Data Structures & Algorithms

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Dept of Computer Sc. & Engg.

Amortized Analysis

- Vector:
  - Array, with size doubled when filled
- Count cost of each usage = 2 steps
  - 1 unit pays for the usage
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- When doubling the size from n to $2n$
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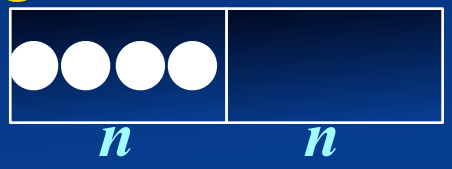

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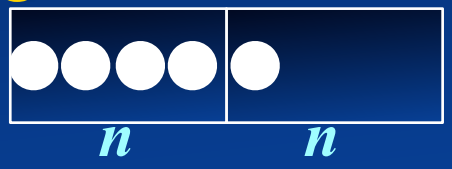

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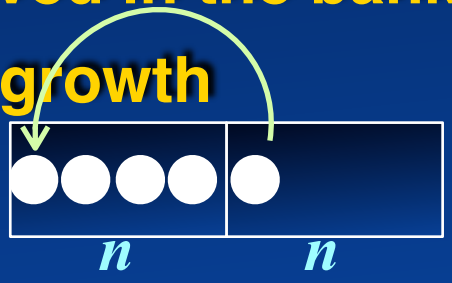

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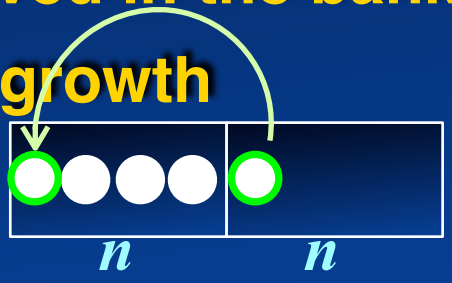

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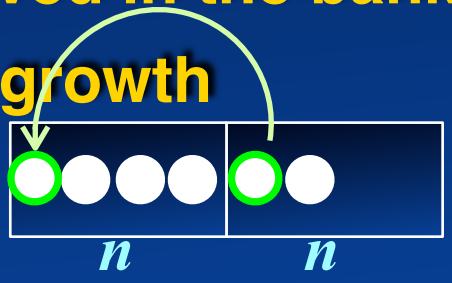

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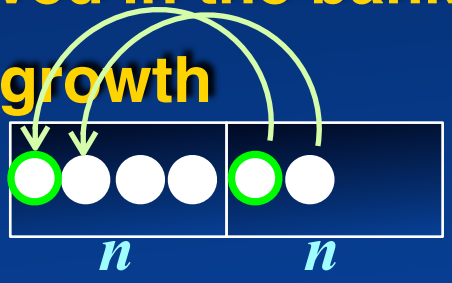

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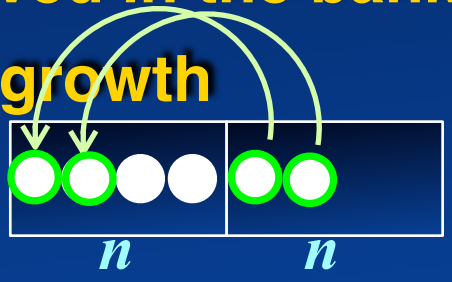

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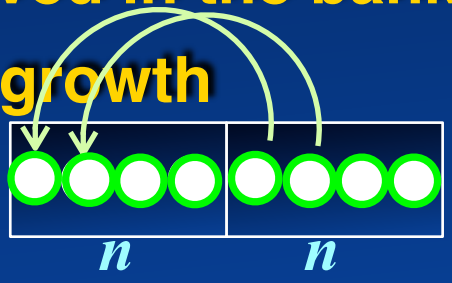

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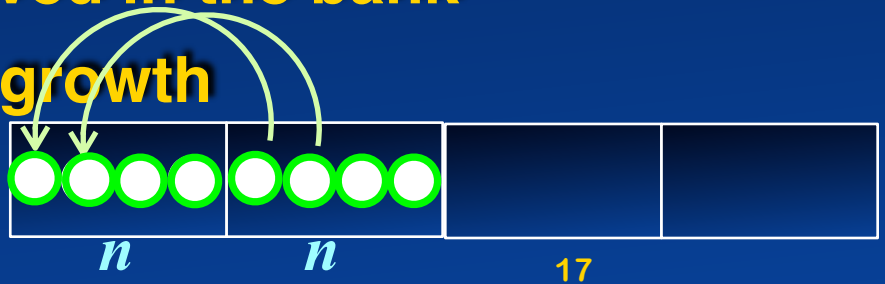

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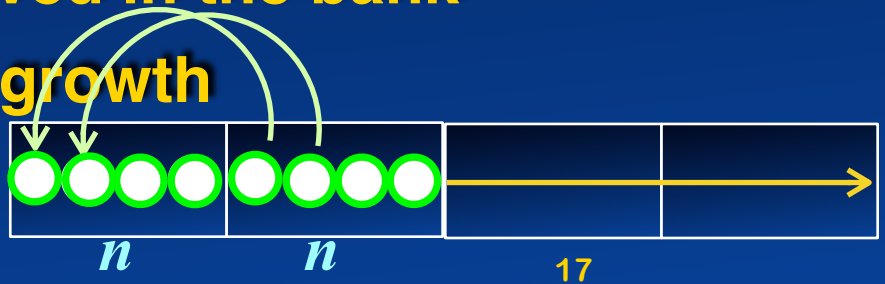

Amortized Analysis

- Vector:
  - Array, with size doubled when filled
- Count 3 units for usage of each index i
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  - 1 unit banked for later copy of index i - $n/2$
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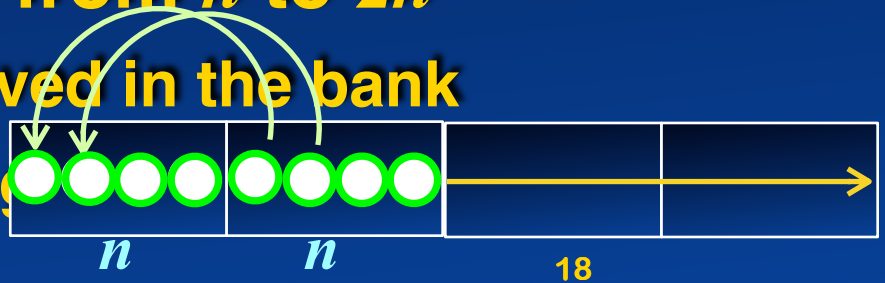

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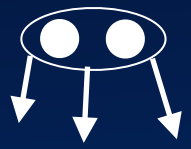

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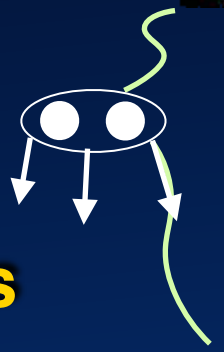

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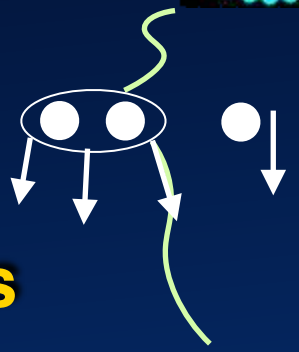

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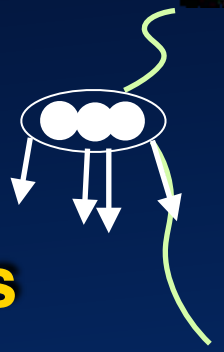

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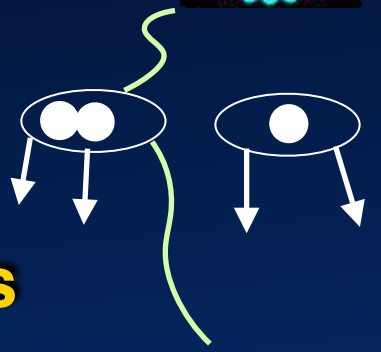

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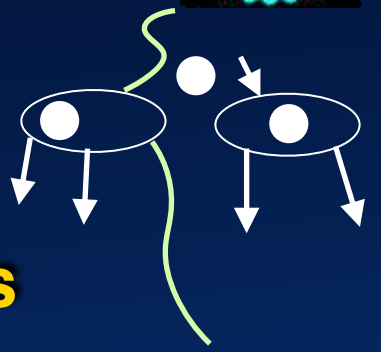

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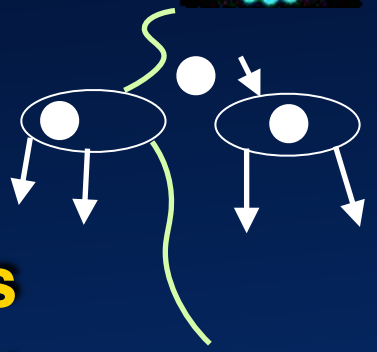

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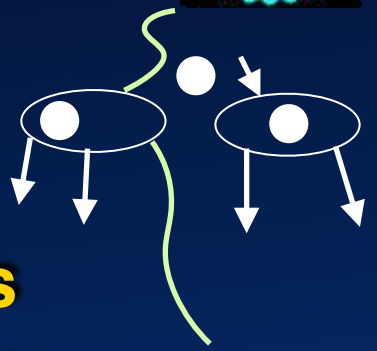

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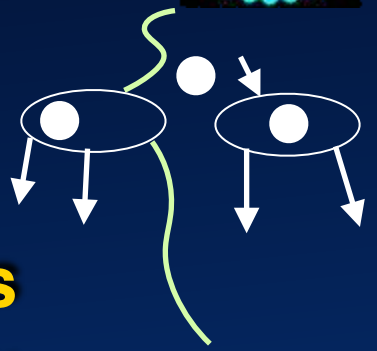

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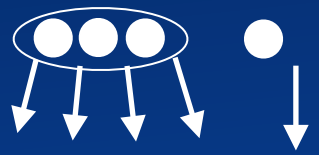

2-4 Tree

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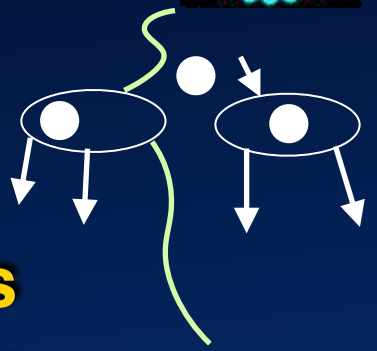

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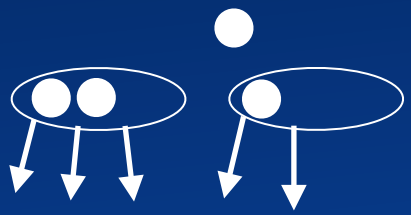

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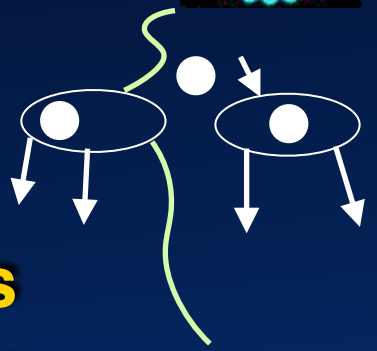

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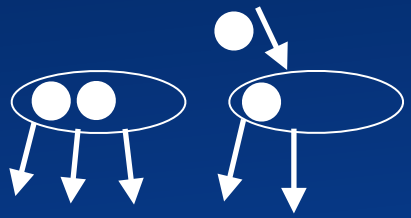

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Insert/Delete in 2-5 Tree

- 5-nodes can split on insertion
- 2 nodes can merge on deletion
- Φ = No. of 5 nodes + No. of 2 nodes
- After insertion, ΔΦ ≤ 1 – *No. of splits*
- After deletion, ΔΦ ≤ 1 – 2x *No. of merges*
  - ΔΦ ≤ n – *No. of splits* – 2x *No. of merges* = Φ
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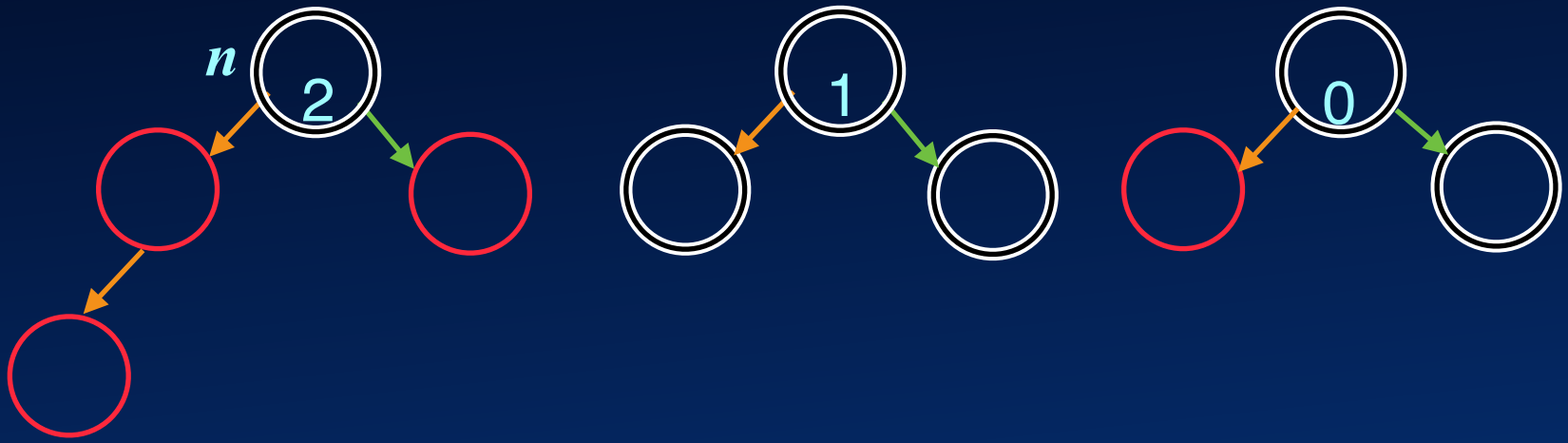
$$b = 2a + 1$$

R-B Tree Color changes

Insert



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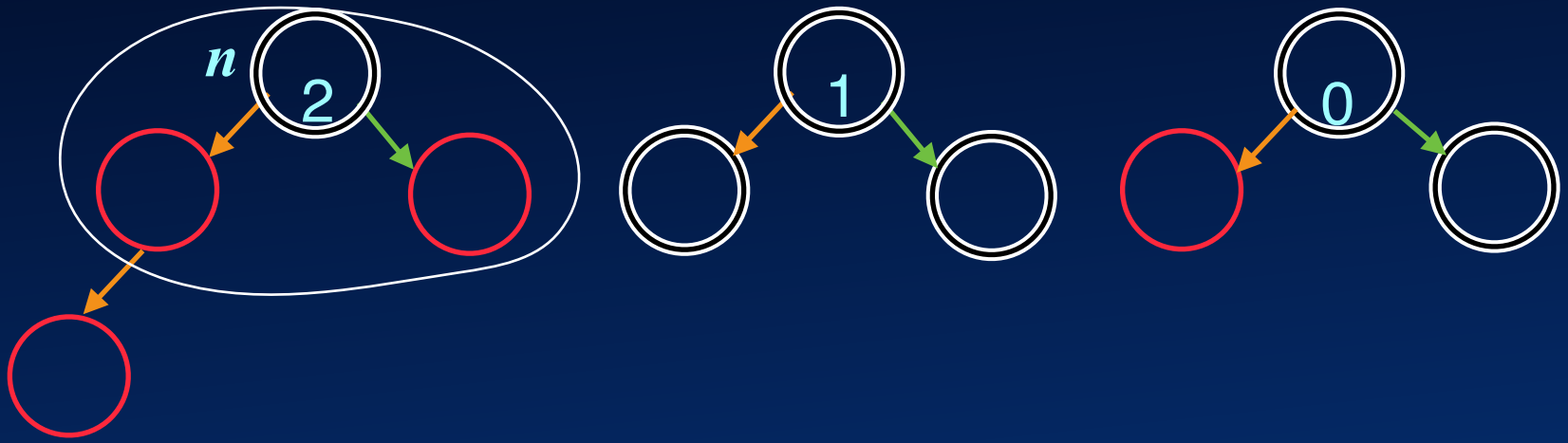

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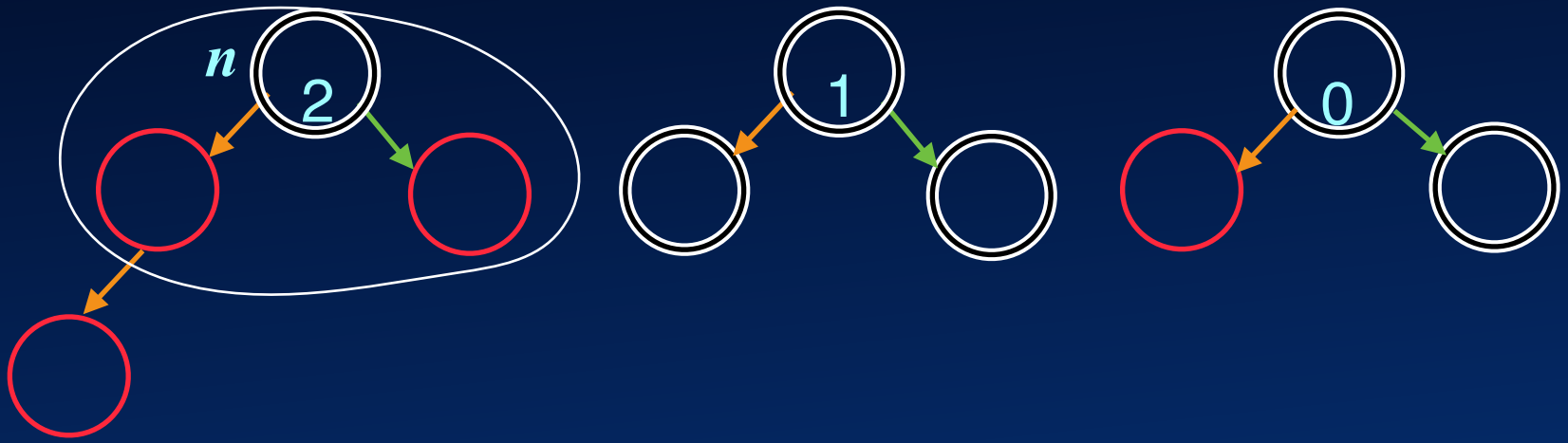

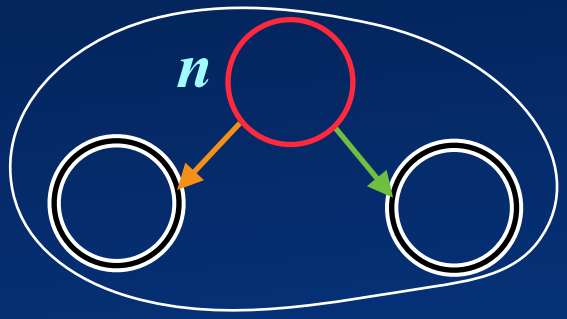

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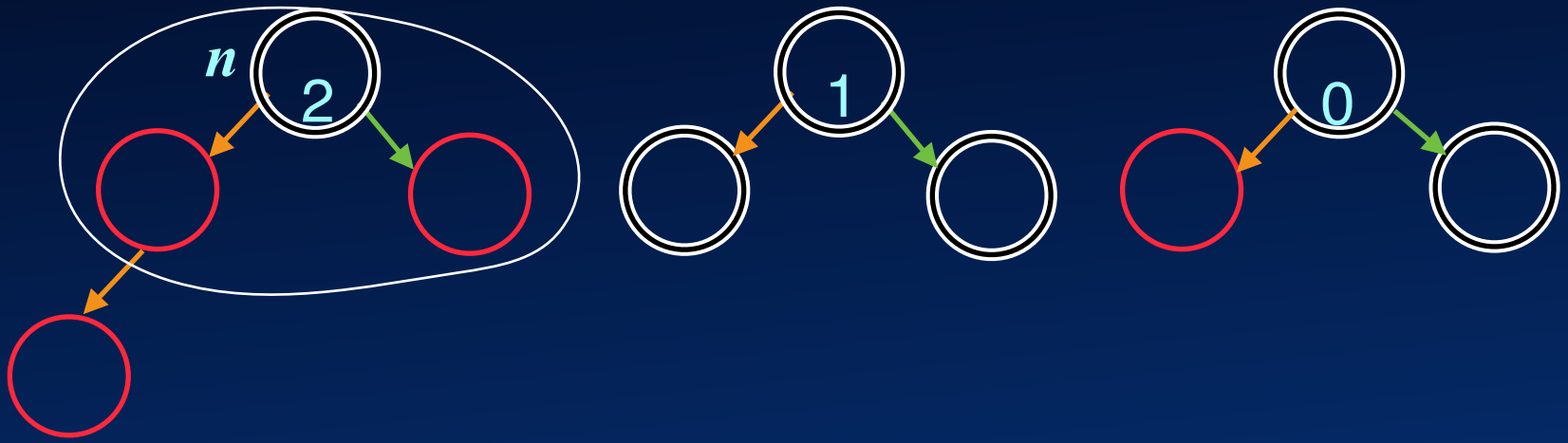

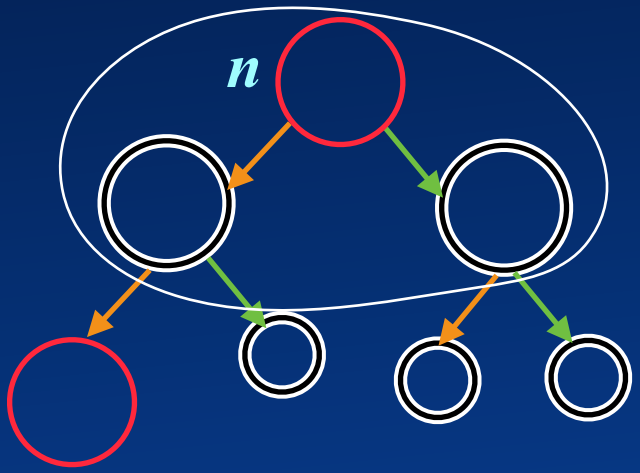

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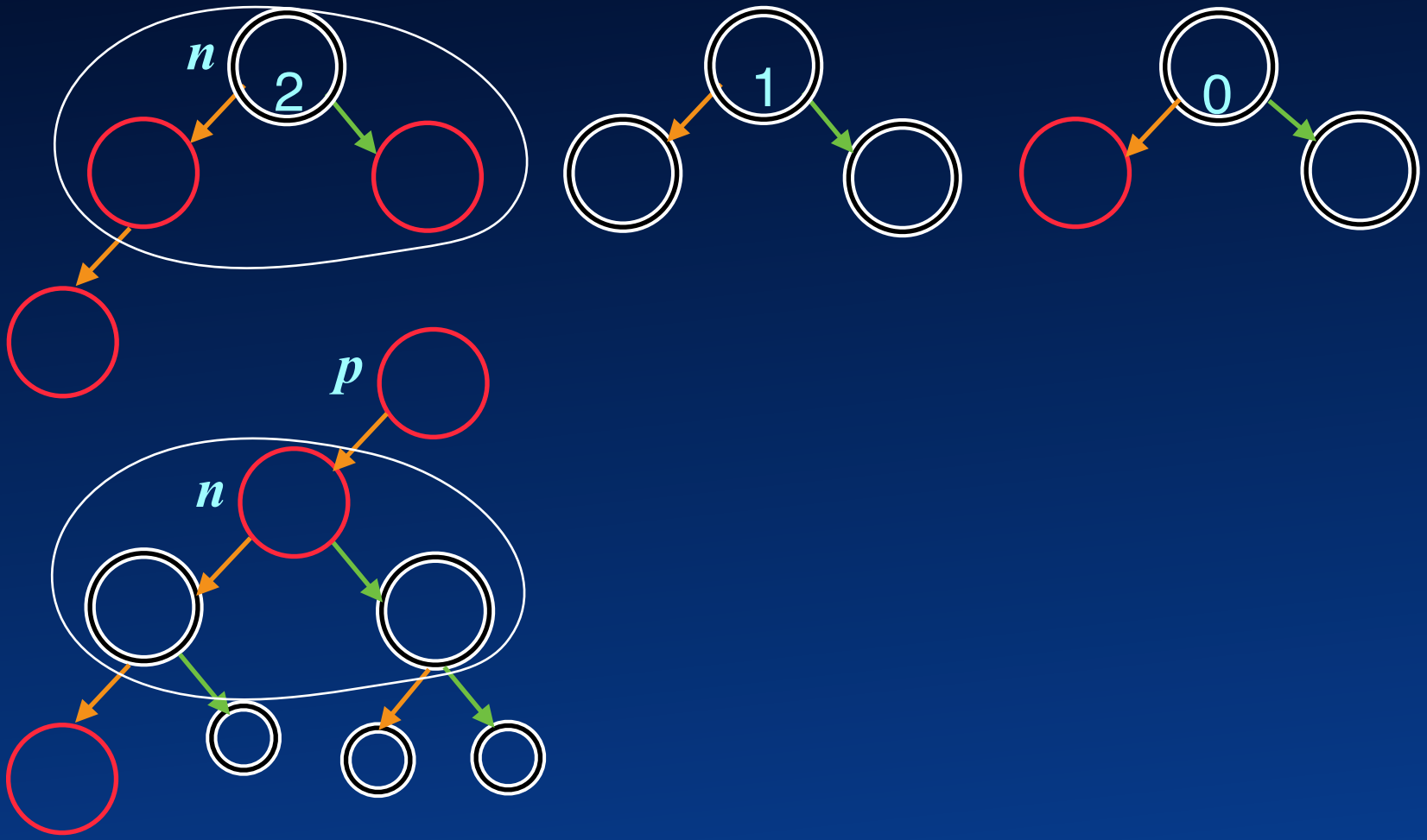

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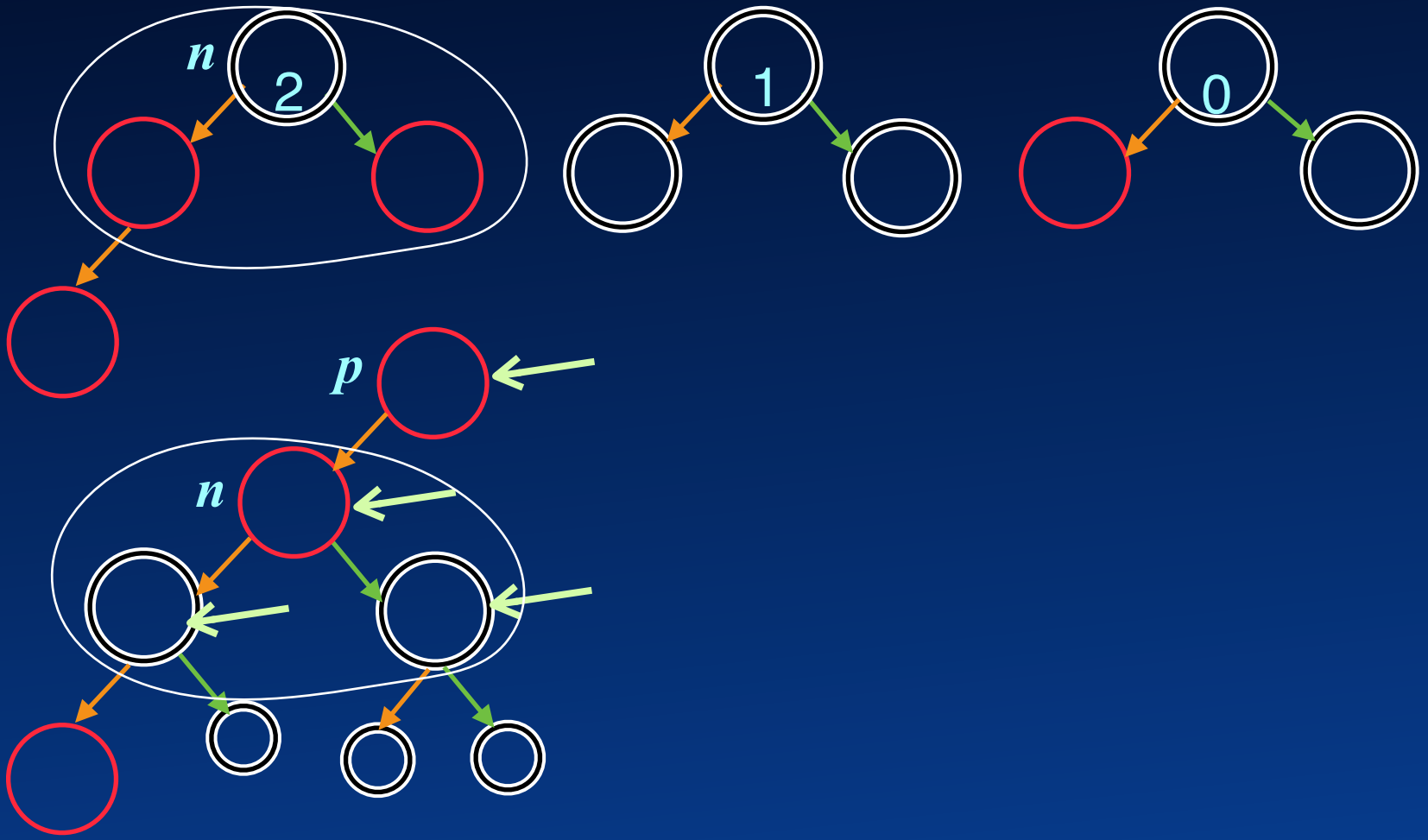
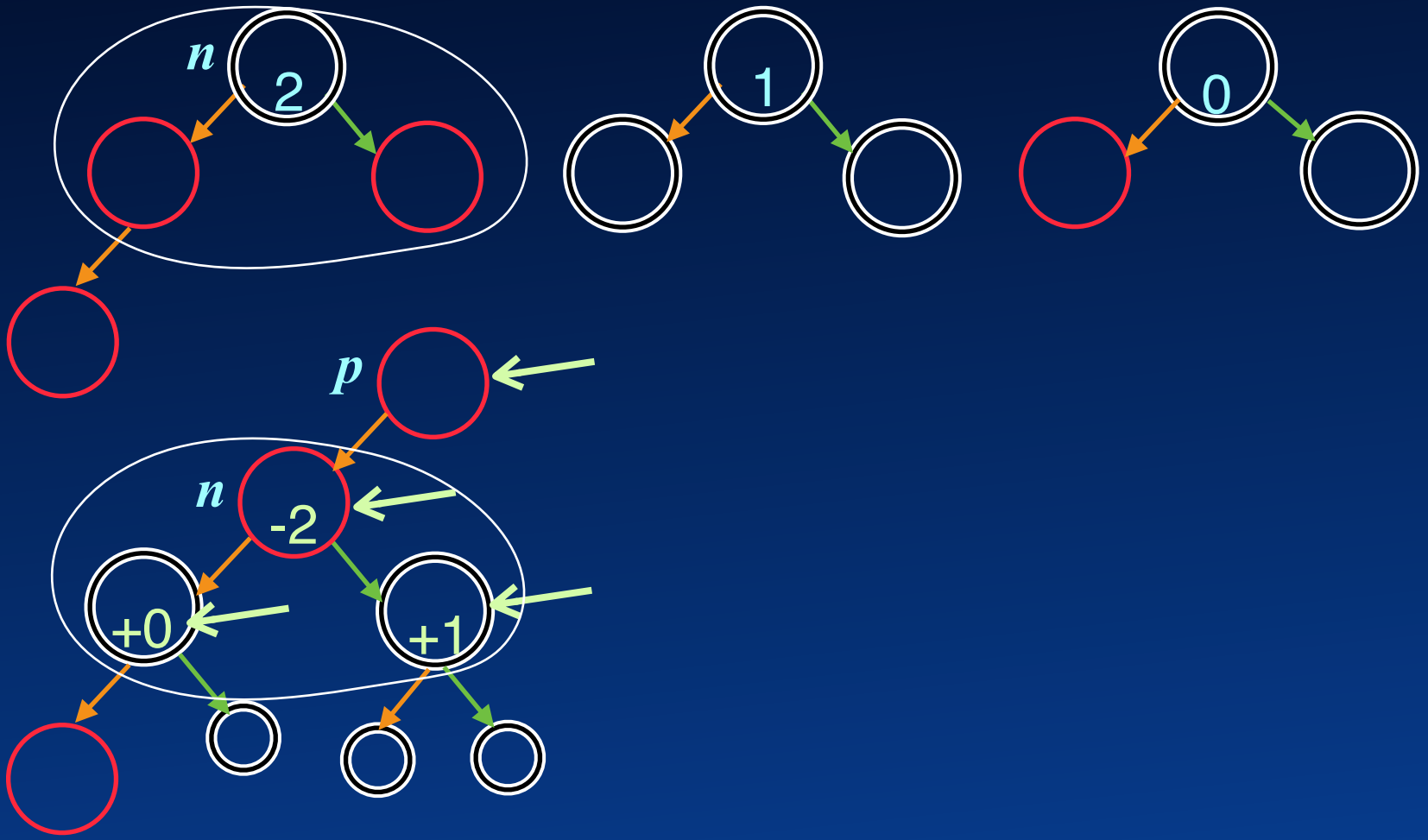


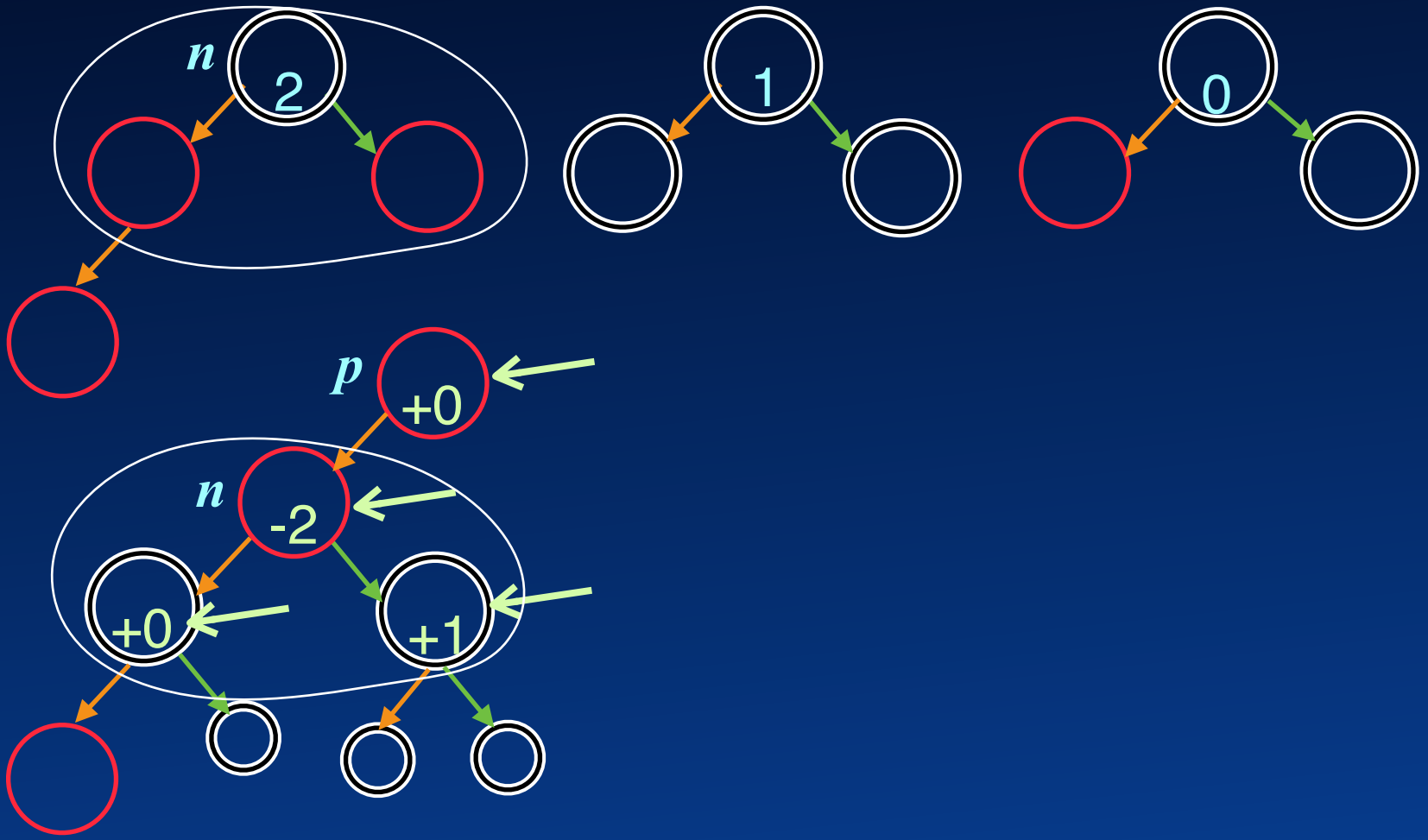
R-B Tree Color changes

Insert



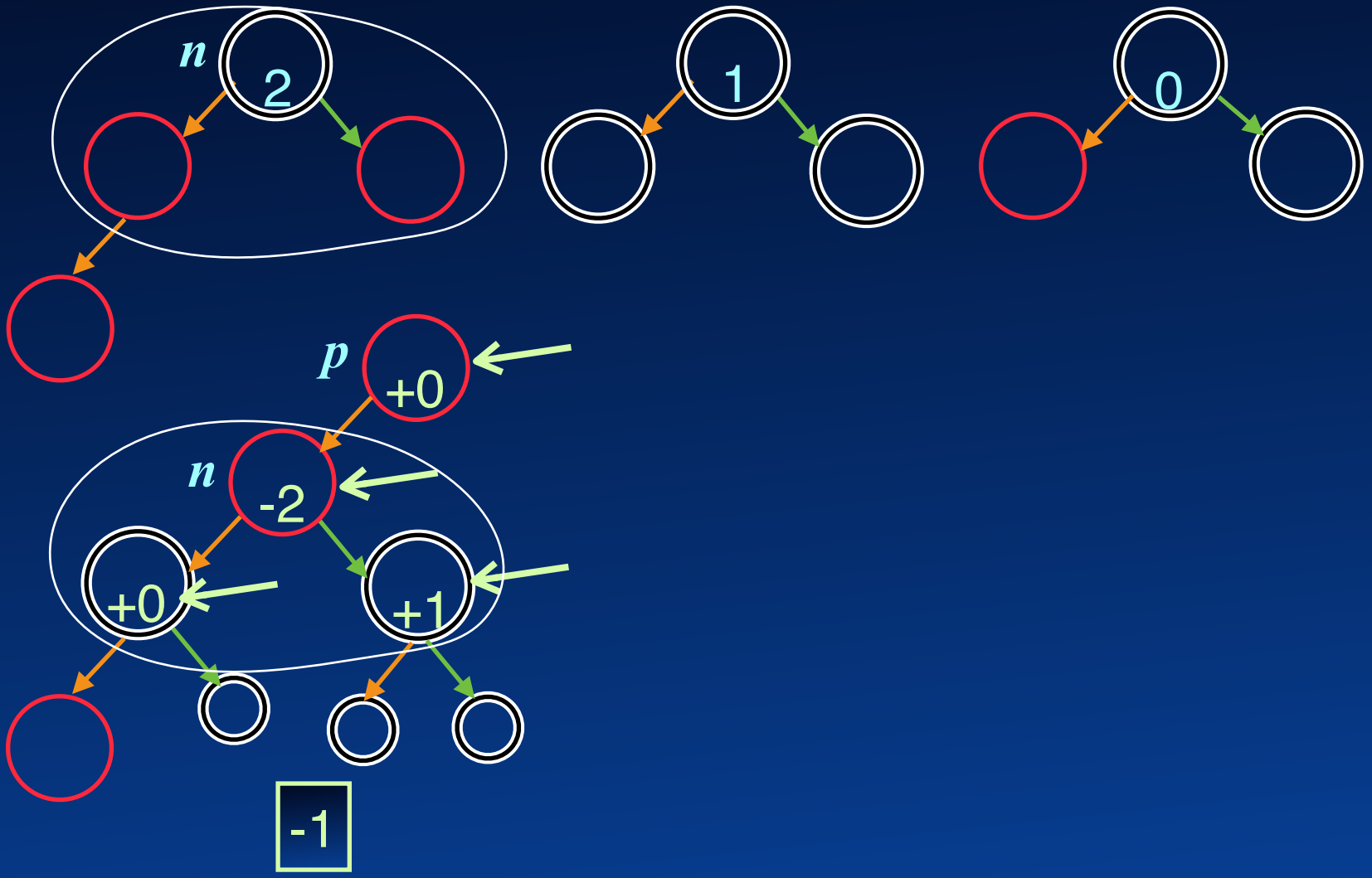
R-B Tree Color changes

Insert



R-B Tree Color changes

Insert



R-B Tree Color changes

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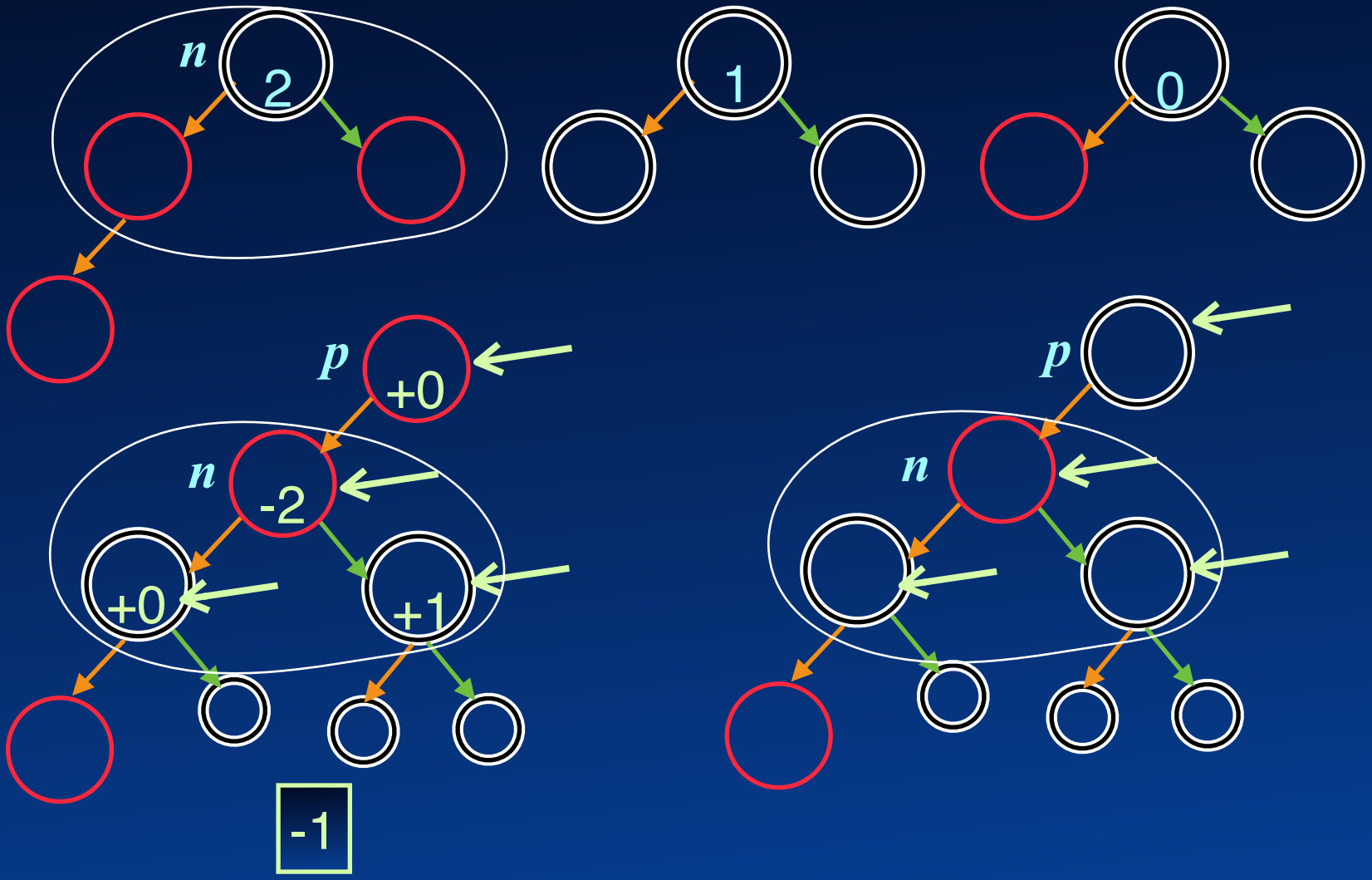

R-B Tree Color changes

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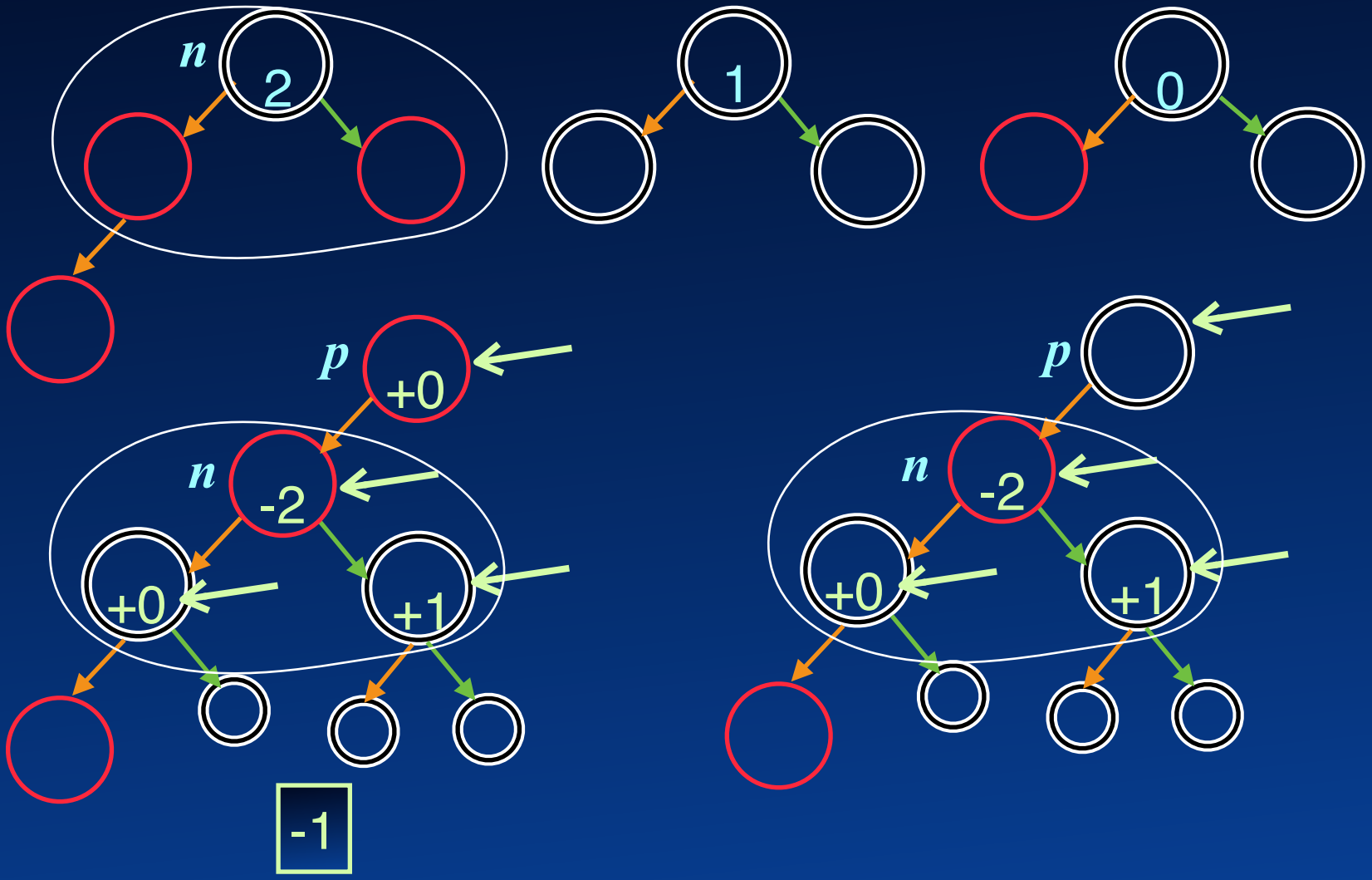

R-B Tree Color changes

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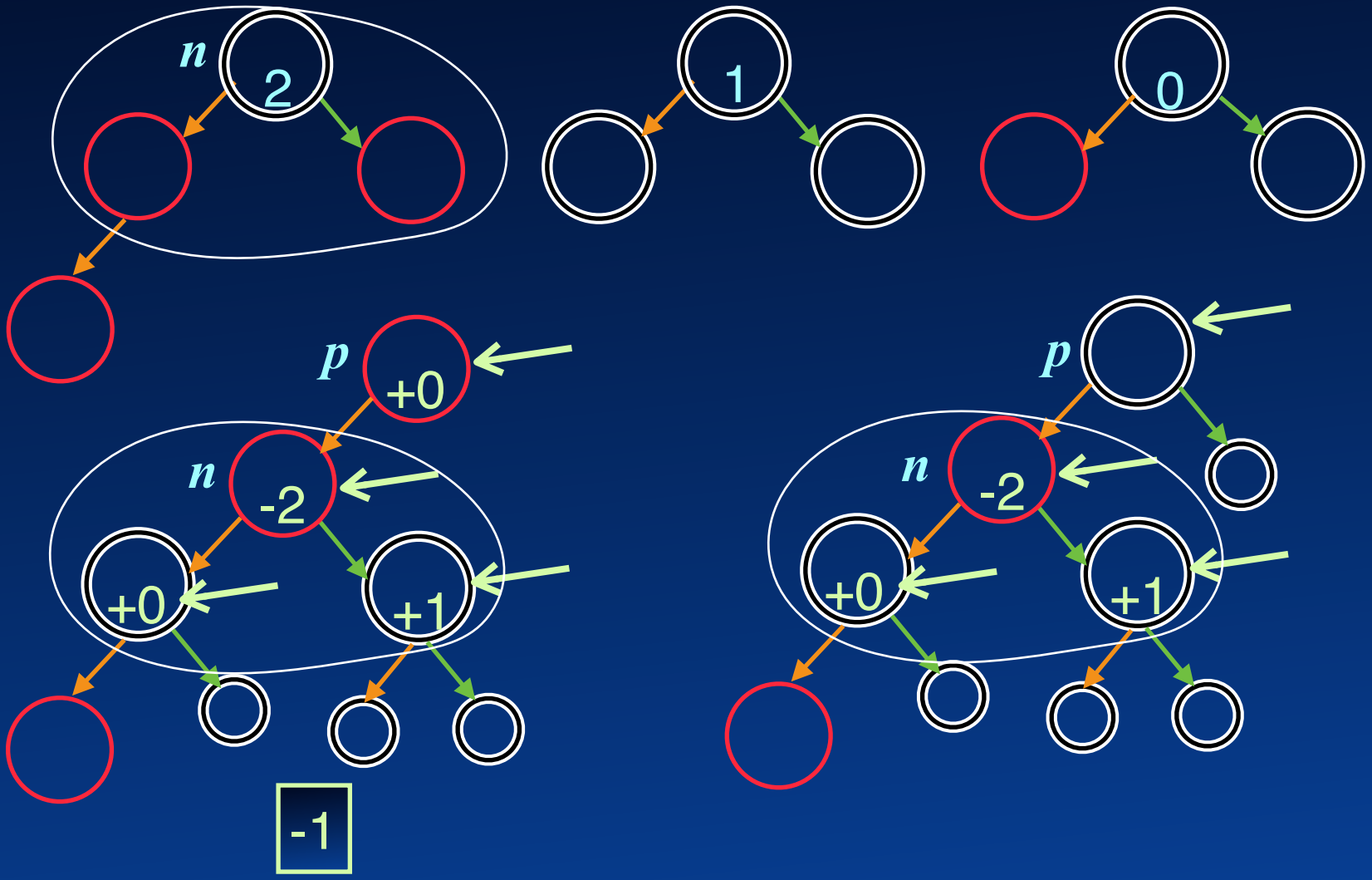

R-B Tree Color changes

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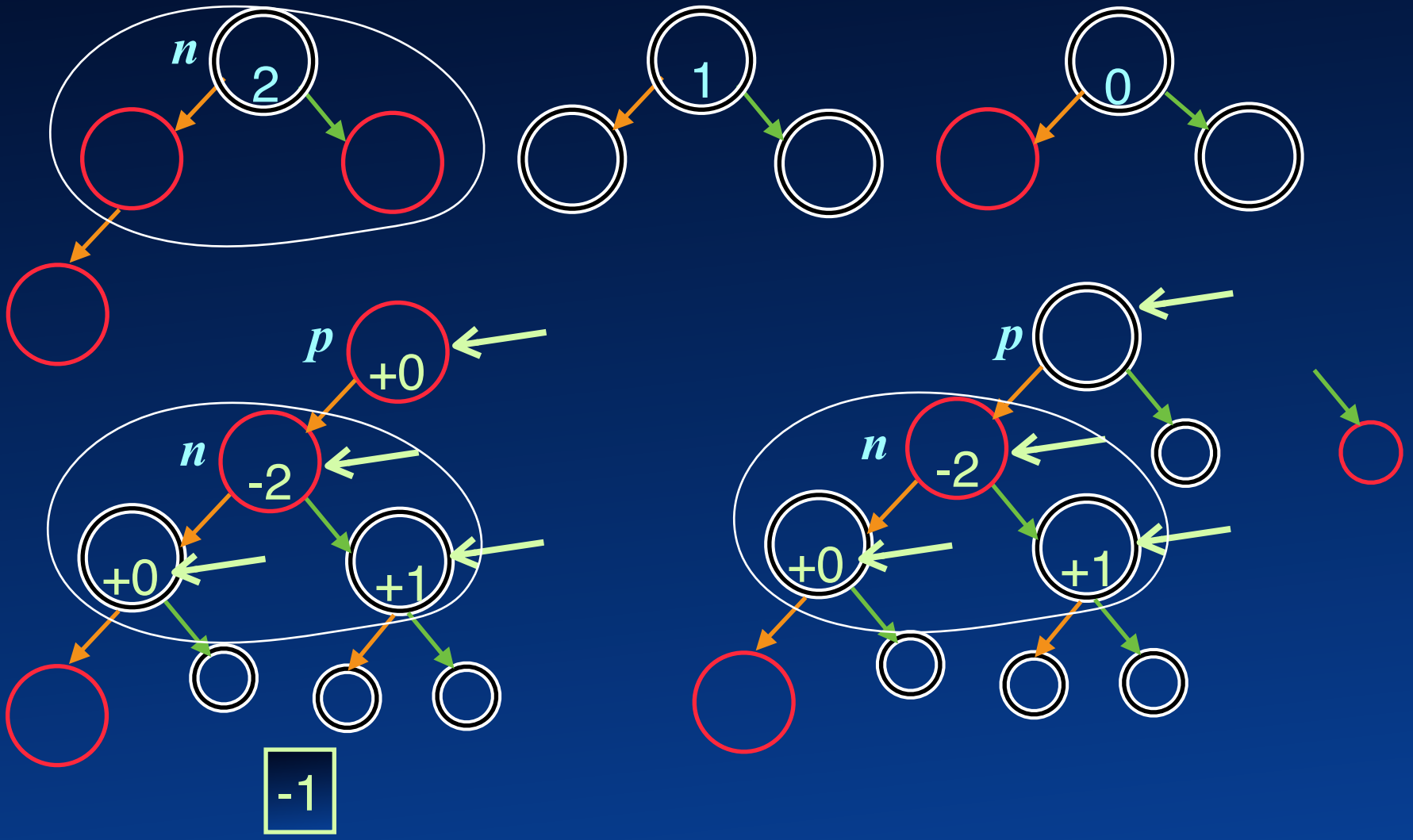

R-B Tree Color changes

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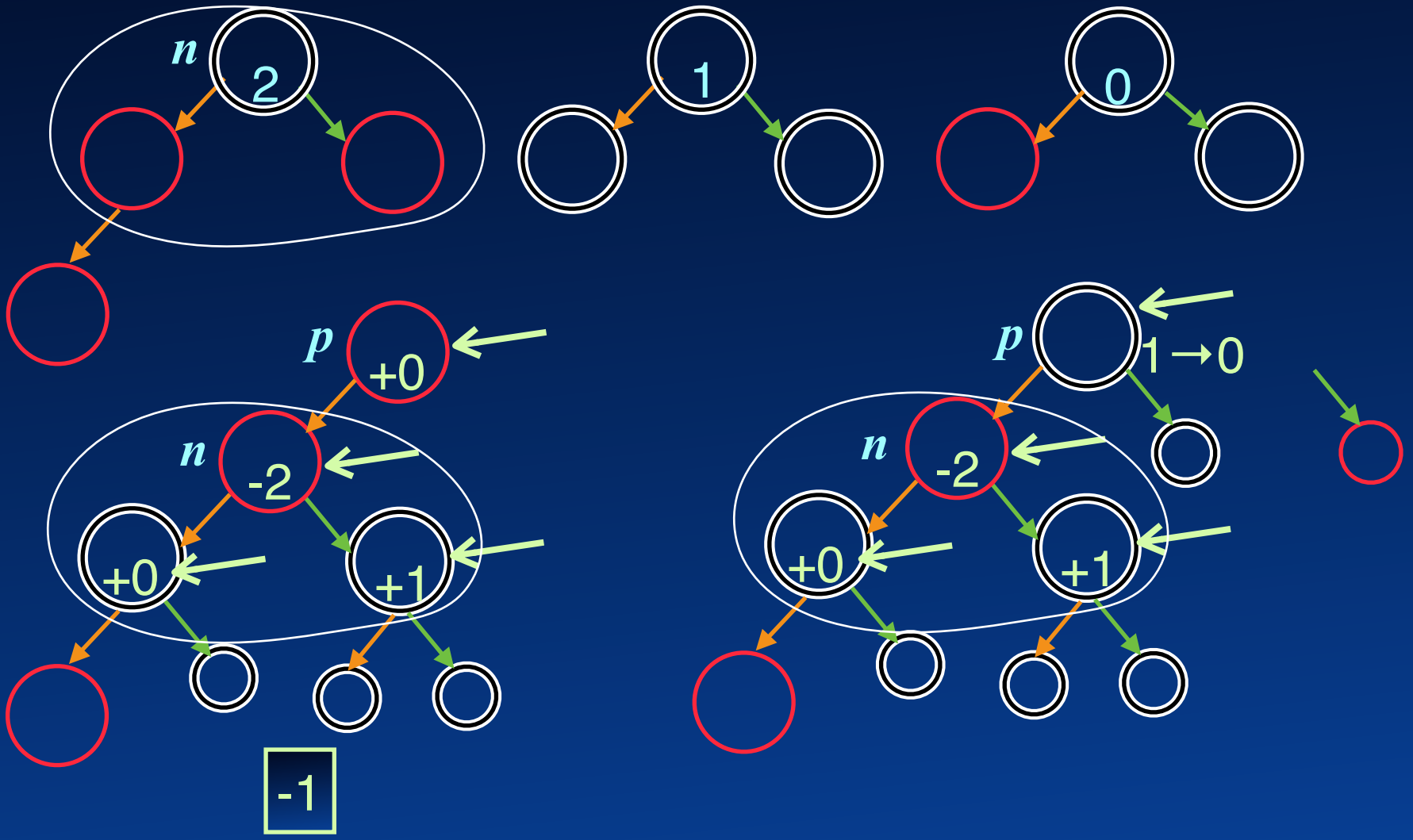

R-B Tree Color changes

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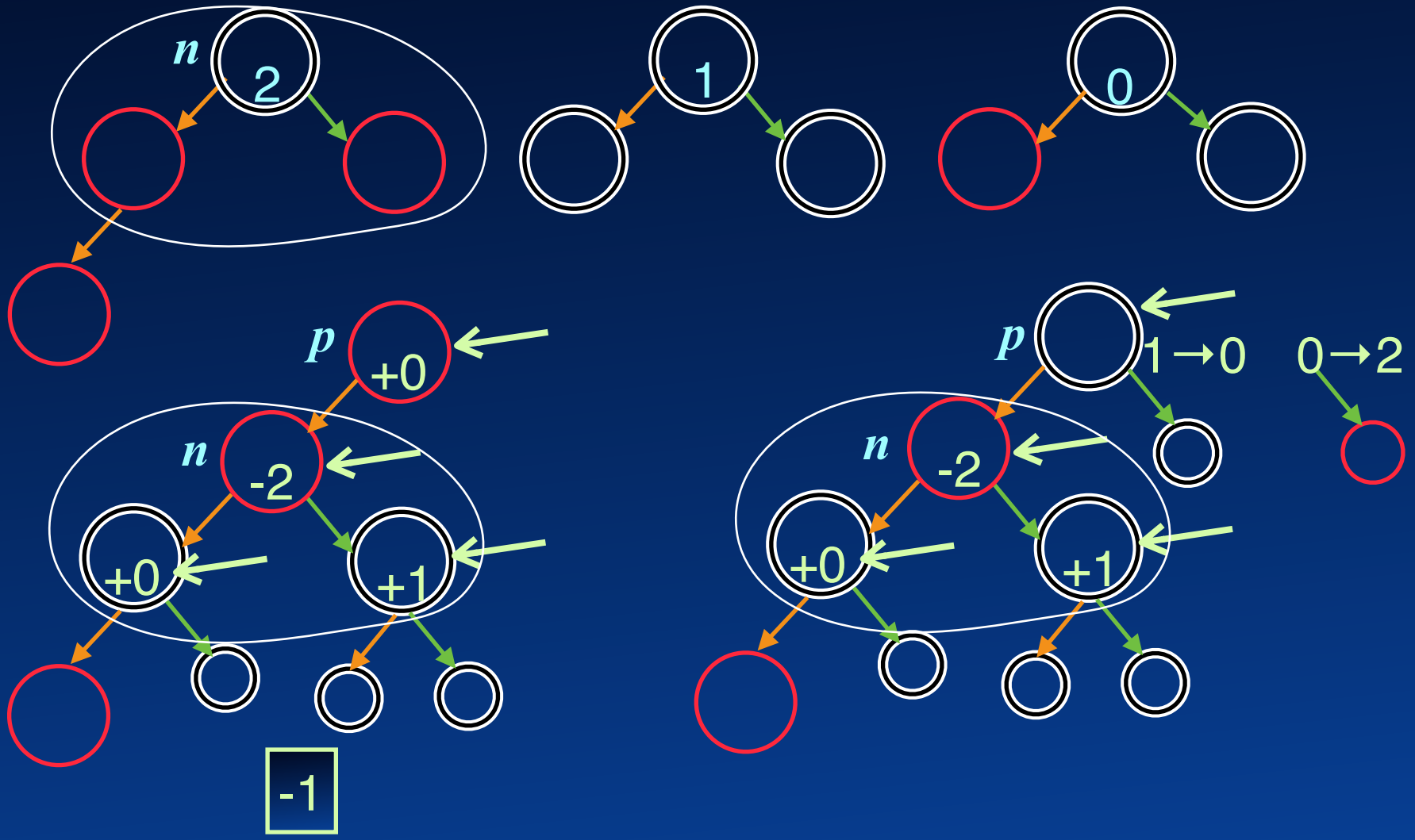

R-B Tree Color changes

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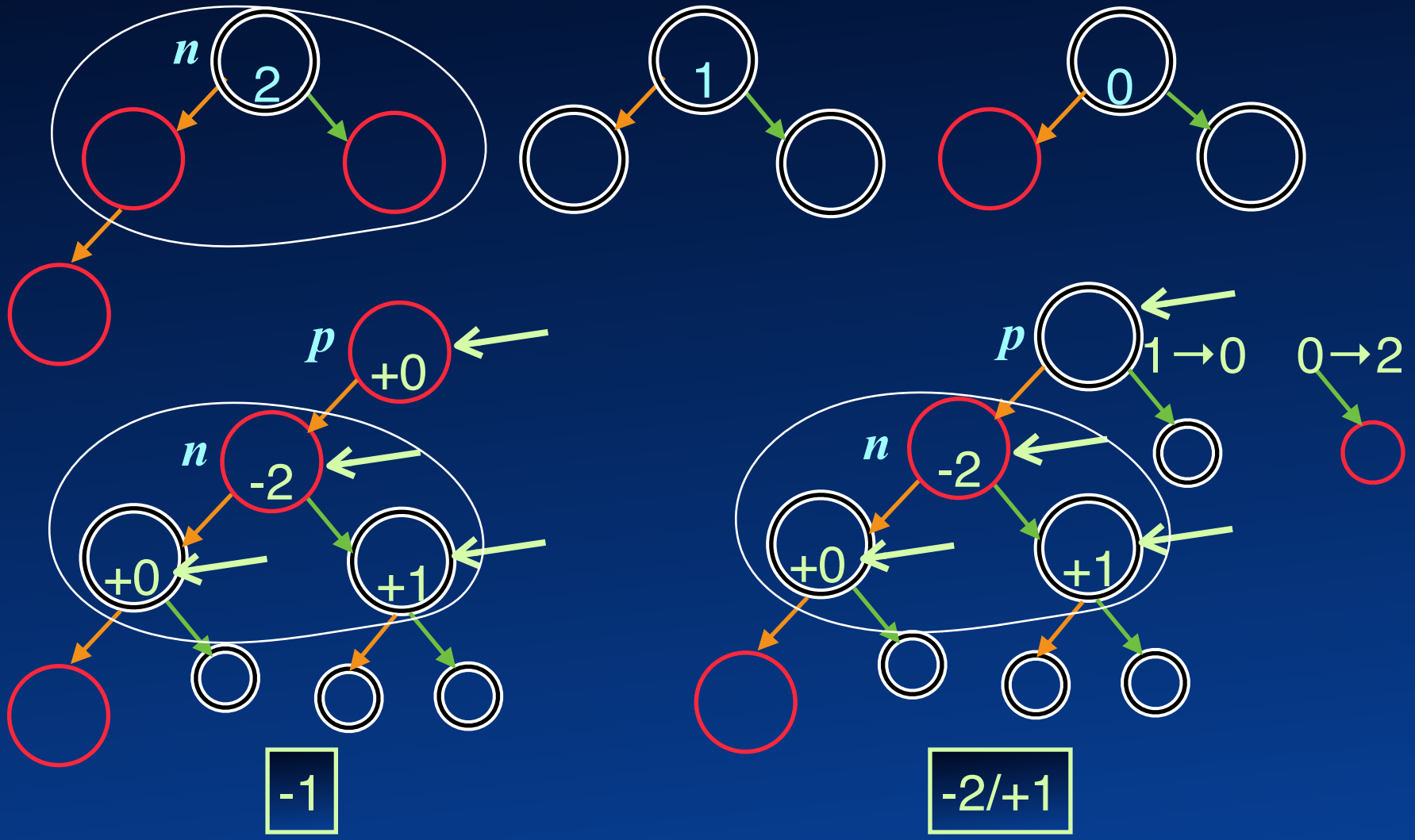

R-B Tree Color changes

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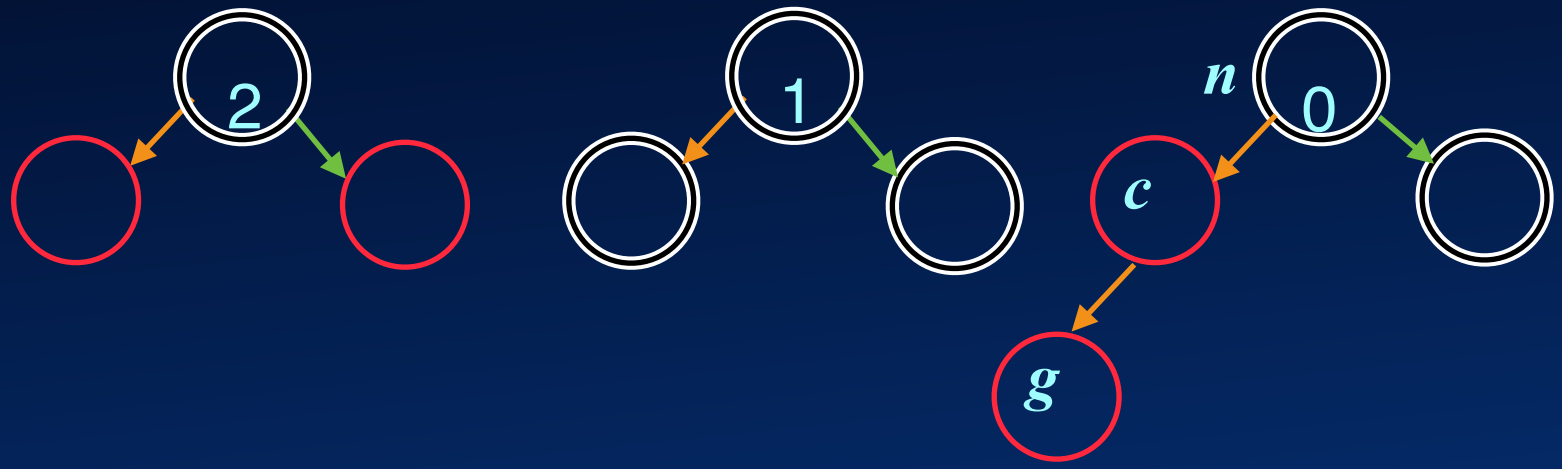

R-B Tree Color changes

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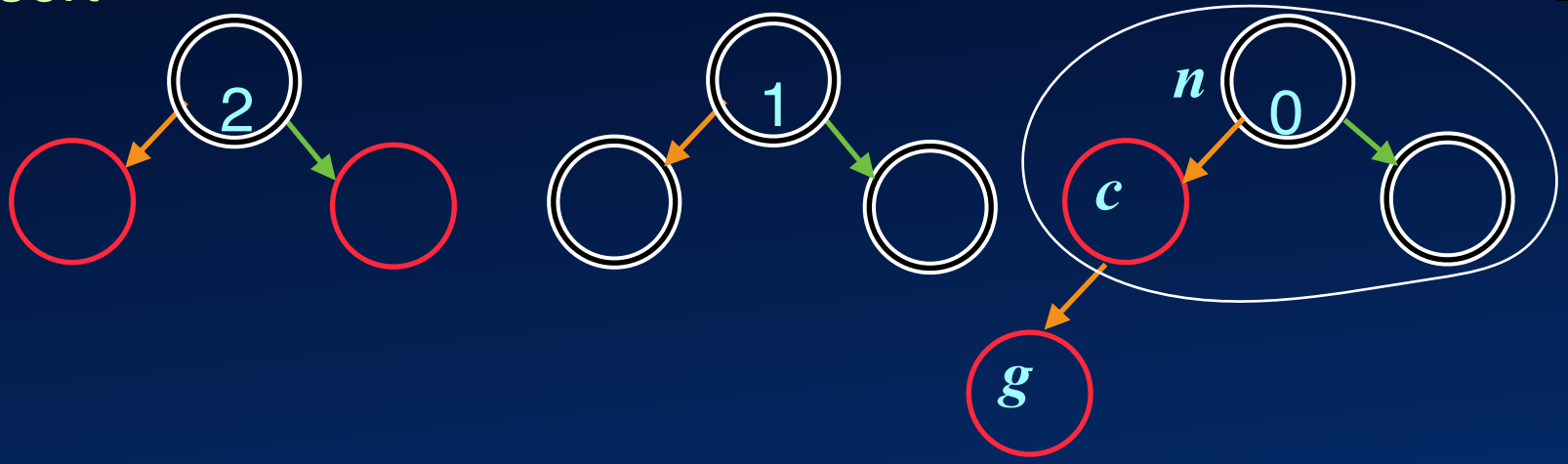

R-B Tree Color changes

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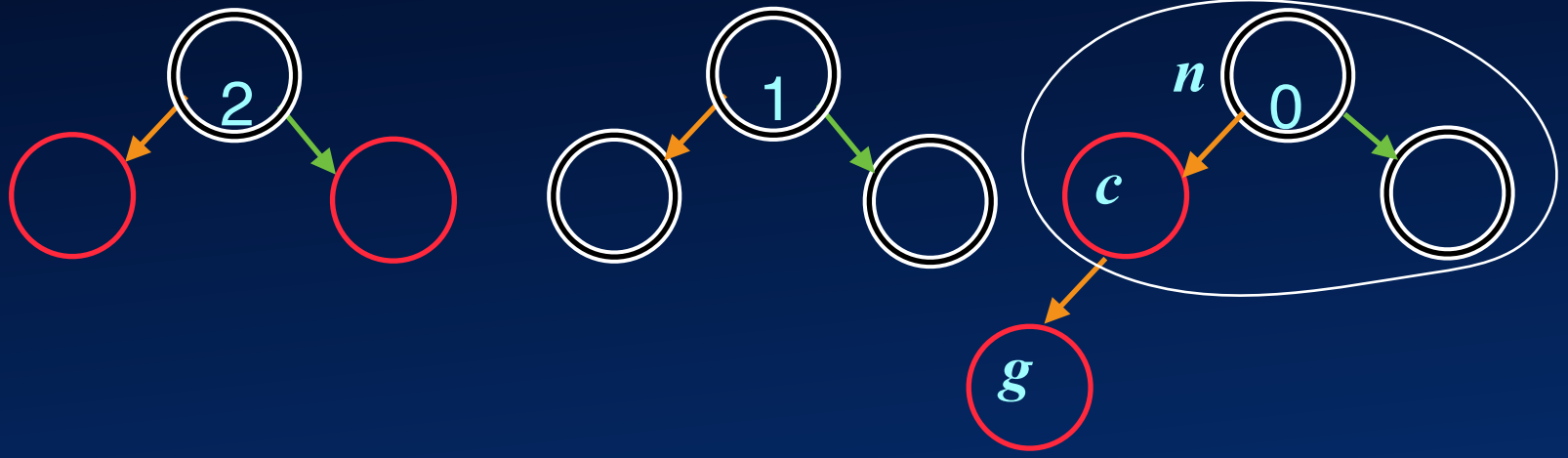

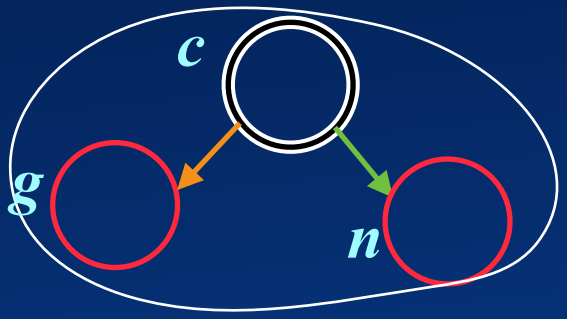

R-B Tree Color changes

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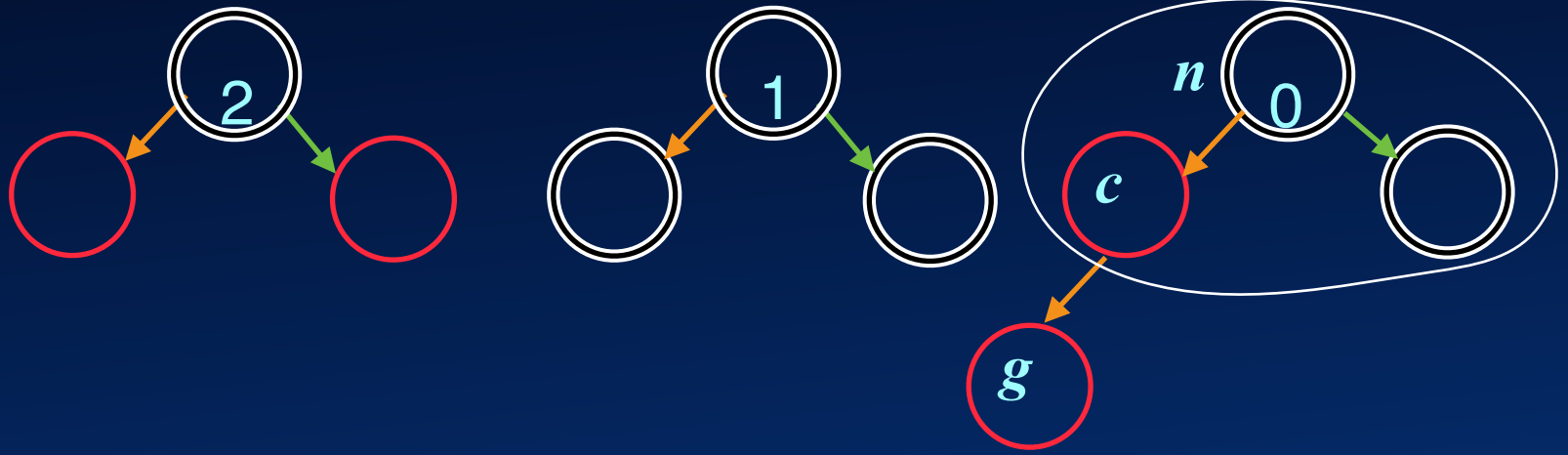

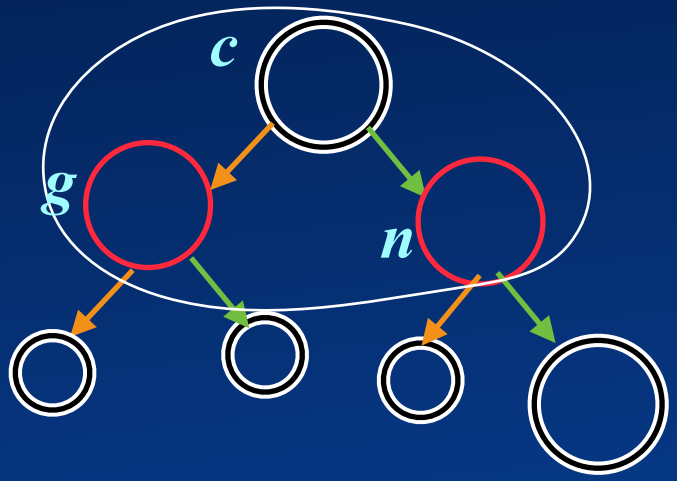

R-B Tree Color changes

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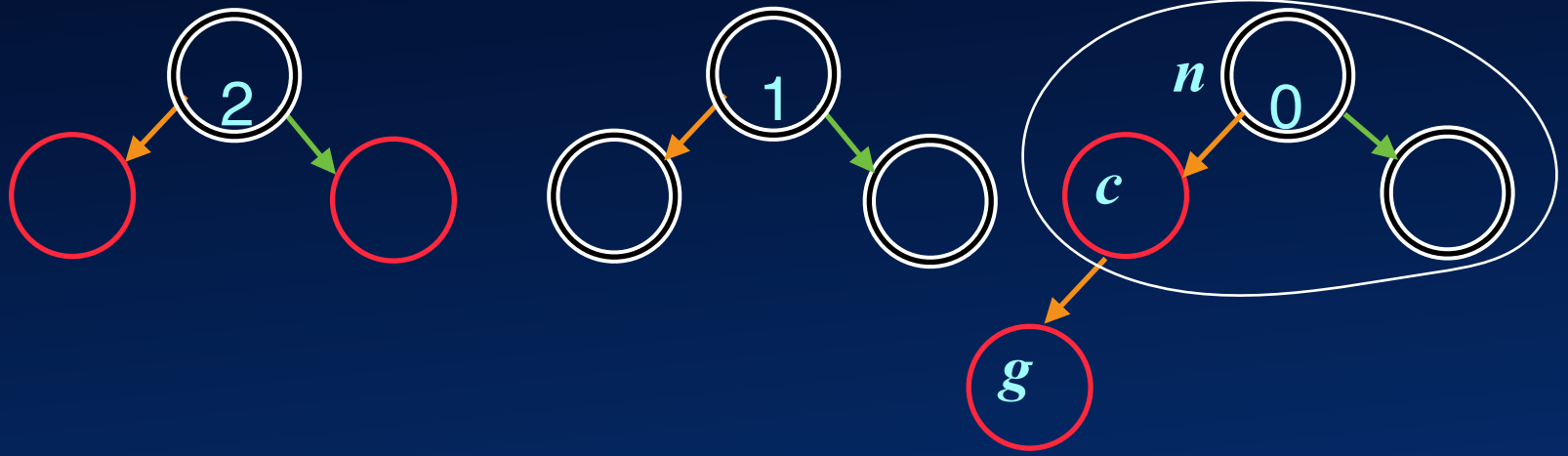

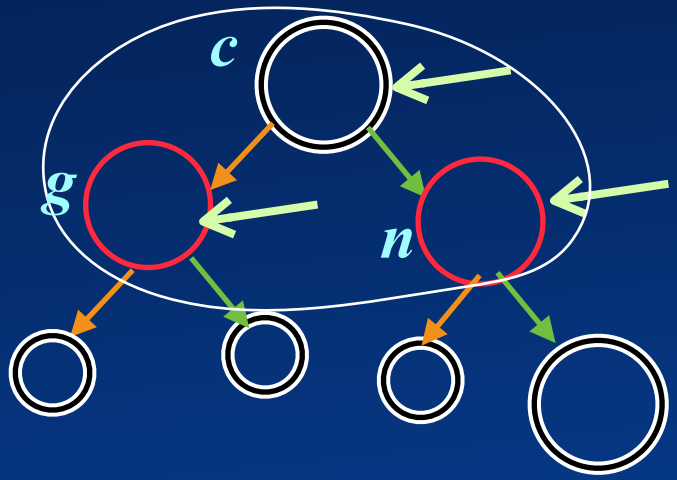

R-B Tree Color changes

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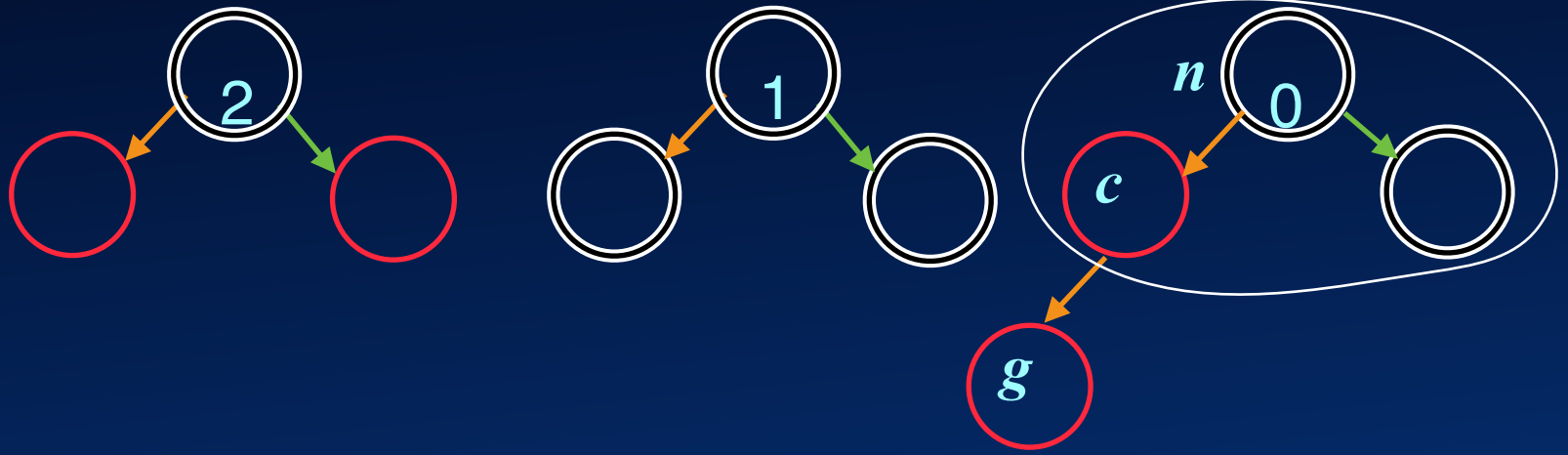

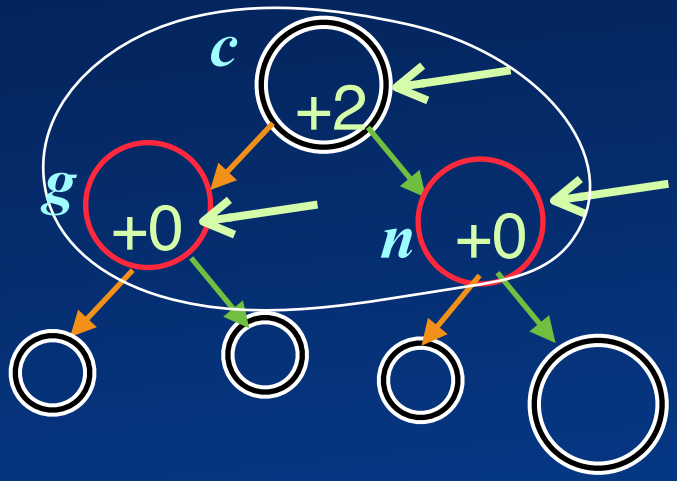

R-B Tree Color changes

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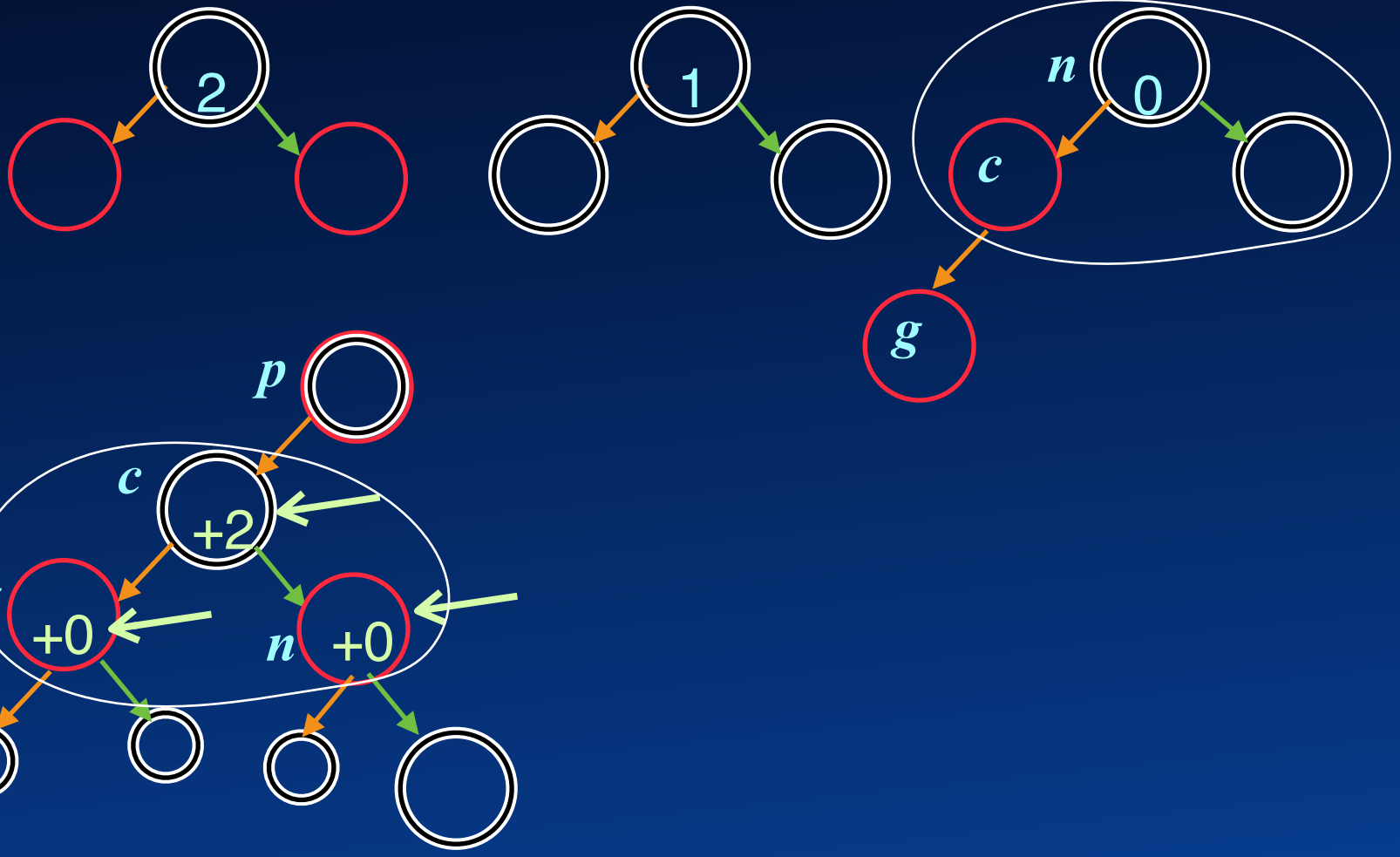

R-B Tree Color changes

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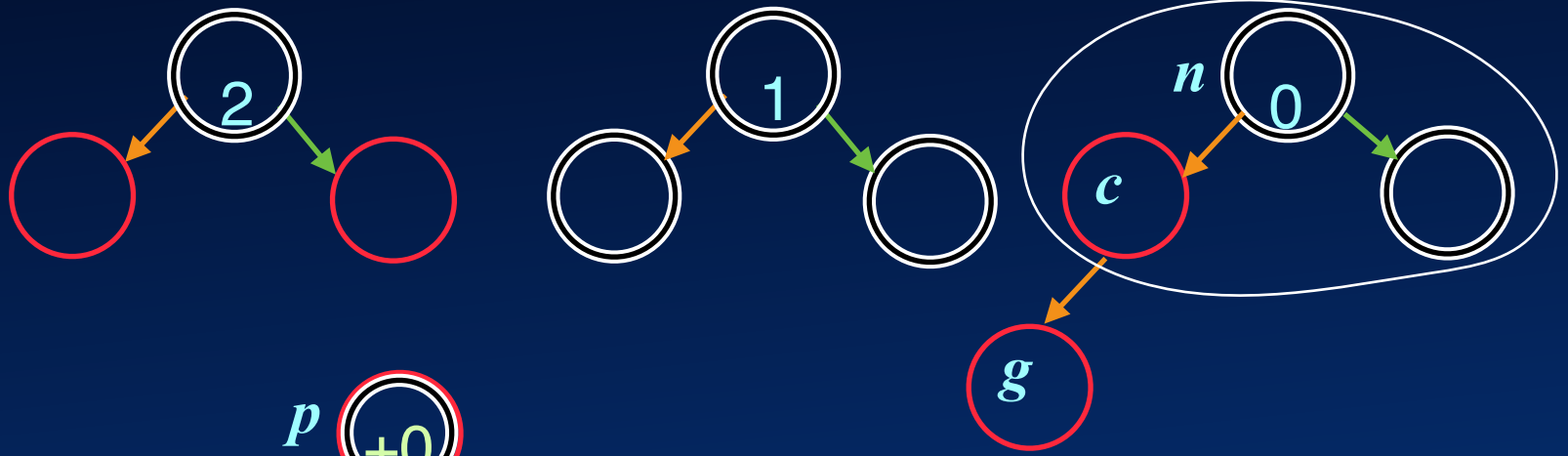

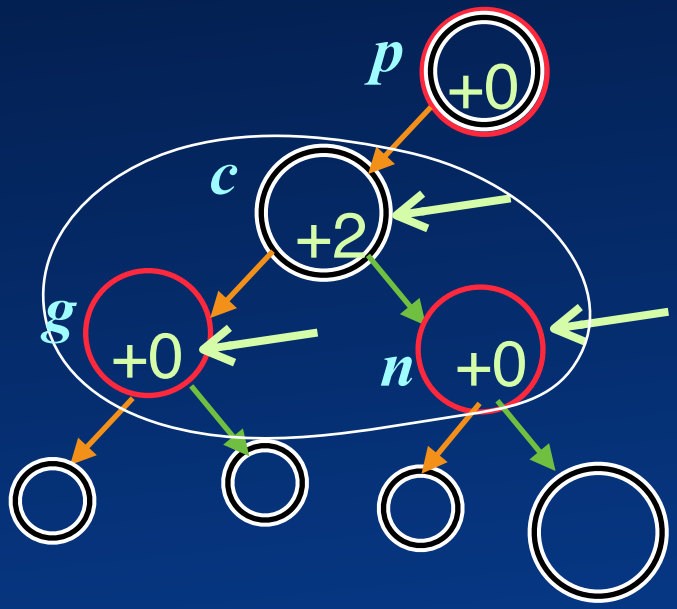

R-B Tree Color changes

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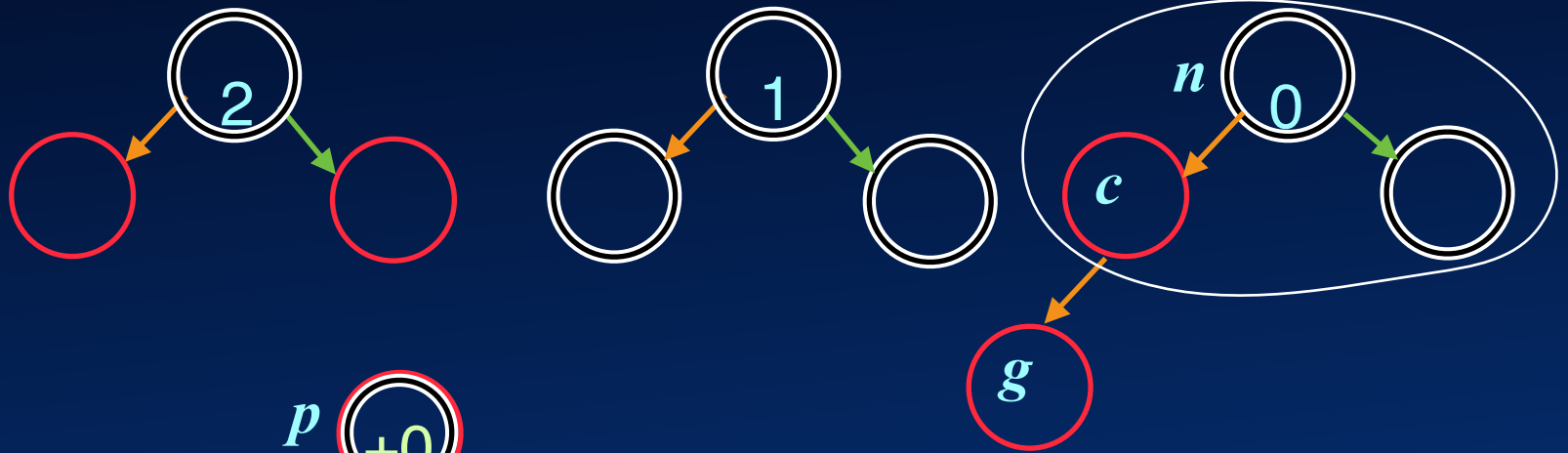

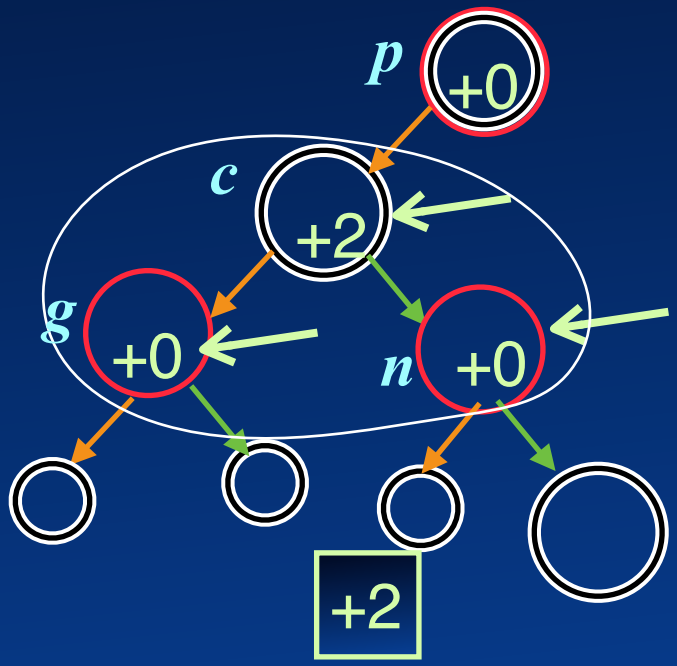

R-B Tree Color changes

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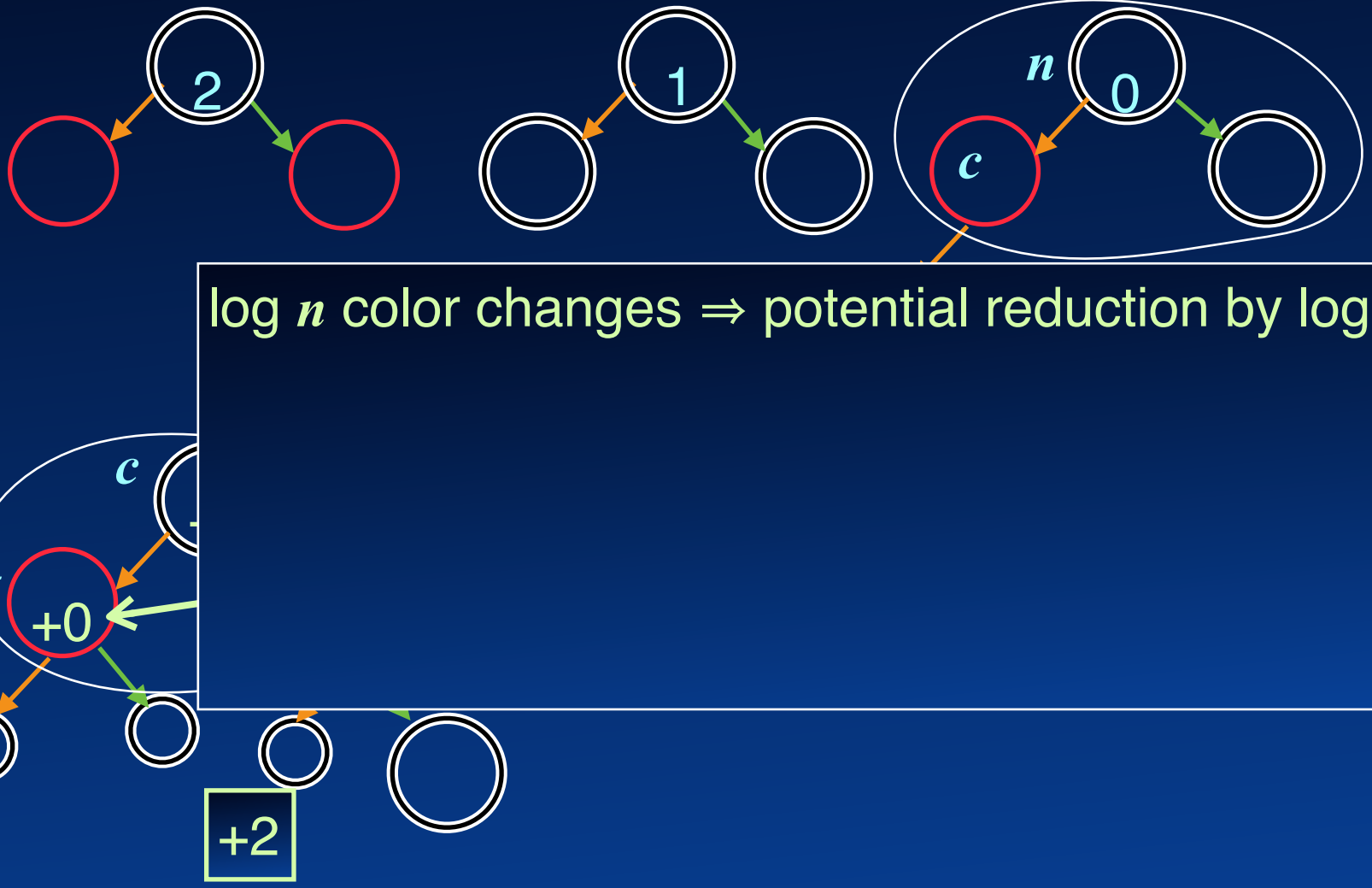

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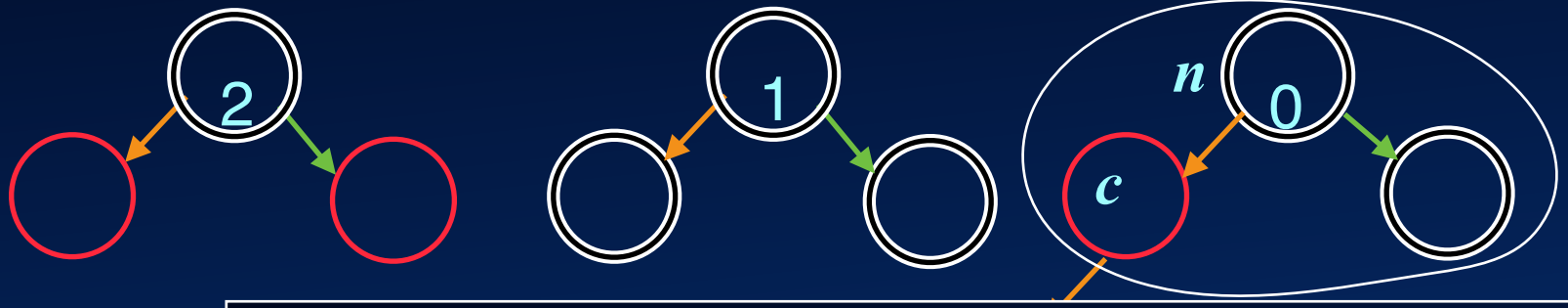
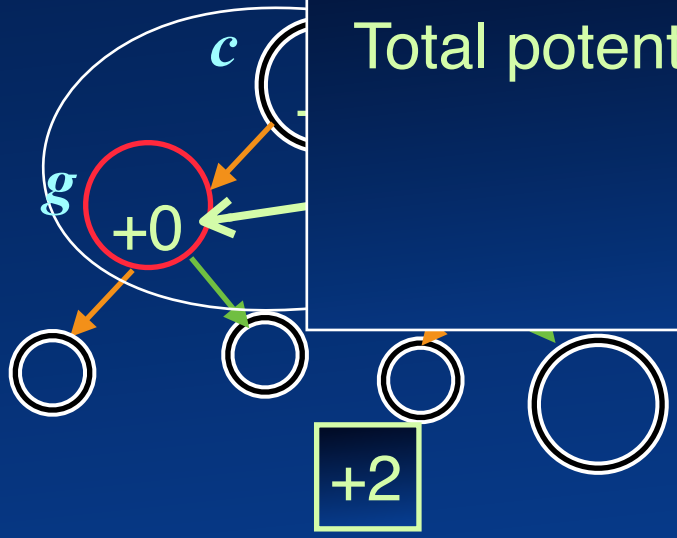


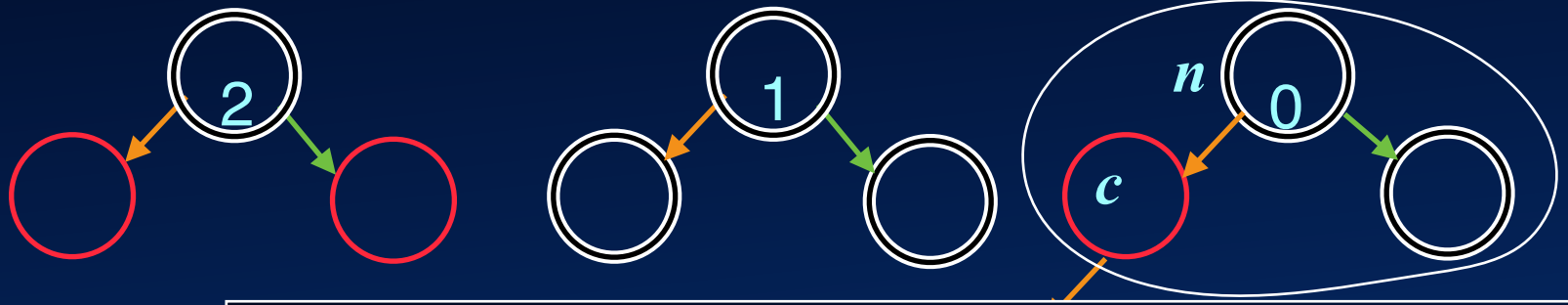
log n color changes ⇒ potential reduction by log n

After n insertions (starting from a null tree):
Total potential ≤ n



R-B Tree Color changes

Insert

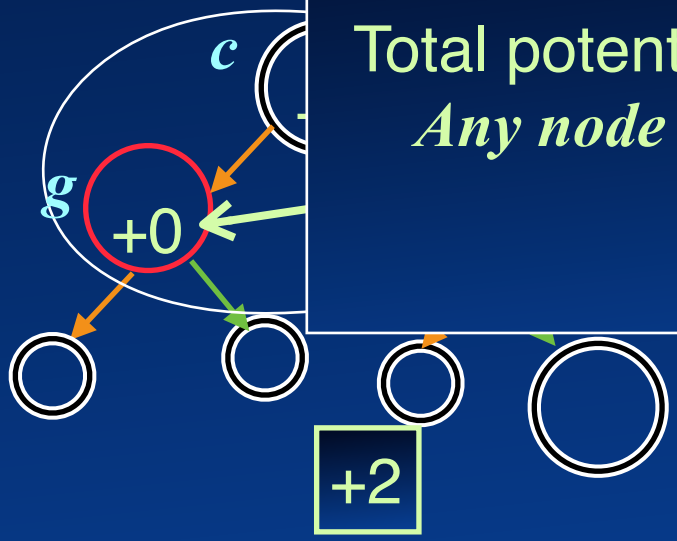


log n color changes ⇒ potential reduction by log n

After n insertions (starting from a null tree):

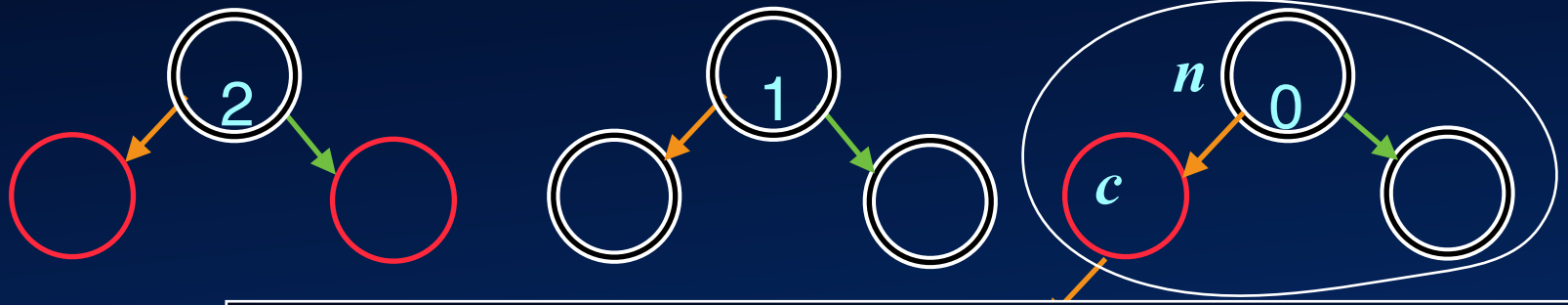
Total potential ≤ n

Any node with $\Phi = 2$ implies 2 red nodes



R-B Tree Color changes

Insert



log n color changes ⇒ potential reduction by log n

After n insertions (starting from a null tree):

Total potential ≤ n

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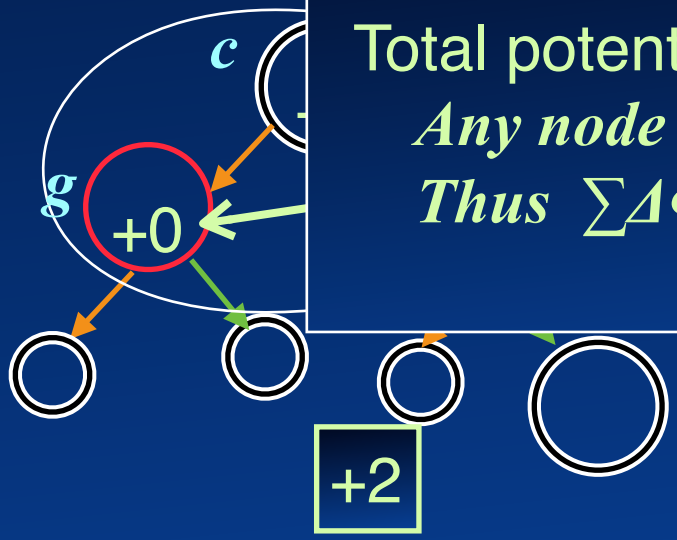
Thus $\sum \Delta\Phi \le n \Rightarrow$ +2 .. -Recolors - Recolors.. $\le n$



R-B Tree Color changes

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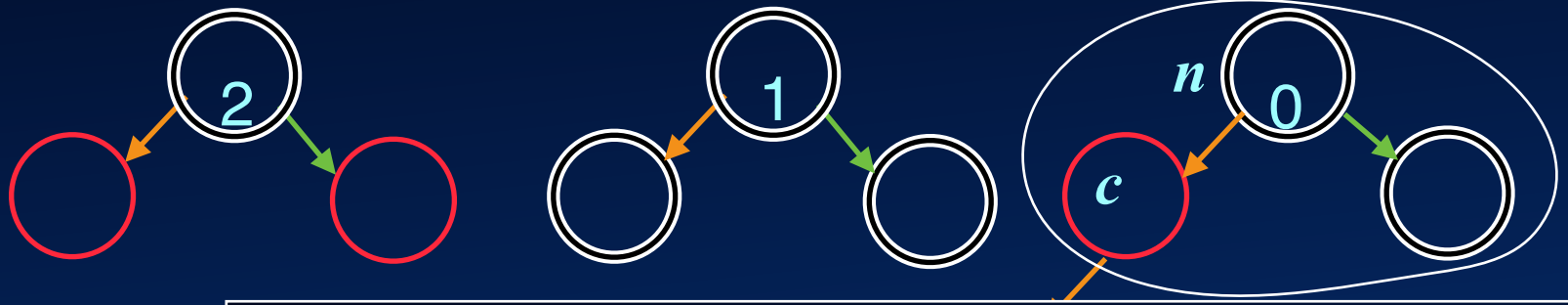

log n color changes ⇒ potential reduction by log n

After n insertions (starting from a null tree):

Total potential ≤ n

Any node with $\Phi = 2$ implies 2 red nodes

Thus $\sum \Delta\Phi \le n \Rightarrow$ +2 .. -Recolors - Recolors.. $\le n$

$\sum$ other cases $\le 2n \Rightarrow \sum$ bad cases = O(n)