Datawave Marine Solutions

OFreq Dynamics

Fundamental Mathematics

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1 Introduction

oFreq makes use of three major concepts in advanced mathematics: complex numbers; linear algebra; and differential equations. This document covers the fundamentals of these math concepts, as they are applied to oFreq. It does not provide a very extensive introduction, just the essentials necessary for application for oFreq.

2 Complex Numbers

A long time ago, some mathematician asked the embarrasing question of what is the meaning of:

$$\sqrt{-1} \qquad \text{Equation 2.1}$$

The problem was with the rules of mathematics, this number could not exist in reality. So we called it an imaginary number. We use the letter i to represent an imaginary number.

$$i=\sqrt{-1} \qquad \text{Equation 2.2}$$

The one other catch is that if you square an imaginary number, you get back to a simple real number.

$$i^2=\sqrt{-1}^2=-1 \qquad \text{Equation 2.3}$$

If we combine an imaginary number with a real number, we get a complex number. Taken together, they form a pair of coordinates. Only this time, instead of giving coordinates for an x,y graph, we have coordinates for a real, imaginary graph. Real numbers along the bottom, that is your normal number line. And the imaginary numbers form a vertical number line. (Figure 2.1)

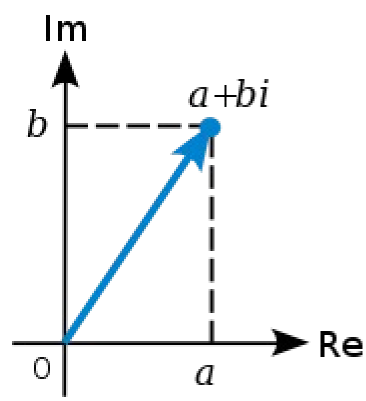

Figure 2.1: Complex Number

It is not essential to know the details of the mathematics for complex numbers. If you are interested, Wikipedia provides a thorough article on the algorithms for complex number mathematics. [1] I do need to clarify some notation at this point. Very often, the mathematics will represent a complex number as a single variable, but it is understand that the complex number contains both an real and imaginary component.

$$\underline{Z}=a+i\,b \qquad \text{Equation 2.4}$$

Notice that the complex number was shown with an underline; this is the indicator that it complex, and not a normal number. To carry this example a little further, if I wrote an equation to multiply two complex numbers together, I really mean to multiply the individual components.

$$\begin{aligned}\underline{Z_1}&=a+i\,b\\ \underline{Z_2}&=c+i\,d\end{aligned} \qquad \text{Equation 2.5}$$

$$\begin{aligned}\underline{Z_1}\,\underline{Z_2}&=(a+ib)(c+i\,d)\\ &=ac+i(ad+bc)+(i)^2bd\\ &=(ac-bd)+i(ad+bc)\end{aligned} \qquad \text{Equation 2.6}$$

A good C++ library for complex numbers will already contain all the normal algorithms with suitable operator overloads to treat complex numbers just like any other number. So you could just multiply the two complex variables $\underline{Z_1}$ and $\underline{Z_2}$, and the library will know to use the algorithm for multiplying individual components.

2.1 Rectangular And Polar Form

So far, I have shown complex numbers in rectangular form. This is the natural state for the numbers, and it is similar to coordinates for an x,y graph. There is an alternative way to represent complex numbers: polar form. Polar form would be similar to representing the complex numbers as a pair of polar coordinates on a graph. Complex numbers are represented in polar form with the following notation.

$$\underline{Z_1}=R\,e^{i\epsilon} \qquad \text{Equation 2.7}$$

The conversion from rectangular to polar form is very simple. If I start with a complex number in rectangular form

$$\underline{Z_1}=a+i\,b \qquad \text{Equation 2.8}$$

I can convert it to polar form with the following two formulae.

$$\begin{aligned}R&=\sqrt{a^2+b^2}\\ \epsilon&=\tan^{-1}\left(\frac{b}{a}\right)\end{aligned} \qquad \text{Equation 2.9}$$

And now the complex number is in polar form. Notice that it still contains two pieces of information.

$$\underline{Z_1}=R\,e^{i\epsilon} \quad \text{Equation 2.10}$$

In polar form, we call them the amplitude and phase. You can not easily separate out which part is real or imaginary. Complex numbers should always be used in their base rectangular form. But the polar form is often used as a form of user input in programs, which I will show later in Section 2.2.

As I said, complex numbers should always be converted to their base rectangular form. To convert from polar to rectangular form, use the following equations.

$$\underline{Z_1}=R\,e^{i\epsilon} \quad \text{Equation 2.11}$$

$$\begin{aligned} a&=R\cos(\epsilon)\\ b&=R\sin(\epsilon)\end{aligned} \quad \text{Equation 2.12}$$

$$\underline{Z_1}=a+i\,b \quad \text{Equation 2.13}$$

2.2 Application To OFreq

For oFreq, it is not as important to understand the mechanics of complex numbers, but rather what they represent inside oFreq. Complex numbers are very useful because each number contains two pieces of information. In the dynamics analysis of oFreq, we also need to track two different components with each math operation. So we use complex numbers as a container to hold both pieces of information within a single mathematical object.

The two pieces of information to track are the amplitude and phase of motion. oFreq involves solving sinusoidal motions. We have lots of sinusoidal forces interacting. And when that happens, there are two important parts to the interaction: the amplitude (how big the force is); and the phase (how closely are the forces synchronized with each other).

3 Linear Algebra

Linear algebra is a system for solving simultaneous equations. For examples, let us assume we have the following three equations.

$$\begin{aligned} &3\mathrm{x}+2\mathrm{y}+-1z=1\\ &2\mathrm{x}+-2\mathrm{y}+4\mathrm{z}=-2\\ &-1\mathrm{x}+\frac{1}{2}y+-1\mathrm{z}=0\end{aligned} \quad \text{Equation 3.1}$$

Notice that each equation depends on the other two. They are all linked and must be solved simultaneously. There are several methods to do this. But imagine if there were 12 equations, with 12 variables. That becomes a lot of information to track. Linear algebra provides a means of shorthand to track this information. Notice that all the equations have the same format. And what is really important are the coefficients in front of each variable. So we can separate this out to hold all the coefficients in one matrix, the variables in another matrix, and the constants in one last matrix. Now, the set of equations turns into the following format.

$$\begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & 4 \\ -1 & \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$ Equation 3.2

This is still the same equation set, just in a more concise notation. Notice how all the coefficients are in the first matrix. Each row in the matrix represents a single equation. And each column in the matrix represents a single variable. So, all the numbers in the first row correspond to the first equation.

$$\begin{bmatrix} 3 & 2 & -1 \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3x + 2y + -1z$$ Equation 3.3

In the same way, all the numbers in a single column on the first matrix correspond to a single variable.

$$\begin{bmatrix} 3 & \square & \square \\ 2 & \square & \square \\ -1 & \square & \square \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{matrix} 3x + \square y + \square z \\ 2x + \square y + \square z \\ -1x + \square y + \square z \end{matrix}$$ Equation 3.4

That covers the basic use of matrices. At the heart, they are just a means of shorthand to concisely track all the information for a series of simultaneous equations.

3.1 Matrix Mathematics

Matrices get represented as their own self-contained objects within mathematics. As such, they have their own set of rules and algorithms for how to perform math operations. I won't cover the algorithms here. The algorithms should be built into any good linear algebra library. If you are interested in the details, I recommend the following free resources. [2][3]

I will cover a few of the basic rules for matrix mathematics. The matrix is designed as a variable name with brackets.

$$[A] = \begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & 4 \\ -1 & \frac{1}{2} & -1 \end{bmatrix}$$ Equation 3.5

The notation of [A] doesn't specify the size of the matrix. The math assumes that the equations are setup with the correct size to allow any specified operations. This gets into a few of the rules for matrix operations.

3.1.1 Transpose

The transpose is a useful operation for matrices. The transpose operation is designed in the following manner.

$$[A]^T \qquad \text{Equation 3.6}$$

The algorithm for a matrix transpose is to swap each element in a matrix with a element of the opposite index.

$$a_{ij} = a_{ji} \qquad \text{Equation 3.7}$$

Notice the sequence of the index *I* and j in Equation 3.7. The order of the indices switched. This is better shown through an example. If we start with the following matrix.

$$[A] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \text{Equation 3.8}$$

Then the transpose of the matrix becomes.

$$[A]^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \qquad \text{Equation 3.9}$$

Notice how each element in the matrix has swapped with the element of the opposite index. Also notice how a transpose operation can change the dimensions of a matrix. The original [A] matrix had dimensions of 3x2, but the transposed matrix $[A]^T$ has dimensions of 2x3.

3.1.2 Addition And Subtraction

For addition and subtraction, the matrices must be the same size; their rows and columns must match. Take the following example matrices.

$$[A] = \begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & 4 \\ -1 & \frac{1}{2} & -1 \end{bmatrix} \quad [B] = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \quad [C] = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad [D] = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix} \qquad \text{Equation 3.10}$$

Matrix [A] has 3 rows and 3 columns: 3x3 matrix

Matrix [B] has 1 row and 3 columns: 1x3 matrix

Matrix [C] has 3 rows and 1 column: 3x1 matrix

Matrix [D] has 3 rows and 1 column: 3x1 matrix

Of all these matrices, only [C] and [D] can combine to form a valid operation.

$$[C]+[D]=\begin{bmatrix}4\\5\\6\end{bmatrix}+\begin{bmatrix}-1\\-2\\-3\end{bmatrix}=\begin{bmatrix}3\\3\\3\end{bmatrix}$$ Equation 3.11

$$[C]+[A]=ERROR$$ Equation 3.12

3.1.3 Multiplication

When multiplying matrices, the inner two dimensions of the matrix must match. This is best shown by example.

[A] [C] = (3x3 matrix) (3x1 matrix) = Valid

The columns for [A] match the rows for [C].

[B] [A] = (1x3 matrix) (3x3 matrix) = Valid

The columns for [B] match the rows for [A].

[A] [B] = (3x3 matrix) (1x3) matrix) = ERROR

The columns for [A] *do not* match the rows for [B].

This brings up another point for matrix math. Order of operations matters with matrices.

$$[A][B]\neq[B][A]$$ Equation 3.13

3.1.4 Division

Matrix division does not really exist. There is only matrix multiplication. To perform a division operation, we must take the inverse of a matrix, which is designed like this.

$[A]^{-1}=$ Matrix Inversion Equation 3.14

You would then multiply the inverted matrix to perform the equivalent of division.

$$[A]^{-1}[C]$$ Equation 3.15

The operations for performing matrix inversion can get fairly complicated, and very computationally intense. It warrants a moment to discuss the complexity of matrix inversion.

3.2 Matrix Inversion – Computational Complexity

Matrix inversion requires $O(n^3)$ operations, for an NxN matrix. [4] So a 3x3 matrix requires 27 operations. The situation gets worse if you involve complex numbers. Any operation involving multiplication of N complex numbers requires $O(n^2)$ operations. And all operations in matrix inversion require multiplication. So inversion of a matrix full of complex numbers of size NxN requires O(n^5) operations. Inversion of a 6x6 matrix will require 7,776 operations!

Due to this computational effort, mathematicians have invented many different ways to perform matrix inversion, each optimized for a specific situation. This large computational cost is why efficient calculation of the matrix inversion is essential, and why the anticipated matrix size should remain in the forefront of any developer's mind.

3.3 Application To OFreq

How does oFreq use matrices and linear algebra? oFreq basically forms a series of simultaneous equations composed of different forces. It then adds all these forces to form a single matrix of coefficients that relate the forces to the dynamic motions. The oFreq problem will look something like this:

$$\left[[K_1]+[K_2]+[K_3]+[K_4]\right][X]=[F]$$
$$[A]=[K_1]+[K_2]+[K_3]+[K_4]$$

Equation 3.16

$$[A][X]=[F]$$

oFreq then inverts the [A] matrix and pre-multiplies to the forcing [F] matrix to solve for the body motions [X].

$$[X]=[A]^{-1}[F]$$

Equation 3.17

4 Differential Equations

There are many physical quantities in this world that we can not directly measure. Things like energy. However, we can measure the change that it creates in the world. And calculus [5] developed a set of mathematics so that if we can understand the change created by a thing, we can understand and predict the thing which generated that change in the first place.

Now take the idea one step further. Say we have two things interacting. And we can only measure some parts of each. For example, if you have two cars traveling straight towards each other. You know the speed of one car. You know the acceleration of the other car. And you know the point when they impacted. There is not enough information about either car to truly understand its path of travel. However, we have a relationship between the two in the form of when they impacted. This where differential equations comes in.

Differential equations allows us to relate objects and events together, based only on partial information from each object. With only that limited information on the relationship, we can successfully predict the controlling behavior of all items. In the example of those cars, we can use differential equations to relate the two and predict their exact distance apart when they started, and the time history of how fast they accelerated. That is the power of differential equations.

5 Conclusion

This covers the fundamental mathematics for the program oFreq. I left out many of the details of because each of these concepts is a full class worth of knowledge in its own right. A full explanation of the abilities and rules for all subjects would be overwhelming. But this should provide a basic foundation on the purpose of complex numbers, linear algebra, and differential equations, as they apply to oFreq.

6 References

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