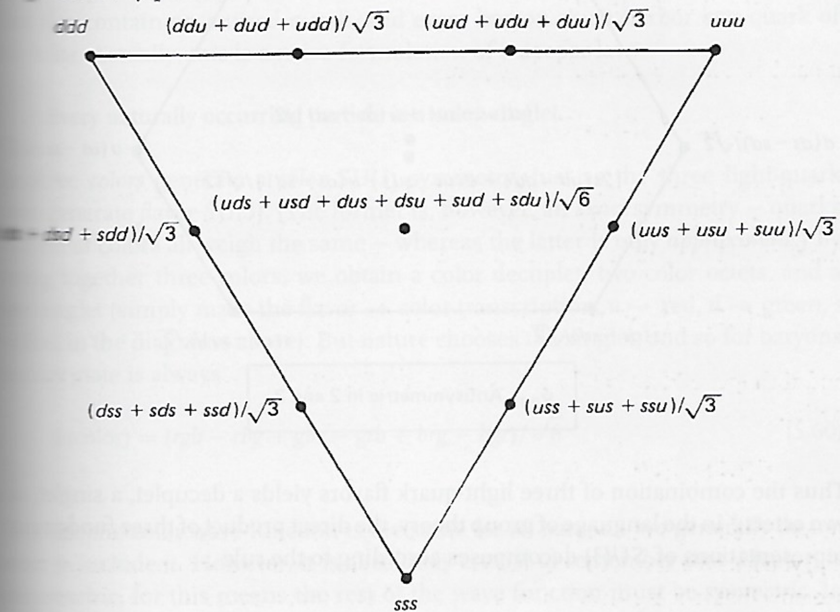
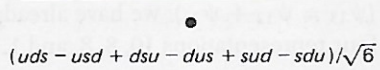
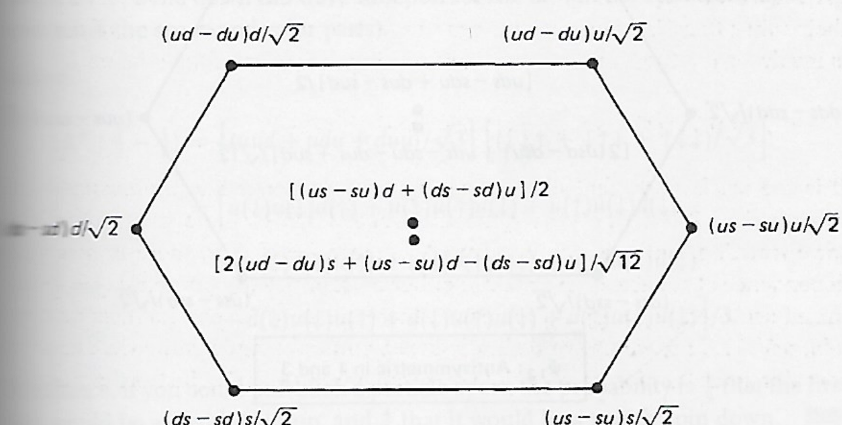
combinations form representations of $SU(2)$. These are conveniently displayed in Eightfold-Way patterns:



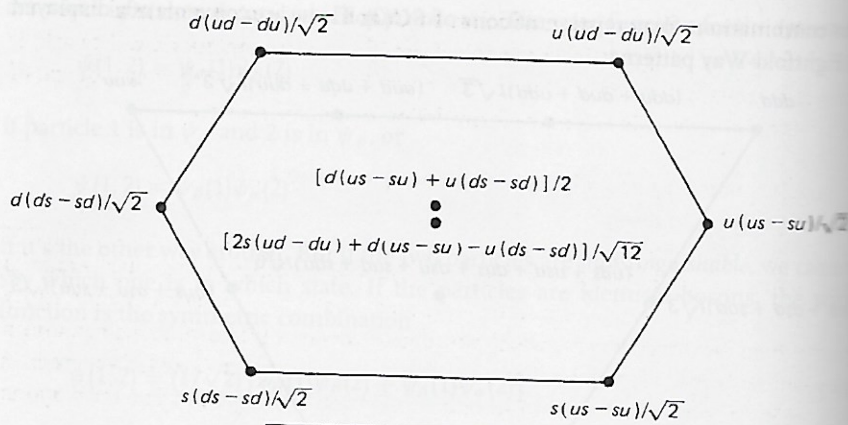
ψ_s: Completely symmetric states



ψ_A: Completely antisymmetric state



ψ_{12}: Antisymmetric in 1 and 2

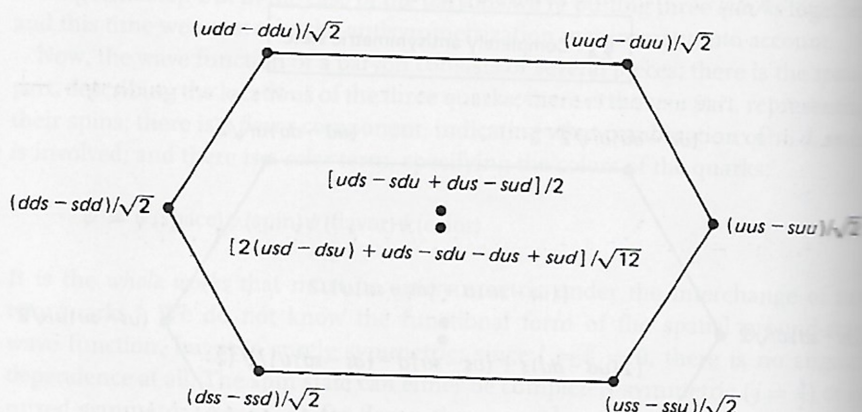


ψ_{23}: Antisymmetric in 2 and 3

Thus the combination of three light-quark flavors yields a decuplet, a singlet, and two octets;* in the language of group theory, the direct product of three fundamental representations of $SU(3)$ decomposes according to the rule

$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1 \tag{5.59}$$

Incidentally, we can also construct an octet that is antisymmetric in 1 and 3, but this is not independent ($\psi_{13} = \psi_{12} + \psi_{23}$); we have already used up the 27 states available in making the four representations 10, 8, 8, and 1.



ψ_{13}: Antisymmetric in 1 and 3

* As always in octet (and nonet) diagrams, I put the isotriplet ('Σ^0') above, and the isosinglet(s) ('Λ') beneath it, in the center.

Equation 7.125) is replaced by a v, the corresponding mass on the right-hand side switches sign (see Problem 7.28).

Example 7.5 In the case of electron–muon scattering (Equation 7.115), $\Gamma_2 = \gamma^\nu$, and hence $\overline{\Gamma}_2 = \gamma^0\gamma^{\nu\dagger}\gamma^0 = \gamma^\nu$ (Problem 7.29). Applying Casimir's trick twice, we find

$$\langle|\mathcal{M}|^2\rangle = \frac{g_e^4}{4(p_1 - p_3)^4}\mathrm{Tr}[\gamma^\mu(\not{p}_1 + mc)\gamma^\nu(\not{p}_3 + mc)] \times \mathrm{Tr}[\gamma_\mu(\not{p}_2 + Mc)\gamma_\nu(\not{p}_4 + Mc)] \tag{7.126}$$

where m is the mass of the electron and M is the mass of the muon. The factor of $\frac{1}{4}$ is included because we want the *average* over the initial spins; since there are two particles, each with two possible orientations, the average is a quarter of the sum. ■

Casimir's trick reduces everything to an exercise in calculating the trace of some complicated product of γ matrices. This algebra is facilitated by a number of theorems, which I shall now list (I'll leave the proofs to you – see Problems 7.31–7.34). First of all, I should mention three facts about traces in general: if A and B are any two matrices, and α is any number

1. $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$
2. $\mathrm{Tr}(\alpha A) = \alpha\,\mathrm{Tr}(A)$
3. $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

It follows from number 3 that $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA)$, but this is not equal, in general, to the trace of the matrices taken in the other order: $\mathrm{Tr}(ACB) = \mathrm{Tr}(BAC) = \mathrm{Tr}(CBA)$. You can 'peel' matrices off the back end of a product and move them around to the front, but you must preserve the ordering. It is useful to note that

4. $g_{\mu\nu}g^{\mu\nu} = 4$

and to recall the fundamental anticommutation relation for the γ matrices (together with an associated rule for 'slash' products):

5. $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ | 5′. $\not{a}\not{b} + \not{b}\not{a} = 2a \cdot b$

From these there follows a sequence of 'contraction theorems':

6. $\gamma_\mu\gamma^\mu = 4$
7. $\gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu$ | 7′. $\gamma_\mu\not{a}\gamma^\mu = -2\not{a}$

8. $\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu$	$=$	$4g^{\nu\lambda}$	8′. $\gamma_\mu \not{a} \not{b} \gamma^\mu$	$=$	$4(a \cdot b)$
9. $\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^\mu$	$=$	$-2\gamma^\sigma \gamma^\lambda \gamma^\nu$	9′. $\gamma_\mu \not{a} \not{b} \not{c} \gamma^\mu$	$=$	$-2\not{c}\not{b}\not{a}$

and a collection of 'trace theorems':

10. The trace of the product of an *odd* number of gamma matrices is *zero*.

11. $\mathrm{Tr}(1) = 4$

12. $\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$ 12′. $\mathrm{Tr}(\not{a}\not{b}) = 4(a \cdot b)$

13. $\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma)$ 13′. $\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d})$

$= 4(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda})$ $= 4(a \cdot b\, c \cdot d - a \cdot c\, b \cdot d + a \cdot d\, b \cdot c)$

Finally, since $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is the product of an *even* number of γ matrices, it follows from Rule 10 that $\mathrm{Tr}\,(\gamma^5\gamma^\mu) = \mathrm{Tr}\,(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda) = 0$. When γ^5 is multiplied by an *even* number of γ's, we find

14. $\mathrm{Tr}(\gamma^5) = 0$

15. $\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu) = 0$ 15′. $\mathrm{Tr}(\gamma^5 \not{a}\not{b}) = 0$

16. $\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) = 4i\epsilon^{\mu\nu\lambda\sigma}$ 16′. $\mathrm{Tr}(\gamma^5 \not{a}\not{b}\not{c}\not{d}) = 4i\epsilon^{\mu\nu\lambda\sigma} a_\mu b_\nu c_\lambda d_\sigma$

where*

$$\epsilon^{\mu\nu\lambda\sigma} \equiv \begin{cases} -1, & \text{if } \mu\nu\lambda\sigma \text{ is an } even \text{ permutation of } 0123, \\ +1, & \text{if } \mu\nu\lambda\sigma \text{ is an } odd \text{ permutation}, \\ 0, & \text{if any two indices are the same.} \end{cases} \tag{7.127}$$

Example 7.6 Evaluate the traces in electron–muon scattering (Equation 7.126):

$$\mathrm{Tr}[\gamma^\mu(\not{p}_1 + mc)\gamma^\nu(\not{p}_3 + mc)]$$
$$= \mathrm{Tr}(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3) + mc\left[\mathrm{Tr}(\gamma^\mu \not{p}_1 \gamma^\nu) + \mathrm{Tr}(\gamma^\mu \gamma^\nu \not{p}_3)\right] + (mc)^2 \mathrm{Tr}(\gamma^\mu \gamma^\nu)$$

*By 'even permutation' I mean an even number of interchanges of two indices. Thus $\epsilon^{\mu\nu\lambda\sigma} = -\epsilon^{\nu\mu\lambda\sigma} = \epsilon^{\nu\lambda\mu\sigma} = -\epsilon^{\nu\lambda\sigma\mu}$, and so on – in other words, $\epsilon^{\mu\nu\lambda\sigma}$ is antisymmetric in every pair of superscripts. It might seem strange that ϵ^{0123} is *minus* 1; why not make it *plus* 1? It's purely conventional, of course – evidently, whoever established the definition wanted ϵ_{0123} to be plus 1, and from that it follows that $\epsilon^{0123} = -1$, since three spatial indices are raised. By the way, if you are used to working with the three-dimensional Levi–Civita symbol ϵ_{ijk} (Problem 4.19), be warned that although an even permutation on *three* indices corresponds to preservation of cyclic order ($\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$), this is *not* the case for *four* indices: $\epsilon^{\mu\nu\lambda\sigma} = -\epsilon^{\nu\lambda\sigma\mu} = \epsilon^{\lambda\sigma\mu\nu} = -\epsilon^{\sigma\mu\nu\lambda}$.

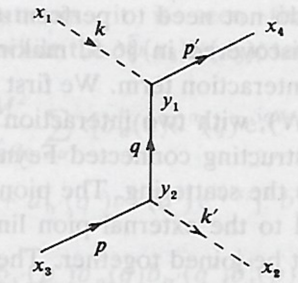


Fig. 6.9. $\pi^+ p$ scattering.

This is to be inserted into (6.169). Recalling that

$$K_x \Delta_F(x-y) = (\Box_x + m^2)\Delta_F(x-y) = -\delta^4(x-y),$$
$$D_x S(x-y) = (\mathrm{i}\gamma\cdot\partial_x - M)S(x-y) = \delta^4(x-y),$$
$$S(x-y)\overleftarrow{D}_x = S(x-y)(-\mathrm{i}\gamma\cdot\overleftarrow{\partial}_x - M) = \delta^4(x-y),$$

we obtain

$$\begin{aligned} S_{\mathrm{fi}} &= -2\mathrm{i}g^2 \int \mathrm{d}x_1 \ldots \mathrm{d}x_4\, \mathrm{d}y_1\, \mathrm{d}y_2\, \mathrm{e}^{\mathrm{i}k'x_2}\, \mathrm{e}^{\mathrm{i}p'x_4}\, \bar{u}^{s'}(p')\delta^4(x_2-y_2) \\ &\quad \times \delta^4(x_4-y_1)\gamma_5 S(y_1-y_2)\gamma_5\delta^4(y_2-x_3) \\ &\quad \times \delta^4(y_1-x_1)u^s(p)\,\mathrm{e}^{-\mathrm{i}px_3}\,\mathrm{e}^{-\mathrm{i}kx_1} \\ &= -2\mathrm{i}g^2 \int \mathrm{d}y_1\, \mathrm{d}y_2\, \mathrm{e}^{\mathrm{i}(p'-k)y_1}\, \mathrm{e}^{\mathrm{i}(k'-p)y_2}\, \bar{u}^{s'}(p')\gamma_5 S(y_1-y_2)\gamma_5 u^s(p). \end{aligned}$$

Combining equations (6.135) for $S(x)$, and (6.14) for $\Delta_F(x)$, we have

$$S(y_1-y_2) = \frac{1}{(2\pi)^4}\int \mathrm{d}q \frac{\gamma^\mu q_\mu + M}{q^2 - M^2}\,\mathrm{e}^{-\mathrm{i}q(y_1-y_2)}. \tag{6.172}$$

Inserting this into the above equation, we may first carry out the integration over y_1 and y_2:

$$\begin{aligned} \int \mathrm{d}y_1\, \mathrm{d}y_2\, \mathrm{e}^{\mathrm{i}(p'-k)y_1}\, \mathrm{e}^{\mathrm{i}(k'-p)y_2}\, \mathrm{e}^{-\mathrm{i}q(y_1-y_2)} &= (2\pi)^8\delta^4(p'-k-q)\delta^4(k'-p+q) \\ &= (2\pi)^8\delta^4(q-p+k')\delta^4(p'+k'-p-k) \\ &= (2\pi)^8\delta^4(q-p+k')\delta^4(P_{\mathrm{f}}-P_{\mathrm{i}}), \end{aligned}$$

showing that the intermediate neutron has 4-momentum q, and 4-momentum is conserved at each vertex, as well as overall. The integration over q may now be carried out, as long as q is replaced by $p-k' = p'-k$. Therefore

$$S_{\mathrm{fi}} = -\mathrm{i}\delta^4(P_{\mathrm{f}}-P_{\mathrm{i}})2g^2(2\pi)^4\bar{u}^{s'}(p')\gamma_5\frac{\gamma\cdot(p-k')+M}{(p-k')^2-M^2}\gamma_5 u^s(p). \tag{6.173}$$

Now using the Dirac equation (2.140) in momentum space and the properties of the γ matrices, this expression may be simplified to give

$$S_{\text{fi}} = \mathrm{i}\delta^4(P_\text{f} - P_\text{i})2g^2(2\pi)^4\bar{u}^{s'}(p')\gamma\cdot k'u^s(p)\frac{1}{2pk' - m^2}. \qquad (6.174)$$

From the scattering amplitude, the next task is to find the scattering cross section, and this is dealt with in the next section. Before finishing this section, however, we shall summarise briefly the *Feynman rules* for calculating the scattering amplitude for an interaction involving scalar (or pseudoscalar) and spinor particles. It will immediately be seen that they yield the expression (6.173) for the particular case of $\pi^+ p$ scattering to lowest order perturbation theory.

1. To nth order, perturbation theory corresponds to a diagram with n vertices. The amplitude for a particular process (i.e. with particular ingoing and outgoing external lines) to a particular order is obtained by adding the amplitudes of all topologically inequivalent connected diagrams. Fig. 6.8 shows the two diagrams for (pseudo-)scalar spinor scattering to second order. Fig. 6.10 shows some fourth order diagrams. Spinor lines are continuous, scalar lines dotted.
2. For each incoming spinor particle write $u(p)$ ($v(p)$) for its antiparticle), and for each outgoing spinor particle $\bar{u}(p)$.
3. For each vertex write $\mathrm{i}g$ (for scalar interaction) or $\mathrm{i}g\gamma_5$ (for pseudoscalar), where g is the relevant coupling constant read off from the interaction Lagrangian, and multiply by $(2\pi)^4\delta^4$ (incoming momenta).
4. For each spinor propagator (internal line) of momentum p write

$$\frac{1}{(2\pi)^4}\frac{\mathrm{i}}{(\gamma\cdot p - M)}\mathrm{d}^4p.$$

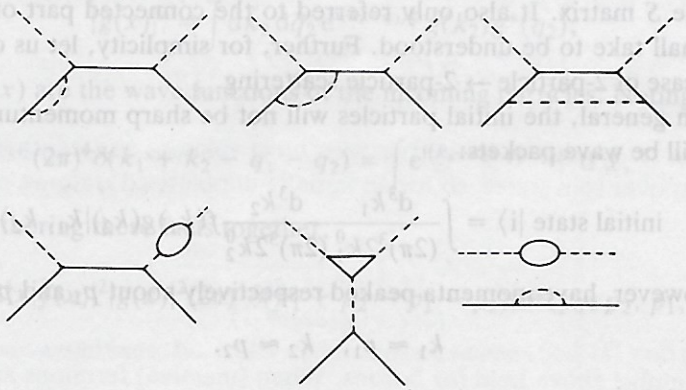


Fig. 6.10. Some fourth order diagrams for scalar/pseudoscalar-spinor (e.g. pion–nucleon) scattering. The last diagram is disconnected and is not counted.

Table 6.1. *Feynman rules for scalar (or pseudoscalar) and spinor theories*

Factor in S matrix	Diagrammatic representation
$u(p)$	incoming fermion (external line)
$\bar{u}(p)$	outgoing fermion (external line)
$ig(\gamma_5)(2\pi)^4\delta^4(p+k-p')$	vertex (k, p', p)
$\dfrac{i}{(2\pi)^4}\dfrac{1}{\gamma\cdot p - M}\left(\text{with} \int d^4p\right)$	$\xrightarrow[p]{}$ fermion propagator (internal line)
$\dfrac{i}{(2\pi)^4}\dfrac{1}{p^2 - m^2}\left(\text{with} \int d^4p\right)$	$\dashrightarrow$ p scalar propagator (internal line)

5. For each (pseudo-) scalar propagator write

$$\frac{1}{(2\pi)^4}\frac{i}{p^2 - m^2}d^4p.$$

6. Integrate over internal momenta.

These rules are summarised in Table 6.1.

6.10 Scattering cross section

We now show how to calculate the scattering cross section from the scattering amplitude. First, we define the Lorentz invariant amplitude M by

$$\langle p_1', p_2', \ldots, |S-1|p_1, p_2, \ldots\rangle$$
$$= (2\pi)^4\delta^4(P_f - P_i)iM(p_1', p_2', \ldots, p_1, p_2, \ldots), \quad (6.175)$$

recalling that what we calculated above did not include the identity contribution to the S matrix. It also only referred to the connected part of $S-1$, but this we shall take to be understood. Further, for simplicity, let us consider the simplest case of 2-particle $\rightarrow$ 2-particle scattering.

Now, in general, the initial particles will not be sharp momentum states p_1, p_2, but will be wave packets:

$$\text{initial state } |i\rangle = \int \frac{d^3k_1}{(2\pi)^3 2k_1^0}\frac{d^3k_2}{(2\pi)^3 2k_2^0} f(k_1)g(k_2)|k_1, k_2\rangle$$

which, however, have momenta peaked respectively about p_1 and p_2:

$$k_1 \approx p_1, \quad k_2 \approx p_2.$$

The final state is

$$|f\rangle = |p_1', p_2'\rangle$$

so the transition amplitude is (with $\mathrm{d}\widetilde{k} = \mathrm{d}^3k/(2\pi)^3 2k^0$)

$$\int \mathrm{d}\widetilde{k}_1\,\mathrm{d}\widetilde{k}_2 f(k_1)g(k_2)\langle p_1' p_2'|S-1|k_1 k_2\rangle$$
$$= (2\pi)^4 \mathrm{i} \int \mathrm{d}\widetilde{k}_1\,\mathrm{d}\widetilde{k}_2 f(k_1)g(k_2)\delta(p_1' + p_2' - k_1 - k_2)M(p_1', p_2', k_1, k_2)$$

and the transition probability is

$$W = (2\pi)^8 \int \mathrm{d}\widetilde{k}_1\,\mathrm{d}\widetilde{k}_2\,\mathrm{d}\widetilde{q}_1\,\mathrm{d}\widetilde{q}_2 f(k_1)g(k_2)f^*(q_1)g^*(q_2)$$
$$\times\, \delta(p_1' + p_2' - k_1 - k_2)\delta(p_1' + p_2' - q_1 - q_2)$$
$$\times\, M(p_1', p_2', k_1, k_2)M^*(p_1', p_2', q_1, q_2).$$

Because of the first delta function, the second one may be written as $\delta(k_1 + k_2 - q_1 - q_2)$. Also, since f and g are peaked around p_1 and p_2, we may approximate M above by $M(p_1', p_2', p_1, p_2)$. Finally, to get W into a comprehensible form, we introduce the Fourier transforms of f and g

$$\tilde{f}(x) = \int \mathrm{d}\widetilde{q}\, \mathrm{e}^{\mathrm{i}qx} f(q)$$

where, as above,

$$\mathrm{d}\widetilde{q} = \frac{\mathrm{d}^3 q}{(2\pi)^3 2q_0}.$$

So

$$|\tilde{f}(x)|^2 = \int \mathrm{d}\widetilde{k}_1\,\mathrm{d}\widetilde{q}_1\, \mathrm{e}^{\mathrm{i}(k_1 - q_1)x} f(k_1)f^*(q_1)$$

and similarly

$$|\tilde{g}(x)|^2 = \int \mathrm{d}\widetilde{k}_2\,\mathrm{d}\widetilde{q}_2\, \mathrm{e}^{\mathrm{i}(k_2 - q_2)x} g(k_2)g^*(q_2);$$

$\tilde{f}(x)$ and $\tilde{g}(x)$ are the wave functions of the incoming particles. Noting that

$$(2\pi)^4\delta(k_1 + k_2 - q_1 - q_2) = \int \mathrm{e}^{\mathrm{i}(k_1 + k_2 - q_1 - q_2)x}\,\mathrm{d}^4x,$$

we have, gathering these facts together,

$$W = \int \mathrm{d}^4x\, |\tilde{f}(x)|^2 |\tilde{g}(x)|^2 (2\pi)^4 \delta(p_1' + p_2' - p_1 - p_2)|M(p_1', p_2', p_1, p_2)|^2. \tag{6.176}$$

The first factor is the overlap of the wave functions, a precondition for scattering. This factor is unity when the initial state is a momentum eigenstate.

The second factor is the familiar expression of Fermi's Golden Rule. We can then write the transition probability per unit volume per unit time as

$$\frac{\mathrm{d}W}{\mathrm{d}V\,\mathrm{d}t} = |\tilde{f}(x)|^2|\tilde{g}(x)|^2(2\pi)^4\delta(p_1' + p_2' - p_1 - p_2)|M(p_1', p_2', p_1, p_2)|^2.$$

Using our covariant normalisation, which corresponds to $2p_0$ particles per unit volume (see §4.1), we have, for our wave packets, that the number of particles '1' per unit volume is $|\tilde{f}(x)|^2 2p_1^0$, and the corresponding number of particles '2' is $|\tilde{g}(x)|^2 2p_2^0$. Now assume that particle 2 is initially at rest, $p_2^0 = m_2$. The incident flux is the relative velocity $v = |\mathbf{p}_1|/p_1^0$ times the particle density $2p_1^0|\tilde{f}(x)|^2$, which is $2|\mathbf{p}_1||\tilde{f}(x)|^2$. The target density, similarly, is $2m_2|\tilde{g}(x)|^2$.

Now the scattering cross section $\mathrm{d}\sigma$ is defined in terms of $\mathrm{d}W/\mathrm{d}V\,\mathrm{d}t$, the number of transitions per unit time and unit volume, by

$$\frac{\mathrm{d}W}{\mathrm{d}V\,\mathrm{d}t} = (\text{incident flux}) \times (\text{target density}) \times \mathrm{d}\sigma$$

which gives

$$\mathrm{d}\sigma = (2\pi)^4\delta^4(p_1' + p_2' - p_1 - p_2)\frac{1}{4m_2|\mathbf{p}_1|}|M|^2.$$

There is a Lorentz invariant generalisation of the quantity $m_2|\mathbf{p}_1|$, namely

$$\begin{aligned} B &= [(p_1 \cdot p_2)^2 - m_1^2 m_2^2]^{1/2} \\ &= m_2|\mathbf{p}_1| \quad \text{in laboratory frame.} \end{aligned}$$

As it stands, this expression for the cross section refers to an undefined final state, subject only to the condition $p_1' + p_2' = p_1 + p_2$. What is measured in the laboratory, however, is the differential cross section, for scattering into a particular solid angle $\mathrm{d}\Omega$; and therefore with final momentum in a range $\mathrm{d}\mathbf{p}_1$. We therefore write the cross section for the final momenta in the momentum-space interval $\mathrm{d}^3p_1'\,\mathrm{d}^3p_2'$ as

$$\mathrm{d}\sigma = \frac{(2\pi)^4}{4B}\frac{\mathrm{d}^3p_1'}{(2\pi)^3 2(p_1')_0}\frac{\mathrm{d}^3p_2'}{(2\pi)^3 2(p_2')_0}\delta^4(p_\mathrm{f} - p_\mathrm{i})|M|^2. \qquad (6.177)$$

In the case in which the initial particles have spins s_1 and s_2, there will be a summation over spin states, so for an initially unpolarised state we replace $|M|^2$ by

$$|M|^2 \to \frac{1}{(2s_1 + 1)(2s_2 + 1)}\sum_{s_\mathrm{i}, s_\mathrm{f}}|M_\mathrm{fi}|^2.$$

The formulae above hold for bosons. When (massive) fermions are involved, the normalisation of our Dirac spinors is equivalent to p_0/m particles per unit volume, so if the target is a fermion, the target density is $|\tilde{g}(x)|^2$. For the final state fermion, the Lorentz invariant phase space $\mathrm{d}^3p/(2\pi)^3 2p_0$ is replaced by $(m/p_0)[\mathrm{d}^3p/(2\pi)^3]$. Putting these things together, the cross section for pion–nucleon scattering is given by

$$\begin{aligned}\mathrm{d}\sigma &= \frac{1}{(2\pi)^2}\frac{\mathrm{d}^3p_1'}{2E_1'}\frac{\mathrm{d}^3p_2'}{E_2'/M}\frac{M}{2B}\delta^4(p_\mathrm{f}-p_\mathrm{i})\tfrac{1}{2}\sum_{\text{spin}}|M_{\mathrm{fi}}|^2\\ &= \frac{1}{32\pi^2}\frac{\mathrm{d}^3p_1'}{E_1'}\frac{\mathrm{d}^3p_2'}{E_2'}\frac{M^2}{B}\delta(E_1'+E_2'-E_\mathrm{i})\delta^3(p_1'+p_2'-p_\mathrm{i})\sum_{\text{spin}}|M_{\mathrm{fi}}|^2.\end{aligned}$$

Let us now work in the centre of mass system

$$\mathbf{p}_1' = -\mathbf{p}_2' = \mathbf{p}_\mathrm{f}.$$

Then the phase space integral becomes

$$\begin{aligned}I &= \int\frac{\mathrm{d}^3p_1'}{E_1'}\frac{\mathrm{d}^3p_2'}{E_2'}\delta(E_1'+E_2'-E_\mathrm{i})\delta^3(p_1'+p_2'-p_\mathrm{i})\\ &= \int\frac{\mathrm{d}^3p_\mathrm{f}}{E_1'E_2'}\delta(E_1'+E_2'-E_\mathrm{i})\\ &= \int\frac{p_\mathrm{f}^2\,\mathrm{d}p_\mathrm{f}\,\mathrm{d}\Omega_\mathrm{f}}{E_1'E_2'}\delta(E_1'+E_2'-E_\mathrm{i}).\end{aligned}$$

Invoking the formula

$$\delta(f(x)) = [f'(x_0)]^{-1}\delta(x-x_0),\ \text{where } f(x_0)=0,$$

we then see that

$$I = \frac{p_\mathrm{f}}{E_\mathrm{i}}\int\mathrm{d}\Omega_\mathrm{f},$$

so the differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{32\pi^2}\frac{M^2p_\mathrm{f}}{BE_\mathrm{i}}\sum_{\text{spin}}|M_{\mathrm{fi}}|^2.$$

In the centre-of-mass system, $B = p_\mathrm{f}(E_1'+E_2') = p_\mathrm{f}W$ where $W = E_\mathrm{i}$ is the total energy of the system, so

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{32\pi^2}\left(\frac{M}{W}\right)^2\sum_{\text{spin}}|M_{\mathrm{fi}}|^2. \tag{6.178}$$

The invariant amplitude M_{fi} is, by comparing equations (6.174) and (6.175),

$$M_{\mathrm{fi}} = 2g^2\bar{u}^{s'}(p')\gamma\cdot k'u^s(p)\frac{1}{2p\cdot k'-m^2}. \tag{6.179}$$

We now have to evaluate a quantity of the form

$$\sum_{\text{spins}} |\bar{u}' A u|^2$$

where $u' = u^{s'}(p')$ and A is a Dirac matrix operator. We have

$$\begin{aligned}(\bar{u}' A u)^* &= u'^{\mathrm{T}} \gamma^{0*} A^* u^* \\ &= u^\dagger A^\dagger \gamma^{0\dagger} u' \\ &= \bar{u} \bar{A} u'\end{aligned}$$

with $\bar{A} = \gamma^0 A^\dagger \gamma^0$. Hence

$$\begin{aligned}|\bar{u}' A u|^2 &= (\bar{u}' A u)^* (\bar{u}' A u) \\ &= \bar{u} \bar{A} u' \bar{u}' A u \\ &= \bar{u}'_k A_{kl} u_l \bar{u}_m \bar{A}_{mi} u'_i \\ &= u'_i \bar{u}'_k A_{kl} u_l \bar{u}_m \bar{A}_{mi}.\end{aligned}$$

Summing over spin, we now appeal to equation (2.145)

$$\sum_\alpha u_i^\alpha(p) \overline{u_j^\alpha}(p) = \left(\frac{\gamma \cdot p + M}{2M}\right)_{ij},$$

to give

$$\sum_{\text{spins}} |\bar{u}' A u|^2 = \mathrm{Tr}\left\{\left(\frac{\gamma \cdot p' + M}{2m}\right) A \left(\frac{\gamma \cdot p + M}{2M}\right) \bar{A}\right\} \tag{6.180}$$

where the trace is over the Dirac matrices. We also know that (see (2.147) and (2.151))

$$\mathrm{Tr}\,(\gamma \cdot a)(\gamma \cdot b) = 4a \cdot b$$

$$\begin{aligned}\mathrm{Tr}\,(\gamma \cdot a)(\gamma \cdot b)(\gamma \cdot c)(\gamma \cdot d) &= -\mathrm{Tr}\,(\gamma \cdot b)(\gamma \cdot a)(\gamma \cdot c)(\gamma \cdot d) \\ &\quad + 2a \cdot b\, \mathrm{Tr}\,(\gamma \cdot c)(\gamma \cdot d)\end{aligned}$$

and that the trace of an odd number of γ matrices vanishes ((2.150)). This gives

$$\begin{aligned}\sum_{\text{spin}} |M_{\text{fi}}|^2 &= 4g^4 \left(\frac{1}{2p \cdot k' - m^2}\right)^2 \frac{1}{4M^2} \mathrm{Tr}\,\{(\gamma \cdot p' + M)\gamma \cdot k' (\gamma \cdot p + M) \gamma \cdot k'\} \\ &= \frac{g^4}{M^2}\left(\frac{1}{2p \cdot k' - m^2}\right)^2 4\{2(p \cdot k')(p' \cdot k') + m^2[M^2 - (p \cdot p')]\}.\end{aligned}$$

In the centre-of-mass system

$$|\mathbf{p}| = |\mathbf{p}'| = |\mathbf{k}| = |\mathbf{k}'| = q$$

$$p = ((q^2 + M^2)^{1/2}, \mathbf{q}), \qquad k = ((q^2 + m^2)^{1/2}, -\mathbf{q})$$

$$p' = ((q^2 + M^2)^{1/2}, \mathbf{q}'), \quad k' = ((q^2 + m^2)^{1/2}, -\mathbf{q}').$$

At low energies, m, $M \gg q$ and $p \cdot k' \approx Mm$, $p' \cdot k' \approx Mm$, $p \cdot p' \approx M^2$, giving

$$\sum_{\text{spin}} |M_{\text{fi}}|^2 \approx \frac{8g^4}{(2M - m)^2}$$

and hence, since $W \approx M + m$ in equation (6.178),

$$\frac{d\sigma}{d\Omega} = \frac{g^4}{4\pi^2}\left(\frac{M}{M + m}\right)^2 \frac{1}{(2M - m)^2} \approx \frac{g^4}{16\pi^2} \cdot \frac{1}{M^2} \tag{6.181}$$

if m is neglected in comparison with M.

If we know g we can arrive at a prediction, from first order perturbation theory, of the pion–nucleon scattering cross section; g may be found by two distinct methods, and they give very different values. First, it may be found from the deuteron binding energy. The Yukawa potential between nucleons a distance r apart is

$$V = \frac{g\, e^{-\eta r}}{r}$$

where $\eta = \lambda\!\!\!^{-}{}^{-1}$, where $\lambda\!\!\!^{-}$ is the pion Compton wavelength, 1.4×10^{-15} m. At a separation of $r = 2.8 \times 10^{-15}$ m, the exponential term may be neglected, and the r dependence of the potential is the same as the Coulomb law. We know from the theory of the deuteron that the potential well between two nucleons at a distance of 2.8×10^{-15} m is about 20 MeV deep. On the other hand, the electrostatic interaction is only about 0.5 MeV at the same distance. Hence

$$\frac{g^2}{e^2} \approx \frac{20}{0.5} = 40$$

where e is the electric charge of the proton in e.s.u. Hence

$$\frac{g^2}{\hbar c} \approx 40 \times \frac{e^2}{\hbar c} \approx 0.3. \tag{6.182}$$

Putting this value in (6.181), we predict

$$\begin{aligned} \sigma &= 4\pi\left(\frac{d\sigma}{d\Omega}\right) \\ &= 4\pi\left(\frac{g^2}{4\pi}\right)^2 \frac{1}{M^2} \\ &= \frac{1}{4\pi}\left(\frac{g^2}{\hbar c}\right)^2\left(\frac{\hbar}{Mc}\right)^2 \\ &\simeq 120\ \mu\text{b}^{\ddagger} \end{aligned} \tag{6.183}$$

‡ $1\ \text{b} = 10^{-28}\ \text{m}^2$.

which is an order of magnitude smaller than the low energy pion–nucleon scattering cross section ≈ a few mb.

A second determination of g can be made from a knowledge of the phase shifts in π–N scattering. By using a *derivative* coupling (m = pion mass)

$$\mathscr{L}_{\text{int}} = \frac{f}{m}\bar{\psi}\gamma_5\gamma^{\mu}\boldsymbol{\tau}\psi\cdot\partial_{\mu}\boldsymbol{\phi} \tag{6.184}$$

it may be shown using a static theory with cut-off, that a resonant behaviour is to be expected for the π–N scattering amplitude. This is, of course, observed (see, for example, Fig. 1.2). Comparison of the theory with the measured phase shifts gives a value for f (see, for example, Sakurai 1964, p. 215):

$$\frac{f^2}{4\pi} = 0.08. \tag{6.185}$$

On the other hand, it may be shown (see, for example, Schweber, Bethe & de Hoffmann 1956, vol. 1, §26) that to first order, the pseudovector coupling (6.184) and the pseudoscalar coupling (6.162) are equivalent if $g = (2M/m)f$ (M = nucleon mass). This gives

$$\frac{g^2}{4\pi} \approx 15. \tag{6.186}$$

The corresponding cross section is

$$\sigma = 4\pi\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \approx 48\text{ b}, \tag{6.187}$$

which is far in excess of the experimental value.

In conclusion, neither value for g produces agreement with experiment. The value (6.186), which tends to be taken more seriously by particle physicists, is so high that a perturbation series in g^2 will diverge. It may reasonably be objected that this makes the theory meaningless. From a contemporary perspective, however, g is not a fundamental parameter. The basic strong interaction is the one between quarks and gluons, which is a gauge interaction. We shall start to consider these interactions in the next chapter, and in Chapter 9 will see that the coupling constant concerned actually changes, depending on the energy at which it is measured. The present exercise, then, is seen to be a rather academic one, but has served the useful purpose of illustrating the application of the Feynman rules.

Summary

[1]The generating functional $Z[J]$ for scalar fields is written down, and converted into a form involving the Feynman propagator Δ_{F}, which is also written down in Euclidean space. [2]Functional integration is introduced and it is shown

question. The phase space factor is purely *kinematic*; it depends on the masses, energies, and momenta of the participants, and reflects the fact that a given process is more likely to occur the more 'room to maneuver' there is in the final state. For example, the decay of a heavy particle into light secondaries involves a large phase space factor, for there are many different ways to apportion the available energy. By contrast, the decay of the neutron ($n \to p + e + \overline{\nu}_e$), in which there is almost no extra mass to spare, is tightly constrained and the phase space factor is very small.*

The ritual for calculating reaction rates was dubbed the *Golden Rule* by Enrico Fermi. In essence, Fermi's Golden Rule says that a transition rate is given by the *product* of the phase space and the (absolute) square of the amplitude. You may have encountered the nonrelativistic version, in the context of time-dependent perturbation theory [2]. We need the relativistic version, which comes from quantum field theory [3]. I can't *derive* it here; what I will do is *state* the Golden Rule and try to make it plausible. Actually, I'll do it twice: once in a form appropriate to decays and again in a form suitable for scattering.

6.2.1 Golden Rule for Decays

Suppose particle 1 (at rest)† decays into several other particles 2, 3, 4, ..., n:

$$1 \to 2 + 3 + 4 + \cdots + n \tag{6.14}$$

The decay rate is given by the formula

$$\Gamma = \frac{S}{2\hbar m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4 \left(p_1 - p_2 - p_3 \cdots - p_n\right) \times \prod_{j=2}^{n} 2\pi\, \delta\left(p_j^2 - m_j^2 c^2\right) \theta\left(p_j^0\right) \frac{d^4 p_j}{(2\pi)^4} \tag{6.15}$$

where m_i is the mass of the ith particle and p_i is its four-momentum. S is a statistical factor that corrects for double-counting when there are identical particles in the final state: for each such group of s particles, S gets a factor of $(1/s!)$. For instance, if $a \to b + b + c + c + c$, then $S = (1/2!)(1/3!) = 1/12$. If there are no identical particles in the final state (the most common circumstance), then $S = 1$.

Remember: The *dynamics* of the process is contained in the *amplitude*, $\mathcal{M}(p_1, p_2, \ldots, p_n)$, which is a function of the various momenta; we'll calculate it (later

* For a more extreme case, consider the (kinematically forbidden) decay $\Omega^- \to \Xi^- + \overline{K}^0$. Since the final products weigh more than the Ω, there is no phase space available at all and the decay rate is zero.

† There is no loss of generality in assuming particle 1 is at rest; this is simply an astute choice of reference frame.

evaluating the appropriate Feynman diagrams. The rest is *phase space*; it tells us to *integrate over all outgoing four-momenta*, subject to three kinematical constraints:

1. Each outgoing particle lies on its mass shell: $p_j^2 = m_j^2 c^2$ (which is to say, $E_j^2 - \mathbf{p}_j^2 c^2 = m_j^2 c^4$). This is enforced by the delta function $\delta\left(p_j^2 - m_j^2 c^2\right)$, which is zero unless its argument vanishes.*
2. Each outgoing energy is positive: $p_j^0 = E_j/c > 0$. Hence the θ function.†
3. Energy and momentum must be conserved: $p_1 = p_2 + p_3 \cdots + p_n$. This is ensured by the factor $\delta^4(p_1 - p_2 - p_3 \cdots - p_n)$.

The Golden Rule (Equation 6.15) may *look* forbidding, but what it actually *says* could hardly be simpler: all outcomes consistent with the three natural kinematic constraints are a priori equally likely. To be sure, the *dynamics* (contained in $\mathcal{M}$) may favor some combinations of momenta over others, but with that modulation you just *add up all the possibilities*. How about all those factors of 2π? These are easy to keep track of if you adhere scrupulously to the following rule:‡

$$\text{Every } \delta \text{ gets } (2\pi); \text{ every d gets } 1/(2\pi). \tag{6.16}$$

Four-dimensional 'volume' elements can be split into spatial and temporal parts:

$$d^4 p = dp^0\, d^3\mathbf{p} \tag{6.17}$$

(I'll drop the subscript j, for simplicity – this argument applies to *each* of the outgoing momenta). The p^0 integrals§ can be performed immediately, by exploiting the delta function

$$\delta\left(p^2 - m^2c^2\right) = \delta\left[(p^0)^2 - \mathbf{p}^2 - m^2c^2\right] \tag{6.18}$$

Now

$$\delta(x^2 - a^2) = \frac{1}{2a}\left[\delta(x-a) + \delta(x+a)\right] \quad (a > 0) \tag{6.19}$$

* If you are unfamiliar with the Dirac delta function, you *must* study Appendix A carefully before proceeding.

† θ is the (Heaviside) step function: 0 if $x < 0$ and 1 if $x > 0$ (see Appendix A).

‡ Some of these factors eventually cancel out, and you might wonder if there is a more efficient way to manage them. I don't think so. Feynman is supposed to have shouted in exasperation (at a graduate student who 'couldn't be bothered with such trivial matters') 'If you can't get the 2π's right, you don't know *nothing!*'

§ The integral sign in Equation 6.15 actually stands for $4(n-1)$ integrations – one for each component of the $n-1$ outgoing momenta.

(see Problem A.7), so

$$\theta\left(p^0\right)\delta\left[(p^0)^2-\mathbf{p}^2-m^2c^2\right]=\frac{1}{2\sqrt{\mathbf{p}^2+m^2c^2}}\,\delta\left(p^0-\sqrt{\mathbf{p}^2+m^2c^2}\right) \tag{6.20}$$

(the theta function kills the spike at $p^0=-\sqrt{\mathbf{p}^2+m^2c^2}$, and it's 1 at $p^0=\sqrt{\mathbf{p}^2+m^2c^2}$). Thus Equation 6.15 reduces to

$$\Gamma=\frac{S}{2\hbar m_1}\int|\mathcal{M}|^2(2\pi)^4\delta^4\left(p_1-p_2-p_3\cdots-p_n\right)$$
$$\times\prod_{j=2}^{n}\frac{1}{2\sqrt{\mathbf{p}_j^2+m_j^2c^2}}\frac{\mathrm{d}^3\mathbf{p}_j}{(2\pi)^3} \tag{6.21}$$

with

$$p_j^0\to\sqrt{\mathbf{p}_j^2+m_j^2c^2} \tag{6.22}$$

wherever it appears (in $\mathcal{M}$ and in the remaining delta function). This is a more useful way to express the Golden Rule, though it obscures the physical content.*

6.2.1.1 Two-particle Decays

In particular, if there are only two particles in the final state

$$\Gamma=\frac{S}{32\pi^2\hbar m_1}\int|\mathcal{M}|^2\frac{\delta^4(p_1-p_2-p_3)}{\sqrt{\mathbf{p}_2^2+m_2^2c^2}\sqrt{\mathbf{p}_3^2+m_3^2c^2}}\,\mathrm{d}^3\mathbf{p}_2\,\mathrm{d}^3\mathbf{p}_3 \tag{6.23}$$

The four-dimensional delta function is a product of temporal and spatial parts:

$$\delta^4(p_1-p_2-p_3)=\delta\left(p_1^0-p_2^0-p_3^0\right)\delta^3\left(\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_3\right) \tag{6.24}$$

But particle 1 is at rest, so $\mathbf{p}_1=\mathbf{0}$ and $p_1^0=m_1c$. Meanwhile, p_2^0 and p_3^0 have been replaced (Equation 6.22), so†

$$\Gamma=\frac{S}{32\pi^2\hbar m_1}\int|\mathcal{M}|^2\frac{\delta\left(m_1c-\sqrt{\mathbf{p}_2^2+m_2^2c^2}-\sqrt{\mathbf{p}_3^2+m_3^2c^2}\right)}{\sqrt{\mathbf{p}_2^2+m_2^2c^2}\sqrt{\mathbf{p}_3^2+m_3^2c^2}}$$
$$\times\,\delta^3\left(\mathbf{p}_2+\mathbf{p}_3\right)\mathrm{d}^3\mathbf{p}_2\,\mathrm{d}^3\mathbf{p}_3 \tag{6.25}$$

* You might recognize the quantity $\sqrt{\mathbf{p}_j^2+m_j^2c^2}$ as E_j/c, and many books write it this way. It's dangerous notation: $\mathbf{p}_j$ is an integration variable, so E_j is not some constant you can take outside the integral. Use it as shorthand, if you like, but remember that E_j is a function of $\mathbf{p}_j$, not an independent variable.

† We can drop the minus sign in the final delta function, since $\delta(-x)=\delta(x)$.

The $\mathbf{p}_3$ integral is now trivial: in view of the final delta function it simply makes the replacement

$$\mathbf{p}_3 \to -\mathbf{p}_2 \tag{6.26}$$

leaving

$$\Gamma = \frac{S}{32\pi^2 \hbar m_1} \int |\mathcal{M}|^2 \frac{\delta\left(m_1 c - \sqrt{\mathbf{p}_2^2 + m_2^2 c^2} - \sqrt{\mathbf{p}_2^2 + m_3^2 c^2}\right)}{\sqrt{\mathbf{p}_2^2 + m_2^2 c^2}\sqrt{\mathbf{p}_2^2 + m_3^2 c^2}} \, \mathrm{d}^3\mathbf{p}_2 \tag{6.27}$$

For the remaining integral we adopt spherical coordinates, $\mathbf{p}_2 \to (r,\ \theta,\ \phi)$, $\mathrm{d}^3\mathbf{p}_2 \to r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi$ (this is *momentum* space, of course: $r = |\mathbf{p}_2|$).

$$\Gamma = \frac{S}{32\pi^2 \hbar m_1} \int |\mathcal{M}|^2 \frac{\delta\left(m_1 c - \sqrt{r^2 + m_2^2 c^2} - \sqrt{r^2 + m_3^2 c^2}\right)}{\sqrt{r^2 + m_2^2 c^2}\sqrt{r^2 + m_3^2 c^2}}$$
$$\times\, r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi \tag{6.28}$$

Now, $\mathcal{M}$ was originally a function of the four-momenta p_1, p_2, and p_3, but $p_1 = (m_1 c, \mathbf{0})$ is a constant (as far as the integration is concerned), and the integrals already performed have made the replacements $p_2^0 \to \sqrt{\mathbf{p}_2^2 + m_2^2 c^2}$, $p_3^0 \to \sqrt{\mathbf{p}_3^2 + m_3^2 c^2}$, and $\mathbf{p}_3 \to -\mathbf{p}_2$, so by now $\mathcal{M}$ depends only on $\mathbf{p}_2$. As we shall see, however, amplitudes must be *scalars*, and the only scalar you can make out of a vector is the dot product with itself:* $\mathbf{p}_2 \cdot \mathbf{p}_2 = r^2$. At this stage, then, $\mathcal{M}$ is a function only of r (not of θ or ϕ). That being the case we can do the angular integrals

$$\int_0^{\pi} \sin\theta \, \mathrm{d}\theta = 2, \qquad \int_0^{2\pi} \mathrm{d}\phi = 2\pi \tag{6.29}$$

and there remains only the r integral:

$$\Gamma = \frac{S}{8\pi \hbar m_1} \int_0^{\infty} |\mathcal{M}(r)|^2 \frac{\delta\left(m_1 c - \sqrt{r^2 + m_2^2 c^2} - \sqrt{r^2 + m_3^2 c^2}\right)}{\sqrt{r^2 + m_2^2 c^2}\sqrt{r^2 + m_3^2 c^2}} r^2 \, \mathrm{d}r \tag{6.30}$$

To simplify the argument of the delta function, let

$$u \equiv \sqrt{r^2 + m_2^2 c^2} + \sqrt{r^2 + m_3^2 c^2} \tag{6.31}$$

* If the particles carry spin, then $\mathcal{M}$ might depend also on $(\mathbf{p}_i \cdot \mathbf{S}_j)$ and $(\mathbf{S}_i \cdot \mathbf{S}_j)$. However, since experiments rarely measure the spin orientation, we almost always work with the spin-averaged amplitude. In that case, and of course in the case of spin 0, the only vector in sight is $\mathbf{p}_2$ and the only scalar variable is $(\mathbf{p}_2)^2$.

so

$$\frac{du}{dr} = \frac{ur}{\sqrt{r^2 + m_2^2c^2}\sqrt{r^2 + m_3^2c^2}} \tag{6.32}$$

Then

$$\Gamma = \frac{S}{8\pi\hbar m_1}\int_{(m_2+m_3)c}^{\infty} |\mathcal{M}(r)|^2 \delta(m_1c - u)\frac{r}{u}\,du \tag{6.33}$$

The last integral sends* u to m_1c, and hence r to

$$r_0 = \frac{c}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2} \tag{6.34}$$

(Problem 6.5). Remember that r was short for the variable $|\mathbf{p}_2|$; r_0 is the *particular* value of $|\mathbf{p}_2|$ that is consistent with conservation of energy, and Equation 6.25 simply reproduces the result we obtained back in Chapter 3 (Problem 3.19). In more comprehensible notation, then,

$$\Gamma = \frac{S|\mathbf{p}|}{8\pi\hbar m_1^2c}|\mathcal{M}|^2 \tag{6.35}$$

where $|\mathbf{p}|$ is the magnitude of either outgoing momentum, given in terms of the three masses by Equation 6.34, and $\mathcal{M}$ is evaluated at the momenta dictated by the conservation laws. The various substitutions (Equations 6.22, 6.26, and 6.34) have systematically enforced these conservation laws – hardly a surprise, since they were built into the Golden Rule.

The two-body decay formula (Equation 6.35) is surprisingly simple; we were able to carry out all the integrals *without ever knowing the functional form of* $\mathcal{M}$. Mathematically, there were just enough delta functions to cover all the variables; physically, two-body decays are *kinematically determined*: the particles have to come out back-to-back with opposite three-momenta – the *direction* of this axis is not fixed, but since the initial state was symmetric, it doesn't matter. We will use Equation 6.35 frequently. Unfortunately, when there are three or more particles in the final state, the integrals cannot be done until we know the specific functional form of $\mathcal{M}$. In such cases (of which we shall encounter mercifully few), you have to go back to the Golden Rule and work it out from scratch.

6.2.2
Golden Rule for Scattering

Suppose particles 1 and 2 collide, producing particles 3, 4, ..., n:

$$1 + 2 \rightarrow 3 + 4 + \cdots + n \tag{6.36}$$

* This assumes $m_1 > (m_2 + m_3)$; otherwise the delta function spike is outside the domain of integration and we get $\Gamma = 0$, recording the fact that a particle cannot decay into heavier secondaries.

The scattering cross

$\sigma = \frac{}{4\sqrt{(p_1 \cdot p}}$

$\times \prod_{j=3}^{n} 2\pi$

where p_i is the fou
S is the same as be
as before: integrate
constraints (every co
positive, and energ
delta and theta fun
integrals:

$\sigma = \frac{}{4\sqrt{(p_1 \cdot p}}$

$\times \prod_{j=3}^{n} 2$

with

$p_j^0 = \sqrt{\mathbf{p}_j^2 +}$

wherever it occurs in

6.2.2.1 Two-body Sca
Consider the proces

$1 + 2 \rightarrow 3 +$

In the CM frame, $\mathbf{p}$

$\sqrt{(p_1 \cdot p_2)^2 -}$

(Problem 6.7). In th

$\sigma = \frac{S}{64\pi^2(E}$

The scattering cross section is given by the formula

$$\sigma = \frac{S\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 \cdots - p_n)$$
$$\times \prod_{j=3}^{n} 2\pi\delta\left(p_j^2 - m_j^2 c^2\right)\theta\left(p_j^0\right)\frac{d^4 p_j}{(2\pi)^4} \tag{6.37}$$

where p_i is the four-momentum of particle i (mass m_i) and the statistical factor (S) is the same as before (Equation 6.15). The phase space is essentially the same as before: integrate over all outgoing momenta, subject to the three kinematical constraints (every outgoing particle is on its mass shell, every outgoing energy is positive, and energy and momentum are conserved), which are enforced by the delta and theta functions. Once again, we can simplify matters by performing the p^0 integrals:

$$\sigma = \frac{S\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 \cdots - p_n)$$
$$\times \prod_{j=3}^{n} \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2 c^2}} \frac{d^3 \mathbf{p}_j}{(2\pi)^3} \tag{6.38}$$

with

$$p_j^0 = \sqrt{\mathbf{p}_j^2 + m_j^2 c^2} \tag{6.39}$$

wherever it occurs in $\mathcal{M}$ and the delta function.

6.2.2.1 Two-body Scattering in the CM Frame

Consider the process

$$1 + 2 \rightarrow 3 + 4 \tag{6.40}$$

In the CM frame, $\mathbf{p}_2 = -\mathbf{p}_1$ (Figure 6.5), where

$$\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2} = (E_1 + E_2)|\mathbf{p}_1|/c \tag{6.41}$$

(Problem 6.7). In this case, Equation 6.38 reduces to

$$\sigma = \frac{S\hbar^2 c}{64\pi^2 (E_1 + E_2)|\mathbf{p}_1|} \int |\mathcal{M}|^2 \frac{\delta^4(p_1 + p_2 - p_3 - p_4)}{\sqrt{\mathbf{p}_3^2 + m_3^2 c^2}\sqrt{\mathbf{p}_4^2 + m_4^2 c^2}} \, d^3\mathbf{p}_3 \, d^3\mathbf{p}_4 \tag{6.42}$$

The scattering cross section is given by the formula

$$\sigma = \frac{S\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 \cdots - p_n)$$
$$\times \prod_{j=3}^{n} 2\pi\,\delta\left(p_j^2 - m_j^2 c^2\right)\theta\left(p_j^0\right)\frac{d^4 p_j}{(2\pi)^4} \tag{6.37}$$

where p_i is the four-momentum of particle i (mass m_i) and the statistical factor (S) is the same as before (Equation 6.15). The phase space is essentially the same as before: integrate over all outgoing momenta, subject to the three kinematical constraints (every outgoing particle is on its mass shell, every outgoing energy is positive, and energy and momentum are conserved), which are enforced by the delta and theta functions. Once again, we can simplify matters by performing the p^0 integrals:

$$\sigma = \frac{S\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 \cdots - p_n)$$
$$\times \prod_{j=3}^{n} \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2 c^2}}\frac{d^3 \mathbf{p}_j}{(2\pi)^3} \tag{6.38}$$

with

$$p_j^0 = \sqrt{\mathbf{p}_j^2 + m_j^2 c^2} \tag{6.39}$$

wherever it occurs in $\mathcal{M}$ and the delta function.

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$$\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2} = (E_1 + E_2)|\mathbf{p}_1|/c \tag{6.41}$$

(Problem 6.7). In this case, Equation 6.38 reduces to

$$\sigma = \frac{S\hbar^2 c}{64\pi^2 (E_1 + E_2)|\mathbf{p}_1|}\int |\mathcal{M}|^2 \frac{\delta^4(p_1 + p_2 - p_3 - p_4)}{\sqrt{\mathbf{p}_3^2 + m_3^2 c^2}\sqrt{\mathbf{p}_4^2 + m_4^2 c^2}}\, d^3\mathbf{p}_3\, d^3\mathbf{p}_4 \tag{6.42}$$

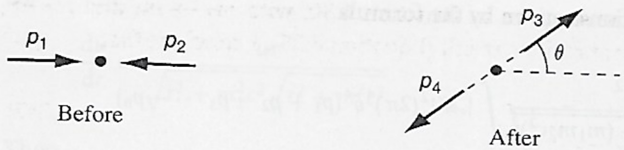


Fig. 6.5 Two-body scattering in the *CM* frame.

As before, we begin by rewriting the delta function:*

$$\delta^4(p_1 + p_2 - p_3 - p_4) = \delta\left(\frac{E_1 + E_2}{c} - p_3^0 - p_4^0\right)\delta^3(\mathbf{p}_3 + \mathbf{p}_4) \tag{6.43}$$

Next we insert Equation 6.39 and carry out the $\mathbf{p}_4$ integral (which sends $\mathbf{p}_4 \to -\mathbf{p}_3$):

$$\sigma = \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc}{(E_1 + E_2)|\mathbf{p}_1|}\int |\mathcal{M}|^2 \times \frac{\delta\left[(E_1 + E_2)/c - \sqrt{\mathbf{p}_3^2 + m_3^2c^2} - \sqrt{\mathbf{p}_3^2 + m_4^2c^2}\right]}{\sqrt{\mathbf{p}_3^2 + m_3^2c^2}\sqrt{\mathbf{p}_3^2 + m_4^2c^2}}\, \mathrm{d}^3\mathbf{p}_3 \tag{6.44}$$

This time, however, $|\mathcal{M}|^2$ depends on the *direction* of $\mathbf{p}_3$ as well as its magnitude,† so we cannot carry out the angular integration. But that's all right – we didn't really want σ in the first place; what we're after is $d\sigma/d\Omega$. Adopting spherical coordinates, as before,

$$\mathrm{d}^3\mathbf{p}_3 = r^2\, \mathrm{d}r\, \mathrm{d}\Omega \tag{6.45}$$

(where r is shorthand for $|\mathbf{p}_3|$ and $\mathrm{d}\Omega = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi$), we obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc}{(E_1 + E_2)|\mathbf{p}_1|}\int_0^\infty |\mathcal{M}|^2 \times \frac{\delta\left[(E_1 + E_2)/c - \sqrt{r^2 + m_3^2c^2} - \sqrt{r^2 + m_4^2c^2}\right]}{\sqrt{r^2 + m_3^2c^2}\sqrt{r^2 + m_4^2c^2}}\, r^2\, \mathrm{d}r \tag{6.46}$$

* Observe that $\mathbf{p}_1$ and $\mathbf{p}_2$ are *fixed* vectors (related by our choice of reference frame: $\mathbf{p}_2 = -\mathbf{p}_1$), but at this stage $\mathbf{p}_3$ and $\mathbf{p}_4$ are integration variables. It is only *after* the $\mathbf{p}_4$ integration that they are restricted ($\mathbf{p}_4 = -\mathbf{p}_3$), and after the $|\mathbf{p}_3|$ integration that they are determined by the scattering angle θ.

† In general, $|\mathcal{M}|^2$ depends on all four-momenta. However, in this case $\mathbf{p}_2 = -\mathbf{p}_1$ and $\mathbf{p}_4 = -\mathbf{p}_3$, so it remains a function only of $\mathbf{p}_1$ and $\mathbf{p}_3$ (assuming again that spin does not come into it). From these vectors we can construct three scalars: $\mathbf{p}_1 \cdot \mathbf{p}_1 = |\mathbf{p}_1|^2$, $\mathbf{p}_3 \cdot \mathbf{p}_3 = |\mathbf{p}_3|^2$, and $\mathbf{p}_1 \cdot \mathbf{p}_3 = |\mathbf{p}_1||\mathbf{p}_3|\cos\theta$. But $\mathbf{p}_1$ is fixed, so the only *integration* variables on which $|\mathcal{M}|^2$ can depend are $|\mathbf{p}_3|$ and θ.

The integral over r is the same as in Equation 6.30, with $m_2 \to m_4$ and $m_1 \to (E_1 + E_2)/c^2$. Quoting our previous result (Equation 6.35), I conclude that

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1 + E_2)^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|} \tag{6.47}$$

where $|\mathbf{p}_f|$ is the magnitude of either outgoing momentum and $|\mathbf{p}_i|$ is the magnitude of either incoming momentum.

As in the case of decays, the two-body final state is peculiarly simple, in the sense that we are able to carry the calculation through to the end *without knowing the explicit functional form of* $\mathcal{M}$. We will be using Equation 6.47 frequently in later chapters.

By the way, lifetimes obviously carry the dimensions of *time* (seconds); decay rates ($\Gamma = 1/\tau$), therefore, are measured in inverse seconds. Cross sections have dimensions of *area* – cm^2, or, more conveniently, 'barns':

$$1\ \text{b} = 10^{-24}\ \text{cm}^2 \tag{6.48}$$

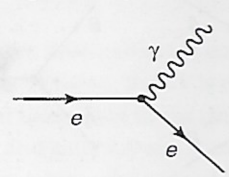
Differential cross sections, $d\sigma/d\Omega$, are given in barns per steradian or simply barns (steradians, like radians, being dimensionless). The amplitude, $\mathcal{M}$, has units that depend on the number of particles involved: if there are n external lines (incoming plus outgoing), the dimensions of $\mathcal{M}$ are those of momentum raised to the power $4 - n$:

$$\text{Dimensions of } \mathcal{M} = (mc)^{4-n} \tag{6.49}$$

For example, in a three-body process ($A \to B + C$), $\mathcal{M}$ has dimensions of momentum; in a four-body process ($A \to B + C + D$ or $A + B \to C + D$), $\mathcal{M}$ is dimensionless. You can check for yourself that the two Golden Rules then yield the correct units for Γ and σ.

6.3
Feynman Rules for a Toy Theory

In Section 6.2, we learned how to calculate decay rates and scattering cross sections, in terms of the amplitude $\mathcal{M}$ for the process in question. Now I'll show you how to determine $\mathcal{M}$ itself, using the 'Feynman rules' to evaluate the relevant diagrams. We could go straight to a 'real-life' system, such as quantum electrodynamics, with electrons and photons interacting via the primitive vertex:



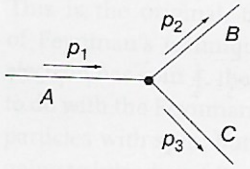


Fig. 6.7 Lowest-order contribution to $A \to B + C$.

5. *Integration over internal momenta*: For each internal line, write down a factor*

$$\frac{1}{(2\pi)^4}\,d^4 q_j$$

and integrate over all internal momenta.

6. *Cancel the delta function*: The result will include a delta function

$$(2\pi)^4 \delta^4(p_1 + p_2 + \cdots - p_n)$$

reflecting overall conservation of energy and momentum. Erase this factor† and multiply by i. The result is $\mathcal{M}$.

6.3.1
Lifetime of the A

The simplest possible diagram, representing the lowest-order contribution to $A \to B + C$, has no internal lines at all (Figure 6.7). There is one vertex, at which we pick up a factor of $-ig$ (Rule 2) and a delta function

$$(2\pi)^4 \delta^4(p_1 - p_2 - p_3)$$

(Rule 4), which we promptly discard (Rule 6). Multiplying by i, we get

$$\mathcal{M} = g \tag{6.50}$$

This is the *amplitude* (to lowest order); the decay rate is found by plugging $\mathcal{M}$ into Equation 6.35:

$$\Gamma = \frac{g^2 |\mathbf{p}|}{8\pi \hbar m_A^2 c} \tag{6.51}$$

* Notice (again) that every δ gets a factor of (2π) and every d gets a factor of $1/(2\pi)$.

† Of course, the Golden Rule immediately puts this factor back in Equations 6.15 and 6.37, and you might wonder why we don't just keep it in $\mathcal{M}$. The problem is that $|\mathcal{M}|^2$, not $\mathcal{M}$, comes into the Golden Rule and the *square* of a delta function is undefined. So you have to remove it here, even though you'll be putting it back at the next stage.

where $|\mathbf{p}|$ (the magn

$$|\mathbf{p}| = \frac{c}{2m_A}\sqrt{\cdots}$$

The lifetime of the A

$$\tau = \frac{1}{\Gamma} = \frac{8\pi \cdots}{g^2 \cdots}$$

You should check f

6.3.2
A + A → B + B Scat

The lowest-order co
In this case, there
with the propagator

$$\frac{i}{q^2 - m_C^2 c^2}$$

two delta functions

$$(2\pi)^4 \delta^4(p_1 - \cdots$$

and one integratio

$$\frac{1}{(2\pi)^4}\,d^4 q$$

Rules 1–5, then, yi

$$-i(2\pi)^4 g^2 \int \cdots$$

Doing the integral

$$-ig^2 \frac{\cdots}{(p_4 - p\cdots}$$

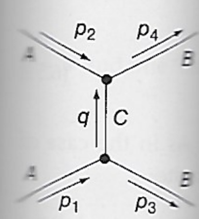


Fig. 6.8 Lowest-orde

where $|\mathbf{p}|$ (the magnitude of either outgoing momentum) is

$$|\mathbf{p}| = \frac{c}{2m_A}\sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2} \tag{6.52}$$

The lifetime of the A, then, is

$$\tau = \frac{1}{\Gamma} = \frac{8\pi\hbar m_A^2 c}{g^2|\mathbf{p}|} \tag{6.53}$$

You should check for yourself that τ comes out with the correct units.

6.3.2
$A + A \to B + B$ Scattering

The lowest-order contribution to the process $A + A \to B + B$ is shown in Figure 6.8. In this case, there are two vertices (hence two factors of $-ig$), one internal line, with the propagator

$$\frac{i}{q^2 - m_C^2 c^2}$$

two delta functions:

$$(2\pi)^4\delta^4(p_1 - p_3 - q) \quad and \quad (2\pi)^4\delta^4(p_2 + q - p_4)$$

and one integration:

$$\frac{1}{(2\pi)^4}\mathrm{d}^4 q$$

Rules 1–5, then, yield

$$-i(2\pi)^4 g^2 \int \frac{1}{q^2 - m_C^2 c^2}\,\delta^4(p_1 - p_3 - q)\,\delta^4(p_2 + q - p_4)\,\mathrm{d}^4 q$$

Doing the integral, the second delta function sends $q \to p_4 - p_2$, and we have

$$-ig^2 \frac{1}{(p_4 - p_2)^2 - m_C^2 c^2}(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4)$$

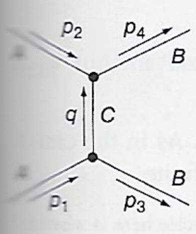


Fig. 6.8 Lowest-order contribution to $A + A \to B + B$.

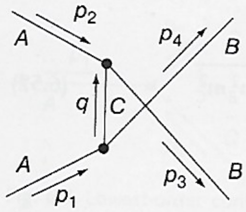


Fig. 6.9 Second diagram contributing in lowest order to $A + A \to B + B$.

As promised, there is one remaining delta function, reflecting overall conservation of energy and momentum. Erasing it and multiplying by i (Rule 6), we are left with

$$\mathcal{M} = \frac{g^2}{(p_4 - p_2)^2 - m_C^2c^2} \tag{6.54}$$

But that's not the whole story, for there is another diagram of order g^2, obtained by 'twisting' the B lines (Figure 6.9).* Since this differs from Figure 6.8 only by the interchange $p_3 \leftrightarrow p_4$, there is no need to compute it from scratch; quoting Equation 6.54, we can write down immediately the total amplitude (to order g^2) for the process $A + A \to B + B$:

$$\mathcal{M} = \frac{g^2}{(p_4 - p_2)^2 - m_C^2c^2} + \frac{g^2}{(p_3 - p_2)^2 - m_C^2c^2} \tag{6.55}$$

Notice, incidentally, that $\mathcal{M}$ is a Lorentz-invariant (scalar) quantity. This is *always* the case; it is built into the Feynman rules.

Suppose we are interested in the differential cross section $(\mathrm{d}\sigma/\mathrm{d}\Omega)$ for this process, in the CM system (Figure 6.10). Say, for simplicity, that $m_A = m_B = m$ and $m_C = 0$. Then

$$(p_4 - p_2)^2 - m_C^2c^2 = p_4^2 + p_2^2 - 2p_2 \cdot p_4 = -2\mathbf{p}^2(1 - \cos\theta) \tag{6.56}$$

$$(p_3 - p_2)^2 - m_C^2c^2 = p_3^2 + p_2^2 - 2p_3 \cdot p_2 = -2\mathbf{p}^2(1 + \cos\theta) \tag{6.57}$$

(where $\mathbf{p}$ is the incident momentum of particle 1), and hence

$$\mathcal{M} = -\frac{g^2}{\mathbf{p}^2 \sin^2\theta} \tag{6.58}$$

According to Equation 6.47, then,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2}\left(\frac{\hbar c g^2}{16\pi E\,\mathbf{p}^2 \sin^2\theta}\right)^2 \tag{6.59}$$

(there are two identical particles in the final state, so $S = 1/2$). As in the case of Rutherford scattering (Example 6.4), the *total* cross section is infinite.

* You don't get yet another new diagram by twisting the A lines; the only choice here is whether p_3 connects to p_1 or to p_2.

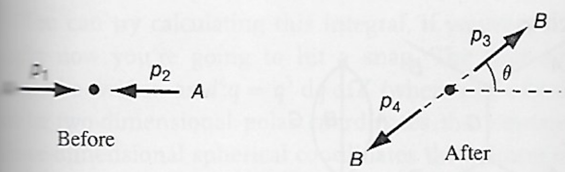
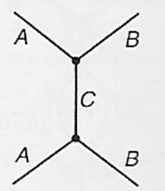


Fig. 6.10 $A + A \to B + B$ in the CM frame.

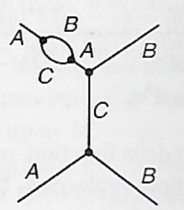
6.3.3 Higher-order Diagrams

So far we have looked only at lowest-order ('tree level') Feynman diagrams; in the case of $A + A \to B + B$, for instance, we considered the graph:

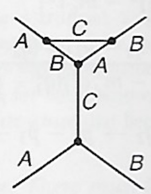


This diagram has two vertices, so $\mathscr{M}$ is proportional to g^2. But there are eight diagrams with *four* vertices (and eight more with the external B lines 'twisted'):

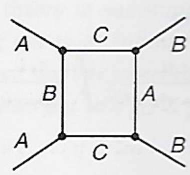
- five 'self-energy' diagrams, in which one of the lines sprouts a loop:



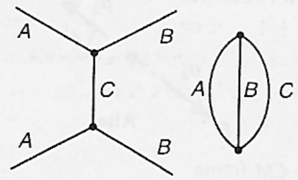
- two 'vertex corrections', in which a vertex becomes a triangle:



- and one 'box' diagram:

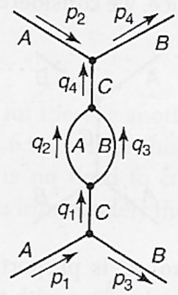


(Disconnected diagrams, such as



don't count.)

I am certainly not going to evaluate all these 'one-loop' diagrams (or even think about *two*-loop diagrams), but I would like to take a closer look at *one* of them – the one with a bubble on the virtual C line:



Applying Feynman rules 1–5, we obtain

$$g^4 \int \frac{\delta^4(p_1 - q_1 - p_3)\,\delta^4(q_1 - q_2 - q_3)\,\delta^4(q_2 + q_3 - q_4)\,\delta^4(q_4 + p_2 - p_4)}{(q_1^2 - m_C^2c^2)(q_2^2 - m_A^2c^2)(q_3^2 - m_B^2c^2)(q_4^2 - m_C^2c^2)} \times d^4q_1\, d^4q_2\, d^4q_3\, d^4q_4 \tag{6.60}$$

Integration over q_1, using the first delta function, replaces q_1 by $(p_1 - p_3)$; integration over q_4, using the last delta function, replaces q_4 by $(p_4 - p_2)$:

$$\frac{g^4}{[(p_1 - p_3)^2 - m_C^2c^2][(p_4 - p_2)^2 - m_C^2c^2]} \times \int \frac{\delta^4(p_1 - p_3 - q_2 - q_3)\,\delta^4(q_2 + q_3 - p_4 + p_2)}{(q_2^2 - m_A^2c^2)(q_3^2 - m_B^2c^2)}\, d^4q_2\, d^4q_3 \tag{6.61}$$

Here, the first delta function sends $q_2 \to p_1 - p_3 - q_3$, and the second delta function becomes

$$\delta^4(p_1 + p_2 - p_3 - p_4)$$

which, by Rule 6, we erase, leaving

$$\mathcal{M} = i\left(\frac{g}{2\pi}\right)^4 \frac{1}{[(p_1 - p_3)^2 - m_C^2c^2]^2} \int \frac{1}{[(p_1 - p_3 - q)^2 - m_A^2c^2](q^2 - m_B^2c^2)}\, d^4q \tag{6.62}$$

(I drop the subscript on q_3 at this point.)

You can try calculating this integral, if you've got the energy, but I'll tell you right now you're going to hit a snag. The four-dimensional volume element could be written as $d^4q = q^3\, dq\, d\Omega'$ (where $d\Omega'$ stands for the angular part), just as in two-dimensional polar coordinates the element of area is $r\, dr\, d\theta$ and in three-dimensional spherical coordinates the volume element is $r^2\, dr \sin\theta\, d\theta\, d\phi$. At large q the integrand is essentially just $1/q^4$, so the q integral has the form

$$\int^{\infty} \frac{1}{q^4} q^3\, dq = \ln q|^{\infty} = \infty \tag{6.63}$$

The integral is logarithmically divergent at large q. This disaster, in one form or another, held up the development of quantum electrodynamics for nearly two decades, until, through the combined efforts of many great physicists – from Dirac, Pauli, Kramers, Weisskopf, and Bethe through Tomonaga, Schwinger, and Feynman – systematic methods were developed for 'sweeping the infinities under the rug'. The first step is to *regularize* the integral, using a suitable cutoff procedure that renders it finite without spoiling other desirable features (such as Lorentz invariance). In the case of Equation 6.62, this can be accomplished by introducing a factor

$$\frac{-M^2c^2}{(q^2 - M^2c^2)} \tag{6.64}$$

under the integral sign. The *cutoff* mass M is assumed to be very large, and will be taken to infinity at the end of the calculation (note that the 'fudge factor', Equation 6.64, goes to 1 as $M \to \infty$).* The integral can now be calculated [6] and separated into two parts: a finite term, independent of M, and a term involving (in this case) the logarithm of M, which blows up as $M \to \infty$.

At this point, a miraculous thing happens: all the divergent, M-dependent terms appear in the final answer in the form of *additions to the masses and the coupling constant*. If we take this seriously, it means that the *physical* masses and couplings

* No one would deny that this procedure is artificial. Still, it can be argued that the inclusion of Equation 6.64 merely confesses our ignorance of the high-energy (short distance) behavior of quantum field theory. Perhaps the Feynman propagators are not quite right in this regime, and this is simply a crude way of accounting for the unknown modification. (This would be the case, for example, if the 'particles' have substructure that becomes relevant at extremely close range.) Dirac said, of renormalization,†

> *It's just a stop-gap procedure. There must be some fundamental change in our ideas, probably a change just as fundamental as the passage from Bohr's orbit theory to quantum mechanics. When you get a number turning out to be infinite which ought to be finite, you should admit that there is something wrong with your equations, and not hope that you can get a good theory just by doctoring up that number.*

† P. Buckley and F. D. Peat, *A Question of Physics* (Toronto: University of Toronto Press, 1979), page 39.