Spanning Tree Problem With Constraint on the Number of Leaves

17050{02,17,66,92,94}
Department of CSE, BUET

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Outline

Spanning Tree

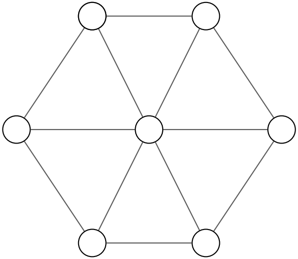


Figure: The Graph W_7

Spanning Tree

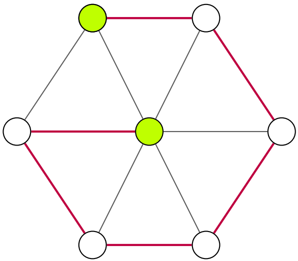


Figure: A Spanning Tree of W_7 with 2 leaves

Spanning Tree

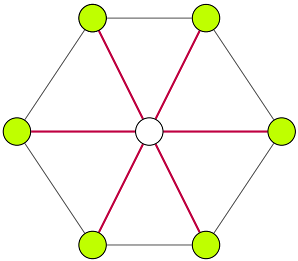


Figure: A Spanning Tree of W_7 with 6 leaves

Problems: Optimization Versions

Minimum Leaf Spanning Tree Problem

Given a connected graph G, find a spanning tree of G with the **minimum** number of leaves.

Maximum Leaf Spanning Tree Problem

Given a connected graph G, find a spanning tree of G with the **maximum** number of leaves.

Problems: Decision Versions

Minimum Leaf Spanning Tree Problem (MinLST)

Given a connected graph G and an integer k, does G have a spanning tree with **at most** k leaves?

Maximum Leaf Spanning Tree Problem (MaxLST)

Given a connected graph G and an integer k, does G have a spanning tree with **at least** k leaves?

Outline

Section Overview

MinLST and MaxLST: Both problems are in $\mathcal{NP}$

For either problem, given a YES-certificate, we can verify in polynomial time.

Now we will reduce known $\mathcal{NP}$-hard problems to our problems to show them to be $\mathcal{NP}$-hard.

1. HamPath $\leq_P$ MinLST
2. VertexCover $\leq_P$ MinConDomSet $\leq_P$ MaxLST

Outline

HAMPATH $\leq_P$ MINLST

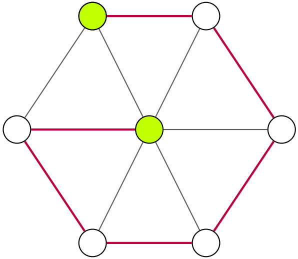


Figure: A Spanning Tree of W_7 with at most 2 leaves which is also a Hamiltonian Path

HamPath $\leq_P$ MinLST

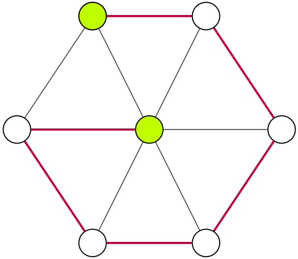


Figure: A Spanning Tree of W_7 with at most 2 leaves which is also a Hamiltonian Path

Lemma

A connected graph G has a Hamiltonian path if and only if it has a spanning tree of at most two leaves.

Outline

Connected Dominating Set

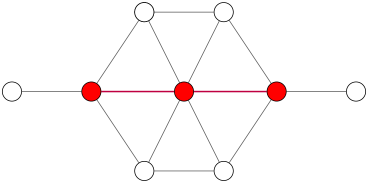


Figure: The red vertices form a Connected Dominating Set

Connected Dominating Set

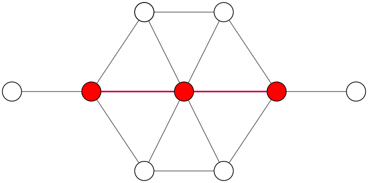


Figure: The red vertices form a Connected Dominating Set

Definition

Given a graph $G = (V, E)$, a connected dominating set $S \subseteq V$ is such that

1. For every vertex $u \in V$, either $u \in S$ or u has a neighbor $v \in S$.
2. The subgraph of G induced by S is connected.

MinCDS

Minimum Connected Dominating Set: Decision Version

Given a connected graph G and an integer k, does G have a connected dominating set of size **at most** k?

MinCDS $\leq_P$ MaxLST

A connected graph G has a connected dominating set of size at most k if and only if it has a spanning tree of at least $n - k$ leaves.

Minimum Connected Dominating Set

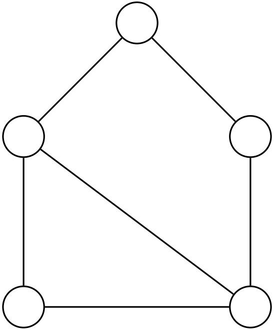
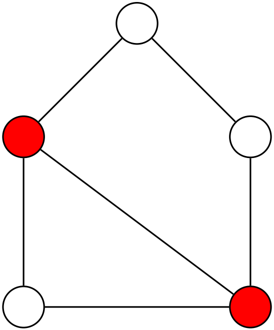


Figure: The maximum leaf spanning tree(MaxLST) is equivalent to the minimum connected dominating set(MinCDS)

Minimum Connected Dominating Set



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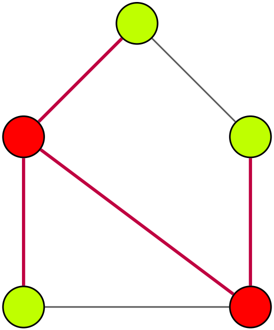

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Given a connected graph G and an integer k, does G have a connected dominating set of size **at most** k?

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VertexCover $\leq_P$ MinCDS

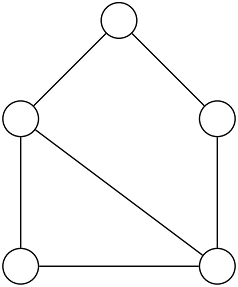


Figure: An instance of VertexCover

- An instance $(G(V, E), k)$ of VertexCover

concepts

VertexCover $\leq_P$ MinCDS

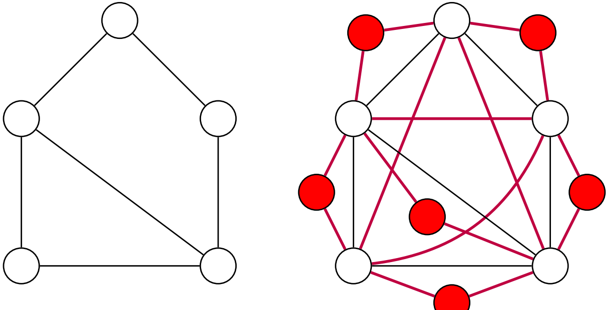


Figure: Constructed instance of MinCDS

- Constructed instance $(G'(V', E'), k)$ of MinCDS

VertexCover $\leq_P$ MinCDS

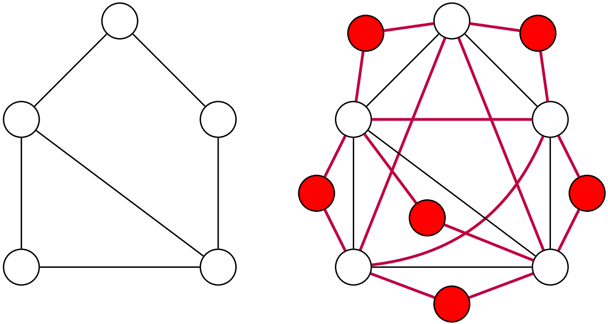


Figure: Constructed instance of MinCDS

- $V' = V \cup \{x_{uv} : (u, v) \in E\}$
- $E' = E \cup \{(u, v) : u, v \in V, u \neq v, (u, v) \notin E\} \cup \{(x_{u,v}, u) : (u, v) \in E\}$

VertexCover $\leq_P$ MinCDS

Lemma

concepts G has a vertex cover of size at most k if and only if G' has a connected dominating set of size at most k.

Necessity

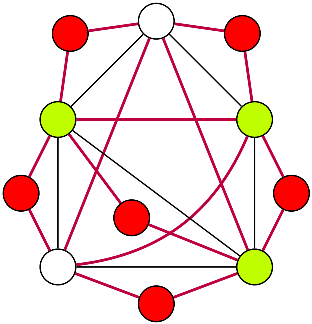
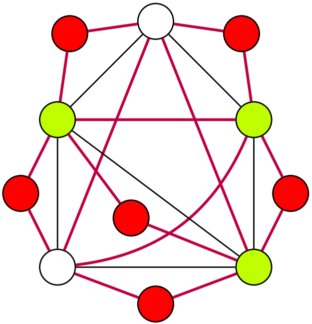


Figure: VertexCover of G

- S is a VertexCover of G

Necessity

- for any $v \in V' \setminus S$



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- for any $v \in V' \setminus S$
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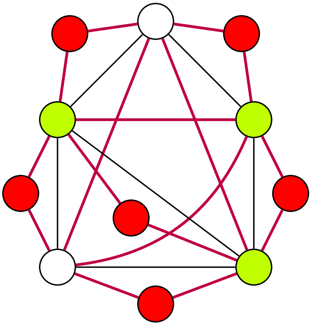

Figure: VertexCover of G

Necessity

- for any $v \in V' \setminus S$
- Case 1: $v \in V$
- Case 2: $v = v_{xy}$, an edge vertex of some edge $(x, y) \in E$

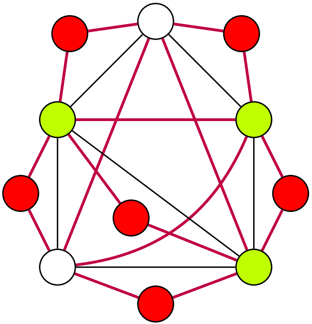

Figure: VERTEXCOVER of G

Sufficiency

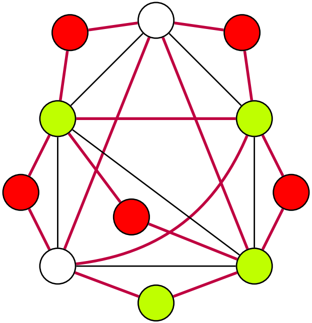

Figure: CDS of G'

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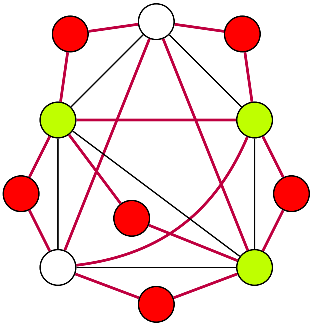
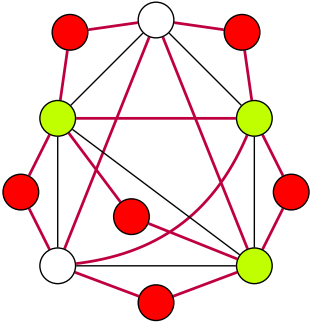


Figure: CDS of G'

- For any edge vertex $v_{xy} \in S$, we replace it with x or y.

Sufficiency



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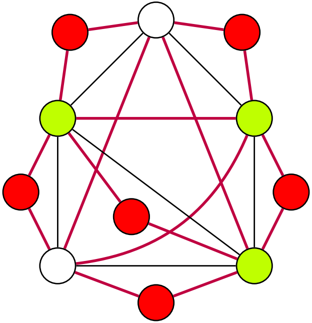

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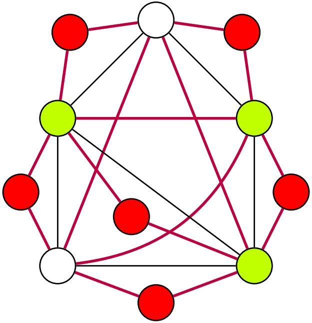

Figure: CDS of G'

- S contains no edge vertex
- every edge vertex is dominated by S
- S is a vertex cover for G

Outline

MaxLST

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5. Genetic Algorithm Based Heuristic Approach by Farvaresh et al. (2022)
6. A modified version of the problem, where instead of an integer k, a subset of vertices is specified, is polynomially solvable. (Rahman and Kaykobad, 2005)

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3. An $\mathcal{O}^*(1.8916^n)$ exact algorithm by Fernau et al. (2009) for graph with $\Delta \leq 3$.
4. Algorithm that finds a spanning tree of G with $\geq k$ internal vertices $(2 < k < n-2)$ if it exists in $\mathcal{O}^*(2^{3.5klogk})$.

Outline

Applications

Forest Fire Detection using MaxLST (Farvaresh et al. (2022))

As a realistic application of MaxLST, a forest fire detection system was developed and successfully simulated for the Iranian Arasbaran forest using a wireless sensor network (WSN).

Applications

The WSN is built using MaxLST concepts-

1. A connected network with a clustered hierarchical topology is developed
2. The number of cluster nodes is controlled
3. Which nodes should lie on the communication backbone

Applications

Figure: Arasbaran forest and the highlighted simulation area

Applications

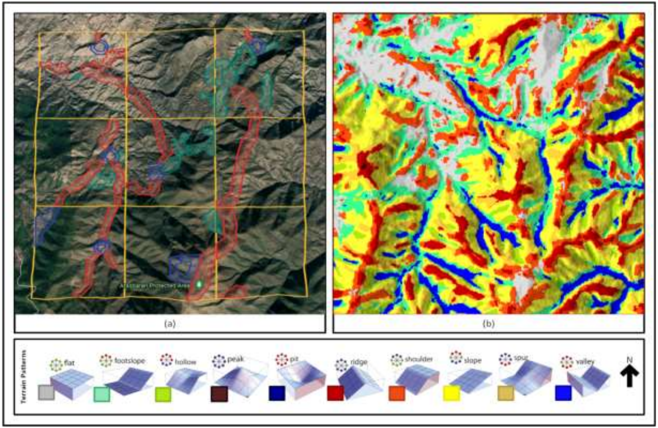

Figure: 3D satellite image of the simulation area with high risk of fire ignition

Applications

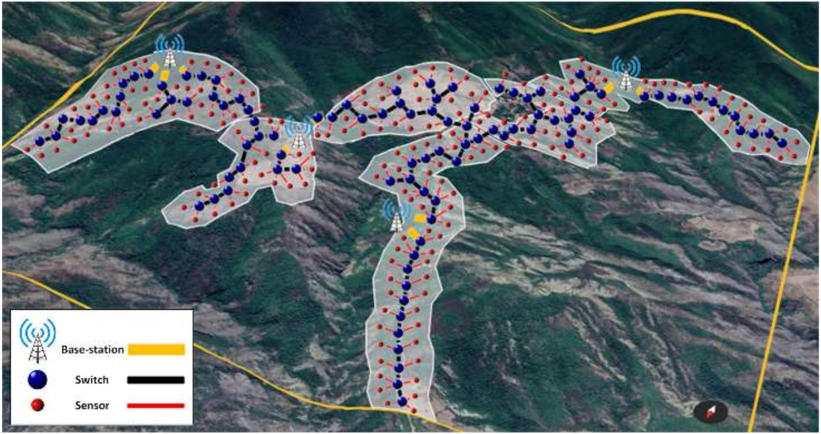

Figure: Optimum output of the sensor network of the simulation area

Thank You