Explanation of Gaussian Quadrature in Finite Element Method

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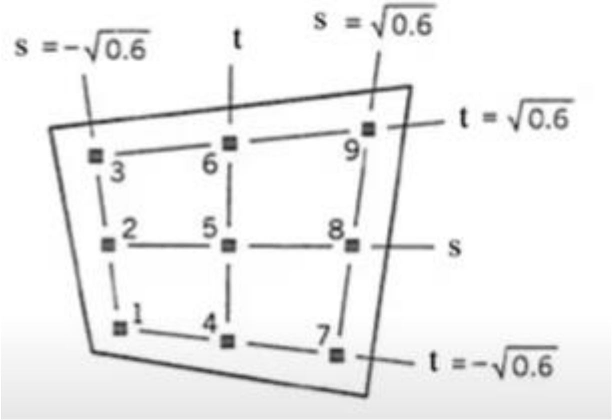
Contents

Abstract

Gaussian Quadrature:

- One of the Numerical Integration Method
- Applicable in solving FEA problems using Natural co-ordinates.
- Gauss quadrature, often known as "the" Gaussian quadrature or Legendre quadrature, is a numerical integration method. A Gaussian quadrature with a weighting function,W(x)=1 over the interval[-1,1]. The roots of the Legendre polynomials, Pn(x), which occur symmetrically around 0, provide the abscissas for quadrature order.



Introduction

Gaussian Quadrature

- In numerical analysis, a quadrature rule is an approximation of the definite integral of a function, usually stated as a weighted sum of function values at specified points within the domain of integration. An n-point Gaussian quadrature rule is a quadrature rule constructed to yield an exact result for polynomials of degree **2n − 1** or less by a suitable choice of the nodes xi and weights wi for i = 1, ..., n. The most common domain of integration for such a rule is taken as [−1, 1], so the rule is stated as

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i),$$

Topic

- **Basic Concept**
- **Problem-solving**
- **Application in FEM(2D)**

BASIC CONCEPT

Gauss Quadrature-Single integral

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i).$$

Gauss Quadrature-Double integral

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dx\,dy = w_1^2 f(x_1,y_1) + w_1 w_2 f(x_1,y_2) + w_2 w_1 f(x_2,y_1) + w_2^2 f(x_2,y_2)$$

Gauss-Jacobi Quadrature

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} g(x)\,dx \approx \sum_{i=1}^{n} w_i' g(x_i') \qquad \alpha, \beta > -1,$$

Table

Number of points, n	Points, x_i		Weights, w_i	
1	0		2	
2	$\pm\frac{1}{\sqrt{3}}$	±0.57735...	1	
3	0		$\frac{8}{9}$	0.888889...
	$\pm\sqrt{\frac{3}{5}}$	±0.774597...	$\frac{5}{9}$	0.555556...
4	$\pm\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}$	±0.339981...	$\frac{18+\sqrt{30}}{36}$	0.652145...
	$\pm\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}$	±0.861136...	$\frac{18-\sqrt{30}}{36}$	0.347855...
5	0		$\frac{128}{225}$	0.568889...
	$\pm\frac{1}{3}\sqrt{5-2\sqrt{\frac{10}{7}}}$	±0.538469...	$\frac{322+13\sqrt{70}}{900}$	0.478629...
	$\pm\frac{1}{3}\sqrt{5+2\sqrt{\frac{10}{7}}}$	±0.90618...	$\frac{322-13\sqrt{70}}{900}$	0.236927...

The weights are given by the formula

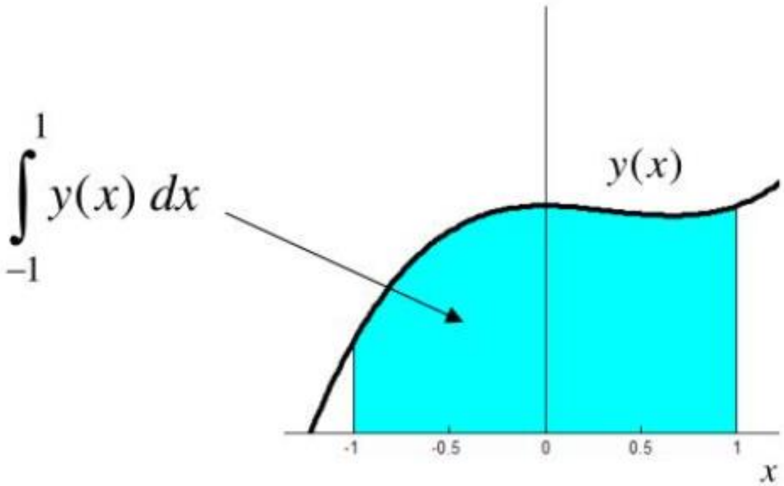
$$w_i = \frac{2}{\left(1-x_i^2\right)\left[P_n'(x_i)\right]^2}$$

BASIC CONCEPT

Gauss Quadrature-Single integral

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i),$$

Consider Sigle Integral of the form



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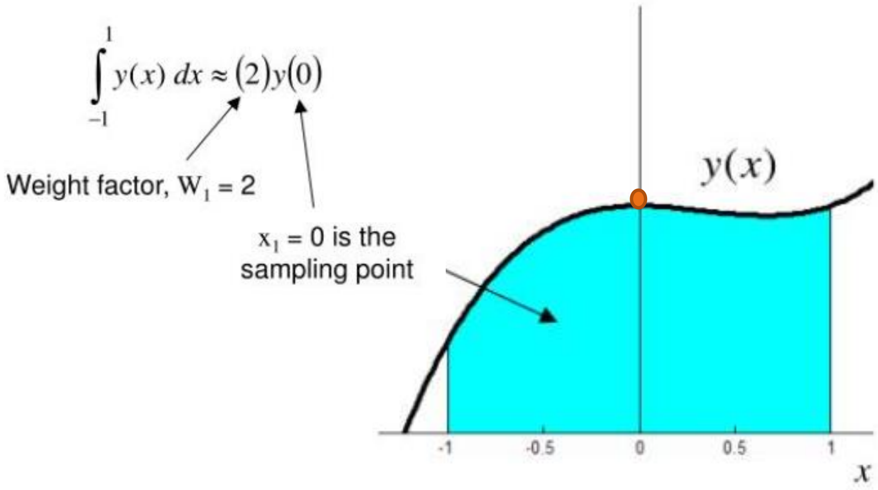
BASIC CONCEPT

Gauss Quadrature-Single integral

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i),$$

Function at one point(n=1)

Result-Y(x) is the first-order polynomial(exact)



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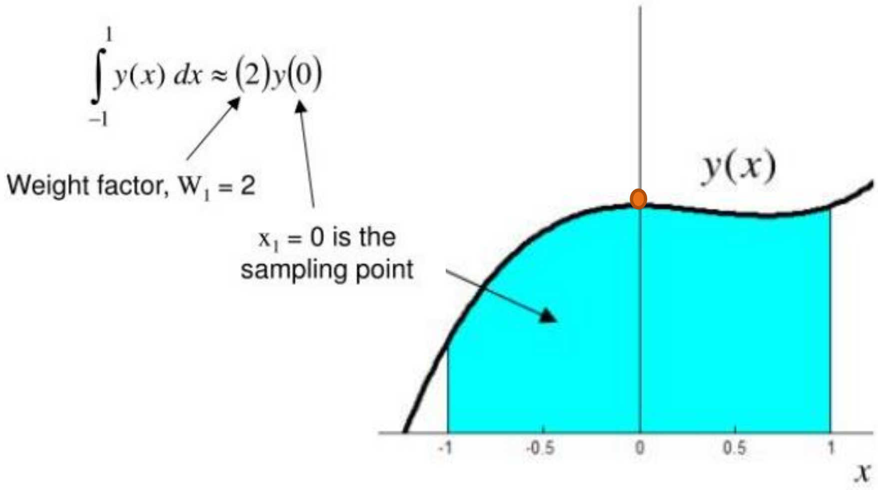


BASIC CONCEPT

Gauss Quadrature-Single integral

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i),$$

Function at two points(n=2)

Result-Y(x) is the 3rd-order polynomial(exact)

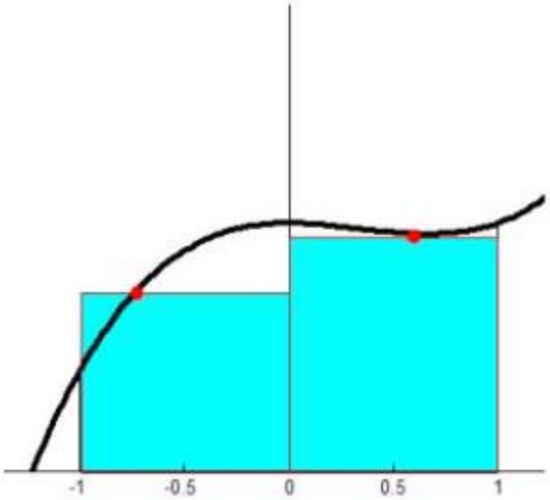
Number of points, n	Points, x_i		Weights, w_i
1	0		2
2	$\pm\frac{1}{\sqrt{3}}$	±0.57735...	1

$$\int_{-1}^{1} y(x)\,dx \approx W_1 f(x_1) + W_2 f(x_2)$$

$$W_1 = W_2 = 1$$

$$x_1 = -.57735$$

$$x_1 = +.57735$$



BASIC CONCEPT

Gauss Quadrature-Single integral

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i),$$

Function at three points(n=3)

Result-Y(x) is the 5th-order polynomial(exact)

Number of points, n	Points, x_i		Weights, w_i	
1	0		2	
2	$\pm\frac{1}{\sqrt{3}}$	±0.57735...	1	
3	0		$\frac{8}{9}$	0.888889...
	$\pm\sqrt{\frac{3}{5}}$	±0.774597...	$\frac{5}{9}$	0.555556...

$$\int_{-1}^{1} y(x)\,dx \approx W_1 f(x_1) + W_2 f(x_2) + W_3\ f(x_3)$$

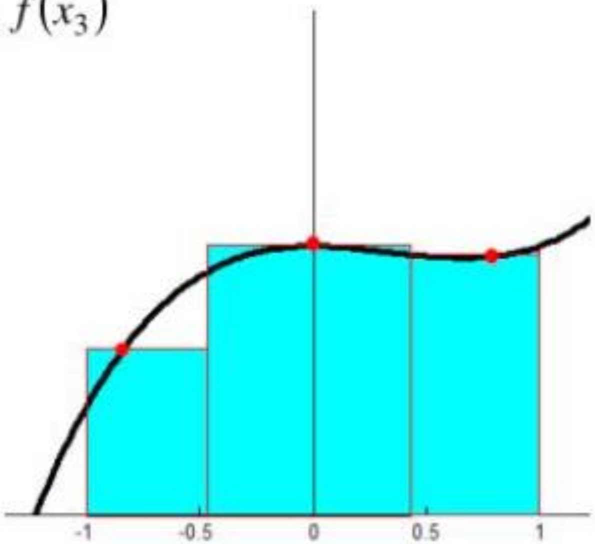
$$W_1 = W_3 = \frac{5}{9} = 0.5555...$$

$$W_2 = \frac{8}{9} = 0.8888...$$

$$x_1 = -.774597$$

$$x_2 = 0$$

$$x_3 = +.774597$$



Example

$$\int_{-1}^{1} (x^4 - 3x + 7)\,dx.$$

problem

Gauss Quadrature-Single integral

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i),$$

Number of points, n	Points, x_i		Weights, w_i	
1	0		2	
2	$\pm\frac{1}{\sqrt{3}}$	±0.57735...	1	
3	0		$\frac{8}{9}$	0.888889...
	$\pm\sqrt{\frac{3}{5}}$	±0.774597...	$\frac{5}{9}$	0.555556...

Example

$$\int_{-1}^{1} (x^4 - 3x + 7)\,dx$$

4=2n-1=Higher-order

n=2.5≈ 3(three point quadrature point)

$F(x)=\int_{-1}^{1} x^4 - 3x + 7$

$x_2=0;\ x_3=+\sqrt{\frac{3}{5}}\ ;\ x_1=-\sqrt{\frac{3}{5}}$

$W_2=8/9;\ W_1,W_3=5/9$

$F(x_1)=\int_{-1}^{1} x_1^4 - 3x_1 + 7 = 7$

$F(x_2)=\int_{-1}^{1} x_2^4 - 3x_2 + 7 = 5.036$

$F(x_3)=\int_{-1}^{1} x_3^4 - 3x_3 + 7 = 9.6837$

$F(x)=\int_{-1}^{1} x^4 - 3x + 7 = W_1F(x_1) + W_2F(x_2) + W_3F(x_3) = \boxed{14.40}$

Problem

Gauss Quadrature-Double integral

$$\int_{-1}^{1}\int_{-1}^{1} F(x)\,dx\,dy \approx \sum_i \sum_j W_i W_j F(dx, dy)$$

Example

$$\int_{-1}^{1}\int_{-1}^{1} (2x^2 + 3xy + 4y^2)\,dx\,dy$$

Number of points, n	Points, x_i		Weights, w_i
1	0		2
2	$\pm\frac{1}{\sqrt{3}}$	±0.57735...	1

2=2n-1=Higher-order

n=1.5≈ 2(two point quadrature point)

$F(x)=2x^2-3xy+4y^2$

$x_2=+\sqrt{\frac{1}{3}}=y_2;\ x_1=-\sqrt{\frac{1}{3}}=y_1$

$W_1, W_2=1$

$F(x_1, y_1)=\int_{-1}^{1} 2x_1^2 - 3x_1y_1 + 4y_1^2 = 3$

$F(x_1, y_2)=\int_{-1}^{1} 2x_1^2 - 3x_1y_2 + 4y_2^2 = 1$

$F(x_2, y_1)=\int_{-1}^{1} 2x_2^2 - 3x_2y_1 + 4y_1^2 = 1$

$F(x_2, y_2)=\int_{-1}^{1} 2x_2^2 - 3x_2y_2 + 4y_2^2 = 3$

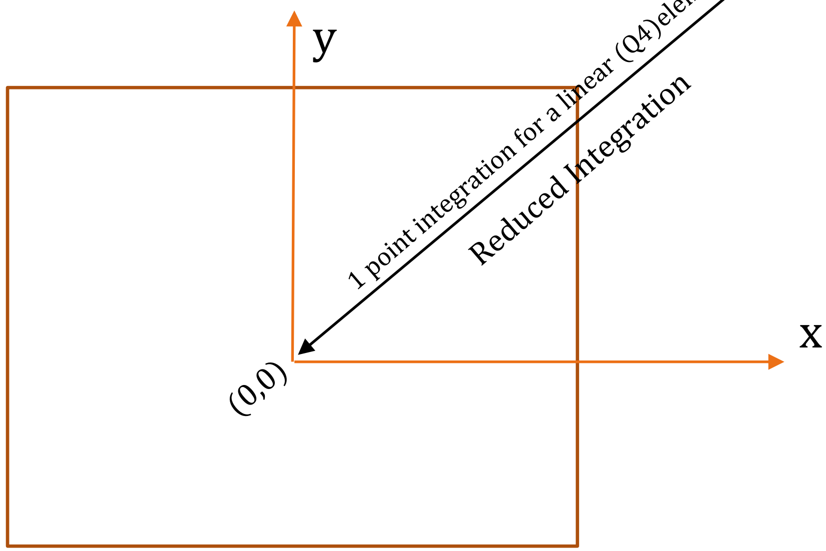
$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dx\,dy = w_1^2 f(x_1,y_1) + w_1 w_2 f(x_1,y_2) + w_2 w_1 f(x_2,y_1) + w_2^2 f(x_2,y_2) = \boxed{8}$$

Application in FEA(2D)

2D Integration- 1 Integration Point

Quadrature approach is applied to both integrals while evaluating both integral.

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i); \qquad = 2.2.\,f(0,0)$$



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Application in FEA(2D)

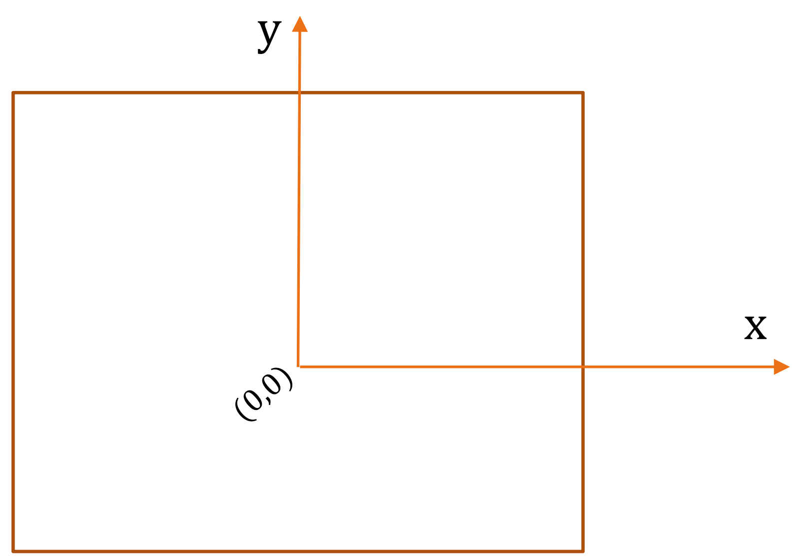
2D Integration-2x2 Integration Point

Full integration for a linear (Q4)element(exact good Result)

Reduced Integration – Q8 element(Incomplete Result)

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dx\,dy = w_1^2 f(x_1,y_1) + w_1 w_2 f(x_1,y_2) + w_2 w_1 f(x_2,y_1) + w_2^2 f(x_2,y_2)$$

4 points locations



Problem

Gauss Quadrature* Double *Integral

$$\int_{-1}^{1}\int_{-1}^{1} F(x)\,dx\,dy \approx \sum_i \sum_j W_i W_j F(dx, dy)$$

Number of points, n	Points, x_i		Weights, w_i
1	0		2
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Example

$$\int_{-1}^{1}\int_{-1}^{1} (2\,x^2 + 3\,x\,y + 4\,y^2)\,dx\,dy$$

2=2n-1=Higher-order

n=1.5≈ 2(two point quadrature point)

$$F(x) = 2x^2 - 3xy + 4y^2$$

$$x_1 = +\sqrt{\frac{1}{3}} = y_1;\ x_2 = -\sqrt{\frac{1}{3}} = y_2$$

$W_1, W_2 = 1$

$$F(x_1, y_1) = \int_{-1}^{1} 2x_1^2 - 3x_1y_1 + 4y_1{}^2 = 3$$

$$F(x_1, y_2) = \int_{-1}^{1} 2x_1^2 - 3x_1y_2 + 4y_2{}^2 = 1$$

$$F(x_2, y_1) = \int_{-1}^{1} 2x_2^2 - 3x_2y_1 + 4y_1{}^2 = 1$$

$$F(x_2, y_2) = \int_{-1}^{1} 2x_2^2 - 3x_2y_2 + 4y_2{}^2 = 3$$

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dx\,dy = w_1^2\, f(x_1, y_1) + w_1\, w_2 f(x_1, y_2) + w_2\, w_1 f(x_2, y_1) + w_2^2\, f(x_2, y_2) = \boxed{8}$$

Application in FEA(2D)

2D Integration-2x2 Integration Point

Full integration for a linear (Q4)element(exact good Result)

Reduced Integration – Q8 element(Incomplete Result)

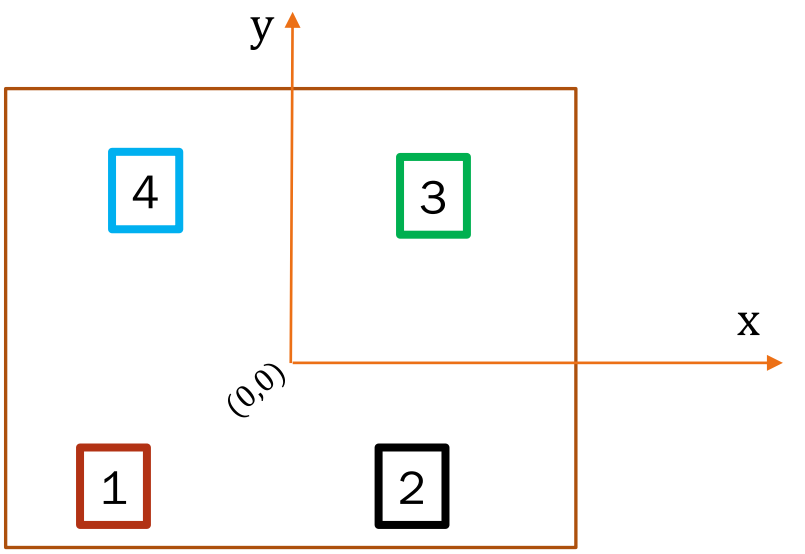
$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dx\,dy = w_1^2 f(x_1,y_1) + w_1 w_2 f(x_1,y_2) + w_2 w_1 f(x_2,y_1) + w_2^2 f(x_2,y_2)$$

$$\boxed{1\cdot 1\cdot f\left(\tfrac{-1}{\sqrt{3}},\tfrac{-1}{\sqrt{3}}\right)} + \boxed{1\cdot 1\cdot f\left(\tfrac{1}{\sqrt{3}},\tfrac{-1}{\sqrt{3}}\right)} + \boxed{1\cdot 1\cdot f\left(\tfrac{1}{\sqrt{3}},\tfrac{1}{\sqrt{3}}\right)} + \boxed{1\cdot 1\cdot f\left(\tfrac{-1}{\sqrt{3}},\tfrac{1}{\sqrt{3}}\right)}$$

4 points locations

$x_2 = +\sqrt{\frac{1}{3}} = y_2$; $x_1 = -\sqrt{\frac{1}{3}} = y_1$

$W_1, W_2 = 1$



Problem

Gauss Quadrature Double ***Integral***

$$\int_{-1}^{1}\int_{-1}^{1} F(x)\,dx\,dy \approx \sum_i \sum_j W_i W_j F(dx, dy)$$

Example

$$\int_{-1}^{1}\int_{-1}^{1} (2\,x^2 + 3\,x\,y + 4\,y^2)\,dx\,dy$$

Number of points, n	Points, x_i		Weights, w_i
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2=2n-1=Higher-order

n=1.5≈ 2(two point quadrature point)

$F(x)=2x^2-3xy+4y^2$

$x_1=+\sqrt{\frac{1}{3}}=y_1$; $x_2=-\sqrt{\frac{1}{3}}=y_2$

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$$F(x_1, y_1)=\int_{-1}^{1} 2x_1^2-3x_1y_1+4{y_1}^2=3$$
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$$F(x_2, y_2)=\int_{-1}^{1} 2x_2^2-3x_2y_2+4{y_2}^2=3$$

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dx\,dy = w_1^2\,f(x_1,y_1)+w_1\,w_2 f(x_1,y_2)+w_2\,w_1 f(x_2,y_1)+w_2^2\,f(x_2,y_2) = \boxed{8}$$

Problem

Gauss Quadrature Double ***ntegral***

$$\int_{-1}^{1}\int_{-1}^{1} F(x)\,dx\,dy \approx \sum_i \sum_j W_i W_j F(dx,dy)$$

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Example

$$F(x)=2x^3-3xy+4y^2$$

3=2n-1=Higher-order

n=2.5≈ 3(two point quadrature point)

$$\int_{-1}^{1}\int_{-1}^{1} F(x)\,dx\,dy \approx \sum_i \sum_j W_i W_j F(dx,dy)$$

$x_1=-\sqrt{\frac{3}{5}}=y_1$; $x_2=0=y_2$; $x_3=+\sqrt{\frac{3}{5}}=y_3$

$W_1=W_3=5/9$; $W_2=8/9$

$$\approx \boxed{\frac{5}{9}\cdot\frac{5}{9}\cdot f\left(-\sqrt{\tfrac{3}{5}},-\sqrt{\tfrac{3}{5}}\right)}+\boxed{\frac{5}{9}\cdot\frac{8}{9}\cdot f\left(-\sqrt{\tfrac{3}{5}},0\right)}+\boxed{\frac{5}{9}\cdot\frac{5}{9}\cdot f\left(-\sqrt{\tfrac{3}{5}},\sqrt{\tfrac{3}{5}}\right)}$$

$$+\frac{8}{9}\cdot\frac{5}{9}\cdot f\left(0,-\sqrt{\tfrac{3}{5}}\right)+\frac{8}{9}\cdot\frac{8}{9}\cdot f(0,0)+\frac{8}{9}\cdot\frac{5}{9}\cdot f\left(0,\sqrt{\tfrac{3}{5}}\right)$$

$$+\frac{5}{9}\cdot\frac{5}{9}\cdot f\left(\sqrt{\tfrac{3}{5}},-\sqrt{\tfrac{3}{5}}\right)+\frac{5}{9}\cdot\frac{8}{9}\cdot f\left(\sqrt{\tfrac{3}{5}},0\right)+\frac{5}{9}\cdot\frac{5}{9}\cdot f\left(\sqrt{\tfrac{3}{5}},\sqrt{\tfrac{3}{5}}\right)$$

Application in FEA(2D)

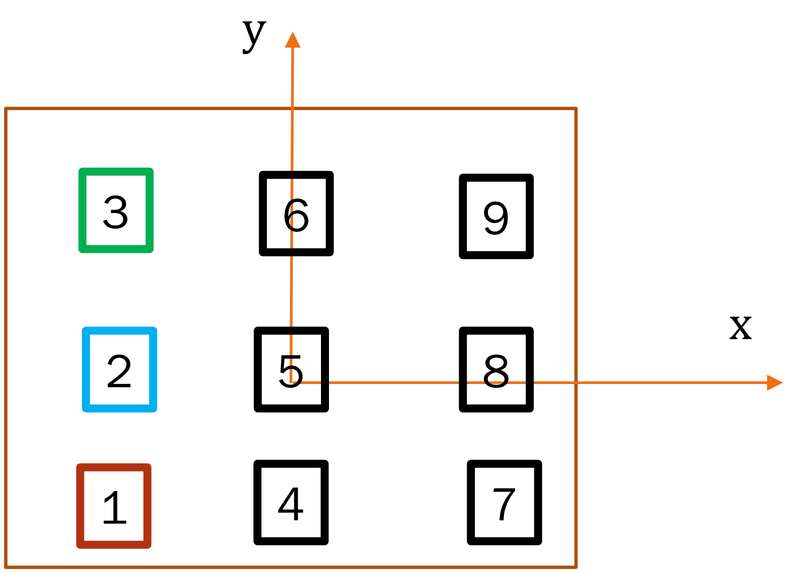
2D Integration-3x3 Integration Point

$$\int_{-1}^{1}\int_{-1}^{1} F(x)\,dx\,dy \approx \sum_i \sum_j W_i W_j F(dx, dy)$$

$$\approx \boxed{\tfrac{5}{9}\cdot\tfrac{5}{9}\cdot f\left(-\sqrt{\tfrac{3}{5}},-\sqrt{\tfrac{3}{5}}\right)} + \boxed{\tfrac{5}{9}\cdot\tfrac{8}{9}\cdot f\left(-\sqrt{\tfrac{3}{5}},0\right)} + \boxed{\tfrac{5}{9}\cdot\tfrac{5}{9}\cdot f\left(-\sqrt{\tfrac{3}{5}},\sqrt{\tfrac{3}{5}}\right)}$$

$$+\tfrac{8}{9}\cdot\tfrac{5}{9}\cdot f\left(0,-\sqrt{\tfrac{3}{5}}\right) + \tfrac{8}{9}\cdot\tfrac{8}{9}\cdot f(0,0) + \tfrac{8}{9}\cdot\tfrac{5}{9}\cdot f\left(0,\sqrt{\tfrac{3}{5}}\right)$$

$$+\tfrac{5}{9}\cdot\tfrac{5}{9}\cdot f\left(\sqrt{\tfrac{3}{5}},-\sqrt{\tfrac{3}{5}}\right) + \tfrac{5}{9}\cdot\tfrac{8}{9}\cdot f\left(\sqrt{\tfrac{3}{5}},0\right) + \tfrac{5}{9}\cdot\tfrac{5}{9}\cdot f\left(\sqrt{\tfrac{3}{5}},\sqrt{\tfrac{3}{5}}\right)$$



Conclusions

- Basic knowledge of Gaussian Quadrature.
- How to integrate area.
- How to locate point.

Reference

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8. https://www.youtube.com/watch?v=jZAWWE0Jo9w&list=WL&index=1&t=12s

Thank You