MTL 106 (Introduction to Probability Theory and Stochastic Processes)

Time allowed: 2 hours **Answer to Major Examination** **Max. Marks: 52**

Name: Entry Number: Signature:

Single Selection Questions: **Section 1** ($7 \times 1 = 7$ **marks)**

Each of the following questions 1 to 7 has four options out of which **one option** can be **correct**. Write A, B, C or D which corresponds to the correct option for the correct answer. **1 mark** is awarded if the correct answer is written, **0 mark** for no answer or any incorrect answer.

1. Let X follows Poisson distribution with parameter λ. Then, the pgf of X, for all $s > 0$, is given by (A) $e^{\lambda(s-1)}$ (B) $e^{-\lambda(s-1)}$ (C) $e^{s\lambda-1}$ (D) $e^{\lambda(s+1)}$ Answer: (A)

2. An urn contains b black balls and r red balls. One of the balls is drawn at random, but when it is put back in the urn, c additional balls of the same colour are put in with it. Now suppose that we draw another ball. Find the probability that the first ball drawn was red given that the second ball drawn was black?
(A) $\frac{r}{r+b+c}$ (B) $\frac{b+c}{r+b+c}$ (C) $\frac{b}{r+b+c}$ (D) $\frac{b}{r+b}$ Answer: (A)

3. n points are uniformly taken within $[0, T]$. Find the probability that k out of n point lie within an interval $[0, X]$ where X is uniform distributed random variable in $[0, T]$. (A) $\frac{1}{n+1}$ (B) $\frac{1}{n}$ (C) $\frac{k}{n}$ (D) $\frac{n-k}{n}$ Answer: (A)

4. Consider an irreducible DTMC with finite state space $S = \{0, 1, 2, \ldots, n\}$ and $p_{ii} > 0$ for at least one $i \in S$. Which one of the following statements is NOT TRUE? (A) The chain is aperiodic (B) There exists a positive recurrent state (C) Infinitely many stationary distribution exists (D) The chain is ergodic Answer: (C)

5. Consider an irreducible DTMC $\{X_n, n = 0, 1, \ldots\}$ with state space S. Let T_i denotes the time spent in a particular state $i \in S$ before transition into any other states. Then, the distribution of T_i is (A) Bernoulli (B) geometric (C) binomial (D) Poisson Answer: (B)

6. Which one of the following transition probability matrices for the ergodic DTMC
(A) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (B) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (C) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ (D) $\begin{pmatrix} 0 & 1 & 0 \\ 0.3 & 0.3 & 0.4 \\ 0 & 1 & 0 \end{pmatrix}$ Answer: (D)

7. Consider New Delhi International Airport. Suppose that, it has only one runway. Air-planes have been found to arrive at the rate of 10 per hour. It is estimated that each landing takes 3 minutes. Assume that a Poisson process for arrivals and an exponential distribution for landing times. Without loss of generality, assume that the system is modeled as a birth and death process. The above situation can be modeled via
(A) $M/M/3$ (B) $M/M/3/3$ (C) $M/M/1$ (D) $M/M/\infty$ Answer: (C)

——————————Space for Rough Work ——————————

Multiple Selection Questions: Section 2 ($5 \times 2 = 10$ **marks**)

Each of the following questions 8 to 12 has four options out of which more than one options can be correct. Write A, B, C or D which corresponds to the correct option for the first correct answer followed by space and the next correct answer and so on. **2 marks** is awarded if all correct answers are written, **0 mark** for no answer or partial correct answers or any incorrect answer.

8. Let X be uniform distributed random variable on the interval $(0, 1)$. Suppose $Y = g(X)$ is uniform distributed random variable on the interval (a, b) with $-\infty < a < b < \infty$. Then, $g(X)$ is (A) $a + (b - a)X$ (B) $b + (a - b)X$ (C) $aX + (b - a)$ (D) $bX + (a - b)$

   Answer: (A) and (B)

9. Which of the following distributions are NOT satisfy the memoryless property?
   (A) Bernoulli (B) exponential (C) Poisson (D) geometric Answer: (A) and (C)

10. 2 points are chosen uniformly and independently in a segment of length L. Let X denotes the distance to the origin of the point closest to the origin. Let Y denotes the distance to the origin of the point farthest to the origin. Then, which of the following statements are TRUE? (A) $f(x) = \frac{1}{L},\ 0 \leq x \leq L$ (B) $f(x) = \frac{2}{L}\frac{L-x}{L},\ 0 \leq x \leq L$ (C) $f(x, y) = \frac{2}{L^2},\ x < y$ (D) $f(x, y) = \frac{2}{L^2},\ y < x$ Answer: (B) and (C)

11. Consider a DTMC with states $\{0, 1, 2, 3, 4\}$. Suppose $p_{0,4} = 1$; and suppose that when the chain is in state i, $i > 0$, the next state is equally likely to be any of the states $0, 1, \ldots, i-1$. Then, which of the following statements are TRUE? (A) Chain is irreducible (B) Chain is reducible (C) All states are aperiodic (D) All states are recurrent

    Answer: (A), (C) and (D)

12. Which of the following matrices are doubly stochastic matrix for a DTMC? (A) $P = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$ (B) $P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (C) $P = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$ (D) $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

    Answer: (A), (B), (C) and (D)

——————————Space for Rough Work ——————————

Short Answer Type Questions: Section 3 ($11 \times 2 = 22$ **marks**)

Write the answer upto 4 decimal places (D) or in fraction (F) or the expression (E) for the following questions 13 to 23. **2 marks** are awarded if answer is correct, and **0 mark** for no answer or partial correct answer or an incorrect answer.

13. A point P uniformly chosen in a square of Side L centered at the origin and the x-axis. Let X denotes the coordinate of the orthogonal projection of P on the horizontal axis. Find the pdf of X.

    Answer(E): $f(x) = \frac{1}{L}, -L/2 \leq x \leq L/2$

14. Let X, Y and Z be iid random variables each having exponential distribution with parameter 1. Find $P(X < Y < Z)$?

    Answer(F/D): $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

15. Two numbers are chosen independently and at random from the interval $(0, 1)$. What is the probability that the two numbers differs by more than 1/2?

    Answer(F/D): 1/4

16. Suppose that you distribute randomly 25 apples over 10 boxes. What is the expected value of the number of boxes that will contain exactly k apples for $k = 0, 1, \ldots, 25$?

    Answer(E): $10 \times (25ck)(1/10)^k(9/10)^{25-k}$, $k = 0, 1, \ldots, 25$

17. The number of storms in the upcoming rainy season is Poisson distributed but with a parameter value that is uniformly distributed over (0, 5). That is, Λ is uniformly distributed over (0, 5), and given that $\Lambda = \lambda$, the number of storms is Poisson with mean λ. Find the probability there are at least two storms this season?

    Answer(E): $1 - \frac{1}{5}\left(2 - 7e^{-5}\right)$

18. Let X be a random variable with cdf $F(x)$. Find the cdf of $X^- = \max\{0, -X\}$ in terms of $F(x)$?

    Answer(E): $P(X^- \leq x) = \begin{cases} 0, & x < 0 \\ 1 - F(-x^-), & x \geq 0 \end{cases}$

———————————Space for Rough Work ———————

19. A certain town never has two sunny days in a row. Each day is classified as being either sunny, cloudy (but dry), or rainy. If it is sunny one day, then it is equally likely to be either cloudy or rainy the next day. If it is rainy or cloudy one day, then there is one chance in two that it will be the same the next day, and if it changes then it is equally likely to be either of the other two possibilities. In the long run, what proportion of days are sunny?

    Answer(F/D): $\frac{1}{5}$

20. Consider a DTMC with two states 0 and 1. Suppose that $p_{0,1} = r$, $p_{1,0} = s$. For which values of r and s do we obtain a reducible Markov chain?

    Answer (E): One of them (or both) should be zero.

21. The transition probability matrix of a DTMC $\{X_n, n = 0, 1, 2, \ldots\}$ having three states 1, 2 and 3 is $P = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$ and the initial distribution is $\Pi(0) = (0.7, 0.2, 0.1)$. Find $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$?

    Answer(F/D): 0.0048

22. Let $X(t) = A_0 + A_1 t + A_2 t^2$, where A_i's are uncorrelated random variables with mean 0 and variance 1. Find the covariance function of $X(t)$.

    Answer(E): $Cov(X(s), X(t)) = 1 + st + s^2t^2$

23. Consider a $M/M/1$ queuing system with arrival rate 2 per hour and average service time 20 minutes. Then, the probability of having no customer in the system in long run is given by

    Answer(F/D): $\frac{1}{3}$

Space for Rough Work

Subjective Type Questions: **Section 4** $(6+7=13$ **marks)**

Write the answer in the same page provided for the questions 24 and 25. **Full marks** are awarded if all the steps are correct, and **partial marks** for an incorrect answer with wrong steps.

24. Consider an $M/M/1/2$ queueing model.

   (a) Find the time-dependent probabilities $(\pi_n(t), n = 0, 1, \ldots)$ for the above queueing model when no customer initially. (3 marks)

   (b) Deduce the steady state probabilities for the above queueing model from the above results. (1 mark)

   (c) Find the mean time spent in the system by any customer in the longer run. (2 marks)

Answer:

(a) Solving the system of differential equations with $\pi_0(0) = 1$

$$\begin{aligned}
\pi_0^{'}(t) &= +-\lambda\pi_0(t) + \mu\pi_1(t) \\
\pi_1^{'}(t) &= \lambda\pi_0(t) - (\lambda+\mu)\pi_1(t) + \mu\pi_2(t) \\
\pi_2^{'}(t) &= \lambda\pi_1(t) - \mu\pi_2(t)
\end{aligned}$$

we get

$$\begin{aligned}
\pi_0(t) &= \frac{\mu^2}{\lambda^2+\lambda\mu+\mu^2} + \ldots e^{-(\lambda+\mu+\sqrt{\lambda\mu})t} + \ldots e^{-(\lambda+\mu-\sqrt{\lambda\mu})t} \\
\pi_1(t) &= \frac{\lambda\mu}{\lambda^2+\lambda\mu+\mu^2} + \ldots e^{-(\lambda+\mu+\sqrt{\lambda\mu})t} + \ldots e^{-(\lambda+\mu-\sqrt{\lambda\mu})t} \\
\pi_2(t) &= \frac{\lambda^2}{\lambda^2+\lambda\mu+\mu^2} + \ldots e^{-(\lambda+\mu+\sqrt{\lambda\mu})t} + \ldots e^{-(\lambda+\mu-\sqrt{\lambda\mu})t}
\end{aligned}$$

(b) Letting $t \to \infty$ in the above solutions, we get

$$\pi_0 = \frac{\mu^2}{\lambda^2+\lambda\mu+\mu^2}; \quad \pi_1 = \frac{\lambda\mu}{\lambda^2+\lambda\mu+\mu^2}; \quad \pi_2 = \frac{\lambda^2}{\lambda^2+\lambda\mu+\mu^2}$$

(C) By using $\lambda_{eff}E(R) = E(N)$ and $\lambda_{eff} = \lambda(1-\pi_2)$, we get

$$E(R) = \frac{\rho + 2\rho^2}{\lambda(1+\rho)}$$

25. Let $\{N(t), t \geq 0\}$ be a Poisson process.

(a) Define the Poisson process $\{N(t), t \geq 0\}$. (2 marks)

(b) Derive the distribution of $N(t)$ for any $t > 0$. (3 marks)

(c) Prove that, conditional on $\{N(t), t \geq 0\}$ having exactly n jumps in the interval $[s, s+t]$, the time at which a jump occurs is uniformly distributed on $[s, s+t]$. (2 marks)

Answer:

(a) Class notes

(b) Class notes (in two ways)

(c) Required conditional probability density function is $n \left(\frac{t+s-u}{t}\right)^{n-1} \times \frac{1}{t}$. When $n = 1$, it is $\frac{1}{t}$ for $u \in (s, s+t)$.