Methods for Solving Linear Equations

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February 2025

1 Introduction

Many real life problems in engineering give rise to a system of linear equations. For example, such systemss occur in certain applications of statistical analysis and in finiding the numerical solution of partial differential equations and so on.

2 Gauss Elimination Method

A system of linear equations consists of multiple equations with multiple unknowns, typically represented in the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \dots \quad & a_{1n}x_n = b_1 \\ a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \dots \quad & a_{1n}x_n = b_2 \\ \vdots\vdots \qquad & \ddots\vdots \\ a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \dots \quad & a_{1n}x_n = b_3 \end{aligned}$$

o solve such systems, we use systematic numerical methods. One of the most commonly used methods is Gaussian Elimination, which transforms the system into an equivalent triangular form, making it easier to solve through back-substitution.

In this presentation, we will explore the Gaussian Elimination method step by step, demonstrating how it simplifies complex systems into solvable forms. Using the matrix notation, the above system can be written in compact form as

$$AX = B$$

Where

$$A = \begin{pmatrix} a_{11} & a_{11} & a_{11} \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} \dots & a_{2n} \\ a_{3n} & a_{n2} & a_{n3} \dots & a_{nn} \end{pmatrix}$$

and

$$X = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{3n} \end{pmatrix}$$

and

$$B = \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{3n} \end{pmatrix}$$

problem 1

$$\begin{aligned} 10x + y + z &= 12 \\ x + 10y + z &= 12 \\ x + y + 10z &= 12 \end{aligned}$$

step 1 create the system

$$AX = B$$

Identify the Matrix A, X and B. Now our aim is to reduce the augmented matrix of (1) to upper triangular matrix.(see key def).

step 2 Create the Augmented matrix

$$(A|B)$$

.

$$augmentedmatrix(A|B) = \begin{pmatrix} 10 & 1 & 1 & \vdots & 12 \\ 1 & 10 & 1 & \vdots & 12 \\ 1 & 1 & 10 & \vdots & 12 \end{pmatrix}$$

step 3 Creating the Pivot.
Make the first element 1. Divide all the elements of First row by the first element.

$$A = \begin{pmatrix} 1 & 1/10 & 1/10 & \vdots & 12/10 \\ 1 & 10 & 1 & \vdots & 12 \\ 1 & 1 & 10 & \vdots & 12 \end{pmatrix}$$

r$'_1 = r'_1 * 1/10$ **step 4** Use calculator to do this calculation.
r$'_2 = r_2 - r_1$

$$\begin{pmatrix} 1 & 1/10 & 1/10 & \vdots & 12/10 \\ 0 & 99/10 & 9/10 & \vdots & 108/10 \\ 1 & 1 & 10 & \vdots & 12 \end{pmatrix}$$

step 5 Then do this calculation like the previous one. $r'_3 = r_3 - r_1$

$$\begin{pmatrix} 1 & 1/10 & 1/10 & \vdots & 12/10 \\ 0 & 99/10 & 9/10 & \vdots & 108/10 \\ 0 & 9/10 & 99/10 & \vdots & 108/10 \end{pmatrix}$$

step 6 Make the second pivot. By dividing all the elements of 2nd row by 99/10.

$$\begin{pmatrix} 1 & 1/10 & 1/10 & \vdots & 12/10 \\ 0 & 1 & 1/11 & \vdots & 108/99 \\ 0 & 9/10 & 99/10 & \vdots & 108/10 \end{pmatrix}$$

step 7 Make the 9/10 in the third row zero to make pivot. Use calculator to find $r'_3 = r_3 - 9/10 * r_2$.

$$\begin{pmatrix} 1 & 1/10 & 1/10 & \vdots & 12/10 \\ 0 & 1 & 1/11 & \vdots & 108/99 \\ 0 & 0 & 108/11 & \vdots & 108/11 \end{pmatrix}$$

step 8 Remove 108/11.

$$\begin{pmatrix} 1 & 1/10 & 1/10 & \vdots & 12/10 \\ 0 & 1 & 1/11 & \vdots & 108/99 \\ 0 & 0 & 1 & \vdots & 1 \end{pmatrix}$$

The final step

$$\begin{aligned} x + 1/10y + 1/10z &= 12 \\ y + 1/11z &= 108/99 \\ z &= 1 \end{aligned}$$

Answer

$$\begin{aligned} x &= 12/10 - 1/10 - 1/10 = 1 \\ y &= 108/99 - 1/11 = 1 \\ z &= 1 \end{aligned}$$

HOME WORK
Problem 2

$$
\begin{aligned}
2x + 3y - z &= 5 \\
4x + 4y - 3z &= 3 \\
-2x + 3y - z &= 1 \\
answer \\
[x = 1, y = 2, z = 3]
\end{aligned}
$$

problem 3

$$
\begin{aligned}
x + y + z &= 7 \\
3x + 3y + 4z &= 24 \\
2x + y + 3z &= 16 \\
answer \\
[x = 3, y = 1, z = 3]
\end{aligned}
$$

3 key definition

Upper triangular matrix An upper triangular matrix is a square matrix in which all the entries below the main diagonal are zero.

$$
\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 7 \end{bmatrix}
$$