MATH 8000 HOMEWORK 4

Due on Thursday, September 14

(1) An element of order two is called an involution. Show that if $u, v \in G$ are two distinct involutions in a finite group G, then the subgroup generated by u and v is isomorphic to a dihedral group.

(2) For each $i = 1, 2, \ldots, n$, let G_i be a group and H_i be a subgroup of G_i. Let $H = \prod_{i=1}^n H_i$ and let $G = \prod_{i=1}^n G_i$.
  (a) Prove that $H \lhd G$ if and only if $H_i \lhd G_i$ for each i.
  (b) If $H \lhd G$, then show that $G/H \cong \prod_{i=1}^n G_i/H_i$.

(3) Let $1 \to H \xrightarrow{\varphi} G \xrightarrow{\psi} K \to 1$ be a short exact sequence of groups. A homomorphism $\sigma \colon K \to G$ is called a *splitting* if $\psi \circ \sigma = \mathrm{id}_K$. (If such a homomorphism exists, then the short exact sequence is said to *split*.)
  (a) Show that G is isomorphic to a semidirect product of H and K if and only if the above short exact sequence splits. Explicitly write down the corresponding homomorphism $\alpha \colon K \to \mathrm{Aut}(H)$.
  (b) Show that for any $m, n \in \mathbb{N}$, there is a short exact sequence

$$1 \to \mathbb{Z}/m \to \mathbb{Z}/mn \to \mathbb{Z}/n \to 1,$$

  which splits if and only if $\gcd(m, n) = 1$.

(4) Recall that $SL_n(\mathbb{C}) = \{M \in GL_n(\mathbb{C}) \mid \det(M) = 1\}$.
  (a) Show that there is a short exact sequence

$$1 \to SL_n(\mathbb{C}) \to GL_n(\mathbb{C}) \to \mathbb{C}^\times \to 1,$$

  where $\mathbb{C}^\times$ is the multiplicative group of complex numbers.
  (b) Show that this short exact sequence is split.

(5) Let $K = \mathbb{Z}/7$ be the cyclic group of order 7, generated by $x \in K$.
  (a) Show that K has an automorphism of order 3. (That is, if composed three times, it gives the identity automorphism.)
  (b) Use the previous part to construct (with proof) a nonabelian group of order 21.

(6) Suppose we have a diagram as follows, where the two horizontal lines are short exact sequences, and each square commutes.

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & H \arrow[r, "\varphi"] \arrow[d, "a"] & G \arrow[r, "\psi"] \arrow[d, "b"] & K \arrow[r] \arrow[d, "c"] & 1 \arrow[d] \\
1 \arrow[r] & H' \arrow[r, "\varphi'"] & G' \arrow[r, "\psi'"] & K' \arrow[r] & 1
\end{tikzcd}$$

Show that if a and c are both isomorphisms, then so is b.