2.4

Question 1

Given that

$$F(x) = 1 - e^{-\theta x} \text{ and } f(x \mid \theta) = \theta e^{-\theta x}$$

So

$$\begin{aligned}
\int_0^m f(x \mid \theta) dx = F(m) = 1 - e^{-\theta m} \\
\Rightarrow \frac{1}{2} = 1 - e^{-\theta m} \\
\therefore \theta = \frac{\log 2}{m}
\end{aligned}$$

Hence

$$\begin{aligned}
g(x \mid m) &= f(x \mid \theta(m)) \\
&= \theta e^{-\theta x} \\
&= \frac{\log 2}{m} e^{-x \log 2/m} \\
&= \frac{\log 2}{m} 2^{-x/m}
\end{aligned}$$

Question 2

Given that

$$u_i = 1 - e^{-\theta_0 x_i}$$

So

$$x_i = \frac{-\log(1 - u_i)}{\theta_0}$$

A program to generate random exponential distribution can be found on page 9.

Running the program, we get

$$x = [1.14230.49130.78950.65541.10510.4857]$$

Now

$$\begin{aligned} l_n(m) &= \log \prod_{i=1}^{n} g(x_i \mid m) \\ &= \log \prod_{i=1}^{n} \frac{\log 2}{m} 2^{-x_i/m} \\ &= n \log\log 2 - n \log m - \frac{\sum x_i}{m} \log 2 \end{aligned}$$

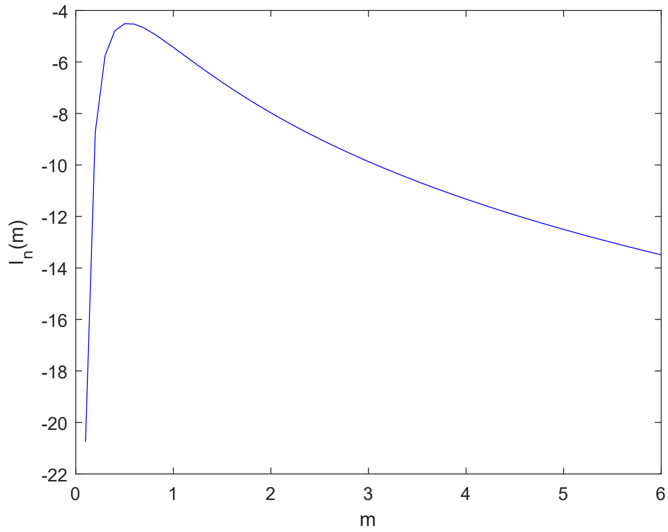

Figure 1: *log-likelyhood* function $l_n(m)$ against m

Figure 1 shows the *log-likelihood* function $l_n(m)$ against m. A program to generate it can be found on page 9.

To find the stationary points of $l_n(m)$, we differentiate it by m,

$$\frac{d}{dm} l_n(m) = -\frac{n}{m} + \frac{\sum x_i}{m^2} \log 2$$

So

$$\begin{aligned} \frac{d}{dm} l_n(m) &= 0 \\ \Rightarrow m &= \log 2 \frac{\sum x_i}{n} \end{aligned}$$

Now

$$l_n(m) = \lim_{m \to \infty} (n \log\log 2 - n \log m - \frac{\sum x_i}{m} \log 2) \longrightarrow -\infty \text{ as } m \to \infty$$

So $m = \log 2 \frac{\sum x_i}{n}$ is indeed a maximum point of $l_n(m)$. Hence

$$\hat{m_n} = \log 2 \frac{\sum x_i}{n}$$

Easy to see that

$$\mathbb{E}(\hat{m_n}) = \log 2 \mathbb{E}(\frac{\sum x_i}{n}) = \frac{\log 2}{\theta} = m_0$$

Hence $\hat{m_n}$ is an unbiased statistic of m_0.

For this random sample, we get $\hat{m_6} = 0.8907$ whereas the true value of the median is $m_0 = \frac{\log 2}{\theta_0} = 0.5776$.

Question 3

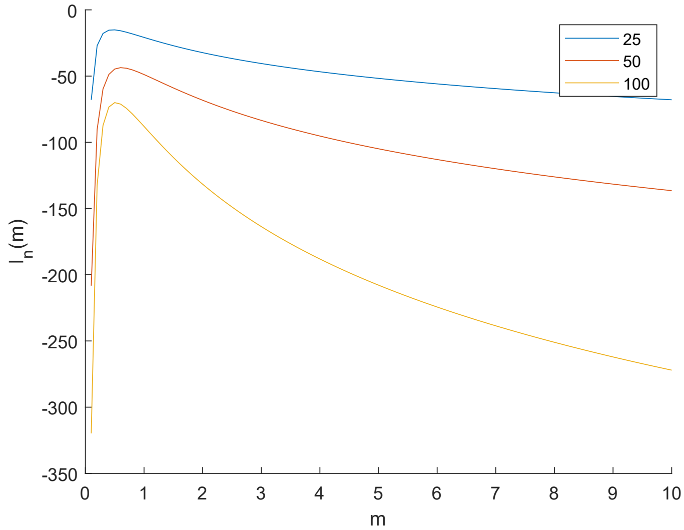

Figure 2: *log-likelyhood* function $l_n(m)$ against m for $n = 25, 50, 100$

The value of the MLE we get here are $\hat{m_n} = 1.0325, 0.7874$ and 0.9358 for $n = 25, 50$ and 100 resp whereas the true value is 0.5776.

Figure 2 shows the graphs of $l_n(m)$ for $n = 25, 50$ and 100. A program to generate it can be found on page 9.

We see that as n increases, $l_n(m)$ tends to zero much faster and becomes highly peaked at the maximum point.

Question 4

Given that X and Y have exponenetial distribution with mean $\frac{1}{\theta}$. So

$$f_X(x) = \theta e^{-\theta x}$$

So the moment generating function for $\lambda < \theta$ is

$$\begin{aligned}
M_X(\lambda) &= E(e^{\lambda X}) \\
&= \int_0^\infty e^{\lambda x} f_X(x) dx \\
&= \int_0^\infty e^{\lambda x} \theta e^{-\theta x} \\
&= \theta \int_0^\infty e^{(\lambda - \theta)x} \\
&= \theta \Big[\frac{e^{(\lambda-\theta)x}}{\lambda - \theta} \Big]_0^\infty \\
&= \frac{\theta}{\theta - \lambda} \\
&= (1 - \frac{\lambda}{\theta})^{-1}
\end{aligned}$$

As X and Y are independent, we have

$$\begin{aligned}
M_{X+Y}(\lambda) &= E(e^{\lambda(X+Y)}) \\
&= E(e^{\lambda X}) E(e^{\lambda Y}) \\
&= (1 - \frac{\lambda}{\theta})^{-2}
\end{aligned}$$

So we must have $X + Y \sim \Gamma(2, \theta)$.

Question 5

Given that

$$f(x \mid \theta) = \theta^2 x e^{-\theta x}$$

So

$$\begin{aligned}
F(x) &= \int_0^x \theta^2 y e^{-\theta y} dy \\
&= \big[-\theta y e^{-\theta y} \big]_0^x - \int_0^x (-\theta) e^{-\theta y} dy \\
&= \big[-\theta y e^{-\theta y} - e^{-\theta y} \big]_0^x \\
&= 1 - e^{-\theta x}(1 + \theta x)
\end{aligned}$$

Question 6

We have

$$\begin{aligned}
l_n(\theta) &= \log \prod_{i=1}^n f(x_i \mid \theta) \\
&= \log (\theta^{2n} \prod x_i e^{-\theta \sum x_i}) \\
&= 2n \log \theta + \sum \log x_i - \theta \sum x_i
\end{aligned}$$

Now

$$\frac{d}{d\theta}l_n(\theta) = 0$$
$$\Rightarrow \frac{2n}{\theta} - \sum x_i = 0$$
$$\therefore \theta = \frac{2n}{\sum x_i}$$

Question 7

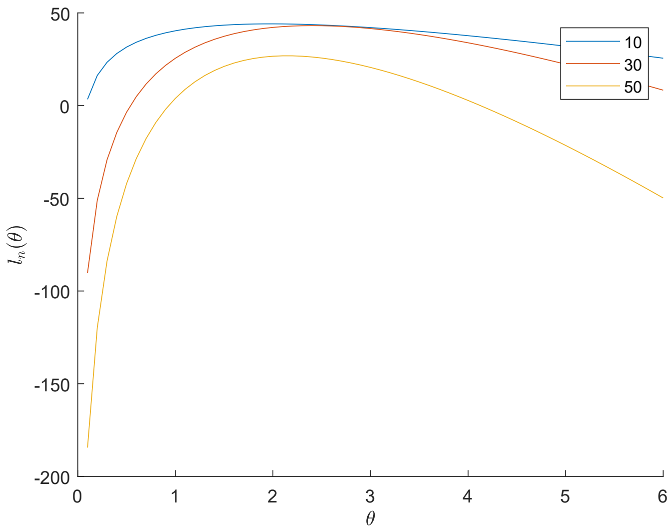

Figure 3: $l_n(\theta)$ *against* θ *for* $n = 10, 30, 50$

The values of maximum likelyhood estimator we get are $1.9758, 2.4056, 2.1469$ for $n = 10, 30, 50$ respectively.

Figure 3 shows $l_n(\theta)$ against θ for $n = 10, 30, 50$. A program to generate it can be found on page 10.

Question 8

Figure 4 and 5 shows the histogram of $\hat{\theta_n}$ for $n = 10$ and 50 resp. A program to generate them can be found on page 11.

As n increases from 10 to 50, we see that there are more values of MLE's near the point 2.2. This is something we would expect as the true value of θ is $\theta_0 = 2.2$ and as n increases, we get more accurate representation of θ.

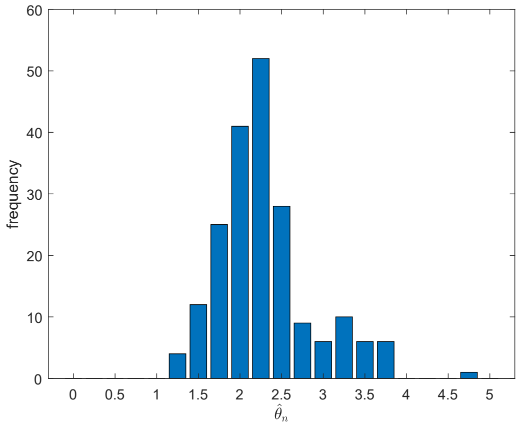

Figure 4: *Histogram of* $\hat{\theta}_n$ *for* $n = 10$

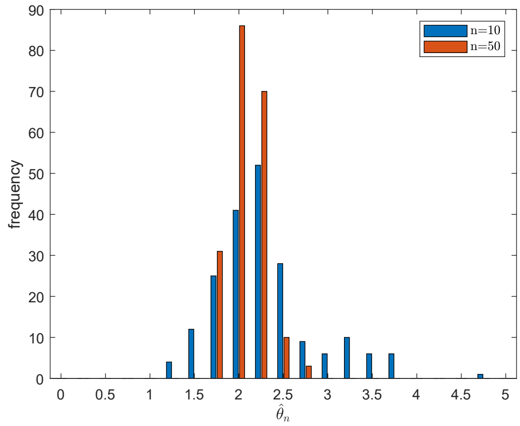

Figure 5: *Histogram of* $\hat{\theta}_n$ *for* $n = 10$ *and* 50

Question 9

Given that

$$X = \mu_1 + \sigma\sqrt{V}\cos\Phi$$
$$Y = \mu_2 + \sigma\sqrt{V}\sin\Phi$$

So

$$(X-\mu_1)^2 + (X-\mu_2)^2 = \sigma^2 V(\cos^2\phi + \sin^2\phi)$$
$$\therefore V = \frac{(X-\mu_1)^2 + (X-\mu_2)^2}{\sigma^2}$$

Also

$$\begin{aligned}
\mid \frac{\partial(\phi, v)}{\partial(x, y)} \mid &= \mid \frac{\partial(x, y)}{\partial(\phi, v)} \mid^{-1} \\
&= \begin{vmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial \phi} \end{vmatrix}^{-1} \\
&= \begin{vmatrix} -\sigma\sqrt{v}\sin\phi & \sigma\sqrt{v}\cos\phi \\ -\frac{-\sigma\cos\phi}{2\sqrt{v}} & -\frac{-\sigma\cos\phi}{2\sqrt{v}} \end{vmatrix}^{-1} \\
&= 2\sigma^{-2}
\end{aligned}$$

So

$$\begin{aligned}
g(x, y) &= f(\phi(x, y), v(x, y)) \mid \frac{\partial(\phi, v)}{\partial(x, y)} \mid \\
&= \frac{1}{4\pi} e^{-\frac{v}{2}} .2\sigma^{-2} \\
&= \frac{1}{2\pi\sigma^2} e^{-(X-\mu_1)^2 + (X-\mu_2)^2/\sigma^2}
\end{aligned}$$

Hence X, Y are independent with distribution $N(\mu_1, \sigma^2), N(\mu_2, \sigma^2)$ respectively.

Question 10

The program to generate a random sample of size n from $N(\mu, 1)$ can be found on page 10.

If $X_1, X_2, \ldots X_n \sim N(\mu, 1)$ and $\bar{X} = \frac{X_1+x_2+\cdots+X_n}{n}$, then

$$\bar{X}\ N(\mu, 1/n) \text{ hence } \sqrt{n}(\bar{X}-\mu)\ N(0,1)$$

Now if $(-\xi, \xi)$ is a symmetric 80% confidence interval of $N(0, 1)$, then
We also have

$$\mathbb{P}(-\xi \le \sqrt{n}(\bar{X}-\mu) \le \xi) = \mathbb{P}(\bar{X} - \frac{\xi}{\sqrt{n}} \le \mu \le \bar{X} + \frac{\xi}{\sqrt{n}}) = 0.8$$

So $(\bar{x} - \frac{1.282}{\sqrt{n}}, \bar{x} + \frac{1.282}{\sqrt{n}})$ is an 80% confidence interval of μ.

Question 11

$\bar{x}$	y_1	y_2	
3.96	3.77	4.14	1
3.92	3.74	4.10	1
3.84	3.66	4.03	1
4.00	3.82	4.18	1
3.95	3.77	4.13	1
3.93	3.75	4.11	1
4.11	3.93	4.29	1
4.07	3.89	4.25	1
4.20	4.02	4.38	0
4.20	4.02	4.38	0
3.84	3.66	4.03	1
4.12	3.94	4.30	1
4.05	3.87	4.24	1
4.07	3.88	4.25	1
3.90	3.72	4.08	1
3.72	3.54	3.90	0
4.16	3.98	4.34	1
3.84	3.66	4.02	1
4.06	3.88	4.25	1
3.91	3.73	4.10	1
3.98	3.80	4.16	1
4.11	3.93	4.29	1
3.80	3.62	3.98	0
3.96	3.78	4.14	1
4.22	4.04	4.40	0

Table 1: Sample distribution of $N(\mu, 1)$ and confidence interval

Table 1 shows a sample distribution of $N(\mu, 1)$ and confidence interval.(y_1, y_2 are the upper and lower bounds of the confidence interval). A program to generate it can be found on page 12.

The interval didn't contain μ in 5 occurences out of 25 times which is exactly what we expected as the it's an 80% confidence interval.

Question 12

Even if we change the values of n and μ, it still remains an 80% confidence interval of μ. So the expected number of times for the confidence interval not to contain μ would be 20% of 25 which is 5.

Question 13

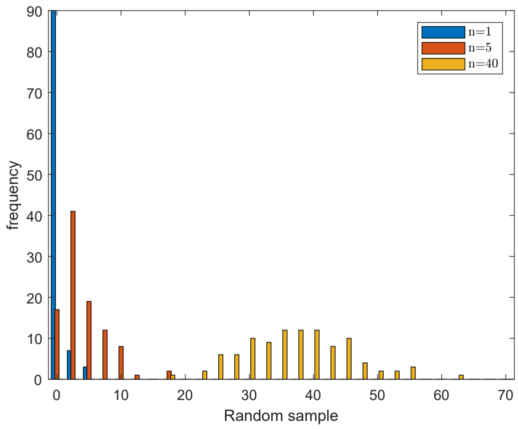

Figure 6: *Histogram of* $\hat{\theta}_n$ *for* $n = 10$

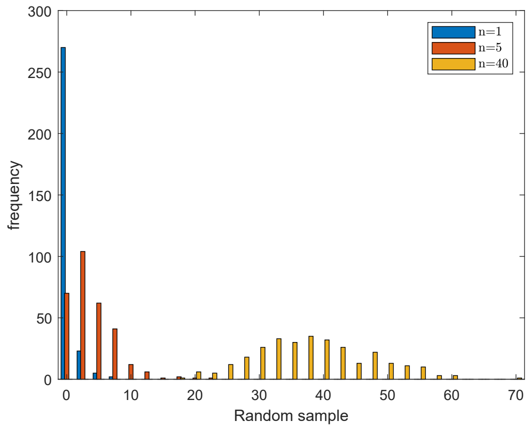

Figure 7: *Histogram of* $\hat{\theta}_n$ *for* $n = 10$

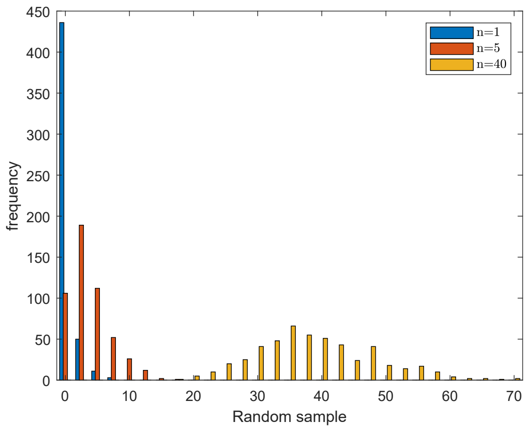

Figure 8: *Histogram of* $\hat{\theta}_n$ *for* $n = 10$

A program to generate sample of chi-square with m degree of freedom (χ^2_m) distributuion can be found in page 13.

Figure 6-8 show the histogram of random chi-square $1, 5$ and 40 degree of freedom for $n = 100, 300, 500$ resp. A program to generate them can be found on page 13.

Program to generate sample of exponential distribution

```
exp_rand.m

function [x] = exp_rand( n , theta_0)

u = rand(n,1) ;

x = -(log(1 - u))/(theta_0);
```

Program for question 2

```
q2plot.m

n = 6;
theta_0 = 1.2;

x = exp_rand( n , theta_0);

m_0 = (log(2))/(theta_0)

m_n = (log(2))/(sum(x)/n)

m = [ 0.1 : 0.1: 6];
l_n =  n*log(log(2)) - n.*log(m) - log(2)*(sum(x))./m;

figure
plot( m, l_n,'b')
xlabel('m')
ylabel('l_n(m)')
```

Program for question 3

```
q3plot.m

theta_0 = 1.2;
hold on
for n = [ 25 50 100 ]

    x = exp_rand( n , theta_0);

    m_0 = (log(2))/(theta_0)

    m_n = (log(2))/(sum(x)/n)

    m = [ 0.1 : 0.1: 10];
    l_n =  n*log(log(2)) - n.*log(m) - log(2)*(sum(x))./m;
```

```
    plot( m, l_n)
    xlabel('m')
    ylabel('l_n(m)')

end

hold off
legend('25','50','100')
```

Program for question 7

```
gamma_2.m

function [x, theta_mle] = gamma_2(n,theta_0)

y1 = exp_rand(n,theta_0);
y2 = exp_rand(n,theta_0);

x = y1 + y2;

theta_mle = (2*n)/(sum(x));
```

Program to generate sample of $\Gamma(2, \theta)$ distribution

```
q7plot.m

theta_0 = 2.2;
hold on
for n = [10, 30, 50]
    [x , theta_mle ] = gamma_2(n, theta_0) ;

    theta_mle

    t = 0.1 : 0.1 : 6;
    l_n = 2*n*log(t) + sum(log(t)) - t.*sum(x);


    plot(t,l_n)
    xlabel('$\theta$','Interpreter','latex')
    ylabel('$l_n(\theta)$','Interpreter','latex')
end


hold off
legend('10','30','50')
```

Program for question 8

q8.m

```
theta_0 = 2.2;
N = 200;


for i = 1:N
    [u , theta_mle10(i) ] = gamma_2(10, theta_0);
    [v , theta_mle50(i) ] = gamma_2(50, theta_0);
end


binRange = 0:0.25:5;
hcx = histcounts(theta_mle10,[binRange Inf]);
hcy = histcounts(theta_mle50,[binRange Inf]);

figure(1)
bar(binRange,hcx')
xlabel('$\hat{\theta}_n$','Interpreter','latex')
ylabel('frequency')

figure(2)
bar(binRange,[hcx;hcy]')

xlabel('$\hat{\theta}_n$','Interpreter','latex')
ylabel('frequency')
legend('n=10','n=50','Interpreter','latex')
```

Program to generate sample of normal distribution

normal.m

```
n=100;
mu = 0;

A = rand(n,1);
B = rand(n,1);

phi = 2*pi.*A;
V = -2*log(1-B);

X = mu + sqrt(V).*cos(phi);

x_ = (sum(X))/n

xi = [x_ - (1.282)/(sqrt(n)) , x_ + (1.282)/(sqrt(n))]
```

Program for question 11

```
(i) q11_0.m

n = 100;
mu = 0;

x = normal(mu,n)

x_bar = (sum(x))/n

xi1 = x_bar - (1.282)/(sqrt(n))
xi2 = x_bar + (1.282)/(sqrt(n))
```

```
(ii)q11.m

n = 100;
mu = 0;

for j=1:25

    x = normal(mu,n);

    x_bar = (sum(x))/n ;

    xi1 = x_bar - (1.282)/(sqrt(n)) ;
    xi2 = x_bar + (1.282)/(sqrt(n));

    if xi1 <= mu && mu <= xi2
        y = 1;
    else
        y = 0;
    end

    fprintf('%4.2f   %4.2f   %4.2f   %d \n', x_bar, xi1, xi2, y);

end
```

Program to generate sample of Chi-squared distribution

chi_square.m

```
function y = chi_square(m,n)

for i = 1:m
    x(i,:) = normal(0,n);
end

for j = 1:n
    y(j) = sum(x(:,j).^2);
end
```

Program for question 13

q13.m

```
for n = [ 100 300 500 ]

        y1  = chi_square(1,n);
        y5  = chi_square(5,n);
        y40 = chi_square(40,n);

        binRange = 0:3.33:70;

        hc1 = histcounts(y1,[binRange Inf]);
        hc5 = histcounts(y5,[binRange Inf]);
        hc40 = histcounts(y40,[binRange Inf]);

        figure(n)
        bar(binRange,[hc1;hc5;hc40]','BarWidth', 1.2)

        xlabel('Random sample')
        ylabel('frequency')
        legend('n=1','n=5','n=40','Interpreter','latex')
end
```