Chapter 1

On Operations Research

This chapter is adapted from Wikipedia article *Operations Research* and [4, Ch. 1].

1.1 What is Operations Research

Definitions

To define anything non-trivial — like beauty or mathematics — is very difficult indeed. Here is a reasonably good definition of Operations Research:

1.1.1 Definition. *Operations Research* (OR) is an interdisciplinary branch of applied mathematics and formal science that uses methods like mathematical modeling, statistics, and algorithms to arrive at optimal or near optimal solutions to complex problems.

Definition 1.1.1 is problematic: to grasp it we already have to know, e.g., what is formal science or near optimality.

From a practical point of view, OR can be defined as an art of optimization, i.e., an art of finding minima or maxima of some objective function, and — to some extend — an art of defining the objective functions. Typical objective functions are

- profit,
- assembly line performance,
- crop yield,
- bandwidth,
- loss,
- waiting time in queue,
- risk.

From an organizational point of view, OR is something that helps management achieve its goals using the scientific process.

The terms OR and Management Science (MS) are often used synonymously. When a distinction is drawn, management science generally implies a closer relationship to Business Management. OR also closely relates to Industrial Engineering. Industrial engineering takes more of an engineering point of view, and industrial engineers typically consider OR techniques to be a major part of their tool set. Recently, the term Decision Science (DS) has also be coined to OR.

More information on OR can be found from the INFORMS web page

`http://www.thescienceofbetter.org/`

(If OR is "the Science of Better" the OR'ists should have figured out a better name for it.)

OR Tools

Some of the primary tools used in OR are

- statistics,
- optimization,
- probability theory,
- queuing theory,
- game theory,
- graph theory,
- decision analysis,
- simulation.

Because of the computational nature of these fields, OR also has ties to computer science, and operations researchers regularly use custom-written software.

In this course we will concentrate on optimization, especially linear optimization.

OR Motto and Linear Programming

The most common OR tool is Linear Optimization, or Linear Programming (LP).

1.1.2 Remark. The "Programming" in Linear Programming is synonym for "optimization". It has — at least historically — nothing to do with computer-programming.

LP is the OR'ists favourite tool because it is

- simple,
- easy to understand,

- robust.

"Simple" means easy to implement, "easy to understand" means easy to explain (to you boss), and "robust" means that it's like the Swiss Army Knife: perfect for nothing, but good enough for everything.

Unfortunately, almost no real-world problem is really a linear one — thus LP is perfect for nothing. However, most real-world problems are "close enough" to linear problems — thus LP is good enough for everything. Example 1.1.3 below elaborates this point.

1.1.3 Example. Mr. Quine sells gavagais. He will sell one gavagai for 10 Euros. So, one might expect that buying x gavagais from Mr. Quine would cost — according to the linear rule — $10x$ Euros.

The linear rule in Example 1.1.3 may well hold for buying 2, 3, or 5, or even 50 gavagais. But:

- One may get a discount if one buys 500 gavagais.
- There are only $1{,}000{,}000$ gavagais in the world. So, the price for $1{,}000{,}001$ gavagais is $+\infty$.
- The unit price of gavagais may go up as they become scarce. So, buying $950{,}000$ gavagais might be considerably more expensive than €$9{,}500{,}000$.
- It might be pretty hard to buy 0.5 gavagais, since half a gavagai is no longer a gavagai (gavagais are bought for pets, and not for food).
- Buying -10 gavagais is in principle all right. That would simply mean selling 10 gavagais. However, Mr. Quine would most likely not buy gavagais with the same price he is selling them.

1.1.4 Remark. You may think of a curve that would represent the price of gavagais better than the linear straight line — or you may even think as a radical philosopher and argue that there is no curve.

Notwithstanding the problems and limitations mentioned above, linear tools are widely used in OR according to the following motto that should — as all mottoes — be taken with a grain of salt:

OR Motto. *It's better to be quantitative and naïve than qualitative and profound.*

1.2 History of Operations Research*

This section is most likely omitted in the lectures. Nevertheless, you should read it — history gives perspective, and thinking is nothing but an exercise of perspective.

Prehistory

Some say that Charles Babbage (1791–1871) — who is arguably the "father of computers" — is also the "father of operations research" because his research into the cost of transportation and sorting of mail led to England's universal "Penny Post" in 1840.

OR During World War II

The modern field of OR arose during World War II. Scientists in the United Kingdom including Patrick Blackett, Cecil Gordon, C. H. Waddington, Owen Wansbrough-Jones and Frank Yates, and in the United States with George Dantzig looked for ways to make better decisions in such areas as logistics and training schedules.

Here are examples of OR studies done during World War II:

- Britain introduced the convoy system to reduce shipping losses, but while the principle of using warships to accompany merchant ships was generally accepted, it was unclear whether it was better for convoys to be small or large. Convoys travel at the speed of the slowest member, so small convoys can travel faster. It was also argued that small convoys would be harder for German U-boats to detect. On the other hand, large convoys could deploy more warships against an attacker. It turned out in OR analysis that the losses suffered by convoys depended largely on the number of escort vessels present, rather than on the overall size of the convoy. The conclusion, therefore, was that a few large convoys are more defensible than many small ones.

- In another OR study a survey carried out by RAF Bomber Command was analyzed. For the survey, Bomber Command inspected all bombers returning from bombing raids over Germany over a particular period. All damage inflicted by German air defenses was noted and the recommendation was given that armor be added in the most heavily damaged areas. OR team instead made the surprising and counter-intuitive recommendation that the armor be placed in the areas which were completely untouched by damage. They reasoned that the survey was biased, since it only included aircraft that successfully came back from Germany. The untouched areas were probably vital areas, which, if hit, would result in the loss of the aircraft.

- When the Germans organized their air defenses into the Kammhuber Line, it was realized that if the RAF bombers were to fly in a bomber stream they could overwhelm the night fighters who flew in individual cells directed to their targets by ground controllers. It was then a matter of calculating the statistical loss from collisions against the statistical loss from night fighters to calculate how close the bombers should fly to minimize RAF losses.

1.3 Phases of Operations Research Study

Seven Steps of OR Study

An OR project can be split in the following seven steps:

Step 1: Formulate the problem The OR analyst first defines the organization's problem. This includes specifying the organization's objectives and the parts of the organization (or system) that must be studied before the problem can be solved.

Step 2: Observe the system Next, the OR analyst collects data to estimate the values of the parameters that affect the organization's problem. These estimates are used to develop (in Step 3) and to evaluate (in Step 4) a mathematical model of the organization's problem.

Step 3: Formulate a mathematical model of the problem The OR analyst develops an idealized representation — i.e. a mathematical model — of the problem.

Step 4: Verify the model and use it for prediction The OR analyst tries to determine if the mathematical model developed in Step 3 is an accurate representation of the reality. The verification typically includes observing the system to check if the parameters are correct. If the model does not represent the reality well enough then the OR analyst goes back either to Step 3 or Step 2.

Step 5: Select a suitable alternative Given a model and a set of alternatives, the analyst now chooses the alternative that best meets the organization's objectives. Sometimes there are many best alternatives, in which case the OR analyst should present them all to the organization's decision-makers, or ask for more objectives or restrictions.

Step 6: Present the results and conclusions The OR analyst presents the model and recommendations from Step 5 to the organization's decision-makers. At this point the OR analyst may find that the decision-makers do not approve of the recommendations. This may result from incorrect definition of the organization's problems or decision-makers may disagree with the parameters or the mathematical model. The OR analyst goes back to Step 1, Step 2, or Step 3, depending on where the disagreement lies.

Step 7: Implement and evaluate recommendation Finally, when the organization has accepted the study, the OR analyst helps in implementing the recommendations. The system must be constantly monitored and updated dynamically as the environment changes. This means going back to Step 1, Step 2, or Step 3, from time to time.

In this course we shall concentrate on Step 3 and Step 5, i.e., we shall concentrate on mathematical modeling and finding the optimum of a mathematical model. We will completely omit the in-between Step 4. That step belongs to the realm of statistics. The reason for this omission is obvious: The statistics needed in OR is way too important to be included as side notes in this course! So, any OR'ist worth her/his salt should study statistics, at least up-to the level of parameter estimization.

Example of OR Study

Next example elaborates how the seven-step list can be applied to a queueing problem. The example is cursory: we do not investigate all the possible objectives or choices there may be, and we do not go into the details of modeling.

1.3.1 Example. A bank manager wants to reduce expenditures on tellers' salaries while still maintaining an adequate level of customer service.

Step 1: The OR analyst describes bank's objectives. The manager's vaguely stated wish may mean, e.g.,

- The bank wants to minimize the weekly salary cost needed to ensure that the average waiting a customer waits in line is at most 3 minutes.
- The bank wants to minimize the weekly salary cost required to ensure that only 5% of all customers wait in line more than 3 minutes.

The analyst must also identify the aspects of the bank's operations that affect the achievement of the bank's objectives, e.g.,

- On the average, how many customers arrive at the bank each hour?
- On the average, how many customers can a teller serve per hour?

Step 2: The OR analyst observes the bank and estimates, among others, the following parameters:

- On the average, how many customers arrive each hour? Does the arrival rate depend on the time of day?

- On the average, how many customers can a teller serve each hour? Does the service speed depend on the number of customers waiting in line?

Step 3: The OR analyst develops a mathematical model. In this example a queueing model is appropriate. Let

$$
\begin{aligned}
W_q &= \text{Average time customer waits in line} \\
\lambda &= \text{Average number of customers arriving each hour} \\
\mu &= \text{Average number of customers teller can serve each hour}
\end{aligned}
$$

A certain mathematical queueing model yields a connection between these parameters:

$$
W_q = \frac{\lambda}{\mu(\mu - \lambda)}. \tag{1.3.2}
$$

This model corresponds to the first objective stated in Step 1.

Step 4: The analyst tries to verify that the model (1.3.2) represents reality well enough. This means that the OR analyst will estimate the parameter W_q, λ, and μ statistically, and then she will check whether the equation (1.3.2) is valid, or close enough. If this is not the case then the OR analyst goes either back to Step 2 or Step 3.

Step 5: The OR analyst will optimize the model (1.3.2). This could mean solving how many tellers there must be to make μ big enough to make W_q small enough, e.g. 3 minutes.

We leave it to the students to wonder what may happen in steps 6 and 7.

Chapter 2

On Linear Programming

This chapter is adapted from [2, Ch. 1].

2.1 Example towards Linear Programming

Very Naïve Problem

2.1.1 Example. Tela Inc. manufactures two product: #1 and #2. To manufacture one unit of product #1 costs €40 and to manufacture one unit of product #2 costs €60. The profit from product #1 is €30, and the profit from product #2 is €20.

The company wants to maximize its profit. How many products #1 and #2 should it manufacture?

The solution is trivial: There is no bound on the amount of units the company can manufacture. So it should manufacture infinite number of either product #1 or #2, or both. If there is a constraint on the number of units manufactured then the company should manufacture only product #1, and not product #2. This constrained case is still rather trivial.

Less Naïve Problem

Things become more interesting — and certainly more realistic — when there are restrictions in the resources.

2.1.2 Example. Tela Inc. in Example 2.1.1 can invest €40,000 in production and use 85 hours of labor. To manufacture one unit of product #1 requires 15 minutes of labor, and to manufacture one unit of product #2 requires 9 minutes of labor.

The company wants to maximize its profit. How many units of product #1 and product #2 should it manufacture? What is the maximized profit?

The rather trivial solution of Example 2.1.1 is not applicable now. Indeed, the company does not have enough labor to put all the €40,000 in product #1.

Since the profit to be maximized depend on the number of product #1 and #1, our decision variables are:

$$\begin{aligned} x_1 &= \text{number of product \#1 produced,} \\ x_2 &= \text{number of product \#2 produced.} \end{aligned}$$

So the situation is: We want to *maximize* (max)

$$\text{profit:} \qquad 30x_1 + 20x_2$$

subject to (s.t.) the constraints

$$\begin{array}{lrcrcr} \text{money:} & 40x_1 & + & 60x_2 & \le & 40{,}000 \\ \text{labor:} & 15x_1 & + & 9x_2 & \le & 5{,}100 \\ \text{non-negativity:} & & & x_1, x_2 & \ge & 0 \end{array}$$

Note the last constraint: $x_1, x_2 \ge 0$. The problem does not state this explicitly, but it's implied — we are selling products #1 and #2, not buying them.

2.1.3 Remark. Some terminology: The unknowns x_1 and x_2 are called *decision variables*. The function $30x_1+20x_2$ to be maximized is called the *objective function*.

What we have now is a *Linear Program* (LP), or a Linear Optimization problem,

$$\begin{array}{lcrcrcr} \max z & = & 30x_1 & + & 20x_2 & & \\ \text{s.t.} & & 40x_1 & + & 60x_2 & \le & 40{,}000 \\ & & 15x_1 & + & 9x_2 & \le & 5{,}100 \\ & & & & x_1, x_2 & \ge & 0 \end{array}$$

We will later see how to solve such LPs. For now we just show the solution. For decision variables it is optimal to produce no product #1 and thus put all the resource to product #2 which means producing 566.667 number of product #2. The profit will then be €11,333.333. In other words, the optimal solution is

$$\begin{aligned} x_1 &= 0, \\ x_2 &= 566.667, \\ z &= 11{,}333.333. \end{aligned}$$

2.1.4 Remark. If it is not possible to manufacture fractional number of products, e.g. 0.667 units, then we have to reformulate the LP-problem above to an *Integer Program* (IP)

$$\begin{array}{lrcrcrcr} \max z & = & 30x_1 & + & 20x_2 & & \\ \text{s.t.} & & 40x_1 & + & 60x_2 & \leq & 40{,}000 \\ & & 15x_1 & + & 9x_2 & \leq & 5{,}100 \\ & & & & x_1, x_2 & \geq & 0 \\ & & & & x_1, x_2 & \multicolumn{2}{l}{\text{are integers}} \end{array}$$

We will later see how to solve such IPs (which is more difficult than solving LPs). For now we just show the solution:

$$\begin{aligned} x_1 &= 1, \\ x_2 &= 565, \\ z &= 11{,}330. \end{aligned}$$

In Remark 2.1.4 above we see the usefulness of the OR Motto. Indeed, although the LP solution is not practical if we cannot produce fractional number of product, the solution it gives is close to the true IP solution: both in terms of the value of the objective function and the location of the optimal point. We shall learn more about this later in Chapter 8.

2.2 Solving Linear Programs Graphically

From Minimization to Maximization

We shall discuss later in Chapter 5, among other things, how to transform a minimization LP into a maximization LP. So, you should skip this subsection and proceed to the next subsection titled "Linear Programs with Two Decision Variables" — unless you want to know the general, and rather trivial, duality between minimization and maximization.

Any minimization problem — linear or not — can be restated as a maximization problem simply by multiplying the objective function by -1:

2.2.1 Theorem. *Let $K \subset \mathbb{R}^n$, and let $g : \mathbb{R}^n \to \mathbb{R}$. Suppose*

$$w^* = \min_{\mathbf{x} \in K} g(\mathbf{x})$$

and $\mathbf{x}^* \in \mathbb{R}^n$ *is a point where the minimum* w^* *is attained. Then, if* $f = -g$ *and* $z^* = -w^*$*, we have that*

$$z^* = \max_{\mathbf{x} \in K} f(\mathbf{x}),$$

and the maximum z^* *is attained at the point* $\mathbf{x}^* \in \mathbb{R}^n$*.*

The mathematically oriented should try to prove Theorem 2.2.1. It's not difficult — all you have to do is to not to think about the constraint-set K or any other specifics, like the space $\mathbb{R}^n$, or if there is a unique optimum. Just think about the big picture! Indeed, Theorem 2.2.1 is true in the greatest possible generality: It is true whenever it makes sense!

Linear Programs with Two Decision Variables

We shall solve the following LP:

2.2.2 Example.

$$\begin{array}{lcrcrcll}
\max z & = & 4x_1 & + & 3x_2 & & & \\
\text{s.t.} & & 2x_1 & + & 3x_2 & \le & 6 & (1) \\
 & & -3x_1 & + & 2x_2 & \le & 3 & (2) \\
 & & & & 2x_2 & \le & 5 & (3) \\
 & & 2x_1 & + & x_2 & \le & 4 & (4) \\
 & & & & x_1, x_2 & \ge & 0 & (5)
\end{array}$$

The LP in Example 2.2.2 has only two decision variables: x_1 and x_2. So, it can be solved graphically on a piece of paper like this one. To solve graphically LPs with three decision variables would require three-dimensional paper, for four decision variables one needs four-dimensional paper, and so forth.

Four-Step Graphical Algorithm

Step 1: Draw coordinate space Tradition is that x_1 is the horizontal axis and x_2 is the vertical axis. Because of the non-negativity constraints on x_1 and x_2 it is enough to draw the 1st quadrant (the NE-quadrant).

Step 2: Draw constraint-lines Each constraint consists of a line and of information (e.g. arrows) indicating which side of the line is feasible. To draw, e.g., the line (1), one sets the inequality to be the equality

$$2x_1 + 3x_2 = 6.$$

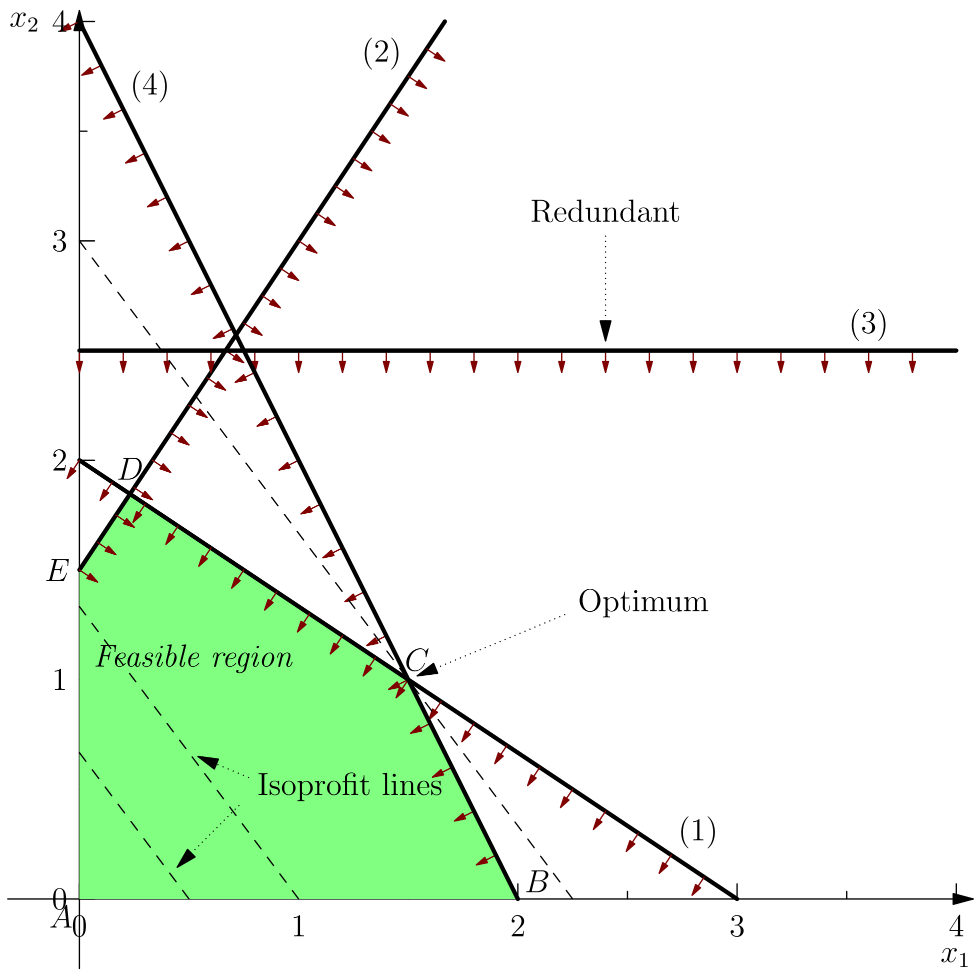
To draw this line we can first set $x_1 = 0$ and then set $x_2 = 0$, and we see that the line goes through points $(0, 2)$ and $(3, 0)$. Since (1) is a $\leq$-inequality, the feasible region must be below the line.

Step 3: Define feasible region This is done by selecting the region satisfied by all the constraints including the non-negativity constraints.

Step 4: Find the optimum by moving the isoprofit line The *isoprofit line* is the line where the objective function is constant. In this case the isoprofit lines are the pairs (x_1, x_2) satisfying

$$z \quad = \quad 4x_1 + 3x_2 \quad = \quad \text{const.}$$

(In the following picture we have drawn the isoprofit line corresponding to const $= 2$ and const $= 4$, and the optimal isoprofit line corresponding to const $= 9$.) The further you move the line from the origin the better value you get (unless the maximization problem is trivial in the objective function, cf. Example 2.2.3 later). You find the best value when the isoprofit line is just about to leave the feasible region completely (unless the maximization problem is trivial in constraints, i.e. it has an unbounded feasible region, cf. Example 2.2.4 later).



From the picture we read — by moving the isoprofit line away from the origin — that the optimal point for the decision variables (x_1, x_2) is

$$C \quad = \quad (1.5, 1).$$

Therefore, the optimal value is of the objective is

$$z \quad = \quad 4\times1.5 + 3\times1 \quad = \quad 9.$$

Example with Trivial Optimum

Consider the following LP maximization problem, where the objective function z does not grow as its arguments x_1 and x_2 get further away from the origin:

2.2.3 Example.

$$\begin{array}{lrcrcrl}
\max z & = & -4x_1 & - & 3x_2 & & \\
\text{s.t.} & & 2x_1 & + & 3x_2 & \leq & 6 \qquad (1)\\
 & & -3x_1 & + & 2x_2 & \leq & 3 \qquad (2)\\
 & & & & 2x_2 & \leq & 5 \qquad (3)\\
 & & 2x_1 & + & x_2 & \leq & 4 \qquad (4)\\
 & & & & x_1, x_2 & \geq & 0 \qquad (5)
\end{array}$$

In this case drawing a graph would be an utter waste of time. Indeed, consider the objective function under maximization:

$$z = -4x_1 - 3x_2$$

Obviously, given the standard constraints $x_1, x_2 \geq 0$, the optimal solution is

$$\begin{aligned} x_1 &= 0, \\ x_2 &= 0, \\ z &= 0. \end{aligned}$$

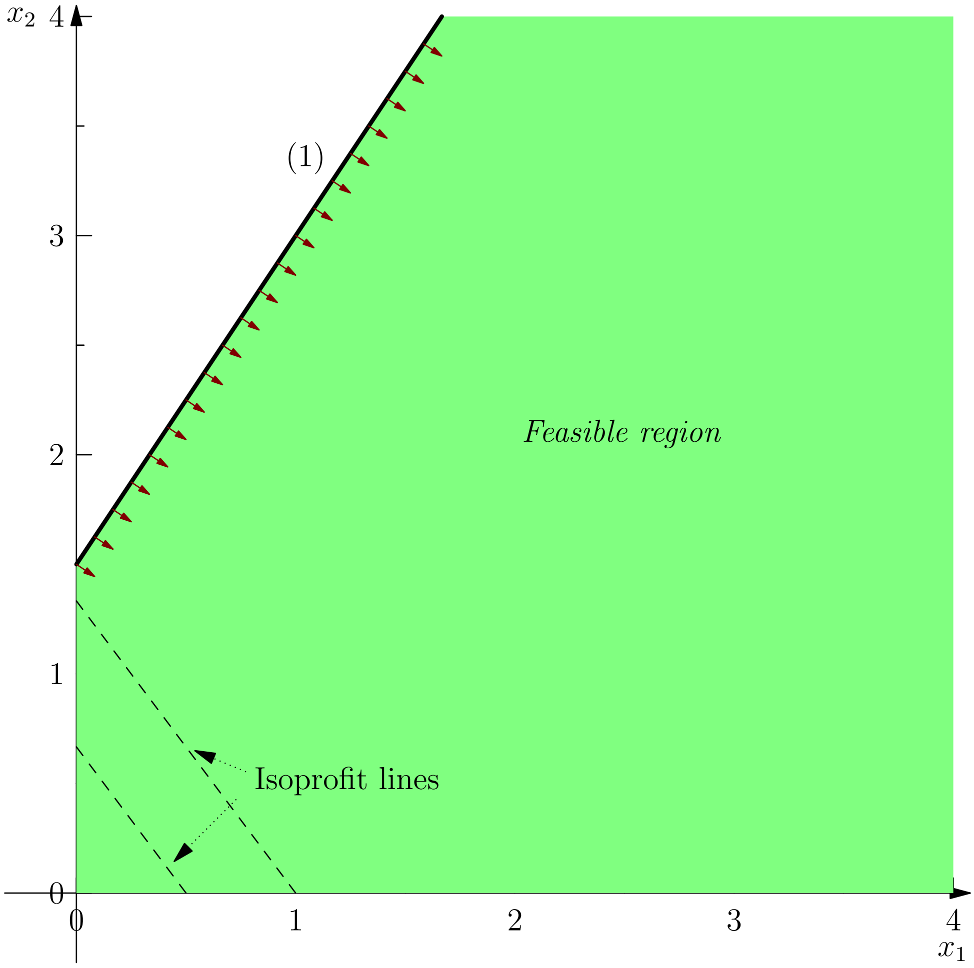
Whenever you have formulated a problem like this you (or your boss) must have done something wrong!

Example with Unbounded Feasible Region

2.2.4 Example.

$$\begin{array}{lrcrcrl}
\max z & = & 4x_1 & + & 3x_2 & & \\
\text{s.t.} & & -3x_1 & + & 2x_2 & \leq & 3 \qquad (1)\\
 & & & & x_1, x_2 & \geq & 0 \qquad (2)
\end{array}$$

From the picture below one sees that this LP has unbounded optimum, i.e., the value of objective function z can be made as big as one wishes.



Whenever you have formulated a problem like this you (or your boss) must have done something wrong — or you must be running a sweet business, indeed!