1. INTRODUCTION TO OR

1.1 TERMINOLOGY

The British/Europeans refer to "operational research", the Americans to "operations research" - but both are often shortened to just "OR" (which is the term we will use).

Another term which is used for this field is "management science" ("MS"). The Americans sometimes combine the terms OR and MS together and say "***OR/MS***" or "ORMS".

Yet other terms sometimes used are "industrial engineering" ("IE"), "decision science" ("DS"), and “problem solving”.

In recent years there has been a move towards a standardization upon a single term for the field, namely the term "OR".

“Operations Research (Management Science) is a scientific approach to decision making that seeks to best design and operate a system, usually under conditions requiring the allocation of scarce resources.”

A system is an organization of interdependent components that work together to accomplish the goal of the system.

1.2 THE METHODOLOGY OF OR

When OR is used to solve a problem of an organization, the following seven step procedure should be followed:

<u>Step 1. Formulate the Problem</u>

OR analyst first defines the organization's problem. Defining the problem includes specifying the organization's objectives and the parts of the organization (or system) that must be studied before the problem can be solved.

<u>Step 2. Observe the System</u>

Next, the analyst collects data to estimate the values of parameters that affect the organization's problem. These estimates are used to develop (in Step 3) and evaluate (in Step 4) a mathematical model of the organization's problem.

Step 3. Formulate a Mathematical Model of the Problem

The analyst, then, develops a mathematical model (in other words an idealized representation) of the problem. In this class, we describe many mathematical techniques that can be used to model systems.

Step 4. Verify the Model and Use the Model for Prediction

The analyst now tries to determine if the mathematical model developed in Step 3 is an accurate representation of reality. To determine how well the model fits reality, one determines how valid the model is for the current situation.

Step 5. Select a Suitable Alternative

Given a model and a set of alternatives, the analyst chooses the alternative (if there is one) that best meets the organization's objectives.

Sometimes the set of alternatives is subject to certain restrictions and constraints. In many situations, the best alternative may be impossible or too costly to determine.

Step 6. Present the Results and Conclusions of the Study

In this step, the analyst presents the model and the recommendations from Step 5 to the decision making individual or group. In some situations, one might present several alternatives and let the organization choose the decision maker(s) choose the one that best meets her/his/their needs.

After presenting the results of the OR study to the decision maker(s), the analyst may find that s/he does not (or they do not) approve of the recommendations. This may result from incorrect definition of the problem on hand or from failure to involve decision maker(s) from the start of the project. In this case, the analyst should return to Step 1, 2, or 3.

Step 7. Implement and Evaluate Recommendation

If the decision maker(s) has accepted the study, the analyst aids in implementing the recommendations. The system must be constantly monitored (and updated dynamically as the environment changes) to ensure that the recommendations are enabling decision maker(s) to meet her/his/their objectives.

1.3 HISTORY OF OR

(Prof. Beasley's lecture notes)

OR is a relatively new discipline. Whereas 70 years ago it would have been possible to study mathematics, physics or engineering (for example) at university it would not have been possible to study OR, indeed the term OR did not exist then. It was only really in the late 1930's that operational research began in a systematic fashion, and it started in the UK.

Early in 1936 the British Air Ministry established Bawdsey Research Station, on the east coast, near Felixstowe, Suffolk, as the centre where all pre-war radar experiments for both the Air Force and the Army would be carried out. Experimental radar equipment was brought up to a high state of reliability and ranges of over 100 miles on aircraft were obtained.

It was also in 1936 that Royal Air Force (RAF) Fighter Command, charged specifically with the air defense of Britain, was first created. It lacked however any effective fighter aircraft - no Hurricanes or Spitfires had come into service - and no radar data was yet fed into its very elementary warning and control system.

It had become clear that radar would create a whole new series of problems in fighter direction and control so in late 1936 some experiments started at Biggin Hill in Kent into the effective use of such data. This early work, attempting to integrate radar data with ground based observer data for fighter interception, was the start of OR.

The first of three major pre-war air-defense exercises was carried out in the summer of 1937. The experimental radar station at Bawdsey Research Station was brought into operation and the information derived from it was fed into the general air-defense warning and control system. From the early warning point of view this exercise was encouraging, but the tracking information obtained from radar, after filtering and transmission through the control and display network, was not very satisfactory.

In July 1938 a second major air-defense exercise was carried out. Four additional radar stations had been installed along the coast and it was hoped that Britain now had an aircraft location and control system greatly improved both in coverage and effectiveness. Not so! The exercise revealed, rather, that a new and serious problem had arisen. This was the need to coordinate and correlate the additional, and often conflicting, information received from the additional radar stations. With the out-break of war apparently imminent, it was obvious that something new - drastic if necessary - had to be attempted. Some new approach was needed.

Accordingly, on the termination of the exercise, the Superintendent of Bawdsey Research Station, A.P. Rowe, announced that although the exercise had again demonstrated the technical feasibility of the radar system for detecting aircraft, its operational achievements still fell far short of requirements. He therefore proposed that a crash program of research into the operational - as opposed to the technical -

aspects of the system should begin immediately. The term "operational research" [RESEARCH into (military) OPERATIONS] was coined as a suitable description of this new branch of applied science. The first team was selected from amongst the scientists of the radar research group the same day.

In the summer of 1939 Britain held what was to be its last pre-war air defense exercise. It involved some 33,000 men, 1,300 aircraft, 110 antiaircraft guns, 700 searchlights, and 100 barrage balloons. This exercise showed a great improvement in the operation of the air defense warning and control system. The contribution made by the OR teams was so apparent that the Air Officer Commander-in-Chief RAF Fighter Command (Air Chief Marshal Sir Hugh Dowding) requested that, on the outbreak of war, they should be attached to his headquarters at Stanmore.

On May 15th 1940, with German forces advancing rapidly in France, Stanmore Research Section was asked to analyze a French request for ten additional fighter squadrons (12 aircraft a squadron) when losses were running at some three squadrons every two days. They prepared graphs for Winston Churchill (the British Prime Minister of the time), based upon a study of current daily losses and replacement rates, indicating how rapidly such a move would deplete fighter strength. No aircraft were sent and most of those currently in France were recalled.

This is held by some to be the most strategic contribution to the course of the war made by OR (as the aircraft and pilots saved were consequently available for the successful air defense of Britain, the Battle of Britain).

In 1941 an Operational Research Section (ORS) was established in Coastal Command which was to carry out some of the most well-known OR work in World War II.

Although scientists had (plainly) been involved in the hardware side of warfare (designing better planes, bombs, tanks, etc) scientific analysis of the operational use of military resources had never taken place in a systematic fashion before the Second World War. Military personnel, often by no means stupid, were simply not trained to undertake such analysis.

These early OR workers came from many different disciplines, one group consisted of a physicist, two physiologists, two mathematical physicists and a surveyor. What such people brought to their work were "scientifically trained" minds, used to querying assumptions, logic, exploring hypotheses, devising experiments, collecting data, analyzing numbers, etc. Many too were of high intellectual caliber (at least four wartime OR personnel were later to win Nobel prizes when they returned to their peacetime disciplines).

By the end of the war OR was well established in the armed services both in the UK and in the USA.

OR started just before World War II in Britain with the establishment of teams of scientists to study the strategic and tactical problems involved in military operations. The objective was to find the most effective utilization of limited military resources by the use of quantitative techniques.

Following the end of the war OR spread, although it spread in different ways in the UK and USA.

You should be clear that the growth of OR since it began (and especially in the last 30 years) is, to a large extent, the result of the increasing power and widespread availability of computers. Most (though not all) OR involves carrying out a large number of numeric calculations. Without computers this would simply not be possible.

2. BASIC OR CONCEPTS

"OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing."

We can also define a mathematical model as consisting of:

- *Decision variables*, which are the unknowns to be determined by the solution to the model.
- *Constraints* to represent the physical limitations of the system
- An *objective* function
- An *optimal solution* to the model is the identification of a set of variable values which are feasible (satisfy all the constraints) and which lead to the optimal value of the objective function.

An optimization model seeks to find values of the decision variables that optimize (maximize or minimize) an objective function among the set of all values for the decision variables that satisfy the given constraints.

Two Mines Example

The Two Mines Company own two different mines that produce an ore which, after being crushed, is graded into three classes: high, medium and low-grade. The company has contracted to provide a smelting plant with 12 tons of high-grade, 8 tons of medium-grade and 24 tons of low-grade ore per week. The two mines have different operating characteristics as detailed below.

Mine	Cost per day (£'000)	Production (tons/day)		
		High	Medium	Low
X	180	6	3	4
Y	160	1	1	6

Consider that mines cannot be operated in the weekend. How many days per week should each mine be operated to fulfill the smelting plant contract?

Guessing

To explore the Two Mines problem further we might simply guess (i.e. use our judgment) how many days per week to work and see how they turn out.

- work one day a week on *X*, one day a week on *Y*

This does not seem like a good guess as it results in only 7 tones a day of high-grade, insufficient to meet the contract requirement for 12 tones of high-grade a day. We say that such a solution is *infeasible*.

- work 4 days a week on *X*, 3 days a week on *Y*

This seems like a better guess as it results in sufficient ore to meet the contract. We say that such a solution is *feasible*. However it is quite expensive (costly).

We would like a solution which supplies what is necessary under the contract at minimum cost. Logically such a minimum cost solution to this decision problem must exist. However even if we keep guessing we can never be sure whether we have found this minimum cost solution or not. Fortunately our structured approach will enable us to find the minimum cost solution.

Solution

What we have is a verbal description of the Two Mines problem. What we need to do is to translate that verbal description into an *equivalent* mathematical description.

In dealing with problems of this kind we often do best to consider them in the order:

- Variables
- Constraints
- Objective

This process is often called *formulating* the problem (or more strictly formulating a mathematical representation of the problem).

<u>Variables</u>

These represent the "decisions that have to be made" or the "unknowns".

We have two decision variables in this problem:

x = number of days per week mine *X* is operated

y = number of days per week mine *Y* is operated

Note here that $x \geq 0$ and $y \geq 0$.

Constraints

It is best to first put each constraint into words and then express it in a mathematical form.

ore production constraints - balance the amount produced with the quantity required under the smelting plant contract

Ore	
High	$6x + 1y \geq 12$
Medium	$3x + 1y \geq 8$
Low	$4x + 6y \geq 24$

days per week constraint - we cannot work more than a certain maximum number of days a week e.g. for a 5 day week we have

$x \leq 5$

$y \leq 5$

Inequality constraints

Note we have an inequality here rather than an equality. This implies that we may produce more of some grade of ore than we need. In fact we have the general rule: given a choice between an equality and an inequality choose the inequality

For example - if we choose an equality for the ore production constraints we have the three equations $6x+y=12$, $3x+y=8$ and $4x+6y=24$ and there are no values of x and y which satisfy all three equations (the problem is therefore said to be "over-constrained"). For example the values of x and y which satisfy $6x+y=12$ and $3x+y=8$ are $x=4/3$ and $y=4$, but these values do not satisfy $4x+6y=24$.

The reason for this general rule is that choosing an inequality rather than an equality gives us more flexibility in optimizing (maximizing or minimizing) the objective (deciding values for the decision variables that optimize the objective).

Implicit constraints

Constraints such as days per week constraint are often called implicit constraints because they are implicit in the definition of the variables.

Objective

Again in words our objective is (presumably) to minimize cost which is given by

$180x + 160y$

Hence we have the ***complete mathematical representation*** of the problem:

minimize

$$180x + 160y$$

subject to

$$6x + y \geq 12$$
$$3x + y \geq 8$$
$$4x + 6y \geq 24$$
$$x \leq 5$$
$$y \leq 5$$
$$x, y \geq 0$$

Some notes

The mathematical problem given above has the form

- all variables continuous (i.e. can take fractional values)
- a single objective (maximize or minimize)
- the objective and constraints are linear i.e. any term is either a constant or a constant multiplied by an unknown (e.g. 24, $4x$, $6y$ are linear terms but xy or x^2 is a non-linear term)

Any formulation which satisfies these three conditions is called a *linear program* (LP).

We have (implicitly) assumed that it is permissible to work in fractions of days - problems where this is not permissible and variables must take integer values will be dealt with under *Integer Programming* (IP).

Discussion

This problem was a *decision problem.*

We have taken a real-world situation and constructed an equivalent mathematical representation - such a representation is often called a mathematical *model* of the real-world situation (and the process by which the model is obtained is called *formulating* the model).

Just to confuse things the mathematical model of the problem is sometimes called the *formulation* of the problem.

Having obtained our mathematical model we (hopefully) have some quantitative method which will enable us to numerically solve the model (i.e. obtain a numerical solution) - such a quantitative method is often called an *algorithm* for solving the model.

Essentially an algorithm (for a particular model) is a set of instructions which, when followed in a step-by-step fashion, will produce a numerical solution to that model.

Our model has an *objective*, that is something which we are trying to *optimize*.
Having obtained the numerical solution of our model we have to translate that solution back into the real-world situation.

"OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing."

3. LINEAR PROGRAMMING

It can be recalled from the Two Mines example that the conditions for a mathematical model to be a linear program (LP) were:

- all variables continuous (i.e. can take fractional values)
- a single objective (minimize or maximize)
- the objective and constraints are linear i.e. any term is either a constant or a constant multiplied by an unknown.

LP's are important - this is because:

- many practical problems can be formulated as LP's
- there exists an algorithm (called the *simplex* algorithm) which enables us to solve LP's numerically relatively easily

We will return later to the simplex algorithm for solving LP's but for the moment we will concentrate upon formulating LP's.

Some of the major application areas to which LP can be applied are:

- Work scheduling
- Production planning & Production process
- Capital budgeting
- Financial planning
- Blending (e.g. Oil refinery management)
- Farm planning
- Distribution
- Multi-period decision problems
  - Inventory model
  - Financial models
  - Work scheduling

Note that the key to formulating LP's is practice. However a useful hint is that common objectives for LP's are maximize profit/minimize cost.

There are four basic assumptions in LP:

- Proportionality
    - The contribution to the objective function from each decision variable is proportional to the value of the decision variable (The contribution to the objective function from making four soldiers ($4\times\$3=\$12$) is exactly four times the contribution to the objective function from making one soldier ($3))
    - The contribution of each decision variable to the LHS of each constraint is proportional to the value of the decision variable (It takes exactly three times as many finishing hours (2hrs$\times$3=6hrs) to manufacture three soldiers as it takes to manufacture one soldier (2 hrs))
- Additivity
    - The contribution to the objective function for any decision variable is independent of the values of the other decision variables (No matter what the value of train (x_2), the manufacture of soldier (x_1) will always contribute $3x_1$ dollars to the objective function)
    - The contribution of a decision variable to LHS of each constraint is independent of the values of other decision variables (No matter what the value of x_1, the manufacture of x_2 uses x_2 finishing hours and x2 carpentry hours)
        - *1st implication*: The value of objective function is the sum of the contributions from each decision variables.
        - *2nd implication*: LHS of each constraint is the sum of the contributions from each decision variables.
- Divisibility
    - Each decision variable is allowed to assume fractional values. If we actually can not produce a fractional number of decision variables, we use IP (It is acceptable to produce 1.69 trains)
- Certainty
    - Each parameter is known with certainty

3.1 FORMULATING LP

3.1.1 Giapetto Example

(Winston 3.1, p. 49)

Giapetto's wooden soldiers and trains. Each soldier sells for $27, uses $10 of raw materials and takes $14 of labor & overhead costs. Each train sells for $21, uses $9 of raw materials, and takes $10 of overhead costs. Each soldier needs 2 hours finishing and 1 hour carpentry; each train needs 1 hour finishing and 1 hour carpentry. Raw materials are unlimited, but only 100 hours of finishing and 80 hours of carpentry are available each week. Demand for trains is unlimited; but at most 40 soldiers can be sold each week. How many of each toy should be made each week to maximize profits?

Answer

Decision variables completely describe the decisions to be made (in this case, by Giapetto). Giapetto must decide how many soldiers and trains should be manufactured each week. With this in mind, we define:

x_1 = the number of soldiers produced per week

x_2 = the number of trains produced per week

Objective function is the function of the decision variables that the decision maker wants to maximize (revenue or profit) or minimize (costs). Giapetto can concentrate on maximizing the total weekly profit (z).

Here profit equals to (weekly revenues) – (raw material purchase cost) – (other variable costs). Hence Giapetto's objective function is:

Maximize $z = 3x_1 + 2x_2$

Constraints show the restrictions on the values of the decision variables. Without constraints Giapetto could make a large profit by choosing decision variables to be very large. Here there are three constraints:

Finishing time per week

Carpentry time per week

Weekly demand for soldiers

Sign restrictions are added if the decision variables can only assume nonnegative values (Giapetto can not manufacture negative number of soldiers or trains!)

All these characteristics explored above give the following ***Linear Programming*** (LP) model

$$
\begin{array}{lll}
\max z = 3x_1 + 2x_2 & & \text{(The Objective function)} \\
\text{s.t.} & 2x_1 + x_2 \leq 100 & \text{(Finishing constraint)} \\
& x_1 + x_2 \leq 80 & \text{(Carpentry constraint)} \\
& x_1 \leq 40 & \text{(Constraint on demand for soldiers)} \\
& x_1, x_2 \geq 0 & \text{(Sign restrictions)}
\end{array}
$$

A value of (x_1, x_2) is in the ***feasible region*** if it satisfies all the constraints and sign restrictions.

Graphically and computationally we see the solution is (x_1, x_2) = (20, 60) at which z = 180. (***Optimal solution***)

Report

The maximum profit is $180 by making 20 soldiers and 60 trains each week. Profit is limited by the carpentry and finishing labor available. Profit could be increased by buying more labor.

3.1.2 Advertisement Example

(Winston 3.2, p.61)

Dorian makes luxury cars and jeeps for high-income men and women. It wishes to advertise with 1 minute spots in comedy shows and football games. Each comedy spot costs $50K and is seen by 7M high-income women and 2M high-income men. Each football spot costs $100K and is seen by 2M high-income women and 12M high-income men. How can Dorian reach 28M high-income women and 24M high-income men at the least cost?

Answer

The decision variables are

x_1 = the number of comedy spots

x_2 = the number of football spots

The model of the problem:

$$
\begin{array}{lrcrl}
\min z = 50x_1 + 100x_2 & & & & \\
\text{st} & 7x_1 & + & 2x_2 & \geq 28 \\
& 2x_1 & + & 12x_2 & \geq 24 \\
x_1, x_2 \geq 0 & & & &
\end{array}
$$

The graphical solution is z = 320 when (x_1, x_2) = (3.6, 1.4). From the graph, in this problem rounding up to (x_1, x_2) = (4, 2) gives the best *integer* solution.

Report

The minimum cost of reaching the target audience is $400K, with 4 comedy spots and 2 football slots. The model is dubious as it does not allow for saturation after repeated viewings.

3.1.3 Diet Example

(Winston 3.4., p. 70)

Ms. Fidan's diet requires that all the food she eats come from one of the four "basic food groups". At present, the following four foods are available for consumption: brownies, chocolate ice cream, cola, and pineapple cheesecake. Each brownie costs 0.5$, each scoop of chocolate ice cream costs 0.2$, each bottle of cola costs 0.3$, and each pineapple cheesecake costs 0.8$. Each day, she must ingest at least 500 calories, 6 oz of chocolate, 10 oz of sugar, and 8 oz of fat. The nutritional content per unit of each food is shown in Table. Formulate an LP model that can be used to satisfy her daily nutritional requirements at minimum cost.

	Calories	Chocolate (ounces)	Sugar (ounces)	Fat (ounces)
Brownie	400	3	2	2
Choc. ice cream (1 scoop)	200	2	2	4
Cola (1 bottle)	150	0	4	1
Pineapple cheesecake (1 piece)	500	0	4	5

Answer

The decision variables:

x_1: number of brownies eaten daily

x_2: number of scoops of chocolate ice cream eaten daily

x_3: bottles of cola drunk daily

x_4: pieces of pineapple cheesecake eaten daily

The objective function (the total cost of the diet in cents):

$$\min w = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

Constraints:

$$400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500 \quad \text{(daily calorie intake)}$$

$$3x_1 + 2x_2 \geq 6 \quad \text{(daily chocolate intake)}$$

$$2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10 \quad \text{(daily sugar intake)}$$

$$2x_1 + 4x_2 + x_3 + 5x_4 \geq 8 \quad \text{(daily fat intake)}$$

$$x_i \geq 0, \; i = 1, 2, 3, 4 \quad \text{(Sign restrictions!)}$$

Report

The minimum cost diet incurs a daily cost of 90 cents by eating 3 scoops of chocolate and drinking 1 bottle of cola ($w = 90$, $x_2 = 3$, $x_3 = 1$)

3.1.4 Post Office Example

(Winston 3.5, p.74)

A PO requires different numbers of employees on different days of the week. Union rules state each employee must work 5 consecutive days and then receive two days off. Find the minimum number of employees needed.

	Mon	Tue	Wed	Thur	Fri	Sat	Sun
Staff Needed	17	13	15	19	14	16	11

Answer

The decision variables are x_i (# of employees starting on day i)

Mathematically we must

$$\begin{array}{llllllllll}
\min z = & x_1 & + x_2 & + x_3 & + x_4 & + x_5 & + x_6 & + x_7 & & \\
\text{s.t.} & x_1 & & & + x_4 & + x_5 & + x_6 & + x_7 & \geq & 17 \\
 & x_1 & + x_2 & & & + x_5 & + x_6 & + x_7 & \geq & 13 \\
 & x_1 & + x_2 & + x_3 & & & + x_6 & + x_7 & \geq & 15 \\
 & x_1 & + x_2 & + x_3 & + x_4 & & & + x_7 & \geq & 19 \\
 & x_1 & + x_2 & + x_3 & + x_4 & + x_5 & & & \geq & 14 \\
 & & + x_2 & + x_3 & + x_4 & + x_5 & + x_6 & & \geq & 16 \\
 & & & + x_3 & + x_4 & + x_5 & + x_6 & + x_7 & \geq & 11 \\
 & x_t \geq 0, \forall t & & & & & & & &
\end{array}$$

The solution is $(x_i) = (4/3, 10/3, 2, 22/3, 0, 10/3, 5)$ giving $z = 67/3$.

We could round this up to $(x_i) = (2, 4, 2, 8, 0, 4, 5)$ giving $z = 25$ (may be wrong!).

However restricting the decision var.s to be integers and using Lindo again gives $(x_i) = (4, 4, 2, 6, 0, 4, 3)$ giving $z = 23$.

3.1.5 Sailco Example

(Winston 3.10, p. 99)

Sailco must determine how many sailboats to produce in the next 4 quarters. The demand is known to be 40, 60, 75, and 25 boats. Sailco must meet its demands. At the beginning of the 1st quarter Sailco starts with 10 boats in inventory. Sailco can produce up to 40 boats with regular time labor at $400 per boat, or additional boats at

\$450 with overtime labor. Boats made in a quarter can be used to meet that quarter's demand or held in inventory for the next quarter at an extra cost of \$20.00 per boat.

Answer

The decision variables are for t = 1,2,3,4

x_t = # of boats in quarter t built in regular time

y_t = # of boats in quarter t built in overtime

For convenience, introduce variables:

i_t = # of boats in inventory at the end quarter t

d_t = demand in quarter t

We are given that $d_1 = 40$, $d_2 = 60$, $d_3 = 75$, $d_4 = 25$, $i_0 = 10$

$x_t \leq 40, \forall t$

By logic $i_t = i_{t-1} + x_t + y_t - d_t, \forall t.$

Demand is met iff $i_t \geq 0, \forall t$

(Sign restrictions $x_t, y_t \geq 0, \forall t$)

We need to minimize total cost z subject to these three sets of conditions where

$$z = 400\ (x_1 + x_2 + x_3 + x_4) + 450\ (y_1 + y_2 + y_3 + y_4) + 20\ (i_1 + i_2 + i_3 + i_4)$$

Report:

Lindo reveals the solution to be $(x_1, x_2, x_3, x_4) = (40, 40, 40, 25)$ and $(y_1, y_2, y_3, y_4) = (0, 10, 35, 0)$ and the minimum cost of \$78450.00 is achieved by the schedule

		Q_1	Q_2	Q_3	Q_4
Regular time (x_t)		40	40	40	25
Overtime (y_t)		0	10	35	0
Inventory (i_t)	10	10	0	0	0
Demand (d_t)		40	60	75	25

3.1.6 Customer Service Level Example

(Winston 3.12, p. 108)

CSL services computers. Its demand (hours) for the time of skilled technicians in the next 5 months is

t	Jan	Feb	Mar	Apr	May
d_t	6000	7000	8000	9500	11000

It starts with 50 skilled technicians at the beginning of January. Each technician can work 160 hrs/month. To train a new technician they must be supervised for 50 hrs by an experienced technician for a period of one month time. Each experienced technician is paid \$2K/mth and a trainee is paid \$1K/mth. Each month 5% of the skilled technicians leave. CSL needs to meet demand and minimize costs.

Answer

The decision variable is

x_t = # to be trained in month t

We must minimize the total cost. For convenience let

y_t = # experienced tech. at start of t^{th} month

d_t = demand during month t

Then we must

min $z = 2000\,(y_1 + ... + y_5) + 1000\,(x_1 + ... + x_5)$

subject to

$160y_t - 50x_t \geq d_t$ for $t = 1,..., 5$

$y_1 = 50$, $d_1 = 6000$, $d_2 = 7000$, $d_3 = 8000$, $d_4 = 9500$, $d_5 = 11000$

$y_t = .95y_{t-1} + x_{t-1}$ for $t = 2,3,4,5$

$x_t,\ y_t \geq 0$

3.2 SOLVING LP

3.2.1 LP Solutions: Four Cases

When an LP is solved, one of the following four cases will occur:

1. The LP has a **unique optimal solution**.
2. The LP has **alternative (multiple) optimal solutions**. It has more than one (actually an infinite number of) optimal solutions
3. The LP is **infeasible**. It has no feasible solutions (The feasible region contains no points).
4. The LP is **unbounded**. In the feasible region there are points with arbitrarily large (in a max problem) objective function values.

3.2.2 The Graphical Solution

Any LP with only two variables can be solved graphically

Example 1. Giapetto

(*Winston 3.1, p. 49*)

Since the Giapetto LP has two variables, it may be solved graphically.

Answer

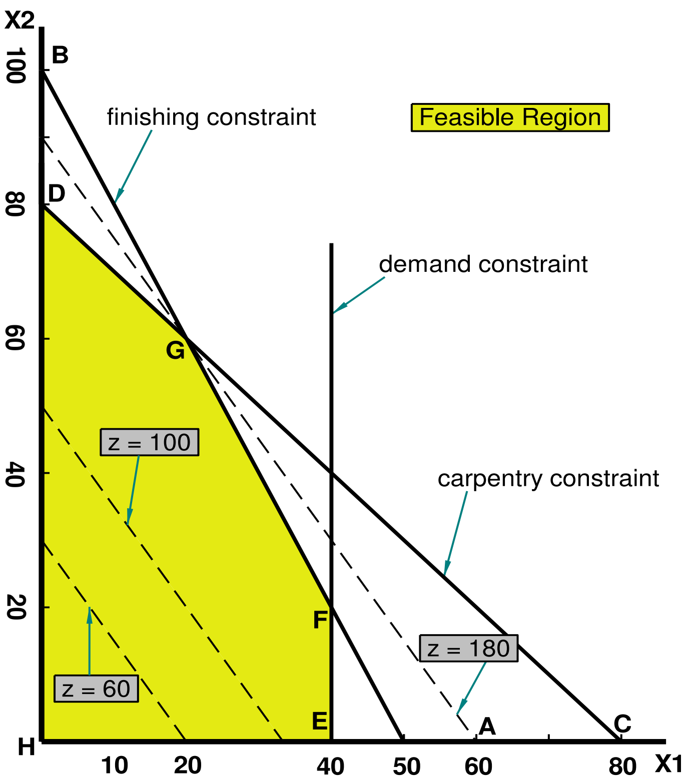
The feasible region is the set of all points satisfying the constraints.

$$\begin{array}{llll}
\max z = 3x_1 + 2x_2 & & & \\
\text{s.t.} & 2x_1 + x_2 & \le 100 & \text{(Finishing constraint)} \\
 & x_1 + x_2 & \le 80 & \text{(Carpentry constraint)} \\
 & x_1 & \le 40 & \text{(Demand constraint)} \\
 & x_1, x_2 & \ge 0 & \text{(Sign restrictions)}
\end{array}$$

The set of points satisfying the LP is bounded by the five sided polygon DGFEH. Any point ***on*** or ***in*** the interior of this polygon (the shade area) is in the ***feasible region***.

Having identified the feasible region for the LP, a search can begin for the ***optimal solution*** which will be the point in the feasible region with the *largest* z-value (maximization problem).

To find the optimal solution, a line on which the points have the same z-value is graphed. In a max problem, such a line is called an ***isoprofit*** line while in a min problem, this is called the ***isocost*** line. (*The figure shows the isoprofit lines for z = 60, z = 100, and z = 180*).



In the unique optimal solution case, isoprofit line last hits a point (vertex - corner) before leaving the feasible region.

The optimal solution of this LP is point G where (x_1, x_2) = (20, 60) giving z = 180.

A constraint is ***binding*** (active, tight) if the left-hand and right-hand side of the constraint are equal when the optimal values of the decision variables are substituted into the constraint.

A constraint is ***nonbinding*** (inactive) if the left-hand side and the right-hand side of the constraint are unequal when the optimal values of the decision variables are substituted into the constraint.

In Giapetto LP, the finishing and carpentry constraints are binding. On the other hand the demand constraint for wooden soldiers is nonbinding since at the optimal solution $x_1 < 40$ ($x_1 = 20$).

Example 2. Advertisement

(*Winston 3.2, p. 61*)

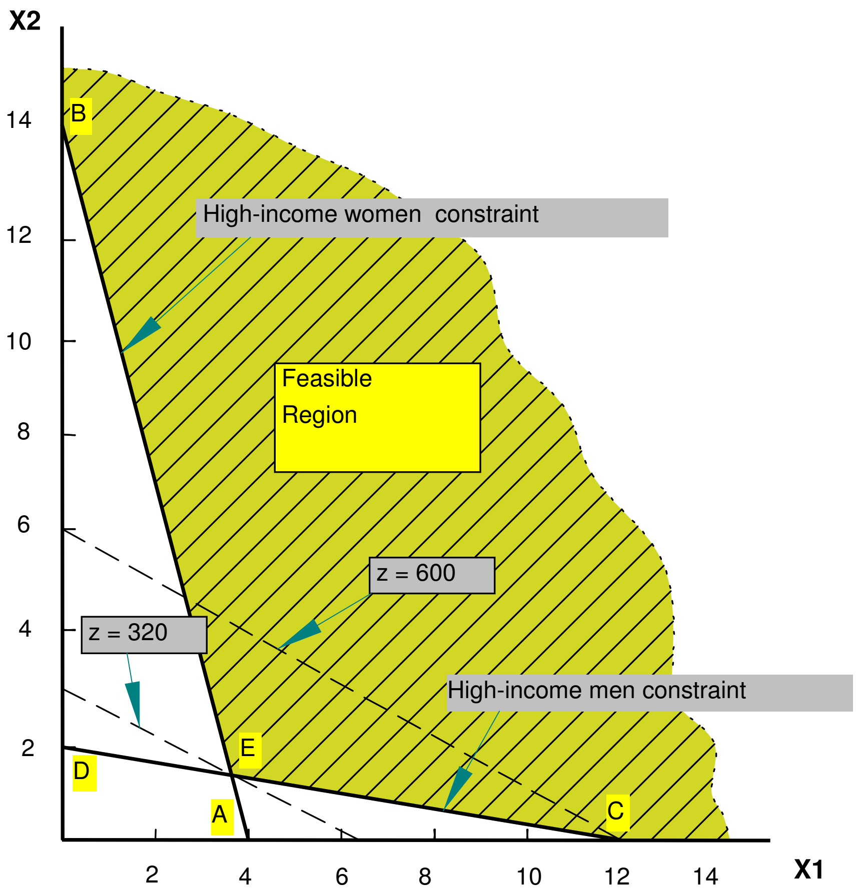
Since the Advertisement LP has two variables, it may be solved graphically.

Answer

The feasible region is the set of all points satisfying the constraints.

$$\min z = 50x_1 + 100x_2$$

$$\begin{aligned} \text{s.t.}\quad & 7x_1 + 2x_2 \geq 28 && \text{(high income women)} \\ & 2x_1 + 12x_2 \geq 24 && \text{(high income men)} \\ & x_1, x_2 \geq 0 \end{aligned}$$



Since Dorian wants to minimize total advertising costs, the optimal solution to the problem is the point in the feasible region with the smallest z value.

An isocost line with the smallest z value passes through point E and is the optimal solution at $x_1 = 3.6$ and $x_2 = 1.4$ giving $z = 320$.

Both the high-income women and high-income men constraints are satisfied, both constraints are binding.

Example 3. Two Mines

$$\min 180x + 160y$$

$$\text{st} \quad 6x + y \geq 12$$

$$3x + y \geq 8$$

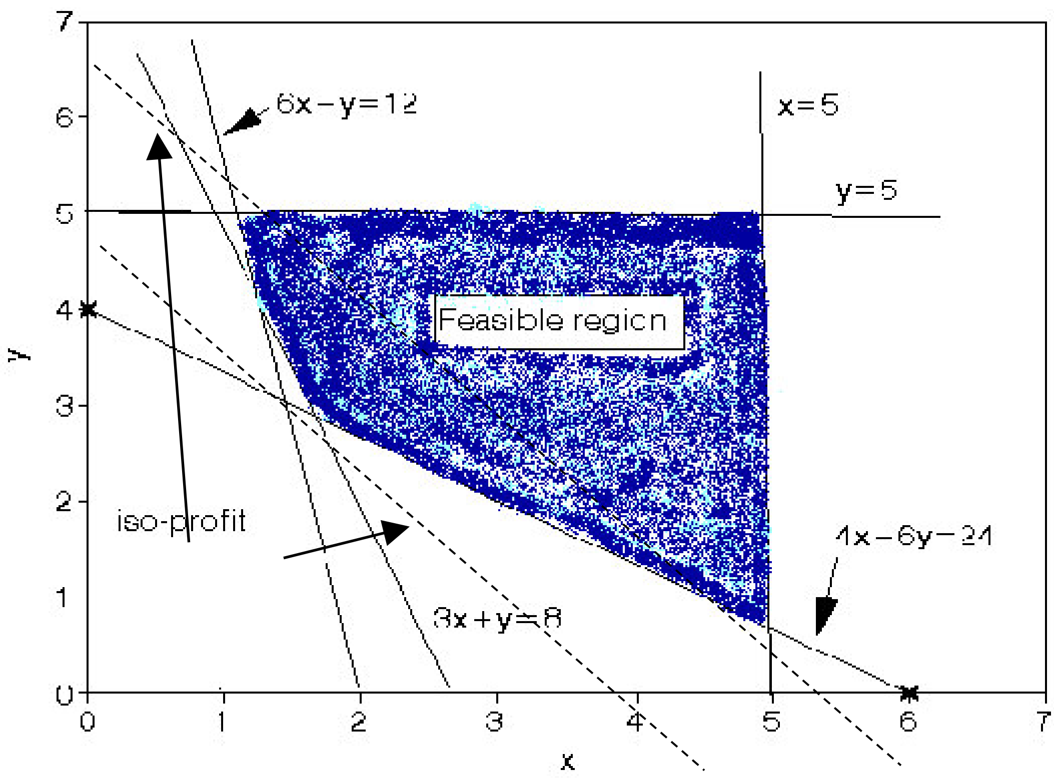
$$4x + 6y \geq 24$$

$$x \leq 5$$

$$y \leq 5$$

$$x, y \geq 0$$

Answer



Optimal sol'n is 765.71. 1.71 days mine *X* and 2.86 days mine *Y* are operated.

Example 4. Modified Giapetto

$$\max z = 4x_1 + 2x_2$$

s.t.	$2x_1 + x_2 \leq 100$	(Finishing constraint)
	$x_1 + x_2 \leq 80$	(Carpentry constraint)
	$x_1 \leq 40$	(Demand constraint)
	$x_1, x_2 \geq 0$	(Sign restrictions)

Answer

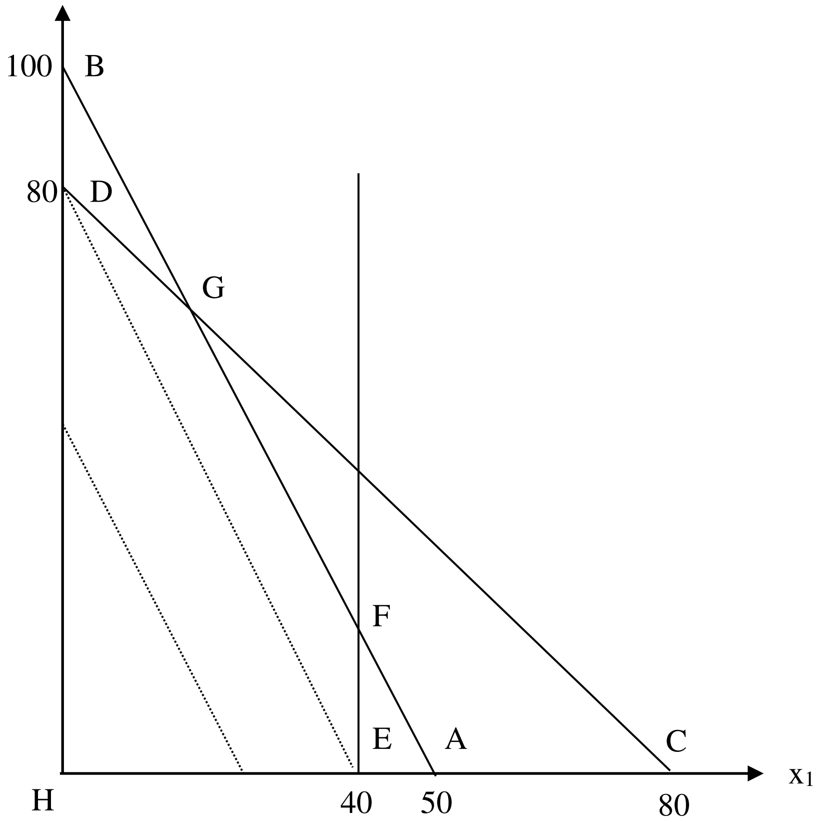
Points on the line between points G (20, 60) and F (40, 20) are the ***alternative optimal solutions*** (see figure below).

Thus, for $0 \leq c \leq 1$,

c [20 60] + (1 - c) [40 20] = **[40 - 20*c*, 20 + 40*c*]**

will be optimal

For all optimal solutions, the optimal objective function value is 200.



Example 5. Modified Giapetto (v. 2)

Add constraint $x_2 \geq 90$ (Constraint on demand for trains).

Answer

No feasible region: ***Infeasible LP***

Example 6. Modified Giapetto (v. 3)

Only use constraint $x_2 \geq 90$

Answer

Isoprofit line never lose contact with the feasible region: ***Unbounded LP***