18 Sep

STOCHASTIC PROCESS

→ Let (Ω, F, P) be a prob space. The collection of r.v.s. $\{X(t), t \in T\}$ defined on (Ω, F, P) is called a stochastic process.

→ may be countably / uncountably infinite.

→ S – sample space
T – parameter space (time)

S \ T	dist	cont
disc.	discrete state, dis. time st. process — no. of students who registered in the nth semester	→ no. of students entering class at the time t.
cont.	X_n = temp at nth time limit (could be hours, day etc.)	X_t = volume of ice melted over time t.

→ $\{X(t); t \geq 0\}$ – Poisson process
↳ no. of events occur upto and including time t, for a fixed t, $X(t) \sim P(\lambda t)$

→ see time series.

Properties of Stoch Process

1. Independent increments.

for arbitrary $0 \leq t_0 < t_1 < \ldots t_n < \ldots$

for every n, if the r.v.s

$X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots\ldots X(t_n) - X(t_{n-1})$ are mutually indep.

Then $\{X(t), t \geq 0\}$ has ind. increment property

2. Stationary property:

1) wide sense: If
1) $E(X(t))$ is not func of t
2) $E(X^2(t)) < \infty$
3) $\mathrm{Cov}(X(t), X(s))$ depends only on $|t-s|$

— also called covariant stationarity

b) strict sense

$t_1 < t_2$ — — $< t_n < \ldots$

for every n, $\forall h > 0$

If $(X(t_1), X(t_2), \ldots X(t_n)) \stackrel{d}{=} (X(t_1+h), X(t_2+h), \ldots X(t_n+h))$

In general, wide sense $\not\Leftrightarrow$ strict sense — in neither direction

3. Memoryless / Markov property

$t_1 < t_2 < \ldots < t_n < \ldots$ $\quad n = 1, 2, \ldots$

$P\{X(t_n) \le x \mid X(t_0) = x_0, X(t_1) = x_1 : \ldots X(t_{n-1}) = x_{n-1}) = P(X(t_n) \le x \mid X(t_{n-1}) = x_{n-1})$

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Poisson Process.

Let $\{N(t), t \ge 0\}$ be a discrete state continuous time stoch. process. defined on a probability space (Ω, F, P).

Assume

1) $N(0) = 0$. — almost surely

2) for all $0 < t_1 < t_2 < \ldots < t_n < \ldots$
$N(t_i) - N(t_{i-1})$ $\quad i = 1, 2, \ldots$ are indepedent and stationary

3) for all $0 \le s < t$
$N(t) - N(s) \sim P(\lambda(t-s))$

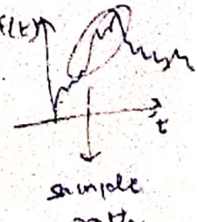
Then $\{N(t), t \ge 0\}$ is a Poisson process

Markov Process (discrete state)

disc time
→ Discrete time Markov chain (DTMC)

cont time
→ cont time Markov process (CTMC)

→ cont. state X — eg Brownian motion, wiener process. stock market prices



Sample path - all possible trajectories
time series - ~~which~~ trajectory actually observed

DTMC

Defn: Let $\{X_n, n=0,1,2 \ldots\}$ be a disc. state discrete time stochastic process with state space $S = \{0,1,2 \ldots\}$

If $\forall\ x, x_0, \ldots x_n$

$$P\{X_{n+1} = x \mid \overbrace{X_0 = x_0, X_1 = x_1, \ldots X_n = x_n}^{\text{known info}}\} = P\{\underbrace{X_{n+1} = x}_{\text{future}} \mid \underbrace{X_n = x_n}_{\text{latest}}\}$$

→ future only depends on the latest event - could be finite no. of such no. which occured i.e from $n-i$ to n, for other types of s.p.

Then $\{X_n, n=0,1,2 \ldots\}$ is said to be a DTMC.

→ For eg temperature forecast at $n = 8{:}00$ am will only depend on that at $X_{n-1} = 5{:}00$ am and not other times before that.

→ **eg 2:** a frog jumping from one "lilly leaf" to another. The leaf it jumps to next only depends on which leaf it is at present or latest and not all leaves it has been to.

eg - let $\{X_1, X_2 \ldots X_n \ldots\}$ be a seq. of iid r.v.s with common pmf

$$P\{X_i = 0\} = 1-p, \quad P\{X_i = 1\} = p \quad 0<p<1$$

Define, $S_0 = 0$

$$S_n = \sum_{i=1}^{n} X_i \quad n = 1,2, \ldots$$

$\{S_n, n=0,1,2 \ldots\}$ is disc state disc. time stoch. process

possible values = $0,1,2 \ldots$ bcz X_i can take 0 or 1.

S_n - stoch pro. with state space $S = \{0,1,2 \ldots\}$
parameter space $T = \{0,1,2 \ldots\}$

$$P\{S_{n+1} = x \mid S_0 = x_0, S_1 = x_1 \ldots S_n = x_n\} = \frac{P\{S_{n+1} = x, S_0 = x_0, S_1 = x_1 \ldots S_n = x_n\}}{P\{S_0 = x_0 \ldots S_n = x_n\}}$$

$= P\{X_1 + X_2 + \ldots + X_{n+1} = x,\ S_0 = x_0, X_1 = x_1, X_1 + X_2 = x_2, X_1 + X_2 + X_3 = x_3 \ldots$

$= P\{X_{n+1} = x - x_n,\ S_0 = x_0, X_1 = x_1, X_2 = x_2 - x_1, X_3 = x_3 - x_2, \ldots X_n = x_n$

$= P\{S_{n+1} = x \mid S_n = x_n\}.\ n = 1,2, \ldots$

$S_n \{0,1,2, \ldots\}$ - DTMC

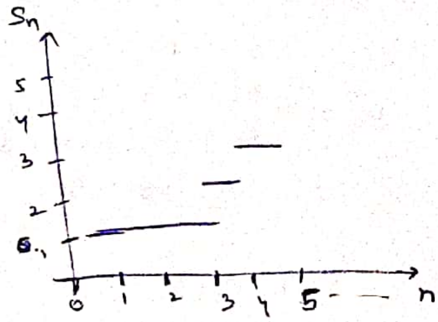
→ Increments are stationary : X_{n+1} $n = 1, 2 \ldots$ iid r.v.s.

$(X_1, X_5, X_7) \xrightarrow[\overset{d}{=}]{\text{shift by } 4} (X_5, X_9, X_{11})$ — both distributions same.

— due to identical X_n s.

3). Markov property

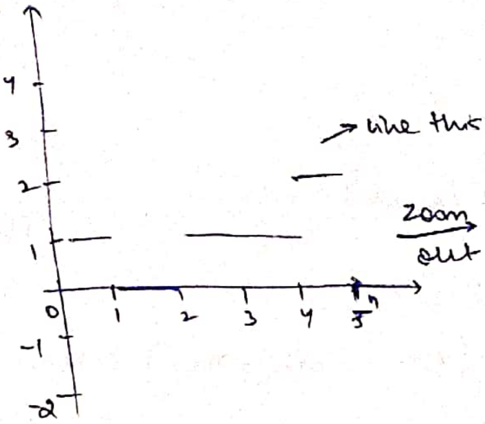
Let $X_1 = 1$
$X_2 = 0$
$X_3 = 0$
$X_4 = 1$
$X_5 = 1$



Sample path.

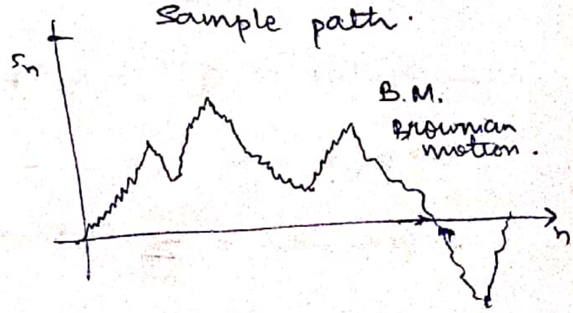
for $P(X_i = 1) = p$
$P(X_i = -1) = 1-p$.

$S_0 = 0$
$S_n = \sum_{i=1}^{n} X_i$



→ like this

Zoom out



B.M.
Brownian motion.

→ The sample path of Brownian motion is nowhere differential.

10th

Def: Initial Distribution vector

$\pi(0) = (\pi_0(0), \pi_1(0), \ldots)$ where $\pi_i(0) = \text{Prob}\{X_0 = i\}$ $i \in S$.

$\{X_n, n = 0, 1, 2, \ldots\}$ — DTMC

$S = \{0, 1, 2, 3 \ldots\}$ $T = \{0, 1, 2, 3 \ldots\}$

Transition prob

$P_{ij}(m, n) = \text{Prob.}\{X_n = j \mid X_m = i\}$ $n, m =$ steps $\leftarrow \{0, 1, 2 \ldots\}$

→ prob of transition from $i \to j$ from the $n \to m$th step. $i, j \in S$, $m > n$

→ Assume that the system is time homogenous — means the interval $(m-n)$ matters and not the values of m and n. (Time invariant of increments (increments are stationary).)

— $P_{ij}(n) = \text{Prob}\{X_{n+m} = j \mid X_m = i\}$ $\forall m$ $n = 1, 2, \ldots$

↳ n step transition prob

one step transition prob...

Let $P = [P_{ij}(1)]$ — one step transition

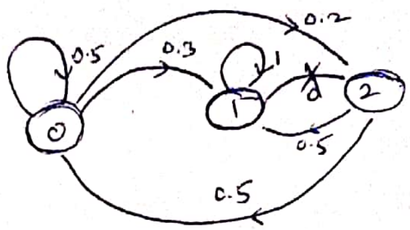
It satisfies -

1) $0 \le P_{ij} \le 1 \quad \forall i,j$

2) $\sum_j P_{ij} = 1 \quad \forall i$

→ graphical representation of state transition diagram

eg
$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0.5 & 0.3 & 0.2 \\ 1 & 0 & 1 & 0 \\ 2 & 0.5 & 0.5 & 0 \end{array}$$



dont draw 0 prob cases

$S = \{0,1,2\}$

— $P_{ij}(n) = \text{prob}\{X_n = j / X_0 = i\}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

→ $P(X_0 = x_0, X_1 = x_1 \text{ ---- } X_n = x_n)$

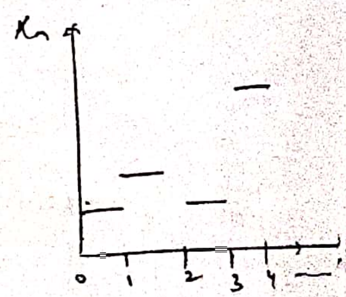
— $P(X_n = x_n) = \sum_{n-1} P(X_n = x_n | X_0 = \text{---} X_{n-1} = x_{n-1}) \times P(X_0 = x_0 \text{ --- } X_{n-1} = x_{n-1})$

$= \sum_0 \text{--} \sum_{n-1} \cdot P_{n-1\, n}\, x_n(1)$

eg $\{X_n,\ n = 0,1,2, \text{---}\}$ — DTMC

i) what is the distribution of X_n for fixed n

ii) Distribution of X_n as $n \to \infty$



initial distribution $\pi(0)$

$\pi(0) = (\pi_0(0), \pi_1(0), \pi_2(0) \text{ --- })$ $i \in S$

Transition probability $P_{ij} = \text{Prob}\{X_{n+1} = j | X_n = i\} \ \forall n$

$P(X_0 = i_0, X_1 = i_1, X \text{ --- } X_n = i_n) = P(X_0 = i_0) \cdot P(X_1 = i_1 / X_0 = i_0) \cdot \text{ --- } P(X_n = i_n / X_{n-1})$

$= \underbrace{\pi_{i_0}(0)}_{\text{initial distribution}} \cdot \underbrace{P_{i_0 i_1} \text{ --- } P_{i_{n-1}, i_n}}_{\text{one step transition prob}}$

Q) $P^{(n)} = P^n \ \forall n$ where

P = one step transition prob. matrix

$P^{(n)}$ = n step transition prob matrix

$n = 2$

$P_{ij}(2) = \sum_k P_{ik} P_{kj}$ $P^{(2)} = P^2$

$P_{ij}^{(2)} = \text{Prob}\{X_{n+2} = j / X_n = i\}$

3) We know that P is stochastic matrix

$\rightarrow P^{(n)}$ is also stochastic matrix.

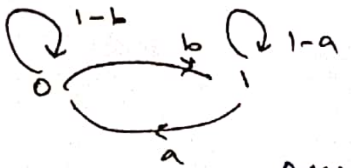
4) Suppose in addition

$\sum_i P_{ij} = 1$

in P matrix $(0 \le P_{ij} \le 1$, $\sum_j P_{ij} = 1)$

Then P is doubly stochastic matrix.

eg State transition diag $\Rightarrow$

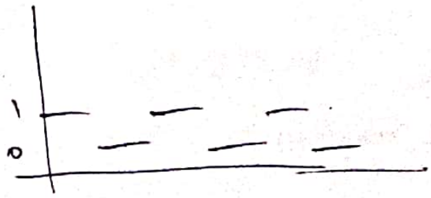


$S = \{0, 1\}$

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1-b & b \\ a & 1-a \end{pmatrix} \end{array}$$

Assume that $0 < a, b < 1$

If $b, a = 0, 1$ the state may stuck up where it starts



$\pi(1) = (\pi_0(1), \pi_1(1))$

distribution of X_n $\pi(n) = (\pi_0(n), \pi_1(n))$

$\pi_j(n) = \text{Prob}\{X_n = j\}$ $n = 1, 2, \ldots$ $j \in S$

$\pi_j(1) = \sum_i \pi_i(0) \cdot P_{ij}(1)$ $j \in S$

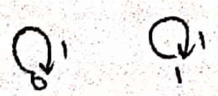
Similarly,

$\pi_j(n) = \text{Prob}\{X_n = j\} = \sum_i \pi_i(0) P_{ij}^{(n)} = \sum_i \pi_i(0) \cdot [P^n]_{ij}$

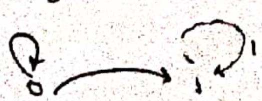
$\pi_j(n) = \sum_i \pi_i(0) [P^n]_{ij}$

In matrix form, $\underset{\text{vector}}{\pi(n)} = \pi(0) P^n$ $\{X_n, n = 0, 1, 2, \ldots\}$ with state space S.

eg2.

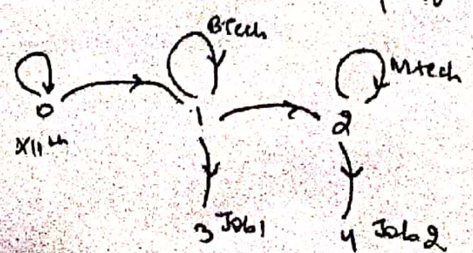


3.



default self loop is not given

4.



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$\rightarrow \{X_n, n=0,1,2 \ldots\}$ is a DTMC with state space S.

defn - i Visit $i \rightarrow j$

if $P_{ij}^{(n)} > 0$ for some n $i,j \in S$

2. communicates $i \leftrightarrow j$.

if $P_{ij}^{(n)} > 0$ and $P_{ji}^{(m)} > 0$ for some m and n — System can go from i to j and j to i.

3 period of state i.

$d_i = \text{g.c.d.}\{ n \geq 1 : P_{ii}^{(n)} > 0 \}$ if $d_i = 1$, then aperiodic

4 First visit :

$f_{ij}^{(n)}$ - conditional prob that the system visits state j in exactly n^{th} step given that $X_0 = i$.

$f_{ij}^{(1)} = P_{ij}$ $f_{ij} = \sum_n f_{ij}^{(n)}$

$P_{ij}^{(n)} = \sum_k f_{ij}^{(k)} P_{jj}^{(n-k)}$ — P can be in terms of f and not other way around

5. Recurrent event :

iff $f_{ii} = 1$

6. Transient state:

iff $f_{ii} < 1$

7. Absorbing state : (special case of Recurrent event)

iff $P_{ii} = 1$

8: Mean Recurrence time :

$\mu_i = \sum_n n . f_{ii}^{(n)}$

9. Positive Recurrence state:

If $\mu_i < \infty$

10 Null Recurrent state.

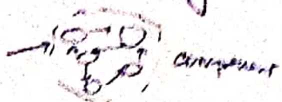
$\mu_i = \infty$

classification of states

- Recurrent
  - positive (absorbing special case)
  - Null
- Transient

11. **Def:** Closed communicating class:

A set of communicating states say $C \subseteq S$ is said to be any c.c.c. if no state outside C can be reached from any state in C. – (can have an incoming edge).



In general, $S = \bigcup_i C_i \cup T$ — set of transient states

S — set of all states

C_i and or T can be empty too.

12. Irreducible.

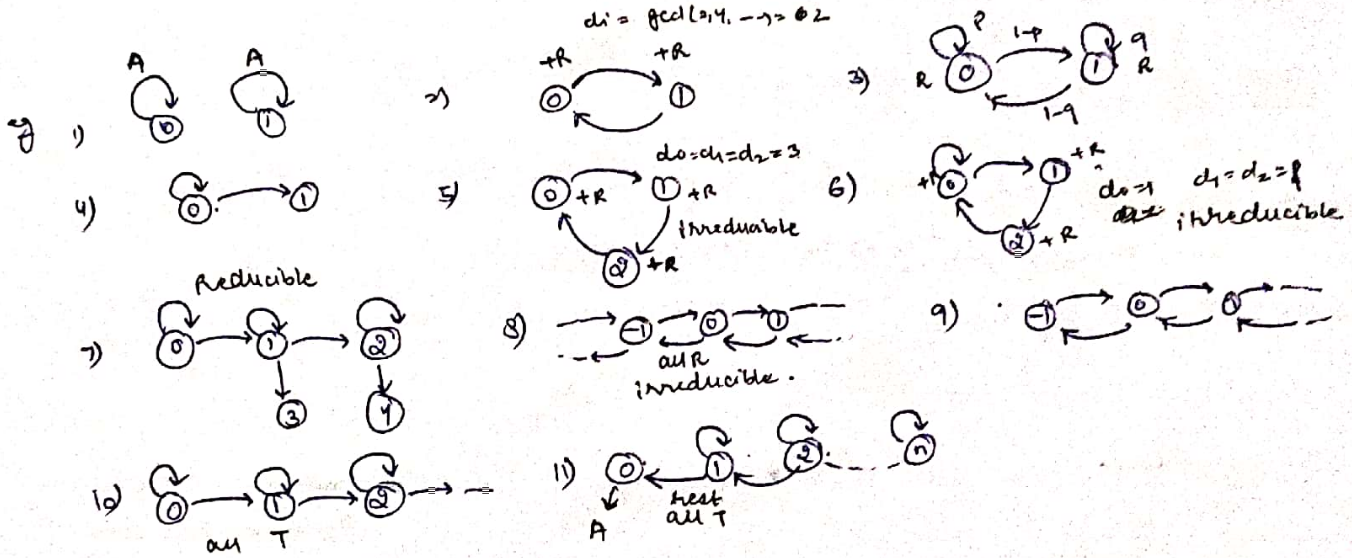
A DTMC is said to be irreducible if

$S = C$

where C is the only one c.c.c with all the states.

Otherwise the system is said to be reducible Markov chain.

eg



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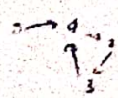
2) $f_{00}^{(1)} = 0 \quad f_{00}^{(2)} = 1 \quad f_{00}^{(3)} = 0 \quad - \quad f_{00}^{(4)} = 0 \rightarrow$ bcoz in 4 steps it is not reaching for first time. It reached at 2nd step.

$\mu_i = 2$. — +ve recurrent.

also $p_{ii} = 0 \quad p_{ii}^{(2)} = 1 \quad \text{--} \quad p_{ii}^{(2n)} = 1 \quad p_{ii}^{(2n+1)} = 0$



3)- Recurrent – only two states 0 and 1 and after some no of steps, there is always a prob that it will come to 0

$f_{ii} = \sum_n f_{ii} = 1.$ –

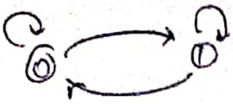
$d_i = 1$ irreducible

$\rightarrow$ ~~gcd~~ ~~ds~~ $d_0 = \gcd\{\underbrace{1}_{\text{self loop}}, 2, 2 \text{ —}\} = 1.$

4) state 0 - transient state - +ve prob it is not comeback.

16 Oct

Limiting Distribution and stat distribution.

eg 

$P = \begin{vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{vmatrix}$ $\quad P^4 = \begin{bmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{bmatrix}$

$P^2 = \begin{vmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{vmatrix}$ $\quad P^8 = \begin{vmatrix} 0.572 & 0.428 \\ 0.570 & 0.430 \end{vmatrix}$

As $n \to \infty$ (large n)

$$P^n = \begin{vmatrix} 0.57 & 0.43 \\ 0.57 & 0.43 \end{vmatrix} \; j$$

Let $v_j = \lim_{n\to\infty} P_{ij}^{(n)}$, exist $\cdot j \in S$ $\quad v = (v_0, v_1) = [0.57 \;\; 0.43]$

This is called $\{v_j\}_{j\in S}$ limiting distribution.

Stationary distribution:

suppose $\pi = \pi P$ and $\sum_{i \in S} \pi_i = 1$ where $\pi = (\pi_i)_{i \in S}$

Then π is called the stationary distribution of the stmc $\{X_n ; n \geq 0,1,2 \text{———}\}$ with state space S.

$$\pi_j = \sum_i \cancel{\pi p} \, \pi_i P_{ij}$$

---

eg $P = \begin{vmatrix} 0.7 & 0.2 & 0.1 \\ 0.1 & 0 & 0 \\ 0.4 & 0.2 & 0.4 \end{vmatrix}$ $\quad \pi = (\pi_0, \pi_1, \pi_2)$

$\underbrace{\pi = \pi P}_{\text{homogenous eq}^n} \cdot \sum_{i=0}^{2} \pi_i = 1$

~~Cal~~ Find stationary distribution for the given p-matrix.

$(\pi_0, \pi_1, \pi_2) = 0.7\pi_0 + \pi_1 + 0.4\pi_2$.

$0.3\pi_0 = \pi_1 + 0.4\pi_2$ $\quad \pi_1 = 0.2\pi_0 + 0.2\pi_2$ $\quad 0.6\pi_2 = 0.1\pi_0$

$\pi_1 = 1.4\pi_2$ $\quad \boxed{6\pi_2 = \pi_0}$

$\boxed{5\pi_1 = 7\pi_2}$

$\frac{7}{5}\pi_2 + \pi_2 + 6\pi_2 = 1$

$\pi_2 = \frac{5}{42}$ $\quad \frac{1}{8.4}$

Scanned by CamScanner

Results

called an ergodic system
1) When the DTMC $\{X_n, n=0,1,2,\ldots\}$ is irreducible, aperiodic and the recurrent, then the limiting distribution is same as the stationary dist. eg-3,6

2) For a finite state space DTMC, if the state is recurrent, then it must be positive recurrent.

3) When two states are communicating, then both are having the same property. (+R, aperiodic, null R etc) $\boxed{\text{not for transient}}$

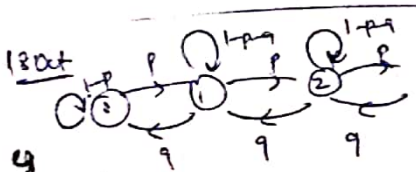
4) In an irreducible MC, (DTMC or CTMC), all the states are of the same type.

5) Steady state or equilibrium dist is name as stationary dist.

BIRTH – Death Process

made by forward arcs — by Backward arcs

] in any small interval, either birth or death occurs — both dont occur saath mein



eg

→ new sum = 0 in matrix.

Case 1: $q=0$ $p>0$

all are transient states $T=\{0,1,2,\ldots\}$ – set of transient states

$$\lim_{n\to\infty} P_{ij}^{n} = 0 \quad \forall\, i,j \in S.$$

Case 2: $p=0$ $q>0$.

→ self loop prob for $0=1$.

Closed communicating class $C=\{0\}$. $T=\{1,2,\ldots\}$

$$\lim_{n\to\infty} P_{ij}^{(n)} = \begin{cases} 1 & j=0 \\ 0 & j\neq 0 \end{cases} \quad \forall i$$

| at the end of the day, state always ends up at state 0.

Case 3: $0<p,q<1$

→ irreducible → whole s is one C.C.C

→ aperiodic

→ $f_{ii}=1, i\in S$ → recurrent.

Assume the MC is positive recurrent.

→ limiting distribution exists and is $\pi=\pi P$ and $\sum_{i\in S}\pi=1$

$$(\pi_0, \pi_1 \ldots) = (\pi_0, \pi_1 \ldots) \begin{array}{c|cccc} & & & & \\ 0 & 1-p & p & 0 & \cdots \\ 1 & q & 1-p-q & p & \cdots \\ 2 & 0 & q & 1-p-q & p \cdots \\ \vdots & & & & \end{array}$$

$\pi_0 = (1-p)\pi_0 + q\pi_1 \Rightarrow \quad \pi_1 = \frac{p}{q}\pi_0$

$\pi_1 = p\pi_0 + (1-p-q)\pi_1 + q\pi_2$. $\quad \pi_1 = q\pi_1 + (1-p-q)\pi_1 + q\pi_2$

$\frac{p}{q}\pi_1 = \pi_2 = \frac{p^2}{q^2}\pi_0$

$\pi_n = \frac{p}{q}\pi_{n-1} = \cdots = \frac{p^n}{q^n}\pi_0 \quad n = 1, 2, \ldots$

Use $\sum_{i\in S}\pi_i = 1$ — normalisation constant — ~~do~~ normalising

$\Rightarrow \pi_0 = \frac{1}{1+\frac{p}{q}+\frac{p^2}{q^2}\cdots}$ — provided denominator is finite

$\Rightarrow \frac{p}{q} < 1$

→ This is an iff condition for +ve recurrent. $\frac{p}{q} < 1$.
Intuitively $p<q$ thus tendency of system to move towards

→ when $\frac{p}{q} > 1$, then it will be null or Transient.
Thus there's a positive probability that it wont come back

Remark.

Case 3: $0<p<q<1$

$\pi_0 = \frac{1}{1+\frac{p}{q}+\frac{p^2}{q^2}\cdots\frac{p^N}{q^N}}$ — always den finite
always +ve recurrent

eg **Reducible MC with finite states space** — having some absorbing states.

→ Reorder the states with all absorbing ones in the beginning.

$$P = \begin{array}{c|c|c|} & \text{Abs} & \text{Trans} \\ \hline \text{Ab} & I_{m\times m} & 0_{m\times n-m} \\ \hline \text{Trans} & A_{n-m\times m} & B_{n\times n} \end{array}$$

I = Identity matrix

0 = null matrix.

<u>Define:</u> $M = (I-B)^{-1}$ — fundamental matrix.

$= I + B + B^2 + B^3 \cdots$ — possible if reducible finite state MC.

<u>Thm</u>. Let

μ_{ij} = mean no. of visits of the system to state j, given that $X_0 = i$ before absorption $i,j \in T$.

g_{ij} – cond prob that the system absorbs in state j, given that $X_0 = i$.

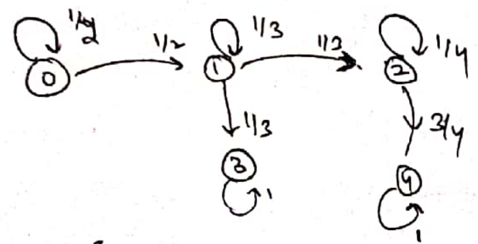
$i \in T$, $j \in S \setminus T$

$$G = (g_{ij}) = (I-B)^{-1} A$$

$$M = (\mu_{ij}) = (I-B)^{-1}$$

— No proof

eg.



$$P = \begin{array}{c|ccccc} & 3 & 4 & 0 & 1 & 2 \\ \hline 3 & 1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 1 & 1/3 & 0 & 0 & 1/3 & 1/3 \\ 2 & 0 & 3/4 & 0 & 0 & 1/4 \end{array}$$

$$A = \begin{pmatrix} 0 & 0 \\ 1/3 & 0 \\ 0 & 3/4 \end{pmatrix} \qquad B = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/3 & 1/3 \\ 0 & 0 & 1/4 \end{pmatrix}$$

$B = $.

$$M = (I-B)^{-1} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 2 & 3/2 & 2/3 \\ 1 & 0 & 3/2 & 2/3 \\ 2 & 0 & 0 & 4/3 \end{array}$$

$\mu_{ij} = 0$ means starting from i, we can't go ~~far~~ to j.

$\mu_{ij} = x$ means x ~~time~~ avg spent in going from i to j before getting absorbed in some state.

$$G = (I-B)^{-1} A = \begin{array}{c|cc} & 3 & 4 \\ \hline 0 & 1/2 & 1/2 \\ 1 & 1/2 & 1/2 \\ 2 & 0 & 1 \end{array}$$

<u>29 oct</u>

eg for discrete state ~~continuous~~ continuous type

- no. of strikes till time t in a baseball game

– ~~[illegible]~~

→ **CTMC**

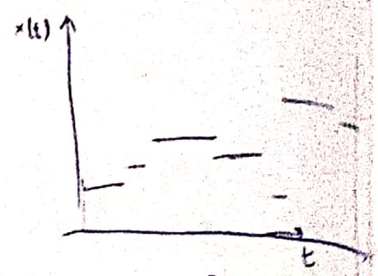
Defn: Let $\{X(t), t \geq 0\}$ be a discrete state space S and cont. time stoch. process. For arbitrary $0 \leq t_0 < t_1 < \dots < t_n < t$ if

$$P(\underbrace{X(t) \leq x}_{\text{future}} \mid \underbrace{X(t_0) = x_0, X(t_1) = x_1, \dots}_{\text{past}} \underbrace{x(t_n) = x_n}_{\text{latest}}) = P(X(t) \leq x \mid \underbrace{X(t_n) = x_n}_{\text{latest}})$$

x. – First order dependency

→ AR(n) – latest n info matters. (Auto Recursive) – AR(1). –

Then $\{X(t), t \geq 0\}$ is CTMC



CTMC

→ ~~Time~~ The interval in any state can vary now being continuous time space.

→ Time homogeneous – dont bother about the actual time, only time interval matters.

→ Trajectory
→ sample path
→ Time series.

Sample ques:

Given	To find
Initial dist	dist of $X(t)$ for fixed t
transition	——— as $t \to \infty$

① Initial distn: $\pi(0) = (\pi_0(0), \pi_1(0), \pi_2(0) \text{———})$

where $\pi_i(0) = \text{Prob}\{X(0) = i \mid i \in S\}$

② Transition prob

$P_{i,j}(s,t) = \text{Prob}(X(t) = j \mid X(s) = i\}$ $\quad i,j \in S \quad 0 \leq s < t < \infty$

Assume that $\{X(t), t \geq 0\}$ is a time-homogenous

$P_{i,j}(t) = \text{Prob.}\{X(s+t) = j \mid X(s) = i\} \quad \forall s$

$P(t) = [P_{i,j}(t)]_{i,j \in S}$

Dist of $X(t)$

$\pi_j(t) = \text{Prob}(X(t) = j) \quad j \in S$

$\pi(t) = [\pi_j(t)]_{j \in S}$

Defn: generator of matrix

$Q = [q_{ij}]_{i,j \in S}$ $\qquad i = j \quad q_{ii} = -\sum_{i \neq j} q_{ij}$

For $i \neq j$

$\underbrace{q_{ij}}_{\text{rate}} = \frac{d}{dt} P_{ij}(t) \Big|_{t=0} \quad i,j \in S$ $\qquad$ – indicates rate of going to state j.

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① $p_{ij}(\Delta t) = q_{ij}\Delta t + o(\Delta t) \quad i \neq j$ $\qquad$ $o(\Delta t)$ = function with order Δt. $o(\Delta t) \to 0$ as $\Delta t \to 0$.

② $p_{ii}(\Delta t) = 1 + q_{ii}\Delta t + o(\Delta t)$

q_{ij} satisfies: ① $q_{ij} \geq 0 \quad \forall i \neq j$

② $q_{ii} \leq 0$

③ $\sum_j q_{ij} = 0$

$$Q = [q_{ij}] = \begin{pmatrix} q_{00} & q_{01} & q_{02} & \cdots \\ q_{10} & \ddots & & \\ q_{20} & & \ddots & \\ \vdots & & & \end{pmatrix}$$

Chapmann Kolmogorov Eqn.

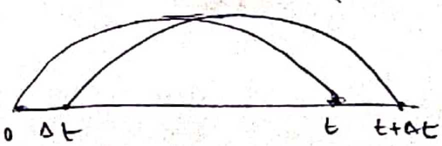
$p_{ij}(t+T) = \sum_k p_{ik}(t)\,p_{kj}(T)$ $\qquad$ DTMC

Diff wrt t.

$p'_{ij}(t+T) = \sum_k p_{ik}(t) \cdot p'_{kj}(T)$, put $T=0$

$p'_{ij}(t) = \sum_k p_{ik}(t) q_{kj}$ $\Rightarrow$ $P'(t) = P(t)Q$ — forward kolmogorov equation.

forward $p'_{ij}(t) = P(t)Q$ $\qquad$ backward — —

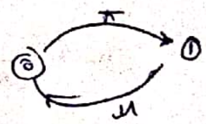


given $P(0)$

$\pi(t) = \pi(0)P(t)$

~~Stand~~ State Transition Diagram. CTMC

$Q = q_{ij} = \begin{array}{c} 1 \\ 0 \end{array}\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} \quad \lambda, \mu > 0$

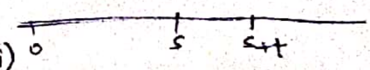


random amt of time spent in each state.

→ Time spent in the system before jumping into other states.

let τ be the rv. — — —

$P(\tau > s+t \mid X(0)=i) = P(\tau > s+t \mid X(s)=i) \cdot P(\tau > s \mid X(0)=i)$



$F^c_\tau(s+t) = F^c_\tau(t) \cdot F^c_\tau(s)$

↳ s to s+t can be shifted to 0 to t — time homogenous

→ This property is had by exponential funcs

$e^{-\lambda_i(s+t)} = e^{-\lambda_i t} \cdot e^{-\lambda_i s}$. for some $\lambda_i > 0$

The cdf of τ $= \begin{cases} 0, & -\infty < t < 0 \\ 1 - e^{-\lambda_i t}, & 0 \leq t < \infty \end{cases}$

Time spent in 0 before jumping to any other state ~ exp(λ)

$\tau \sim \exp(\lambda_i)$

——— 1 ——— ~exp(μ)

23 oct

$$\pi(0) = (\pi_0(0), \pi_1(0)) = (0, 1)$$

$$Q = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & -\mu & \mu \\ 1 & \lambda & -\lambda \end{array}$$

$$\pi'(t) = \pi(t)Q$$

$$(\pi_0'(t)\ \pi_1'(t)) = (\pi_0(t), \pi_1(t))\,Q \qquad \pi_0(t) = ? \quad \pi_1(t) = ?$$

$$(a'\ b') = (a\ b)\begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix}$$

$$x' = -x\mu + y\lambda$$
$$y' = x\mu - y\lambda$$

$$\frac{y'' + y'\lambda}{\mu} = -y' - \frac{y\lambda}{\mu} + y\lambda \qquad x = \frac{y' + y\lambda}{\mu}$$

$$y'' = -y'(\mu+\lambda) \qquad \frac{y''}{y'} = -(\mu+\lambda) \qquad \ln y' = -(\mu+\lambda)t$$

$$y' = e^{-(\mu+\lambda)t}$$

$$\boxed{y = \frac{-e^{-(\mu+\lambda)t}}{(\mu+\lambda)}}$$

$$\pi_1(t) = \frac{-e^{-(\mu+\lambda)t}}{\mu+\lambda} + c$$

Ans- $\pi_0(t) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}$

$\pi_1(t) = \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda}e^{-(\lambda+\mu)t}$

↳ Time dependent solution — states func. of time.
Transient solution

→ **stationary dist.**

$$0 = \pi Q \quad . \sum_{i \in S} \pi_i = 1 \qquad \pi = (\pi_i)_{i \in S}$$

where $\pi_i = \lim_{t\to\infty} \text{prob}\{X(t) = i\}$

for $\pi(0) = (\pi_0, \pi_1)\cdot\begin{pmatrix} -\mu, \mu \\ \lambda & -\lambda \end{pmatrix} \qquad \pi_0 + \pi_1 = 1.$

$\pi_0 = \frac{\lambda}{\lambda+\mu} \qquad \pi_1 = \frac{\mu}{\lambda+\mu}$ — same as the limiting distribution

↳ bcuz model — irreducible, +ve recurrent] → only condition for CTMC

μ = repair rate → after repair, upstate is reached. (1)
λ = failure rate → after repair, downstate is reached (0).

for two states only:

→ **Availability** — prob that the system is up/working/available at any time t. — $\pi_1(t)$

Unavailability — $\pi_0(t)$ — prob that the system is down at any time t.

↳ for multi state — atleast one component up → availability.

$$\pi_0 = \frac{1/\mu}{\frac{1}{\mu} + \frac{1}{\lambda}} \qquad \pi_1 = \frac{1/\lambda}{1/\mu + 1/\lambda}$$

$\frac{1}{\mu}$ or $\frac{1}{\lambda}$ = average time spent in that state

→ unavailability = ratio of time the system is not available

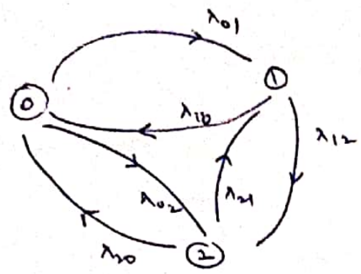
$\frac{1}{\mu}$ = avg time unavailable $\frac{1}{\mu}+\frac{1}{\lambda}$ = one unit av. time.

→ similarly availability.

→ Thus without knowing the distribution, we can use these ratios to find av. time of availability / unavailability.

---

eg:



→ Time spent in 0 before going to 1 is a rv with exp. dist.

$T_{01} \sim Exp(\lambda_{01})$
$T_{02} \sim Exp(\lambda_{02})$] independent.

→ Time spent in 0 before jumping into any state

$T_0 =$ ~~$T_{01}+T_{02}$~~ $\min(T_{01}, T_{02})$ —??

$T_0 \sim Exp(\lambda_{01}+\lambda_{02})$

$T_2 \sim Exp(\lambda_{20}+\lambda_{21})$

→ Time spent in a state = min of time spent before jumping into any state

---

BIRTH DEATH PROCESS (BDP)

Def: · A BDP is a CTMC $\{X(t), t\geq 0\}$ with state space

$S = \{0,1,2,\ldots\}$ s.t.

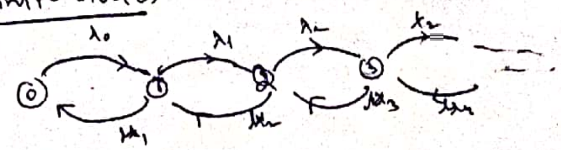
$q_{i,i+1} = \lambda_i$ $i=0,1,2,\ldots$

$q_{i,i-1} = \mu_i$ $i=1,2,\ldots$

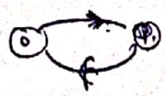
$q_{i,j} = 0$ $|i-j|>1$

$q_{i,i} = \begin{cases} -\lambda_0 & i=0 \\ -(\lambda_i+\mu_i) & i=1,2,\ldots \end{cases}$

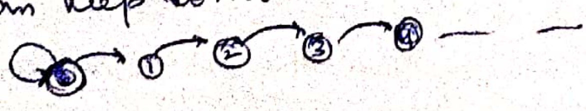
Infinite states



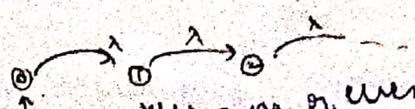
finite states



→ can keep some λ_i or $\mu_i = 0$



eg



$X(t)$ = no. of events occur upto time t.

$T_i \sim Exp(\lambda)$ $i=0,1,2,\ldots$ independent rvs

$\{X(t), t\geq 0\}$ — PP

$\pi'(t) = \pi(t)\,Q$

$(\pi_0'(t), \pi_1'(t), \ldots) = (\pi_0(t)\cdot\pi_1(t)\ldots)$

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$\pi_0(0) = 1$ $\quad (\pi_0'(t), \pi_1'(t) \ldots) = (\pi_0(t), \pi_1(t) \ldots) \begin{pmatrix} -\lambda & \lambda & 0 \\ 0 & -\lambda & \lambda \\ 0 & & \end{pmatrix}$

$\pi_0'(t) = -\lambda \pi_0(t) \Rightarrow e^{-\lambda t} = \pi_0(t)$.

$\pi_0(0) = 1 \quad \pi_0(1) = e^{-\lambda}$ ______

$\pi_1'(t) = \lambda \pi_0'(t) - \lambda \pi_1(t)$

$\pi_1'(t) = -\lambda^2 e^{-\lambda t} - \lambda \pi_1(t)$ — linear. $\Rightarrow \boxed{\pi_1(t) = \lambda t e^{-\lambda t}}$

$$\boxed{\pi_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}}$$ $n = 0, 1, 2 \ldots$

→ CTMC model with a matrix,
→ a time dependent solution for the model

0 → 1 → 2 ...

→ called a pure birth process — no deaths
↳ for $i = 0, 1, 2 \ldots$ — system always goes to next process

25 Oct

eg for birth death process
- no of cars in the parking lot of a mall → finite state space (limited parking slots)
  → only one car enters or exits.

Special cases

1) $\lambda_i > 0, \quad i = 0, 1, 2 \ldots$
$\mu_i = 0 \quad i = 1, 2 \ldots$ — Pure birth process

eg — $\lambda_i = \lambda \quad i = 0, 1, \ldots$
$\mu_i = 0 \quad i = 1, 2, \ldots$ — Poisson process

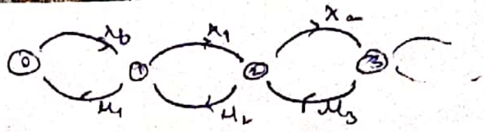
2) $\lambda_i = 0 \quad \mu_i > 0$
$i = 0, 1, 2, \ldots \quad i = 1, 2, 3 \ldots$ — Pure death process

eg $X(0) = n$ $\quad 0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \ldots \leftarrow n$ — there are n tubelights ($\rightarrow Exp(\mu)$).
and any can fail.
But fails dont occur at same time
Thus one fail at a time.

→ Rate from $n \to n-1$ = now anyone of $x_0\ x_1 \ldots x_n$ can fail. Thus rate $= \min(\lambda_1, \lambda_2, \ldots)$
$= n\mu$

Similarly from $k \to k-1$ rate $= k\mu$. $\quad \mu$ = parameter for each exponential tubelight ~~lifetime~~ lifetime (iid) $\sim Exp(\mu)$

⑤ $\lambda_i > 0 \quad \mu_i > 0$



eg ~~any~~ time spent in 2 before jumping to any other state

$T_2 \sim \text{Exp}(\lambda_2 + \mu_2)$.

Steady State prob

Assume the recurrent. We'll find condition later.

$0 = \pi Q$ and $\sum_{i \in S} \pi_i = 1$

$$(0, 0, 0 - \text{——}) = (\pi_0\ \pi_1\ \pi_2 \text{———}) \begin{bmatrix} -\lambda_0 & \lambda_0 & - & \text{—} & - \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & - & - \\ 0 & \mu_2 & \cdot & - & - \end{bmatrix}$$

$0 = -\lambda_0\pi_0 + \mu_1\pi_1 \Rightarrow \boxed{\pi_1 = \frac{\lambda_0}{\mu_1}\pi_0}$

$0 = \lambda_0\pi_0 - (\lambda_1 + \mu_1)\pi_1 + \mu_2\pi_2 \qquad \pi_2 = \frac{\lambda_1}{\mu_2}\pi_1 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2}\pi_0$

ergodic system
the recurrent
irreducible
$\frac{\lambda}{\mu} < 1$

observing the pattern-

$$\boxed{\pi_n = \frac{\lambda_0\lambda_1\lambda_2 \text{——} \lambda_{n-1}}{\mu_1\mu_2 \text{——} \mu_n}\pi_0}$$

use $\sum \pi_i = 1$

$$\pi_0\left(1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \text{——}\right) = 1$$

$$\pi_0 = \frac{1}{\left(1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} \text{——}\right)}$$

Suppose
$\lambda_i = \lambda \quad \forall i$
$\mu_i = \mu \quad \forall i$
then condition for steady state prob existing
$= \boxed{\frac{\lambda}{\mu} < 1}$ – only condition for the recurrent

$\frac{\lambda}{\mu} \geq 1$ – null recurrent or transient.

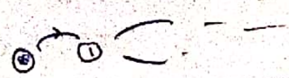
Suppose $|S| = n$

$$\pi_0 = \frac{1}{1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} \text{——} + \frac{\lambda_0\lambda_1 \text{——} \lambda_{n-1}}{\mu_1\mu_2 \text{——} \mu_n}}$$

→ system ~~is~~ irreducible $\Rightarrow \lambda_i > 0 \quad \mu_i > 0 \quad \forall i$ (bcuz nahi toh beech mein break ho jayega)

→ Thus for a fixed n, the denominator is always finite $\Rightarrow$ always ~~finite~~ the rec.

→ dotted arcs



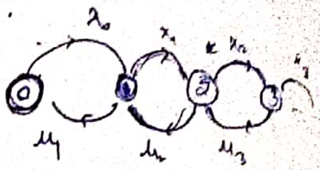
Because finite state space, but arrivals ~~are~~ still coming

eg – parking lot with finite seats and a separate space for ~~to~~ waiting cars. – If parking full, then cars wait there.

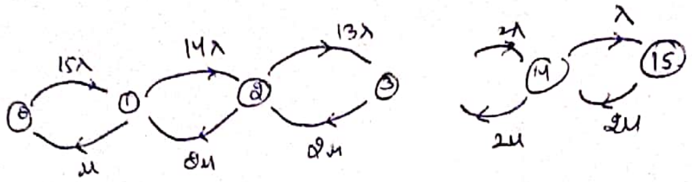
eg: a lab has 15 PCs and 2 printer
Printers ~~send~~ receive commands and one by one printing is done.

optimisation problem : how ~~man~~ ~~no~~ for much av. ~~time~~ is ~~the~~ the no. of jobs o (no commands for printer to print)

$X(t)$ = No. of jobs in the printer at time t.



$S = \{0, 1, 2, \ldots 15\}$



- all 2μ bcoz only 2 printers working. We will have to see when one of them get over ($\min(\mu_1, \mu_2)$)

others jobs are in the waiting queue

find π_0

$0 = \pi Q$ ———

29 oct

Queing Models

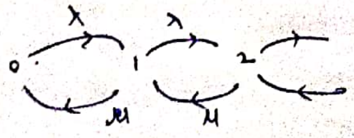
we only study simple queueing models.

Assume:

→ Arrival follows poisson process with parameter λ.
→ Service follows Exponential distribution $\text{Exp}(\mu)$
→ No. of ~~services~~ servers is 1.
→ Infinite capacity of the system (queue + inservice)
→ FIFO | LIFO | RO | priority
→ Population size is infinite.

$X(t)$ = No. of customers in the system at time t.
$\{X(t), t \geq 0\}$ is a stochastic process with cont time and discrete state space

$S = \{0, 1, 2 \ldots\}$ ↳ BDP with $\lambda_i = \lambda$ $i = 0, 1, 2 \ldots$
$\mu_i = \mu$ $i = 1, 2, \ldots$



This queueing model is called M|M|1 queueing model.

Kendall Notation: A|B|c|D|E|X|Y

A: Arrival pattern
B: Service pattern
C: No. of servers
D - capacity of system
X - service discipline
Y - population size

eg - M|G|1 ~~M~~ G|G|1 M|M|1|50
M|M|500|500 M|G|1600|1600
M|M|∞ M|M|3|8

M: exponential
G = non-exponential.

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1 Nov

queueing model

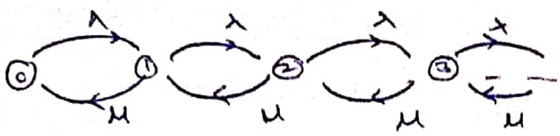
M | M | 1 | ∞

inter arrival $\sim Exp(\lambda)$ — service $\sim Exp(\mu)$ — No. of servers — capacity of system

$X(t)$ = no of customers in the system at time t (both server and queue)

→ This is a birth death process. Some relaxations may result in a non birth-death process.

$\{X(t) ; t \geq 0\}$ is a BDP



$\lambda_i = \lambda$ $i = 0, 1, 2, \ldots$ - arrival rates

$\mu_i = \mu$ $i = 0, 1, 2 \ldots$ departure rates.

→ here λ is const does n't mean each arrival is happening after a particular time. what it means is arrival follows exp (λ) with a constant mean λ. λ could also be a function of t.

$$\pi_n(t) = \text{prob}\{X(t) = n\} \quad n = 0, 1, 2 \ldots \quad t \geq 0$$

In queing theory model, this means prob of n ~~cu~~ customers in the system.

$\pi(0) = (\pi_0(0), \pi_1(0), \pi_2(0) \ldots)$ — initial distribution vector.

$Q = (q_{ij})_{i,j \in S}$

$$\pi'(t) = \pi(t) \circ Q \quad \text{with given } \pi(0)$$

— this is a differential ~~diff eq.~~ difference eq.

Stationary distribution.

called the long run

$$0 = \pi Q \quad \text{and} \quad \sum_{i \in S} \pi_i = 1$$

$$\pi_0 = \frac{1}{1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2} + \ldots} \quad \frac{\lambda}{\mu} < 1 \quad \lambda < \mu \quad \pi_n = \left(\frac{\lambda}{\mu}\right)^n \pi_0 \quad n = 0, 1, 2 \ldots$$

→ Thus at average, we have to do faster services ($\mu > \lambda$) for the system to remain in control

$$\frac{\lambda}{\mu} = \rho = \text{Intensity}$$

π_n in terms of ρ -

$$\pi_0 = \frac{1-\rho}{1} \quad \text{(gp sum)}$$

$$\boxed{\pi_n = (1-\rho)\,\rho^n}$$

Average no of customers in the system.

$$E(N) = \sum_{n=0}^{\infty} n \pi_n = \frac{\rho}{1-\rho}$$

All calculation for longer run

Average no of customers in the queue.

$$E(Q) = \sum_{n=1}^{\infty} (n-1)\pi_n = \sum_{n=0}^{\infty} n\pi_{n+1}$$

Prob that no customer in the system $= \pi_0 = 1-\rho$

Prob of the server busy $= 1-(1-\rho) = \rho$.

R = Time spent in the system by any customer.

- cont time R.V → time in the queue + his own service

- when a customer enters (x_n), someone already undergoing service (partially complete) and $n-1$ people will undergo complete service

$\underbrace{x_1}_{\text{partially done}} + \underbrace{x_2 \; x_3 \; \ldots}_{\text{will take full time}} + \underbrace{x_N}_{\text{full service time}}$ = total wait time

— partial wala dont matter as it is $\exp(\lambda)$ thus prob func remains same.

gamma dist.

$$f_R(t) = \sum_n f_{R/N}(t/n) \cdot \underbrace{P(N=n)}_{\text{discrete type}}$$

$$F_R(t) = \begin{cases} 0 & -\infty < t < 0 \\ 1-e^{-\mu(1-\rho)t} & 0 < t < \infty \end{cases}$$

$$E(R) = \frac{1}{\mu-\lambda} \qquad \lambda < \mu$$

Avg time spent in the queue. $E(Q) = E(R) - $ av. service time.

$$= \frac{1}{\mu-\lambda} - \frac{1}{\mu} = \frac{\lambda}{\mu-\lambda}$$

This rv is of mixed type. — Because there's a +ve prob that he may not wait at all → 0 or cont. dist. in $(0,\infty)$] thus mixed type due to finite prob at 0.

$$P\{Q=0\} = \pi_0 = 1-\rho$$

5 Nov

Q) ~~other~~ Models other than $M|M|1|\infty$

1) $M|M|1|\infty$
2) $M|M|1|N$ $\quad N \geq 1$
3) $M|M|c|N$ $\quad 1 \leq c \leq N$
4) $M|M|c|c$ $\quad c \geq 1$
5) $M|M|c|\infty$
6) $M|M|\infty$
7) $M|M|c|N$ with finite population

1) $\underline{M|M|1|\infty}$

$\pi_n = (1-\rho)\rho^n \quad n = 0,1,2,\ldots$

$E(N) = \frac{\rho}{1-\rho} \qquad \rho = \frac{\lambda}{\mu} < 1$

$E(R) = \frac{1}{\mu - \lambda}$

verifying the formula -

$\therefore \frac{1}{\mu-\lambda} \times \lambda = \frac{\lambda}{\mu-\lambda} = \frac{\lambda/\mu}{1-\frac{\lambda}{\mu}} = \frac{\rho}{1-\rho}$

LITTLE'S Formula

$\boxed{\lambda E(R) = E(N)}$ always true for $M|M|1|\infty$.

rate → av. arrival rate

time → av. time spent

expected no. in the system. → av. no.

These results are used at level of **mean**

- mean arrival rate
- mean time spent
- mean no. of —

know any two calculate the third one.

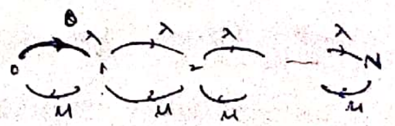
☆

→ Little's formula is actually applicable to all other models too listed above, with small changes.

2) $\underline{M|M|1|N}$

$\pi_n = ?$

At stationary $\quad 0 = \pi Q$



$Q =$

$\pi_0 = \frac{1}{1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2} + \cdots \frac{\lambda^n}{\mu^n}}$

from birth-death process notes

$\pi_n = \frac{\lambda^n}{\mu^n} \cdot \pi_0 = \frac{\lambda^n}{\mu^n} \cdot \frac{1}{\frac{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}{1-\frac{\lambda}{\mu}}}$

$\rho^n \cdot \frac{1-\rho}{1-\rho^{N+1}} = \pi_n \qquad \rho = \frac{\lambda}{\mu}$

$n = 0,1,2,\ldots,N$

→ doesn't matter if $\rho < 1$ or ≥ 1 - -

→ when $\underline{\rho = 1}$ $\quad \boxed{\pi_n = \frac{1}{N+1}}$

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→ probability that people are not allowed anymore = ~~[illegible]~~ = prob that system is full = π_n.

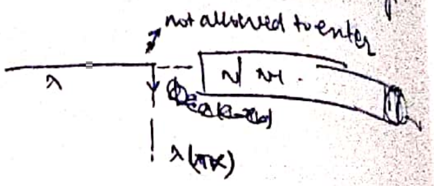
arrival rate $= \lambda$ (~~that is~~ at N to $N+1$ doesn't exist)

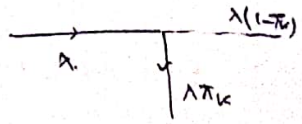
thus $\lambda\pi_n$ people not allowed to enter on avg.

$$\boxed{\lambda_{eff} = \lambda(1-\pi_k)}$$

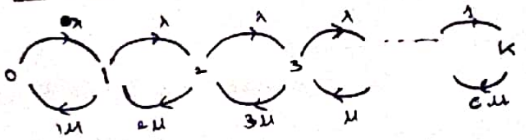
~~When~~ when $M/M/1/\infty$, $\lambda_{eff} = \lambda$ bcuz no restriction on entering

$$\boxed{\lambda_{eff} \cdot E(X) = E(N)}$$





3) $\underline{M/M/c/K}$ $c \le K$



death rates $= c\mu$ for $n \ge c$
$= n\mu$ for $n \le c$

→ c servers. if customers $\ge c$,
then as soon as any one of c
gets done, death occurs $\min(1 - - c) = c\mu$

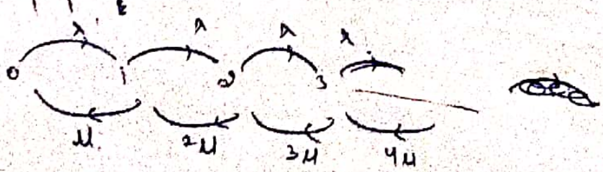
→ when customers $< c$, then $\min(- - n) = n\mu$.
↓
all are getting served.

→ λ same for all only bcuz population is infinite here.

→ we have implicitly assumed that no two servers finish together
Not assumed actually, for an exp distrib this prob is always 0

$\pi_n = \{ - -$ $i \le n < c$
$c \le n \le K$

4) $M/M/\infty$



6 Nov

Steady state prob.

$0 = \pi Q$

$0 = (\pi_0\ \pi_1\ \pi_2\ \pi_3\ \pi_4 \ldots) Q$

$-(\lambda+\mu)\pi_0 + \mu\pi_1$

$$\boxed{\pi_0 = \frac{\mu}{\lambda+\mu}\pi_1}$$

$-(\lambda+\mu)$ λ 0 0 0
μ $-(\lambda+2\mu)$ λ 0 0
0 2μ $(-\lambda+3\mu)$ λ ...

$\lambda\pi_0 - (\lambda+2\mu)\pi_1 + 2\mu\pi_2 = 0$

$\lambda\pi_0 \cdots$

$$\pi_0 = \frac{1}{1+\rho+\frac{\rho^2}{2!}+\frac{\rho^3}{3!}\cdots} \qquad \pi_n = \frac{\pi_1 \pi_2 \cdots \pi_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}\pi_0$$

$$\pi_n = \frac{e^{-\rho}\rho^n}{n!} \quad n=0,1,2,\ldots$$ — Poisson dist with parameter ρ.

$E(N) = \rho \qquad var(N) = \rho$

$E(R)$ = av ~~w~~ spending time

— ~~so~~ servers, thus waiting time $=0$

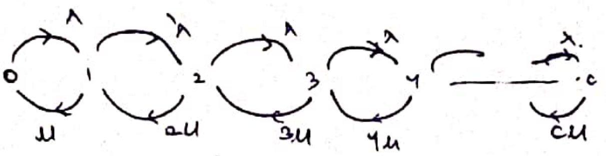
$E(R)$ = av service time.

all services happen with exp (μ). thus $E(R) = av(exp(\mu)) = \frac{1}{\mu}$.

— ~~R~~ distribution of R = dist of service = exp(μ)

— $E(Q)=0$ (waiting time)

5) M|M|c|c



$$\pi_0 = \frac{1}{1+\rho+\frac{\rho^2}{2!}+\cdots\frac{\rho^c}{c!}} \qquad \pi_n = \frac{\rho^n}{n!}\cdot\frac{1}{\ldots} \quad n=0,1,2,3\ldots$$

$$\boxed{\lambda_{eff} = \lambda(1-\pi_c)}$$

Tut 10

Q16. M|M|c|c — M|M|4|4 $\quad \lambda = \quad$ (take in min) $\quad \mu = \quad \frac{1}{\mu} = 2.5$

Q17. M|M|10|∞ condition for ergodic $\frac{\lambda}{10\mu} < 1$

d) — $\frac{\lambda}{12\mu} < 1$ — or show fast service should be done so that system under control.

$$\text{var}(X_1 + \_\_ + X_n) = \sum_{i=1}^{n} \text{var}(X_i) + \sum_i \sum_{j \neq i} \text{cov}(X_i, X_j)$$

If X, Y are independent r.v.s. — mutually ind.

$$\text{var}(X_1 + \_\_\_ + X_n) = \sum_{i=1}^{n} \text{var}(X_i)$$

~~INEQUALITIES~~ CONDITIONAL EXPECTATION

Defn: Let (X,Y) be a 2-dim r.v. Then the ~~exp~~ condition expectation of the r.v X given $Y=y$ is defined as–

$$E(X/Y=y) = \begin{cases} \sum x_i\, P(X=x_i \mid Y=y) & (X,Y) - \text{discrete} \\ \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx & (X,Y) - \text{continuous.} \end{cases}$$

provided RHS exists

① f^n of y, $E(X|y)$ is the value of rv $E(X|Y)$

② $E(E(X/Y)) = \int_{-\infty}^{\infty} E(X/y)\, f_Y(y)\, dy$ $\quad (X,Y)$ - cont

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx \right) f_Y(y)\, dy.$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x\, \frac{f_{X,Y}(x,y)}{f_Y(y)}\, dx \right) f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x\, f_{X,Y}(x,y)\, dx \right) dy$$

$$= E(X)$$

③ When X, Y are independent r.v.s–

$$E(X/y) = E(X)$$

4) $X_1, X_2, X\_\_\_ \; X_n \_\_\_$

$$E(X_{n+1} / X_1 = x_1, X_2 = x_2, X_3 = x_3 \_\_\_ X_n = x_n) \quad n = 1, 2, \_\_\_.$$

$$= x_n$$

$\{X_n, n=1,2,\_\_\_\}$ satisfy martingale property

→ Stochastic process satisfying the Martingale property is called martingale ___ ..

→ **Random sum :**

$X = ~~X_0~~, X_1 + X_2 + \text{———} X_n$ $\quad$ +ve Integer r.v. $1, 2 \text{———}$

↓ last

$X = X_1 + X_2 + \text{— —} X_N$

↳ R.V. it takes value n $\quad n = 1, 2, 3 \text{———}$

distribution of X?

eg. $X_i \sim \mathrm{Exp}(\lambda)$ $\quad i = 1, 2, \text{— —}$ $\quad$ iid

$P(N=n) = (1-p)^{n-1} p \cdot n = 1, 2, \text{——}$

$f_X(x) = \sum_n f_{X/N=n}(X/N)\, P(N=n)$

$E(X) = E[E(X/N)] = \sum_n \cdot E(X/N=n) \cdot P(N=n))$

INEQUALITY

$(\Omega, \mathcal{F}, P)$

$\rightarrow P(A)$

$- P(X \in B)$ dist of X is known exactly

Now $P(X \in B)$ — inequality ($\leq, \geq$)

defn: let X be a non-negative r.v. with $E(X)$ exist and is known.

for fixed $t > 0$,

$$\boxed{P(X > t) \leq \frac{E(X)}{t}}$$ — Markov inequality

for t,

Define $Y = \begin{cases} 0 & x \leq t \\ t & x > t \end{cases}$

$P(Y=0) = P(X \leq t)$

$P(Y=t) = P(X > t)$

$E(Y) = t\,P(X > t)$

$X \geq Y$

$E(X) \geq E(Y)$

$E(X) \geq t\,P(X > t)$

Chebyshev Inequality

Let X be a rv with ~~E(X)~~ $E(X) = \mu$

$var(X) = \sigma^2$ exists and are known

Then, for any +ve no. t,

$$\boxed{P\{|X - c| > t\} \leq \frac{E\{(X-c)^2\}}{t^2}}$$

suppose $c = \mu$

$$P\{|X - \mu| > t\} \leq \frac{Var(X)}{t^2}$$

or

$$P\{|X - \mu| < t\} > 1 - \frac{\sigma^2}{t^2}$$

Take $Y = (X-\mu)^2$ then apply mark inequality.

~~also ...~~

~~$|X-\mu| < t \cup |X-\mu| > t$~~

eg. let x be a rv with $E(x) = \frac{1}{2}$ and $var(X) = \frac{1}{12}$

Find lower bound for

$P\{|x - \frac{1}{2}| < 2\sqrt{\frac{1}{12}}\}$ $\quad 1 - \frac{E(x-\mu)^2}{t^2} = 1 - \frac{1/12}{4/12}$ $\quad \frac{3}{4}$

$P \text{ - - } > (3/4)$

X = cont.

$X = U(0,1)$

$f_X = 1$ $\quad F_X = x$ $\quad x \in (0,1)$

$P\{|X - \frac{1}{2}| < 2\sqrt{1/12}\}$

$P\left(\frac{1}{2} - 2\sqrt{\frac{1}{12}} \le X < 2\sqrt{\frac{1}{12}} + \frac{1}{2}\right) = 1$ as $\quad 2\sqrt{\frac{1}{12}} + \frac{1}{12} > 1$

$\frac{1}{2} - \sqrt{\frac{1}{3}} < 0$

Limiting Probability.

central limit Theorem:

Thm. let ~~E(x), E(x)~~ $X_1, X_2 \text{---}$ be a sequence of indepenpendent rvs defined on the probability space (Ω, F, P) with $E(X_i) = \mu_i$ $i = 1, 2, \text{---}$

$var(X_i) = \sigma^2 > 0$ $\quad i = 1, 2, \text{---}$

Define,

$$Z_n = \frac{\sum_{i=1}^{n} X_i - E\left(\sum_{i=1}^{n} x_i\right)}{\sqrt{var\left(\sum_{i=1}^{n} x_i\right)}} \quad n = 1, 2, \text{---}$$

— ~~seq~~ superimposition of all rv.

— makes it simpler

☆ in conv. distribution

Then, for large n, Z_n approaches (approximately) standard normal distribution.

i.e $\quad P(Z_n \le z) \simeq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz$ $\quad [Z_n \longrightarrow Z]$ $\quad Z \sim N(0,1)$

13Sep

Limiting Distributions

— useful in control system studies.

E Ω $P(A)$

$X_1, X_2 \text{---} X_n, \text{---}$

what is the distribution of X_n as $n \to \infty$ or for large n.

Idea similar to seq. of real no., $a_1, a_2, a_3 \text{---} a_n$ - as what value does ~~a_n~~ a_n converge to

here diff is we have attached a distribution too.

$X_1, X_2 \text{---} X_n \longrightarrow X$. (converges to X)

→ it may not converge

→ if converges then converges to X.

$X_n \longrightarrow X$

Notation

1) $X_n \xrightarrow{P} X$ — X_n converges to X in probability

given $\varepsilon > 0$

if $P\{|X_n - X| > \varepsilon\} \to 0$ as $n \to \infty$

2) $X_n \xrightarrow{d} X$ — in distribution

if $F_{X_n} \to F_X$ as $n \to \infty$

Maybe after applying $\lim_{n\to\infty}$, the func we receive may not be a distribution func..

3) $X_n \xrightarrow[\text{moment}]{r^{th}} X$ — in r^{th} moment.

if $\lim_{n\to\infty} E\left(|X_n - X|^r\right) = 0$

4) $X_n \xrightarrow{a.s.} X$ — in almost surely

if $P\{\lim_{n\to\infty} X_n = X\} = 1$

— diff from first. in 1) we find seq. $A_n = \{w \mid X_n - X > \varepsilon\}$ ~~$\{w \mid X_n = X\}$~~ and then take $\lim_{n\to\infty} A_n$

in 4) $A_n = \{w \mid X_n = X\}$

So in 1) we are finding the prob of that sequence and then taking limit. ~~to to~~ Thus we eventually take limit on a sequence of real no.s.

in 4) we find limit on r.v and at the end take probability.

1) - weak convergence - weak law of

4) - strong convergence SLLN

↳ prob will either turn out to be 0 or 1. — $\boxed{0/1 \text{ law}}$ *

bcuz either $\lim_{n\to\infty} X_n = X$ or not.

eg. let $\{X_n ; n = 1, 2, \ldots\}$ be a sequence of r.v.s s.t.

$P\{X_n = 0\} = 1 - \frac{1}{n}$; $P\{X_n = n\} = \frac{1}{n}$ $n = 1, 2, \ldots$

check $X_n \xrightarrow{P} 0$ $P(X = 0) = 1$

for $\varepsilon > 0$ $P\{|X_n - 0| > \varepsilon\} = \begin{cases} \frac{1}{n} & \varepsilon < n \\ 0 & \varepsilon \geq n \end{cases}$

as $n \to \infty$

$\lim_{n\to\infty} P\{|X_n - 0| > \varepsilon\} = 0$

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eg- let $\{X_n, n = 1,2 \ldots\}$ be a seq of iid. rvs with .

$E(X_i) = \mu$ and $Var(X_i) = \sigma^2$ $i = 1,2 \ldots$

check $\frac{S_n}{n} \to \mu$ in 2nd order moment.

$$S_n = \sum_{i=1}^{n} x_i$$

$$E\left(\frac{S_n}{n}\right) = \frac{E(\sum x_i)}{n} = \frac{n\mu}{n} = \mu$$

$$\lim_{n\to\infty} E\left(\left|\frac{S_n}{n} - \mu\right|^2\right) = 0$$

So $E\left|\frac{S_n}{n} - \mu\right|^2 = Var\left(\frac{S_n}{n}\right)$ $= \cancel{\Sigma\left(\frac{S_n}{n} - \mu\right)^2}$ $n-\mu$

Σ

$$E\left(\left(\frac{S_n}{n}\right)^2\right) - \mu^2 \qquad \frac{E\left((\Sigma x_i)^2\right)}{n^2} =$$

---

11 Sep

LAW OF LARGE NUMBERS

1) Bernoulli Law of large no.s

Thm let E be a random experiment. and A be an event. Consider n independent trials.

$$P(A) = p.$$

Define n_A = no. of times the event A occurs in n trials. $n_A \in [0,n]$

$$f_A = \frac{n_A}{n} \qquad E(f_A) = p$$

$$n_A \sim B(n,p) \qquad Var(f_A) = \frac{p(1-p)}{n}$$

Apply Chebyshev Inequality -

$$P\{|f_A - p| > \epsilon\} \le \frac{p(1-p)}{n\epsilon^2}$$

$$\lim_{n\to\infty} P\{|f_A - p| > \epsilon\} \le \lim_{n\to\infty} \frac{p(1-p)}{n\epsilon^2}$$

$$\underbrace{f_A}_{\text{relative frequency}} \xrightarrow{P} \underbrace{p}_{\text{theoretical probability}}$$

eg:- E: throw unbiased die infinitely
A: getting '6'

what is the min 'n' s.t $P\{|f_A - p| < \varepsilon\} \geq 0.95$ when $\varepsilon = 0.01$ $p = 1/6$

Using inequality $P\{ \text{——} \} \geq \frac{E(f_A - p)^2}{\varepsilon^2}$ $\frac{\text{Var } f_A}{\varepsilon^2}$

$P(\text{——} > \varepsilon\} \leq 0.95$

$f_A = \frac{n_A}{n}$

$p = 1/6$

$n_A \sim B(n, p)$

Mean $= p = 1/6$

Var $= \frac{\text{Var}(n_A)}{n^2} = \frac{p(1-p)}{n}$

$P \text{——} \geq \frac{p(1-p)}{n\varepsilon^2}$

≥ 0.95

$n < \frac{p(1-p)}{\varepsilon^3}$

$= \frac{5}{36 \times} \frac{10^6}{0.950}$

$n = \frac{p(1-p)}{(0.05)\varepsilon^2} = \boxed{27778}$

Proof - Central Limit Th

Assumptions- ① X_i $i = 1, 2$ —— ~~i.i.d.~~ iid r.v.s

$E(X_i) = \mu$

$\text{Var}(X_i) = \sigma^2$ $\forall i$

② MGF of X_i exists $\forall i$ — all moments exist.
— very restrictive r.v.

$$M_{Z_n}(t) = M_{\frac{\Sigma X_i - n\mu}{\sigma\sqrt{n}}}(t)$$

$$= E\left(e^{\left(\frac{\Sigma X_i - n\mu}{\sigma\sqrt{n}}\right)t}\right)$$

$$= e^{-\frac{n\mu t}{\sqrt{n}\sigma}} \cdot E\left(e^{\frac{1}{\sqrt{n}\sigma}(\Sigma X_i)t}\right)$$

$$= e^{-\frac{n\mu t}{\sqrt{n}\sigma}} \cdot \left[E\left(e^{\frac{1}{\sqrt{n}\sigma} X_1 t}\right)\right]^n = e^{-\frac{n\mu t}{\sqrt{n}\sigma}} \cdot \left[M_X\left(\frac{t}{\sqrt{n}\sigma}\right)\right]^n$$

$$M_X(t) = \frac{x}{1 + \frac{\mu t}{1!} + \frac{E(X^2)}{n!}t^2 + \text{——}}$$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \text{——}$

$$\ln M_{Z_n}(t) = -\frac{\sqrt{n}\mu t}{\sigma} + n\ln\left(1 + \frac{\mu t}{\sqrt{n}\sigma} + \text{——}\right)$$

$$n\left[\frac{\mu t}{\sigma\sqrt{n}} + \frac{1}{2!} \cdot \frac{(\sigma^2+\mu^2)t^2}{n\sigma^2} - \cancel{\frac{1}{2}\frac{\mu^2 t^2}{n\sigma^2}} + \text{——}\right.$$

$= \frac{t^2}{2} + (\text{——}) \rightarrow$ have n in denominator

As $n \to \infty$ $\ln M_{Z_n}(t) = t^2/2$

$M_{Z_n}(t) = e^{t^2/2}$

$Z_n \longrightarrow Z \sim N(0,1)$

→ If two mgfs are same, then distributions will be same.

18 Sep

Relaxing second assumption.

→ If mgf doesn't exist, then go with characteristic func cuz that only exists.

$t \to it$

then we get $e^{-t^2/2}$ at end which is charac func of standard normal

→ central limit th only says that it converges to a normal distribution we're done for standard.

Remarks:

1) Assumptions can be relaxed - non identical, MGF doesn't exist.
2) $X = \sum_{i=1}^{n} X_i$; for large n
3) $n \geq 30$.
4) Independent can be relaxed. → $Co(X_i, X_j) = 0$

eg- Let $X_1, X_2 ---- X_n$ --- be a sequence of iid rvs with

$P(X_i = 0) = 1-p$, $P(X_i = 1) = p$ $0 < p < 1$

Define $X = \sum_{i=1}^{10} X_i$ $p = 1/2$

Find i) $P(X \leq 8)$. exactly

ii) $P(X \leq 8)$ lower bound

iii) $P(X \leq 8)$ approximately using CLT.

$P(\sum_{1}^{10} X_i \leq 8)$

Choose any two to be zero.

$^{10}C_2 \; p^8 (1-p)^2$

i) $1 - P(X \geq 9)$

$1 - (^{10}C_1 \cdot p^9 (1-p) + p^{10})$

ii) $P(X \leq 8) \geq$ ~~$\frac{E(X-0)^2}{t^2}$~~ - ~~$P(X-8 \leq 0) >$~~ $P(|X - \text{mean}| < t) > 1 - \frac{\sigma^2}{t^2}$

Mean = follows Binomial $\sim B(10, 1/2)$

Mean = $np = 5$.

$P(|X-5| \leq 3) > 1 - \frac{\text{var}}{9}$

var $\frac{5}{2}$ $\quad 1 - \frac{5}{18}$ $\quad \frac{13}{18}$

according to chebyshev $|X-5| \leq 3$

$\Rightarrow X \in (2, 8)$

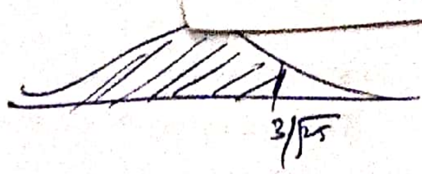
But we want 0 to 8.

Apply Markov instead

$$P(X \le t) \ge 1 - \frac{E(X)}{t} \ge 1 - \frac{5}{8} = 3/8$$

3) $$Z = \frac{\Sigma x_i - E(\Sigma x_i)}{\sqrt{var(\Sigma x_i)}} = \frac{x-5}{\sqrt{2.5}}$$

$$P(X \le 8) = P\left(\frac{x-5}{\sqrt{2.5}} \le \frac{8-5}{\sqrt{2.5}}\right)$$



Apply CLT.

$$\simeq P\left(Z \le 3/\sqrt{2.5}\right) = \phi\left(3/\sqrt{25}\right) = \phi(1.89) = 0.9706.$$

eg. Find the approx value using CLT.

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{e^{-n} n^k}{k!} \sim P(n)$$

$x_n \sim P(n)$ $\quad X_n(k) = \frac{e^{-n} n^k}{k!}$

we know $x_1 + x_2$ when $x_1 \sim P(\lambda)$, $x_2 \sim P(\lambda_2)$

$\Sigma x_i \sim P(\Sigma n)$

$\Sigma x_i \sim P\left(\frac{n(n+1)}{2}\right)$

$$\Sigma x_i \sim \frac{e^{-\frac{n(n+1)}{2}} \left(\frac{n(n+1)}{2}\right)^k}{k!}$$

Mean $= \frac{n(n+1)}{2}$

Variance $= \frac{n^2(n+1)^2}{4}$

~~X~~ No. we are not summing as seen closely k is getting summed up

$$Z_n = \lim_{n\to\infty} \frac{e^{-\frac{n(n+1)}{2}} \left(\frac{n(n+1)}{2}\right)^k - \frac{n(n+1)}{2}}{\frac{n(n+1)}{2}}$$