Answers

MTL 106 (Introduction to Probability Theory and Stochastic Processes)

Time allowed: 1 hour Minor 1 Examination Max. Marks: 25

Name: Entry Number: Signature:

Multiple Selection Questions: Section 1 ($5 \times 1 = 5$ marks)

Each of the following questions 1 to 5 has four options out of which one or more options can be correct. Write A, B, C or D which corresponds to the correct option for the first correct answer followed by space and the next correct answer and so on. **1 mark** is awarded if all correct answers are written, **0 mark** for no answer or partial correct answers or any incorrect answer.

1. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let A and B be two events. For $P(B) > 0$, $P(A/B)$ denotes the conditional probability of A given B. Then,
(A) $0 \leq P(A/B) \leq 1$ (B) $P(\emptyset/B) = 0$ (C) $P(\Omega/B) = 1$ (D) $P(B/B) = 1$ Answer: A B C D

2. Let A and B be two independent events. which of the following option(s) is/are TRUE?
(A) A^c and B^c be two independent events. (B) $A \cup B$ and B be two independent events.
(C) A and B^c be two independent events. (D) A^c and B be two independent events.
Answer: A C D

3. Let X follows exponential distribution with parameter 5. Then, for $t, x > 0$,
(A) $P(X > t/X > t+x) = P(X > t)$ (B) $P(X > t+x/X > t) = P(X > t)$
(C) $P(X > t+x/X > t) = P(X > x)$ (D) $P(X > t/X > t+x) = P(X > x)$ Answer: B

4. Suppose X is a random variable with cdf F. Then, the value of $P(X = 1)$ is
(A) $F(1) - F(0)$ (B) $F(1^+) - F(1^-)$ (C) $F(1) - F(1^-)$ (D) $F(1^+) - F(1)$ Answer: B C

5. Let Φ be the characteristic function of a random variable X. Then (A) $\Phi(0) = 1$
(B) $\Phi(0) = 0$ (C) $\mid \Phi(t) \mid \leq 1$ for all t (D) $\mid \Phi(t) \mid \leq 0$ for all t Answer: A C

Space for Rough Work

Short Answer Type Questions: **Section 2** $(5 \times 2 = 10$ marks)

Write the answer upto 4 decimal places (D) or in fraction (F) or the expression (E) for the following questions 6 to 10. **2 marks** are awarded if answer is correct, and **0 mark** for no answer or partial correct answer or an incorrect answer.

6. Let A and B be two events such that $P(A^c \cap B) = 0$ and $P(B) \neq 0$. Then, the value of $P(A/B)$ is
   Answer(F/D): 1

7. Let X be a continuous type random variable with pdf $f(X) = \begin{cases} \alpha x^{-2}, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$. Then, the value of α is
   Answer(F/D): 2

8. Find the 2019^{th} moment of Y about the origin, if the mgf of Y is $\frac{1}{1+t}$. Answer(E): $(-1)^{2019}\, 2019!$

9. Let Y be a random variable having a geometric distribution with parameter 0.5 . Find the value of $P(Y = 8/Y > 5)$.
   Answer(F/D): $\frac{1}{8}$

10. Let X be an random variable with pdf $f_X(x) = \begin{cases} \overset{\lambda}{3}e^{-3x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$. Let $Y = (X - \frac{1}{3})^2$. Find the value of $E(Y)$.
    Answer(F/D): $\frac{1}{9}$

Space for Rough Work

$\left(\frac{1}{\lambda^2}\right)$

Subjective Type Questions: **Section 3**

Write the answer in the same page provided for the questions 11 and 12. **Full marks** are awarded if all the steps are correct, and **partial marks** for an incorrect answer with wrong steps. **(2 × 5 = 10 marks)**

11. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let X be a random variable. Define

$$F(x) = P\{-\infty < X \le x\}, \quad -\infty < x < \infty$$

Prove that F is a distribution function.

Answer: (5 marks)

(1) Let $x_1 < x_2 \Rightarrow (-\infty, x_1] \subset (-\infty, x_2]$

$\Rightarrow P(X \le x_1) \le P(X \le x_2)$

$\Rightarrow F(x_1) < F(x_2) \Rightarrow$ Hence F is non-decreasing.

(2) Let $\{x_n\}$ be a sequence of real no.s decreasing to $-\infty$.

Then for $x_n \geq x_{n+1}$

$\{X \le x_n\} \supseteq \{X \le x_{n+1}\}$

$\lim\limits_{n\to\infty} \{X \le x_n\} = \bigcap\limits_{n=1}^{\infty} \{X \le x_n\} = \phi.$

$\lim\limits_{n\to\infty} F(x_n) = \lim\limits_{n\to\infty} P(X \le x_n) = P(\lim\limits_{n\to\infty}(X \le x_n)) = P(\phi) = 0$ — (a)

(3) Let $\{x_n\}$ be a sequence of real no.s increasing to ∞.

Then for $x_n \le x_{n+1}$

$\lim\limits_{n\to\infty} \{X \le x_n\} = \bigcup\limits_{n=1}^{\infty} \{X \le x_n\} = \Omega$

$\lim\limits_{n\to\infty} F(x_n) = \lim\limits_{n\to\infty} P(X \le x_n) = P(\lim\limits_{n\to\infty}(X \le x_n)) = P(\Omega) = 1$ — (b)

As (a) & (b) holds for any sequence $\{x_n\}$ Such that $x_n \to -\infty$ or $x_n \to \infty$

Then $\lim\limits_{x\to\infty} F(x) = 0$; $\lim\limits_{x\to\infty} F(x) = 1$.

(4) Let $\{x_n\}$ be a decreasing sequence of real no.s S.t. $x_n \to x$ as $n \to \infty$.

$\Rightarrow \{X \le x_n\} \supseteq \{X \le x\} \ \forall n$

$\Rightarrow \bigcap\limits_{n=1}^{\infty} \{X \le x_n\} = \{X \le x\}. \Rightarrow \lim\limits_{n\to\infty} \{X \le x_n\} = \{X \le x\}$

3

$\lim\limits_{n\to\infty} F(x_n) = \lim\limits_{n\to\infty} P(X \le x_n) = P(\lim\limits_{n\to\infty}(X \le x_n)) = P[X \le x) = F(x)$

as $\lim\limits_{n\to\infty} x_n \to x$ we get $\lim\limits_{n\to\infty} F(x_n) \to F(x)$

This holds for any sequence $\{x_n\} \to x$

$\Rightarrow \lim\limits_{\delta\to 0} F(x+\delta) = F(x) \ \forall x.$

$\Rightarrow$ F is right Continuous.

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12. Let X be a continuous type random variable having the pdf $f(x) = \frac{1}{\pi(1+x^2)}$, $-\infty < x < \infty$.

Define $Z = \begin{cases} X, & \mid X \mid > 2 \\ \mid X \mid, & \mid X \mid \leq 2 \end{cases}$. Find the distribution of the random variable Z.

(5 marks)

Answer: $Z \in (-\infty, -2) \cup [0, \infty)$

For $z \in (-\infty, -2) \cup (2, \infty)$

$Z = X$

$f_Z(z) = f_X(z) = \frac{1}{\pi(1+z^2)}$, $z \in (-\infty, -2) \cup (2, \infty)$

For $z \in [0, 2]$

$$Z = \begin{cases} X & X \in [0, 2] \\ -X & X \in [-2, 0] \end{cases}$$

$P(Z \leq z) = P(|X| \leq z) + P(X \leq -2) = P(-z \leq X \leq z) + P(X \leq -2)$

$= F_X(z) - F_X(-z) + \cancel{P(X} F_X(-2)$

$f_Z(z) = f_X(z) + f_X(-z) + 0$, $z \in [0, 2]$

$= \frac{2}{\pi(1+z^2)}$, $z \in [0, 2]$

$$f_Z(z) = \begin{cases} \frac{1}{\pi(1+z^2)}, & z \in (-\infty, -2) \cup (2, \infty) \\ \frac{2}{\pi(1+z^2)}, & z \in [0, 2] \end{cases}$$

$$F_Z(z) = \begin{cases} \frac{1}{2} + \frac{\tan^{-1} z}{\pi}, & -\infty < z < -2 \\ \frac{2\tan^{-1} z - \tan^{-1} 2}{\pi} + \frac{1}{2}, & 0 \leq z \leq 2 \\ \frac{1}{2} + \frac{\tan^{-1} z}{\pi}, & 2 \leq z < \infty \end{cases}$$

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