MATH 8000 HOMEWORK 8

Due on Thursday, October 26

(1) Let D be the set of complex numbers of the form $m + n\sqrt{-3}$ where m and n are either both integers or both half-integers (halves of odd integers).
  (a) Check that D is a ring.
  (b) Show that D is a Euclidean domain with respect to the Euclidean function $\delta(m+n\sqrt{-3}) = m^2 + 3n^2$.

(2) Prove that if $f(x)$ is a monic polynomial with integer coefficients, then any rational root of $f(x)$ is an integer.

(3) Let F be a finite field with q elements. Prove that the number of irreducible monic quadratic polynomials over F equals $q(q-1)/2$, and the number of irreducible cubics is $q(q^2-1)/3$.

(4) Let F be a subfield of E and let $u \in E$ be algebraic over E of odd degree over F. Show that $F(u) = F(u^2)$.

(5) Let E/F be an algebraic (not necessarily finite) field extension: this means that every $x \in E$ is algebraic over F. Show that any subring of E containing F is a field.

(6) Let $E = F(u)$ where u is transcendental over F. Let $K \neq F$ be a subfield of E containing F. Show that E is algebraic over K.

(7) Let F be a subfield of E and let $u, v \in E$ be algebraic over F. Suppose that the degrees of the minimum polynomials of u and v oven F are relatively prime. Show that the minimum polynomial of v is irreducible over $F(u)$.

(8) Show that if E is a splitting field of $f(x)$ over F, and if $f(x)$ has degree n, then $[E : F] \leq n!$.