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MTL 106 (Introduction to Probability Theory and Stochastic Processes)

Time allowed: 1 hour Minor 2 Examination Max. Marks: 25

Name: Entry Number: Signature:

Multiple Selection Questions: Section 1 ($5 \times 1 = 5$ marks)

Each of the following questions 1 to 5 has four options out of which one or more options can be correct. Write A, B, C or D which corresponds to the correct option for the first correct answer followed by space and the next correct answer and so on. **1 mark** is awarded if all correct answers are written, **0 mark** for no answer or partial correct answers or any incorrect answer.

1. Aditya and Aayush work independently on a problem in assignmnet sheet of a particular course. The time for Aditya to complete the problem is exponential distributed with mean 5 minutes. The time for Aayush to complete the problem is exponential distributed with mean 3 minutes. What is the probability that one of them finishes the problem first before the other one? (A) $\frac{3}{5}$ (B) $\frac{2}{5}$ (C) 0 (D) 1 Answer: D

2. Let $X_1, \cdots, X_{10}$ be independent random variables. Then, which of the following statements are always TRUE?
(A) $X_1X_2$ and $X_2^2$ are independent random variables
(B) $X_1, \cdots, X_{10}$ are identically distributed random variables
(C) $X_1X_4 + X_9$ and $e^{X_5} \cos X_2$ are independent random variables
(D) $Cov(X_1, X_2) = 0$ Answer: C , D

3. Suppose that $X$ has exponential distribution with the parameter $\lambda$. Which of the following statements are NOT TRUE?
(A) $E(X-3/X>3) = 1 + \frac{1}{\lambda}$ (B) $E(X-3/X>3) = \frac{1}{\lambda}$
(C) $E(X-2/X>3) = -1 - \frac{1}{\lambda}$ (D) $E(X-4/X>3) = 1 - \frac{1}{\lambda}$ Answer: A, C, D.

4. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with $E(X_1) = \mu$ and $Var(X_1) = \sigma^2$. Define $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Which of the following statements are NOT TRUE?
(A) $E(\overline{X}) = n\mu$ (B) $Var(\overline{X}) = \sigma^2$ (C) $E(\overline{X}) = \mu$ (D) $Var(\overline{X}) = n\sigma^2$ Answer: A, B, D.

5. Let $X$ and $Y$ be continuous random variables with joint pdf given by

$$f(x,y) = \begin{cases} e^{-(x+y)}, & 0 < x < \infty,\ 0 < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

The correlation coefficient between $X$ and $Y$ is given by
(A) $-1$ (B) 1 (C) 0 (D) 0.5 Answer: C

---------------------------------Space for Rough Work ---------------------------------

**Short Answer Type Questions:** **Section 2** **($5 \times 2 = 10$ marks)**

Write the answer upto 4 decimal places (D) or in fraction (F) or the expression (E) for the following questions 6 to 10. **2 marks** are awarded if answer is correct, and **0 mark** for no answer or partial correct answer or an incorrect answer.

6. Let $X$ and $Y$ be independent exponential distributed random variables with parameter 3 and 4 respectively. Define $Z = \max\{X, Y\}$. Find the distribution of $Z$?
   Answer(E): $F_Z(z) = \begin{cases} 0, & z < 0 \\ (1-e^{-4z})(1-e^{-3z}), & z \geq 0 \end{cases}$ . $f_Z(z) = 3e^{-3z} + 4e^{-4z} - 7e^{-7z}, \; z \geq 0$

7. Let $X_1, X_2, \ldots, X_n$ be a sequence of i.i.d. random variables such that $X_i \sim N(0, 2)$. Define $S_n = \sum_{i=1}^{n} \frac{X_i^2}{n}$, $n \geq 1$. Find the value of $\lim_{n\to\infty} nVar(S_n)$? Answer(F/D/E): 8

8. Let $X$ and $Y$ be iid random variables, each having uniform distribution in the interval $(1, 2)$. Find $E\left(\frac{5Y}{4X}\right)$.
   Answer(E): $\frac{15}{8} Log(2)$

Cancelled. ~~9.~~ Two numbers are chosen independently and at random from the interval $(0, 1)$. What is the probability that the two numbers differs by more than $1/2$? Answer(F/D):

~~Cancelled~~ 10. Suppose the arrival at a counter form a Poisson process with parameter $\lambda$ and suppose each arrival is of type 0 or of type 1 with respective probabilities $p$ and $1 - p$. Let $X(t)$ be the type of the last arrival before time $t$. Write the state space and the parameter space of the stochastic process $\{X(t), t \geq 0\}$.
   Answer(E): state space $= \{0, 1\}$

———————— Space for Rough Work ———— Parameter space $= [0, \infty)$

**Subjective Type Questions:** **Section 3** **(2 × 5 = 10 marks)**

Write the answer in the same page provided for the questions 11 and 12. **Full marks** are awarded if all the steps are correct, and **partial marks** for an incorrect answer with wrong steps.

11. Let $(X, Y)$ be a discrete type random vector with joint pmf

$$Prob\{X = i, Y = j\} = e^{-(a+bi)} \frac{(bi)^j a^i}{j! i!}, \quad i \geq 0; j \geq 0 .$$

Find the conditional distribution of $Y$ given $X = i$. Also, find $E(Y/X = i)$. (3 + 2 marks)

Answer: $P(Y=j \mid X=i) = \dfrac{P_{X,Y}(i,j)}{P_X(i)}$

$$P_X(i) = \sum_{j=0}^{\infty} e^{-(a+bi)} \frac{(bi)^j a^i}{j!\, i!}$$

$$= \frac{a^i e^{-(a+bi)}}{i!} \sum \frac{(bi)^j}{j!}$$

$$= \frac{a^i e^{-a}}{i!} e^{bi}$$

$$P(Y_j / X=i) = \frac{e^{-bi} (bi)^j}{j!}, \quad j = 0, 1, 2, \ldots$$
$$i = 0, 1, 2, \ldots$$

$Y / X=i \sim \text{pois}(bi)$

$E(Y / X=i) = bi$

12. (a) State the central limit theorem.

(b) Consider a certain consumer organization which reports the number of major defects for each new automobile that it tests. Suppose that the number of such defects for a certain model is a random variable with mean 3.2 and standard deviation 2.4. Among 100 randomly selected cars of this model, what is the probability that the average number of defects exceeds 4? Final answer can be in terms of $\phi(z)$ where $\phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt$.

(2 marks)

(3 marks)

Answer: CLT:

Theorem: let $X_1, X_2, \ldots X_n, \ldots$ be a sequence of independent random variables with $E(X_i) = \mu_i$ and $Var(X_i) = \sigma_i^2 > 0$ for $i = 1, 2, 3, \ldots$

Define

$$Z_n = \frac{\sum_{i=1}^{n} X_i - E(\sum_{i=1}^{n} X_i)}{\sqrt{Var(\sum X_i)}}$$

For large $n$, $Z_n$ approaches $N(0,1)$

or

$$\lim_{n\to\infty} P(Z_n \le z) \simeq \phi(z).$$

Note: if $X_1, X_2, \ldots X_n, \ldots$ are identically distributed then $E(X_i) = \mu$ & $Var(X_i) = \sigma_i^2 > 0$

b.) let $X_i$ be the number of defects for $i^{m}$ car

$E(X_i)$ ~~ao~~ $= 3.2$ $Var(X_i) = 2.4^2$.

Required probability $= P\left(\frac{\sum_{i=1}^{100} X_i}{100} > 4\right)$

$= \cancel{P(\bar{X} > 4)}\ P\left(\sum_{i=1}^{100} X_i > 400\right)$

~~$E(\bar{X}) = E(\sum X_i$~~ $= P\left(\frac{\sum_{i=1}^{100} X_i - 100\times 3.2}{\sqrt{100\times 2.4^2}} > \frac{400 - 100\times 3.2}{\sqrt{100\times 2.4^2}}\right)$

$\approx P\left(Z > \frac{80}{2.4\times 10}\right)$

$= P(Z > 3.33)$

$= 1 - \phi(3.33)$.