Properties of the Gamma function

The purpose of this paper is to become familiar with the gamma function, a very important function in mathematics and statistics. The gamma function is a continuous extension to the factorial function, which is only defined for the nonnegative integers. While there are other continuous extensions to the factorial function, the gamma function is the only one that is convex for positive real numbers.

The gamma function, $\Gamma(\alpha)$, for $\alpha > 0$, is defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx, \text{ where } \alpha > 0. \tag{1}$$

The gamma function satisfies the recursive property

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1), \tag{2}$$

which can be proved using intgration by parts and L'Hôpital's Rule. Note that the recursive relationship in (2) can be used to extend the definition of the gamma function to all real numbers except the nonpositive integers.

When $\alpha = n$ and n is a positive integer, then the gamma function is related to the factorial function:

$$\Gamma(n) = (n - 1)! \tag{3}$$

For specific values of α, exact values of $\Gamma(\alpha)$, exist. For the positive integers, $\Gamma(n)$ is defined by (3). The gamma function evalated at $\alpha = \frac{1}{2}$ is

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{4}$$

The recursive relationship in (2) can be used to compute the value of the gamma function of all real numbers (except the nonpositive integers) by knowing only the value of the gamma function between 1 and 2. Table 2 contains the gamma function for arguments between 1 and 1.99. To illustrate, the following three examples show how to evaluate the gamma function for positive integers, fractional positive numbers and a negative noninteger values. First, suppose that α is the positive integer five. Then it follows that

$$\Gamma(5) = 4\Gamma(4) = 4 \cdot 3\Gamma(3) = 4 \cdot 3 \cdot 2 \cdot 1 \cdot \Gamma(1) = 4! = 24.$$

Next, suppose that $\alpha = \frac{23}{3}$, a positive fraction. It follows that

$$\Gamma\left(\frac{23}{3}\right) = \left(\frac{20}{3}\right)\Gamma\left(\frac{20}{3}\right) = \left(\frac{20}{3}\right)\left(\frac{17}{3}\right)\Gamma\left(\frac{17}{3}\right)$$

$$= \left(\frac{20}{3}\right)\left(\frac{17}{3}\right)\left(\frac{14}{3}\right)\left(\frac{11}{3}\right)\left(\frac{8}{3}\right)\left(\frac{5}{3}\right)\Gamma\left(\frac{5}{3}\right)$$

$$= \frac{2094400}{729}\Gamma\left(\frac{5}{3}\right)$$

$$\doteq \frac{2094400}{729}\Gamma(1.67)$$

$$= \frac{2094400}{729}(.90330)$$

$$= 2595.1598.$$

The actual answer (to 8 decimal places) for $\Gamma\left(\frac{23}{3}\right)$ is 2593.56617499. The discrepancy between the actual answer and the approximation is quite large since we approximated $\Gamma\left(\frac{5}{3}\right)$ with $\Gamma\left(1.67\right)$. A better approach would use interpolation of some kind. (Linear or quadratic interpolation would work well). For example, you can approximate y for a given x, when you have the table values (x_1, y_1) and (x_2, y_2). Using the point slope formula for a line, the approximate value for y given x can be derived as follows:

$$y = y_1 + m\left(x - x_1\right) = y_1 + \frac{y_2 - y_1}{x_2 - x_1}\left(x - x_1\right)$$

$$= y_1 + (y_2 - y_1)\frac{x - x_1}{x_2 - x_1}$$

The approximate value for $\Gamma\left(\frac{5}{3}\right)$ is therefore

$$\Gamma\left(\frac{5}{3}\right) = .90167 + (.90330 - .90167)\frac{\frac{5}{3} - 1.66}{1.67 - 1.66}$$

$$= .90275666667$$

Using this value, a better approximation of $\Gamma\left(\frac{23}{3}\right)$ is 2593.59885, which is quite close.

Next, the gamma function can be computed for negative, noninteger arguments. Suppose that $\alpha = -\frac{5}{6}$. Subsitituting $\alpha + 1$ for α in equation (2) yields

$$\Gamma\left(\alpha + 1\right) = \alpha\Gamma\left(\alpha\right) \tag{5}$$

Solving (5) for $\Gamma\left(\alpha\right)$ yields

$$\Gamma\left(\alpha\right) = \frac{\Gamma\left(\alpha + 1\right)}{\alpha} \tag{6}$$

Therefore, by (6), $\Gamma\left(-\frac{5}{6}\right)$ is

$$\Gamma\left(-\frac{5}{6}\right) = \frac{\Gamma\left(-\frac{5}{6} + 1\right)}{-\frac{5}{6}} = -\frac{6}{5}\Gamma\left(\frac{1}{6}\right) = -\frac{6}{5}\frac{\Gamma\left(\frac{7}{6}\right)}{\frac{1}{6}} = -\frac{36}{5}\Gamma\left(\frac{7}{6}\right)$$

Using linear interpolation between 1.16 and 1.17 yields $\Gamma\left(\frac{7}{6}\right) = .927733333333$. Hence $\Gamma\left(-\frac{5}{6}\right) = -6.67968$, which is close to the more exact answer of -6.679579.

Using quadratic interpolation instead of linear interpolation gives a better approximation. It uses three points instead of two. How to implement quadratic interpolation is left to the reader; however, a comparison of the results for linear and quadratic interpolation are summariezed in Table 1.

α	Exact	Linear Approx	Linear Error	Quadratic Approx	Quadratic Error
$\frac{5}{3}$	.902745292951	.90275666667	1.137×10^{-5}	.90274777777	2.485×10^{-6}
$\frac{7}{6}$	.927719333629	.92773333333	1.400×10^{-5}	.92771777777	1.556×10^{-6}

Table 1: Comparison of Linear and Quadratic interpolation to the exact answer

Various kinds of formulas can be derived using the recurrence relationship in (2). For example, the gamma function for $n + \frac{1}{2}$, where n is an integer, can be derived as follows:

$$\Gamma\left(n + \frac{1}{2}\right) = \left(n - \frac{1}{2}\right)\Gamma\left(n - \frac{1}{2}\right)$$

$$= \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right)\Gamma\left(n - \frac{3}{2}\right)$$

$$= \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right)\cdots\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$$

$$= \left(\frac{2n - 1}{2}\right)\left(\frac{2n - 3}{2}\right)\cdots\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}$$

$$= \frac{(2n - 1)(2n - 3)\cdots(3)(1)}{2^{\frac{n+1}{2}}}\sqrt{\pi} \tag{7}$$

$$= \frac{(2n - 1)(2n - 2)(2n - 3)(2n - 4)\cdots(3)(2)(1)}{2^n(2n - 2)(2n - 4)\cdots 2}\sqrt{\pi}$$

$$= \frac{(2n - 1)(2n - 2)(2n - 3)(2n - 4)\cdots(3)(2)(1)}{2^{n-1}2^n(n - 1)(n - 2)\cdots 1}\sqrt{\pi}$$

$$= \frac{(2n - 1)!}{2^{2n-1}(n - 1)!}\sqrt{\pi}$$

$$= \frac{{}_{2n-1}P_n}{2^{2n-1}}\sqrt{\pi}, \quad n \in \mathcal{Z}^+ \tag{8}$$

By (2), the gamma function evaluated at $\frac{9}{2}$ is

$$\Gamma\left(\frac{9}{2}\right) = \left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{105}{16}\sqrt{\pi}.$$

By (8), the gamma function evaluated at $\frac{9}{2}$ is

$$\Gamma\left(\frac{9}{2}\right) = \Gamma\left(4 + \frac{1}{2}\right) = \frac{7!}{2^7 3!}\sqrt{\pi} = \frac{105}{16}\sqrt{\pi}$$

Equation (8) only works if n is an integer. Other non-integral values do not work.

Gamma Function $\Gamma(\alpha)$										
α	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
1.0	1.00000	0.99433	0.98884	0.98355	0.97844	0.97350	0.96874	0.96415	0.95973	0.95546
1.1	0.95135	0.94740	0.94359	0.93993	0.93642	0.93304	0.92980	0.92670	0.92373	0.92089
1.2	0.91817	0.91558	0.91311	0.91075	0.90852	0.90640	0.90440	0.90250	0.90072	0.89904
1.3	0.89747	0.89600	0.89464	0.89338	0.89222	0.89115	0.89018	0.88931	0.88854	0.88785
1.4	0.88726	0.88676	0.88636	0.88604	0.88581	0.88566	0.88560	0.88563	0.88575	0.88595
1.5	0.88623	0.88659	0.88704	0.88757	0.88818	0.88887	0.88964	0.89049	0.89142	0.89243
1.6	0.89352	0.89468	0.89592	0.89724	0.89864	0.90012	0.90167	0.90330	0.90500	0.90678
1.7	0.90864	0.91057	0.91258	0.91467	0.91683	0.91906	0.92137	0.92376	0.92623	0.92877
1.8	0.93138	0.93408	0.93685	0.93969	0.94261	0.94561	0.94869	0.95184	0.95507	0.95838
1.9	0.96177	0.96523	0.96877	0.97240	0.97610	0.97988	0.98374	0.98768	0.99171	0.99581

Table 2: Gamma function, $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$