

[M E N U]

- [Bio-Sketch](#)
- [My Books](#)
- [Publications](#)
- [Projects](#)
- [Professional Activities](#)
- [Software](#)
- [Students](#)
- [Virtual Lab](#)
- [Grants/Awards](#)
- [Courses](#)
- [Events](#)
- [Contact](#)
- [Home](#)

DEPARTMENT OF MATHEMATICS

MTL 106 (Introduction to Probability Theory and Stochastic Processes) 4 Credits (3-1-0)

II Semester 2015-16

Lecture Classes: Tuesday, Wednesday and Friday between 9:00 AM and 9:50 AM in LH 121.**Tutorial Classes: MZ 163 on Mon, Tue, Thu and Fri days between 2:00 PM and 2:50 PM.****For the students who registered I Semester 2015-16, Re Major (for both I grade and E Grade) is scheduled on Jan. 8th, 2016 (Friday) between 12 noon and 2 PM. Venue will be MZ 158 (opposite to my office).****INFORMATION SHEET**

Probability Theory: Axioms of probability, Probability space, Conditional probability, Independence, Baye's rule, Random variable, Some common discrete and continuous distributions, Distribution of Functions of Random Variable, Moments, Generating functions, Two and higher dimensional distributions, Functions of random variables, Order statistics, Conditional distributions, Covariance, correlation coefficient, conditional expectation, Modes of convergences, Law of large numbers, Central limit theorem.

(No. of Lectures - 28)

Stochastic Processes: Definition of Stochastic process, Classification and properties of stochastic processes, Simple stochastic processes, Stationary processes, Discrete and continuous time Markov chains, Classification of states, Limiting distribution, Birth and death process, Poisson process, Steady state and transient distributions, Simple Markovian queuing models (M/M/1, M/M/1/N, M/M/c/N, M/M/N/N).

(No. of Lectures - 14)

Main Text Books

1. [Introduction to Probability and Stochastic Processes with Applications, Liliana Blanco Castaneda, Viswanathan Arunachalam, Selvamuthu Dharmaraja, Wiley, New Jersey, June 2012.](#)
2. [Probability and Statistics with Reliability, Queueing and Computer Science Applications, Kishor S. Trivedi, John Wiley, second edition, 2001.](#)
3. [Introduction to Probability Models, Sheldon M. Ross, Academic Press, tenth edition, 2009.](#)

<http://web.iitd.ac.in/~dharmar/mlt106/main.html>

Course Web Page

Definition : Probability Space

Let E be a random experiment

Let Ω be the collection of all possible outcomes.

Let F be a σ -field (σ -algebra) on Ω

The set function P defined on F satisfying :

$$\text{i)} P(A) \geq 0 \quad \forall A \in F$$

$$\text{ii)} P(\Omega) = 1$$

$$\text{iii)} \text{ If } A_i, i=1, 2, \dots \text{ disjoint events, then } P(\cup A_i) = \sum P(A_i)$$

Then the function P is called probability. (kolmogorov axiomatic defⁿ of probability)
 (Ω, F, P) is called a probability space.

σ - field :

- i) $\emptyset \in F$
- ii) If $A \in F$
 $\Rightarrow A^c \in F$
- iii) If $A_i \in F$
 $\cup A_i \in F$

Ex:- $\Omega = \{a, b, c\}$

$$\textcircled{1} \quad F_1 = \{\emptyset, \{a\}, \{b, c\}, \Omega\}$$

Both are σ field on Ω

$$F_2 = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a\}, \Omega\}$$

\textcircled{2} $\Omega = [0, 1]$

$$F_1 = \{\emptyset, \{0\}, (0, 1], \Omega\}$$

💡 Largest σ -field on R - "Borel" σ field [all subsets of R]
 \vdots
 $(-\infty, \infty)$

ex: E : Drawing a card from a deck of 52 cards

$$\textcircled{3} \quad \Omega = \{A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2\} \quad P(a) = 1/13 \quad \forall a \in \Omega$$

$$F_1 = \{\emptyset, \Omega, \{A, K, Q, J\}, \{10, 9, 8, 7, 6, 5, 4, 3, 2\}\}$$

$$P(\emptyset) = 0 \quad P(\Omega) = 13 \times \frac{1}{13} = 1 \quad P(\{A, K, Q, J\}) = \frac{4}{13} \quad \& \quad P(\{10, 9, 8, 7, 6, 5, 4, 3, 2\}) = \frac{9}{13}$$

\Rightarrow Probability Space

08/01/2016

LECTURE 3

Conditional Probability

Definition: (Ω, \mathcal{F}, P) , $P(B) > 0$

$$P(A|B) = \frac{P(AB)}{P(B)} \quad (\text{P of A given B occurs})$$

It satisfies : (i) $P(A|B) \geq 0$ (ii) $P(B|B) = 1$ (iii) $P\left(\bigcup_i P(A_i|B)\right) = \sum_i P(A_i|B)$
 A_i are disjoint

Probability Space: $(\Omega_B, \mathcal{F}_B, P_B)$

Remark: (i) When A & B are independent events , $P(A|B) = P(A)$

Definition: Total Probability Rule

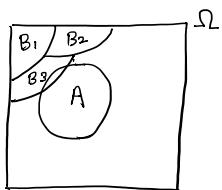
For any event $A \in \mathcal{F}$

$$P(A) = \sum_i P(A|B_i) \cdot P(B_i)$$

Condition :

$$B_i \cap B_j = \emptyset$$

$$\bigvee \beta_i = \square$$



Definition : Baye's Rule

$$P(B_i | A) = \frac{P(A|B_i) P(B_i)}{\sum_j P(A|B_j) P(B_j)}$$

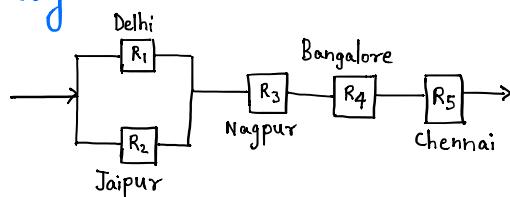
Example :

$$\textcircled{2} \quad \Omega = (-\infty, \infty) \quad P([a, b]) = \int_a^b \frac{1}{2\pi} e^{-x^2/2} \cdot dx$$

$P([a, b]) = 0 \quad \text{if } a = b \quad P([a, b]) = 1 \quad \text{if } a = -\infty \text{ & } b = \infty$
 $\text{i.e. } [a, b] = \Omega$

Reliability

eg:-



A_1, A_2, A_3, A_4 & A_5 are mutually independent.

$$R_i = P(A_i)$$

Reliability of system

$$= P((A_1 \cup A_2) \cap A_3 \cap A_4 \cap A_5)$$

$$= [R_1 + R_2 - R_1 R_2] \times R_3 \times R_4 \times R_5$$

THEOREM

Let (Ω, \mathcal{F}, P) be a probability space. let $\{A_n, n=1, 2, \dots\}$ be an increasing sequence of elements in \mathcal{F} i.e. $A_n \in \mathcal{F}$ & $A_n \leq A_{n+1}$ for $n=1, 2, \dots$

$$\text{Then } P(\lim_{n \rightarrow \infty} A_n) = P(\bigcup_{n=1}^{\infty} A_n) = \lim_{n \rightarrow \infty} P(A_n)$$

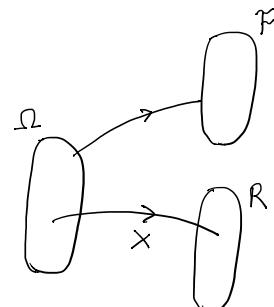
Similarly, if $\{A_n, n=1, 2, \dots\}$ is a decreasing seq. of elements in \mathcal{F} i.e. $A_n \in \mathcal{F}$ & $A_n \geq A_{n+1}$ for $n=1, 2, \dots$, then

$$P(\lim_{n \rightarrow \infty} A_n) = P(\bigcap_{n=1}^{\infty} A_n) = \lim_{n \rightarrow \infty} P(A_n)$$

Random Variable

X is a real valued function such that

$x^{-1}\{(-\infty, x]\} \in \mathcal{F} \quad \forall x \in \mathbb{R}$. Then X is a random variable w.r.t. \mathcal{F} .



$$\text{eg:- } \Omega = \{a, b, c\}, \mathcal{F} = \{\emptyset, \{a\}, \{b, c\}, \Omega\}$$

$$\text{Define } X(\omega) = \begin{cases} -10 & \omega = \{a\} \\ 5 & \omega = \{b\} \\ 5 & \omega = \{c\} \end{cases} \Rightarrow x^{-1}\{(-\infty, x]\} = \begin{cases} \emptyset & -\infty < x < -10 \\ \{a\} & -10 \leq x < 5 \\ \Omega & 5 \leq x < \infty \end{cases}$$

\Rightarrow Random variable.

Similary for not a random variable, (for some \mathcal{F})

$$\text{Define } X(\omega) = \begin{cases} 5 & \omega = \{a\} \\ -10 & \omega = \{b\} \\ 5 & \omega = \{c\} \end{cases} \Rightarrow X^{-1}\{(-\infty, x]\} = \begin{cases} \emptyset & -\infty < x < -10 \\ \{b\} & -10 \leq x < 5 \\ \Omega & 5 \leq x < \infty \end{cases}$$

$\not\Rightarrow$ Random variable.

Remark: Any real function will become a random variable if we take Power set as the σ -field

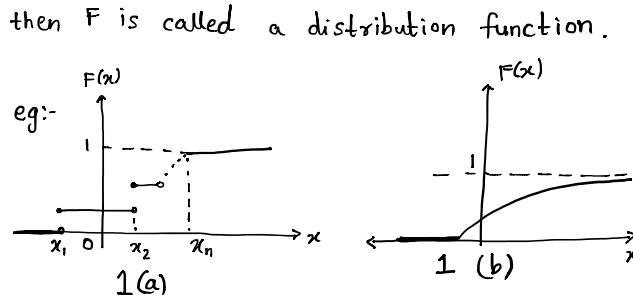
13/01/2016 LECTURE 5

Distribution Function

Any real valued function satisfying:

- i) $0 \leq F(x) \leq 1 \quad \forall -\infty < x < \infty$
- ii) Monotonically increasing in x , i.e., if $x_1 < x_2$ then $F(x_1) \leq F(x_2)$
- iii) $\lim_{x \rightarrow -\infty} F(x) = 0$ & $\lim_{x \rightarrow \infty} F(x) = 1$
- iv) F is (right) continuous function,

eg:-



Cumulative Distribution Function (C.D.F.)

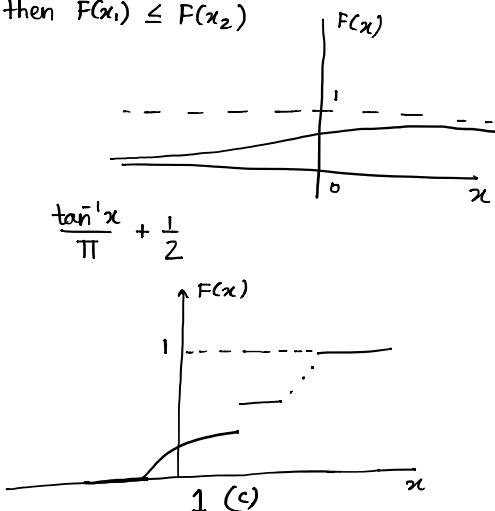
let (Ω, \mathcal{F}, P) be a probability space.

$$F(x) = P(X \leq x) \quad -\infty < x < \infty$$

is said to be C.D.F. of the random variable X .

$$P\{X \leq x\} = P\{\omega \mid X(\omega) \leq x\}$$

/
short hand notation



Classification of Random Variable

1.) Discrete Type Random Variable

C.D.F. has countable no. of discontinuities

Fig. 1(a) above

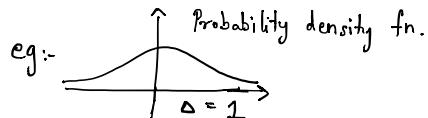
2.) Continuous Type Random Variable

If its C.D.F. is a continuous fn. in x .

Fig 1(b)

$$F(x) = \int_{-\infty}^x f(t) dt \quad \text{probability density fn. (pdf)} \\ \text{of r.v. } X.$$

$$\begin{aligned} f(x) &\geq 0 \quad \forall x \\ \int_{-\infty}^{\infty} f(t) dt &= 1 \end{aligned} \quad \left[\begin{array}{l} \text{Properties of} \\ \text{probability density function} \end{array} \right]$$



3.) Mixed Type Random Variable. Fig 1(c)

If it's C.D.F. has countable discontinuities as well as continuous function in some subintervals.

$$\text{i.e. } \sum_i P(x=x_i) + \int_{-\infty}^{\infty} f(x) dx = 1$$

15/01/2016

LECTURE 6

Example :

① Discrete Type Random Variable

$$a) \quad P(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{10} & 0 \leq x < 10 \\ 1 & 10 \leq x \end{cases}$$

$$b) \quad F(x) = \sum_{n=1}^x \frac{1}{2^n} = 1$$

$$P(x) = \begin{cases} \frac{1}{2^x} & x = 1, 2, 3, \dots \\ 0 & \text{others} \end{cases}$$

② Continuous Type Random Variable

$$a) \quad F(x) = \frac{\tan^{-1} x}{\pi} + \frac{1}{2}$$

$$f(x) = \frac{d}{dx} F(x) = \frac{1}{\pi(1+x^2)}$$

$$b) \quad \int_0^{\infty} e^{-x} dx = 1$$

$$f(x) = \begin{cases} e^{-x} & 0 < x < \infty \\ 0, \text{others} & \end{cases}$$

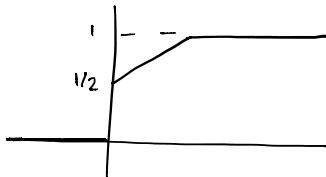
$$c) \sum_{k=0}^n C_k \left(\frac{1}{2}\right)^k = 1$$

$$P(x) = \begin{cases} {}^n C_k \left(\frac{1}{2}\right)^k & k=0, 1, \dots, n \\ 0 & \text{others.} \end{cases}$$

$$c) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

(Mixed Type Random Variable)

$$a) F(x) = \begin{cases} 0 & -\infty < x < 0 \\ \frac{1}{2} + \frac{x}{2} & 0 \leq x < 1 \\ 1 & 1 \leq x < \infty \end{cases}$$



$$P(x) = \begin{cases} \frac{1}{2} & x=0 \\ 0 & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{2} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_k P(x=k) + \int_0^1 \frac{1}{2} dx \\ \frac{1}{2} + \frac{1}{2} = 1$$

19/01/2016 LECTURE 7

Example : ϵ - Random Experiment

Ω - Collection of all possible outcomes

\mathcal{F} - Largest σ -field on Ω

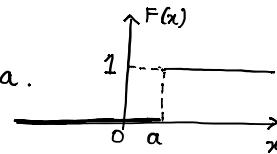
$X: \Omega \rightarrow \mathbb{R}$ s.t. $X^{-1}\{(-\infty, x]\} \in \mathcal{F} \quad \forall x \in \mathbb{R}$

Standard Distributions

Discrete Type Random Variable

① Constant Random Variable

X takes only one value. Suppose the point is a .
 $\Rightarrow P\{X=a\} = 1$

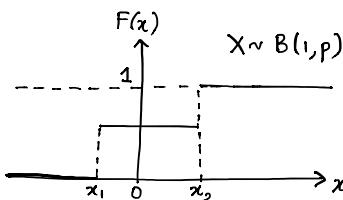


② Bernoulli Distributed Random Variable

X takes values x_1 & x_2

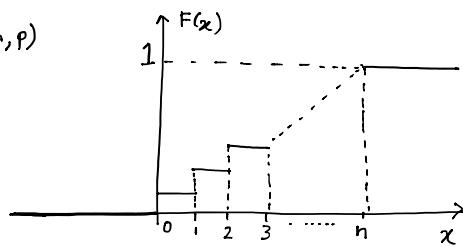
$$P\{X=x_1\} = 1 - P\{X=x_2\} = p$$

$$0 < p < 1$$

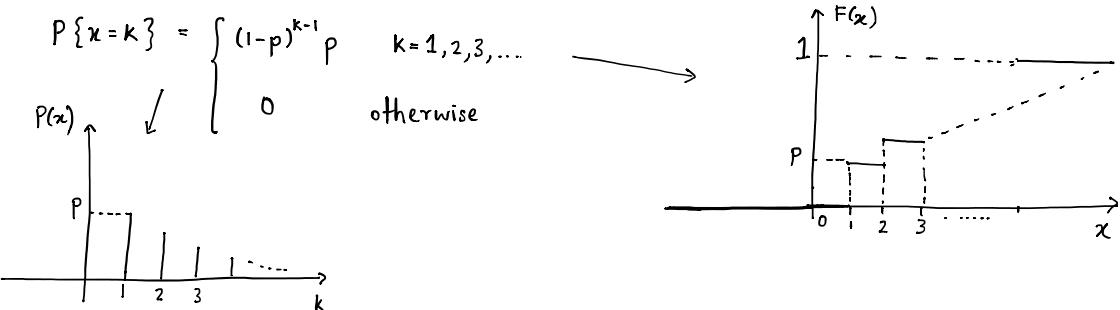


③ Binomial Distributed Random Variable $X \sim B(n, p)$

$$P\{X=k\} = \begin{cases} {}^n C_k p^k (1-p)^{n-k}, & k=0, 1, 2, \dots, n \\ 0, & \text{otherwise} \end{cases}$$



④ Geometric Distribution



⑤ Negative Binomial or Pascal $X \sim NB(r, p)$

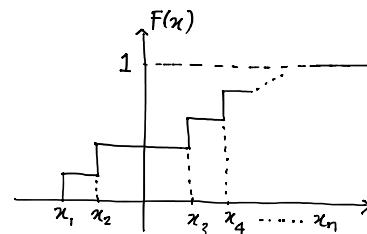
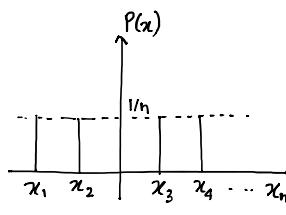
$$P\{x=n\} = \begin{cases} {}^n C_{r-1} \cdot p^r (1-p)^{n-r} & n=r, r+1, \dots \\ 0 & \text{otherwise} \end{cases}$$

if $r=1 \rightarrow$ Geometric Distribution

20/01/2016 LECTURE 8

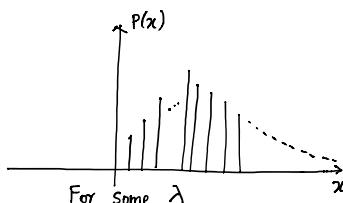
⑥ Discrete Uniform

$$P\{x=k\} = \begin{cases} \frac{1}{n} & k=x_1, x_2, \dots, x_n \\ 0 & \text{otherwise} \end{cases}$$



⑦ Poisson Distribution

$$P\{x=k\} = \begin{cases} \frac{e^{-\lambda} \lambda^k}{k!} & k=0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$



⑧ Poisson Process

$\{X(t), t \in T\}$ - Stochastic Process

For fixed t , $X(t) \sim$ Poisson distributed $X(t) \sim P(\lambda t)$

Assume 1.) $X(0) = 0$

$$2.) P\{X(t+\Delta t) = 1 / X(t) = 0\} = \lambda \Delta t + o(\Delta t)$$

$$3.) P\{X(t+\Delta t) = 0 / X(t) = 0\} = 1 - \lambda \Delta t + o(\Delta t)$$

$$4.) P\{X(t+\Delta t) > 1 / X(t) = 0\} = o(\Delta t)$$

5.) Non-overlapping intervals are independant.

Divide the interval $[0, t]$ into n equal parts.

$$P\{X(t) = k\} = \begin{cases} {}^n C_k \left(\frac{\lambda t}{n}\right)^k \left(1 - \frac{\lambda t}{n}\right)^{n-k} & k=0, 1, \dots, n \\ 0 & \text{otherwise} \end{cases} \xrightarrow{\text{Binomial}} \lim_{n \rightarrow \infty} P\{X(t) = k\} = \begin{cases} e^{-\lambda t} \frac{(\lambda t)^k}{k!} & k=0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \xrightarrow{\text{Poisson Distribution}}$$

22/01/2016 LECTURE 9

• Continuous Type Random Variable

① Uniform Distribution

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0, \text{others} & \end{cases}$$

$$F(x) = \begin{cases} 0 & -\infty < x < a \\ \frac{x-a}{b-a} & a \leq x < b \\ 1 & b \leq x < \infty \end{cases}$$

② Exponential Distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & 0 < x < \infty \\ 0, \text{others} & \end{cases}$$

$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ 1 - e^{-\lambda x} & 0 \leq x < \infty \end{cases}$$

$$P(X > x+t | X > t) = \frac{P(X > x+t)}{P(X > t)} = \frac{e^{-\lambda(x+t)}}{e^{-\lambda t}} = e^{-\lambda x} \quad (\text{independent of } t \Rightarrow \text{Memoryless/Marcov Property})$$

as $P(X > t) = 1 - F(x=t) = e^{-\lambda t}$

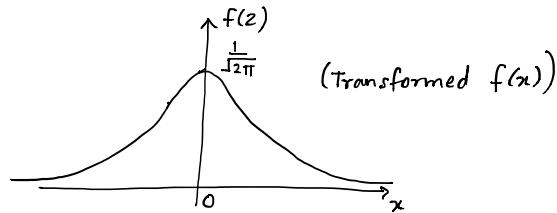
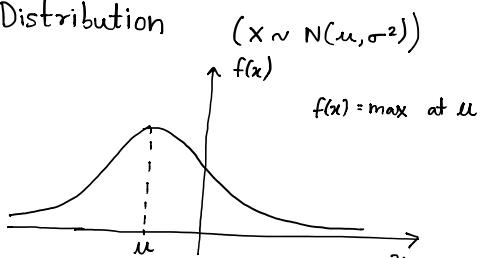
③ Normal Distribution or Gaussian Distribution

$$f(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} \left(\frac{x-\mu}{\sigma}\right)^2}$$

$-\infty < x < \infty$
 $-\infty < \mu < \infty$
 $\sigma > 0$

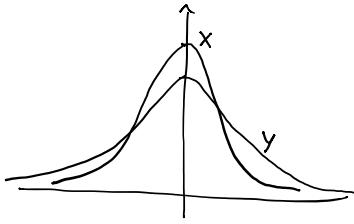
If we take random variable to be $z = \frac{x-\mu}{\sigma}$

$$\Rightarrow f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad -\infty < z < \infty$$



$$X \sim N(0, 1)$$

$$Y \sim N(0, 2)$$



$$f(y)|_{y=0} = \frac{1}{\sqrt{2\pi}} \quad f(x)|_{x=0} = \frac{1}{\sqrt{2\pi}}$$

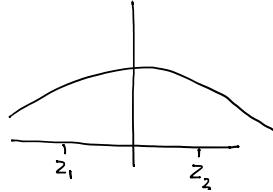
x will have higher peak than y .
 x will intersect y since area = 1 for both.

27/01/2016

LECTURE 10

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-t^2/2} dt$$

$$\begin{aligned} P(a < x \leq b) &= F_x(b) - F_x(a) \\ &= P\left\{\frac{a-u}{\sigma} < \frac{x-u}{\sigma} \leq \frac{b-u}{\sigma}\right\} \\ &= P\{z_1 < z \leq z_2\} \\ &= P(z \leq z_2) - P(z \leq z_1) \\ &= F_z(z_2) - F_z(z_1) = 0.5 + \phi(z_2) - [1 - \phi(z_1) - 0.5] \end{aligned}$$



④ $X \sim \text{Gamma}(\tau, \lambda)$

$$f(x) = \frac{\lambda^\tau x^{\tau-1} e^{-\lambda x}}{\Gamma(\tau)}$$

29/01/2016 LECTURE II

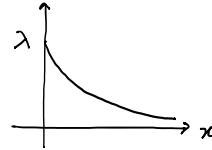
Definition: Let X be a continuous type r.v. with p.d.f $f(x)$

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx \quad \text{provided} \quad \int_{-\infty}^{\infty} |x| f(x) dx < \infty$$

i.e., $E(|x|) < \infty$

e.g:- $X \sim Exp(\lambda)$

$$\textcircled{1} \quad E(x) = \int_{-\infty}^{\infty} x f(x) dx = \frac{1}{\lambda} \quad (f(x) = \lambda e^{-\lambda x}) \quad 0 < x < \infty$$



$$\textcircled{2} \quad X \sim U(x_1, \dots, x_n) \quad [\text{Uniformly Distributed}]$$

$$E(x) = \sum_i x_i P(x=x_i) = \sum_i x_i \frac{1}{n} = \frac{x_1 + x_2 + \dots + x_n}{n} \quad (\text{Average})$$

$$\textcircled{3} \quad X \sim P(\lambda)$$

$$E(x) = \sum x_i P(x=x_i) = \lambda$$

$$\textcircled{4} \quad X \sim \text{Geometric}(p)$$

$$P(x=k) = (1-p)^{k-1} p \quad k=1, 2, \dots$$

$$E(x) = \sum k P(x=k)$$

$$= 1 \cdot P(x=1) + 2 \cdot P(x=2) + 3 \cdot P(x=3) + \dots$$

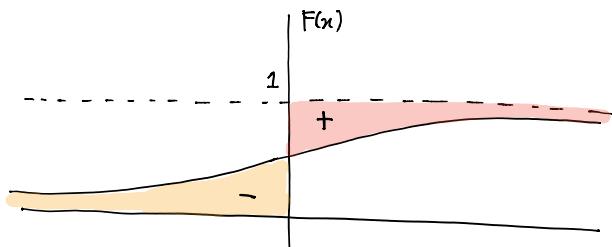
$$\begin{aligned} &= P\{x=1\} + P\{x=2\} + P\{x=3\} + \dots = \sum_{k=1}^{\infty} P\{x \geq k\} = \sum_{k=0}^{\infty} P\{x > k\} \\ &\quad + P\{x=2\} + P\{x=3\} + \dots \\ &\quad + P\{x=3\} + \dots = \sum_{k=0}^{\infty} (1 - F_x(k)) \end{aligned}$$

In general,

if X is a continuous type r.v.

$$E(x) = \int_0^{\infty} (1 - F_x(x)) dx - \int_{-\infty}^0 F_x(x) dx \quad (\text{provided R.H.S. exist})$$

● - ● Area wise



Definition : 2nd order moment

$$\sigma^2 = E[(x-\mu)^2] \quad \text{about the mean (i.e. } \mu = E(x) \text{)}$$

 (Variance of r.v. x)

Remark ① $\sigma^2 \geq 0$

② $\sigma^2 = 0 \Rightarrow \text{Var}(x) = 0 \Rightarrow P(x = \mu) = 1$

③ $\text{Var}(Ax + B) = A^2 \text{Var}(x)$

⑥ $E\left(\frac{1}{x}\right) \neq \frac{1}{E(x)}$

⑦ n^{th} order moment about the mean

$$\mu_n' = E((x-\mu)^n)$$

provided R.H.S. exist

⑧ n^{th} order moment $\not\Rightarrow (n+1)^{\text{th}}$ order moment
exist exist

④ $\sigma^2 = E[(x-\mu)^2] = E(x^2) - (E(x))^2$

⑤ $E(g(x)) = \int_{-\infty}^{\infty} g(x) P\{x=x\}$

$$= \int_{-\infty}^{\infty} g(x) f_x(x) dx$$

If $g(x)$ is Borel measurable function
then $g(x)$ will become a random variable
from R to R

For $g(x)$ to be Borel measurable, every
piece wise continuous fn. should be Borel
Measurable.

02/02/2016

LECTURE 12

Given a random variable X , $x^{-1}\{(-\infty, x]\} \in F \quad \forall x \in R$

g is a function $R \rightarrow R$ such that $y = g(x)$ is a random variable.

i.e. $y^{-1}\{(-\infty, y]\} \in F \quad \forall y \in R$

What is the distribution of y ?

g is a Borel Measurable Function in $R \rightarrow R$

(Special case : Every piecewise continuous function is Borel Measurable)

e.g.: $y = x^2, \sin x, 2x, \begin{cases} x & x < 0 \\ 3 & x > 0 \end{cases}$

2.) $X \sim B(n, p), Y = n - X$

$$P\{Y=k\} = P\{n-X=k\} = P\{X=n-k\} = {}^n C_{n-k} (p)^{n-k} (1-p)^k \Rightarrow Y \sim B(n, 1-p)$$

$$= {}^n C_k (1-p)^k p^{n-k}$$

3.) $X \quad Y = g(x) \quad (\text{Also r.v.}) \quad \star\star$

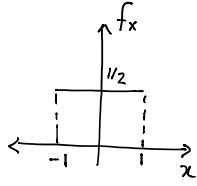
discrete 

continuous 

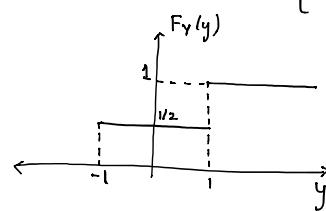
mixed 



$$4.) X \sim U(-1, 1)$$



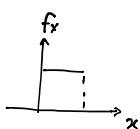
$$Y = \begin{cases} 1 & x < 0 \\ -1 & x \geq 0 \end{cases} \Rightarrow F_Y(y) = \begin{cases} 0 & y < -1 \\ P(X \geq 0) & -1 \leq y < 1 \\ 1 & y \geq 1 \end{cases} = \begin{cases} 0 & y < -1 \\ \int_0^1 dx & -1 \leq y < 1 \\ 1 & y \geq 1 \end{cases}$$



$$= \begin{cases} 0 & y < -1 \\ \frac{1}{2} & -1 \leq y < 1 \\ 1 & y \geq 1 \end{cases}$$

$$5.) X \sim U(0, 1)$$

$$Y = -\frac{1}{\lambda} \ln(1-x)$$



$$F_Y(y) = P(Y \leq y) = P(X \leq 1 - e^{-\lambda y}) = \begin{cases} 0 & y < 0 \\ \int_0^{1-e^{-\lambda y}} 1 dx & y \geq 0 \end{cases} = \begin{cases} 0 & y < 0 \\ 1 - e^{-\lambda y} & y \geq 0 \end{cases}$$

03/02/2016 LECTURE 13

Theorem : *

Let X be a continuous type random variable with p.d.f. $f_X(x)$. If h is strictly monotonic function & differentiable

Then the p.d.f. of $Y = h(x)$ is given by

$$f_Y(y) = \begin{cases} f_X(h^{-1}(y)) \left| \frac{d}{dy} h^{-1}(y) \right| & , \text{ if } y = h(x) \\ 0 & y \neq h(x) \end{cases}$$

Eg :- $X \sim U(0, 1)$, $Y = -\frac{1}{\lambda} \ln(1-x)$

①

$$f_Y(y) = f_X(1 - e^{-\lambda y}) \left| \frac{d}{dy} (1 - e^{-\lambda y}) \right| = \begin{cases} 1 \cdot \lambda \cdot e^{-\lambda y}, & 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Proof :

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(h(x) \leq y) = P(X \leq h^{-1}(y)) \quad (\text{if } h(x) \text{ is monotonically increasing}) \\ &= F_X(h^{-1}(y)) \end{aligned}$$

On differentiating, $f_Y(y) = f_X(h^{-1}(y)) \frac{d}{dy} h^{-1}(y)$

Eg:- $X \sim N(\mu, \sigma^2)$ $Z = \frac{x-\mu}{\sigma}$ $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$ $-\infty < x < \infty$

②

$$h(x) = \frac{x-\mu}{\sigma} \Rightarrow h^{-1}(z) = \sigma z + \mu \Rightarrow f_y(y) = f_x(h^{-1}(y)) \left| \frac{d}{dz} h^{-1}(y) \right| = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2}} \quad \forall z \in \mathbb{R}$$

$$= \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$$

$$\textcircled{3} \quad X \sim U(-1, 1) \quad Y = |X|$$

$$F_Y(y) = P(|X| \leq y) = P(-y \leq X \leq y) = F_X(y) - F_X(-y) \Rightarrow f_Y(y) = f_X(y) + f_X(-y)$$

$$\Rightarrow f_Y(y) = \begin{cases} 1 & y > 0 \\ 0 & y \leq 0 \end{cases}$$

Alternate :

Corollary : Let h be a piecewise strictly monotonic function & differentiable. Further there exists intervals I_1, I_2, \dots, I_n

Then the pdf of $Y = h(X)$ is given by

$$f_Y(y) = \begin{cases} \sum_{k=1}^n f_X(h_k^{-1}(y)) \left| \frac{d}{dy} (h_k^{-1}(y)) \right| & \text{otherwise} \\ 0 & \end{cases}$$

$$I_1 \rightarrow (-\infty, 0) \quad I_2 \rightarrow [0, \infty)$$

$$y = -x \quad h^{-1}(y) = -y \quad y = x \quad h^{-1}(y) = y$$

$$f_Y(y) = \begin{cases} f_X(-y) |-1| + f_X(y) |1| & y > 0 \\ 0 & y \leq 0 \end{cases} = \begin{cases} 1 & y \geq 0 \\ 0 & y < 0 \end{cases}$$

$$\textcircled{4} \quad X \sim N(\mu, \sigma^2), \quad Y = e^X \quad \text{Log normal}$$

$$\textcircled{5} \quad X \sim \text{Exp}(1), \quad Y = \begin{cases} 1 & X \leq 1 \\ \frac{1}{X} & X > 1 \end{cases}$$

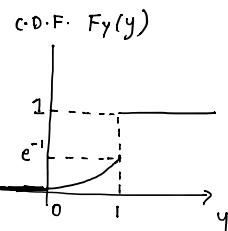
$$f_X(x) = e^{-x}$$

$$F_Y(y) = P(Y \leq y) = P\left(\frac{1}{X} \leq y\right) \quad \text{if } (y < 1) \quad / \quad 1 \quad \text{if } (y \geq 1)$$

$$= P\left(X \geq \frac{1}{y}\right) \quad \text{if } (0 < y < 1)$$

$$= 0 \quad \text{if } y \leq 0$$

$$= \begin{cases} \frac{1}{y} \int_0^{\frac{1}{y}} e^{-x} dx & y > 1 \\ 0 & y \leq 0 \end{cases} \quad \approx \begin{cases} 1 & y > 1 \\ e^{-1/y} & 0 < y \leq 1 \\ 0 & y \leq 0 \end{cases}$$



$$\Rightarrow f_Y(y) = \begin{cases} \frac{e^{-1/y}}{y^2} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$P_Y(y) = \begin{cases} 1 - e^{-1} & y = 1 \\ 0 & \text{otherwise} \end{cases}$$

GENERATING FUNCTIONS

Uniqueness Theorem :

- | | | |
|-----------|-------------|---------------------------------|
| 1. P.G.F. | $G_x(t)$ | Probability Generating Function |
| 2. M.G.F. | $M_x(t)$ | Moment Generating Function |
| 3. C.F. | $\phi_x(t)$ | Characteristic Function |

If $G/M/\phi_x(t) = G/M/\phi_y(t) \quad \forall t \Rightarrow X \stackrel{d}{=} Y$

□ Probability Generating Function (P.G.F.)

Let X be a non-negative integer values r.v. with p.m.f. $p_k = P\{X=k\}, k=0,1,2,\dots$

Then the prob. gen. func. (pgf) be defined as

$$G_x(t) = \sum_{k=0}^{\infty} P\{X=k\} t^k \quad |t| \leq 1$$

e.g:- $X \sim B(n,p) \Rightarrow G_x(t) = (pt + 1-p)^n$

Remarks : (1) $G_x(1) = 1$

 (2) $G_x(t) = E(t^k)$

$$(3) P_k = P\{X=k\} = \left. \frac{d^k G_x(t)}{(dt)^k} \right|_{t=0}$$

$$(4) E(X(X-1)) = \left. \frac{d^2 G_x(t)}{dt^2} \right|_{t=1}$$

↓
Factorial moments

□ Moment Generating Function (M.G.F.)

Let X be a r.v. s.t. $E(e^{tx})$ is finite for some interval in $(-\infty, \infty)$ including the point 0. Then the M.G.F. of X is defined as

$$M_x(t) = E(e^{tx}) = E\left(1 + \frac{tx}{1!} + \frac{t^2 x^2}{2!} + \dots + \frac{t^n x^n}{n!} + \dots\right)$$

e.g:- $X \sim P(\lambda)$

$$\textcircled{1} \quad M_x(t) = E(e^{tx}) = \sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} e^{tk} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{-\lambda} e^{\lambda e^t} = e^{-\lambda(1-e^t)} \quad t \in (-\infty, \infty)$$

② $X \sim Exp(\lambda)$

$$M_X(t) = \int_0^\infty e^{xt} \lambda e^{-\lambda x} \cdot dx = \frac{\lambda}{t-\lambda} = \frac{\lambda}{\lambda-t} \quad \forall t \in (-\infty, \lambda)$$

Remarks : (1.) $E(X^k) = \left. \frac{d^k M_X(t)}{(dt)^k} \right|_{t=0}$

③ Find distribution of X if $M_X(t) = \frac{e^t}{2} + \frac{e^{-t}}{3} + \frac{1}{6}$

x	-1	0	1
$P\{X=k\}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$

□ Characteristic Function (C.F.)

$$\psi_X(t) = E(e^{itX}) \quad i = \sqrt{-1}$$

Remarks : (1.) $\psi_X(0) = 1$

(2.) $|\psi_X(t)| \leq 1$

(3.) If $E(X^k)$ exist,

$$E(X^k) = \left. \frac{1}{i^k} \frac{d^k \psi_X(t)}{dt^k} \right|_{t=0}$$

Ex:- ① $X \sim N(\mu, \sigma^2)$

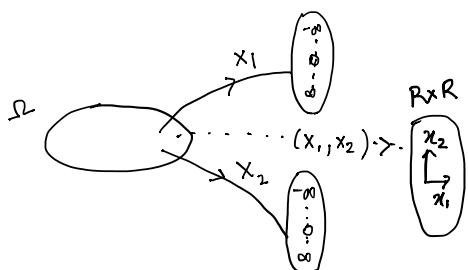
$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot dx \dots = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$x \longrightarrow x \longrightarrow x$

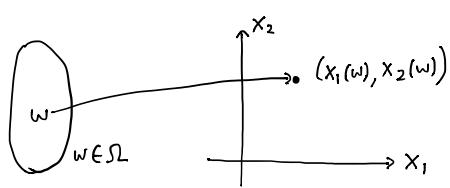
MINOR 1 SYLLABUS

09/02/2016 LECTURE 15

Two & HIGHER DIMENSIONAL RANDOM VARIABLES



If $x^{-1} \{(-\infty, x]\} \in \mathcal{F} \quad \forall x \in R$
 $X = (x_1, x_2)$ random vector of 2 dimension



In general,
 (X_1, X_2, \dots, X_n) n-dim random variable

C.D.F. $F(x_1, x_2)$

$$F_{x_1, x_2}(x_1, x_2) = \text{Prob} \{X_1 \leq x_1, X_2 \leq x_2\}$$

$$= \text{Prob} \left\{ \omega \mid X_1(\omega) \leq x_1, X_2(\omega) \leq x_2 \right\}$$

$$\omega \in \Omega \quad -\infty < x_1 < \infty \\ -\infty < x_2 < \infty$$

If satisfies

$$1.) 0 \leq F(x_1, x_2) \leq 1 \quad \forall x_1, x_2$$

$$2.) \lim_{x_1 \rightarrow -\infty, x_2 \rightarrow -\infty} F(x_1, x_2) = 0$$

$$3.) \lim_{x_1 \rightarrow \infty, x_2 \rightarrow \infty} F(x_1, x_2) = 1$$

4.) $F(x_1, x_2)$ is monotonically increasing in both x_1 & x_2 .

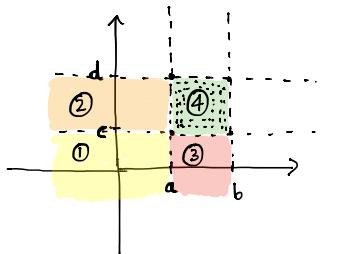
5.) $\forall a < b, c < d$

$$F_{x_1, x_2}(b, d) - F(a, d) - F(b, c) + F(a, c) \geq 0 \quad (\text{similar for multi-dimension})$$

$$\downarrow \\ \text{i.e. } (1+2+3+4) - (1+2) - (1+3) + 1$$

$$= 4 \\ F(4) \geq 0$$

$$\text{eg:- } \textcircled{1} \quad F(x_1, x_2) = \begin{cases} 1 - e^{-(x_1+y)} & 0 \leq x_1 \leq \infty \\ 0 & 0 \leq y \leq \infty \\ 0 & \text{otherwise.} \end{cases}$$



$$\textcircled{2} \quad F(x_1, x_2) = \frac{\tan^{-1}(x_1+x_2)}{\pi} + \frac{1}{2} \quad x \in \mathbb{R}, y \in \mathbb{R}$$

● TWO DIMENSIONAL DISCRETE TYPE RANDOM VARIABLE

$$P_{X_1, X_2}(x_1, x_2) = P_{\text{prob}} \{X_1 = x_1, X_2 = x_2\} \quad (\text{Mass function}) \\ = P_{\text{prob}} \{\omega \mid X_1(\omega) = x_1, X_2(\omega) = x_2\} \quad \omega \in \Omega$$

If satisfies

$$1.) 0 \leq P(x_1, x_2) \leq 1 \quad \forall x_1, x_2$$

$$2.) \sum_{x_1} \sum_{x_2} P(x_1, x_2) = 1$$

$$P_{X_1}(x_1) = \sum_{x_2} P(x_1, x_2); \quad P_{X_2}(x_2) = \sum_{x_1} P(x_1, x_2)$$

- Two DIMENSIONAL CONTINUOUS TYPE RANDOM VECTOR (x_1, x_2)

$$F_{x_1, x_2}(x_1, x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f_{x_1, x_2}(t, s) ds dt$$

joint pdf

Joint pdf satisfies

$$1.) f(x_1, x_2) \geq 0 \quad \forall x_1, x_2$$

$$2.) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, s) ds dt = 1$$

eg :- 1.) $f(x, y) = \begin{cases} e^{-(x+y)} & x > 0 \\ 0 & y > 0 \\ 0 & \text{otherwise} \end{cases}$

2.) $f(x, y) = \begin{cases} \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} & -\infty < x < \infty \\ 0 & -\infty < y < \infty \end{cases}$

eg:-

E : Tossing a coin Thrice

X : # of heads

Y : Difference (in absolute) of # of heads & # of tails

$$\Omega = \{(HHH), \dots, (T, T, T)\} \quad N_q(\Omega) = 8$$

joint p.m.f.
of (x, y)

		y →		$P(x=x)$
		1	3	
x →	0	1/8	1/8	
	1	3/8	0	3/8
	2	3/8	0	3/8
	3	0	1/8	1/8
$P(y=y)$		6/8	2/8	

Individually : (Marginal Distribution of X, Y)

y	p	x	p
0	1/8	0	1/8
1	2/8	1	3/8
3	6/8	2	3/8
		3	1/8

- INDEPENDENT RANDOM VARIABLE

We say that two random variables are independent iff

$$F_{x,y}(x, y) = F_x(x) F_y(y) \quad \forall x, y$$

i.e. if (x, y) - discrete type $\Rightarrow P_{x,y}(x, y) = P_x(x) P_y(y) \quad \forall x, y$

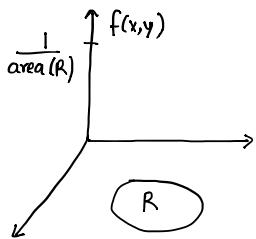
if (x, y) - continuous type $\Rightarrow f_{x,y}(x, y) = f_x(x) \cdot f_y(y) \quad \forall (x, y)$

eg:- Multivariable Normal Distribution (x, y, z)

$$f_{x,y,z}(x,y,z) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{1}{2}(x^2+y^2+z^2)} \quad \begin{matrix} -\infty < x < \infty \\ -\infty < y < \infty \\ -\infty < z < \infty \end{matrix} \quad (\text{Independent})$$

eg:- (x, y) - joint pdf ' R ' in $x-y$ plane

Uniform Distribution. $\quad (\text{Independent})$



3.) $f_{x,y}(x,y) = \begin{cases} 2 & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{Not independent.}$

16/02/2016

LECTURE 17

Password - Markov

CONDITIONAL DISTRIBUTION

$$P(A|B) ; P(B) > 0$$

Defn: Let (x, y) be a 2 dimension discrete type R.V. with joint pmf $P_{x,y}(x,y)$. Then the conditional dist. of X given $y=y_j$ is defined as

$$\boxed{P\{X=x_i / Y=y_j\} = \frac{P\{X=x_i, Y=y_j\}}{P\{Y=y_j\}}} \quad \forall i \quad \text{provided } P\{Y=y_j\} > 0$$

Note:- $\sum_i P\{X=x_i / Y=y_j\} = \frac{\sum_i P\{X=x_i, Y=y_j\}}{P\{Y=y_j\}} = 1$

Defn: Let (x, y) be a 2 dimension continuous type R.V. with joint pdf $f_{x,y}(x,y)$. Then the conditional dist. of X given $y=y_j$ is defined as

$$f_{x/y}(x/y) = \frac{f_{x,y}(x,y)}{f_y(y)}$$

for x
provided $f_y(y) > 0$

$$\text{C.D.F. of } X/Y = y_j = \int_{-\infty}^y f_{x/y}(t/y_j) dt$$

eg:- $f_{x,y}(x,y) = \begin{cases} 2 & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

P.d.f. of

$$(i) X/Y \Rightarrow f_{x/y}(x,y) = \frac{f_{x,y}(x,y)}{f_y(y)} = \begin{cases} \frac{2}{2(1-y)} & y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \Rightarrow X/Y \sim U(y,1)$$

$$f_y(y) = \int_y^1 2 dx = 2(1-y)$$

$$(ii) f_x(x) = \int_0^x 2 dy = 2x$$

$$Y/X = \frac{f_{x,y}(x,y)}{f_x(x)} = \begin{cases} \frac{2}{2x} & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad Y \sim U(0,x)$$

$x \longrightarrow x \longrightarrow x$

17/02/2016 LECTURE 18

FUNCTIONS OF RANDOM VARIABLES

eg:- Let $X \sim B(n,p)$, $Y \sim B(n,p)$ Assume that X & Y are independent r.v.'s.

Define $Z = X+Y$, dist of Z ?

$$\begin{aligned} P(Z=k) &= \sum_i P(X=i, Y=k-i) = \sum_i P(X=i) \cdot P(Y=k-i) \\ &= \sum_i {}^n C_i p^i (1-p)^{n-i} \cdot {}^n C_{k-i} p^{k-i} (1-p)^{n-k+i} \\ &= \sum_i {}^n C_i {}^n C_{k-i} p^k (1-p)^{2n-k} \\ &= \left(\sum_i {}^n C_i {}^n C_{k-i} \right) p^k (1-p)^{2n-k} \\ &= {}^{2n} C_k p^k p^{2n-k} \end{aligned}$$

In general, $X_i \sim B(n_i, p)$, $i = 1, 2, 3, \dots, k$ & X_i 's are independent r.v.
 $Z = \sum_{i=1}^k X_i \sim B\left(\sum_{i=1}^k n_i, p\right)$ iid - independent + identically distributed

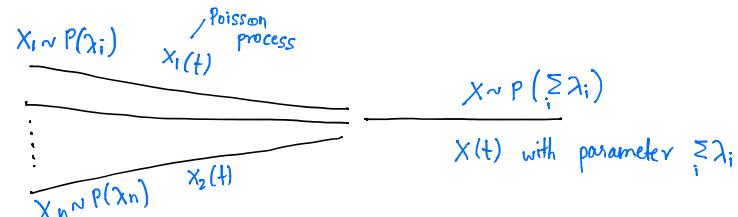
② $X_i \sim B(1, p)$, $i = 1, \dots, n$ X_i - iid's
 $Z = \sum_{i=1}^n X_i \sim B(n, p)$
 $\sum X_i$'s ~ same distribution
 \Rightarrow Reproductive Property

③ $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$
 X & Y are independent
 $Z = X + Y \sim N(\mu_1 + \mu_2, \dots)$

④ $X \sim P(\lambda_1)$, $Y \sim P(\lambda_2)$
 $Z = X + Y$
 $P(Z=k) = \sum_i P(X=i, Y=k-i)$
 $= \sum_i P(X=i) P(Y=k-i) = \sum_i \frac{e^{-\lambda_1} (\lambda_1)^i}{i!} \cdot \frac{e^{-\lambda_2} (\lambda_2)^{k-i}}{(k-i)!}$
 $= \frac{e^{-(\lambda_1+\lambda_2)} (\lambda_1+\lambda_2)^k}{k!} \Rightarrow Z \sim P(\lambda_1+\lambda_2)$ (reproductive)

In general, $X_i \sim P(\lambda_i)$ $i = 1, 2, \dots, n$ X_i 's are iid.

$$Z = \sum_i X_i \sim P\left(\sum_i \lambda_i\right)$$



● MOMENTS OF RANDOM VARIABLE

Defn: Let x_1, x_2, \dots, x_n be R.V.'s

$$E\left(\sum_{i=1}^n x_i\right) = \sum_{i=1}^n E(x_i)$$

$$\text{Var}\left(\sum x_i\right) = \sum_{i=1}^n \text{Var}(x_i) + 2 \sum_{i < j} \text{Cov}(x_i, x_j)$$

Defn: $\text{Cov}(x, y)$

$$\text{Cov}(x, y) = E((x - E(x))(y - E(y)))$$

1st order cross moment about the mean.

$$\mu_{i,j} = E((x - E(x))^i (y - E(y))^j) \quad (i, j)^{\text{th}} \text{ order cross moment about the mean.}$$

$E(x_i, x_j) \rightarrow$ about the zero.

$$E((x - E(x))^2) \rightarrow \text{Var}(x) \quad [\mu_{(2,0)}]$$

Variance Covariance Matrix of n-variables

$$\begin{bmatrix} \text{Var}(x_1) & \text{Cov}(x_1, x_2) & \dots & \text{Cov}(x_1, x_n) \\ \text{Cov}(x_2, x_1) & \dots & \dots & \vdots \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \text{Cov}(x_n, x_1) & \dots & \dots & \text{Cov}(x_{n-1}, x_n) \end{bmatrix} \quad \text{Cov}(x, y) = \text{Cov}(y, x)$$

Remark: If X & Y are independent r.v.'s

$$\begin{aligned} \text{Cov}(x, y) &= E((x - E(x))(y - E(y))) \\ &\quad \downarrow \quad \downarrow \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_1)(y - \mu_2) f_{x,y}(t, s) dt ds \\ &= E(x - E(x)) E(y - E(y)) \\ &= [E(x) - E(x)][E(y) - E(y)] \\ &= 0 \end{aligned}$$

\Rightarrow If X & Y are independent $\Rightarrow \text{Cov}(x, y) = 0$



Vice-versa is not true.

Defn :- Correlation Coefficient

$$P_{x,y} = \frac{\text{Cov}(x,y)}{\sqrt{\text{Var}(x)} \sqrt{\text{Var}(y)}} \quad \text{provided R.H.S. exist.}$$

$x \& y$ are independent, then $P_{x,y} = 0$



Remark:-

- (i) $|P_{x,y}| \leq 1$
- (ii) If $P_{x,y} = \pm 1$
 $\Rightarrow y = ax + b$
 $a > 0 \Rightarrow P_{x,y} = 1$
 $a < 0 \Rightarrow P_{x,y} = -1$

23/02/2016

LECTURE 19

Eq:- $X \sim \text{Exp}(\lambda)$ Game over whenever X or Y reaches 1.

① $Y \sim \text{Exp}(\mu)$ Assume X & Y are independent.

Define $Z = \min\{x, y\}$

$$\begin{aligned} P(Z > z) &= P(\min\{x, y\} > z) = P(x, y > z) \\ &= P(x > z) \cdot P(y > z) = \int_z^\infty \lambda e^{-\lambda x} dx \int_z^\infty \mu e^{-\mu y} dy = e^{-(\lambda+\mu)z} \end{aligned}$$

$$P(Z \leq z) = 1 - e^{-(\lambda+\mu)z}$$

$$F_Z(z) = \begin{cases} 1 - e^{-(\lambda+\mu)z} & z \geq 0 \\ 0 & z < 0 \end{cases} \Rightarrow Z \sim \text{Exp}(\lambda+\mu)$$

In general, $X_i \sim \text{Exp}(\lambda_i)$, $i = 1, 2, \dots, n$, X_i independent r.v.'s

$$Z = \min\{X_i, i=1, 2, \dots, n\} \sim \text{Exp}\left(\sum_{i=1}^n \lambda_i\right)$$

$$W = \max\{X_i, \dots\} \not\sim \text{Exp}\left(\frac{1}{(1-e^{-\lambda_W})(1-e^{-\mu_W})}\right)$$

(2) A camera runs on 2 batteries. We have 6 batteries (Battery life $X_i \sim \text{Exp}(\lambda)$)

a) What is average usage time

Suppose battery 1 & 2 are running. For one to fail, $Y = \min\{X_1, X_2\} \sim \text{Exp}(2\lambda)$

\Rightarrow Avg Time for one battery to fade = $\frac{1}{2\lambda}$. Suppose battery 2 fails. Then put

battery 3, then using memoryless property on battery 1, again $\frac{1}{2\lambda}$.

$$\Rightarrow \text{Total avg. time} = \frac{1}{2\lambda} + \frac{1}{2\lambda} + \frac{1}{2\lambda} + \frac{1}{2\lambda} + \frac{1}{2\lambda} = \frac{5}{2\lambda} \quad (\text{till 5 batteries fail})$$

(b) C.D.F. of $Z = Y_1 + Y_2 + Y_3 + Y_4 + Y_5 = \text{Gamma}(5, 2\lambda)$

$$\downarrow$$

$$\sim \text{Exp}(5\lambda)$$

cont. type

\square **Theorem:** Let (x, y) be a two dimensional random variable with a joint p.d.f. $f_{x,y}(x, y)$. Define $Z = h_1(x, y)$ & $W = h_2(x, y)$

Assume (i) $z = h_1(x, y)$ & $w = h_2(x, y)$ can be solved uniquely for x, y in terms of z & w , i.e. $x = g_1(z, w)$, $y = g_2(z, w)$

(ii) $\frac{\partial x}{\partial z}, \frac{\partial x}{\partial w}, \frac{\partial y}{\partial z}, \frac{\partial y}{\partial w}$ exist and are continuous function

Then (z, w) is 2-dim. cont. type r.v. with joint p.d.f.

$$\star f_{z,w}(z, w) = \begin{cases} f_{x,y}(g_1(z, w), g_2(z, w)) \left| J(z, w) \right| & x = g_1(z, w) \\ & y = g_2(z, w) \\ 0 & \text{otherwise} \end{cases}$$

where $J(z, w) = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} \neq 0$

eg:- 1) Let (x, y) - 2 dim cont. type. X & Y are iid r.v.'s

$$f_x = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$z = x + y \Rightarrow x = \frac{z+w}{2} \quad \& \quad y = \frac{z-w}{2}$$

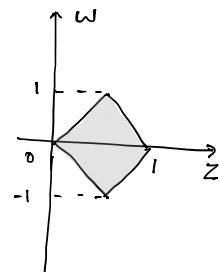
$$x \in (0, 1) \quad \& \quad y \in (0, 1)$$

$$J = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$$

$$f_{z,w}(z, w) = \begin{cases} 1 \cdot 1 \cdot \frac{1}{2} & 0 < \frac{z+w}{2} < 1 \quad \& \quad 0 < \frac{z-w}{2} < 1 \\ 0 & \text{otherwise} \end{cases}$$

Note: $f_{x,y}(x, y) = f_x(x) \cdot f_y(y)$

But $f_{z,w}(z, w) \neq f_z(z) \cdot f_w(w) \Rightarrow z$ & w are not independent



24/02/2016 **LECTURE 20**

ex: 2) $x_i \sim N(0, 1)$, $i=1, 2, \dots, n$, iid r.v.'s

$$Z = \sum_{i=1}^n X_i$$

Assume that M.G.F. of Z exists

$$M_Z(t)$$

$$M_Z(t) = M_{\sum_{i=1}^n X_i}(t) = E\left(e^{\sum_{i=1}^n \mu_i t_i + \frac{1}{2}\sigma_i^2 t_i^2}\right) = \left(e^{\mu t + \frac{1}{2}\sigma^2 t^2}\right)^n$$

$$\Rightarrow Z \sim N(0, n) \quad (\text{Other alternatives lengthy !})$$

CONDITIONAL EXPECTATION

Defn. Let (X, Y) be a 2 dim. r.v.

$$E(X|Y) = \begin{cases} \sum_i x_i P\{X=x_i | Y=y\} & (X, Y) \text{ - disc.} \\ \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx & (X, Y) \text{ - cont.} \end{cases}$$

provided in absolute sense,
R.H.S. exist.

Remark : ① If X & Y are independent random variables, $E(X|Y) = E(X)$;
 $E(Y|X) = E(Y)$

$$\begin{aligned} ② E(E(X|Y)) &= \int_{-\infty}^{\infty} E(X|y) f_Y(y) dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx f_Y(y) dy \\ &= \int_{-\infty}^{\infty} x \underbrace{\int_{-\infty}^{\infty} f_{X|Y}(x|y) f_Y(y) dy}_{dx} = \int_{-\infty}^{\infty} x f_X(x) dx = E(X) \\ \Rightarrow E(E(X|Y)) &= E(X) \end{aligned}$$

③ $E(X|Y)$ is a function of Y ; $E(X|Y)$ is a r.v.

④ $E(XY|Y) = Y E(X|Y)$

⑤ $\{X_1, X_2, \dots, X_n, \dots\}$

Martingale
Property

$$E(X_{n+1} | X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) = \begin{cases} > x_n \\ < x_n \\ = x_n \end{cases} \quad \text{Fair game}$$

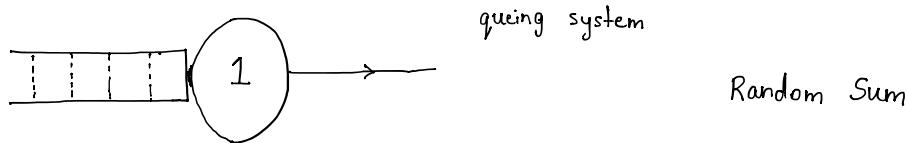
where

$$X_i = X_{i-1} + Y_i$$

$\downarrow i^{\text{th}}$ game

Y_i 's are iid r.v.'s

26/02/2016

LECTURE 21

X : total time spent in the system
 $= X_1 + X_2 + \dots + X_n$
 \downarrow Residual service time \downarrow his own service time $X_i \sim \text{Exp}(\lambda)$
where even n is a random variable.
 iid r.v.'s

$$\begin{aligned} E(x) &= E(E(x|N=n)) \\ &= \sum_n E(x/n) P(N=n) \quad (\text{But } x_i \text{ are independent with } N) \\ &\qquad\qquad\qquad \Rightarrow E(x/n) = n E(x_i) \\ &= \sum_n n E(x_i) P(N=n) \end{aligned}$$

$$\begin{aligned} E(x) &= E(x_i) \cdot E(N) \quad \text{Var}(x) = E(x^2) - [E(x)]^2 \\ &E(x^2) = E(E(x^2|N)) \end{aligned}$$

dist of X , given dist of x_i, N

$$P(N=n) = (1-p)p^n$$

$$f_x(x) = \sum_n \underbrace{f_{x/N=n}(x/n)}_{\substack{\text{cont.} \\ \text{Gamma}}} \cdot \underbrace{P(N=n)}_{\text{discrete}} \quad n=0, 1, 2, \dots$$

$$f_{(x,n)}(x,n) = f_{x/N}(x/n) \cdot P(N=n)$$

$$f_{x,y}(x,y) = f_{x/y}(x/y) \cdot f_y(y) \quad \boxed{\text{Always} \quad \forall x,y}$$

08/03/2016 **LECTURE 22**

- | | |
|--------------------------|---------------------------|
| 1.) Inequalities | 3.) Limiting Distribution |
| 2.) Law of Large Numbers | 4.) Central Limit Theorem |

Defn. : Markov Inequality

Let X be non-negative random variable with $E(X)$ exists and is known.

For fixed $t > 0$

$$\boxed{P(X > t) \leq \frac{E(X)}{t}}$$

↓
event
 $\{w | X(w) > t \quad w \in \Omega\}$

Define $Y = \begin{cases} 0 & x \leq t \\ t & x > t \end{cases}$

$$P\{y=0\} = P\{x \leq t\} \quad Y \text{ is a discontinuous type r.v.}$$

$$P\{y=t\} = P\{x > t\}$$

$$E(Y) = t \cdot P\{x > t\} \quad \text{--- (1)}$$

Now $x \geq y$

$$\therefore E(x) \geq E(y) \quad \text{--- (2)}$$

$$\Rightarrow P\{x > t\} \leq \frac{E(x)}{t}$$

Defn. : Chebychev's Inequality

Let X be a r.v. with $E(x) = \mu$, $\text{Var}(x) = \sigma^2$ exist & are known.

Then for any positive number t ,

$$\boxed{P\{|x-\mu| > t\} \leq \frac{\sigma^2}{t^2}}$$

or $P\{|x-\mu| \leq t\} \geq 1 - \frac{\sigma^2}{t^2}$

Proof:- X is a r.v. $\Rightarrow (x-\mu)^2$ is a non neg. r.v.

$$\text{where } E((x-\mu)^2) = \sigma^2$$

Apply Markov Inequality. Hence Proved.

eg:- Let X be a r.v. with $E(x) = \frac{1}{2}$, $\text{Var}(x) = \frac{1}{12}$. Find the lower bound for

$$P\left\{|x - \frac{1}{2}| < 2\sqrt{\frac{1}{12}}\right\} \Rightarrow \frac{3}{4}$$

Let X be a uniform distribution in interval $[0, 1]$.

$$\frac{1}{2} + 2\sqrt{\frac{1}{12}} > 1 \quad \& \quad \frac{1}{2} - 2\sqrt{\frac{1}{12}} < 0 \quad \Rightarrow \quad P\{|X - \frac{1}{2}| < 2\sqrt{\frac{1}{12}}\} = 1$$

Remark : In Chebyshev's inequality.



μ can be replaced by any number c .

In that case, $\sigma^2 \rightarrow E((X-c)^2)$

09/03/2016 LECTURE 23

LIMITING DISTRIBUTIONS / PROBABILITIES / THEOREMS

$X_1, X_2, X_3, \dots \rightarrow X \quad (\Omega, \mathcal{F}, P)$

① $X_i \sim U(-1, 1)$, $i=1, 2, \dots$ iid's

Define $Y_1 = X_1, Y_2 = X_1 + X_2, \dots, Y_n = \sum_{i=1}^n X_i$

What is the convergence of Y series.

Y 's will be polygons. \Rightarrow Increasing no. of sides \Rightarrow Normal distribution.

② $X_i \sim B(1, p)$ $i=1, 2, \dots$ iid r.v.s

$Y_n = \sum_{i=1}^n X_i$, $n=1, 2, \dots$

For fixed n , $Y_n \sim B(n, p)$

For large n , $n \rightarrow \infty$
normal distribution

Theorem : Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of independent random variables

defined on (Ω, \mathcal{F}, P) with $E(X_i) = \mu_i, i=1, 2, \dots$ and $\text{Var}(X_i) = \sigma_i^2 > 0, i=1, 2, \dots$

Define $Z_n = \frac{\sum_{i=1}^n X_i - E(\sum_{i=1}^n X_i)}{\sqrt{\text{Var}(\sum_{i=1}^n X_i)}}$ $n=1, 2, \dots$

Then for larger n , Z_n approaches standard normal distribution.
(approximately)

i.e. $P(Z_n \leq x) \underset{n \rightarrow \infty}{\approx} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt \quad x \in \mathbb{R}$ CENTRAL LIMIT THEOREM

MODES OF CONVERGENCE

1) IN PROBABILITY

for given $\epsilon > 0$, $X_n \xrightarrow{P} X$
if $P\{|X_n - X| > \epsilon\} \rightarrow 0$ as $n \rightarrow \infty$

2.) IN DISTRIBUTION

$X_n \xrightarrow{d} X$
if $F_{X_n}(x) \rightarrow F(x) \quad \forall x \in R$

3.) IN RTH MOMENT

$X_n \xrightarrow{\gamma^{\text{th moment}}} X \quad \gamma = 1, 2, 3, \dots$
 $E(|X_n - X|^\gamma) \rightarrow 0 \quad \text{as } n \rightarrow \infty$

4.) IN ALMOST SURELY

$X_n \xrightarrow{\text{a.s.}} X$
if $P\left\{\lim_{n \rightarrow \infty} X_n = X\right\} = 1$

e.g.: Let $\{X_n, n=1, 2, 3, \dots\}$ be a sequence of r.v.'s

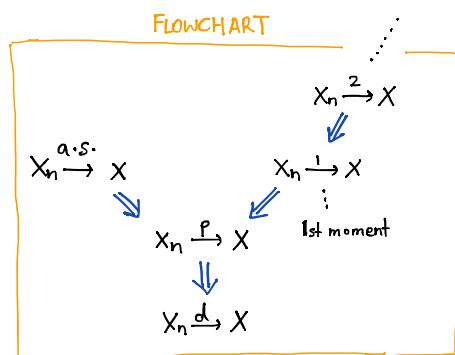
$$\text{s.t. } P\{X_n = 0\} = 1 - \frac{1}{n} \quad ; \quad P\{X_n = n\} = \frac{1}{n} \quad n=1, 2, 3, \dots$$

check $X_n \xrightarrow{P} 0$

$$\lim_{n \rightarrow \infty} P\{|X_n - 0| > \epsilon\} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

check $X_n \xrightarrow{d} 0$

H.W.



eg :- $\{X_n, n=1, 2, 3, \dots\}$ be a seq. of iid r.v.'s with $E(X_i) = \mu$; $\text{Var}(X_i) = \sigma^2$, $i=1, 2, \dots$

Define $S_n = X_1 + X_2 + \dots + X_n$

Check (1) $\frac{S_n}{n} \xrightarrow{\text{2nd order}} \mu$ (2) $\frac{S_n}{n} \xrightarrow{P} \mu$

$$\lim_{n \rightarrow \infty} E\left(\left|\frac{S_n}{n} - \mu\right|^2\right) = \lim_{n \rightarrow \infty} \left[\frac{1}{n^2} E(S_n^2) - 2\frac{\mu}{n} E(S_n) + \mu^2 \right] \\ = \sigma^2$$

15/03/2016 LECTURE 25

• LAW OF LARGE NUMBERS (Bernoulli Law)

Let ϵ be a random experiment and A be an event. Consider n independent trials.

Define $n_A = \#$ of times event A occurs in n -trials

Let $f_A = \frac{n_A}{n}$; $p(A) = p$, constant.

Then $f_A \xrightarrow{P} p$

Proof :

$$n_A = B(n, p)$$

$$E(n_A) = \sum n_A \cdot {}^n C_{n_A} p^{n_A} (1-p)^{n-n_A} = n p \sum {}^{n-1} C_{n_A-1} p^{n_A-1} (1-p)^{n-n_A} \\ = np$$

$$E(f_A) = E\left(\frac{n_A}{n}\right) = \frac{1}{n} \cdot np = p$$

$$\text{Var}(f_A) = \text{Var}\left(\frac{n_A}{n}\right) = \frac{1}{n^2} \text{Var}(n_A) = \frac{p(1-p)}{n}$$

Apply Chebychev's Inequality,

$$P\{|f_A - p| > \epsilon\} \leq \frac{p(1-p)}{n\epsilon^2}$$

$$\lim_{n \rightarrow \infty} P\{|f_A - p| > \epsilon\} = 0 \Rightarrow f_A \xrightarrow{P} p$$

eg :- ϵ : die what is min. n for given $\epsilon = 0.01$ s.t.

A : getting #6

$$P\{|f_A - p| < \epsilon\} \geq 0.95$$

$$\text{Sol}^n : \quad p = \frac{1}{6} \quad P\{|f_A - p| > \epsilon\} < 0.05$$

$$\Rightarrow n = \frac{p(1-p)}{(0.05)\epsilon^2} \approx 27,778$$

Proof of CENTRAL LIMIT THEOREM

Assume : (i) X_i 's are iid r.v.'s
(ii) M.G.F. of X_i 's exist

$$M_{Z_n}(t) = M_{\frac{\sum X_i - n\mu}{\sigma\sqrt{n}}}(t) = E\left(e^{\left(\frac{\sum X_i - n\mu}{\sigma\sqrt{n}}\right)t}\right) = e^{-\frac{\sqrt{n}\mu t}{\sigma}} E\left(e^{\frac{\sum X_i}{\sigma\sqrt{n}}t}\right)$$

$$= e^{-\frac{\sqrt{n}\mu t}{\sigma}} \left[M_{X_i}\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$$

$$M_X(t) = 1 + \underbrace{\frac{\mu t}{1!}}_{\chi} + \underbrace{\frac{E(X^2)t^2}{2!}}_{\chi} + \dots$$

$$\ln(1+\chi) = \chi - \frac{\chi^2}{2} + \frac{\chi^3}{3} - \dots \quad |\chi| < 1$$

$$\begin{aligned} \ln M_{Z_n}(t) &= -\frac{\sqrt{n}\mu t}{\sigma} + n \ln \left[1 + \underbrace{\frac{\mu t}{\sigma\sqrt{n}}}_{\chi} + \underbrace{\frac{(\sigma^2 + \mu^2)t^2}{2!n\sigma^2}}_{\chi} \dots \right] \\ &= -\frac{\sqrt{n}\mu t}{\sigma} + \underbrace{n \left[\frac{\mu t}{\sigma\sqrt{n}} + \frac{(\sigma^2 + \mu^2)t^2}{2n\sigma^2} \dots \right]}_{\chi} - \underbrace{\frac{1}{2} \left(\frac{\mu^2 t^2}{n\sigma^2} + \dots \right)}_{-\chi^2/2} \\ &= \frac{(\sigma^2 + \mu^2)t^2}{2\sigma^2} - \frac{1}{2} \frac{\mu^2 t^2}{\sigma^2} = \frac{t^2}{2} \\ \Rightarrow \lim_{n \rightarrow \infty} \ln M_{Z_n}(t) &= \frac{t^2}{2} \quad \text{or} \quad \lim_{n \rightarrow \infty} M_{Z_n}(t) = e^{\frac{t^2}{2}} \end{aligned}$$

$$Z_n \sim N(0,1)$$

16/03/2016 LECTURE 26

e.g.: let X_1, X_2, \dots, X_n be a sequence of iid r.v. with common p.m.f.

$$P(X_1) = \begin{cases} p & X_1 = 1 \\ 1-p & X_1 = 0 \end{cases} \quad p = 1/2$$

$$\text{Let } X = X_1 + X_2 + \dots + X_{10}$$

(a) Find $P(X \leq 8)$ exactly (distribution)

(b) $P\{X \leq 8\}$ lower bound (Inequalities)

(c) $P\{X \leq 8\}$ approximately (CLT)

Solⁿ (a) $P(X \leq 4) = \sum_0^8 {}^{10}_{C_i} p^i (1-p)^{10-i}$

(b) Markov's inequality

$$P\{X \leq 8\} \geq 1 - \frac{E(X)}{8} = 1 - \frac{5}{8} = \frac{3}{8}$$

(c) $P\{X \leq 8\} = P\left\{\frac{X-5}{\sqrt{2.5}} \leq \frac{8-5}{\sqrt{2.5}}\right\} = P\left\{Z \leq \frac{3}{\sqrt{2.5}}\right\} = F_z(1.89) = 0.9698$
from table
 $\mu = np$
 $\sigma^2 = np(1-p)$
normal

eg :- Find $\lim_{n \rightarrow \infty} \sum_{k=0}^n e^{-n} \frac{n^k}{k!}$ approximately using CLT.

Solⁿ $X_i \sim P(1)$

$$X = \sum_{i=1}^n X_i \sim P(n) \quad P(X \leq n) = P\left(\frac{X-n}{\sqrt{n}} \leq \frac{n-n}{\sqrt{n}}\right) = P(Z \leq 0) = \frac{1}{2}$$

 $Z = \frac{X-n}{\sqrt{n}}$

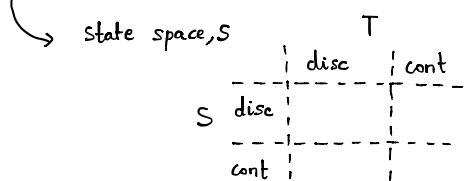
18/03/2016 LECTURE 27

TUTORIAL 5

28/03/2016 LECTURE 28

Defn:

Let (Ω, \mathcal{F}, P) be a prob space. The collection of random variables $\{x(t), t \in T\}$ be fixed on the probability space (Ω, \mathcal{F}, P) is called a stochastic process.
Parameter Space, T



eg:- Temp at time t in a city
 ① $\{x(t), t \geq 0\}$ $T = \{t \mid 0 \leq t \leq \infty\}$ cont. state, cont. time stoch. process
 $S = \{x \mid a \leq x \leq b\}$

② # of vehicles parked in the main gate upto t
 $\{Y(t), t \geq 0\}$ $T = \{t \mid 0 \leq t \leq \infty\}$ $S = \{0, 1, 2, \dots\}$ disc. state cont. time stoch. process

Time Series :

- Observed information / data over the time.

Sample Path is right continuous.

29/03/2016 LECTURE 29

PROPERTIES OF STOCHASTIC PROCESS

1.) Independent Increments $\{x(t), t \geq 0\}$

for arbitrarily $0 < t_0 < t_1 < \dots < t_n < \dots$
 of the rvs $\forall n$

$x(t_1) - x(t_0), x(t_2) - x(t_1), \dots, x(t_n) - x(t_{n-1})$ are mutually independent.

2.) Wide sense stationary / Covariance stationary

If (i) $E(x(t))$ is not a function of time t

(ii) $E(x^2(t)) < \infty$

(iii) $Cov(x(t), x(s))$ depends only on $|t-s|$.

then process is wide sense stationary

3.) Strict sense stationary

$t_1 < t_2 < t_3 \dots < t_n$

$(x(t_1), x(t_2), \dots, x(t_n)) \stackrel{d}{=} (x(t_1+h), x(t_2+h), \dots, x(t_n+h)) \quad \forall h > 0$

Then it is time variant.

4.) Markov Property

$0 \leq t_0 < t_1 < \dots < t_n < t \quad \forall n$

If $P\{x(t) \leq x \mid x(t_0) = x_0, x(t_1) = x_1, \dots, x(t_n) = x_n\} = P\{x(t) \leq x \mid x(t_n) = x_n\}$

Then it is Markov Property.

Ex:- Let $\{X_1, X_2, \dots\}$ be a sequence of iid r.v.s with common p.m.f's.

$$P\{X_i=0\} = 1-p = 1 - P\{X_i=1\} \quad 0 < p < 1$$

Define $S_0 = 0$

$$S_n = \sum_{i=1}^n X_i$$

$\{S_n, n=0, 1, 2, \dots\}$ is a stochastic process

For fixed n , $S_n \sim B(n, p)$

S_n will satisfy only Independent Increment & Markov Property.

$$P\{S_{n+m}=k \mid S_0=0, S_1=i_1, S_2=i_2, \dots, S_n=i_n\}$$

$$= \frac{P\{S_{n+m}=k, S_0=0, \dots, S_n=i_n\}}{P\{S_0=0, S_1=i_1, \dots, S_n=i_n\}} = \frac{P\{S_{n+m}-S_n=k-i_n, S_n-S_{n-1}=i_n-i_{n-1}, \dots, S_1-S_0=i_1-0\}}{P\{S_n-S_{n-1}=i_n-i_{n-1}, \dots, S_1-S_0=i_1-0\}}$$

But $S_n - S_{n-1}, \dots, S_1 - S_0$ are independent, terms will cancel out to

$$P\{S_{n+m}=k \mid S_n=i_n\}$$

$S_n = S_{n-1} + X_n$

↓

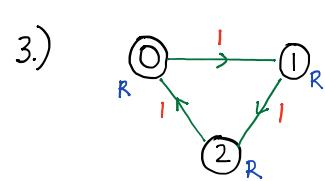
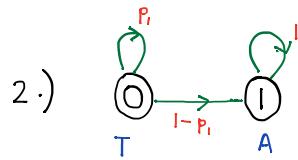
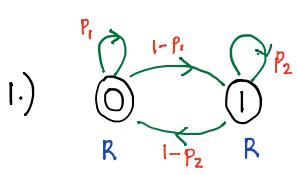
First order dependent or Markov process Auto regressive
AR(1)
Independent increment

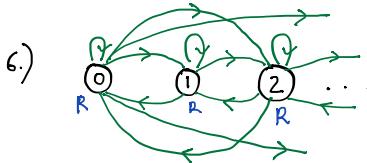
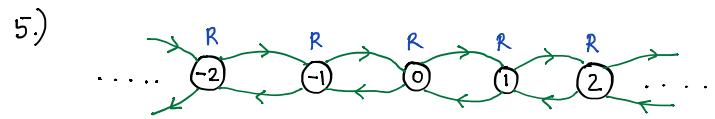
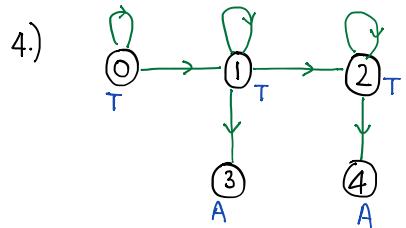
01/09/2016

LECTURE 31

05/04/2016 LECTURE 32

CLASSIFICATION OF STATE





Definitions :

1.) Visit $i \rightarrow j$, if $p_{ij}^{(n)} > 0$ for some n .

2.) Communicate $i \rightarrow j$ if $p_{i,j}^{(m)} > 0$ for some m

$$\& p_{j,i}^{(m)} > 0 \text{ for some } m$$

3.) Periodicity of state i

$$d_i = \text{g.c.d} \{ x \geq 1 : p_{ii}^{(x)} > 0 \}$$

4.) First visit

$f_{i,j}^{(n)}$ - conditional probability that the system first visit to state j in exactly n steps (with initial state i)

$$P \{ X_n = j / X_0 = i, X_k \neq j \text{ } k=1, 2, \dots, n-1 \}$$

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)} \quad \text{ever visiting state } j \text{ starting from state } i$$

$$P_{ij}^{(n)} = \sum_k f_{ij}^{(k)} P_{jj}^{(n-k)}$$

5.) Recurrent state

$$\text{iff } f_{ii} = 1$$

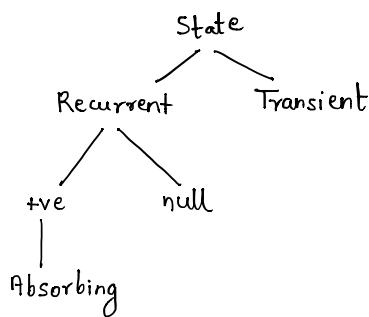
6.) Transient State iff $f_{ii} < 1$

7.) Absorbing State iff $p_{ii} = 1$

8.) Mean recurrence time $\mu_i = \sum n f_{ii}$

9.) +ve recurrent state if $\mu_i < \infty$

10.) null recurrent state if $\mu_i = \infty$



06/04/2016 LECTURE 33

Definition : 1) Closed Communicating Class (C.C.C.)

A set of communicating states, say $C \subseteq S$

If no state outside C can be reached from any state in C , we say the set C is C.C.C.

2) Aperiodic state

$$d_i = 1$$

3.) Irreducible

if $S = C_1$, where C_1 is only one C.C.C.

Defn: Stationary Distribution

Let P be the one-step transition probability matrix of a (time-homogeneous) DTMC.

A probability distribution $\{\pi_i\}_{i \in S}$ is said to be stationary distribution (or time invariant) for the given DTMC or steady state distribution or equilibrium distribution

if $\pi_j = \sum_i \pi_i p_{ij}, j \in S \quad P = [p_{ij}]_{i,j \in S}$

$$\text{s.t. } \pi_i \geq 0 \quad \sum_i \pi_i = 1$$

$$\begin{matrix} \text{matrix} & \text{matrix} & \text{matrix} \\ \pi_j & = & \pi_i \times p_{ij} \\ \downarrow & \downarrow & \downarrow \\ 1 \times n & & 1 \times n \\ & & = 1 \times n \end{matrix}$$

Defn : Limiting Distribution

Let $\pi_j = \lim_{n \rightarrow \infty} \text{Prob}\{X_n=j\}$, $j \in S$
 if it exist
 $\{\pi_j\}$, $j \in S$

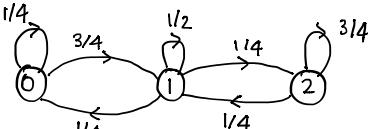
Theorem :

For an irreducible, aperiodic, +ve recurrent DTMC, the limiting distribution exist and is independent of initial distribution.

This is same as the stationary distribution and is given by

$$\pi P = \pi \quad \text{with } \sum_{j \in S} \pi_j = 1$$

Eg:- $P = \begin{matrix} 0 & \begin{bmatrix} 0 & 1 & 2 \\ 1/4 & 3/4 & 0 \\ 1 & 1/4 & 1/2 & 1/4 \\ 2 & 0 & 1/4 & 3/4 \end{bmatrix} \end{matrix}$



$$S = \{0, 1, 2\} = C_1 \Rightarrow \text{Irreducible}$$

$$f_{00} = f_{00}^{(1)} + f_{00}^{(2)} + \dots$$

$$= \frac{1}{4} + \frac{3}{4} \times \frac{1}{4} + \frac{3}{4} \times \frac{1}{2} \times \frac{1}{4} + \dots$$

$$f_{11} = 1 \quad \& \quad f_{22} = 1$$

Check for aperiodic, +ve recurrent, irreducible to apply the theorem.

$$\Rightarrow (\pi_0, \pi_1, \pi_2) = (\pi_0, \pi_1, \pi_2) \begin{pmatrix} 1/4 & 3/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/4 & 3/4 \end{pmatrix} \quad \& \quad \sum \pi_j = 1$$

Limiting dist. same
as stationary distribution

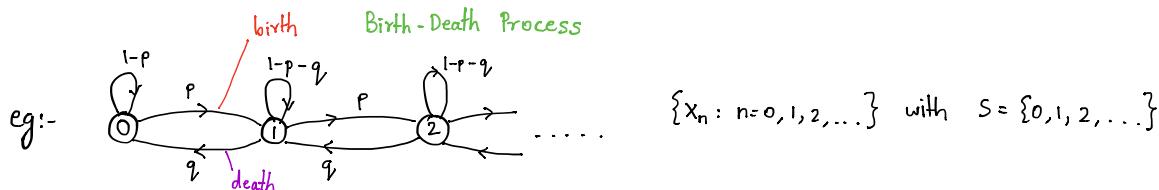
$$\pi_0 = \frac{1}{4} (\pi_0 + \pi_1) \quad \text{--- (1)} \quad \Rightarrow \quad \pi_0 = \frac{\pi_1}{3}, \quad \pi_1 = \pi_2$$

$$\pi_1 = \frac{3}{4} \pi_0 + \frac{\pi_1}{2} + \frac{\pi_2}{4} \quad \text{--- (2)} \quad \frac{\pi_0}{3} + \pi_1 + \pi_1 = 1 \Rightarrow \pi_1 = \frac{3}{7} = \pi_2, \quad \pi_0 = \frac{1}{7}$$

$$\pi_2 = \frac{\pi_1}{4} + \frac{3\pi_2}{4} \quad \text{--- (3)} \quad \left(\frac{1}{7}, \frac{3}{7}, \frac{3}{7} \right)$$

FEW IMPORTANT RESULTS

- 1.) For a finite state space DTMC, if the state is recurrent, then it has to be tve recurrent.
- 2.) In an irreducible MC, all the states are either tve recurrent or null recurrent. All states are having the same period.
- 3.) For an irreducible finite state space DTMC, all the states are tve recurrent.
- 4.) For an irreducible aperiodic DTMC, if the states are null recurrent, then $\Pi_j = 0, j \in S$



Case (i) $q = 0, p > 0$

$$T = \{0, 1, 2, \dots\}$$

$$\begin{aligned} \Pi_j &= \lim_{n \rightarrow \infty} \text{Prob}\{X_n = j\}, j = 0, 1, 2, \dots \\ &= 0 \end{aligned}$$

Case (ii) $p = 0, q > 0$

$$C_1 = \{0\}, T = \{1, 2, \dots\}$$

$$\Pi_0 = 1, \Pi_j = 0, j \in S \setminus \{0\}$$

Case (iii) $0 < p, q < 1$

Assume that all states are tve recurrent.

$$C = \{0, 1, 2, \dots\} = S$$

\Rightarrow Stationary dist. exist

irreducible, recurrent, aperiodic

It can be tve/hull recurrent

$$\Pi = \Pi P \quad \& \quad \sum_{i \in S} \Pi_i = 1$$

$$(\Pi_0, \Pi_1, \dots) = (\Pi_0, \Pi_1, \dots) \begin{pmatrix} 1-p & p & 0 & \dots \\ q & 1-p-q & p & \dots \\ 0 & q & 1-p-q & p \dots \\ \vdots & & & \ddots \end{pmatrix}$$

$$\Pi_0 = (1-p)\Pi_0 + q\Pi_1 \Rightarrow \Pi_1 = \frac{p}{q}\Pi_0$$

$$\Pi_1 = p\Pi_0 + (1-p-q)\Pi_1 + q\Pi_2 \Rightarrow \Pi_2 = \frac{p}{q}\Pi_1 = \frac{p^2}{q^2}\Pi_0$$

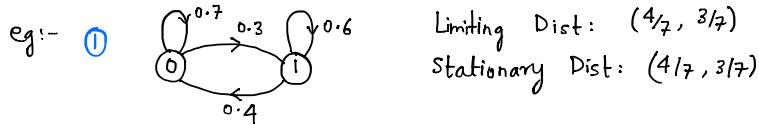
$$\therefore \text{by } \Pi_n = \frac{p}{q}\Pi_{n-1} = \dots = \left(\frac{p}{q}\right)^n \Pi_0 \Rightarrow$$

$$\text{Now, } \sum \Pi_i = 1$$

For this to occur, $\frac{p}{q} < 1$

or $P < q$ Condition for tve recurrent

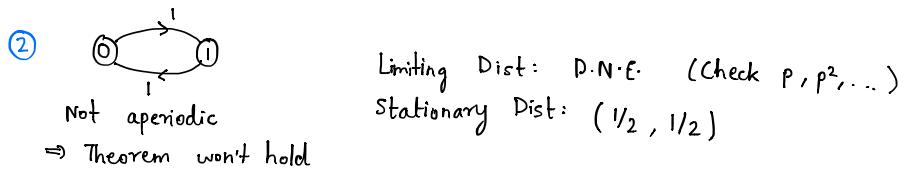
$$\& \quad \Pi_0 \times \frac{1}{1 - \frac{p}{q}} = 1 \quad \text{or} \quad \Pi_0 = \frac{q-p}{q}$$



$C = \{0, 1\} = S$ Irreducible
Aperiodic \Rightarrow Stationary = Limiting

Finite DTMC recurrent \Rightarrow tve recurrent

$$(\Pi_0 \quad \Pi_1) = (\Pi_0 \quad \Pi_1) \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} \Rightarrow \Pi_0 = \frac{4}{7}, \Pi_1 = \frac{3}{7}$$



③

Remarks : When the stationary distribution exists,

1) $X_n \xrightarrow{d} \Pi$

2) $\Pi p^n = \Pi p \cdot p^{n-1} = \Pi \cdot p^{n-1} = \dots = \Pi \quad (n \geq 2)$

3) When $\sum_i P_{ij} = 1$, $P = [P_{ij}]$ is doubly stochastic matrix.

When the DTMC is irreducible finite state space & tve recurrent & P is doubly stochastic, then

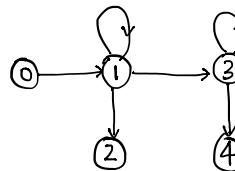
$\Pi_j = \frac{1}{N}, j \in S$ Π is uniformly distributed.

12/04/2016

LECTURE 35

REDUCIBLE MARKOV CHAIN

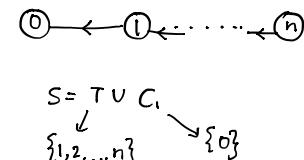
①

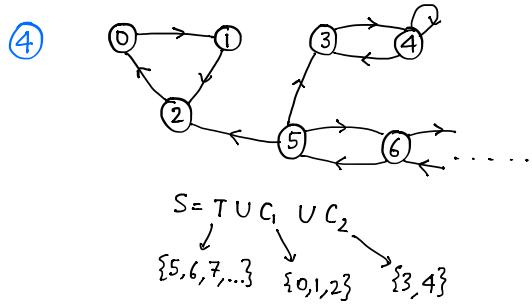
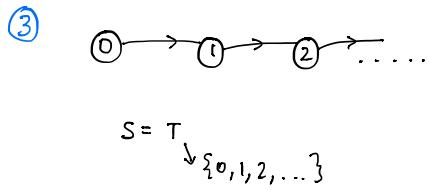


$$S = T \cup C_1 \cup C_2$$

$$\{0, 1, 3\} \quad \{2\} \quad \{4\}$$

②





FINITE STATE SPACE \wedge REDUCIBLE MARKOV CHAIN

$$P = \begin{matrix} & \text{Ab/Rec} & \text{Transient} \\ \text{Absorbing} & R_{R \times R} & O_{R \times n-R} \\ \text{Recurrent} & A_{n-R \times R} & B_{n-R \times n-R} \end{matrix}$$

If all recurrent states are absorbing state
 $R_R = I_R$

$$S = \underbrace{C_1 \cup C_2 \cup \dots \cup C_k}_{\text{Recurrent state}} \cup T$$

Define $M = (I - B)^{-1}$ (fundamental matrix)
 $= I + B + B^2 + B^3 + \dots$

Theorem : Define

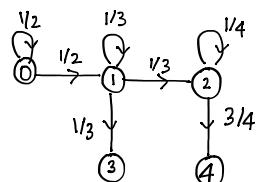
- 1.) μ_{ij} - Mean # of visits of the system to the state j before reaching an absorbing state given that $X_0 = i$, $i \in T$, $j \in T$
- 2.) g_{ij} - Conditional property that the system in absorbing state j given that $X_0 = i$, $i \in T$, $j \in S \setminus T$

Then

$$G = (g_{ij}) = (I - B)^{-1} A$$

$$M = (\mu_{ij}) = (I - B)^{-1}$$

Verification :



$$P = \begin{matrix} & 3 & 4 & 0 & 1 & 2 \\ 3 & \left[\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 3/4 & 0 & 0 & 1/4 \end{matrix} \right] \\ 4 & \end{matrix} \quad 5 \times 5$$

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 0 \\ 1/3 & 0 \\ 0 & 3/4 \end{bmatrix} \quad B = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/3 & 1/3 \\ 0 & 0 & 1/4 \end{bmatrix}$$

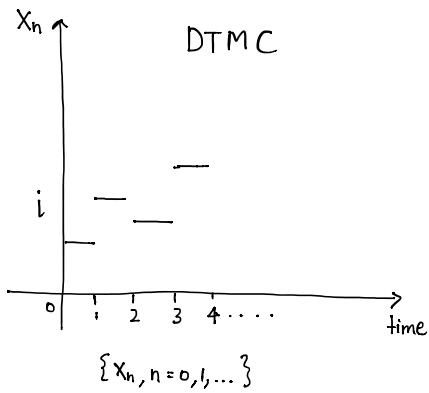
$$I - B = \begin{bmatrix} 1/2 & -1/2 & 0 \\ 0 & 2/3 & -1/3 \\ 0 & 0 & 3/4 \end{bmatrix}$$

$$(I - B)^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 3/8 & 3/8 & 0 \\ 1/6 & 1/6 & 1/3 \end{bmatrix}^T \cdot 4 = \begin{bmatrix} 2 & 3/2 & 2/3 \\ 0 & 3/2 & 2/3 \\ 0 & 0 & 4/3 \end{bmatrix}$$

$$Q = \begin{bmatrix} 2 & 3/2 & 2/3 \\ 0 & 3/2 & 2/3 \\ 0 & 0 & 4/3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1/3 & 0 \\ 0 & 3/4 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

Similarly solve for M.

→ CTMC

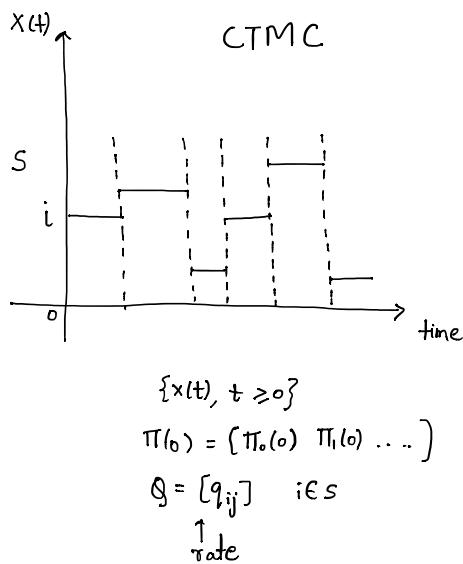


Given $\pi(0)$ - initial dist.

$$\pi(0) = (\pi_0(0) \ \pi_1(0) \ \dots)$$

$$P = [P_{ij}]$$

13/04/2016 LECTURE 36



Definitions :

1.) Initial Distribution

$$\pi(0) = [\pi_0(0) \ \pi_1(0) \ \dots] \quad \text{where } \pi_i(t) = \text{Prob}\{X(t) = i\}, t \geq 0$$

2.) Transmission Probability

$$P_{ij}(s, t) = \text{Prob}\{X(t+s) = j / X(s) = i\} \quad i, j \in S$$

Since system is time homogeneous,

$$P_{ij}(t) = \text{Prob}\{X(t) = j / X(0) = i\} \quad i, j \in S$$

$$= \text{Prob}\{X(t+s) = j / X(s) = i\} \quad \forall s > 0$$

3.) Transmission Probability Matrix

$$P(t) = [P_{ij}(t)] \quad , t \geq 0$$

$$\Pi_j(t) = \underset{\text{unconditional}}{\downarrow} \text{Prob}\{X(t) = j\} = \sum_{i \in S} \underset{\text{conditional}}{\downarrow} \Pi_i(0) P_{ij}(t)$$

$$\Pi(t) = \Pi(0) P(t)$$

4.) Generator Matrix

$$Q = [q_{ij}]$$

$$\text{for } i \neq j \quad q_{ij} = \frac{d}{dt} P_{ij}(t) \Big|_{t=0}$$

$$i=j \quad q_{ii} = -\sum_{i \neq j} q_{ij}$$

$$P_{ij}(\Delta t) = q_{ij} \Delta t + O(\Delta t), \quad j \neq i$$

$$P_{ii}(\Delta t) = 1 + q_{ii} \Delta t + O(\Delta t), \quad j=i$$

q_{ij} satisfies

$$(1) \quad q_{ij} \geq 0$$

$$(2) \quad q_{ii} \leq 0$$

$$(3) \quad \sum_j q_{ij} = 0$$

We know that

$$P_{ij}(t+T) = \sum_{k \in S} P_{ik}(t) \cdot P_{kj}(T) \quad \text{CHAPMAN-KOLMOGOROV EQUATION}$$

Differentiate w.r.t. T & put T=0

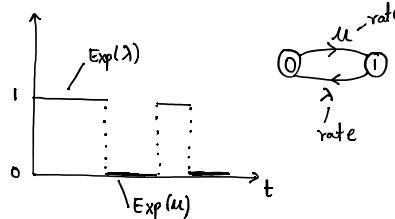
$$P'_{ij}(t) = \sum_{k \in S} P_{ik}(t) q_{kj}$$

$$\Rightarrow P'(t) = P(t) \cdot Q \quad \text{KOLMOGOROV FORWARD EQUATION}$$

Since it is a differential equation, initial cond'n $P(0)$ is required.

$$\Pi'(t) = \Pi(t) Q$$

e.g:-



$$Q = \begin{bmatrix} 0 & 1 \\ -\mu & \mu \end{bmatrix} \quad \Pi(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\Pi'(t) = \Pi(t) \cdot Q \quad \pi_0(t) + \pi_1(t) = 1$$

$$\Rightarrow \pi'_0(t) = -\mu \pi_0(t) + \lambda \pi_1(t)$$

$$\Rightarrow \pi'_0(t) = (-\mu - \lambda) \pi_0(t) + \lambda$$

$$\Rightarrow \frac{\pi'_0(t)}{\pi_0(t) - \frac{\lambda}{\mu+\lambda}} = -(\mu + \lambda)$$

$$\Rightarrow \pi_0(t) = e^{-(-\mu-\lambda)t} \cdot C' + \frac{\lambda}{\mu+\lambda} \Rightarrow \pi_0(t) = \frac{\lambda}{\mu+\lambda} \left[1 - e^{-(\mu+\lambda)t} \right]$$

$$\Rightarrow \ln \left(\pi_0(t) - \frac{\lambda}{\mu+\lambda} \right) + C = -(\mu + \lambda)t$$

$$\text{At } t=\infty, \pi_0(0)=0 \Rightarrow C' = -\frac{\lambda}{\mu+\lambda}$$

Theorem: Time spent in any state i before moving to another state for (time homogeneous) CTMC is always independent exponential distribution with parameter λ_i , $i \in S$

Proof :

ζ - r.v.

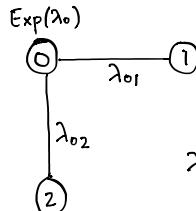
$$\begin{aligned} P(\zeta > s+t / X(t_0) = i) &= P(\zeta > s+t / X(s) = i) \cdot P(\zeta > t / X(s) = i) \\ &= P(\zeta > t / X(s) = i) \cdot P(\zeta > s / X(s) = i) \end{aligned}$$

$$F_\zeta^c(s+t) = F_\zeta^c(t) \cdot F_\zeta^c(s), \quad s, t > 0$$

$$\Rightarrow e^{-\lambda_i(s+t)} = e^{-\lambda_i s} \cdot e^{-\lambda_i t}$$

$$\Rightarrow F_\zeta(s) = 1 - e^{-\lambda_i s} \Rightarrow \zeta \sim \text{Exp}(\lambda_i)$$

eg:-



$$\lambda_0 = \lambda_{01} + \lambda_{02}$$

$$\text{eg:- } \{x(t), t \geq 0\}, \quad S = \{0, 1, 2, \dots\} \quad \pi_0(0) = 1$$

$$\textcircled{0} \xrightarrow{\lambda} \textcircled{1} \xrightarrow{\lambda} \textcircled{2} \xrightarrow{\lambda} \dots \quad \pi'(t) = \pi(t) \cdot Q$$

$$\pi_j(t) = \text{Prob}\{X(t) = j\}$$

$$\Rightarrow (\pi'_0(t) \quad \pi'_1(t) \quad \pi'_2(t) \dots) = (\pi_0(t) \quad \pi_1(t) \quad \pi_2(t) \dots) \cdot \begin{bmatrix} 0 & 1 & 2 & \dots \\ -\lambda & \lambda & 0 & \dots \\ 0 & -\lambda & \lambda & \dots \\ 0 & 0 & -\lambda & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

$$\pi'_0(t) = -\lambda \pi_0(t)$$

$$\ln(c \pi_0(t)) = -\lambda t$$

$$\text{At } t=0, \pi_0(0) = 1 \Rightarrow c=1 \Rightarrow \pi_0(t) = e^{-\lambda t}$$

$$\pi'_1(t) = \lambda \pi_0(t) - \lambda \pi_1(t) \Rightarrow \pi_1(t) = \lambda t e^{-\lambda t}$$

:

$$\pi_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n=1, 2, \dots$$

For fixed t , the dist of $X(t)$ is Poisson with parameter λt

$\{X(t), t \geq 0\} \rightarrow$ Poisson Process (C-T.M.C.)

Consider the previous example

eg:-

Diagram of a birth-death process with states 0 and 1. State 0 has a self-loop arrow labeled μ rate. State 1 has a self-loop arrow labeled λ rate.

$$Q = \begin{bmatrix} 0 & 1 \\ -\mu & \mu \end{bmatrix} \quad \Pi(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\Pi'(t) = \Pi(t) \cdot Q \quad \Pi_0(t) + \Pi_1(t) = 1$$

$$\Rightarrow \Pi'_0(t) = -\mu \Pi_0(t) + \lambda \Pi_1(t)$$

$$\Rightarrow \Pi'_0(t) = (-\mu - \lambda) \Pi_0(t) + \lambda$$

$$\Rightarrow \Pi_0(t) = e^{-(\mu+\lambda)t} \cdot C' + \frac{\lambda}{\mu+\lambda} \Rightarrow \Pi_0(t) = \frac{\lambda}{\mu+\lambda} [1 - e^{-(\mu+\lambda)t}]$$

$$\text{At } t=0, \Pi_0(0)=0 \Rightarrow C' = -\frac{\lambda}{\mu+\lambda}$$

$$\Rightarrow \ln \left(\Pi_0(t) - \frac{\lambda}{\mu+\lambda} \right) + C = -(\mu+\lambda)t$$

$$\text{Availability} = \Pi_1(t)$$

$$\text{steady state availability } \lim_{t \rightarrow \infty} = \Pi_1 = \frac{\mu}{\mu+\lambda} = \frac{1/\lambda}{1/\mu + 1/\lambda} = \frac{\text{Mean Failure Time}}{\text{Mean F Time} + \text{Mean Repair Time}}$$

22/04/2016 LECTURE 38

BIRTH DEATH PROCESS

A B.D.P. is a CTMC $\{x(t), t \geq 0\}$ with a state space $S (= \{0, 1, 2, \dots\})$ s.t.

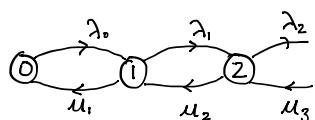
$$q_{i,i+1} = \lambda_i, \quad i = 0, 1, 2, \dots$$

$$q_{i,i-1} = \mu_i, \quad i = 1, 2, \dots$$

$$q_{i,j} = 0 \quad |i-j| > 1$$

$$q_{i,i} = \begin{cases} -(\lambda_i + \mu_i) & i = 1, 2, \dots \\ -\lambda_i & i = 0 \end{cases}$$

$$Q = \begin{bmatrix} \ddots & & 0 \\ \ddots & \ddots & \\ 0 & & \ddots \end{bmatrix}$$



$\lambda_i \rightarrow$ birth rate

$\mu_i \rightarrow$ death rate

eg:- $\{x(t), t \geq 0\}$ - P.P. (λ) is a BDP $(\lambda_i = \lambda \quad i = 0, 1, 2, \dots)$
① Poisson Process $\mu_i = 0 \quad i = 1, 2, \dots$

Pure birth process

② Pure death process It should be finite (Has to start from somewhere)

$$\lambda_i = 0, i=0,1,2,\dots,n$$

$$\mu_i = \alpha, i=1,2,\dots,n$$

→ Steady state Probability

$$\pi'(t) = \pi(t) \cdot Q \quad F.K.E.$$

For steady state, $\pi'(t) = 0 \quad \& \quad \pi(t) = \pi$

$$\Rightarrow 0 = \pi \cdot Q \quad ; \quad \sum_{i \in S} \pi_i = 1$$

$$\pi = (\pi_i)_{i \in S}$$

$$\Rightarrow 0 = -\lambda_0 \pi_0 + \mu_1 \pi_1$$

$$\pi_1 = \frac{\lambda_0}{\mu_1} \pi_0$$

$$0 = \lambda_0 \pi_0 - (\lambda_1 + \mu_1) \pi_1 + \mu_2 \pi_2$$

$$\pi_2 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} \pi_0$$

⋮

π_1, π_2, \dots in terms of π_0

$$\sum \pi_i = 1 \quad \text{Solve for } \pi_0$$

$$\pi_n = \frac{\lambda_{n-1} \lambda_{n-2} \dots \lambda_0}{\mu_n \mu_{n-1} \dots \mu_1} \pi_0$$

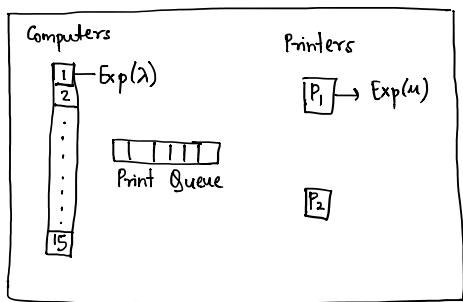
$$\pi_0 \left(1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} + \dots \right) = 1 \quad \Rightarrow \quad \pi_0 = \frac{1}{1 + \dots} > 0$$

Special case:

$\lambda_i \rightarrow \lambda$ Condition for steady state to exist:

$\mu_i \rightarrow \mu$ $\frac{\lambda}{\mu} < 1$ or $\lambda < \mu$ (common ratio of G.P.)

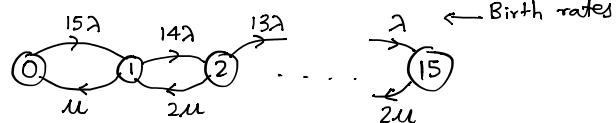
Ex:-



$$\{X(t), t \geq 0\}$$

of print tasks in system at time t

$$S = \{0, 1, \dots, 15\}$$



Since there are finite terms,
steady state will always exist.

Probability that system is always busy = $\sum_{i=2}^{15} \pi_i$

26/04/2016

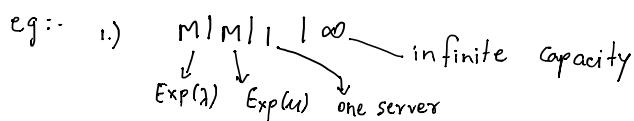
LECTURE 39

QUEUEING MODELS

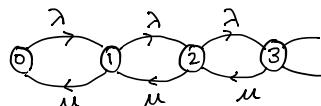
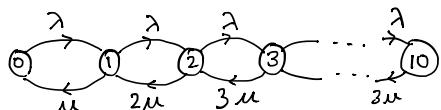
Parameters

- 1.) Inter arrival
 — deterministic
 — probabilistic
- 2.) service
- 3.) # of servers
- 4.) Capacity

$x(t)$ = no. of people in system
(queue + service)

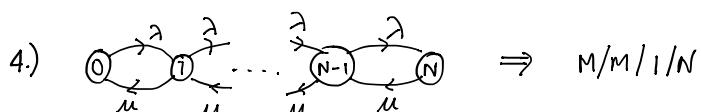
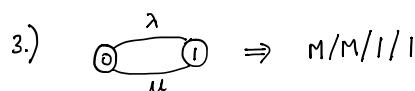


For this to be B.D.P.,
at a small interval of time, maximum of
1 person can enter / leave the system

2.) $M/M/3/10$ 

27/04/2016

LECTURE 40



Note: For a B.D.P.,

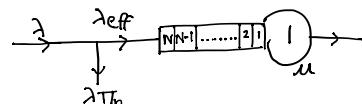
$$\boxed{\lambda \cdot E(R) = E(N)}$$

Little's Formula
 mean arrival rate
 mean spending time
 mean # of customers

\Rightarrow Mean spending time in queue = $E(Q) = E(R) - \frac{1}{\mu}$
 ↓
 mean service time

★ For a N capacity system, at capacity N people are not allowed to enter the system.

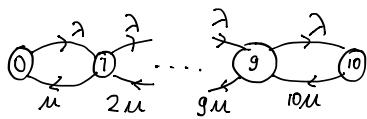
\Rightarrow Effective Arrival Rate $\lambda_{eff} = \lambda(1-\pi_{th})$



eg:- What is average queue & spending time for M/M/10/10

$$\text{Soln: } E(Q) = 0$$

$$E(R) = \frac{1}{\mu}$$



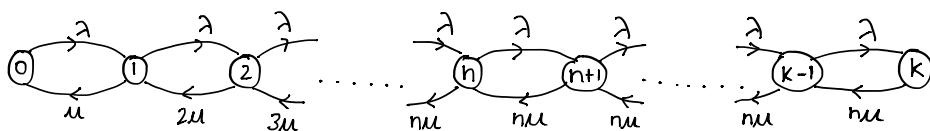
$$\Pi' = \Pi Q = 0$$

29/04/2016

LECTURE 41

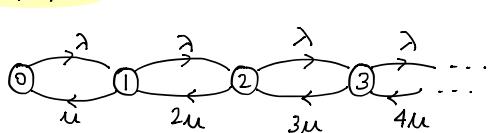
M/M/n/k

$$k > n$$



$$\Pi_n = \frac{1}{1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} + \dots + \frac{\prod \lambda_{k-1}}{\prod \mu_k}} \times \frac{\prod \lambda_{k-1}}{\prod \mu_k}$$

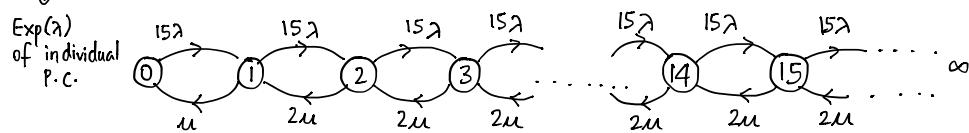
M/M/∞



$$\Pi_0 = \frac{1}{1 + \rho + \frac{\rho^2}{2!} + \dots} = e^{-\rho} \Rightarrow \star$$

\Rightarrow Poisson distribution with parameter ρ
 $\Pi_n \sim P(\rho^n)$

M/M/2/∞ with population size 15

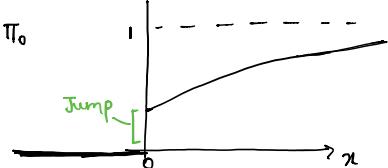


15 printers each giving print command with rate 15 with any P.C. can give subsequently multiple print command.

M/M/1/∞

Let W be the waiting time by any customer. \Rightarrow Mixed Type R.V. (Jump at 0)
when no one in service

$$P\{W=0\} = \Pi_0$$



$$P\{0 < w \leq t\}$$

$$= \sum_{n=1}^{\infty} P\{0 < w \leq t / N=n\} \cdot P\{N=n\}$$

$$= \rho (1 - e^{-(\mu-\lambda)t})$$

Erlang distribution