Particle Physics Phenomenology
2. Phase space and matrix elements

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Four-vectors

$$
\begin{aligned}
\text{four}-\text{vector}: \quad & p = (E; \mathbf{p}) = (E; p_x, p_y, p_z) \\
\text{vector sum}: \quad & p_1 + p_2 = (E_1 + E_2; \mathbf{p}_1 + \mathbf{p}_2) \\
\text{vector product}: \quad & p_1 p_2 = E_1 E_2 - \mathbf{p}_1 \mathbf{p}_2 \\
& = E_1 E_2 - p_{x1} p_{x2} - p_{y1} p_{y2} - p_{z1} p_{z2} \\
& = E_1 E_2 - |\mathbf{p}_1| \, |\mathbf{p}_2| \, \cos\theta_{12} \\
\text{square}: \quad & p^2 = E^2 - \mathbf{p}^2 = E^2 - p_x^2 - p_y^2 - p_z^2 = m^2 \\
\text{transverse mom.}: \quad & p_\perp = \sqrt{p_x^2 + p_y^2} \\
\text{transverse mass}: \quad & m_\perp = \sqrt{m^2 + p_x^2 + p_y^2} = \sqrt{m^2 + p_\perp^2} \\
& E^2 = m^2 + \mathbf{p}^2 = m^2 + p_\perp^2 + p_z^2 = m_\perp^2 + p_z^2
\end{aligned}
$$

Warning: No standard to distinguish $p = (E; p_x, p_y, p_z)$ and $p = |\mathbf{p}| = \sqrt{p_x^2 + p_y^2 + p_z^2}$, but usually clear from context. When we remember, we will try to use $\underline{p} = |\mathbf{p}|$, since $\overline{p} = \mathbf{p}$.

Decay widths and cross sections

Decay width at rest, $1 \to n$:

$$\mathrm{d}\Gamma = \frac{|\mathcal{M}|^2}{2M}\,\mathrm{d}\Phi_n$$

Integrated it gives exponential decay rate

$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t} = \Gamma e^{-\Gamma t} \quad \text{and} \quad \langle\tau\rangle = 1/\Gamma$$

Collision process cross section, $2 \to n$:

$$\mathrm{d}\sigma = \frac{|\mathcal{M}|^2}{4\sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}}\,\mathrm{d}\Phi_n$$

Integrated it gives collision rate

$$N = \sigma \int \mathcal{L}(t)\,\mathrm{d}t \quad \text{with} \quad \mathcal{L} \approx f\,\frac{n_1 n_2}{A}$$

in a theorist's approximation of the luminosity $\mathcal{L}$ for a collider.

Phase space

n-body phase space:

$$\mathrm{d}\Phi_n = (2\pi)^4 \delta^{(4)}(P - \sum_{i=1}^{n} p_i) \prod_{i=1}^{n} \frac{\mathrm{d}^3 p_i}{(2\pi)^3 2E_i}$$

Lorentz covariant:

$$\begin{aligned} \mathrm{d}^4 p_i \, \delta(p_i^2 - m_i^2) \, \theta(E_i) &= \mathrm{d}^4 p_i \, \delta(E_i^2 - (\mathbf{p}_i^2 + m_i^2)) \, \theta(E_i) \\ &= \frac{\mathrm{d}^3 p_i}{2E_i} \end{aligned}$$

with $E_i = \sqrt{\mathbf{p}_i^2 + m_i^2}$ and using

$$\delta(f(x)) = \sum_{x_j, f(x_j)=0} \frac{1}{|f'(x_j)|} \, \delta(x - x_j)$$

Application: Lorentz invariant production cross sections $E \, \mathrm{d}\sigma / \mathrm{d}^3 p$

Spherical symmetry

Spherical coordinates:

$$\frac{\mathrm{d}^3 p}{E} = \frac{\mathrm{d}p_x \,\mathrm{d}p_y \,\mathrm{d}p_z}{E} = \frac{\underline{p}^2 \,\mathrm{d}\underline{p} \,\mathrm{d}\Omega}{E} = \frac{\underline{p}\, E \mathrm{d}E \,\mathrm{d}\Omega}{E} = \underline{p} \,\mathrm{d}E \,\mathrm{d}\Omega$$

where Ω is the unit sphere,

$$\begin{aligned}
\mathrm{d}\Omega &= \mathrm{d}(\cos\theta)\,\mathrm{d}\phi = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
p_x &= \underline{p}\,\sin\theta\,\cos\varphi \\
p_y &= \underline{p}\,\sin\theta\,\sin\varphi \\
p_z &= \underline{p}\,\cos\theta
\end{aligned}$$

and $E^2 = \underline{p}^2 + m^2 \Rightarrow E\,\mathrm{d}E = \underline{p}\,\mathrm{d}\underline{p}$.

Convenient for use e.g. in resonance decays,
but not for standard QCD physics in pp collisions.
Instead:

Cylindrical symmetry and rapidity

Cylindrical coordinates:

$$\frac{\mathrm{d}^3 p}{E} = \frac{\mathrm{d}p_x \, \mathrm{d}p_y \, \mathrm{d}p_z}{E} = \frac{\mathrm{d}^2 p_\perp \, \mathrm{d}p_z}{E} = \mathrm{d}^2 p_\perp \, \mathrm{d}y$$

with rapidity y given by

$$\begin{aligned} y &= \frac{1}{2} \ln \frac{E + p_z}{E - p_z} = \frac{1}{2} \ln \frac{(E + p_z)^2}{(E + p_z)(E - p_z)} = \frac{1}{2} \ln \frac{(E + p_z)^2}{m^2 + p_\perp^2} \\ &= \ln \frac{E + p_z}{m_\perp} = \ln \frac{m_\perp}{E - p_z} \end{aligned}$$

The relation $\mathrm{d}y = \mathrm{dp}_z / E$ can be shown by

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}p_z} &= \frac{\mathrm{d}}{\mathrm{d}p_z} \left(\ln \frac{E + p_z}{m_\perp} \right) = \frac{\mathrm{d}}{\mathrm{d}p_z} \left(\ln(\sqrt{m_\perp^2 + p_z^2} + p_z) - \ln m_\perp \right) \\ &= \frac{\frac{1}{2} \frac{2p_\perp}{\sqrt{m_\perp^2 + p_z^2}} + 1}{\sqrt{m_\perp^2 + p_z^2} + p_z} = \frac{\frac{p_z + E}{E}}{E + p_z} = \frac{1}{E} \end{aligned}$$

Lightcone kinematics and boosts

Introduce (lightcone) $p^+ = E + p_z$ and $p^- = E - p_z$.
Note that $p^+ p^- = E^2 - p_z^2 = m_\perp^2$.
Consider boost along z axis with velocity β, and $\gamma = 1/\sqrt{1-\beta^2}$.

$$
\begin{aligned}
p'_{x,y} &= p_{x,y} \\
p'_z &= \gamma(p_z + \beta\, E) \\
E' &= \gamma(E + \beta\, p_z) \\
p'^+ &= \gamma(1+\beta)p^+ = \sqrt{\frac{1+\beta}{1-\beta}}\, p^+ = k\, p^+ \\
p'^- &= \gamma(1-\beta)p^+ = \sqrt{\frac{1-\beta}{1+\beta}}\, p^- = \frac{p^-}{k} \\
y' &= \frac{1}{2}\ln\frac{p'^+}{p'^-} = \frac{1}{2}\ln\frac{k\, p^+}{p'^-/k} = y + \ln k \\
y'_2 - y'_1 &= (y_2 + \ln k) - (y_1 + \ln k) = y_2 - y_1
\end{aligned}
$$

Pseudorapidity

If experimentalists cannot measure m they may assume $m = 0$.
Instead of rapidity y they then measure pseudorapidity η:

$$y = \frac{1}{2} \ln \frac{\sqrt{m^2 + \mathbf{p}^2} + p_z}{\sqrt{m^2 + \mathbf{p}^2} - p_z} \quad \Rightarrow \quad \eta = \frac{1}{2} \ln \frac{|\mathbf{p}| + p_z}{|\mathbf{p}| - p_z} = \ln \frac{|\mathbf{p}| + p_z}{p_\perp}$$

or

$$\begin{aligned} \eta &= \frac{1}{2} \ln \frac{\underline{p} + \underline{p} \cos\theta}{\underline{p} - \underline{p} \cos\theta} = \frac{1}{2} \ln \frac{1 + \cos\theta}{1 - \cos\theta} \\ &= \frac{1}{2} \ln \frac{2\cos^2\theta/2}{2\sin^2\theta/2} = \ln \frac{\cos\theta/2}{\sin\theta/2} = -\ln\tan\frac{\theta}{2} \end{aligned}$$

which thus only depends on polar angle.
η is **not** simple under boosts: $\eta'_2 - \eta'_1 \neq \eta_2 - \eta_1$.
You may even flip sign!
Assume $m = m_\pi$ for all charged $\Rightarrow y_\pi$; intermediate to y and η.

The pseudorapidity dip

By analogy with $\mathrm{d}y/\mathrm{d}p_z = 1/E$ it follows that $\mathrm{d}\eta/\mathrm{d}p_z = 1/\underline{p}$.

Thus

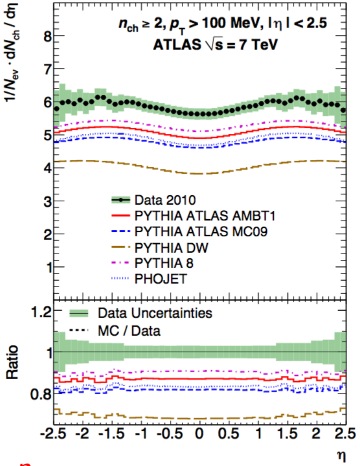
$$\frac{\mathrm{d}\eta}{\mathrm{d}y} = \frac{\mathrm{d}\eta/\mathrm{d}p_z}{\mathrm{d}y/\mathrm{d}p_z} = \frac{E}{\underline{p}} > 1$$

with limits

$$\frac{\mathrm{d}\eta}{\mathrm{d}y} \rightarrow \frac{m_\perp}{p_\perp} \text{ for } p_z \rightarrow 0$$

$$\frac{\mathrm{d}\eta}{\mathrm{d}y} \rightarrow 1 \text{ for } p_z \rightarrow \pm\infty$$

so if $\mathrm{d}n/\mathrm{d}y$ is flat for $y \approx 0$
then $\mathrm{d}n/\mathrm{d}\eta$ has a dip there.



$$\eta - y = \ln \frac{\underline{p} + p_z}{p_\perp} - \ln \frac{E + p_z}{m_\perp} = \ln \frac{\underline{p} + p_z}{E + p_z} \frac{m_\perp}{p_\perp} \rightarrow \ln \frac{m_\perp}{p_\perp} \text{ when } p_z \gg m_\perp$$

Two-body phase space

Evaluate in rest frame, i.e. $P = (E_{\rm cm}, \mathbf{0})$.

$$
\begin{aligned}
\mathrm{d}\Phi_2 &= (2\pi)^4 \delta^{(4)}(P - p_1 - p_2)\,\frac{\mathrm{d}^3 p_1}{(2\pi)^3 2E_1}\,\frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_2} \\
&= \frac{1}{16\pi^2}\delta(E_{\rm cm} - E_1 - E_2)\frac{\mathrm{d}^3 p_1}{E_1 E_2} \\
&= \frac{1}{16\pi^2}\delta\left(\sqrt{m_1^2 + \underline{p}^2} + \sqrt{m_2^2 + \underline{p}^2} - E_{\rm cm}\right)\frac{\underline{p}^2\,\mathrm{d}\underline{p}\,\mathrm{d}\Omega}{E_1 E_2} \\
&= \frac{1}{16\pi^2}\,\frac{\delta(\underline{p} - \underline{p}^*}{|\frac{\underline{p}}{E_1} + \frac{\underline{p}}{E_2}|}\,\frac{\underline{p}^2\,\mathrm{d}\underline{p}\,\mathrm{d}\Omega}{E_1 E_2} \\
&= \frac{1}{16\pi^2}\,\frac{E_1 E_2}{E_1 + E_2}\,\frac{\underline{p}\,\mathrm{d}\Omega}{E_1 E_2} \\
&= \frac{\underline{p}\,\mathrm{d}\Omega}{16\pi^2\, E_{\rm cm}}
\end{aligned}
$$

The Källén function – 1

$$\sqrt{m_1^2 + \underline{p}^2} + \sqrt{m_2^2 + \underline{p}^2} = E_{\mathrm{cm}}$$

gives solution

$$\begin{aligned}
E_1 &= \frac{E_{\mathrm{cm}}^2 + m_1^2 - m_2^2}{2E_{\mathrm{cm}}} \\
E_2 &= \frac{E_{\mathrm{cm}}^2 + m_2^2 - m_1^2}{2E_{\mathrm{cm}}} \\
\underline{p} &= \frac{1}{2E_{\mathrm{cm}}}\sqrt{(E_{\mathrm{cm}}^2 - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2} = \frac{1}{2E_{\mathrm{cm}}}\sqrt{\lambda(E_{\mathrm{cm}}^2, m_1^2, m_2^2)}
\end{aligned}$$

where Källén λ function is

$$\begin{aligned}
\lambda(a^2, b^2, c^2) &= (a^2 - b^2 - c^2)^2 - 4b^2c^2 \\
&= a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 \\
&= (a^2 - (b+c)^2)(a^2 - (b-c)^2) \\
&= (a+b+c)(a-b-c)(a-b+c)(a+b-c)
\end{aligned}$$

The Källén function – 2

Hides everywhere in kinematics, e.g.

$$\mathrm{d}\sigma = \frac{|\mathcal{M}|^2}{4\sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}}\,\mathrm{d}\Phi_n$$

has

$$\begin{aligned} 4((p_1 p_2)^2 - m_1^2 m_2^2) &= (p_1^2 + 2p_1 p_2 + p_2^2 - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2 \\ &= ((p_1 + p_2)^2 - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2 \\ &= \lambda(E_{\mathrm{cm}}^2, m_1^2, m_2^2) \end{aligned}$$

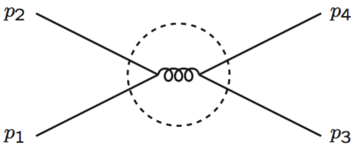
so

$$\mathrm{d}\sigma = \frac{|\mathcal{M}|^2}{2\sqrt{\lambda(E_{\mathrm{cm}}^2, m_1^2, m_2^2)}}\,\mathrm{d}\Phi_n$$

Mandelstam variables

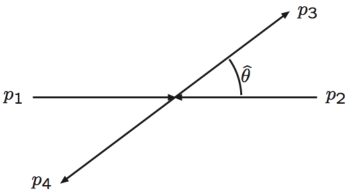
For process $1 + 2 \to 3 + 4$

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\
t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 \\
u &= (p_1 - p_4)^2 = (p_2 - p_3)^2
\end{aligned}
$$



In rest frame, massless limit: $m_1 = m_2 = m_3 = m_4 = 0$,

$$
\begin{aligned}
p_{1,2} &= \frac{E_{\mathrm{cm}}}{2}(1; 0, 0, \pm) \\
p_{3,4} &= \frac{E_{\mathrm{cm}}}{2}(1; \pm \sin\hat{\theta}, 0, \pm\cos\hat{\theta}) \\
s &= E_{\mathrm{cm}}^2 \\
t &= -2p_1 p_3 = -\frac{s}{2}(1 - \cos\hat{\theta}) \\
u &= -2p_2 p_4 = -\frac{s}{2}(1 + \cos\hat{\theta})
\end{aligned}
$$



$$s + t + u = 0$$

Mandelstam variables with masses

$$
\begin{aligned}
\beta_{34} &= \frac{\sqrt{\lambda(s, m_3^2, m_4^2)}}{s} \\
p_{3,4} &= \frac{\sqrt{s}}{2} \left(1 \pm \frac{m_3^2 - m_4^2}{s}; \pm\beta_{34} \sin\hat{\theta}, 0, \pm\beta_{34} \cos\hat{\theta} \right) \\
t &= m_1^2 + m_3^2 - \frac{s}{2} \left(1 + \frac{m_1^2 - m_2^2}{s} \right) \left(1 + \frac{m_3^2 - m_4^2}{s} \right) \\
&\quad + \frac{s}{2} \, \beta_{12} \, \beta_{34} \cos\hat{\theta} \\
\mathrm{d}\sigma &= \frac{|\mathcal{M}|^2}{2\sqrt{\lambda(s, m_1^2, m_2^2)}} \, \frac{p_{34}}{\sqrt{s}} \, \frac{\mathrm{d}\cos\hat{\theta} \, \mathrm{d}\varphi}{16\pi^2} = \frac{|\mathcal{M}|^2}{2s\beta_{12}} \, \frac{\beta_{34}}{2} \, \frac{\mathrm{d}\cos\hat{\theta}}{8\pi}
\end{aligned}
$$

assuming no polarization $\Rightarrow$ no φ dependence

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\mathrm{d}\sigma}{\mathrm{d}cos\hat{\theta}} \, \frac{\mathrm{d}cos\hat{\theta}}{\mathrm{d}t} = \frac{|\mathcal{M}|^2}{16\pi s^2 \beta_{12}^2}
$$

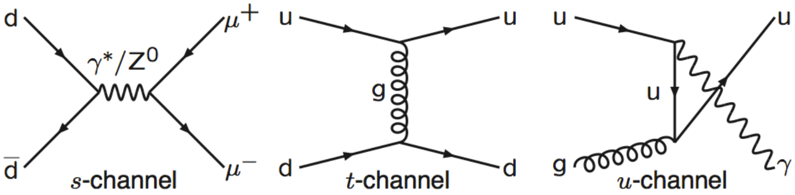
Mandelstam variables with final-state masses

Usually $m_{1,2} \approx 0$, while often $m_{3,4}$ non-negligible

$$
\begin{aligned}
t, u &= -\frac{1}{2}\left[s - m_3^2 - m_4^2 \mp s\beta_{34}\cos\hat{\theta}\right] \\
\frac{\mathrm{d}\sigma}{\mathrm{d}t} &= \frac{|\mathcal{M}|^2}{16\pi s^2} \\
s + t + u &= m_3^2 + m_4^2 \\
tu &= \frac{1}{4}\left[(s - m_3^2 - m_4^2)^2 - s^2\beta_{34}^2\cos^2\hat{\theta}\right] \\
&= \frac{1}{4}\left[s^2\beta_{34}^2 + 4m_3^2m_4^2 - s^2\beta_{34}^2\cos^2\hat{\theta}\right] \\
&= \frac{1}{4}s^2\beta_{34}^2\sin^2\hat{\theta} + m_3^2m_4^2 = sp_\perp^2 + m_3^2m_4^2 \\
p_\perp^2 &= \frac{tu - m_3^2m_4^2}{s}
\end{aligned}
$$

s-, t- and u-channel processes

Classify $2 \to 2$ diagrams by character of propagator, e.g.



Singularities reflect channel character, e.g. pure t-channel:

$$\frac{\mathrm{d}\sigma(\mathrm{qq}' \to \mathrm{qq}')}{\mathrm{d}t} = \frac{\pi}{s^2}\frac{4}{9}\alpha_\mathrm{s}^2\frac{s^2+u^2}{t^2}$$

peaked at $t \to 0 \Rightarrow u \approx -s$, so

$$\frac{\mathrm{d}\sigma(\mathrm{qq}' \to \mathrm{qq}')}{\mathrm{d}t} \approx \frac{8\pi\alpha_\mathrm{s}^2}{9t^2} = \frac{32\pi\alpha_\mathrm{s}^2}{9s^2(1-\cos\hat{\theta})^2} = \frac{8\pi\alpha_\mathrm{s}^2}{9s^2\sin^4\hat{\theta}/2} \approx \frac{8\pi\alpha_\mathrm{s}^2}{9p_\perp^4}$$

i.e. Rutherford scattering!

Order-of-magnitude cross sections

With masses neglected:

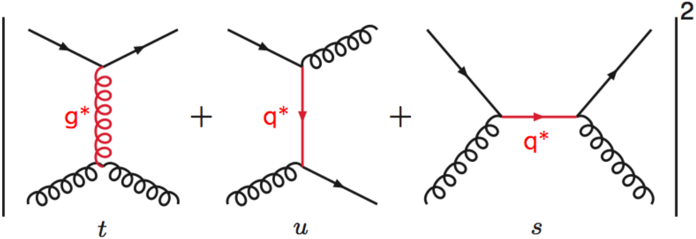
$$
\begin{aligned}
s-\text{channel} &: \quad \frac{\mathrm{d}\sigma}{\mathrm{d}t} \sim \frac{\pi}{s^2} \\
t-\text{channel, spin } 1 &: \quad \frac{\mathrm{d}\sigma}{\mathrm{d}t} \sim \frac{\pi}{t^2} \\
t-\text{channel, spin } \frac{1}{2} &: \quad \frac{\mathrm{d}\sigma}{\mathrm{d}t} \sim \frac{\pi}{-st} \\
u-\text{channel} &: \quad \text{same with } t \to u
\end{aligned}
$$

Add couplings at vertices:

$$
\begin{aligned}
\mathrm{qqg} &: \quad C_F \alpha_{\mathrm{s}} \\
\mathrm{ggg} &: \quad N_c \alpha_{\mathrm{s}} \\
\mathrm{f\,f}\gamma &: \quad e_{\mathrm{f}}^2 \alpha_{\mathrm{em}} \\
\mathrm{f\,f'W} &: \quad |V_{\mathrm{ff'}}|^2 \frac{\alpha_{\mathrm{em}}}{4 \sin^2\theta_W} \\
\mathrm{f\,f'Z} &: \quad (v_{\mathrm{f}}^2 + a_{\mathrm{f}}^2) \frac{\alpha_{\mathrm{em}}}{16 \sin^2\theta_W \cos^2\theta_W}
\end{aligned}
$$

Closeup: $\mathrm{qg} \to \mathrm{qg}$

Consider $\mathrm{q}(1)\,\mathrm{g}(2) \to \mathrm{q}(3)\,\mathrm{g}(4)$:

$|\mathcal{M}|^2 =$ 

$t : p_{\mathrm{g}^*} = p_1 - p_3 \Rightarrow m^2_{\mathrm{g}^*} = (p_1 - p_3)^2 = t \Rightarrow \mathrm{d}\sigma/\mathrm{d}t \sim 1/t^2$

$u : p_{\mathrm{q}^*} = p_1 - p_4 \Rightarrow m^2_{\mathrm{q}^*} = (p_1 - p_4)^2 = u \Rightarrow \mathrm{d}\sigma/\mathrm{d}t \sim -1/su$

$s : p_{\mathrm{q}^*} = p_1 + p_2 \Rightarrow m^2_{\mathrm{q}^*} = (p_1 + p_2)^2 = s \Rightarrow \mathrm{d}\sigma/\mathrm{d}t \sim 1/s^2$

Contribution of each sub-graph is gauge-dependent, only sum is well-defined:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\pi\alpha_{\mathrm{s}}^2}{s^2}\left[\frac{s^2+u^2}{t^2} + \frac{4}{9}\frac{s}{(-u)} + \frac{4}{9}\frac{(-u)}{s}\right]$$

Scale choice

What Q^2 scale to use for $\alpha_{\mathrm{s}} = \alpha_{\mathrm{s}}(Q^2)$?
Should be characteristic virtuality scale of process!
But e.g. for $\mathrm{q}\,\mathrm{g} \to \mathrm{q}\,\mathrm{g}$: both s-, t- and u-channel + interference.
At small t the t-channel graph dominates $\Rightarrow Q^2 \sim |t|$,
at small u the u-channel graph dominates $\Rightarrow Q^2 \sim |u|$,
in between all graphs comparably important $\Rightarrow Q^2 \sim s \sim |t| \sim |u|$.
Suitable interpolation:

$$Q^2 = p_{\perp}^2 = \frac{tu}{s} \quad \begin{array}{ll} \rightarrow & -t \text{ for } t \rightarrow 0 \\ \rightarrow & -u \text{ for } u \rightarrow 0 \\ \rightarrow & \dfrac{s}{4} \text{ for } t = u = -\dfrac{s}{2} \end{array}$$

but could equally well be multiple of $p_{\perp}^2$, or more complicated
$\Rightarrow$ one limitation of LO calculations.

Resonances

Resonance shape given by Breit-Wigner

$$
\begin{aligned}
1 \quad &\mapsto \quad \rho(s) = \frac{1}{\pi} \frac{m\Gamma}{(s-m^2)^2 + m^2\Gamma^2} \\
&\mapsto \quad \frac{1}{\pi} \frac{s\Gamma(m)/m}{(s-m^2)^2 + s^2\Gamma^2(m)/m^2}
\end{aligned}
$$

where $m \mapsto \sqrt{s}$ in phase space and $\Gamma(s) \mapsto \Gamma(m)\sqrt{s}/m$
for gauge bosons, neglecting thresholds.
Latter shape suppressed below and enhanced above peak; tilted.
For $s \to 0$ $\rho(s)$ goes to constant or like s.
PDF's tend to be peaked at small x: convolution enhances small s.
Can give secondary mass-spectrum "peak" in $s \to 0$ region.
But note that

$$|\mathcal{M}|^2 = |\mathcal{M}_{\mathrm{signal}} + \mathcal{M}_{\mathrm{background}}|^2$$

so in many cases Breit-Wigner cannot be trusted except in the neighbourhood of the peak, where signal should dominate.

Three-body phase space

Three-body final states has $3 \cdot 3 - 4$ degrees of freedom.
In massless case straightforward to show that, in CM frame,

$$\begin{aligned} \mathrm{d}\Phi_3 &= (2\pi)^4 \delta^{(4)}(P - p_1 - p_2 - p_3) \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_2} \frac{\mathrm{d}^3 p_3}{(2\pi)^3 2E_3} \\ &= \frac{1}{8(2\pi)^5} \mathrm{d}E_1 \, \mathrm{d}E_2 \, \mathrm{d}\cos\theta_1 \, \mathrm{d}\varphi_1 \, \mathrm{d}\varphi_{21} \end{aligned}$$

with θ_1, φ_1 polar coordinates of 1 and
φ_{21} azimuthal angle of 2 around 1 axis (Euler angles).
Phase space limits $0 \leq E_{1,2} \leq E_{\mathrm{cm}}/2$ and
$E_1 + E_2 = E_{\mathrm{cm}} - E_3 > E_{\mathrm{cm}}/2$.

Same simple phase space expression holds in massive case,
but phase space limits much more complicated!

Higher multiplicities increasingly difficult to understand.
One solution: recursion!

Factorized three-body phase space

Drop factors of 2π, and don't write implicit integral signs.
Introduce intermediate "particle" $12 = 1 + 2$.

$$
\begin{aligned}
& \mathrm{d}\Phi_3(P; p_1, p_2, p_3) \\
\sim\; & \delta^{(4)}(P - p_1 - p_2 - p_3) \frac{\mathrm{d}^3 p_1}{2E_1} \frac{\mathrm{d}^3 p_2}{2E_2} \frac{\mathrm{d}^3 p_3}{2E_3} \, \delta^{(4)}(p_{12} - p_1 - p_2) \, \mathrm{d}^4 p_{12} \\
=\; & \delta^{(4)}(P - p_{12} - p_3) \, \mathrm{d}^4 p_{12} \frac{\mathrm{d}^3 p_3}{2E_3} \left[\delta^{(4)}(p_{12} - p_1 - p_2) \frac{\mathrm{d}^3 p_1}{2E_1} \frac{\mathrm{d}^3 p_2}{2E_2} \right] \\
=\; & \delta^{(4)}(P - p_{12} - p_3) \, \mathrm{d}^4 p_{12} \, \delta(p_{12}^2 - m_{12}^2) \, \mathrm{d}m_{12}^2 \frac{\mathrm{d}^3 p_3}{2E_3} \mathrm{d}\Phi_2(p_{12}; p_1, p_2) \\
=\; & \mathrm{d}m_{12}^2 \left[\delta^{(4)}(P - p_{12} - p_3) \frac{\mathrm{d}^3 p_{12}}{2E_{12}} \frac{\mathrm{d}^3 p_3}{2E_3} \right] \mathrm{d}\Phi_2(p_{12}; p_1, p_2) \\
=\; & \mathrm{d}m_{12}^2 \, \mathrm{d}\Phi_2(P; p_{12}, p_3) \, \mathrm{d}\Phi_2(p_{12}; p_1, p_2)
\end{aligned}
$$

Note: here 4 angles + 1 mass^2; last slide 3 angles + 2 energies.

Recursive phase space

Generalizes to

$$\begin{aligned} \mathrm{d}\Phi_n(P; p_1, \ldots, p_n) &= \mathrm{d}m^2_{12\ldots(n-1)}\, \mathrm{d}\Phi_2(P; p_{12\ldots(n-1)}, p_n) \\ &\times \mathrm{d}\Phi_{n-1}(P; p_1, \ldots, p_{(n-1)}) \end{aligned}$$

Can be viewed as a sequential decay chain, with undetermined intermediate masses.

$$\text{Recall } \mathrm{d}\Phi_2(P; p_1, p_2) \propto \frac{\sqrt{\lambda(M^2, m_1^2, m_2^2)}}{M^2}\, \mathrm{d}\Omega_{12}$$

where $\mathrm{d}\Omega_{12}$ is the unit sphere *in the 1+2 rest frame*.
Now can write down e.g. 4-body phase space:

The M-generator

$$\mathrm{d}\Phi_4(P; p_1, p_2, p_3, p_4) \propto \frac{\sqrt{\lambda(M^2; m_4^2, m_{123}^2)}}{M^2} \, m_{123} \, \mathrm{d}m_{123} \, \mathrm{d}\Omega_{1234}$$

$$\times \frac{\sqrt{\lambda(m_{123}^2; m_3^2, m_{12}^2)}}{m_{123}^2} \, m_{12} \, \mathrm{d}m_{12} \, \mathrm{d}\Omega_{123} \, \frac{\sqrt{\lambda(m_{12}^2; m_1^2, m_2^2)}}{m_{12}^2} \, \mathrm{d}\Omega_{12}$$

Mass limits coupled, but can be decoupled: pick two random numbers $0 < R_{1,2} < 1$ and order them $R_1 < R_2$. Then

$$\begin{aligned}
\Delta &= M - (m_1 + m_2 + m_3 + m_4) \\
m_{12} &= m_1 + m_2 + R_1 \Delta \\
m_{123} &= m_1 + m_2 + m_3 + R_2 \Delta
\end{aligned}$$

uniformly covers $\mathrm{d}m_{12} \, \mathrm{d}m_{123}$ space with weight

$$\frac{\sqrt{\lambda(M^2; m_4^2, m_{123}^2)}}{M} \, \frac{\sqrt{\lambda(m_{123}^2; m_3^2, m_{12}^2)}}{m_{123}} \, \frac{\sqrt{\lambda(m_{12}^2; m_1^2, m_2^2)}}{m_{12}}$$

RAMBO

For massless case a smart solution is RAMBO (RAndom Momenta and BOosts), which is 100% efficient:

RAMBO

1. Pick n massless 4-vectors p_i according to

$$E_i e^{-E_i} \, \mathrm{d}\Omega_i$$

2. boost all of them by a common boost vector that brings them to their overall rest frame
3. rescale them by a common factor that brings them to the desired mass M

Can be modified for massive cases, but then no longer 100% efficiency; gets worse the bigger $\sum m_i/M$ is.
MAMBO: workaround for high multiplicities

Efficiency troubles

Even if you can pick phase space points uniformly, $|\mathcal{M}|^2$ is not!
A n-body process receives contributions from a large number of Feynman graphs, plus interferences.
Can lead to extremely low Monte Carlo efficiency.
Intermediate resonances $\Rightarrow$ narrow spikes when $(p_i + p_j)^2 \approx M_{\mathrm{res}}^2$.
t-channel graphs $\Rightarrow$ peaked at small $p_\perp$.

Multichannel techniques:

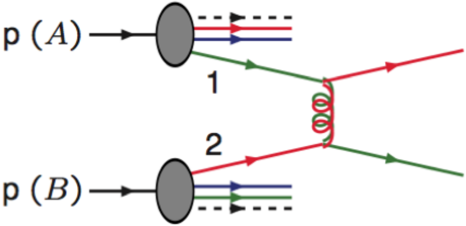
$$|\mathcal{M}|^2 = \frac{|\sum_i \mathcal{M}_i|^2}{\sum_i |\mathcal{M}_i|^2} \sum_i |\mathcal{M}_i|^2$$

so pick optimized for either $|\mathcal{M}_i|^2$ according to their relative integral, and use ratio as weight.
Still major challenge in real life!

Composite beams

In reality all beams are composite:

$\mathrm{p} : \mathrm{q}, \mathrm{g}, \overline{\mathrm{q}}, \ldots$
$e^- : \mathrm{e}^-, \gamma, \mathrm{e}^+, \ldots$
$\gamma : \mathrm{e}^{\pm}, \mathrm{q}, \overline{\mathrm{q}}, \mathrm{g}$



Factorization

$$\sigma^{AB} = \sum_{i,j} \iint \mathrm{d}x_1 \, \mathrm{d}x_2 \, f_i^{(A)}(x_1, Q^2) \, f_j^{(B)}(x_2, Q^2) \, \hat{\sigma}_{ij}$$

x: momentum fraction, e.g. $p_i = x_1 p_A$; $p_j = x_2 p_B$
Q^2: factorization scale, "typical momentum transfer scale"

Factorization only proven for a few cases, like γ^*/Z^0 prodution, and strictly speaking not correct e.g. for jet production,

but good first approximation and unsurpassed physics insight.

Subprocess kinematics

If $p_A + p_B = (E_{\mathrm{cm}}; \mathbf{0})$, A, B along $\pm z$ axis, and $1, 2$ collinear with A, B then convinently put them massless:

$$\begin{aligned} p_1 &= (E_{\mathrm{cm}}/2)(1; 0, 0, 1) \\ p_2 &= (E_{\mathrm{cm}}/2)(1; 0, 0, -1) \end{aligned}$$

such that $\hat{s} = (p_1 + p_2)^2 = x_1\, x_2\, s = \tau\, s$. Velocity of subsystem is

$$\beta_z = \frac{p_z}{E} = \frac{x_1 - x_2}{x_1 + x_2}$$

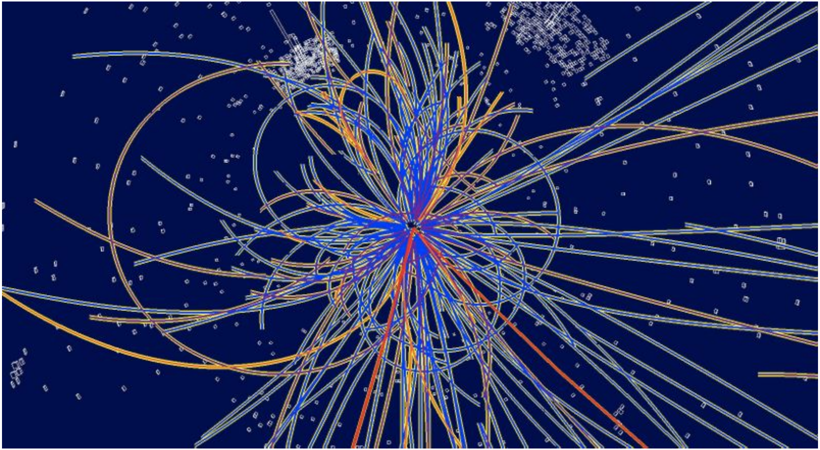
and its rapidity

$$y = \frac{1}{2} \ln \frac{E + p_z}{E - p_z} = \frac{1}{2} \ln \frac{x_1}{x_2}$$

$\mathrm{d}x_1\, \mathrm{d}x_2 = \mathrm{d}\tau\, \mathrm{d}y$ convenient for Monte Carlo.
Historically $x_F = 2p_z/E_{\mathrm{cm}} = x_1 - x_2$.
Subprocess $2 \to 2$ kinematics for $\hat{\sigma}$: $\hat{s}, \hat{t}, \hat{u}$..

Matrix Elements and Their Usage

$\mathcal{L} \Rightarrow$ Feynman rules $\Rightarrow$ Matrix Elements $\Rightarrow$ Cross Sections
$+$ Kinematics $\Rightarrow$ Processes $\Rightarrow \ldots \Rightarrow$



(Higgs simulation in CMS)

QCD at Fixed Order

Distribution of observable: O

In production of X + anything

Sum over identical amplitudes, then square

Momentum configuration

Fixed Order (all orders)

$$\left.\frac{d\sigma}{d\mathcal{O}}\right|_{\mathrm{ME}} = \sum_{k=0} \int d\Phi_{X+k} \left| \sum_{\ell=0} M_{X+k}^{(\ell)} \right|^2 \delta(\mathcal{O} - \mathcal{O}(\{p\}_{X+k}))$$

Phase Space

Cross Section differentially in O

Sum over "anything" ≈ legs

Matrix Elements for X+k at (l) loops

Evaluate observable → differential in O

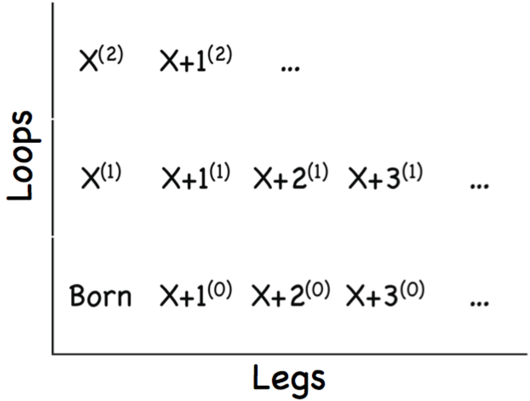
Truncate at k=n, l=0

→ **Leading Order** for X + n

Lowest order at which X + n happens

Loops and Legs

Another representation

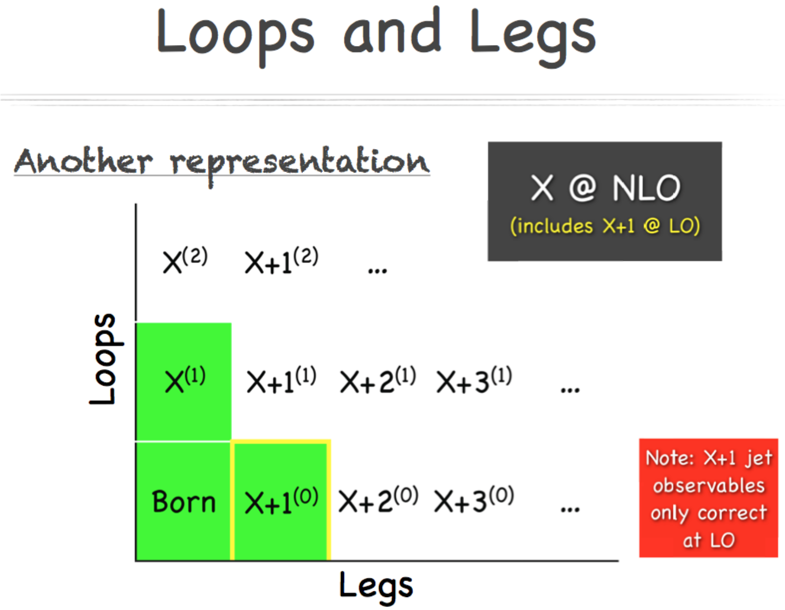


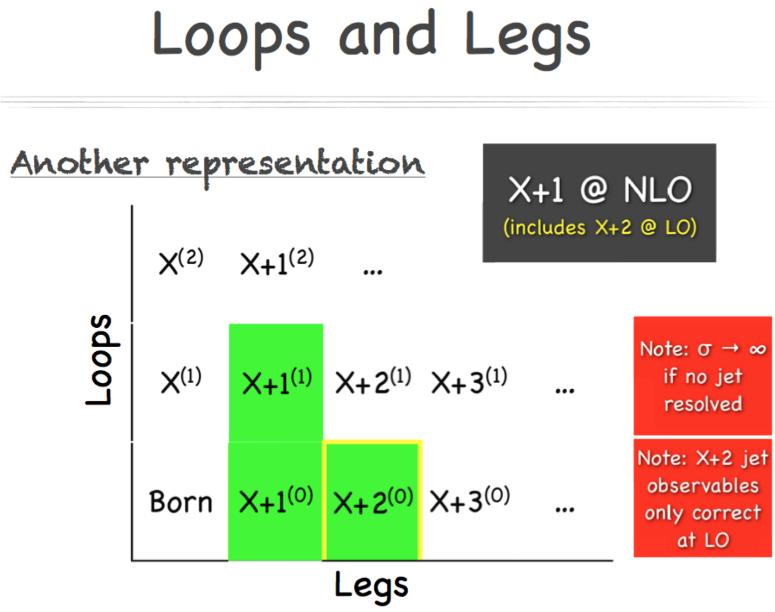
Loops and Legs

Another representation

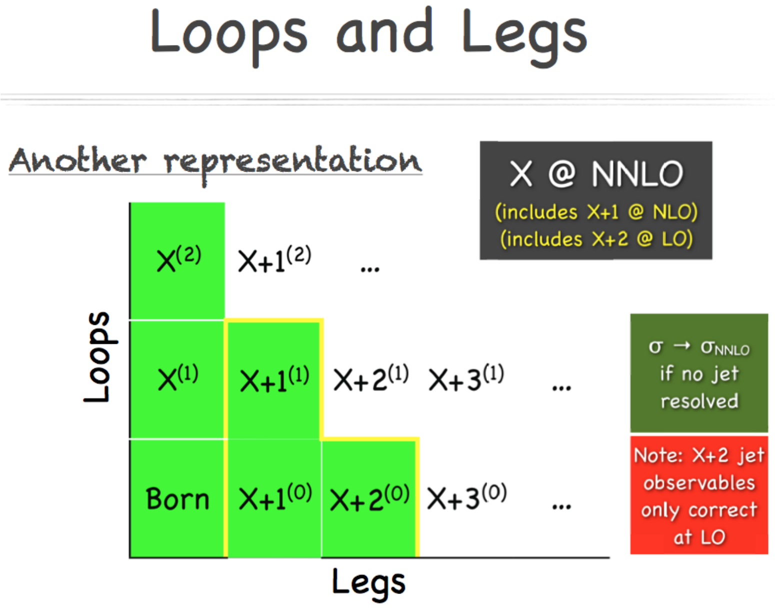
X+2 @ LO

Note: $\sigma \to \infty$ if both jets not resolved


Loops and Legs
Another representation
X @ NLO
(includes X+1 @ LO)
Loops
$X^{(2)}$ $X+1^{(2)}$ ...
$X^{(1)}$ $X+1^{(1)}$ $X+2^{(1)}$ $X+3^{(1)}$ ...
Born $X+1^{(0)}$ $X+2^{(0)}$ $X+3^{(0)}$ ...
Legs
Note: X+1 jet observables only correct at LO


Loops and Legs
Another representation
X+1 @ NLO
(includes X+2 @ LO)
Loops
Legs
X(2) X+1(2) ...
X(1) X+1(1) X+2(1) X+3(1) ...
Born X+1(0) X+2(0) X+3(0) ...
Note: σ → ∞ if no jet resolved
Note: X+2 jet observables only correct at LO

Loops and legs – 6 (Peter Skands)



Stating the problem(s)

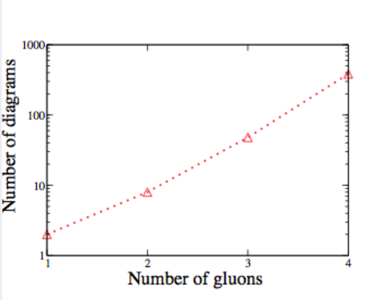
- Multi-particle final states for signals & backgrounds.
- Need to evaluate $\mathrm{d}\sigma_N$:

$$\int_{\mathrm{cuts}} \left[\prod_{i=1}^{N} \frac{\mathrm{d}^3 q_i}{(2\pi)^3 2E_i}\right] \delta^4\left(p_1 + p_2 - \sum_i q_i\right) |\mathcal{M}_{p_1 p_2 \to N}|^2 .$$

- Problem 1: Factorial growth of number of amplitudes.
- Problem 2: Complicated phase-space structure.
- Solutions: Numerical methods.

Example for factorial growth: $e^+e^- \to q\bar{q} + ng$

n	$\#_{\text{diags}}$
0	1
1	2
2	8
3	48
4	384



Remember: to be squared for number of squared MEs.

Born level calculations – 3 (Frank Krauss)

Basic ideas of efficient ME calculation

Need to evaluate $|\mathcal{M}|^2 = \left|\sum_i \mathcal{M}_i\right|^2$

- Obvious: Traditional textbook methods (squaring, completeness relations, traces) fail
  $\Longrightarrow$ result in proliferation of terms $(\mathcal{M}_i\mathcal{M}_j^*)$
  $\Longrightarrow$ Better: Amplitudes are complex numbers,
  $\Longrightarrow$ add them before squaring!
- Remember: spinors, gamma matrices have explicit form could be evaluated numerically (brute force)
  But: Rough method, lack of elegance, CPU-expensive

Helicity method

- Introduce basic helicity spinors (needs to "gauge"-vectors)
- Write everything as spinor products, e.g.

$$\bar{u}(p_1, h_1)u(p_2, h_2) = \text{complex numbers.}$$

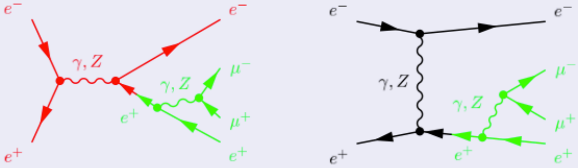
Completeness rel'n: $(\not{p} + m) \implies \frac{1}{2}\sum_h \left[\left(1 + \frac{m^2}{p^2}\right)\bar{u}(p, h)u(p, h) + \left(1 - \frac{m^2}{p^2}\right)\bar{v}(p, h)v(p, h)\right]$

- There are other genuine expressions ...
- Translate Feynman diagrams into "helicity amplitudes": complex-valued functions of momenta & helicities.
- Spin-correlations etc. nearly come for free.

Taming the factorial growth

- In the helicity method
  - Reusing pieces: Calculate only once!
  - Factoring out: Reduce number of multiplications!

  Implemented as a-posteriori manipulations of amplitudes.



- Better method: Recursion relations (recycling built in).
  Best candidate so far: Off-shell recursions

(Dyson-Schwinger, Berends-Giele etc.)

Efficient phase space integration

("Amateurs study strategy, professionals study logistics")

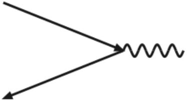
- Democratic, process-blind integration methods:
  - Rambo/Mambo: Flat & isotropic

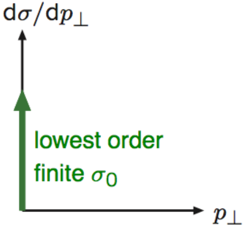
    R.Kleiss, W.J.Stirling & S.D.Ellis, Comput. Phys. Commun. **40** (1986) 359;
  - HAAG/Sarge: Follows QCD antenna pattern

    A.van Hameren & C.G.Papadopoulos, Eur. Phys. J. C **25** (2002) 563.
- Multi-channeling: Each Feynman diagram related to a phase space mapping (= "channel"), optimise their relative weights. R.Kleiss & R.Pittau, Comput. Phys. Commun. **83** (1994) 141.
- Main problem: practical only up to $\mathcal{O}(10\mathrm{k})$ channels.
- Some improvement by building phase space mappings recursively: More channels feasible, efficiency drops a bit.

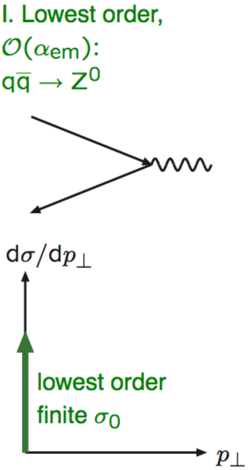
Next-to-leading order (NLO) graphs

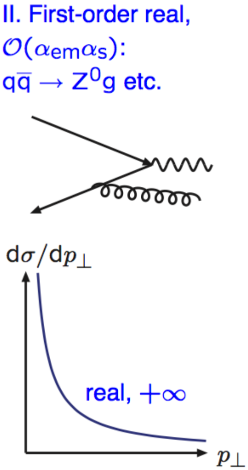
I. Lowest order,
$\mathcal{O}(\alpha_{em})$:
$q\overline{q} \to Z^0$





Next-to-leading order (NLO) graphs





Next-to-leading order (NLO) graphs

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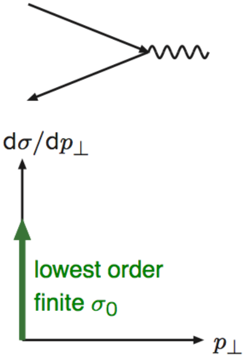
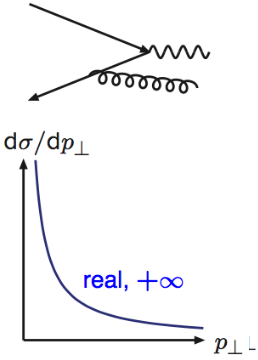
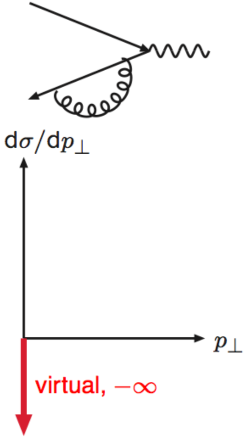


II. First-order real,
$\mathcal{O}(\alpha_{\mathrm{em}}\alpha_{\mathrm{s}})$:
$\mathrm{q}\overline{\mathrm{q}} \to \mathrm{Z}^0\mathrm{g}$ etc.



III. First-order virtual,
$\mathcal{O}(\alpha_{\mathrm{em}}\alpha_{\mathrm{s}})$:
$\mathrm{q}\overline{\mathrm{q}} \to \mathrm{Z}^0$ with loops



NLO calculations – 1

$$\sigma_{\mathrm{NLO}} = \int_n \mathrm{d}\sigma_{\mathrm{LO}} + \int_{n+1} \mathrm{d}\sigma_{\mathrm{Real}} + \int_n \mathrm{d}\sigma_{\mathrm{Virt}}$$

Simple one-dimensional example: $x \sim p_{\perp}/p_{\perp\mathrm{max}}$, $0 \leq x \leq 1$
Divergences regularized by $d = 4 - 2\epsilon$ dimensions, $\epsilon < 0$

$$\sigma_{\mathrm{R+V}} = \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} M(x) + \frac{1}{\epsilon} M_0$$

KLN cancellation theorem: $M(0) = M_0$

NLO calculations – 1

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KLN cancellation theorem: $M(0) = M_0$

Phase Space Slicing:
Introduce arbitrary *finite* cutoff $\delta \ll 1$ (so $\delta \gg |\epsilon|$)

$$\begin{aligned}
\sigma_{\mathrm{R+V}} &= \int_\delta^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} M(x) + \int_0^\delta \frac{\mathrm{d}x}{x^{1+\epsilon}} M(x) + \frac{1}{\epsilon} M_0 \\
&\approx \int_\delta^1 \frac{\mathrm{d}x}{x} M(x) + \int_0^\delta \frac{\mathrm{d}x}{x^{1+\epsilon}} M_0 + \frac{1}{\epsilon} M_0 \\
&= \int_\delta^1 \frac{\mathrm{d}x}{x} M(x) + \frac{1}{\epsilon}\left(1 - \delta^{-\epsilon}\right) M_0 \approx \int_\delta^1 \frac{\mathrm{d}x}{x} M(x) + \ln\delta \, M_0
\end{aligned}$$

NLO calculations – 2

Alternatively **Subtraction**:

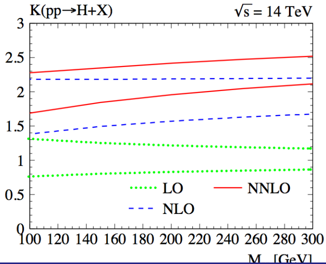
$$
\begin{aligned}
\sigma_{\mathrm{R+V}} &= \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} M(x) - \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} M_0 + \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} M_0 + \frac{1}{\epsilon} M_0 \\
&= \int_0^1 \frac{M(x) - M_0}{x^{1+\epsilon}} \mathrm{d}x + \left(-\frac{1}{\epsilon} + \frac{1}{\epsilon} \right) M_0 \\
&\approx \int_0^1 \frac{M(x) - M_0}{x} \mathrm{d}x + \mathcal{O}(1) M_0
\end{aligned}
$$

NLO calculations – 2

Alternatively **Subtraction**:

$$\begin{aligned}
\sigma_{\mathrm{R+V}} &= \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} M(x) - \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} M_0 + \int_0^1 \frac{\mathrm{d}x}{x^{1+\epsilon}} M_0 + \frac{1}{\epsilon} M_0 \\
&= \int_0^1 \frac{M(x) - M_0}{x^{1+\epsilon}} \mathrm{d}x + \left(-\frac{1}{\epsilon} + \frac{1}{\epsilon} \right) M_0 \\
&\approx \int_0^1 \frac{M(x) - M_0}{x} \mathrm{d}x + \mathcal{O}(1) M_0
\end{aligned}$$

NLO provides a more accurate answer for an integrated cross section:



Warning!
Neither approach operates with positive definite quantities.
No obvious event-generator implementation.
No trivial connection to physical events

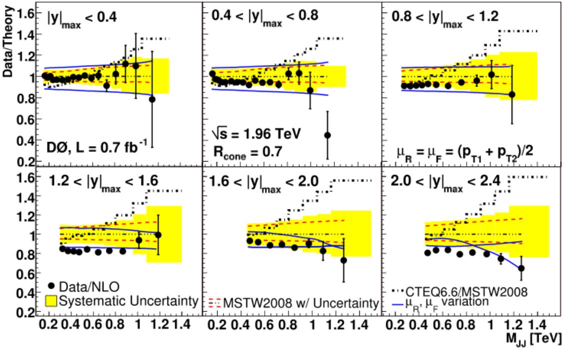
Scale choices

Cross section depends on factorization scale μ_F and renormalization scale μ_R:

$$\sigma^{AB} = \sum_{i,j} \iint \mathrm{d}x_1 \, \mathrm{d}x_2 \, f_i^{(A)}(x_1, \mu_F) \, f_j^{(B)}(x_2, \mu_F) \, \hat{\sigma}_{ij}(\alpha_\mathrm{s}(\mu_R), \mu_F, \mu_R)$$

Historically common to put $Q = \mu_F = \mu_R$ but nowadays varied independently to gauge undertainty of cross section prediction.

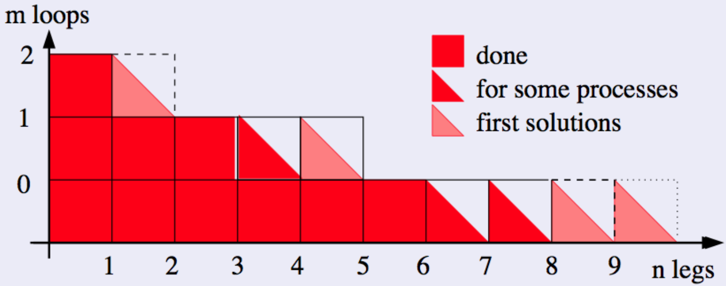
Typical variation factor $2^{\pm 1}$ around "natural value", but beware



Current status (N)(N)LO (Frank Krauss)

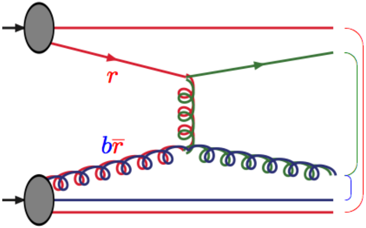
Availability of exact calculations (hadron colliders)

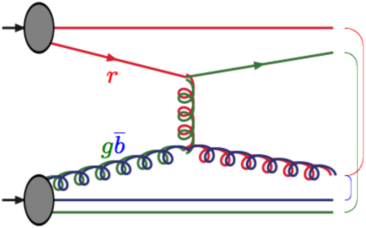
- Fixed order matrix elements ("parton level") are exact to a given perturbative order. (and often quite a pain!)
- Important to understand limitations:
  Only tree-level fully automated, 1-loop-level ongoing.



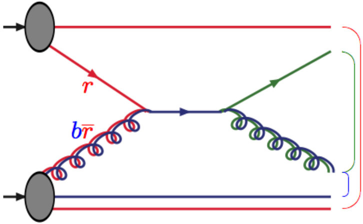
Colour flow in hard processes – 1

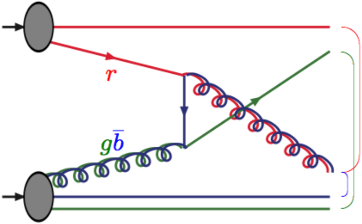
One Feynman graph can correspond to several possible colour flows, e.g. for $\mathrm{qg} \to \mathrm{qg}$:





while other $\mathrm{qg} \to \mathrm{qg}$ graphs only admit one colour flow:





Colour flow in hard processes – 2

so nontrivial mix of kinematics variables $(\hat{s}, \hat{t})$
and colour flow topologies I, II:

$$\begin{aligned} |\mathcal{A}(\hat{s},\hat{t})|^2 &= |\mathcal{A}_{\mathrm{I}}(\hat{s},\hat{t}) + \mathcal{A}_{\mathrm{II}}(\hat{s},\hat{t})|^2 \\ &= |\mathcal{A}_{\mathrm{I}}(\hat{s},\hat{t})|^2 + |\mathcal{A}_{\mathrm{II}}(\hat{s},\hat{t})|^2 + 2\,\mathcal{R}e\left(\mathcal{A}_{\mathrm{I}}(\hat{s},\hat{t})\mathcal{A}^*_{\mathrm{II}}(\hat{s},\hat{t})\right) \end{aligned}$$

with $\mathcal{R}e\left(\mathcal{A}_{\mathrm{I}}(\hat{s},\hat{t})\mathcal{A}^*_{\mathrm{II}}(\hat{s},\hat{t})\right) \neq 0$
$\Rightarrow$ indeterminate colour flow, while
• showers *should* know it (coherence),
• hadronization *must* know it (hadrons singlets).
Normal solution:

$$\frac{\mathrm{interference}}{\mathrm{total}} \propto \frac{1}{N_{\mathrm{C}}^2 - 1}$$

so split $\mathrm{I} : \mathrm{II}$ according to proportions in the $N_{\mathrm{C}} \to \infty$ limit, i.e.

$$\begin{aligned} |\mathcal{A}(\hat{s},\hat{t})|^2 &= |\mathcal{A}_{\mathrm{I}}(\hat{s},\hat{t})|^2_{\mathrm{mod}} + |\mathcal{A}_{\mathrm{II}}(\hat{s},\hat{t})|^2_{\mathrm{mod}} \\ |\mathcal{A}_{\mathrm{I(II)}}(\hat{s},\hat{t})|^2_{\mathrm{mod}} &= |\mathcal{A}_{\mathrm{I}}(\hat{s},\hat{t}) + \mathcal{A}_{\mathrm{II}}(\hat{s},\hat{t})|^2 \left(\frac{|\mathcal{A}_{\mathrm{I(II)}}(\hat{s},\hat{t})|^2}{|\mathcal{A}_{\mathrm{I}}(\hat{s},\hat{t})|^2 + |\mathcal{A}_{\mathrm{II}}(\hat{s},\hat{t})|^2}\right)_{N_{\mathrm{C}} \to \infty} \end{aligned}$$