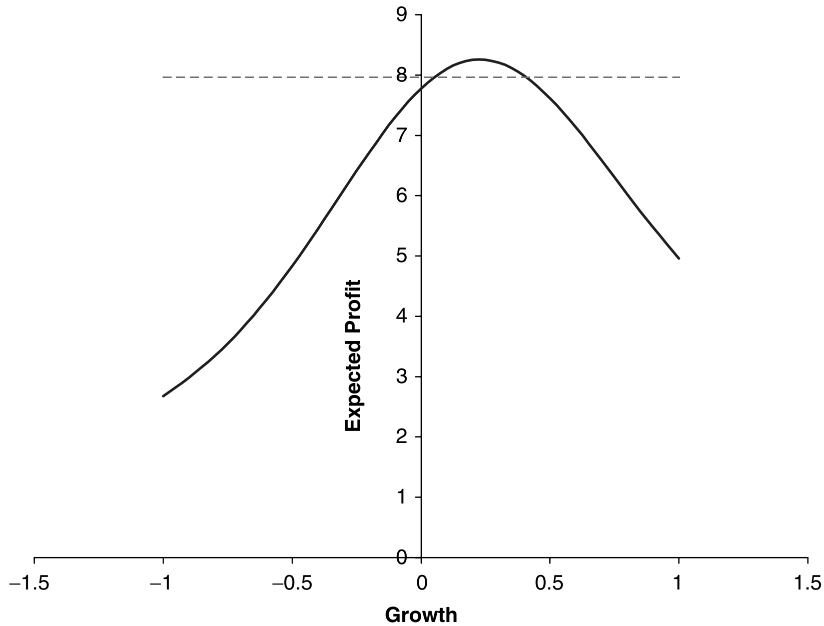

Figure 12.4 Expected profit, hedging using implied volatility, versus growth rate μ; $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $E = 110$, $T = 1$, $\tilde{\sigma} = 0.2$. The dashed line is the profit to be made when hedging with actual volatility.

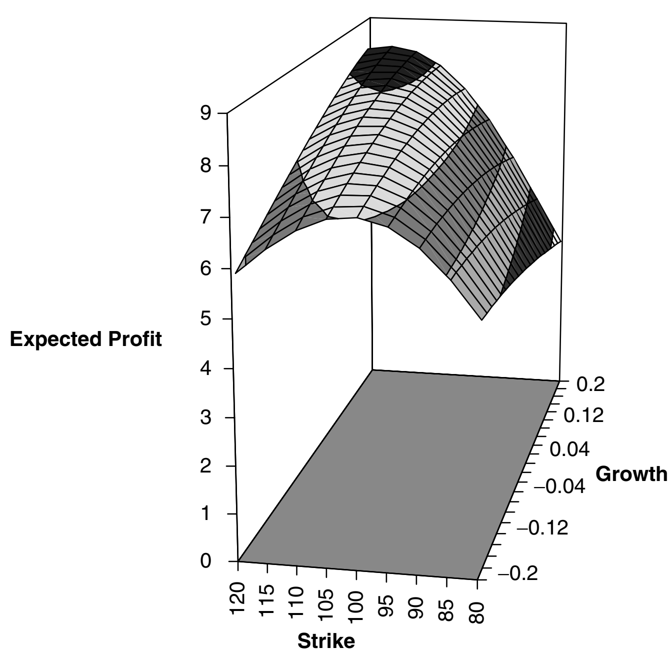

Figure 12.5 Expected profit, hedging using implied volatility, versus growth rate μ and strike E; $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $T = 1$, $\tilde{\sigma} = 0.2$.

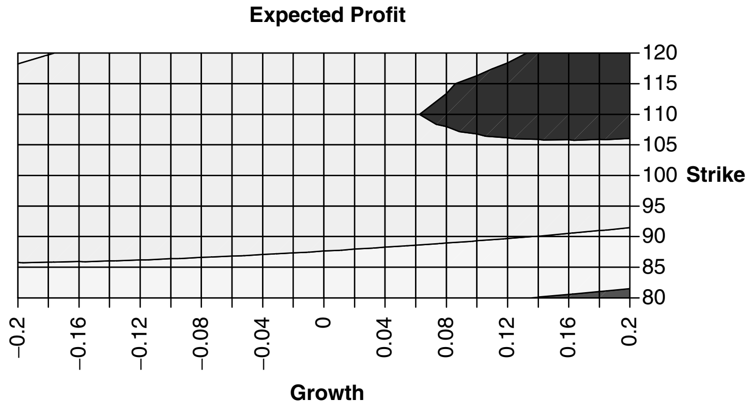

Figure 12.6 Contour map of expected profit, hedging using implied volatility, versus growth rate μ and strike E; $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $T = 1$, $\tilde{\sigma} = 0.2$.

The effect of skew is shown in Figure 12.7. Here I have used a linear negative skew, from 22.5% at a strike of 75, falling to 17.5% at the 125 strike. The at-the-money implied volatility is 20% which in this case is the actual volatility. This picture changes when you divide the expected profit by the price of the option (puts for lower strikes, call for higher), see Figure 12.8. There is no maximum, profitability increases with distance away from the money. Of course, this doesn't take into account the risk, the standard deviation associated with such trades.

12.5.2 The Variance of Profit after Hedging using Implied Volatility

Once we have calculated the expected profit from hedging using implied volatility we can calculate the variance in the final profit. Using the above notation, the variance will be the expected value of I^2 less the square of the average of I. So we will need to calculate $v(S, I, t)$ where

$$\frac{\partial v}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + \mu S \frac{\partial v}{\partial S} + \tfrac{1}{2}(\sigma^2 - \tilde{\sigma}^2)\, e^{-r(t-t_0)} S^2 \Gamma^i \,\frac{\partial v}{\partial I} = 0,$$

with

$$v(S, I, T) = I^2.$$

The details of finding this function v are rather messy, but a solution can be found of the form

$$v(S, I, t) = I^2 + 2I\;H(S, t) + G(S, t).$$

The initial variance is $G(S_0, t_0) - F(S_0, t_0)^2$, where

$$G(S_0, t_0) = \frac{E^2(\sigma^2 - \tilde{\sigma}^2)^2 e^{-2r(T-t_0)}}{4\pi\sigma\tilde{\sigma}} \int_{t_0}^{T} \int_{s}^{T}$$

$$\times \frac{e^{p(u,s;S_0,t_0)}}{\sqrt{s-t_0}\sqrt{T-s}\sqrt{\sigma^2(u-s)+\tilde{\sigma}^2(T-u)}\sqrt{\frac{1}{\sigma^2(s-t_0)} + \frac{1}{\tilde{\sigma}^2(T-s)} + \frac{1}{\sigma^2(u-s)+\tilde{\sigma}^2(T-u)}}}\, du\, ds \qquad (12.3)$$

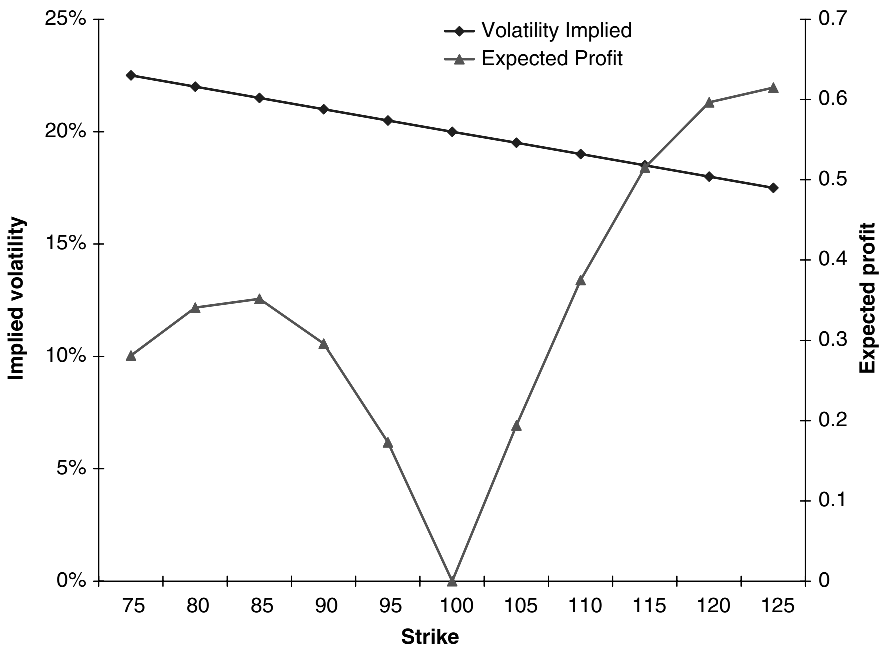

Figure 12.7 Effect of skew, expected profit, hedging using implied volatility, versus strike E; $S = 100$, $\mu = 0$, $\sigma = 0.2$, $r = 0.05$, $D = 0$, $T = 1$.

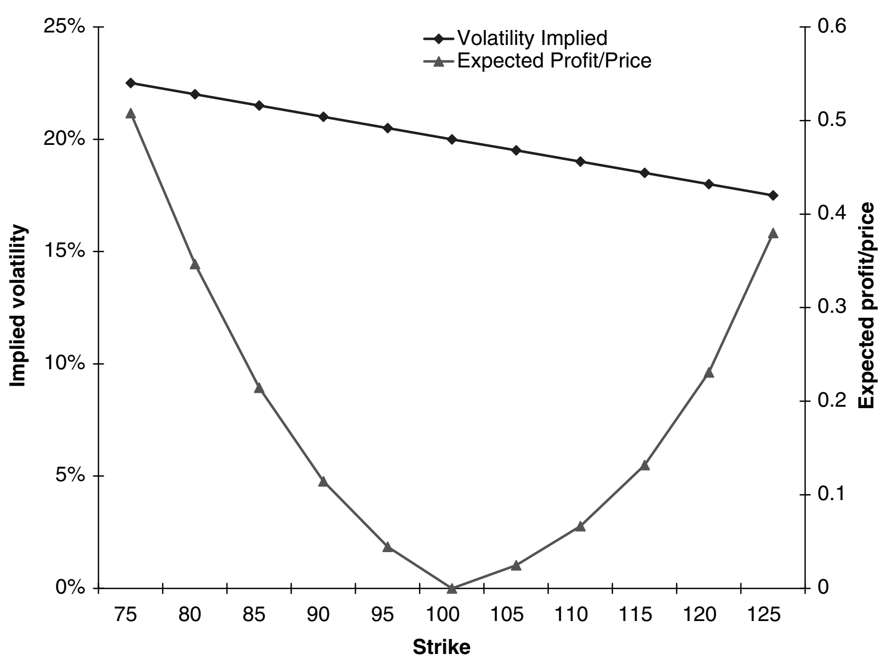

Figure 12.8 Effect of skew, ratio of expected profit to price, hedging using implied volatility, versus strike E; $S = 100$, $\mu = 0$, $\sigma = 0.2$, $r = 0.05$, $D = 0$, $T = 1$.

where

$$p(u,s;S_0,t_0) = -\tfrac{1}{2}\frac{(x+\alpha(T-s))^2}{\tilde{\sigma}^2(T-s)} - \tfrac{1}{2}\frac{(x+\alpha(T-u))^2}{\sigma^2(u-s)+\tilde{\sigma}^2(T-u)}$$

$$+\tfrac{1}{2}\frac{\left(\dfrac{x+\alpha(T-s)}{\tilde{\sigma}^2(T-s)} + \dfrac{x+\alpha(T-u)}{\sigma^2(u-s)+\tilde{\sigma}^2(T-u)}\right)^2}{\dfrac{1}{\sigma^2(s-t_0)} + \dfrac{1}{\tilde{\sigma}^2(T-s)} + \dfrac{1}{\sigma^2(u-s)+\tilde{\sigma}^2(T-u)}}$$

and

$$x = \ln(S_0/E) + \left(\mu - \tfrac{1}{2}\sigma^2\right)(T-t_0), \quad \text{and} \quad \alpha = \mu - \tfrac{1}{2}\sigma^2 - r + D + \tfrac{1}{2}\tilde{\sigma}^2.$$

The derivation of this can be found in the appendix to this chapter.

In Figure 12.9 is shown the standard deviation of profit versus growth rate, $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $E = 110$, $T = 1$, $\tilde{\sigma} = 0.2$. Figure 12.10 shows the standard deviation of profit versus strike, $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $\mu = 0.1$, $T = 1$, $\tilde{\sigma} = 0.2$.

Note that in these plots the expectations and standard deviations have not been scaled with the cost of the options.

In Figure 12.11 is shown expected profit divided by cost versus standard deviation divided by cost, as both strike and expiration vary. In these plots $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $\mu = 0.1$, $\tilde{\sigma} = 0.2$. To some extent, although we emphasize only *some*, these diagrams can be interpreted in a classical mean-variance manner, see Chapter 18. The main criticism is, of course, that we are not working with Normal distributions, and, furthermore, there is no downside, no possibility of any losses.

Figure 12.12 completes the earlier picture for the skew, since it now contains the standard deviation.

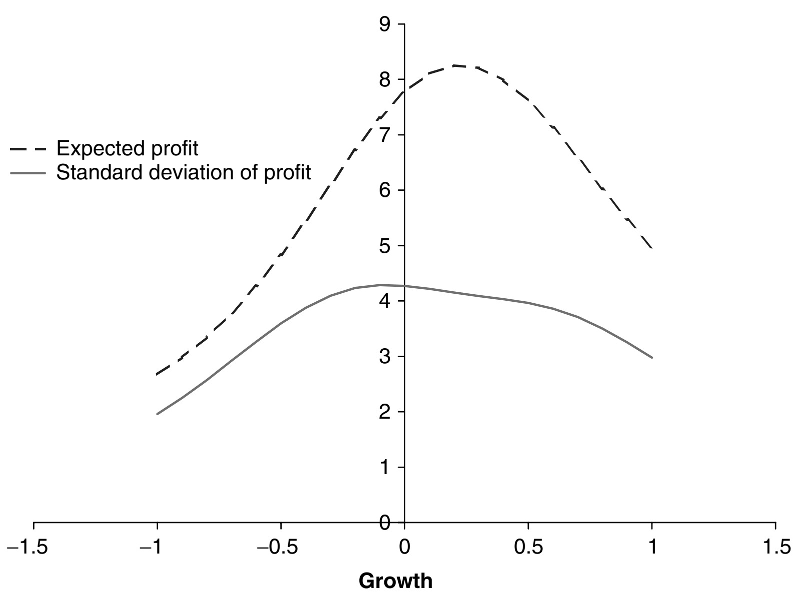

Figure 12.9 Standard deviation of profit, hedging using implied volatility, versus growth rate μ; $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $E = 110$, $T = 1$, $\tilde{\sigma} = 0.2$. (The expected profit is also shown.)

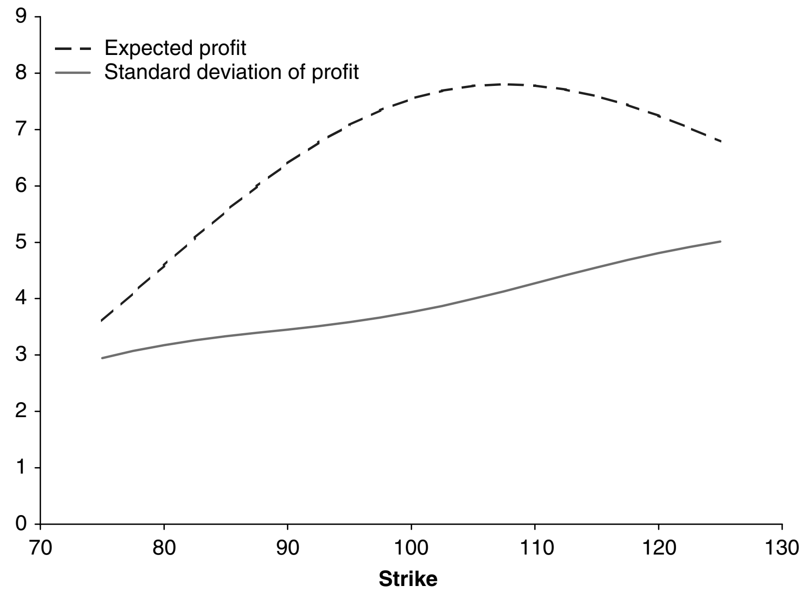

Figure 12.10 Standard deviation of profit, hedging using implied volatility, versus strike E; $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $\mu = 0$, $T = 1$, $\tilde{\sigma} = 0.2$. (The expected profit is also shown.)

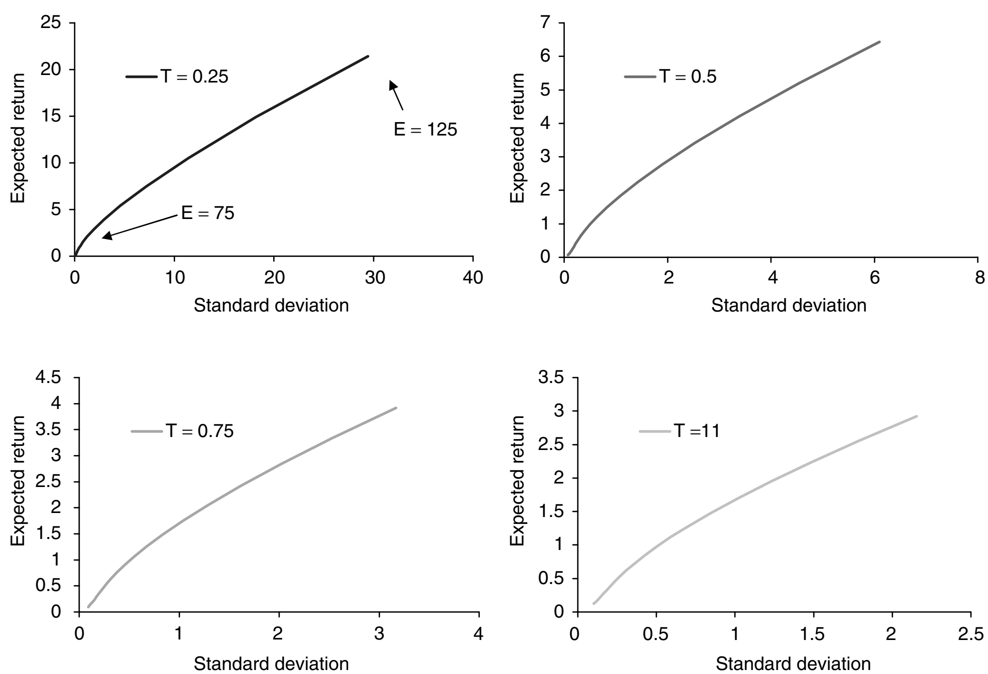

Figure 12.11 Scaled expected profit versus scaled standard deviation; $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $\mu = 0.1$, $\tilde{\sigma} = 0.2$. Four different expirations, varying strike.

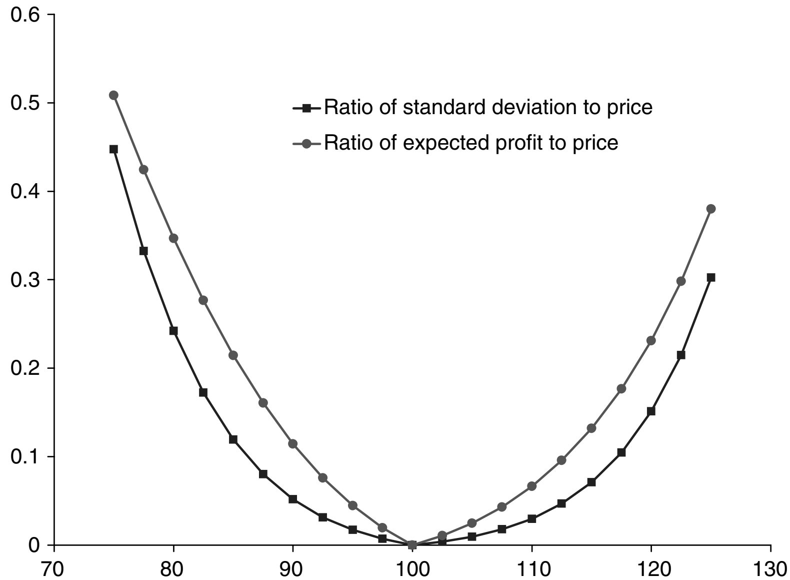

Figure 12.12 Effect of skew, ratio of expected profit to price, and ratio of standard deviation to price, versus strike E; $S = 100$, $\mu = 0$, $\sigma = 0.2$, $r = 0.05$, $D = 0$, $T = 1$.

12.5.3 Hedging with Different Volatilities

We will briefly examine hedging using volatilities other than actual or implied, using the general expression for profit given by (12.2).

The expressions for the expected profit and standard deviations now must allow for the $V(S, t; \sigma_h) - V(S, t; \tilde{\sigma})$, since the integral of gamma term can be treated as before if one replaces $\tilde{\sigma}$ with σ_h in this term. Results are presented in the next two figures.

In Figure 12.13 is shown the expected profit and standard deviation of profit when hedging with various volatilities. The thin, dotted lines, continuing on from the bold lines, represent hedging with volatilities outside the implied-actual range. The chart also shows standard deviation of profit, and minimum and maximum. Parameters are $E = 90$, $S = 100$, $\mu = -0.1$, $\sigma = 0.4$, $r = 0.1$, $D = 0$, $T = 1$, and $\tilde{\sigma} = 0.2$. Note that it is possible to lose money if you hedge at below implied, but hedging with a higher volatility you will not be able to lose until hedging with a volatility of approximately 70%. In this example, the expected profit decreases with increasing hedging volatility.

Figure 12.14 shows the same quantities but now for an option with a strike price of 110. The upper hedging volatility, beyond which it is possible to make a loss, is now slightly higher. The expected profit now increases with increasing hedging volatility.

In practice which volatility one uses is often determined by whether one is constrained to mark to market or mark to model. If one is able to mark to model then one is not necessarily concerned with the day-to-day fluctuations in the mark-to-market profit and loss and so it is natural to hedge using actual volatility. This is usually not far from optimal in the sense of possible expected total profit, and it has no standard deviation of final profit. However, it is common to have to report profit and loss based on market values. This constraint may be imposed by a risk management department, by prime brokers, or by investors who may monitor