MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Tutorial Sheet No. 2 (Random Variable)

1. Consider a probability space $(\Omega, \mathcal{F}, P)$ with $\Omega = \{0, 1, 2\}$, $\mathcal{F} = \{\emptyset, \{0\}, \{1, 2\}, \Omega\}$, $P(\{0\}) = 0.5 = P(\{1, 2\})$. Give an example of a real-valued function on Ω that is NOT a random variable. Justify your answer.

2. Let $\Omega = \{1, 2, 3\}$. Let $\mathcal{F}$ be a σ-algebra on Ω, so that $X(w) = w + 2$ is a random variable. Find $\mathcal{F}$

3. Do the following functions define distribution functions.

   (a) $F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq \frac{1}{2} \\ 1, & x > \frac{1}{2} \end{cases}$ (b) $F(x) = (\frac{1}{\pi})\tan^{-1}\mathrm{x}, -\infty < x < \infty$ (c) $F(x) = \begin{cases} 0, & -\infty < x < 0 \\ 1 - e^{-x}, & 0 \leq x < \infty \end{cases}$

4. Consider the random variable X that represents the number of people who are hospitalized or die in a single head-on collision on the road in front of a particular spot in a year. The distribution of such random variables are typically obtained from historical data. Without getting into the statistical aspects involved, let us suppose that the cumulative distribution function of X is as follows:

x	0	1	2	3	4	5	6	7	8	9	10
$F(x)$	0.250	0.546	0.898	0.932	0.955	0.972	0.981	0.989	0.995	0.998	1.000

   Find (a) $P(X = 10)$ (b) $P(X \leq 5/X > 2)$.

5. Let X be a rv having the cdf: $F(x) = \begin{cases} 0, & x < -1 \\ \frac{1+x}{9}, & -1 \leq x < 0 \\ \frac{2+x^2}{9}, & 0 \leq x < 2 \\ 1, & x \geq 2 \end{cases}$ Find $P(X \in E)$ where E is $(-1, 0] \cup (1, 2)$.

6. Let X be a random variable with cumulative distribution function given by: $F_X(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{25}, & 1 \leq x < 2 \\ \frac{x^2}{10}, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$.

   Determine the cumulative discrete distribution functions F_d and one continuous F_c such that: $F_X(x) = \alpha F_d(x) + \beta F_c(x)$.

7. Let X be a rv such that $P(X = 2) = \frac{1}{4}$ and its CDF is given by $F_X(x) = \begin{cases} 0, & x < -3 \\ \alpha(x+3), & -3 \leq x < 2 \\ \frac{3}{4}, & 2 \leq x < 4 \\ \beta x^2, & 4 \leq x < 8/\sqrt{3} \\ 1, & x \geq 8/\sqrt{3} \end{cases}$.

   (a) Find α, β if 2 is the only jump discontinuity of F.

   (b) Compute $P(X < 3/X \geq 2)$.

8. An airline knows that 5 percent of the people making reservation on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold only 50 passengers. Assume that passengers come to airport are independent with each other. What is the probability that there will be a seat available for every passenger who shows up?

9. The probability of hitting an aircraft is 0.001 for each shot. Assume that the number of hits when n shots are fired is a random variable having a binomial distribution. How many shots should be fired so that the probability of hitting with two or more shots is above 0.95?

10. A reputed publisher claims that in the handbooks published by them misprints occur at the rate of 0.0024 per page. What is the probability that in a randomly choosen handbook of 300 pages, the third misprint will occur after examining 100 pages?

11. Let $0 < p < 1$ and N be a positive integer. Let $X \sim B\left(N, \frac{p}{N}\right)$. Find $\lim_{N\to\infty}\left(1 - \frac{p}{N}\right)^N$, if it exists.

12. In a torture test a light switch is turned on and off until it fails. If the probability that the switch will fail any time it is turned 'on' or 'off' is 0.001, what is probability that the switch will fail after it has been turned on or off 1200 times?.

13. Let X be a Poisson random variable with parameter λ. Show that $P(X = i)$ increases monotonically and then decreases monotonically as i increases, reaching its maximum when i is the largest integer not exceeding λ.

14. For what values of α, p does the following function represent a probability mass function $p_X(x) = \alpha p^x, x = 0, 1, 2, \ldots$. Prove that the random variable having such a probability mass function satisfies the following memoryless property $P(X > a + s/X > a) = P(X \geq s)$.

15. Consider a random experiment of choosing a point in the annular disc of inner radius r_1 and outer radius r_2 $(r_1 < r_2)$. Let X be the distance of chosen point from the center of annular disc. Find the pdf of X.

16. Let X be an absolutely continuous random variable with density function f. Prove that the random variables X and $-X$ have the same distribution function if and only if $f(x) = f(-x)$ for all $x \in \Re$.

17. The life time (in hours) of a certain piece of equipment is a continuous random variable X, having pdf $f_X(x) = \begin{cases} \frac{xe^{-x/100}}{10^4}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$. If four pieces of this equipment are selected independently of each other from a lot, what is the probability that atleast two of them have life length more than 20 hours?.

18. Suppose that f and g are density function and that $0 < \lambda < 1$ is a constant. (a) Is $\lambda f + (1-\lambda)g$ a probability density function? (b) Is fg (i.e., $fg(x) = f(x)g(x)$) a probability density function? Explain.

19. A system has a very large number (can be assumed to be infinite) of components. The probability that any one of these component will fail in the interval (a, b) is $e^{-a/T} - e^{-b/T}$, independent of others, where $T > 0$ is a constant. Find the mean and variance of the number of failures in the interval $(0, T/4)$.

20. A student arrives to the bus stop at 6:00 AM sharp, knowing that the bus will arrive in any moment, uniformly distributed between 6:00 AM and 6:20 AM.
    (a) What is the probability that the student must wait more than five minutes?
    (b) If at 6:10 AM the bus has not arrived yet, what is the probability that the student has to wait at least five more minutes?

21. The time to failure of certain units is exponentially distributed with parameter λ. At time $t = 0$, n identical units are put in operation. The units operate, so that failure of any unit is not affected by the behavior of the other units. For any $t > 0$, let N_t be the random variable whose value is the number of units still in operation time t. Find the distribution of the random variable N_t.

22. Consider the marks of MTL 106 examination. Suppose that marks are distributed normally with mean 76 and standard deviation 15. 15% of the best students obtained A as grade and 10% of the worst students fail in the course. (a) Find the minimum mark to obtain A as a grade. (b) Find the minimum mark to pass the course.

23. Suppose that the life length of two electronic devices say D_1 and D_2 have normal distributions $\mathcal{N}(40, 36)$ and $\mathcal{N}(45, 9)$ respectively. (a) If a device is to be used for 45 hours, which device would be preferred? (b) If it is to be used for 42 hours which one should be preferred?

24. Let X be a rv with cdf $F(x) = \begin{cases} 0, & -\infty < x < 0 \\ p + (1-p)(1 - e^{-\lambda x}), & 0 \leq x < 4 \\ 1, & 4 \leq x < \infty \end{cases}$ with $0 < p < 1$ and $\lambda > 0$. Find the mean of X.

25. Let X be a random variable having a Poisson distribution with parameter λ. Prove that, for $n = 1, 2, \ldots$ $E[X^n] = \lambda E[(X+1)^{n-1}]$.

26. Prove that for any random variable X, $E[X^2] \geq [E[X]]^2$. Discuss the nature of X when one have equality?

27. Let X be the random variable such that $P(a \leq X \leq b) = 1$, where $-\infty < a < b < \infty$. Show that $Var(X) \leq \frac{(b-a)^2}{4}$.

28. Consider a random variable X with $E(X) = 1$ and $E(X^2) = 1$.

    (a) Find $E[(X - E(X))^4]$ if it exists.

    (b) Find $P(0.4 < X < 1.7)$ and $P(X = 0)$.

29. Let X be a continuous type random variable with pdf $f(x) = \begin{cases} \alpha + \beta x^2, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$ If $E(X) = 3/5$, find the value of α and β.

30. Let $X \sim \mathrm{P}(\lambda)$ such that $P(X = 0) = e^{-1}$. Find $Var(X)$.