MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Tutorial Sheet No. 3 (Function of a Random Variable)

1. X has a uniform distribution over the set of integers $\{-n, -(n-1), \ldots, -1, 0, 1, \ldots, (n-1), n\}$. Find the distribution of (i) $|X|$ (ii) X^2 (iii) $1/1+|X|$.

2. Let $P[X \leq 0.49] = 0.75$, where X is a continuous type RV with some CDF defined over (0,1). If $Y = 1 - X$, Find k so that $P[Y \leq k] = 0.25$.

3. Let X be uniformly distributed random variable on the interval $(0, 1)$. Define $Y = a + (b-a)X, \quad a < b$. Find the distribution of Y.

4. If X has $\mathcal{N}(\mu, \sigma^2)$, find the distribution of $Y = a + bX$, and $Z = \left(\frac{X-\mu}{\sigma}\right)^2$.

5. Let X be a random variable with pdf $f(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}, \quad x > 0$ where $\theta > 0$ and $\alpha > 0$. Find the distribution of random variable $Y = \ln\left(\frac{X}{\theta}\right)$.

6. Suppose that X is a random variable having a geometric distribution with parameter p. Find the probability mass function of X^2 and $X + 3$.

7. Let X be an random variable having an exponential distribution with parameter $\lambda > 0$. Let $Y = \left(X - \frac{1}{\lambda}\right)^2$. Find the pdf of Y.

8. Suppose that X is a continuous random variable with pdf $f_X(x) = e^{-x}$ for $x > 0$. Define $Y = \begin{cases} X, & X < 1 \\ \frac{1}{X}, & X \geq 1 \end{cases}$.

   (a) Discuss whether the distribution of Y is discrete or continuous or mixed type.

   (b) Determine the pmf/pdf as applicable to this case.

9. Let X be uniformly distributed random variable over the interval $[0, 10]$. Find the CDF of $Y = \max\{2, \min\{4, X\}\}$.

10. Let X be the life length of an electron tube and suppose that X may be represented as a continuous random variable which is exponentially distributed with parameter λ. Let $p_j = P(j \leq X < j+1)$. Show that p_j is of the form $(1-\alpha)\alpha^j$ and determine α.

11. Consider a nonlinear amplifier whose input X and output Y are related by its transfer characteristic

$$Y = \begin{cases} X^{\frac{1}{2}}, & X > 0 \\ -|X|^{\frac{1}{2}}, & X < 0 \end{cases}$$

    Find pdf of Y if X has $\mathcal{N}(0, 1)$ distribution.

12. A point X is chosen at random in the interval $[-1, 3]$. Find the pdf of $Y = X^2$.

13. Let the phase X of a sine wave be uniformly distributed in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Define $Y = \sin X$. Find the distribution of Y.

14. Let X be a random variable with uniform distribution in the interval $(-\pi/2, \pi/2)$. Define

$$Z = \begin{cases} -1 & X \leq -\pi/4 \\ \tan(X) & -\pi/4 < X < \pi/4 \\ 1 & X \geq \pi/4. \end{cases}$$

    Find the distribution of the random variable Z.

15. Find the probability distribution of a binomial random variable X with parameter n, p, truncated to the right at $X = r$, where r is a positive integer.

16. Find pdf of a doubly truncated normal $\mathcal{N}(\mu, \sigma^2)$ random variable, truncated to the left at $X = \alpha$ and to the right at $X = \beta$, where $\alpha < \beta$.

17. Assume that, the number of years a car will run is exponential distributed with mean 8. Suppose, Mr. Alok buys a four years running used car today, what is the probability that it will still run after 3 years?

18. Let U be a uniform distributed rv on the interval $[0, 1]$. Find the probability that the quadratic equation $x^2 + 4Ux + 1 = 0$ has two distinct real roots x_1 and x_2?

19. A club basketball team will play a 50-game season. Twenty four of these games are aginst class A teams and 26 are against class B teams. The outcomes of all the games are independent. The team will win each game against a class A opponent with probability 0.4 and it will win each game against a class B opponent with probability 0.6. Let X_A and X_B denote, respectively, the number of victories against class A and class B teams. Let X denote its total victories in the season.

    (a) What is the distributions of X_A?

    (b) What is the relationship between X_A, X_B and X?

    (c) What is the distribution of X?

20. Let X be a continuous type random variable with strictly increasing CDF F_X.

    (a) What is the distributions of X?

    (b) What is the distribution of the random variable $Y = -\ln(F_X(X))$?.

21. Let X be a continuous type random variable having the pdf $f(x) = \begin{cases} 0, & -\infty < x \leq 0 \\ \frac{1}{2}, & 0 < x \leq 1 \\ \frac{1}{kx^2}, & 1 < x < \infty \end{cases}$

    (a) Find k?

    (b) Find the pdf of $Y = \frac{1}{X}$?

22. In the claim office with one employee of a public service enterprise, it is known that the time (in minutes) that the employee takes to take a claim from a client is a random variable which is exponential distribution with mean 15 minutes. If you arrive at 12 noon to the claim office and in that moment there is no queue but the employee is taking a claim from a client, what is the probability that you must wait for less than 5 minutes to talk to the employee? What is the mean spending time (including waiting time and claim time) of you?

23. Let X be exponentially distributed random variable with parameter $\lambda > 0$.

    (a) Find $P\left(\mid X - 1 \mid > 1 \mid X > 1\right)$

    (b) Explain whether there exists a random variable $Y = g(X)$ such that the cumulative distribution function of Y has uncountably infinite discontinuity points. Justify your answer.

24. Let X be a continuous random variable having the probability density function (pdf)

$$f_X(x) = kx^2 e^{-x+1}, \quad x > 1$$

    (a) Find k?

    (b) Determine $E(X)$

    (c) Find the pdf of $Y = X^2$?

25. Let X be a random variable with $N(0, \sigma^2)$. Find the moment generating function for the random variable X. Deduce the moments of order n about zero for the random variable X from the above result.

26. The moment generating function (MGF) of a random variable X is given by $M_X(t) = \frac{1}{6} + \frac{1}{4}e^t + \frac{1}{3}e^{2t} + \frac{1}{4}e^{3t}$. If μ is the mean and σ^2 is the variance of X, what is the value of $P(\mu - \sigma < X < \mu)$?

27. Let X be a rv such that $P(X > x) = \begin{cases} q^{[x]}, & x \geq 0 \\ 1, & x < 0 \end{cases}$ where $0 < q < 1$ is a constant and $[x]$ is integral part of x. Determine the pmf/pdf of X as applicable to this case.

28. Let X be a normal distributed random variable with mean 0 and variance 2. Find the pdf of $\ln(X)$?

29. Prove that, if X is a continuous type rv such that $E(X^r)$ exists, then $E(X^s)$ exists for all $s < r$.

30. Suppose that two teams are plying a series of games, each of which is independently won by team A with probability 0.5 and by team B with probability 0.5. The winner of the series is the first team to win four games. Find the expected number of games that are played.

31. Let Φ be the characteristic function of a random variable X. Prove that $1- \mid \Phi(2u) \mid^2 \leq 4\left(1- \mid \Phi(u) \mid^2\right)$.

32. (a) Let X be a uniformly distributed random variable on the interval $[a, b]$ where $-\infty < a < b < \infty$. Find the distribution of the random variable $Y = \frac{X-\mu}{\sigma}$ where $\mu = E(X)$ and $\sigma^2 = Var(X)$. Also, find $P(-2 < Y < 2)$.

    (b) Suppose Shimla's temperature is modeled as a random variable which follows normal distribution with mean 10 Celcius degrees and standard deviation 3 Celcius degrees. Find the mean if the temperature of Shimla were expressed in Fahreneit degrees.

33. Let X be a random variable having a binomial distribution with parameters n and p. Prove that

$$E\left(\frac{1}{X+1}\right) = \frac{1-(1-p)^{n+1}}{(n+1)p}.$$

34. Let X be a continuous random variable with CDF $F_X(x)$. Define $Y = F_X(X)$.

    (a) Find the distribution of Y.

    (b) Find the variance of Y, if it exist?

    (c) Find the characteristic function of Y?

35. Suppose that X is a continuous random variable having the following pdf:

$$f(x) = \begin{cases} \frac{e^x}{2}, & x \leq 0 \\ \frac{e^{-x}}{2}, & x > 0. \end{cases}$$

    Let $Y = |X|$. Obtain $E(Y)$ and $Var(Y)$.

36. The mgf of a r.v. X is given by $M_X(t) = exp(\mu(e^t - 1))$.
    (a) What is the distribution of X? (b) Find $P(\mu - 2\sigma < X < \mu + 2\sigma)$, given $\mu = 4$.

37. Let X be a discrete type rv with moment generating function $M_X(t) = a + be^{2t} + ce^{4t}$, $E(X) = 3$, $\text{Var}(X) = 2$. Find $E(2^X)$?

38. Let X be a random variable with Poisson distribution with parameter λ. Show that the characteristic function of X is $\varphi_X(t) = exp\left[\lambda(e^{it} - 1)\right]$. Hence, compute $E(X^2)$, $Var(X)$ and $E(X^3)$.