Legendre Duplication Formula

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For more information, see Eric's *MathWorld* entry http://mathworld.wolfram.com/LegendreDuplicationFormula.html.

Derivation

$$\text{Integrate}\left[u^{m-1}\,(1-u)^{n-1},\ \{u,0,1\}\right]$$

$$\text{If}\left[\text{Re}[m]>0\ \&\&\ \text{Re}[n]>0,\ \frac{\text{Gamma}[m]\ \text{Gamma}[n]}{\text{Gamma}[m+n]},\ \text{Integrate}\left[(1-u)^{-1+n}\,u^{-1+m},\ \{u,0,1\},\ \text{Assumptions}\to\text{Re}[m]\le 0\ ||\ \text{Re}[n]\le 0\right]\right]$$

$$\left\{2\ \text{FullSimplify}\left[\text{Integrate}\left[x^{2m-1}\,(1-x^2)^{n-1},\ \{x,0,1\},\ \text{Assumptions}\to\text{Re}[m]>0\ \&\&\ \text{Re}[n]>0\right]\right],\ \text{FunctionExpand}[\text{Beta}[m,n]]\right\}$$

$$\left\{\frac{\text{Gamma}[m]\ \text{Gamma}[n]}{\text{Gamma}[m+n]},\ \frac{\text{Gamma}[m]\ \text{Gamma}[n]}{\text{Gamma}[m+n]}\right\}$$

$$\text{Integrate}\left[\left(\frac{1+x}{2}\right)^{z-1}\left(1-\frac{1+x}{2}\right)^{z-1}\Big/2,\ \{x,-1,1\},\ \text{Assumptions}\to\text{Re}[z]>0\right]$$

$$\frac{2^{1-2z}\sqrt{\pi}\ \text{Gamma}[z]}{\text{Gamma}\left[\frac{1}{2}+z\right]}$$

$$2\times 2^{1-2z}\ \text{Integrate}\left[(1-x\text{^}2)^{z-1},\ \{x,0,1\},\ \text{Assumptions}\to\text{Re}[z]>0\right]$$

$$\frac{2^{1-2z}\sqrt{\pi}\ \text{Gamma}[z]}{\text{Gamma}\left[\frac{1}{2}+z\right]}$$

```
Solve[Gamma[z] Gamma[z]/Gamma[2 z] == 2^(1-2 z) Gamma[1/2] Gamma[z]/Gamma[z + 1/2], Gamma[2 z]]
```

$$\left\{\left\{\text{Gamma}[2\,z] \to \frac{2^{-1+2\,z}\,\text{Gamma}[z]\,\text{Gamma}\left[\frac{1}{2}+z\right]}{\sqrt{\pi}}\right\}\right\}$$

```
(2 π)^(-1/2) 2^(2 z-1/2) Gamma[z] Gamma[z + 1/2]
```

$$\frac{2^{-1+2\,z}\,\text{Gamma}[z]\,\text{Gamma}\left[\frac{1}{2}+z\right]}{\sqrt{\pi}}$$

```
Table[(2 π)^(-1/2) 2^(2 z-1/2) Gamma[z] Gamma[z + 1/2] - Gamma[2 z] /.
   z → Random[Complex, {-2 - 2 I, 2 + 2 I}], {10}] // Chop
```

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
FunctionExpand[Gamma[2 z]]
```

Gamma[2 z]

```
FullSimplify[(2 π)^(-1/2) 2^(2 z-1/2) Gamma[z] Gamma[z + 1/2]]
```

Gamma[2 z]