Datawave Marine Solutions

OFreq Dynamics

Equation Of Motion Solver

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1 Introduction

This document provides detailed description for the generalized formation, assembly, and solution of the equations of motion in the oFreq application. It assumes familiarity with all background theory on dynamics, mathematics, and hydrodynamics.

1.1 Setup Assumptions

The actual assembly of equations of motion can change greatly depending on how many bodies are included and which models for equation of motion each body uses. For the assembly and solution of each equation of motion, I will assume the following situation with two bodies.

Body0, with 6 degrees of freedom.

Body1, with 3 degrees of freedom.

2 Assembly Of Body Matrices From User Forces

The first step is to assemble the matrices of user forces within each body. This includes three types of forces: active, reactive, and cross-body forces.

2.1 Active Forces

The active forces do not vary with any variable in the equations of motion. So they are fairly simple. The user defines a series of constant forces. These may vary with wave frequency. Each force is defined as a complex number and associated with a specific equation. For Body0, this assembled into a column matrix of complex numbers, with dimensions of 6,1. The notation shows real and imaginary components for each force.

$$\underline{[F0_{UserActive}]_u} = \begin{bmatrix} (F_R + iF_I)_{00} \\ (F_R + iF_I)_{10} \\ \dots \\ (F_R + iF_I)_{50} \end{bmatrix}$$ Equation 2.1

With Body1, this assembled into a column matrix with dimensions of 3,1. The underline denotes that this is a matrix of complex variables.

$$\underline{[F1_{UserActive}]_u} = \begin{bmatrix} (F_R + iF_I)_0 \\ (F_R + iF_I)_1 \\ (F_R + iF_I)_2 \end{bmatrix}$$ Equation 2.2

Notice that each of the matrices is marked with an index u. This indicates that the user may define multiple matrices for each body. The summation of these multiple matrices is handled in Section 2.4. For now, each force is defined individually and we move on to definition of the reactive forces.

2.2 Reactive Forces

The user provided inputs that give a series of force coefficients and the corresponding order of derivative for each coefficient, and the corresponding variable associated with each equation. Any coefficients not provided are assumed to be zero. For Body0, the program assembles these coefficients into a series of matrices. Each matrix has dimensions of 6 x 6, since the body has six degrees of freedom.

$$[F0_{UserReact}]_{uv}=\begin{bmatrix} k_{00} & k_{01} & \dots & k_{05} \\ k_{10} & k_{11} & \dots & k_{15} \\ \dots & \dots & \dots & \dots \\ k_{60} & k_{61} & \dots & k_{66} \end{bmatrix} \quad \text{Equation 2.3}$$

Where:
First subscript = The index of the equation of motion that the coefficient is associated with.
Second subscript = The index of the variable that the coefficient is associated with.
u = The index of the force defined by the user. The user may define multiple forces for each body.
v = The order of the derivative that the coefficient matrix is associated with.

This produces a series of matrices, up to the highest order of the derivative (index *i).*

For Body1, the program assembles a matrix along the same methodology. But the second matrix only has dimensions of 3 x 3, since the the body has 3 degrees of freedom.

$$[F1_{UserReact}]_{uv}=\begin{bmatrix} k_{00} & k_{01} & k_{02} \\ k_{10} & k_{11} & k_{12} \\ k_{30} & k_{31} & k_{32} \end{bmatrix} \quad \text{Equation 2.4}$$

Again, this produces a series of matrices, up to the highest order of the derivative. The two bodies may have different orders of derivatives.

2.3 Cross-Body Forces

If multiple bodies are included in the run, the user may define some cross forces. These cross forces were defined with a combination of variables between two bodies. The body forces are defined as originating from one body, and reference the variables of another body. The user only needs to define the link within a single body; the program will understand that this force is shared by both bodies. The user may define multiple cross-body forces, but each cross-body force can only reference one other body. To create references to multiple bodies, the user will need to define multiple cross-body forces.

The definition of cross-body forces creates a body interaction matrix. The dimensions of the matrix depend on the number of equations within each body. The matrix will have the number of rows as equations within the body that owns the cross-body term. And it will have the same number of rows as the equations of the linking body. Once again, the user has defined a derivative associated with each cross-body force. A separate body interaction matrix is defined for each order of derivative. So for a cross-body force defined within Body0, and linking to Body1, the following matrix gets assembled.

$$[F01_{UserCross}]_{uv}=\begin{bmatrix} k_{00} & k_{01} & k_{02} \\ k_{10} & k_{11} & k_{12} \\ \ldots & \ldots & \ldots \\ k_{60} & k_{61} & k_{62} \end{bmatrix} \qquad \text{Equation 2.5}$$

Where:
First subscript = The index of the equation of motion that the coefficient is associated with, from Body0.
Second subscript = The index of the equation of motion that the coefficient is associated with, linking to Body1.
u = The index of the force defined by the user. The user may define multiple forces for each body.
v = The order of the derivative that the coefficient matrix is associated with.

2.4 Summation Of User Defined Forces

Normally, the user will just define one series of forces for each equation. However, the user may wish to separate out the forces to track different elements separately. In that case, the program will add the coefficients for all forces that are associated with each equation and variable. This combines individual forces into a single force, which is what the program cares about. Each combined force is a single matrix, where each matrix is handled in a different manner. The individual matrices are summarized as follows.

2.4.1 Active Force Summation

The active forces for each body are summed with the following formula.

$$[F0_{UserActive}]=\sum_{u=0}^{n}[F0_{UserActive}]_u \qquad \text{Equation 2.6}$$

Where:
u = Index of user defined active force.

underline = Matrix of complex variables

Notice that each body now only contains a single active force matrix.

2.4.2 Reactive Force Summation

The reactive forces are summed in a similar manner. Each body will contain a series of reactive force matrices, one matrix for each order of derivative.

$$[F0_{UserReact}]_v = \sum_{u=0}^{n} [F0_{UserReact}]_{uv}$$ Equation 2.7

Where:
u = Index of user defined active force.
v = Order of equation derivative.

There are now a series of matrices defined for each body, each matrix corresponding to an order of derivative.

2.4.3 Cross-Body Force Summation

The cross-body forces are summed in the same manner as the reaction forces.

$$[F01_{UserCross}]_v = \sum_{u=0}^{n} [F01_{UserCross}]_{uv}$$ Equation 2.8

Where:
u = Index of user defined active force.
v = Order of equation derivative.

3 Definition Of Hydrodynamic Matrices

Each body will have a series of hydrodynamic forces associated with it. These were calculated by another piece of software and written in a hydrodynamic database. Each of these forces varies by wave frequency. The oFreq program is aware of the wave frequency and uses this to interpolate within the hydrodynamic database generate the forces for each frequency.

The hydrodynamic forces fall into three categories: active forces, reactive forces, and cross-body forces.

3.1 Active Forces

Active forces are typically called wave forces. This is the actual force generated by the waves impacting on the body. These forces are assembled into a matrix for each body and defined the same way as the user defined active forces.

$$\underline{[F0_{HydroActive}]}=\begin{bmatrix}(F_R+i\,F_I)_{00}\\(F_R+i\,F_I)_{10}\\\dots\\(F_R+i\,F_I)_{50}\end{bmatrix}$$ Equation 3.1

Each of the terms in the matrix will be defined for each equation of motion and vary with frequency. The underline denotes that this is a matrix of complex variables.

3.2 Reactive Forces

The reactive forces are defined the same way as the user reactive forces. But in this case, they only go up to a second order derivative. Each order of derivative has a specific term used to describe it. They are: hydrostatic force, added damping, and added mass.

3.2.1 Hydrostatic Force

The hydrostatic force is associated with the 0th order derivative. (That is, no derivative is used.)

$$[F0_{HydroReact}]_0=\begin{bmatrix}H_{00} & H_{01} & \dots & H_{05}\\H_{10} & H_{11} & \dots & H_{15}\\\dots & \dots & \dots & \dots\\H_{60} & H_{61} & \dots & H_{66}\end{bmatrix}$$ Equation 3.2

Where:
H_{00} … H_{66} = Hydrostatic values for the body.

Notice that the matrix is defined with the index 0. This shows that the hydrostatic reactive force is only associated with the first derivative.

3.2.2 Added Damping

The added damping force is associated with the 1st order derivative.

$$[F0_{HydroReact}]_1=\begin{bmatrix}B_{00} & B_{01} & \dots & B_{05}\\B_{10} & B_{11} & \dots & B_{15}\\\dots & \dots & \dots & \dots\\B_{60} & B_{61} & \dots & B_{66}\end{bmatrix}$$ Equation 3.3

Where:
B_{00} … B_{66} = Added damping values for body.

The added damping force is only associated with the first order derivative.

3.2.3 Added Mass

The added damping force is associated with the 1st order derivative.

$$[F0_{HydroReact}]_2 = \begin{bmatrix} A_{00} & A_{01} & \dots & A_{05} \\ A_{10} & A_{11} & \dots & A_{15} \\ \dots & \dots & \dots & \dots \\ A_{60} & A_{61} & \dots & A_{66} \end{bmatrix}$$ Equation 3.4

Where:
$A_{00} \dots A_{66}$ = Added damping values for body.

The added damping force is only associated with the first order derivative. Hydrodynamic forces still fall into the larger category of reactive forces. Except that hydrodynamic forces only go up the second order derivative.

3.3 Cross-Body Forces

The cross-body forces are defined the same way as a user based cross-body force. Each force matrix is associated with a specific body, and derived from the motions of a single reference body.

$$[F01_{HydroCross}]_v = \begin{bmatrix} C_{00} & C_{01} & C_{02} \\ C_{10} & C_{11} & C_{12} \\ \dots & \dots & \dots \\ C_{60} & C_{61} & C_{62} \end{bmatrix}$$ Equation 3.5

Where:
$C_{00} \dots C_{62}$ = Added damping values for body.

4 Definition Of Body Momentum Matrix

There is one final matrix to define for the body forces. This is another reactive force, solely associated with the second order derivative. It comes purely from the physical mass of the body, and that mass distribution. It has the following definition for a 6 degree of freedom body.

$$[F0_{Mass}]_2 = \begin{bmatrix} M & 0 & 0 & 0 & -M\,y_g & M\,x_g \\ 0 & M & 0 & M\,y_g & 0 & -M\,z_g \\ 0 & 0 & M & -M\,x_g & M\,z_g & 0 \\ 0 & -M\,y_g & M\,x_g & I_{xx} & I_{xy} & I_{xz} \\ M\,y_g & 0 & -M\,z_g & I_{xy} & I_{yy} & yz \\ -M\,x_g & M\,z_g & 0 & I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$ Equation 4.1

Where:
M = Mass of body.
x_g = X-axis center of gravity of body, in body local coordinates.
y_g = Y-axis center of gravity of body, in body local coordinates.
z_g = Z-axis center of gravity of body, in body local coordinates.
I_{xx} = X-axis moment of inertia of body, in body local coordinates.
I_{yy} = Y-axis moment of inertia of body, in body local coordinates.
I_{zz} = Z-axis moment of inertia of body, in body local coordinates.
I_{xy} = XY-axis cross product of inertia, in body local coordinates.
I_{xz} = XZ-axis cross product of inertia, in body local coordinates.
I_{yz} = YZ-axis cross product of inertia, in body local coordinates.

5 Assembly Of Global Matrices For All Bodies

5.1 Global Matrix For User Forces

Each body matrix is not assembled into a single global matrix that contains all bodies. The reaction forces and cross-body forces form one global matrix. This includes the user defined and hydrodynamic forces. Remember that you get one of these matrices for every derivative present.

$$[F_{React}]_v = \begin{bmatrix} [F0_{UserReact}]_v + [F0_{HydroReact}]_v & [F01_{UserCross}]_v + [F01_{HydroCross}]_v \\ [F01_{UserCross}]_v^T + [F01_{HydroCross}]_v^T & [F1_{UserReact}]_v + [F1_{HydroReact}]_v \end{bmatrix}$$ Equation 5.1

Where:
v = Index of the derivative the matrix is associated with.
$[F0_{UserReact}]_v$ = User reaction forces matrix for body 0, associated with derivative v.
$[F01_{UserCross}]_v$ = User cross-body matrix owned by body 0, linked to body 1, for derivative v.
$[F1_{UserReact}]_v$ = User reaction forces matrix for body 1, associated with derivative v.
$[F0_{HydroReact}]_v$ = Hydrodynamic reaction forces matrix for body 0, associated with

derivative v.
$[F01_{HydroCross}]_v$ = Hydrodynamic cross-body matrix owned by body 0, linked to body 1, for derivative v.
$[F1_{HydroReact}]_v$ = Hydrodynamic reaction forces matrix for body 1, associated with derivative v.

5.2 Global Matrix For Body Momentum

The body momentum matrix is also assembled. There is only one body momentum matrix, associated with the 2nd order derivative.

$$[F_{Mass}]_2 = \begin{bmatrix} [F0_{Mass}]_2 & [0] \\ [0] & [F1_{Mass}]_2 \end{bmatrix}$$ Equation 5.2

Where:
subscript 2 = Second order derivative that forces are associated with.
[0] = Matrix of zeros.
$[F0_{Mass}]_2$ = Body moment matrix for body 0.
$[F1_{Mass}]_2$ = Body moment matrix for body 1.

This mass matrix is then added to the second order reactive force matrix.

$$[F_{React}]_2 = [F_{React}]_2 + [F_{Mass}]_2$$ Equation 5.3

5.3 Assembly Of Global Matrix For Active Forces

Finally, the active forces are assembled into a single global force matrix. There is only one active force matrix. It is not associated with any derivative.

$$\underline{[F_{Active}]} = \begin{bmatrix} \underline{[F0_{UserActive}]} + \underline{[F0_{HydroActive}]} \\ \underline{[F1_{UserActive}]} + \underline{[F1_{HydroActive}]} \end{bmatrix}$$ Equation 5.4

Where:
$[F0_{UserActive}]$ = User defined active force for body 0.
$[F0_{HydroActive}]$ = Hydrodynamic active force for body 0.
$[F1_{UserActive}]$ = User defined active force for body 1.
$[F1_{HydroActive}]$ = Hydrodynamic active force for body 1.

Notice that this is a single column matrix. It only expands in rows for each body addition.

This completes definition of the global force matrices. There is now one active force matrix. And one reactive force matrix defined for each order of derivative. The next step is to add all the reactive forces into a single matrix.

6 Addition Of Reactive Forces

The only thing that differentiates the different reactive forces for each derivative is an appropriate coefficient that must be multiplied to each matrix. The algorithm proceeds through each matrix, adding it to a total.

$$[F_{React}]=\sum_{v=0}^{n}[F_{React}]_v (i)^v \omega^v$$ Equation 6.1

Where:
$[F_{react}]_v$ = Reaction force matrix for entire global system.
(i) = Imaginary number
ω = Wave frequency

Notice that now the reactive forces are a complex variable matrix. This summation is all that is required to apply the differential equation for this dynamic motion solver.

We now have a single reactive force matrix (Notice that the subscript is no longer present for the matrix.) This matrix, and the active forces matrix are the only

7 Solution Of Equation Of Motion

The first step is to take the inverse of the reaction force matrix. Remember that inversion of this matrix can become very intensive. Thankfully, the entire algorithm for matrix inversion is captured within any good C++ library.

$$[F_{React}]^{-1}=inverse([F_{React}])$$ Equation 7.1

For the purposes of error capturing, it is useful to know a matrix is invertible only if the determinant of the matrix equals zero. The determinant is another matrix operation provided by any good C++ library.

$$|[F_{React}]|=determinant([F_{React}])$$ Equation 7.2

After the reaction force matrix is inverted, it gets pre-multiplied to the active force matrix. Remember that order of multiplication matters for matrix operations. That is why I say pre-multiply, to mean that the invert reaction matrix comes first in the multiplication sequence. This provides the solution to the equation of motion.

$$[X]=[F_{React}]^{-1}[F_{Active}]$$ Equation 7.3

Where:
[X] = Global solution of equation of motion (complex variables).

8 Conclusion

That completes the mathematics for the motion solver. The solver assembles all these matrices from the individual bodies. It combines the matrices and and inverts the reaction force matrix. This is pre-multiplied to the active force matrix to produce the solution. Results for motions are returned to their respective bodies.

9 References