Explanation of Shape Function in Finite Element Method

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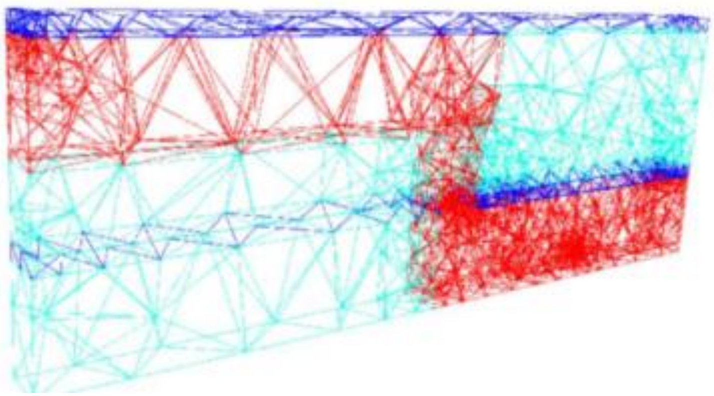
Contents

Abstract

Shape Function:

- In the finite element method, continuous models are approximated using the information at a finite number of discrete locations. Dividing the structure into discrete elements is called **discretization.**
- Interpolation within the elements is achieved through **shape functions,**
- Basically, the shape function is a function that interpolates the value between discrete mesh nodes. Lower order polynomials are chosen as shape functions. The shape function is a displacement function as well as an interpolation function.

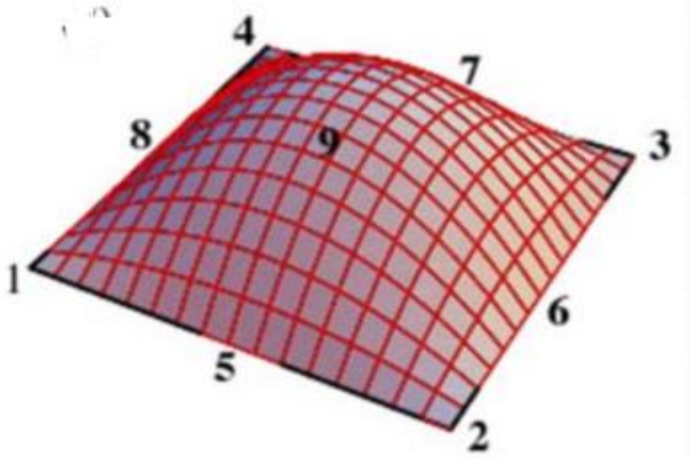


Finite element mesh of a three-dimensional interconnect structure discretized with tetrahedrons

Introduction

Shape Function

- The shape function is the function that interpolates the the solution between the discrete values obtained at the mesh nodes.



Shape Function Plots

Requirements

- **Interpolation**
- **Local Support**
- **Continuity**
- **Completeness**

Classifications

- **Element Form**
- **Polynomial Degree**
- **Type of Shape Functions**

Properties of Shape Function

- **Kronecker delta Property**
- **Lagrangian Shape Functions**
- **Serendipity Shape Function**

Examples

- **3D**

BASIC CONCEPT

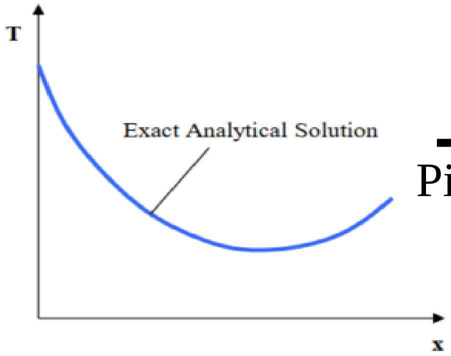
Discretization Concept

Continuous solution field

- Stress
- Displacement
- Temperature
- Pressure etc

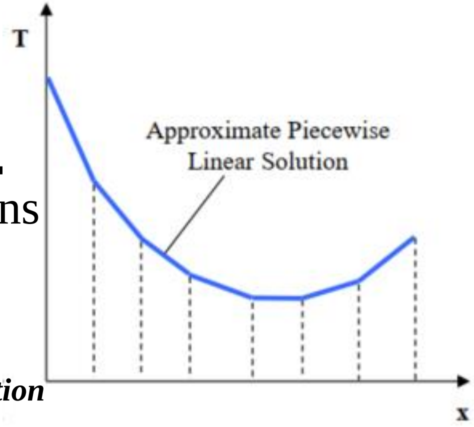
discrete model → Piecewise continuous functions

Finite number of Subdomains

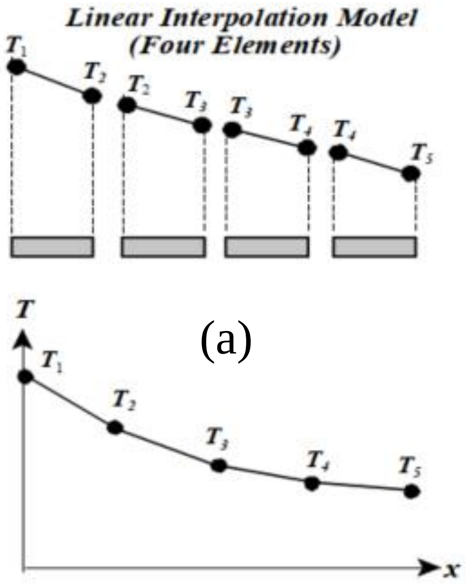


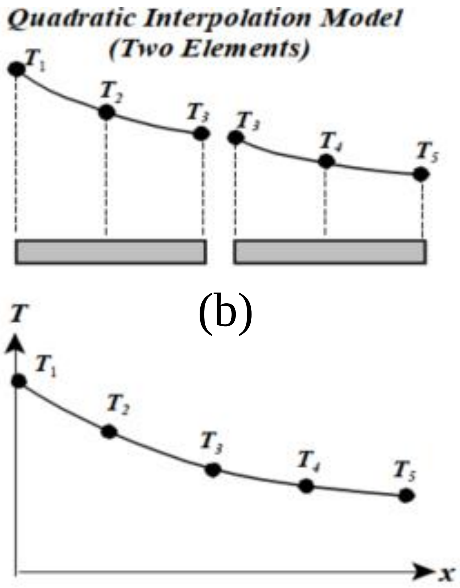
discrete model

Piecewise continuous functions



One-Dimensional Temperature Distribution





(a)Piecewise Linear Approximation

Temperature Continuous but with Discontinuous Temperature Gradients

(b)Piecewise Quadratic Approximation

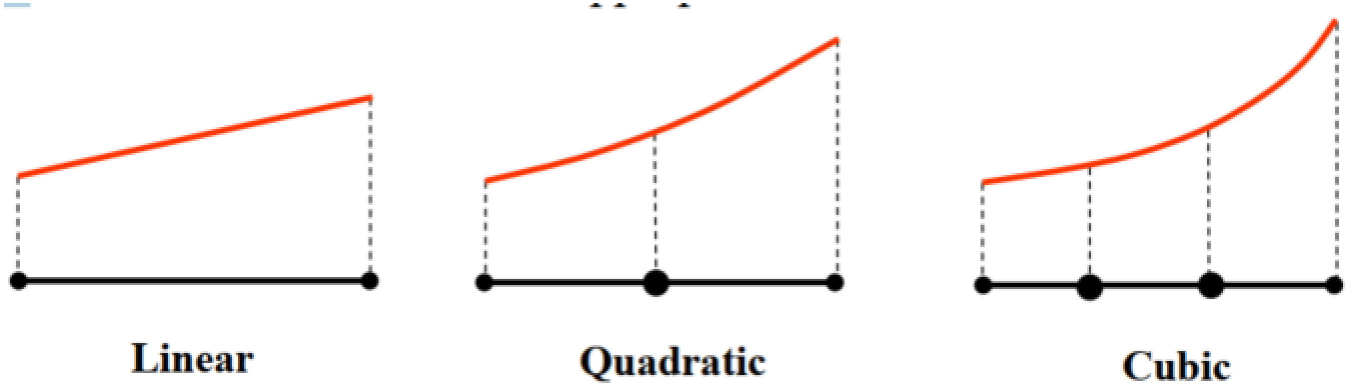
Temperature and Temperature Gradients Continuous

BASIC CONCEPT

Common Approximation Schemes

Polynomials are used to construct approximation functions for each element.

Depends on the order of approximation, different numbers of element parameters



Special Approximation
Example: Infinite elements, crack or other singular elements

Depends on the special properties, determined from theoretical considerations

Requirements

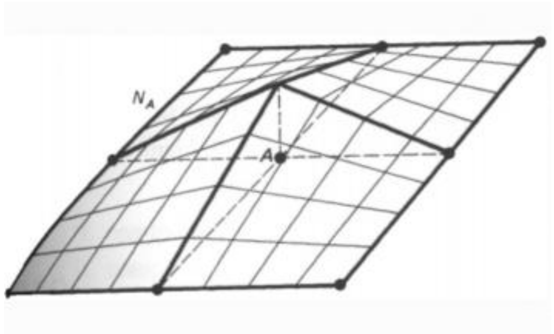
Aim to Convergence

The mesh is refined the FEM solution should approach the analytical solution of the mathematical model

Types of Requirements

- **Interpolation**
- **Local Support**
- **Continuity**
- **Completeness**
- **Compatibility**

(A) ***Interpolation condition.*** Takes a unit value at node *i* , and is zero at all other nodes.



(B) ***Local support condition.*** Vanishes over any element boundary (a side in 2D, a face in 3D) that does not include node *i*.

*(C)**Interelement compatibility** condition.* Satisfies C^0 continuity between adjacent elements over any element boundary that includes node *i*

(D)The requirement for completeness*:* The interpolation has to be able to represent:

1. the rigid body displacement
2. constant strain state

If (C) and (D) are considered together, this case can be called CONSISTENCY

(E)The requirement for compatibility: The interpolation has to be such that field of displacements is :

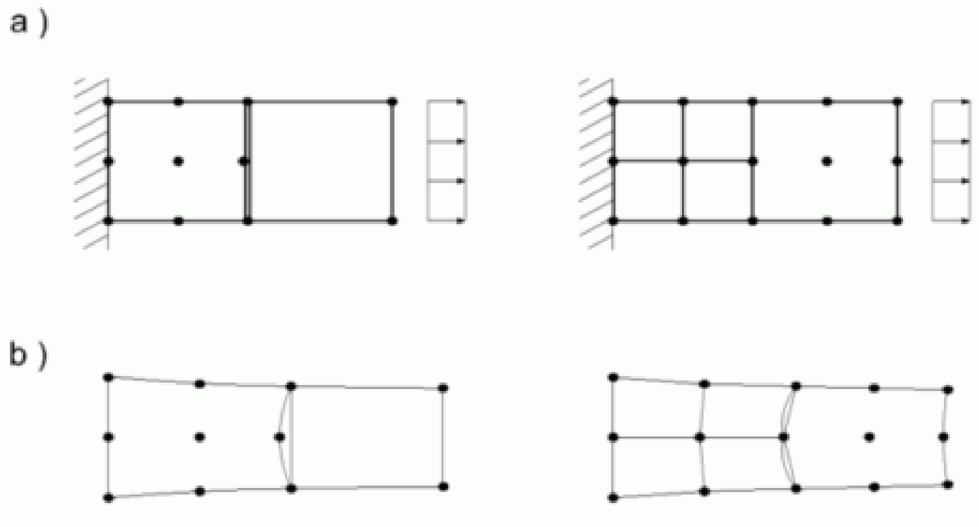
1. continual and derivable inside the element
2. continual across the element border

Requirements

Compatibility

- Provides displacement continuity between elements
- Ensures no material gaps appear as the elements deform

The shape functions must be $C^{(m-1)}$ continuous between elements, and C^{m} piecewise differentiable inside each element.



a) Discretization and load; b) Deformed shape

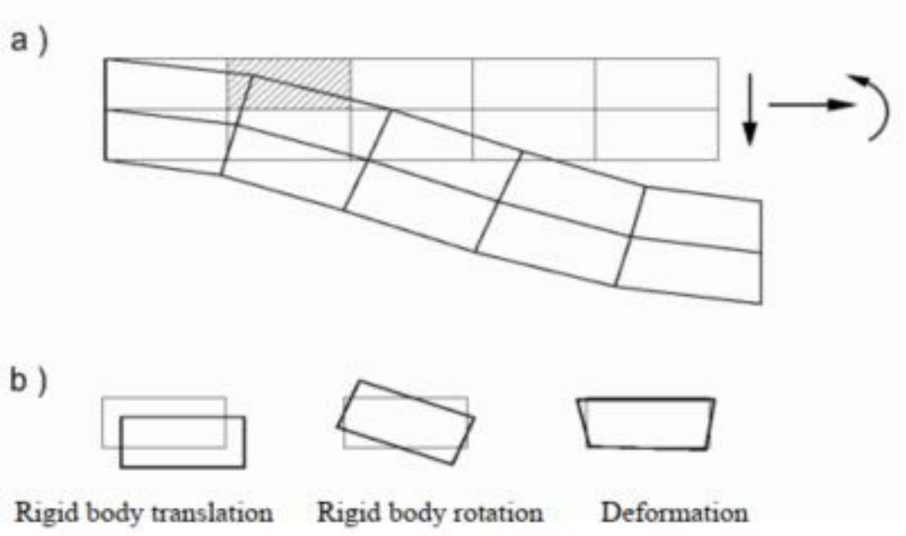
Requirements

Completeness

The interpolation has to be able to represent:

1. The rigid body displacement
2. Constant strain state

The element shape functions must represent exactly all polynomial terms of order ≤ m in the Cartesian coordinates. A set of shape functions that satisfies this condition is called m-complete



a) Deformation of the cantilever beam. B) Rigid body displacement and deformation of the hatched element

Classification

The element form:

– triangular elements,
– rectangular elements.

Type of the shape functions

– Lagrange shape functions
– serendipity shape functions
–Triangular Element Functions

Polynomial degree of the shape functions

– linear
– quadratic
– cubic
–…

Properties

Kronecker delta property

The shape function at any node has a value of 1 at that node and a value of zero at ALL other nodes.

Compatibility + Completeness ⇒ Convergence

Ensure that the solution gets better as more elements are introduced and, in the limit, approaches the exact answer

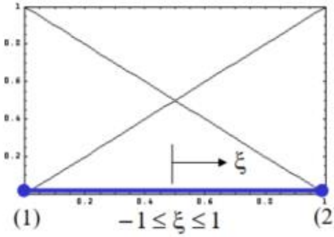
Lagrangian Shape Functions:

$$L_k^{(m)}(\xi)=\frac{(\xi-\xi_0)(\xi-\xi_1)\cdots\boxed{(\xi-\xi_{k-1})(\xi-\xi_{k+1})}\cdots(\xi-\xi_m)}{(\xi_k-\xi_0)(\xi_k-\xi_1)\cdots\boxed{(\xi_k-\xi_{k-1})(\xi_k-\xi_{k+1})}\cdots(\xi_k-\xi_m)}=\prod_{\substack{i=0\\i\neq k}}^{m}\frac{(\xi-\xi_i)}{(\xi_k-\xi_i)}.$$

No ξ-ξ_k term!

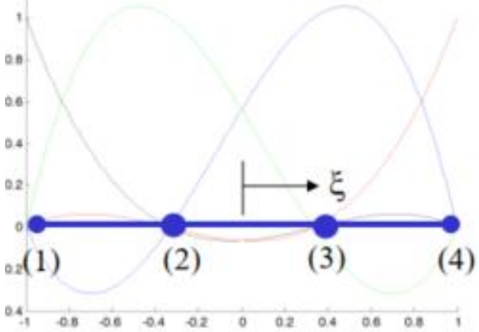
Lagrange polynomial of order m at node k

$$N_i(\xi_j)=\begin{cases}1\ , i=j\\0\ , i\neq j\end{cases}$$



$$N_1=\frac{1}{2}(1-\xi)$$

$$N_2=\frac{1}{2}(1+\xi)$$

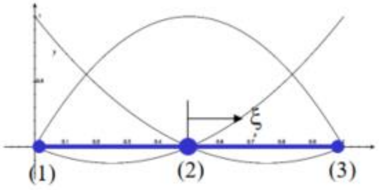


$$N_1=-\frac{9}{16}(1-\xi)(\frac{1}{3}+\xi)(\frac{1}{3}-\xi)$$

$$N_2=\frac{27}{16}(1-\xi)(1+\xi)(\frac{1}{3}-\xi)$$

$$N_3=\frac{27}{16}(1-\xi)(1+\xi)(\frac{1}{3}+\xi)$$

$$N_4=-\frac{9}{16}(\frac{1}{3}+\xi)(\frac{1}{3}-\xi)(1+\xi)$$



$$N_1=-\frac{1}{2}\xi(1-\xi)$$

$$N_2=(1-\xi)(1+\xi)$$

$$N_3=\frac{1}{2}\xi(1+\xi)$$

Properties

Lagrangian Shape Functions:

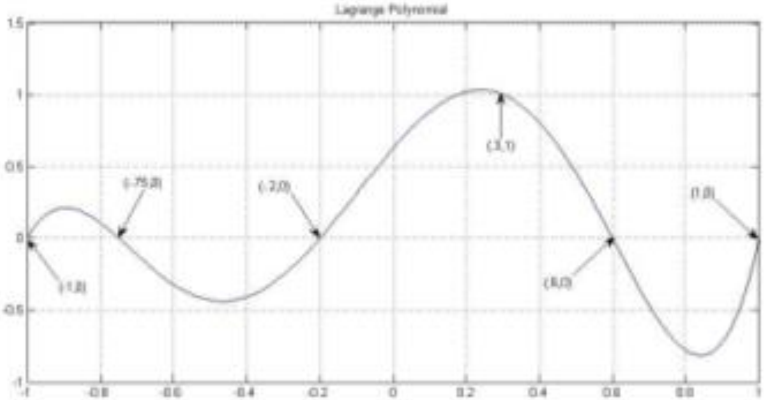
$$L_k^{(m)}(\xi)=\frac{(\xi-\xi_0)(\xi-\xi_1)\cdots\boxed{(\xi-\xi_{k-1})(\xi-\xi_{k+1})}\cdots(\xi-\xi_m)}{(\xi_k-\xi_0)(\xi_k-\xi_1)\cdots\boxed{(\xi_k-\xi_{k-1})(\xi_k-\xi_{k+1})}\cdots(\xi_k-\xi_m)}=\prod_{\substack{i=0\\i\neq k}}^{m}\frac{(\xi-\xi_i)}{(\xi_k-\xi_i)}.$$

No ξ-ξ_k term!

Lagrange polynomial of order m at node k

Automatically satisfies the Kronecker delta property for shape functions

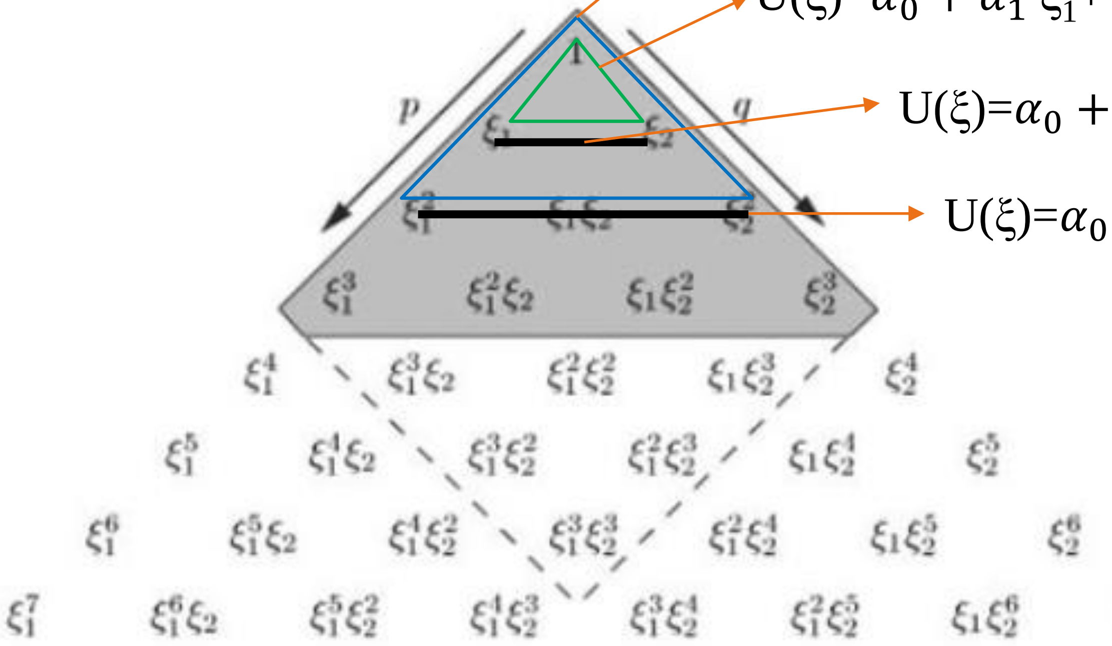
Example of 6 points; want function = 1 at and function = 0 at other designated points:



$$L_3^{(5)}(\xi)=\frac{(\xi-\xi_0)(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_4)(\xi-\xi_5)}{(\xi_3-\xi_0)(\xi_3-\xi_1)(\xi_3-\xi_2)(\xi_3-\xi_4)(\xi_3-\xi_5)}.$$

$\xi_0=-1;$
$\xi_1=-.75;$
$\xi_2=-.2;$
$\xi_3=.3;$
$\xi_4=.6;$
$\xi_5=1.$

Lagrangian Family



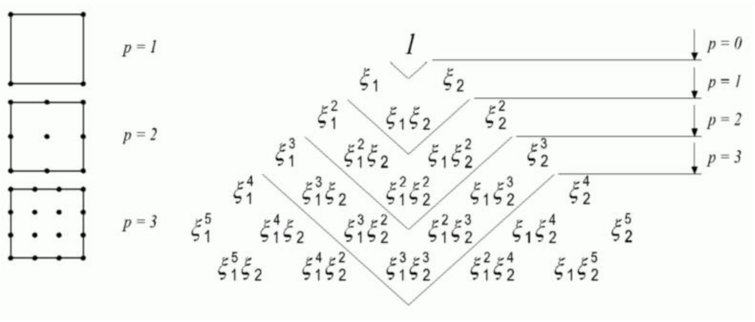
$U(\xi)=\alpha_0 + \alpha_1\ \xi_1 + \alpha_2\ \xi_2 + \alpha_3\ \xi_1^{\ 2} + \alpha_3\ \xi_1\ \xi_2 + \alpha_5\ \xi_2^{\ 2}$

$U(\xi)=\alpha_0 + \alpha_1\ \xi_1 + \alpha_2\ \xi_2$

$U(\xi)=\alpha_0 + \alpha_1\ \xi_1$

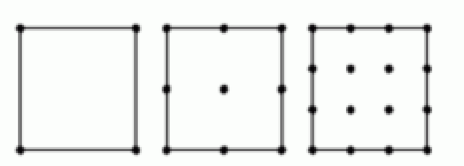
$U(\xi)=\alpha_0 + \alpha_1\ \xi_1 + \alpha_2\ \xi_1^{\ 2}$

Complete two-dimensional Lagrange polynomials in the Pascal triangle



Rectangular Elements

Order n element has $(n+1)^2$ nodes arranged in square symmetric pattern – requires internal nodes



Lagrange interpolation polynomial in one direction :

$$l_k^n(\xi_1) = \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{\xi_1 - \xi_1^i}{\xi_1^k - \xi_1^i}$$

An easy and systematic method of generating shape functions of any order now can be achieved by simple products of Lagrange polynomials in the two coordinates :

$$N_a = N_{IJ} = l_I^n(\xi_1) l_J^m(\xi_2)$$

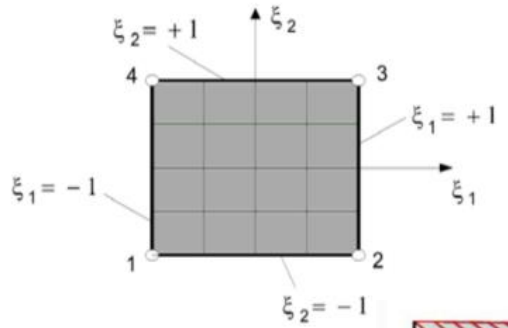
where :

$$\xi_1 = \frac{2(x-x_e)}{a_e} \quad \xi_2 = \frac{2(y-y_e)}{b_e}$$

x_e, y_e are coordinates of the center of the element

a_e, b_e are dimensions of the element

The Four-Node Bilinear Quadrilateral

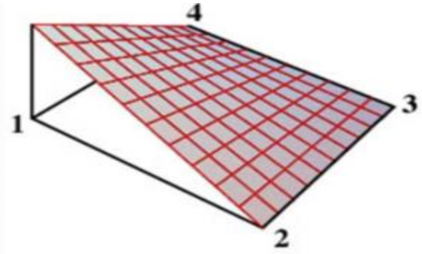


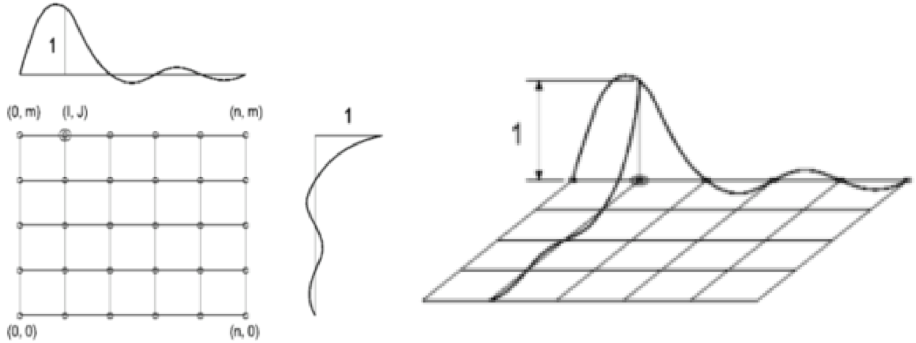
$$N^1(\xi) = \frac{1}{4}(1-\xi_1)(1-\xi_2)$$

$$N^2(\xi) = \frac{1}{4}(1+\xi_1)(1-\xi_2)$$

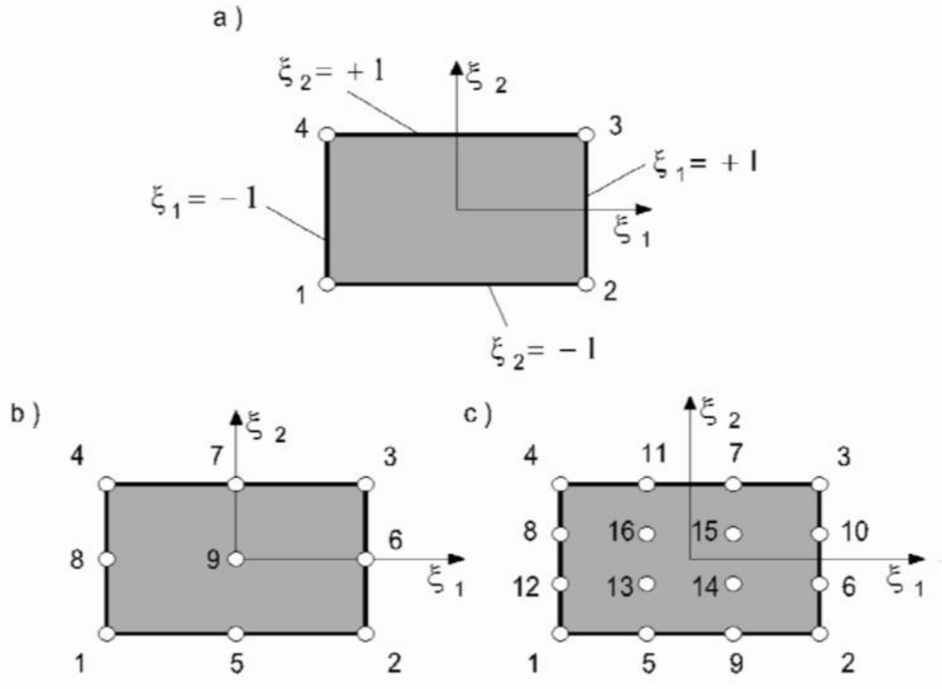
$$N^3(\xi) = \frac{1}{4}(1+\xi_1)(1+\xi_2)$$

$$N^4(\xi) = \frac{1}{4}(1-\xi_1)(1+\xi_2)$$

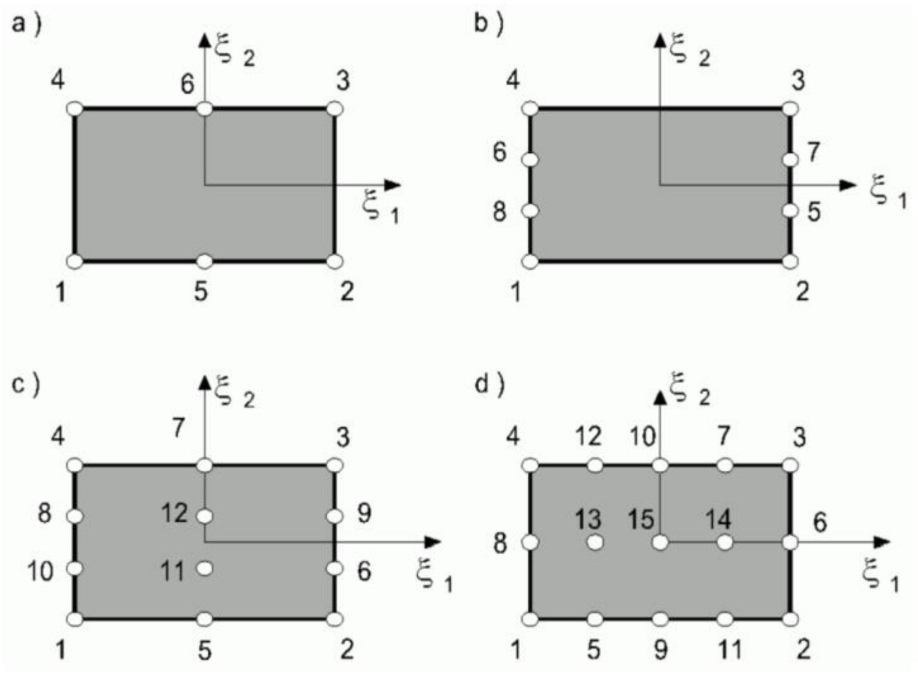




Rectangular Elements



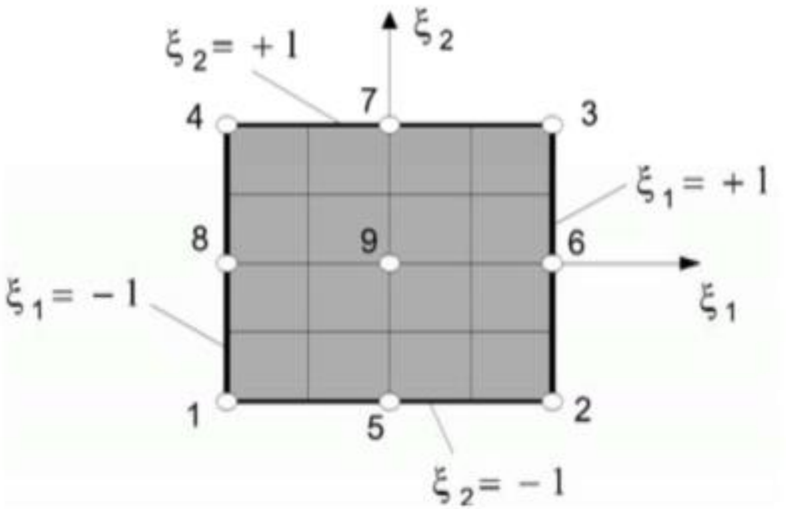
The Quadrilateral Lagrangian elements:
a) bilinear, b) biquadratic c) bicubic

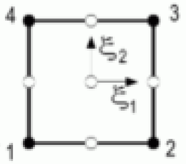


The Quadrilateral Lagrangian elements:
a) quadratic-linear, b) linear-cubic
c)quadratic-cubic, d) quartic-quadratic

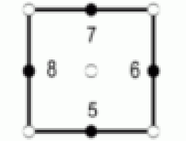
Rectangular Elements

The Nine-Node Biquadratic Quadrilateral



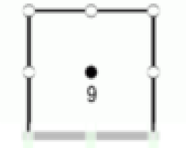


$$N^i = \tfrac{1}{4}(1+\xi_1^i\xi_1)\xi_1^i\xi_1(1+\xi_2^i\xi_2)\xi_2^i\xi_2$$

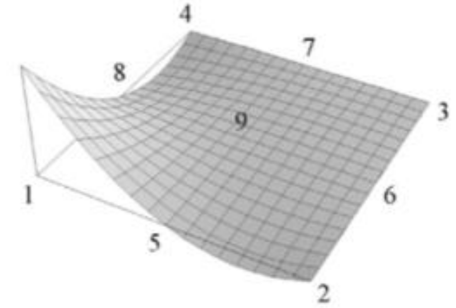


$$N^i = \tfrac{1}{2}(1-\xi_2^2)(1+\xi_1^i\xi_1)\xi_1^i\xi_1\ ,\ \xi_2^i = 0$$

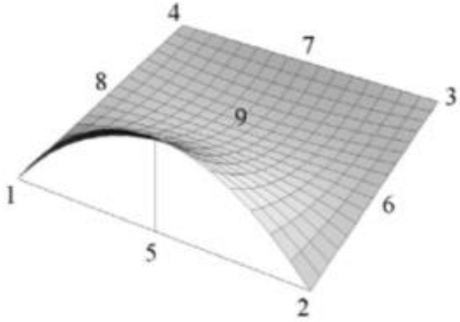
$$N^i = \tfrac{1}{2}(1-\xi_1^2)(1+\xi_2^i\xi_2)\xi_2^i\xi_2\ ,\ \xi_1^i = 0$$



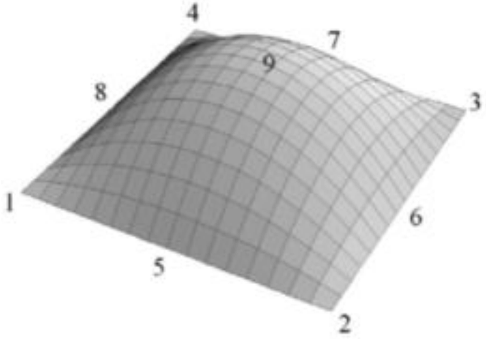
$$N^9 = (1-\xi_1^2)(1-\xi_2^2)$$



$$N^1(\xi) = \frac{1}{4}(1-\xi_1)(1-\xi_2)\,\xi_1\,\xi_2$$



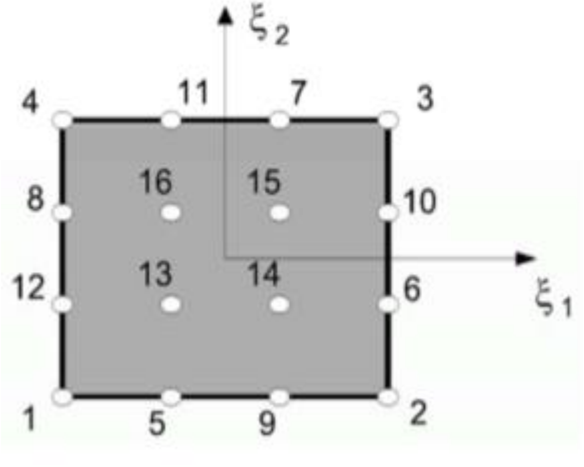
$$N^5(\xi) = \frac{1}{2}(1-\xi_1^2)\,(\xi_2-1)\xi_2$$

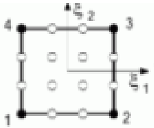


$$N^9(\xi) = (1-\xi_1^2)(1-\xi_2^2)$$

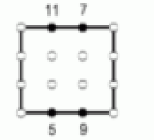
Rectangular Elements

The 16-Node Biquadratic Quadrilateral

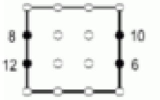




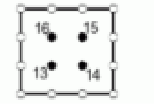
$$N^i = \tfrac{81}{256}(1+\xi_1^i\xi_1)(1+\xi_2^i\xi_2)(\tfrac{1}{9}-\xi_1^2)(\tfrac{1}{9}-\xi_2^2)$$



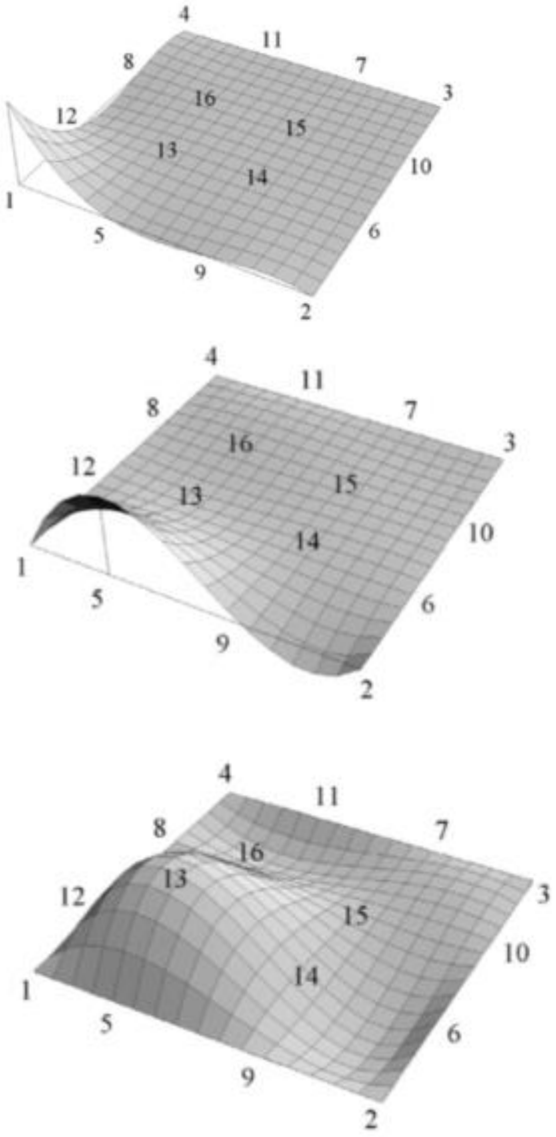
$$N^i = \tfrac{243}{256}(1-\xi_1^2)(\xi_2^2-\tfrac{1}{9})(\tfrac{1}{3}+3\xi_1^i\xi_1)(1+\xi_2^i\xi_2)$$



$$N^i = \tfrac{243}{256}(1-\xi_2^2)(\xi_1^2-\tfrac{1}{9})(\tfrac{1}{3}+3\xi_2^i\xi_2)(1+\xi_1^i\xi_1)$$

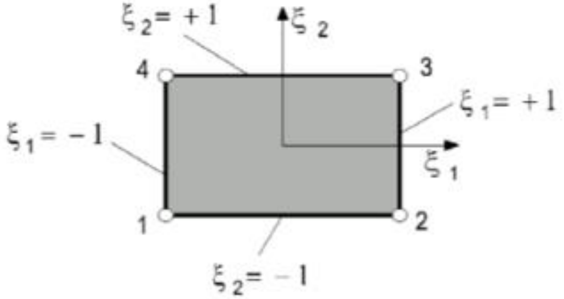


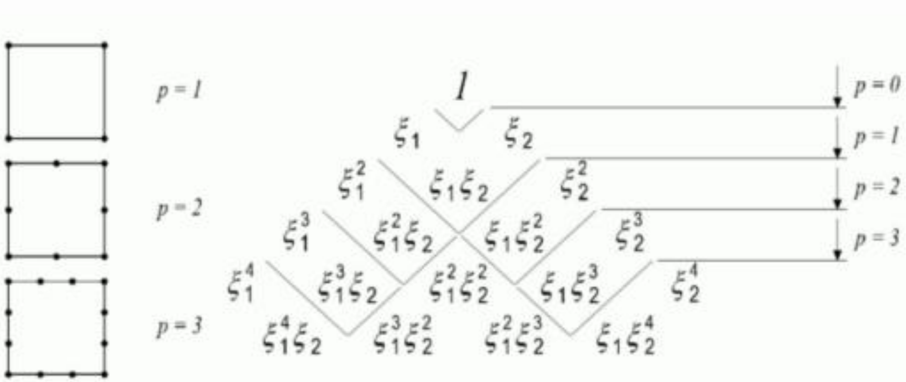
$$N^i = \tfrac{729}{256}(1-\xi_1^2)(1-\xi_2^2)(\tfrac{1}{3}+3\xi_1^i\xi_1)(\tfrac{1}{3}+3\xi_2^i\xi_2)$$



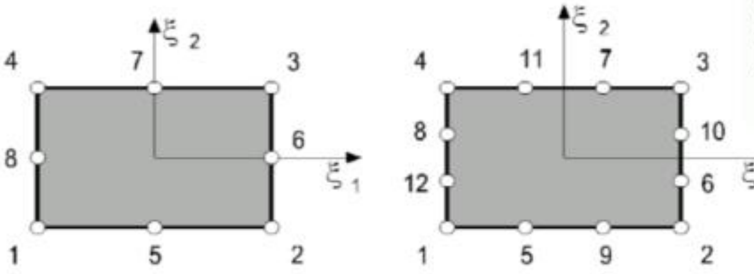
Serendipity Elements

Serendipity elements are constructed with nodes only on the element boundary

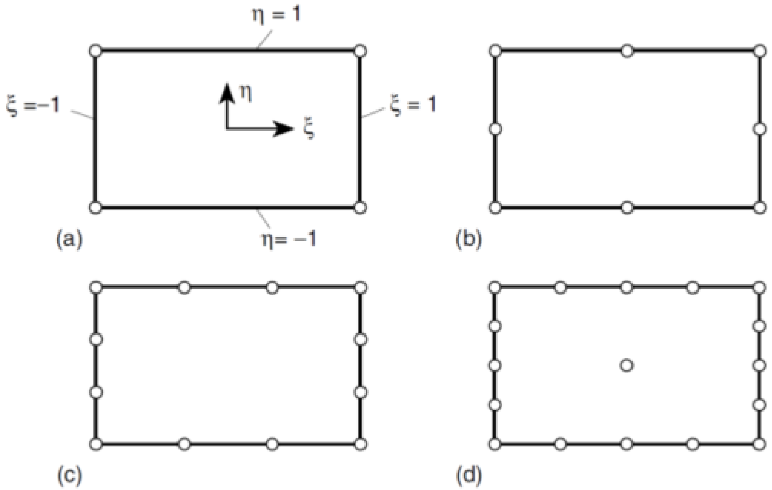




Two dimensional serendipity polynomials of quadrilateral elements in Pascal triangle



Serendipity quadrilateral elements:
a) bilinear , b) biquadratique, c) bicubic



Rectangles of boundary node (serendipity) family:
(a) linear, (b) quadratic, (c) cubic, (d) quartic

Serendipity Elements

In general serendipity shape functions can be obtained with the following expression

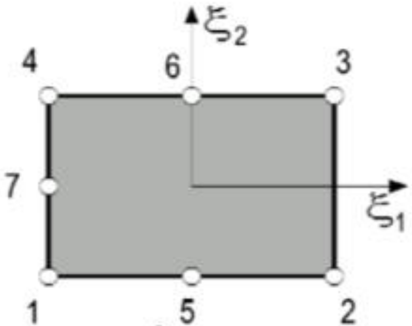
$$
\begin{aligned}
N^i(\xi_1,\xi_2) = {} & \tfrac{1}{2}(1-\xi_2)N^i(\xi_1,-1)+\tfrac{1}{2}(1+\xi_1)N^i(1,\xi_2) \\
& +\tfrac{1}{2}(1+\xi_2)N^i(\xi_1,1)+\tfrac{1}{2}(1-\xi_1)N^i(-1,\xi_2) \\
& -\tfrac{1}{4}(1-\xi_1)(1-\xi_2)N^i(-1,-1)-\tfrac{1}{4}(1+\xi_1)(1-\xi_2)N^i(1,-1) \\
& -\tfrac{1}{4}(1+\xi_1)(1+\xi_2)N^i(1,1)-\tfrac{1}{4}(1-\xi_1)(1+\xi_2)N^i(-1,1)
\end{aligned}
$$

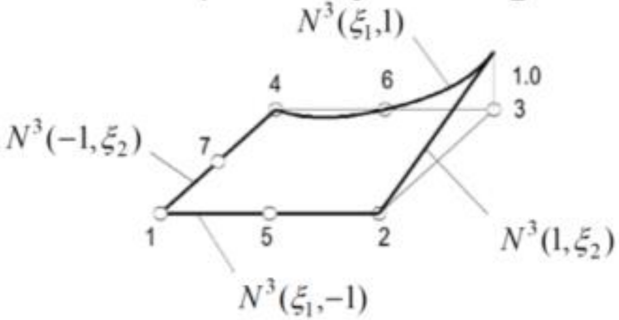
where functions $N_i(\xi_1,-1)$, $N_i(1,\xi_2)$, $N_i(\xi_1,1)$, $N_i(-1,\xi_2)$ are lagrangian interpolations along the corresponding boundary and values $N^i(-1,-1)$, $N^i(1,-1)$, $N^i(1,1)$, $N^i(-1,1)$ have values 0 or 1 and represent values of interpolation on corners

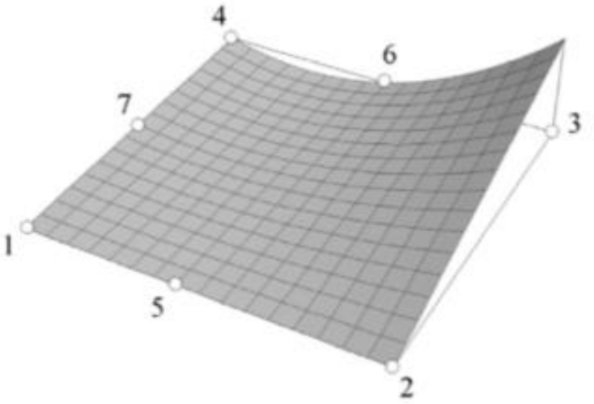
Serendipity Elements

Example: Find shape function of the node N^3

$$\begin{aligned}N^i(\xi_1,\xi_2) &= \tfrac{1}{2}(1-\xi_2)N^i(\xi_1,-1)+\tfrac{1}{2}(1+\xi_1)N^i(1,\xi_2)\\ &+\tfrac{1}{2}(1+\xi_2)N^i(\xi_1,1)+\tfrac{1}{2}(1-\xi_1)N^i(-1,\xi_2)\\ &-\tfrac{1}{4}(1-\xi_1)(1-\xi_2)N^i(-1,-1)-\tfrac{1}{4}(1+\xi_1)(1-\xi_2)N^i(1,-1)\\ &-\tfrac{1}{4}(1+\xi_1)(1+\xi_2)N^i(1,1)-\tfrac{1}{4}(1-\xi_1)(1+\xi_2)N^i(-1,1)\end{aligned}$$





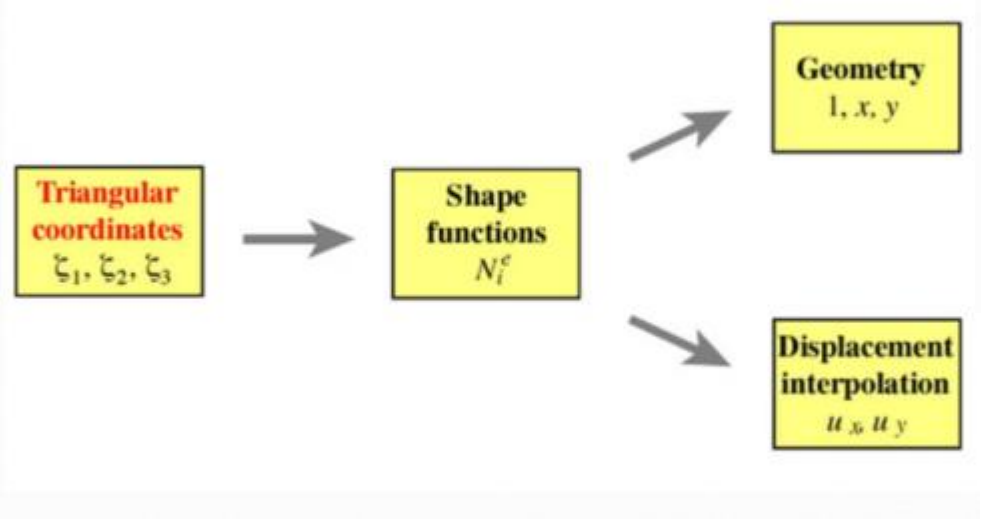


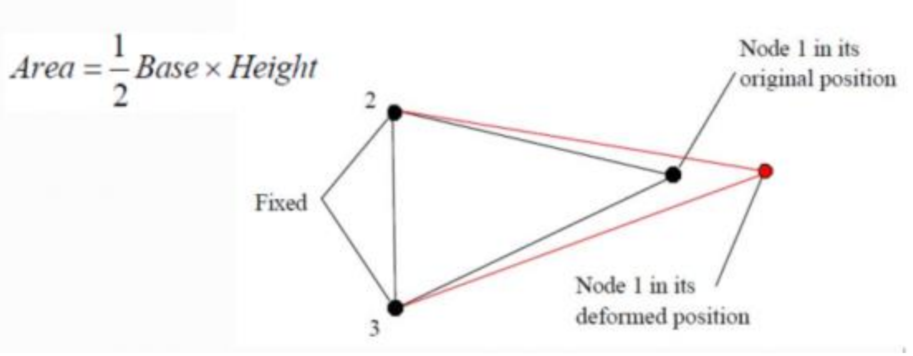
$$N^3(\xi_1,\xi_2)=\tfrac{1}{4}\xi_1(1+\xi_1)(1+\xi_2)$$

$$\begin{aligned}&N^3(\xi_1,\xi_2)=\tfrac{1}{2}(1+\xi_1)N^3(1,\xi_2)+\tfrac{1}{2}(1+\xi_2)N^3(\xi_1,1)-\tfrac{1}{4}(1+\xi_1)(1+\xi_2)N^3(1,1)=\\ &\tfrac{1}{2}(1+\xi_1)\cdot\tfrac{1}{2}(1+\xi_2)+\tfrac{1}{2}(1+\xi_2)\cdot\tfrac{1}{2}\xi_1(1+\xi_1)-\tfrac{1}{4}(1+\xi_1)(1+\xi_2)\cdot 1=\\ &\tfrac{1}{4}\xi_1(1+\xi_1)(1+\xi_2)\end{aligned}$$

Triangular Elements

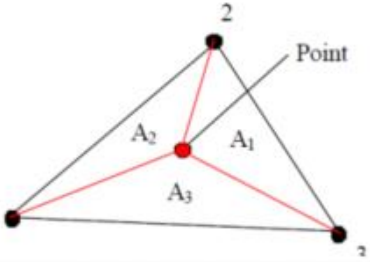
Isoparametric Representation for Triangular





$$Area = \frac{1}{2} Base \times Height$$

$$A = A_1 + A_2 + A_3$$



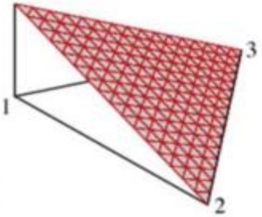
We can derive shape functions;

$$N_1 = \frac{A_1}{A},\ N_2 = \frac{A_2}{A},\ \text{and}\ N_3 = \frac{A_3}{A}$$

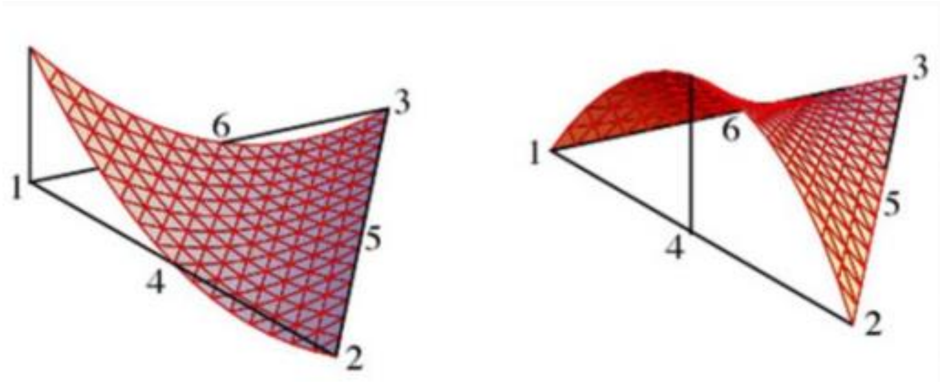
A1, A2, and A3 be the areas of each of triangular regions and A the total area of the element; According to shape function-

$$N_1 + N_2 + N_3 = 1$$

$$N_1 = \xi\ ,\ N_2 = \eta\ ,\ \text{and}\ N_3 = 1 - \xi - \eta$$

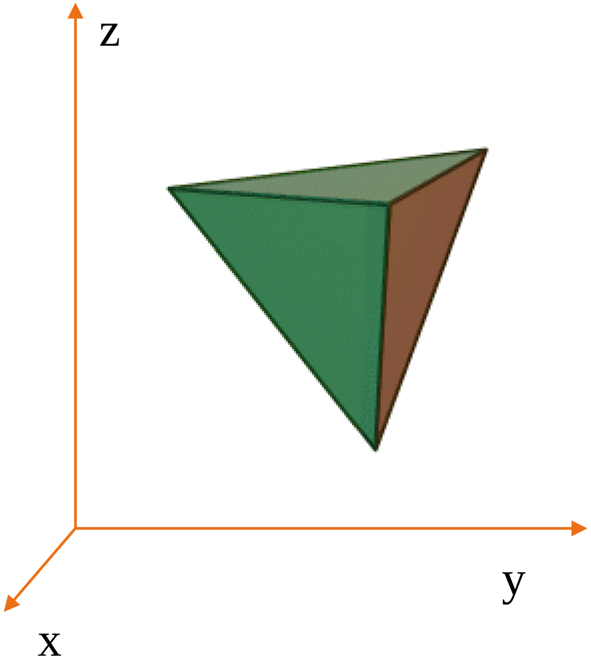


6 Node Triangle:Shape Function Plots



Example:

3D- Linear Tetrahedron element

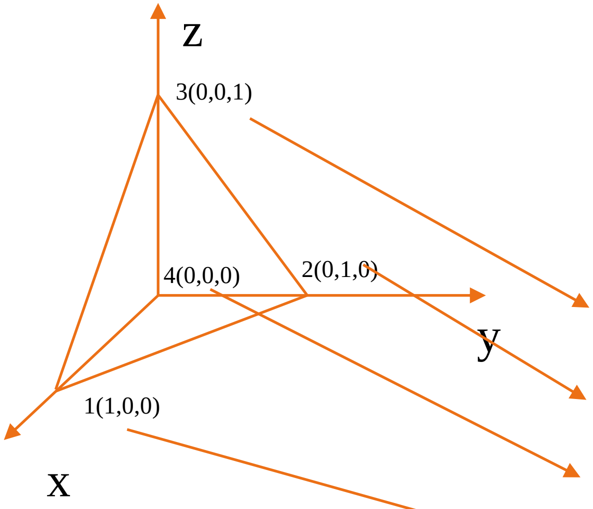


-Global Coordinates

-Simplex 3D element having four triangular faces

- 4 nodes, 3 DOF

Local Coordinates; ξ-Xee; η-eta; ζ-Zeta; (ξ, η, ζ)



$0= a_0+a_3$

$0= a_0+a_2$

$N_1=0$

$$\underbrace{1= a_0+a_1}$$

$$\underbrace{0= a_0=a_2=a_3;\ 1= a}$$

$N_1= ξ$

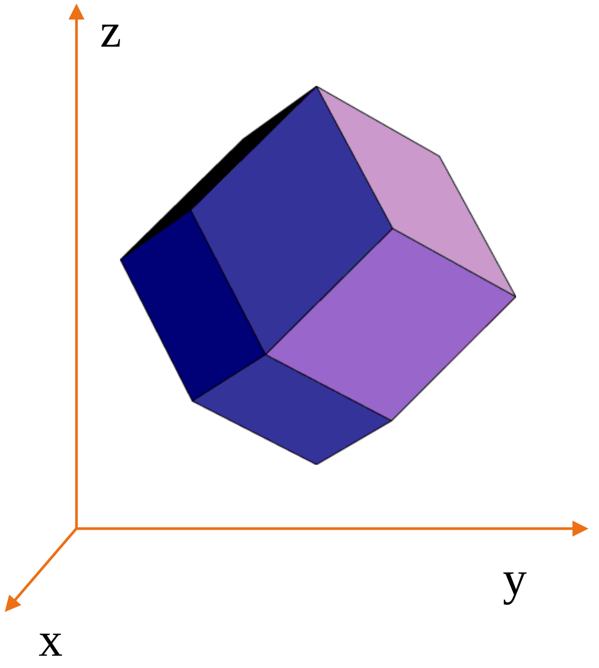
For Shape Function, N1= a_0+a_1 ξ + a_2 η + a_3 η

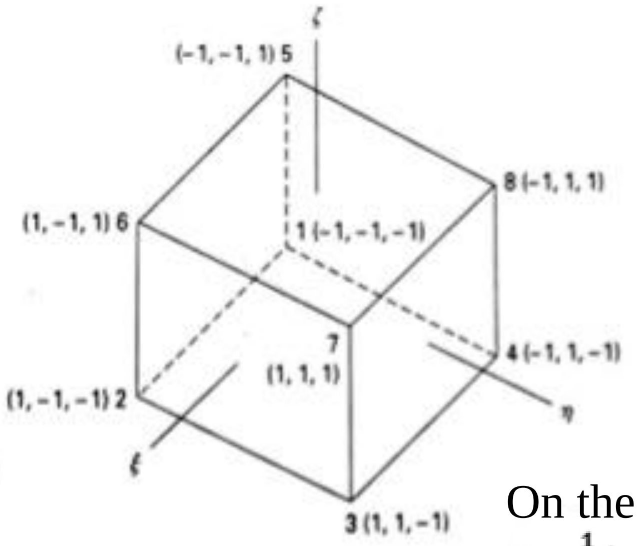
N_1= ξ Similarly, N_2= η N_3= ζ

Shape Func Law, N_4=1- ξ- η- ζ

Example:

3D- Linear Hexahedral element

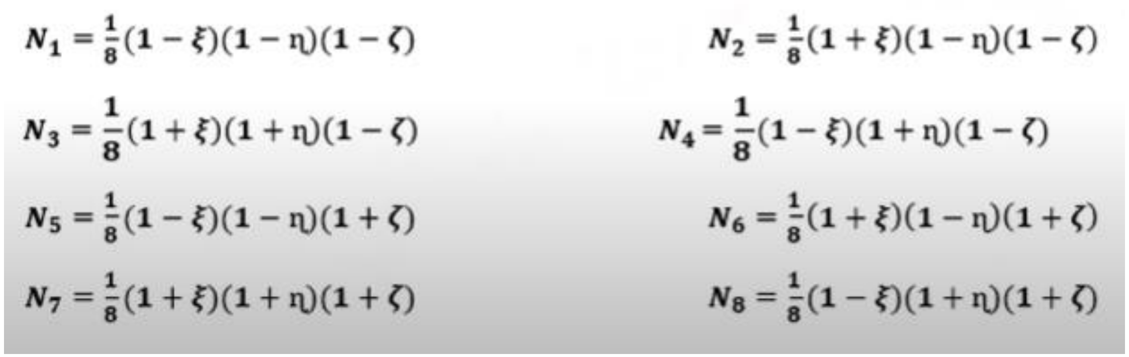




On the master cube-Lagrange Func.

$$N_i = \frac{1}{8}(1 + \xi\, \xi_i)(1 + \eta\, \eta_i)(1 + \zeta\, \zeta_i) \qquad i = 1 \text{ to } 8$$

Shape Function(Plugging the coordinates)-



$$N_1 = \frac{1}{8}(1 - \xi)(1 - \eta)(1 - \zeta) \qquad N_2 = \frac{1}{8}(1 + \xi)(1 - \eta)(1 - \zeta)$$

$$N_3 = \frac{1}{8}(1 + \xi)(1 + \eta)(1 - \zeta) \qquad N_4 = \frac{1}{8}(1 - \xi)(1 + \eta)(1 - \zeta)$$

$$N_5 = \frac{1}{8}(1 - \xi)(1 - \eta)(1 + \zeta) \qquad N_6 = \frac{1}{8}(1 + \xi)(1 - \eta)(1 + \zeta)$$

$$N_7 = \frac{1}{8}(1 + \xi)(1 + \eta)(1 + \zeta) \qquad N_8 = \frac{1}{8}(1 - \xi)(1 + \eta)(1 + \zeta)$$

https://www.youtube.com/watch?v=XRrEI8V0kIw&list=WL&index=3

Conclusions

- Shape Function properties.
- Example.
- Application.

References

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2. www.sanfoundry.com
3. Zienkiewicz O.C. , Taylor R.L. , Zhu J.Z. : The Finite Element Method: Its Basis and Fundamentals, 6. Edition, Elsevier Butterworth-Heinemann, 2005.

Thank You