2.1

Question 1

Given that

$$\theta(x,t) = \theta_0 + (\theta_1 - \theta_0)F(x,t)$$

So $[F] = 0$ where $[P]$ denotes the dimension of quantity P. Also F depends only on the variables x, t and constant K with

$$[x] = L \tag{1}$$
$$[t] = T \tag{2}$$
$$[K] = L^2T^{-1} \tag{3}$$

Here $[x]$ and $[t]$ are independent with $[K]$ being dependent and $[\frac{x}{\sqrt{Kt}}] = 0$.
So by dimensional analysis, we must have $F(x,t) = f(\frac{x}{\sqrt{Kt}})$ for some function f.
Given that

$$\begin{aligned}\theta(x,t) &= \theta_0 + (\theta_1 - \theta_0)F(x,t)\\ \therefore F(0,t) &= 1\\ F(x,0) &= 0\end{aligned}$$

Now assuming $\xi = \frac{x}{\sqrt{Kt}}$, we have

$$\begin{aligned}\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} = \frac{-x}{2\sqrt{Kt^3}}\frac{\partial}{\partial \xi} &= \frac{-\xi}{2t}\frac{\partial}{\partial \xi}\\ \frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} &= \frac{1}{\sqrt{Kt}}\frac{\partial}{\partial \xi}\\ \frac{\partial^2}{\partial x^2} &= \frac{1}{Kt}\frac{\partial^2}{\partial \xi^2}\end{aligned}$$

Therefore the heat equation can be reposed as a differential equation only in ξ:

$$\frac{-\xi}{2t}\frac{df}{d\xi} = \frac{K}{Kt}\frac{d^2f}{d\xi^2} \text{ or } \frac{d}{d\xi}\left(\frac{df}{d\xi}\right) = \frac{-\xi}{2}\frac{df}{d\xi}$$

With the substitution

$$
\begin{aligned}
\zeta &= \frac{df}{d\xi} \\
\frac{d\zeta}{d\xi} &= -\frac{\xi\zeta}{2} \\
\log\zeta &= -(\xi/2)^2 + C_1 \\
\zeta &= C_2 e^{-\xi^2/4} \\
\frac{df}{d\xi} &= C_2 e^{-\xi^2/4} \\
f &= C_3 \int_0^{\xi/2} e^{-u^2} du + D
\end{aligned}
$$

Now as $F(0,t) = 1$, we have $f(0) = 1$ or $D = 1$
Also when $t \to 0, \xi \to \infty$, hence

$$
\begin{aligned}
0 &= \lim_{\xi\to\infty} \left[C_3 \int_0^{\xi/2} e^{-u^2} du + 1 \right] \\
&= C_3 \int_0^{\infty} e^{-u^2} du + 1 \\
&= \frac{C_3\sqrt{\pi}}{2} + 1 \\
\Rightarrow C_3 &= -\frac{2}{\sqrt{\pi}} \\
\Rightarrow f(\xi) &= 1 - \frac{2}{\sqrt{\pi}} \int_0^{\xi/2} e^{-u^2} du \\
&= \frac{2}{\sqrt{\pi}} \int_{\xi/2}^{\infty} e^{-u^2} du
\end{aligned}
$$

Question 2

Let

$$V(X,T) = U(X,T) - 1 + X$$

So

$$
\begin{aligned}
V(X,0) &= X - 1 \\
V(0,T) &= 0 \\
V(1,T) &= 0
\end{aligned}
$$

Hence V has homogenous boundary condition. So by seperation of variables, we can write

$$V(X,T) = p(X)q(T)$$

Therefore

$$
\begin{aligned}
\dot{q} &= -\lambda q \\
p'' &= -\lambda p
\end{aligned}
$$

for some positive eigenvalue λ. The solution for p can be expressed as

$$p(X) = A\cos(\sqrt{\lambda}X) + B\sin(\sqrt{\lambda}X)$$

Applying the boundary conditions

$$p(0) = 0 \Rightarrow A = 0$$
$$p(1) = 0 \Rightarrow B\sin(\sqrt{\lambda}) = 0$$

which implies that

$$\lambda = n^2\pi^2$$
$$A = 0$$

for positive integers n. So we have eigenfunctions

$$p_n(X) = B_n \sin(n\pi X)$$

Therefore

$$\dot{q_n} = -n^2\pi^2 q_n$$
$$q_n = C_n e^{-n^2\pi^2 T}$$

The general solution for V is

$$V(X,T) = \sum_{n=1}^{\infty} b_n \sin(n\pi X)e^{-n^2\pi^2 T}$$

The initial condition is that

$$\sum_{n=1}^{\infty} b_n \sin(n\pi X) = X - 1$$

So

$$\begin{aligned} b_m &= \int_0^1 (X-1)\sin(m\pi X)dX \\ &= [(X-1)\frac{-\cos(m\pi X)}{m\pi}]_0^2 - \int_0^2 \frac{-\cos(m\pi X)}{m\pi}dX \\ &= -\frac{2}{m\pi} + [\frac{\sin(m\pi X)}{m^2\pi^2}]_0^2 \\ &= -\frac{2}{m\pi} \end{aligned}$$

Therefore

$$\begin{aligned} U(X,T) &= 1 - X + V(X,T) \\ &= 1 - X + \sum_{n=1}^{\infty} \frac{-2}{n\pi} e^{-n^2\pi^2 T}\sin(n\pi X) \\ \therefore g_n(T) &= e^{-n^2\pi^2 T} \end{aligned}$$

Now we solve the insulated end-point problem.

The steady function $U_s(x) = 1$ satisfies both boundary conditions: $U(0,T) = 0$ and $U_X(1,T) = 0$.

So let's assume

$$V(X,T) = U(X,T) - 1$$

So

$$\begin{aligned} V(X,0) &= -1 \\ V(0,T) &= 0 \\ V_X(1,T) &= 0 \end{aligned}$$

Hence V has homogenous boundary condition. So by seperation of variables, we can write

$$V(X,T) = p(X)q(T)$$

Therefore

$$\begin{aligned} \dot{q} &= -\lambda q \\ p'' &= -\lambda p \end{aligned}$$

for some positive eigenvalue λ. The solution for p can be expressed as

$$p(X) = A\cos(\sqrt{\lambda}X) + B\sin(\sqrt{\lambda}X)$$

Applying the boundary conditions

$$p(0) = 0 \to A = 0 p_X(1) = 0 \to \sqrt{\lambda}B\cos(\sqrt{\lambda}) = 0$$

which implies that

$$\begin{aligned} \lambda &= (n-1/2)^2\pi^2 \\ A &= 0 \end{aligned}$$

for positive integers n. So we have eigenfunctions

$$p_n(X) = B_n \sin((n-1/2)\pi X)$$

Therefore

$$\begin{aligned} \dot{q_n} &= -(n-1/2)^2\pi^2 q_n \\ q_n &= C_n e^{-(n-1/2)^2\pi^2 T} \end{aligned}$$

The general solution for V is

$$V(X,T) = \sum_{n=1}^{\infty} b_n \sin((n-1/2)\pi X) e^{-(n-1/2)^2\pi^2 T}$$

The initial condition is that

$$\sum_{n=1}^{\infty} b_n \sin((n-1/2)\pi X) = -1$$

Now

So

$$
\begin{aligned}
b_m/2 &= \int_0^1 -1 \sin(m\pi X) dX \\
&= -\left[\frac{-\cos[(m-1/2)\pi X]}{(m-1/2)\pi}\right]_0^1 \\
&= -\frac{2}{(m-1/2)\pi} \\
\therefore b_m &= -\frac{2}{(m-1/2)\pi}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
U(X,T) &= 1 + V(X,T) \\
&= 1 + \sum_{n=1}^{\infty} \frac{-2}{(n-1/2)\pi} e^{-(n-1/2)^2\pi^2 T} \sin((n-1/2)\pi X)
\end{aligned}
$$

So we get

$$
\begin{aligned}
U_s(X) &= 1 \\
G_n(T) &= e^{-(n-1/2)^2\pi^2 T} \\
H_n(X) &= \sin((n-1/2)\pi X)
\end{aligned}
$$

Programming Task

We write U_a and U_b for the functions in equation (18) and (19) respectively.

A program to evaluate the analytic solutions can be found on page 11.
Table 1 shows the three solutuions at $T = 0.25$ and $x = 0.125n$
Figure 1-6 shows the non-dimensionalised temperature profiles, U, against X, for all three at $T = 0.0625, 0.125, 0.25, 0.5, 1.0$ and 2.0.
Figure 7 shows the graph of heat flux $-U_X$ at $X = 0$ against T.
Programs for generating tables and plots can be found on page 11 and 12.

X	U_b	U_a	erfc($\xi/2$)
0.000	1.00000000	1.00000000	1.00000000
0.125	0.85432785	0.86503967	0.85968380
0.250	0.71180789	0.73553911	0.72367361
0.375	0.57510947	0.61665612	0.59588309
0.500	0.44601148	0.51298728	0.47950012
0.625	0.32513275	0.42838185	0.37675912
0.750	0.21184081	0.36583931	0.28884437
0.875	0.10435113	0.32747896	0.21592494
1.000	−0.00000000	0.31455423	0.15729921

Table 1: Table of U_b, U_a and F for $T = 0.25$ and $X = 0.125n$, $0 \le n \le 8$

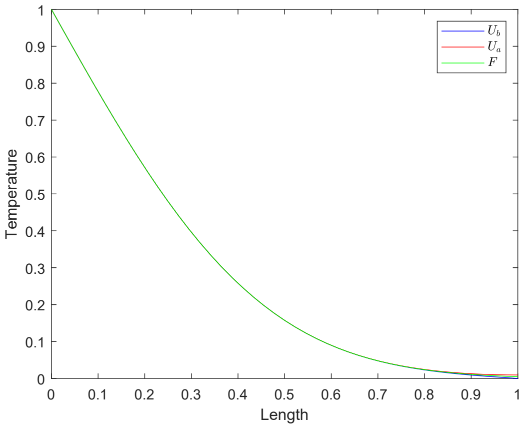

Figure 1: Graph of U_a, U_B and F at $T = 0.0625$

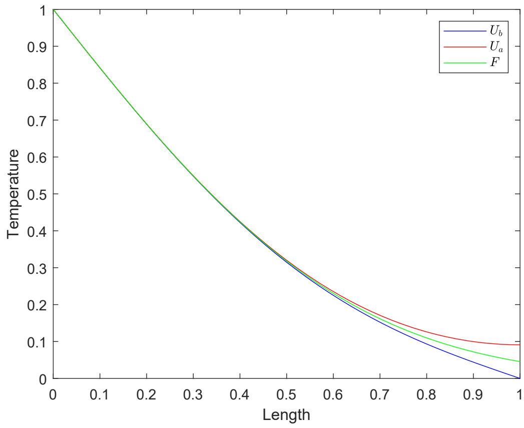

Figure 2: Graph of U_a, U_B and F at $T = 0.125$

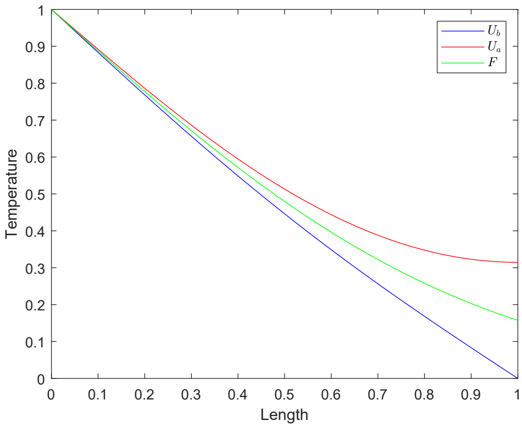

Figure 3: Graph of U_a, U_B and F at $T = 0.25$

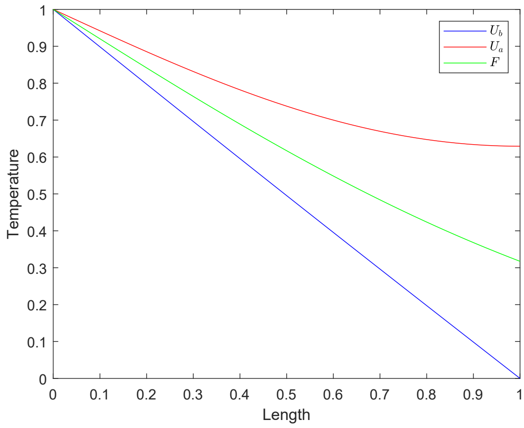

Figure 4: Graph of U_a, U_B and F at $T = 0.5$

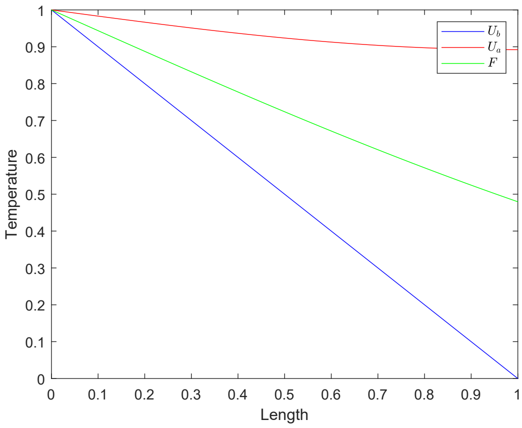

Figure 5: Graph of U_a, U_B and F at $T = 1$

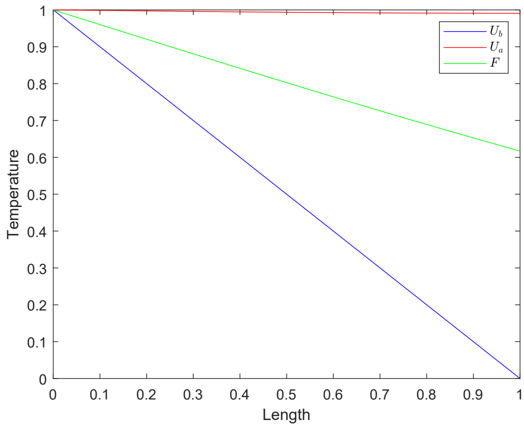

Figure 6: Graph of U_a, U_B and F at $T = 2$

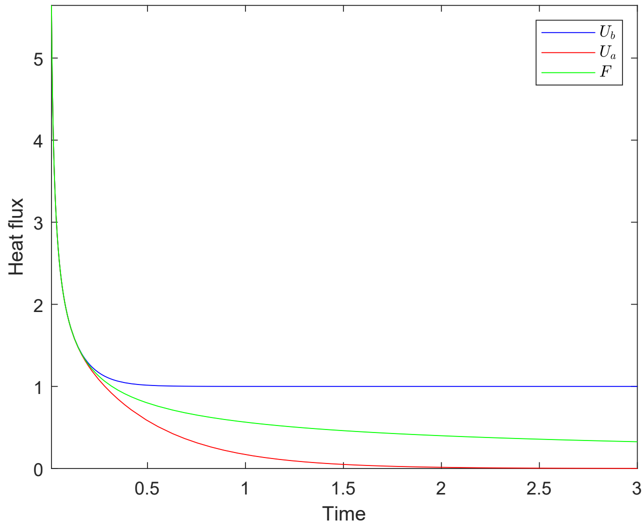

Figure 7: Graph of heat flux at $X = 0$

Now for the fixed endpoint problem, we have

$$\left|\frac{-2}{n\pi}e^{-n^2\pi^2 T}\sin(n\pi X)\right| \leq e^{-n\log 10(\frac{n\pi^2 T}{\log 10})} \leq 10^{-n(\frac{11\times\pi^2\times 0.0625}{\log 10})} \leq 10^{-n}$$

So

$$\begin{aligned}\left|U(X,T) - \left(1 - X + \sum_{n=1}^{10}\frac{-2}{n\pi}e^{-n^2\pi^2 T}\sin(n\pi X)\right)\right| &= \left|\sum_{n=11}^{\infty}\frac{-2}{n\pi}e^{-n^2\pi^2 T}\sin(n\pi X)\right| \\ &\leq \sum_{n=11}^{\infty}10^{-n} \\ &\leq 10^{-10}\sum_{n=1}^{\infty}10^{-n} \\ &\leq 10^{-10}\end{aligned}$$

Similary for the insulated endpoint problem, we can show that

$$\left|\frac{-2}{(n-1/2)\pi}e^{-(n-1/2)^2\pi^2 T}\sin((n-1/2)\pi X)\right| \leq 10^{-n}$$

Hence

$$\left|U(X,T) - \left(1 + \sum_{n=1}^{10}\frac{-2}{(n-1/2)\pi}e^{-(n-1/2)^2\pi^2 T}\sin((n-1/2)\pi X)\right)\right| \leq 10^{-10}$$

So our soulutions are accurate up to at least 10 digits.

From the graphs, we can see that as time evolves, both U_a and U_b tend to their steady state very fast. Whereas the semi-infinite solution always lie between them.

Question 3

A program to implement the numerical scheme can be found on page 13.
Programs for generating tables and plots can be found on page 13.

(i) Table 2-5 show both the analytic and the numerical solutions, and the value of the error, at $T = 0.125, 0.25, 0.5$ and 1.0.

X	Num. Sol	Ana. Sol	Error
0.000	1.00000000	1.00000000	0.00000000
0.125	0.80364990	0.80274281	0.00090710
0.250	0.61831665	0.61753354	0.00078311
0.375	0.45501709	0.45440672	0.00061037
0.500	0.31869507	0.32000973	0.00131466
0.625	0.21429443	0.21725892	0.00296449
0.750	0.14102173	0.14603370	0.00501197
0.875	0.09808350	0.10456725	0.00648375
1.000	0.08419800	0.09100052	0.00680253

Table 2: Table of analytic & numerical solution and error at $T = 0.125$

X	Num. Sol	Ana. Sol	Error
0.000	1.00000000	1.00000000	0.00000000
0.125	0.86494803	0.86503967	0.00009164
0.250	0.73522040	0.73553911	0.00031871
0.375	0.61614143	0.61665612	0.00051469
0.500	0.51199744	0.51298728	0.00098984
0.625	0.42707473	0.42838185	0.00130713
0.750	0.36402237	0.36583931	0.00181694
0.875	0.32548946	0.32747896	0.00198950
1.000	0.31235286	0.31455423	0.00220137

Table 3: Table of analytic & numerical solution and error at $T = 0.25$

X	Num. Sol	Ana. Sol	Error
0.000	1.00000000	1.00000000	0.00000000
0.125	0.92777571	0.92766011	0.00011560
0.250	0.85830080	0.85810127	0.00019953
0.375	0.79432463	0.79399728	0.00032736
0.500	0.73817741	0.73781172	0.00036568
0.625	0.69218931	0.69170327	0.00048604
0.750	0.65791673	0.65744286	0.00047387
0.875	0.63691607	0.63634600	0.00057007
1.000	0.62973373	0.62922257	0.00051116

Table 4: Table of analytic & numerical solution and error at $T = 0.5$

(ii) Figure 8-13 show the analytic and the numerical solutions for $T = 0.0625, 0.125, 0.25, 0.5, 1.0$ and 2.0.

X	Num. Sol	Ana. Sol	Error
0.000	1.00000000	1.00000000	0.00000000
0.125	0.97913568	0.97893472	0.00020096
0.250	0.95906546	0.95867897	0.00038649
0.375	0.94058344	0.94001117	0.00057227
0.500	0.92436283	0.92364870	0.00071413
0.625	0.91107684	0.91022037	0.00085647
0.750	0.90117528	0.90024222	0.00093306
0.875	0.89510798	0.89409770	0.00101027
1.000	0.89303289	0.89202296	0.00100994

Table 5: Table of analytic & numerical solution and error at $T = 1$

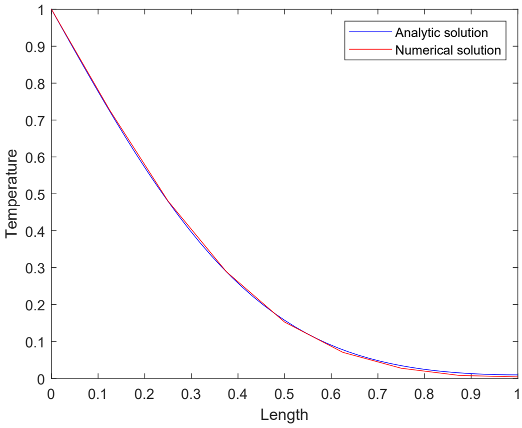

Figure 8: RK4 program with $\omega = 1, \gamma = 0.25$

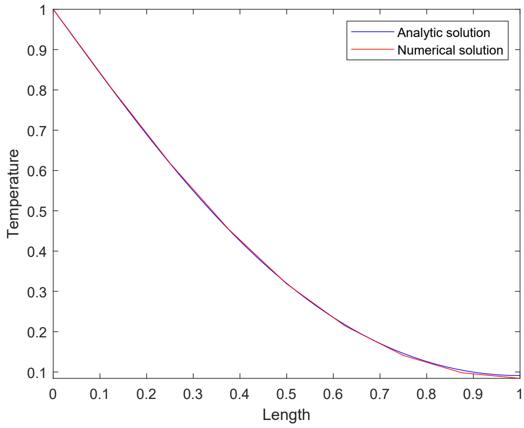

Figure 9: RK4 program with $\omega = 1, \gamma = 1.0$

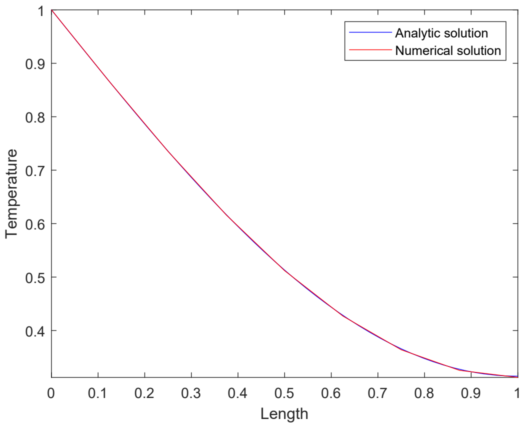

Figure 10: RK4 program with $\omega = 2, \gamma = 0.25$

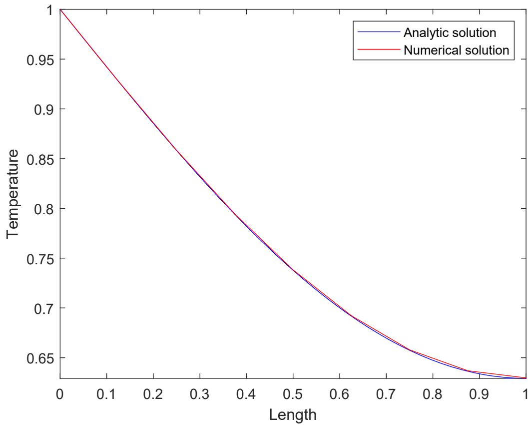

Figure 11: RK4 program with $\omega = 2, \gamma = 1.9$

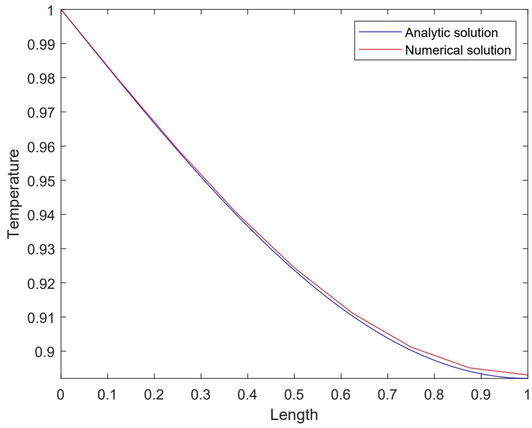

Figure 12: RK4 program with $\omega = 2, \gamma = 0.25$

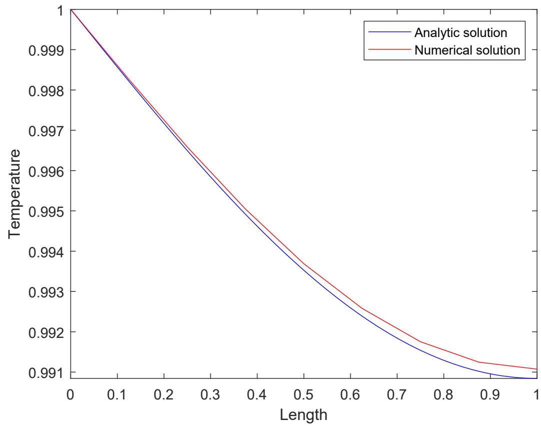

Figure 13: RK4 program with $\omega = 2, \gamma = 1.9$

Program to evaluate analytic solutions

(i) U_a.m

```
function U = U_a(x,t,k)

U = 1 ;
for n = 1:k
    U = U + (-2)*exp(-(n-1/2)^2*(pi)^2*t)*sin((n-1/2)*(pi)*x)/((n-1/2)*(pi));
end
```

(ii) U_b.m

```
function U = U_b(x,t,k)

U = 1 - x;

for n = 1:k
    U = U + (-2)*exp(-(n^2)*(pi)^2*t)*sin(n*(pi)*x)/(n*(pi));
end
```

Program to generate tables and plots of question 2

(i) q2plot1.m

```
for n = 1:9
    X = 0.125*(n-1);
    T = 0.25;
    fprintf( '%.3f %.8f %.8f %.8f \n', X, U_b(X,T,10), U_a(X,T,10), erfc(X/(2*sqrt(T))))

end
```

(ii) q2plot2.m

```
for T = [ 0.0625 0.125 0.25 0.5 1.0 2.0 ]

        figure(1024*T)

        fplot(@(X) U_b(X,T,10),[0 1],'b')
        hold on
        fplot(@(X) U_a(X,T,10),[0 1],'r')
        fplot(@(X) erfc(X/(2*sqrt(T))),[0 1],'g')
        hold off
        xlabel('Length')
        ylabel('Temperature')
        legend('$U_b$','$U_a$','$F$','Interpreter','latex')
end
```

(iii) dU_a.m

```
function dU = dU_a(T,k)

dU = 0;
for n = 1:k
    dU = dU + (-2)*exp(-(n-1/2)^2*(pi)^2*T);
end
```

(iv) dU_b.m

```
function dU = dU_b(T,k)

dU = -1 ;
for n = 1:k
    dU = dU + (-2)*exp(-(n^2)*(pi)^2*T);
end
```

(v) q2plot3.m

```
fplot(@(T) -dU_b(T,10),[0.01 3],'b')
hold on
fplot(@(T) -dU_a(T,10),[0.01 3],'r')
fplot(@(T) 1/(sqrt(pi*T)),[0.01 3],'g')
hold off

xlabel('Time')
ylabel('Heat flux')
legend('$U_b$','$U_a$','$F$','Interpreter','latex')
```

Program to find numerical solution

diff_num.m

```
function U = diff_num(N,M,C)

U = zeros(N+1,M+1);
U(:,1) = 0;
U(1,:) = 1;
U(1,1) = 0.5;

for m = 1:M
    for i = 2:N
        U(i,m+1) = U(i,m) + C*( U(i+1,m) - 2*U(i,m) + U(i-1,m));
    end
      U(N+1,m+1) = U(N+1,m) + C*( U(N,m) - 2*U(N+1,m) + U(N,m));
end
```

Program to generate tables and plots of question 3

(i) q3table.m

```
N = 8;
C = 0.5;

U = diff_num(8,128,0.5);

for m = [ 2^4 2^5 2^6 2^7 ]
    for n = 1:9
        fprintf('%.3f %.8f %.8f \n' ,(n-1)/N, U(n,m+1), U_a((n-1)/N,m/128,5),
        abs(U(n,m+1) - U_a((n-1)/N,m/128,5)))

    end
end
```

(ii)q3plot.m

```
U = diff_num(8,256,0.5);

for m = [ 2^3 2^4 2^5 2^6 2^7 2^8 ]
    figure(m)
    fplot(@(X) U_a(X,m/128,10),[0 1],'b')
    hold on
    x = 0:0.125:1 ;
    u = U(8*x+1,m+1) ;
    plot(x,u,'r')
    hold off

    xlabel('Length')
    ylabel('Temperature')
    legend('Analytic solution','Numerical solution')
end
```