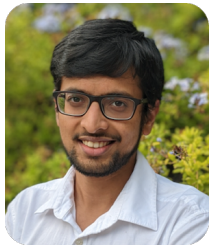
Regularization and Optimal Multiclass Learning

Julian Asilis Siddartha Devic Shaddin Dughmi



Vatsal Sharan Shang-Hua Teng




Context on Classification

Binary classification

Rules:

- Domain $\mathcal{X}$ (arbitrary)
- Label set $\mathcal{Y} = \{0, 1\}$
- Loss function $\ell_{0-1}(y, y') = 1[y \neq y']$

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When to learn? VC($\mathcal{H}$) $< \infty$ [BEHW89]

How to learn? ERM

- Extremely simple
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Learnability and the Vapnik-Chervonenkis Dimension

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When to learn? DS($\mathcal{H}$) $< \infty$ [BCDMY22]

How to learn? Not so clear…

- BCDMY learner is highly complex: subsampling, list PAC learning, sample compression, etc.

[BCDMY22] – Brukhim, Carmon, Dinur, Moran and Yehudayoff.
A Characterization of Multiclass Learnability

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Are there any simple algorithmic templates for optimal multiclass learning?

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Starting point: ERM & SRM

Empirical risk minimization (ERM)

$$A(S) = \operatorname{argmin}_{\mathcal{H}} L_S(h)$$

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Theorem [DS14]: In multiclass classification, there are learnable classes that cannot be learned by *any* proper learner.

Learning $\mathcal{H}$ can require emitting functions outside of $\mathcal{H}$.
(Even in realizable case!)

Dooms ERM & SRM – phrased as optimization problems over $\mathcal{H}$.

Starting point: ERM & SRM

Our motivating question:

What is the minimal augmentation of SRM that allows it to learn all (learnable) multiclass problems?

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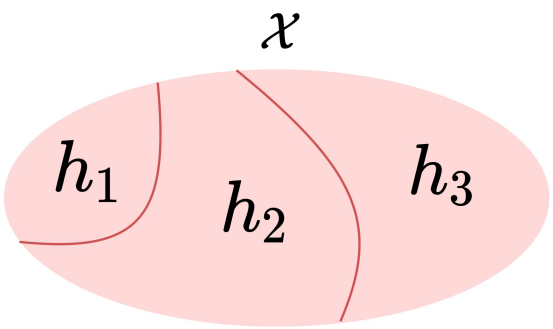

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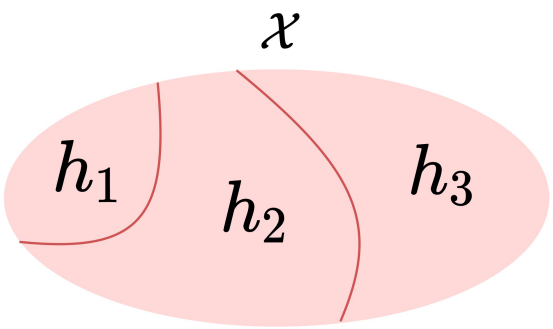

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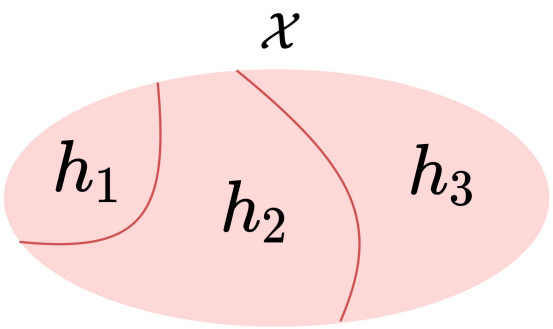

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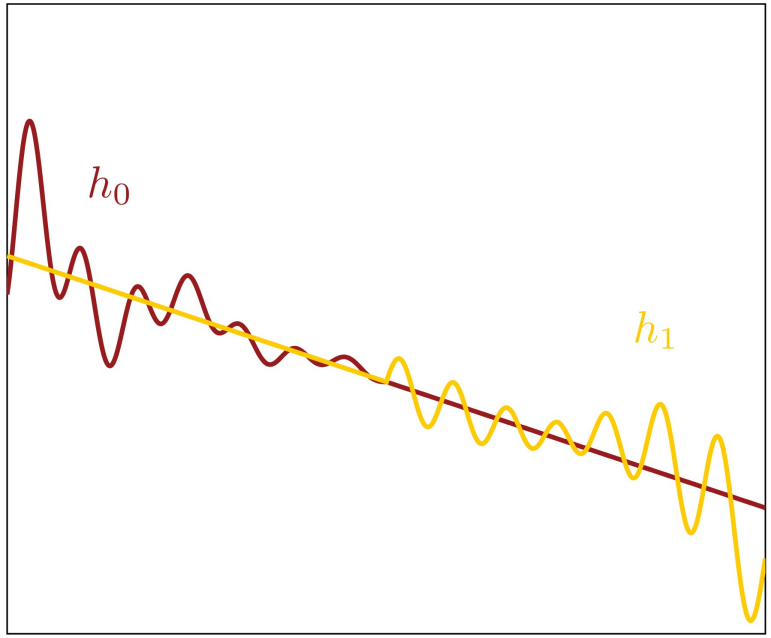

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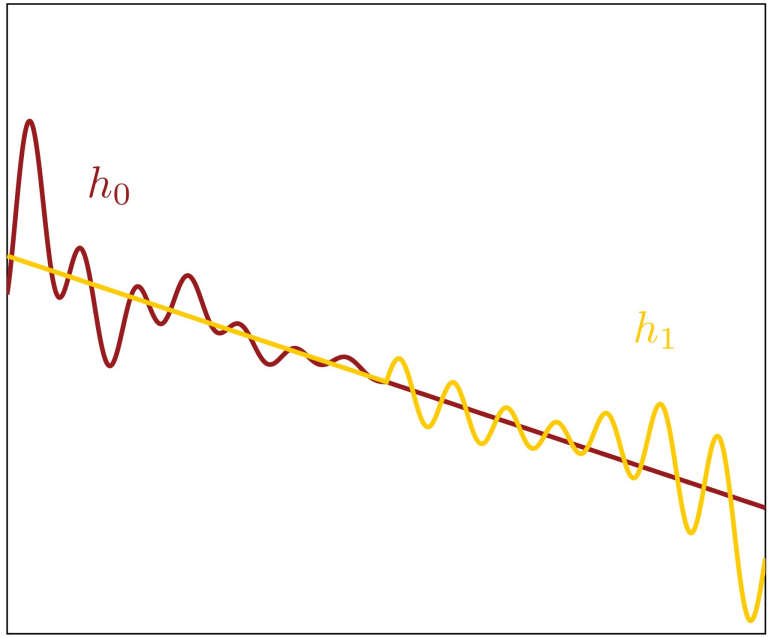

Theorem: Even local regularization fails on learnable multiclass problems.

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Relaxation 2: Unsupervised Learning of Regularizer

- Assigning hypothesis “complexity” may require more context than the test point
  - E.g., h is simple on $\{x_{test}, x_1, \dots, x_n\}$ but complex on $\{x_{test}, x_1', \dots, x_n'\}$

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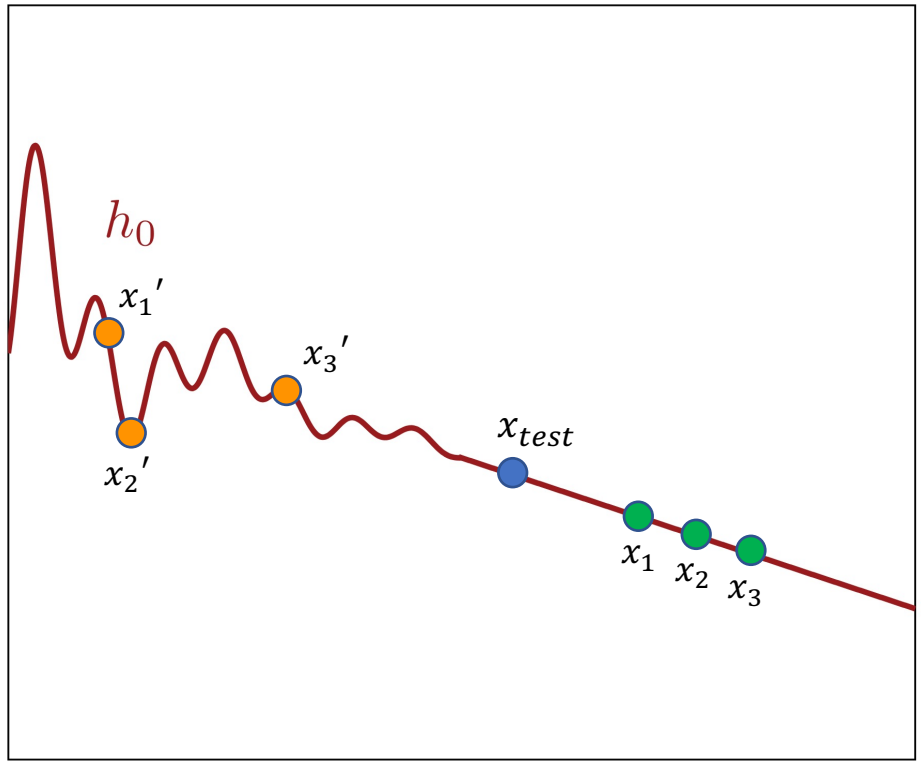

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- Given unlabeled data S_X, learn local regularizer $\psi : \mathcal{H} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$

- Use local regularizer as before:

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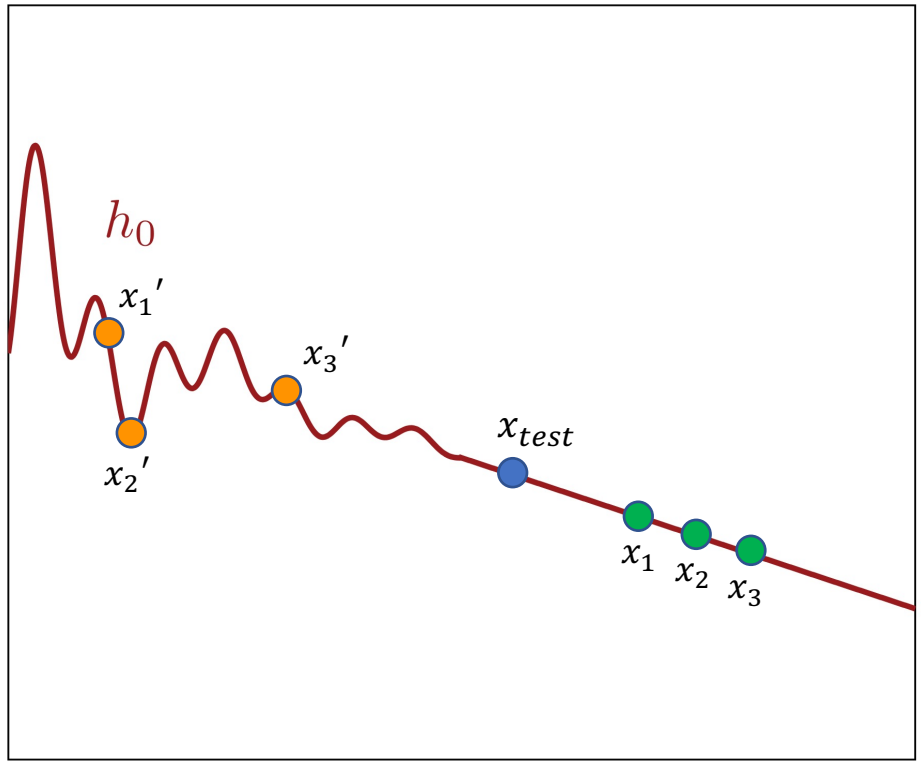

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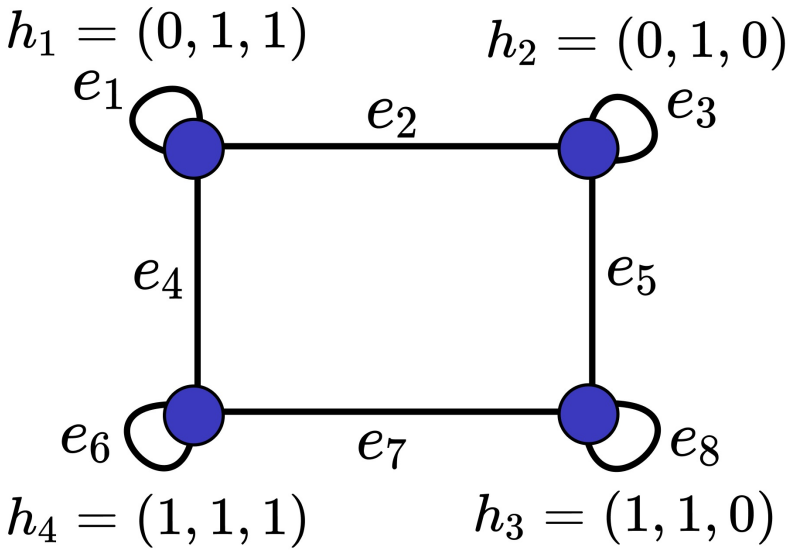
(Results stated for transductive model. Transfer directly to PAC in black-box manner, up to lower-order factors.)

One-inclusion graphs

Throughout, we model learning using the *one-inclusion graph* (OIG) of $\mathcal{H}$ on $S \in \mathcal{X}^n$:

- Vertex set $\mathcal{H}|_S$
- Hyperedges group hypotheses agreeing on $n-1$ points



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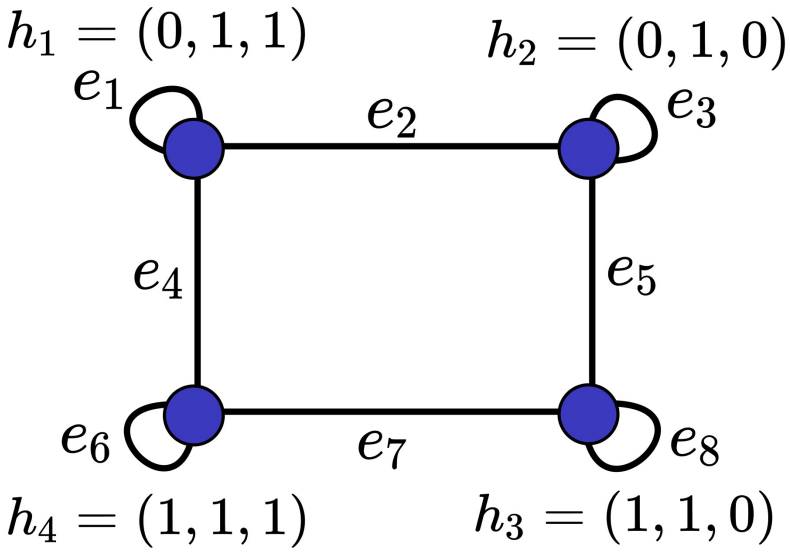
1. Learner ≡ **orientation** of edges
2. Good learner ⇔ low outdegrees



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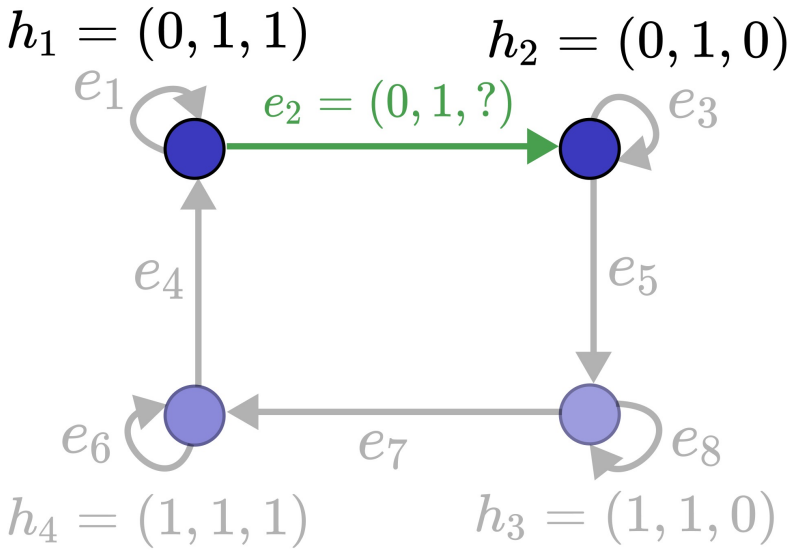

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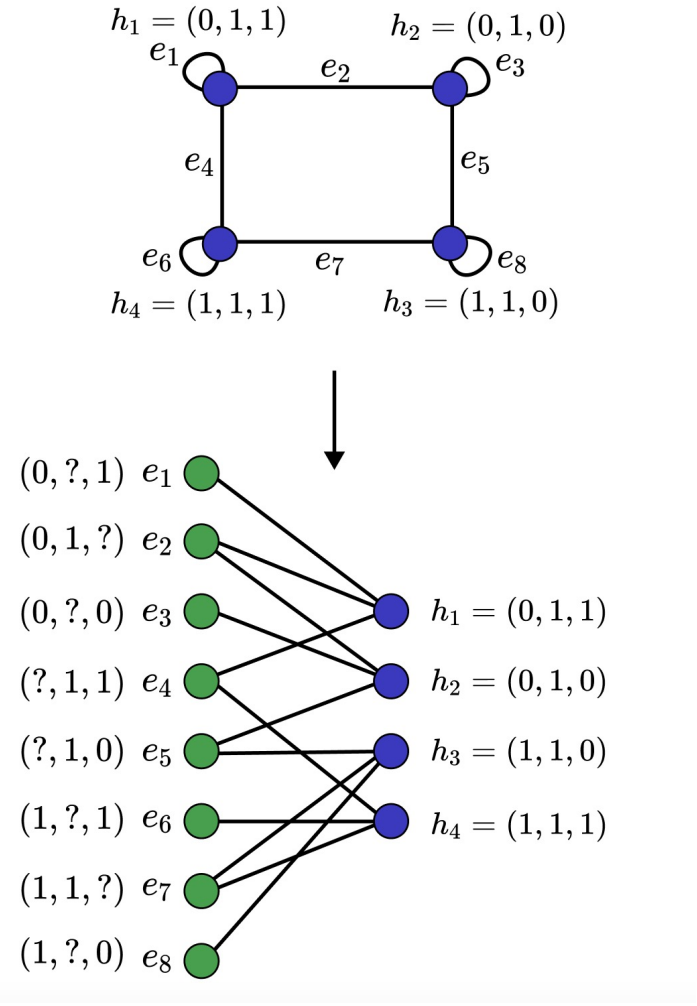

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Useful trick: pass to *bipartite version*

Good learner ≡ good matching/assignment in bipartite OIG

Glance at the Proofs

Theorem: Every realizable classification problem with at most countably many labels admits a near optimal (factor 2) local unsupervised SRM (deterministic).

Proof idea:

local unsupervised SRM $\equiv$ assignment of number to each node in the *one-inclusion graph* G, by which you direct edges

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Key technical result: OIGs have optimal acyclic orientations (up to factor 2)

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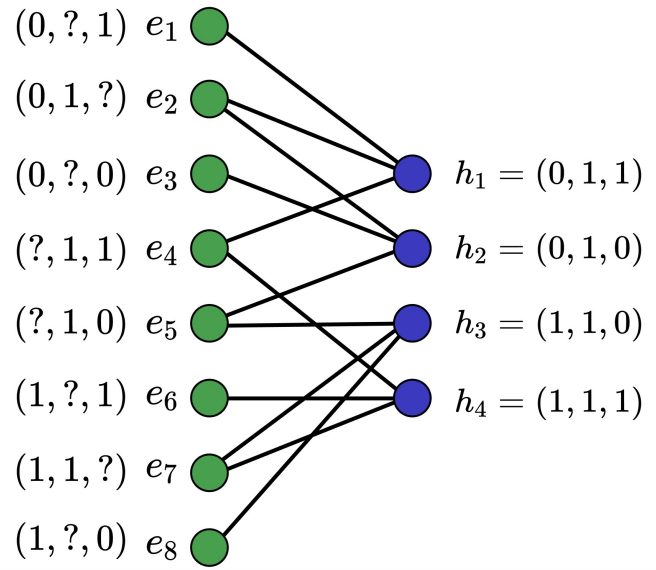
Consider maximum entropy program over matchings in bipartite OIG



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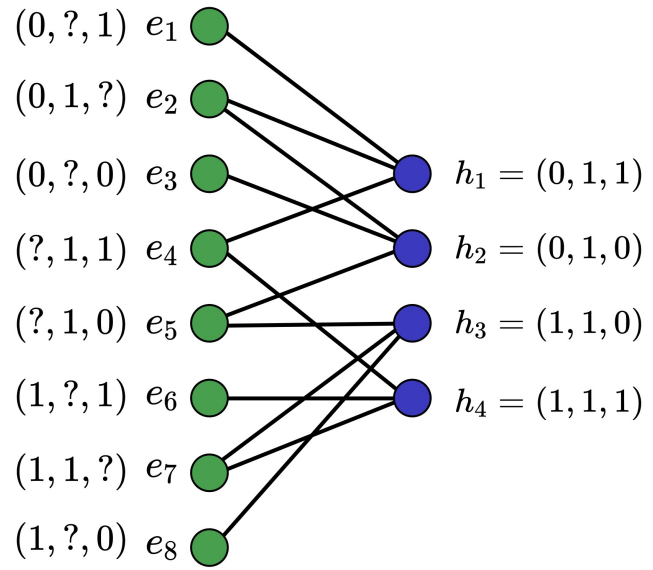

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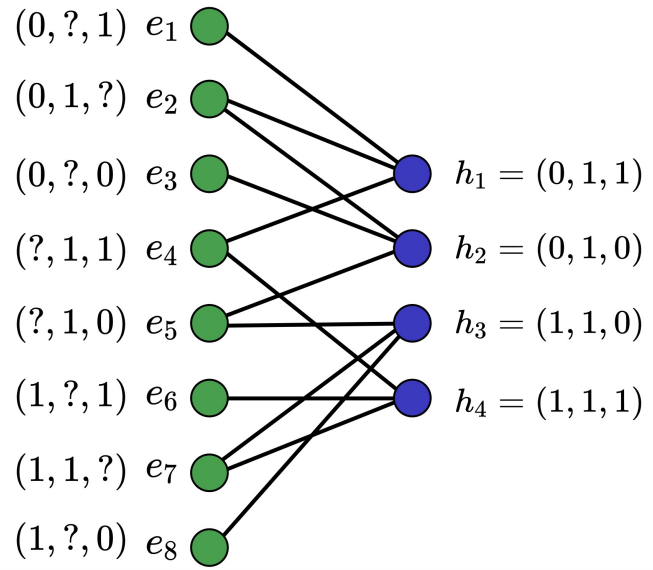
Fractions of dual variables yield *prior probabilities* for RHS nodes. Optimal learner has **Bayesian** structure.

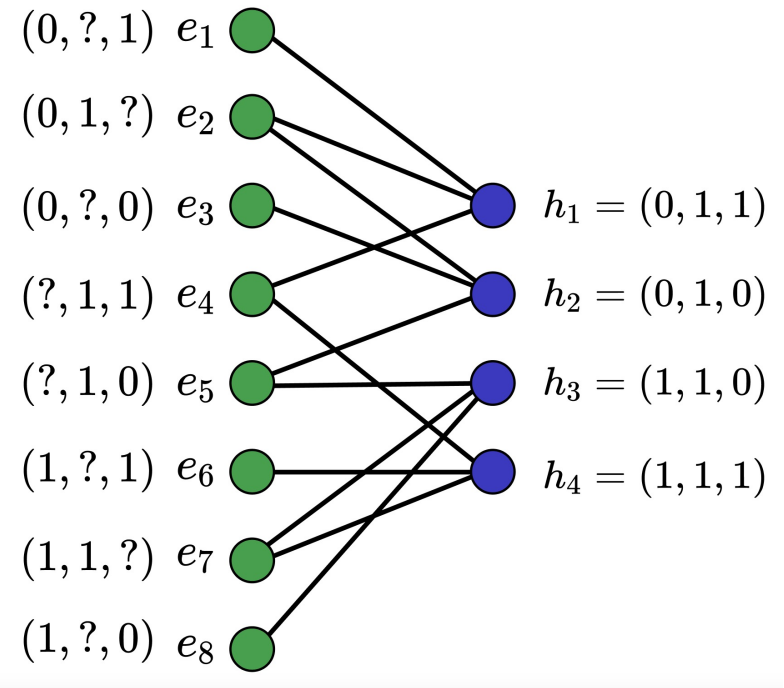


Companion Result: Hall Complexity

Theorem: The Hall complexity of a class $\mathcal{H}$ *exactly* characterizes its transductive error rate.

Hall complexity defined/analyzed by applying Hall's theorem to the bipartite OIGs.

Improves upon maximum subgraph density of [DS14], off by factor 2.



Conclusion

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- Concrete connections to Bayesian learning, maximum entropy principle

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Future work: See our Open Problem talk tomorrow!

Open Problem: Can Local Regularization Learn All Multiclass Problems?

Asilis, Julian; Devic, Siddartha; Dughm, Shaddin

Session: Open Problems 2, Tuesday, Jul 02, 10:00-10:30

Thank you!