The Gamma Function, Beta Function, and Duplication Formula

The title might as well continue — *because I constantly forget them and hope that writing about them will make me remember*. At least afterwards I'll have a centralized repository for my preferred proofs, regardless.

In this note, we will play with the Gamma and Beta functions and eventually get to Legendre's Duplication formula for the Gamma function. This is part reference, so I first will write the results themselves.

1. Results

We define the Gamma function for $s > 0$ by

$$\Gamma(s) := \int_0^\infty t^s e^t \frac{dt}{t}. \tag{1}$$

Similarly, we define the Beta function by

$$B(a, b) := \int_0^1 t^{a-1}(1-t)^{b-1} dt \tag{2}$$

for $a, b > 0$.

From these defininitions, it is not so obvious that these two functions are intimately related – but they are! In fact,

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(x, y) \tag{3}$$

Evaluating the Gamma function at integers is easy. We can use the relation with the Beta function to evalate it at half-integers too.

$$\Gamma(\frac{1}{2}) = \sqrt{\pi} \tag{4}$$

Finally, we can relate the values at half-integers and integers in an intimate way.

$$\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\Gamma(2z) \tag{5}$$

for $\mathrm{Re}(z) > 0$.

1.1. Proof of 3

We begin by writing down a different representation of the Beta function.

$$B(a,b) = \int_0^\infty \frac{u^a}{(1+u)^{a+b}} \frac{du}{u},$$

which is in terms of the Haar measure and is generally more agreeable. *Proof:* Consider the (uninspired) substitution $u = \frac{t}{1-t}$, or equivalently $t = \frac{u}{1+u}$. Then the bounds $0 \mapsto 0$ and $1 \mapsto \infty$, and the integrand transforms exactly into the form in the proposition. $\square$

We will also want a different representation of the Gamma function.

$$\int_0^\infty e^{-pt} t^z \frac{dt}{t} = \frac{\Gamma(z)}{p^z}.$$

Proof: This comes quite quickly. Performing the change of variables $s = pt$ in the integral definition of the Gamma function pops out the extra p^z factor and gives this form of the integral. $\square$

We can now put these together. Rearranging the lemma above gives

$$\frac{1}{p^z} = \frac{1}{\Gamma(z)} \int_0^\infty e^{-pt} t^{z-1}\, dt.$$

Thinking of $p = 1 + u$ and $z = a + b$, we can substitute this expression inside the lemma-given integral expression for the Beta function.

$$\begin{aligned}
B(a,b) &= \frac{1}{\Gamma(a+b)} \int_0^\infty e^{-t} t^{a+b-1} dt \int_0^\infty e^{-ut} u^{a-1} du && (1) \\
&= \frac{\Gamma(a)}{\Gamma(a+b)} \int_0^\infty e^{-t} t^{b-1} dt && (2) \\
&= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, && (3)
\end{aligned}$$

where the first Gamma factor pulled out a factors of t from the first integral. This completes the proof of Prop 3.

1.2. Proof of Prop 4

We begin with another integral representation of the Beta function.

$$B(a,b) = 2\int_0^{\pi/2} (\cos u)^{2a-1}(\sin u)^{2b-1} du$$

for a, b with positive real part. *Proof:* This comes immediately from the change of variables $t = \cos^2 u$ in the integral definition of the Beta function. It's necessary to flip the bounds of integration to cancel the negative sign from the sign of the change of variables. $\square$

In this form, it is particularly easy to see that $B(\frac{1}{2}, \frac{1}{2}) = \pi$, since we integrate the constant function 1 from 0 to $\pi/2$ and multiply the result by 2. And from Prop 1, we know that $B(\frac{1}{2}, \frac{1}{2}) = (\Gamma(\frac{1}{2})^2$ (as $\Gamma(1) = 1$).

Thus $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, and we know it's the positive square root because $\Gamma(\frac{1}{2})$ is clearly positive. This completes the proof.

This is my favorite proof, as it uses neither complex analysis nor multivariable integration – both of which are dear to my heart, but separate from the pleasant theory of the Gamma function.

1.3. Proof of Theorem 5

Start from

$$\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)} = B(z,z) = \int_0^1 u^{z-1}(1-u)^{z-1} du.$$

Perform the substitution $u = \frac{1+x}{2}$, so that $du = dx/2$. This transforms the above integral into

$$2^{1-2z} \cdot 2\int_0^1 (1-x^2)^{z-1} dx.$$

$$B(m,n) = 2\int_0^1 x^{2m-1}(1-x^2)^{n-1} dx$$

Proof: This is immediate upon the change of variables $t = x^2$ in the defining integral for the Beta function. □

This allows us to recognize the integral above

$$2^{1-2z} \cdot 2 \int_0^1 (1-x^2)^{z-1} dx = 2^{1-2z} B(\frac{1}{2}, z).$$

Rewriting in terms of Gammas,

$$B(\frac{1}{2}, z) = \frac{\Gamma(\frac{1}{2})\Gamma(z)}{\Gamma(z+\frac{1}{2})}.$$

In total, we have that

$$\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)} = 2^{1-2z} \frac{\Gamma(\frac{1}{2})\Gamma(z)}{\Gamma(z+\frac{1}{2})}.$$

Rearranging, and using that $\Gamma(1/2) = \sqrt{\pi}$ as above, we see that

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma(z+\frac{1}{2}),$$

which is what we wanted to show.

3 RESPONSES TO *THE GAMMA FUNCTION, BETA FUNCTION, AND DUPLICATION FORMULA*

David Lowry-Duda *says:*

MAY 11, 2015 AT 5:26 PM

I can't believe how often I've come back to this post. Above all others, I refer back here the most.

I always forget the duplication formula. And when I see it, I need to remember why it's true. Perhaps I should find a more meaningful/memorable proof of the duplication formula.

REPLY

Nisarg *says:*

NOVEMBER 11, 2017 AT 5:01 AM

In the proof of 5, where you substituted u=(1+x)/2, you didn't change the value of limits ,

It should be -1 to +1

REPLY

David Lowry-Duda (mixedmath) *says:*

NOVEMBER 12, 2017 AT 10:22 AM

Dear Nisarg,

Thank you for pointing this out. You're right that the bounds should be -1 to 1, but as the integrand is symmetric I went ahead and doubled the integral and replaced the bounds with 0 to 1 in the same step. This is the additional 2 in front of the integral, and the reason why I separated that 2 from the other 2 factors.

REPLY