Proof of $\Gamma(1/2)$

The gamma function is defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

Making the substitution $x = u^2$ gives the equivalent expression

$$\Gamma(\alpha) = 2\int_0^\infty u^{2\alpha-1} e^{-u^2} du$$

A special value of the gamma function can be derived when $2\alpha - 1 = 0$ ($\alpha = \frac{1}{2}$). When $\alpha = \frac{1}{2}$, $\Gamma\left(\frac{1}{2}\right)$ simplifies as

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2} du$$

To derive the value for $\Gamma\left(\frac{1}{2}\right)$, the following steps are used. First, the value of $\Gamma\left(\frac{1}{2}\right)$ is squared. Second, the squared value is rewritten as a double integral. Third, the double integral is evaluated by transforming to polar coordinates. Fourth, the $\Gamma\left(\frac{1}{2}\right)$ is explicitly solved for.
First, square the value for $\Gamma\left(\frac{1}{2}\right)$ and rewrite as a double integral. Hence,

$$\begin{aligned}
\left[\Gamma\left(\frac{1}{2}\right)\right]^2 &= \left[\Gamma\left(\frac{1}{2}\right)\right]\left[\Gamma\left(\frac{1}{2}\right)\right] \\
&= \left[2\int_0^\infty e^{-u^2} du\right]\left[2\int_0^\infty e^{-v^2} dv\right] \\
&= 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)} dv du \qquad (1)
\end{aligned}$$

The region R which defines the first quadrant, is the region of integration for the integral in (1). The bivariate transformation $u = r\cos\theta$, $v = r\sin\theta$ will transform the integral problem from cartesian coordinates to polar coordinates, (r, θ). These new variables will range from $0 \leqslant r \leqslant \infty$ and $0 \leqslant \theta \leqslant \frac{\pi}{2}$ for the first quadrant. The Jacobian of the transformation is

$$|J| = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

Hence, (1) can be written as

$$\begin{aligned}
\left[\Gamma\left(\frac{1}{2}\right)\right]^2 &= 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)} dv du = 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r dr d\theta \\
&= 4\left[\int_0^{\frac{\pi}{2}} d\theta\right]\left[\underset{\substack{u=-r^2 \\ du=-2rdr}}{\int_0^\infty e^{-r^2} r dr}\right] = 4\left[\frac{\pi}{2}\right]\left[-\frac{1}{2}\int_0^{-\infty} e^u du\right] \\
&= -\pi\left[0 - 1\right] \\
&= \pi
\end{aligned}$$

Finally, since $e^{-u^2} > 0$ for all $u \geqslant 0$, then $\Gamma\left(\frac{1}{2}\right) \geqslant 0$. Hence,

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi \Longrightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$