Datawave Marine Solutions

OFreq Dynamics

Theory Of Dynamics

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March 30, 2013

Rev: 1.0

DMS1303-000-110-01

Revision History

Revision	Date	Changes	Approval
1.00	Apr 03, 2013	Initial Issue	Nicholas Barczak

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1 Introduction

This report provides an introduction to the theory of Dynamics, which is the core function of the program oFreq. It covers the engineering concepts behind the core program functions. It does not present any information in a rigorous engineering format. The explanation skips over some of the messy details for the derivations. Instead, this document serves as a guide to an end user on the functional use of the theories and mathematics.

2 Dynamics

Dynamics is the science of predicting the motions of a body from all the forces that act on it. And these forces tend to fall into four major categories:

1. Reactive Force – Acceleration
2. Reactive Force – Velocity
3. Reactive Force – Position
4. Active Force

The reactive forces that depend on acceleration have to do with an object's momentum. Simply put: a big heavy thing takes more force to push faster than a small thing. Acceleration based forces depend on an object's mass, and the distribution of that mass.

The reactive forces that depend on velocity usually have to do with viscosity and friction. The faster the difference in velocity between two surfaces, the strong that friction acts to slow things down.

The reactive forces that depend on position are usually springs. The farther you pull things apart, the harder they pull back.

And lastly, you have active forces, which are completely independent of anything related to motion. These are constants that don't change. The basic idea of dynamics is that you can add all these forces up and find out what the resulting changes are.

It was an oversimplification to say that you can simply add up the forces. Actually, the manner in which you combine them changes depending on the situation. For the application of oFreq, we will use a special subset of dynamics called simple harmonic motion. This relates to situations with repeated motion, like a weight bouncing on a spring. Or for a more complex example, a ship bouncing on ocean waves. The critical thing to remember for simple harmonic motion is that all those forces must add up to equal zero. They must balance.

$$F_{accel}(t)+F_{vel}(t)+F_{pos}(t)+F(t)=0 \qquad \text{Equation 2.1}$$

The *(t)* indicates that each of the forces varies with time, but at any instant in time, all four forces must balance and add to zero. Now this equation needs to be reworked to include the actual motions that we are trying to solve for. I make the following substitutions.

$$F_{accel}(t) = -k_2 a(t)$$ Equation 2.2

k_2 = some coefficient that is known and specific to each problem.
a(t) = acceleration.

$$F_{vel}(t) = -k_1 v(t)$$ Equation 2.3

k_1 = another coefficient that is known and specific to each problem.
v(t) = velocity

$$F_{pos}(t) = k_0 x(t)$$ Equation 2.4

k_0 = another coefficient that is known and specific to each problem.
x(t) = position. *This is the motion variable that we are trying to solve for.*

Using these substitutions, the forces are rearranged into the typical presented form for the equation of motion. We have the reaction forces on the left side of the equation, the active forces on the right side of the equation, and the variable x(t) mixed into the middle.

$$k_2 a(t) + k_1 v(t) + k_0 x(t) = F(t)$$ Equation 2.5

The next problem becomes how to relate x(t) to a(t) and v(t) and solve for everything. To do that, I use differential equations.

2.1 Differential Equations Applied To Dynamics

Differential equations and calculus allow us to relate the acceleration and velocity to position. Thanks to previous scientific work, it is known that the acceleration and velocity are related by the derivatives of position, *x(t)*.

$$a(t) = \frac{d^2 x}{dt^2}$$
$$v(t) = \frac{d\,x}{dt}$$ Equation 2.6

I can substitute those derivatives in for acceleration and velocity. This gives the following differential equation.

$$k_2 \frac{d^2 x}{dt^2} + k_1 \frac{dx}{dt} + k_0 x(t) = F(t)$$ Equation 2.7

The good news is that now all the reactive forces on the left hand side of the equation are in terms of *x(t).* This is the basic differential equation for dynamic motions involving simple harmonic motion. The bad news is that now we need to solve a differential equation.

3 Dynamics With Ocean Waves

There are many ways to solve differential equations. When we apply this equation to the problem of ocean waves, it is fairly easily to solve. One way to solve a differential equation is trial and error. We try a function and calculate its derivatives.

Try a solution of the form

$$x = \underline{X} e^{i\omega t}$$ Equation 3.1

Where:
- $\underline{X}$ = the amplitude of motion, represented as a complex number. (Underline denotes complex number.)
- i = imaginary number
- ω = frequency of the waves
- t = time

This function has the following derivatives.

$$\frac{dx}{dt} = \underline{X} i e^{i\omega t}$$
$$\frac{d^2x}{dt^2} = \underline{X}(i)^2 \omega^2 e^{i\omega t} = \underline{X}(-1)\omega^2 e^{i\omega t}$$ Equation 3.2

I substitute this function and its derivatives into the equation of motion. At the same time, I will make one other substitution. Because I have limited my problem to simple harmonic motion, I know that my active force will also be harmonic. So I can replace it with the following substitution.

$$F(t) = \underline{F} e^{i\omega t}$$ Equation 3.3

All these substitutions give the following equation.

$$k_2 \underline{X}(i)^2 \omega^2 e^{i\omega t} + k_1 \underline{X} i \omega e^{i\omega t} + k_0 \underline{X} e^{i\omega t} = \underline{F} e^{i\omega t}$$ Equation 3.4

$$\left(k_2 \omega^2 (i)^2 + k_1 i \omega + k_0\right) \underline{X} e^{i\omega t} = \underline{F} e^{i\omega t}$$ Equation 3.5

The term $e^{i\omega t}$ is on both sides of the equation and cancels out.

$$\left(k_2 \omega^2 (i)^2 + k_1 i \omega + k_0\right) \underline{X} = \underline{F}$$ Equation 3.6

This leaves a single equation. This equation describes the amplitude of the motion and its phase, in terms of the variable $\underline{X}$, which is a complex number. The variable $\underline{X}$ is what we need to solve for.

The really important part about this is to notice how a differential equation transformed into an algebraic equation. But any program still needs to understand that the coefficients

k_0, k_1, and k_2 were originally associated with derivatives of different orders.

4 Generalization

Equation 3.6 was developed only considering terms up to second order derivatives of position ($\underline{X}$). This was logical because derivatives above this have no physical meaning. However, the intent is to make this program adaptable to future unanticipated situations. For example, certain computer control systems for ships may react based on the third derivative of position. To achieve this flexibility, the equation for a force was generalized to allow any number of derivatives.

$$F_{react}=\left(\sum_{j=0}^{n} k_j (i)^j \omega^j\right)\underline{X}\, e^{i\omega t} \qquad \text{Equation 4.1}$$

Where:
j = Counting variable.
k_j = Some real number coefficient that is user assigned or interpolated from data.
n = Maximum number terms used in the equation.

The counting variable *j* corresponds to the order of the derivative, but we have no need to deal with derivatives specifically. All that we need is for the user to specify the order of the derivative corresponding to each coefficient that they add.

So the equation of motion can be generalized to the following.

$$\left(\sum_{j=0}^{n} k_j (i)^j \omega^j\right)\underline{X}=\underline{F} \qquad \text{Equation 4.2}$$

All equations of motion will use this general notation from this point on.

5 Multiple Degrees Of Freedom

So far, the equations of motion have only used one degree of freedom. They only considered one motion. One variable at any time. But for a body floating at sea, it has six degrees of freedom. And some odd designs can have even more.

The importance of this is that the equations are coupled. They require solution of simultaneous equations, using linear algebra. The generalized equation for a force now becomes the following.

$$F_{react}=\left(\sum_{j=0}^{n} k_{jp}(i)^{j}\omega^{j}\right)\underline{X_0}e^{i\omega t}+$$
$$\left(\sum_{j=0}^{n} k_{jp}(i)^{j}\omega^{j}\right)\underline{X_1}e^{i\omega t}+$$
$$\ldots\left(\sum_{j=0}^{n} k_{jp}(i)^{j}\omega^{j}\right)\underline{X_m}e^{i\omega t}$$

Equation 5.1

Where:
X_0, X_1, X_m = Individual variables for body motion, each a complex number.
p = Counting variable for each individual equation of motion.
m = Number of total equations of motion for body

Most of the coefficients k_{jp} will be values of zero. The user will want to only specify the coefficients with non-zero values and have the program assume a value of zero for any other coefficient.

The equation of motion is now represented as a series of matrices.

$$\left(\sum_{j=0}^{n}[k_p]_j (i)^j\omega^j\right)[\underline{X}]=[\underline{F}]$$

Equation 5.2

Where:
M = Total number of equations of motion.
$[k_p]_j$ = Coefficients for reactive forces. Square matrix of dimensions M x M.
$[\underline{X}]$ = Solutions to equation of motion (unknown variable). Column matrix of dimensions M x 1.
$[\underline{F}]$ = Active forces. Column matrix of dimensions M x 1.

6 Conclusion

That covers the basics of dynamics, as applied within the program oFreq. This introduction skipped over many of the details for data management and how the forces are tracked and combined. But it provided a basis for the program mathematics.

7 References