retrieved deterministically through a finite number of projective measurements. In addition, the number of representable colors and positions in an image does not depend on the actual physical implementation of the quantum system, and a larger class of more complex image processing operations can be applied using this model because the color is represented using a computational basis state that can act as a control for applying value-dependent color transforms and for computing statistics in the image, as addressed in [2]. However, additional qubits are required to encode the color information of images in this complex representation.

## 2.2 Flexible Representation for Quantum Images

The FRQI representation has shown widespread appeal in recent QIMP literature. This representation is now introduced and some of its properties, as well as related transformations, are highlighted [12].

### *2.2.1 FRQI Representation and Initialization*

FRQI is similar to pixel representation of images on traditional computers. It captures the essential information about the colors and the position of every point in an image, and integrates them into a quantum state with the formula:

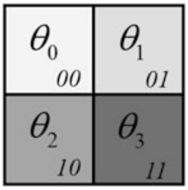
$$|I\rangle = \frac{1}{2^n} \sum_{i=0}^{2^{2n}-1} |c_i\rangle \otimes |i\rangle, \tag{2.1}$$

where

$$|c_i\rangle = \cos\theta_i |0\rangle + \sin\theta_i |1\rangle, \tag{2.2}$$

where $|0\rangle$ and $|1\rangle$ are two-dimensional computational basis states; $|i\rangle$, $i = 0, 1, \ldots, 2^{2n} - 1$, are $2^{2n}$-dimensional computational basis states; and $\theta = (\theta_0, \theta_1, \ldots, \theta_{2^{2n}-1})$, $\theta_i \in [0, \frac{\pi}{2}]$, is the vector of angles encoding colors. The two parts in the FRQI representation of an image, $|c_i\rangle$ and $|i\rangle$, respectively encode information about the colors and corresponding positions in the image. An example of a $2 \times 2$ FRQI image with its quantum state is shown in Fig. 2.1.

In quantum computation, computers are usually initialized in well-prepared states [19]. Hence, a preparation process is necessary to transform quantum computers from the initialized state to the desired quantum image state. The polynomial preparation theorem (PPT), which follows from Theorem 2.1, demonstrates an efficient preparation process [12].




$$|I\rangle = \frac{1}{2}[(\cos\theta_0\,|0\rangle + \sin\theta_0\,|1\rangle)\otimes|00\rangle + (\cos\theta_1\,|0\rangle + \sin\theta_1\,|1\rangle)\otimes|01\rangle$$
$$+ (\cos\theta_2\,|0\rangle + \sin\theta_2\,|1\rangle)\otimes|10\rangle + (\cos\theta_3\,|0\rangle + \sin\theta_3\,|1\rangle)\otimes|11\rangle]$$

**Fig. 2.1** A $2 \times 2$ FRQI image and its quantum state

**Theorem 2.1** *Given a vector* $\theta = (\theta_0, \theta_1, \cdots, \theta_{2^{2n}-1})$ *of angles* ($\theta_i \in [0, \frac{\pi}{2}]$, $i = 0, 1, \cdots, 2^{2n} - 1$)*, there is a unitary transform* $\mathscr{P}$ *that turns quantum computers from the initialized state,* $|0\rangle^{\otimes 2n+1}$*, to the FRQI state in Eq. (2.1), composed of a polynomial number of simple gates.*

***Proof*** There are two steps to achieve the unitary transform $\mathscr{P}$, where Hadamard transforms are first used to change $|0\rangle^{\otimes 2n+1}$ to $|H\rangle$, followed by controlled-rotation transforms from $|H\rangle$ to $|I\rangle$.

Considering the two-dimensional identity matrix $I$ and the $2n$ Hadamard matrices $H^{\otimes 2n}$, applying the transform $\mathscr{H} = I \otimes H^{\otimes 2n}$ on $|0\rangle^{\otimes 2n+1}$ produces the state $|H\rangle$:

$$\mathscr{H}(|0\rangle^{\otimes 2n+1}) = \frac{1}{2^n}|0\rangle \otimes \sum_{i=0}^{2^{2n}-1} |i\rangle = |H\rangle. \tag{2.3}$$

In addition, the rotation matrices $R_y(2\theta_i)$ and controlled-rotation matrices $R_i$ with $i = 0, 1, \cdots, 2^{2n} - 1$ are considered, i.e.,

$$R_y(2\theta_i) = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}, \tag{2.4}$$

$$R_i = \left(I \otimes \sum_{j=0, j\neq i}^{2^{2n}-1} |j\rangle\langle j|\right) + R_y(2\theta_i) \otimes |i\rangle\langle i|. \tag{2.5}$$

The controlled-rotation $R_i$ is a unitary matrix, since $R_i R_i^\dagger = I^{\otimes 2n+1}$. Applying $R_k$ and $R_l R_k$ on $|H\rangle$ gives:

$$\begin{aligned} R_k(|H\rangle) &= R_k\left(\frac{1}{2^n}|0\rangle \otimes \sum_{i=0}^{2^{2n}-1} |i\rangle\right) \\ &= \frac{1}{2^n}\left[I|0\rangle \otimes \left(\sum_{i=0, i\neq k}^{2^{2n}-1} |i\rangle\langle i|\right)\left(\sum_{i=0}^{2^{2n}-1} |i\rangle\right)\right. \end{aligned}$$

$$+R_y(2\theta_k)|0\rangle \otimes |k\rangle\langle k| \left(\sum_{i=0}^{2^{2n}-1} |i\rangle\right)\Bigg] \tag{2.6}$$

$$= \frac{1}{2^n}\left[|0\rangle \otimes \left(\sum_{i=0, i\neq k}^{2^{2n}-1} |i\rangle\right) + (\cos\theta_k|0\rangle + \sin\theta_k|1\rangle) \otimes |k\rangle\right],$$

and

$$\begin{aligned} R_l R_k|H\rangle &= R_l(R_k|H\rangle) \\ &= \frac{1}{2^n}\Bigg[|0\rangle \otimes \left(\sum_{i=0, i\neq k,l}^{2^{2n}-1} |i\rangle\right) + (\cos\theta_k|0\rangle + \sin\theta_k|1\rangle) \otimes |k\rangle \\ &\quad + (\cos\theta_l|0\rangle + \sin\theta_l|1\rangle) \otimes |l\rangle\Bigg]. \end{aligned} \tag{2.7}$$

It is obvious from Eq. (2.7) that

$$\mathscr{R}|H\rangle = \left(\prod_{i=0}^{2^{2n}-1} R_i\right)|H\rangle = |I\rangle. \tag{2.8}$$

Therefore, the unitary transform $\mathscr{P} = \mathscr{R}\mathscr{H}$ turns quantum computers from the initialized state $|0\rangle^{\otimes 2n+1}$ to the FRQI state $|I\rangle$. The computational complexity of the preparation for FRQI image can be calculated as $O(2^{4n})$ [12].

The essential requirements to represent a classical or quantum image are simplicity and efficiency in the storage and retrieval of the image [14]. The storage of an FRQI image is achieved by the preparation process using the PPT as discussed earlier. The measurement of the quantum image state produces a probability distribution that is used for the retrieval of the image. As presented in Sect. 1.1.2, measuring a quantum state produces an outcome and a post-measurement quantum state, which is a projection of the state vector onto the basis vector that corresponds to the outcome obtained. Therefore, to extract the angles encoding the gray levels and the corresponding positions from the image, many identical FRQI states are required. Multiple measurement operations on these identical quantum states give information about the quantum state in the form of a probability distribution [12].

It is noteworthy that with general quantum states, the probability distributions are not enough to clearly understand the states because their coefficients are complex numbers. The FRQI, however, contains only real-valued coefficients that enable retrieval of all of the information about the state [14].